Optimality Criteria without Constraint Qualifications for Linear Semidefinite Problems *

Kostyukova O.I.† Tchemisova T.V.‡

Abstract

We consider two closely related optimization problems: a problem of convex Semi-Infinite Programming with multidimensional index set and a linear problem of Semidefinite Programming. In study of these problems we apply the approach suggested in our recent paper [14] and based on the notions of immobile indices and their immobility orders. For the linear semidefinite problem, we define the *subspace of immobile indices* and formulate the first order optimality conditions in terms of a basic matrix of this subspace. These conditions are explicit, do not use constraint qualifications, and have the form of criterion. An algorithm determining a basis of the subspace of immobile indices in a finite number of steps is suggested. The optimality conditions obtained are compared with other known optimality conditions.

Key words. Semi-Infinite Programming (SIP), Semidefinite Programming (SDP), subspace of immobile indices, Constraint Qualification (CQ), optimality conditions.

AMS subject classification. 90C25, 90C30, 90C34

1 Introduction

In this paper, we are concerned with convex problems of infinite optimization, namely a problem of convex Semi-Infinite Programming (SIP) and a linear problem of Semidefinite Programming (SDP), that are closely related [20].
Semi-Infinite Programming deals with extremal problems that consist in minimization of an objective function of finitely many variables in a set described by an infinite system of constraints. SIP models appear in different fields of modern science and engineering where it is necessary to simulate a behavior of complex processes whose models contain at least one inequality constraint for each value of some parameter (for example, time) varying in a given compact domain (for references see [9], [10] et al).
In SDP, an objective function is minimized under the condition that some matrix valued function is positive semidefinite. When the objective function is convex and the matrix valued

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†Institute of Mathematics, National Academy of Sciences of Belarus, Surganov str. 11, 220072, Minsk, Belarus (kostyukova@im.bas-net.by).

‡Mathematical Department, University of Aveiro, Campus Universitario Santiago, 3810-193, Aveiro, Portugal (tatiana@mat.ua.pt).

function is an affine combination of some symmetric matrices, we get a convex problem. There are many applications of SDP models to combinatorial optimization, control theory, approximation theory, etc. (see [22], [23] et al).
In some cases, a semi-infinite optimization problem can be reduced to a semidefinite problem (see examples in [20]). In turn, any SDP problem can be written in the form of some SIP problem. Therefore, it is natural to expect that similar approaches can be used in study of these two classes of optimization problems.
Optimality conditions for SIP and SDP are of special interest both from theoretical and practical points of view. In literature, a special attention is devoted to the results that do not need additional conditions on the constraints, so called constraint qualifications (CQ). For SIP such optimality conditions were proposed, for example, in [4, 11, 14], and for SDP in [8, 17, 18], and some other papers.

Let us consider an optimization problem in the form

$$\min\ c(x) \quad \text{s.t. } x \in X := \{x \in \mathbb{R}^n : f(x,t) \leq 0,\ \forall t \in T\}, \tag{1}$$

where $T \subset \mathbb{R}^s$ is a given index set; $c(x), f(x,t)$ are given functions with $x \in \mathbb{R}^n$, $t \in T$. This problem is a Nonlinear Programming (NLP) problem when the index set T is finite, and is a SIP problem otherwise.
It was noticed by many authors that CQ may fail for problem (1) in the presence of the constraints that vanish for any feasible solution (see [2, 7, 8, 15, 21] et al.). In different papers these constraints are referred to differently. For example, for NLP problems with finite set T, they are called "always binding constraints" in [2], and *implicit equality constraints* in [8], while in [21] such constraints are said to form the *equality set of constraints*. In [7], where convex semi-infinite systems in the form of the constraints of problem (1) are studied, the indices of the inequalities that vanish for all feasible values of the decision variable, are called *carrier indices*.

In [13], the set of indices of always vanishing constraints was considered for linear SIP problems with one - dimensional index set. In this preprint that was published in Russian, such indices were called *immobile* and the notion of the *immobile order* was introduced for them (in [12], this term is translated from Russian as *motionless degree*). The immobile indices were used to obtain optimality conditions for linear SIP and an algorithm determining these indices together with their immobility orders was suggested. Later, in [14, 15], this approach was generalized to convex SIP problems.
To avoid confusion, further in this paper we will use the following definition.

Definition 1 *An index $t \in T$ is called immobile w.r.t. constraints of problem (1) if $f(x,t) = 0$ for all feasible $x \in X$.*

Given problem (1), it must be noticed that the constraints whose indices are immobile, are not only differently named, but (that is more important) are used for different approaches to solving this problem. Thus in [2] and [21], a procedure of regularization of the convex problem in the form (1) with a finite index set T was suggested. It was shown how the set of immobile indices can be used to reduce this problem to an equivalent one satisfying the Slater CQ, and the algorithms determining this set were suggested. The same approach was used in [6] for the case of problem (1) with infinite index set T. In characterizing the optimality, the authors used the cone of directions of constancy along the "equality set" of constraints. The results obtained

for infinite problems are less constructive, and can be considered mostly as a general abstract conceptual idea rather than concrete optimality conditions easy to be verified.
In [7], the immobile (carrier) indices were used in study of the facial geometry of convex semi-infinite systems and the topological properties of their solution sets, in particular, for characterization of their interior and relative interior. It was proved that in the presence of the carrier indices the Slater condition is not satisfied, and was showed how these indices can be used for linear representation of the convex systems.
In [14], the convex infinite problem (1) with a compact index set $T \subset \mathbb{R}$ was considered and the immobile indices were used to formulate a new approach to optimality conditions for SIP. In this approach, the immobility orders of the immobile indices were used as well, and the new optimality conditions (implicit and explicit) were formulated in terms of the Lagrange multipliers. Notice here that the importance of such optimality conditions have been emphasized in [8].

This paper is the first attempt to generalize the results from [14] to the case of the convex SIP problem with a multidimensional index set $T \subset \mathbb{R}^s$, $s \geq 2$. One simple example of such problem is a linear SDP problem. For the latest problem we will introduce a subspace of immobile indices, using the notion of multidimensional immobile indices in convex SIP. The subspace of immobile indices will permit us to formulate and prove new optimality conditions for linear SDP in the form of criterion.
There exists a large number of papers on linear SDP, but the results from [8, 17, 18] approximate more closely to the aims of our investigation. In the papers mentioned above, so called *no gap dual problems* are formulated for linear SDP problems not satisfying the Slater condition. In [17], a dual problem called an *extended Lagrange-Slater dual problem* (ELSD), is built. The constraints of this problem include recursive systems of additional constraints. In [18], a problem named *regularized dual problem* (DRP) is considered. It is obtained as a dual to the *regularized primal problem* denoted there by (PR). In [18], it is proved that the dual problems (ELSD) and (DRP), obtained using different approaches, are equivalent and the strong duality property is guaranteed for both of them. These gap free dual problems permit to formulate new optimality conditions for linear SDP using the minimal cone of problem (PR) and its polar.
Here we will show that the optimality conditions obtained in this paper using the approach based on immobile indices are equivalent to the conditions formulated in [17, 18], nevertheless they have different forms.
The paper is organized as follows. In section 2, we formulate a convex SIP problem with the multidimensional index set, define the set of immobile indices, and prove that this set is empty if and only if the Slater type CQs are satisfied. In section 3, we consider a linear SDP problem and the equivalent convex SIP problem, define the subspace of immobile indices, and study its properties. Optimality conditions are proved and a simple example is discussed. The comparison of these optimality conditions with the conditions proposed in [8, 17, 18], is carried out in section 4. Additionally, we obtain the explicit expressions in terms of the subspace of immobile indices for certain constructions (such as minimal cone, its polar and others) used in [17, 18] for formulation of dual SDP problems. In section 5, we describe and justify an algorithm that determines a basis of the subspace of immobile indices. The final section 6 contains some conclusions and remarks.

2 Convex semi-infinite programming problem: immobile indices and constraint qualifications

Consider a convex Semi-Infinite Programming problem in the form

$$(P) \qquad \min_{x\in\mathbb{R}^n} c(x) \tag{2}$$
$$\text{s.t.} \quad x \in Q, \;\; f(x,t) \le 0 \;\; \forall\, t \in T, \tag{3}$$

where $Q \subset \mathbb{R}^n$ is a convex set, $c(x), x \in Q$, is a convex function, the function $f(x,t), x \in Q, t \in T$, is convex w.r.t. x; $T \subset \mathbb{R}^s$ is a compact index set. Denote by X the feasible set of problem (2), (3):

$$X = \{x \in Q : \; f(x,t) \le 0, \forall t \in T\}.$$

Consider the following CQs for constraints (3):

- The constraints (3) satisfy *the Slater type condition I* if

$$\text{there exists } \bar{x} \in Q \text{ such that } f(\bar{x},t) < 0, \forall t \in T. \tag{4}$$

- The constraints (3) satisfy *the Slater type condition II* if

$$\begin{array}{l}\text{for any index set } \{t_i \in T, \; i = 1,\dots,n+1\} \text{ there exists} \\ \text{a vector } \tilde{x} \in Q \text{ such that } f(\tilde{x},t_i) < 0, \; i = 1,\dots,n+1.\end{array} \tag{5}$$

Let

$$\tau_1,\dots,\tau_n \in T \tag{6}$$

be a set of indices from T. Consider the nonlinear problem

$$(P_D) \qquad \min_{x\in Q} c(x) \tag{7}$$
$$\text{s.t.} \;\; f(x,\tau_i) \le 0, \;\; i = 1,...,n.$$

Under the Slater type CQs, the following proposition can be formulated for the optimal solutions of problems (P) and (P_D).

Proposition 1 *Consider the convex SIP problem (2), (3). Suppose that $X \neq \emptyset$ and let constraints (3) satisfy the Slater type condition II. Then there exist indices (6) such that*

$$val(P) = val(P_D), \tag{8}$$

where $val(P)$ and $val(P_D)$ are the optimal values of the objective functions in problems (2), (3) and (7) respectively.

To prove Proposition 1, one has to repeat the proof of Proposition 5.105 from [5] replacing the condition $x \in \mathbb{R}^n$ by the following one: $x \in Q$ with convex $Q \subset \mathbb{R}^n$.

According to Definition 1, an index $t \in T$ is called *immobile* w.r.t. constraints (3) if $f(x,t) = 0$ for all feasible $x \in X$. Denote by T^* the set of immobile indices in problem (2), (3):

$$T^* := \{t \in T : \; f(x,t) = 0 \; \forall x \in X\}.$$

Proposition 2 *Let $X \neq \emptyset$ in the convex SIP problem (2), (3). Then the following conditions are equivalent:*
$CQ1$*: The constraints (3) satisfy the Slater type condition I.*
$CQ2$*: The constraints (3) satisfy the Slater type condition II.*
$CQ3$*: The set of immobile indices in problem (2), (3) is empty: $T^* = \emptyset$.*

Proof. Suppose that $X \neq \emptyset$, and note that $CQ1 \Rightarrow CQ3$ and $CQ1 \Rightarrow CQ2$.

1. Let us prove now that $CQ2 \Rightarrow CQ1$. Suppose that constraints (3) satisfy the condition $CQ2$ and do not satisfy the condition $CQ1$. Consider an auxiliary SIP problem

$$(P^{aux}) \qquad \begin{array}{rl} & \min\limits_{(x,x_{n+1})\in\mathbb{R}^{n+1}} x_{n+1} \\ \text{s.t.} & f(x,t) - x_{n+1} \leq 0 \ \forall t \in T \subset \mathbb{R}^s, \\ & x \in Q, \ -2 \leq x_{n+1} \leq 2. \end{array} \tag{9}$$

   Since $X \neq \emptyset$, there exists $\hat{x} \in X$ and evidently, the vector $(\hat{x}, x_{n+1})$ with $x_{n+1} = 0$, is a feasible solution for problem (P^{aux}). The objective function in problem (P^{aux}) is limited and all the functions defining this problem are convex w.r.t. x. Therefore problem (P^{aux}) admits the optimal value $val(P^{aux})$ of the objective function and $-2 \leq val(P^{aux}) \leq 0$. Suppose that $val(P^{aux}) < 0$. In this case, there exists a vector (x^*, x^*_{n+1}) such that $x^* \in Q$, $x^*_{n+1} < 0$, and $f(x^*,t) - x^*_{n+1} \leq 0$ for any $t \in T$. Hence $f(x^*,t) < 0$, $t \in T$, that contradicts the hypothesis that the condition $CQ1$ is not satisfied. Thus $val(P^{aux}) = 0$.

   It is easy to verify that the vector $(\hat{x}, \hat{x}_{n+1})$ with $\hat{x}_{n+1} = 1$, is feasible in problem (P^{aux}) when $\hat{x} \in X$. By construction, for any $t \in T$ we have $f(\hat{x},t) - \hat{x}_{n+1} < f(\hat{x},t) \leq 0$ and $-2 < \hat{x}_{n+1} < 2$. Therefore the convex problem (P^{aux}) satisfies the Slater type condition I and hence the Slater type condition II is satisfied for it as well. Then by Proposition 1 applied to problem (P^{aux}), there exists a set of points $\{\bar{t}_1, \ldots, \bar{t}_{n+1}\} \subset T$ such that the optimal value $val(P^{aux})$ of problem (P^{aux}) is equal to the optimal value $val(P^{aux}_D)$ of its discretization (P^{aux}_D) given by

$$(P^{aux}_D) \qquad \begin{array}{rl} & \min\limits_{(x,x_{n+1})\in\mathbb{R}^{n+1}} x_{n+1} \\ \text{s.t.} & f(x,\bar{t}_i) - x_{n+1} \leq 0, \ i = 1, ..., n+1, \\ & x \in Q, \ -2 \leq x_{n+1} \leq 2. \end{array}$$

   Hence we get $val(P^{aux}) = val(P^{aux}_D) = 0$.

   On the other hand, from the assumption that the condition $CQ2$ is satisfied for constraints (3), we conclude that for the index set $\{\bar{t}_1, \ldots, \bar{t}_{n+1}\}$ there exists the vector $(\tilde{x}, \tilde{x}_{n+1}) \in \mathbb{R}^{n+1}$ such that $\tilde{x} \in Q$, $f(\tilde{x}, \bar{t}_i) - \tilde{x}_{n+1} \leq 0$, $i = 1, \ldots, n+1$, and $-2 < \tilde{x}_{n+1} < 0$. Then $(\tilde{x}, \tilde{x}_{n+1})$ is feasible in (P^{aux}_D) and the corresponding value of the objective function is equal to $\tilde{x}_{n+1} < 0$, which contradicts the equality proved above: $val(P^{aux}_D) = 0$. From the contradiction obtained we conclude that the condition $CQ1$ is satisfied for problem (2), (3).

2. Finally, let us prove that $CQ3 \Rightarrow CQ2$. Suppose that constraints (3) satisfy the condition $CQ3$ and do not satisfy the condition $CQ2$. Let $\{t_1, \ldots, t_{n+1}\} \subset T$ be a set of indices that do not satisfy (5).

From the condition $CQ3$ it follows that for every $t_i \in \{t_1, \ldots, t_{n+1}\}$ there exists $x^{(i)} \in X$ such that $f(x^{(i)}, t_i) < 0$. Hence the following relations take place:

$$f(x^{(i)}, t_i) < 0, \ f(x^{(i)}, t_j) \leq 0, j = 1, ..., n+1, \ j \neq i, \ i = 1, ..., n+1. \tag{10}$$

Let $\tilde{x} := \dfrac{1}{n+1} \sum_{i=1}^{n+1} x^{(i)}$. Taking into account the convexity of the set Q, the convexity w.r.t x of the function $f(x,t)$, and relations (10), we obtain

$$\tilde{x} \in Q, \ f(\tilde{x}, t_i) = f(\frac{1}{n+1} \sum_{i=1}^{n+1} x^{(i)}, t_i) \leq \frac{1}{n+1} \sum_{i=1}^{n+1} f(x^{(i)}, t_i) < 0, \ i = 1, \ldots, n+1.$$

The contradiction obtained proves that the condition $CQ2$ is satisfied for constraints (3) whenever the condition $CQ3$ is valid.

The chain of the implications $CQ1 \Rightarrow CQ3$, $CQ3 \Rightarrow CQ2$, and $CQ2 \Rightarrow CQ1$ proves the Proposition. ■

Remark 1 *Notice that in [7] one can find another proof of the equivalence $CQ1 \Leftrightarrow CQ3$.*

3 Linear semi-definite programming problem

3.1 Equivalent formulation of linear SDP problem. Subspace of immobile indices, its representation and properties

Here and in what follows, we use the following notations: given integers k and p, $\mathbb{R}^{k \times p}$ denotes the set of all $k \times p$ matrices, $\mathcal{S}(k)$ stands for the space of symmetric $k \times k$ matrices, and $\mathcal{P}(k)$ for the cone of positive semidefinite symmetric $k \times k$ matrices.
Given $A \in \mathcal{S}(k)$, we write $A \succ 0$ ($A \succeq 0$) if A is positive definite (positive semi-definite), and $A \prec 0$ ($A \preceq 0$) if A is negative definite (negative semi-definite).
The space $\mathcal{S}(k)$ is considered here as a vector space with the trace inner product

$$A \bullet B := trace(AB) \quad \text{for } A, B \in \mathcal{S}(k),$$

where AB is the conventional matrix product.

Suppose that $s > 1, \ s \in \mathbb{N}$. Consider a linear SDP problem in the form

$$\min_{x \in Q} c^T x \quad \text{s.t. } \mathcal{A}(x) \preceq 0, \tag{11}$$

where $x = (x_1, ..., x_n)^T$ is $n-$vector, Q is a polyhedron defined as

$$Q := \{x \in \mathbb{R}^n : \ h_i^T x = b_i, \ i \in I_1, \ h_i^T x \leq b_i, \ i \in I_2\}, \tag{12}$$

and $\mathcal{A}(x)$ is the matrix valued function

$$\mathcal{A}(x) := \sum_{i=1}^{n} A_i x_i + A_0, \tag{13}$$

matrices $A_i \in \mathcal{S}(s), i = 0, 1, \dots, n$, vectors $c \in \mathbb{R}^n$, $h_i \in \mathbb{R}^n$, and numbers b_i, $i \in I_1 \bigcup I_2$, are given. Here I_1 and I_2 are finite index sets.
It is a well known fact that the SDP problem (11) is equivalent to the problem

$$\min_{x \in Q} c^T x$$
$$\text{s.t.} \quad l^T \mathcal{A}(x) l \le 0, \ \forall\, l \in L = \{l \in \mathbb{R}^s : \|l\| = 1\}, \tag{14}$$

which is a particular case of the convex SIP problem (2), (3) with s - dimensional index set $L \subset \mathbb{R}^s$, $s > 1$.
It is evident that the feasible sets of problems (11) and (14) coincide and admit the following representations:

$$\mathcal{X} = \{x \in Q : \ \mathcal{A}(x) \preceq 0\} = \{x \in Q : \ l^T \mathcal{A}(x) l \le 0, \quad \forall l \in L\}. \tag{15}$$

According to the definition from section 2, the constraints of the SIP problem (14) satisfy the Slater type condition I if

$$\exists\, \bar{x} \in Q \ \text{ such that } \ l^T \mathcal{A}(\bar{x}) l < 0, \quad \forall\, l \in L. \tag{16}$$

In terms of the SDP problem (11), the condition (16) can be written in the equivalent form

$$\exists\, \bar{x} \in Q \ \text{ such that } \ \mathcal{A}(\bar{x}) \prec 0. \tag{17}$$

According to Definition 1, the set of immobile indices in problem (14) is

$$L^* := \{l \in L : l^T \mathcal{A}(x) l = 0, \forall x \in \mathcal{X}\}. \tag{18}$$

The following proposition is a corollary of Proposition 2.

Proposition 3 *Given SDP problem in the form (11), the condition (17) is equivalent to the emptiness of the set L^*.*

Let us study some properties of the multidimensional immobile index set $L^* \subset \mathbb{R}^s$.

Proposition 4 *Consider vectors $\bar{l}$ and $\tilde{l}$ from L^*. Then any vector $\hat{l} \in L \bigcap \{l \in \mathbb{R}^s : l = \alpha_1 \bar{l} + \alpha_2 \tilde{l}, \ \alpha_1, \alpha_2 \in \mathbb{R}\}$ belongs to L^* as well.*

Proof. Indeed, for any $x \in \mathcal{X}$ we have $\mathcal{A}(x) \preceq 0$ and therefore for $x \in \mathcal{X}$ the condition $l^T \mathcal{A}(x) l = 0$ is equivalent to $\mathcal{A}(x) l = 0$. Hence for any $\bar{l} \in L^*$ and $\tilde{l} \in L^*$ the following equalities hold:

$$A(x)\bar{l} = 0, \ A(x)\tilde{l} = 0, \forall x \in \mathcal{X}.$$

Consider vector $\hat{l} = \alpha_1 \bar{l} + \alpha_2 \tilde{l}$, such that $\|\hat{l}\| = 1$, where $\alpha_1, \alpha_2 \in \mathbb{R}$. Then

$$\hat{l}^T \mathcal{A}(x) \hat{l} = (\alpha_1 \bar{l} + \alpha_2 \tilde{l})^T \mathcal{A}(x) (\alpha_1 \bar{l} + \alpha_2 \tilde{l}) =$$

$$\alpha_1^2 \bar{l}^T \mathcal{A}(x) \bar{l} + \alpha_2^2 \tilde{l}^T \mathcal{A}(x) \tilde{l} + \alpha_1 \alpha_2 \tilde{l}^T \mathcal{A}(x) \bar{l} + \alpha_1 \alpha_2 \bar{l}^T \mathcal{A}(x) \tilde{l} = 0, \ \forall x \in \mathcal{X},$$

and the proposition is proved. ■

It follows from Proposition 4, that the set L^* of immobile indices in problem (14) can be represented in the form

$$L^* = L \cap \mathcal{M}, \tag{19}$$

where $\mathcal{M}$ is some subspace of the vector space $\mathbb{R}^s$ and $dim(\mathcal{M}) =: s_* \le s$.

The subspace $\mathcal{M}$ will be called *the subspace of immobile indices* (or *immobile index subspace*) in the SDP problem (11). It follows from (18) and (19) that

$$\mathcal{M} = \{l \in \mathbb{R}^s : l^T \mathcal{A}(x) l = 0, \ \ \forall x \in \mathcal{X}\} = \{l \in \mathbb{R}^s : \mathcal{A}(x) l = 0, \ \ \forall x \in \mathcal{X}\}. \tag{20}$$

Remark 2 *From Proposition 3 one can conclude that problem (11) satisfies the Slater type condition I if and only if* $\mathcal{M} = 0 \in \mathbb{R}^s$.

Denote by $m_i, \ i = 1, ..., s_*$, vectors of a basis of the subspace $\mathcal{M}$. Let $\mathcal{M}^\perp$ be the orthogonal complement of $\mathcal{M}$ in $\mathbb{R}^s$. Then the following lemma holds.

Lemma 1 *Consider the set*

$$\tilde{\mathcal{X}} = \{x \in Q \subset \mathbb{R}^n : \mathcal{A}(x) m_i = 0, \ i = 1, ..., s_*, \ l^T \mathcal{A}(x) l \le 0, \ \forall l \in \mathcal{M}^\perp\} \tag{21}$$

and the feasible set $\mathcal{X}$ *of problem (11). The equality* $\mathcal{X} = \tilde{\mathcal{X}}$ *is true.*

Proof. From the definition of the subspace $\mathcal{M}$ it is evident that for all $x \in \mathcal{X}$ the following equivalence holds:

$$\mathcal{A}(x) m_i = 0, i = 1, ..., s_*, \iff \mathcal{A}(x) l = 0, \ \forall \ l \in \mathcal{M}. \tag{22}$$

Taking into account (15) and (20)-(22), we deduce that the condition $x \in \mathcal{X}$ implies the condition $x \in \tilde{\mathcal{X}}$. Hence $\mathcal{X} \subset \tilde{\mathcal{X}}$.

Suppose now that $\tilde{x} \in \tilde{\mathcal{X}}$. Consider a vector $l \in L$. It can be represented in the form

$$l = l^{(1)} + l^{(2)} \ \text{ with } l^{(1)} \in \mathcal{M}, \ l^{(2)} \in \mathcal{M}^\perp.$$

Then taking into account (22), we have

$$l^T \mathcal{A}(\tilde{x}) l = \begin{cases} l^{(1)T} \mathcal{A}(\tilde{x}) l^{(1)} = 0 \ \text{ if } l^{(2)} = 0, \\ (l^{(1)} + l^{(2)})^T \mathcal{A}(\tilde{x})(l^{(1)} + l^{(2)}) = l^{(2)T} \mathcal{A}(\tilde{x}) l^{(2)} \le 0 \ \text{ if } l^{(2)} \ne 0, \end{cases} \tag{23}$$

therefore for any $l \in \mathbb{R}^s$ the inequality $l^T \mathcal{A}(\tilde{x}) l \le 0$ is true. From the latest inequality and from the fact that $\tilde{x} \in Q$, we get $\tilde{x} \in \mathcal{X}$ and hence $\tilde{\mathcal{X}} \subset \mathcal{X}$.
The inclusions obtained, $\tilde{\mathcal{X}} \subset \mathcal{X}$ and $\mathcal{X} \subset \tilde{\mathcal{X}}$, prove the lemma. ■

Notice here that the set $\tilde{\mathcal{X}}$ in (21) can be written in the form

$$\tilde{\mathcal{X}} = \{x \in Q \subset \mathbb{R}^n : \mathcal{A}(x) M = 0, \ N^T \mathcal{A}(x) N \preceq 0\} = \{x \in \bar{Q} \subset \mathbb{R}^n : N^T \mathcal{A}(x) N \preceq 0\},$$

where $\bar{Q} = \{x \in Q \subset \mathbb{R}^n : \mathcal{A}(x) M = 0\}, \ \ M = (m_i, \ i = 1, ..., s_*)$ is a basic matrix[1] of the subspace $\mathcal{M}$, and $N \in \mathbb{R}^{s \times p_*}$ is a basic matrix of the subspace $\mathcal{M}^\perp$, with $p_* = s - s_*$. Then from Lemma 1 it follows that the set $\mathcal{X}$ defined in (15) admits the equivalent representation

$$\mathcal{X} = \{x \in \bar{Q} \subset \mathbb{R}^n : N^T \mathcal{A}(x) N \preceq 0\} = \{x \in \bar{Q} \subset \mathbb{R}^n : l^T \mathcal{A}(x) l \le 0, \forall l \in \mathcal{M}^\perp \cap L\}.$$

[1]Here and in what follows we equate a basis of a subspace to some matrix whose columns are the vectors of this basis.

Consequently problem (11) is equivalent to the problem

$$\min_{x\in\bar{Q}} c^T x \quad \text{s.t. } N^T\mathcal{A}(x)N \preceq 0, \tag{24}$$

whose semi-infinite form is

$$\min_{x\in\bar{Q}} c^T x \quad \text{s.t. } l^T\mathcal{A}(x)l \le 0, \ \ \forall l \in L\cap\mathcal{M}^{\perp}. \tag{25}$$

Different formulations (24) and (25) of SDP and SIP problems (11), (14) were obtained here at the cost of introducing the polyhedron $\bar{Q}$. In the next section, we will prove that problem (24) satisfies the Slater type condition I

$$\exists\, \bar{x} \in \bar{Q}: \ N^T\mathcal{A}(\bar{x})N \prec 0. \tag{26}$$

This important property will permit us to obtain new optimality conditions for problem (24) and the equivalent problem (11).

3.2 Optimality conditions

In this section, we will formulate optimality conditions for the SDP problem (11).

First, let us consider the case when the constraints of problem (11) satisfy the Slater type conditions. In this case, the optimality criterion for a feasible solution x^0 is well known (see [5]):

Theorem 1 *Suppose that the constraints of the linear SDP problem (11) satisfy condition (17). Then $x^0 \in \mathcal{X}$ is optimal if and only if there exist a matrix $\Omega^0 \in \mathcal{P}(s)$ and a vector $\lambda = (\lambda_i, i \in I_1 \cup I_2)$, such that*

$$\Omega^0 \bullet A_j + c_j + \sum_{i\in I_1\cup I_2} \lambda_i h_{ij} = 0, \ j = 1,\dots,n, \tag{27}$$

$$\Omega^0 \bullet \mathcal{A}(x^0) = 0, \ \ \lambda_i(h_i^T x^0 - b_i) = 0, \ \lambda_i \ge 0, i \in I_2,$$

where $h_i^T = (h_{ij}, j = 1,\dots,n)$, $i \in I_1 \cup I_2$.

Now let us consider optimality conditions without constraint qualifications. The main aim of this section is to prove the following result.

Theorem 2 *A feasible solution x^0 is optimal for the linear SDP problem (11) if and only if there exist matrices $\Pi \in \mathcal{P}(p_*)$, $\Gamma \in \mathbb{R}^{s\times s_*}$, and a vector $\lambda = (\lambda_i, i \in I_1 \cup I_2)$, such that*

$$(\Omega + \Upsilon) \bullet A_j + c_j + \sum_{i\in I_1\cup I_2} \lambda_i h_{ij} = 0, \ j = 1,\dots,n, \tag{28}$$

$$\Omega \bullet \mathcal{A}(x^0) = 0, \ \ \lambda_i(h_i^T x^0 - b_i) = 0, \ \lambda_i \ge 0, i \in I_2, \tag{29}$$

where $\Omega = N\Pi N^T$ and $\Upsilon = M\Gamma^T$.

Notice that Theorem 2 is valid for any SDP problem in the form (11). To prove the theorem we will need the following lemma.

Lemma 2 *The constraints of problem (24) satisfy condition (26).*

Proof. Suppose that on the contrary, the constraints of problem (24) do not satisfy (26). Then the constraints of the equivalent SIP problem (25) do not satisfy the Slater type condition I and, according to Proposition 3, the set of immobile indices in this problem given by

$$\bar{L}^* = \{l \in L \cap \mathcal{M}^\perp : l^T \mathcal{A}(x) l = 0 \text{ for all } x \in \tilde{\mathcal{X}}\}$$

is nonempty.
Let $l^* \in \bar{L}^*$. Since $\mathcal{X} = \tilde{\mathcal{X}}$, it is evident that l^* is an immobile index for problem (14) and hence $l^* \in L \cap \mathcal{M}$. But this contradicts the condition $l^* \in L \cap \mathcal{M}^\perp$ that takes place by construction. Lemma is proved. ■

Now we are ready to prove Theorem 2.

Proof of Theorem 2.
It was shown above that the SDP problem (11) is equivalent to problem (24) whose constraints satisfy the Slater type CQs. Applying Theorem 1 to problem (24) and taking into account the equivalence indicated above, we get the following statement for problem (11):

A feasible solution x^0 is optimal for the linear SDP problem (11) if and only if there exist matrix $\Pi \in \mathcal{P}(p_*)$, and vectors $\gamma_i \in \mathbb{R}^s$, $i = 1, ..., s_*$, $\lambda = (\lambda_i, i \in I_1 \cup I_2)$ such that

$$\Pi \bullet (N^T A_j N) + \sum_{i=1}^{s_*} \gamma_i^T A_j m_i + c_j + \sum_{i \in I_1 \cup I_2} \lambda_i h_{ij} = 0, \; j = 1, \ldots, n, \tag{30}$$

$$\Pi \bullet (N^T \mathcal{A}(x^0) N) = 0, \;\; \lambda_i (h_i^T x^0 - b_i) = 0, \; \lambda_i \geq 0, i \in I_2. \tag{31}$$

The optimality conditions (30), (31) can be rewritten in the form (28), (29) that concludes the proof. ■

It is easy to see that Theorem 2 can be reformulated as follows:

Theorem 3 *A feasible solution x^0 is optimal for the linear SDP problem (11) if and only if there exist vectors $\theta^{(k)} \in \Theta(x^0)$, $k \in I, |I| \leq \min\{n, p_*\}$, $\gamma_i \in \mathbb{R}^s, i = 1, ..., s_*$, and a vector $\lambda = (\lambda_i, i \in I_1 \cup I_2)$, such that*

$$\sum_{k=1}^{p_*} \theta^{(k)T} N^T A_j N \theta^{(k)} + c_j + \sum_{i=1}^{s_*} \gamma_i^T A_j m_i + \sum_{i \in I_1 \cup I_2} \lambda_i h_{ij} = 0, \; j = 1, \ldots, n, \tag{32}$$

$$\lambda_i (h_i^T x^0 - b_i) = 0, \; \lambda_i \geq 0, i \in I_2. \tag{33}$$

Here $\Theta(x) := \{\theta \in \mathbb{R}^{p_*} : \theta \neq 0, \; \theta^T N^T \mathcal{A}(x) N \theta = 0\}$.

The new optimality conditions obtained above for the linear SDP problem (11) use the explicit representation of the subspace of immobile indices by its basic vectors m_i, $i = 1, ..., s_*$. An algorithm that finds these vectors in a finite number of steps, is described later.

Remark 3 *The optimality results of this section were obtained due to the fact that the we have reformulated problem (11) in the form (24) satisfying the Slater type CQs. In its turn, this reformulation was possible due to introduction of the subspace of immobile indices of problem (11). This allows us to draw a conclusion about the important role that immobility plays in characterization of feasible sets not only in SIP, but in SDP.*

3.3 Example

Here we suggest a simple illustrative example of a linear SDP problem whose constraints do not satisfy the Slater type conditions. We consider an optimal feasible solution for which the classical optimality conditions (Theorem 1) are not satisfied and show that nevertheless the optimality conditions obtained here (Theorem 3) are true.

Consider the following linear SDP problem:

$$\begin{aligned} &\min_{x\in\mathbb{R}^n}(x_1+2x_2+\cdots+nx_n)\\ \text{s.t. } &\sum_{j=1}^{n}A_jx_j-A_0\preceq 0,\ \ x_j\geq 0, j=1,\dots,n, \end{aligned} \tag{34}$$

where $A_j=\begin{pmatrix} a_j & 1\\ 1 & 0\end{pmatrix}$, $j=0,\dots,n;\ a_0=1, a_1=0, a_j\in\mathbb{R}, j=2,\dots,n.$
The constraints of this problem do not satisfy the Slater type conditions since the subspace of immobile indices is not empty and has the form $\mathcal{M}=\{l=(l_1,l_2)^T\in\mathbb{R}^2: l_1=0\}$. Hence $M=(m_1=e_2)$, $e_2^T=(0,1)$, $s_*=p_*=1$.
Consider the feasible solution $x^0=(1,0,\dots,0)^T$. The corresponding constraint matrix is $\mathcal{A}(x^0)=diag(-1,0)$. For the given x^0, the optimality conditions of Theorem 1 can be equivalently reformulated as follows:

There exist vectors $l^{(i)}\in\mathbb{R}^2$, $i=1,2$, *and numbers* $\lambda_j\geq 0$, $j=1,\dots,n$ *such that*

$$\begin{aligned} &1)\ \ \mathcal{A}(x^0)l^{(i)}=0,\ i=1,2;\\ &2)\ \ \lambda_jx^0_j=0,\ j=1,\dots,n;\\ &3)\ \ \sum_{i=1}^{2}(l^{(i)})^TA_jl^{(i)}+c_j-\lambda_j=0,\ j=1,\dots,n. \end{aligned}$$

Conditions 1) and 2) imply that $l^{(i)}=(0,l_2^{(i)})^T,\ i=1,2,\ \ \lambda_1=0.$
Let us check condition 3). Notice that

$$(l^{(i)})^TA_jl^{(i)}=(0,l_2^{(i)})\begin{pmatrix} a_j & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0\\ l_2^{(i)}\end{pmatrix}=0 \text{ for all } i=1,2,\ j=1,\dots,n.$$

Hence we have $\sum_{i=1}^{2}(l^{(i)})^TA_jl^{(i)}=0$ for all $j=1,\dots,n$ and condition 3) for $j=1$ is not satisfied since $\sum_{i=1}^{2}(l^{(i)})^TA_1l^{(i)}+c_1-\lambda_1=c_1=1\neq 0$. Hence the (classical) optimality conditions 1) -3) do not permit to conclude about optimality of x^0.

Now let us check for x^0 the optimality conditions obtained in this paper. In our example, we have $s_*=p_*=1, m_1=e_2$, $N^T=(1,0)$, and $\mathcal{A}(x^0)=diag(-1,0)$. Then, evidently, $\Theta(x^0)=\emptyset$, and Theorem 3 can be reformulated as follows:

A feasible solution x^0 is optimal for the linear SDP problem (34) if and only if there exist vectors $\gamma_1\in\mathbb{R}^2$ and $\lambda=(\lambda_j, j=1,\dots,n)$ such that

$$c_j+\gamma_1^TA_je_2-\lambda_j=0,\ \lambda_jx^0_j=0,\ \lambda_j\geq 0, j=1,\dots,n.$$

Taking into account the specific form of matrix A_j and denoting $\gamma_1^T = (\gamma_{11}, \gamma_{12})$, we can rewrite the latest conditions as follows:

$$c_j + \gamma_{11} - \lambda_j = 0,\ \lambda_j x_j^0 = 0,\ \lambda_j \geq 0,\ j = 1, \ldots, n.$$

These conditions are satisfied with $\gamma_{11} = -1, \lambda_1 = 0,\ \lambda_j = c_j + \gamma_{11} = j - 1 \geq 0,\ \ j = 2, \ldots, n.$ Hence according to Theorem 3, the vector x^0 is optimal.

4 Comparison with other forms of optimality conditions

In the previous section, we have proved the new optimality criteria for the linear SDP problem (11). These criteria were formulated with the help of the subspace of immobile indices and its orthogonal complement.

There exists a number of papers where other optimality conditions for SDP are studied. Usually, the optimality conditions for SDP are obtained using duality. Different types of dual problems are known.

The dual problems used for SDP in [3, 16, 19], and some other papers (see references in [3]), either are restricted to a special form of SDP, or assume a kind of the Slater type CQ, therefore the optimality conditions that can be obtained with the help of these duals are more restricted. Here we compare the optimality results of the previous sections with that of papers [8, 17], and [18]. The new formulations of dual problems introduced in these papers permit to eliminate the duality gap whenever the primal is feasible and bounded. The optimality conditions obtained using this duality (*strong or no-gap duality*) do not use CQ and have the form of criteria.

To simplify the comparison and to be more precise and accurate, let us represent SDP problem (11) in the same form as in [8]. Namely, in this section we will suppose that $Q = \mathbb{R}^n$ and consider the problem

$$\text{(P*)} \qquad \min_{x \in \mathbb{R}^n} c^T x \quad \text{s.t.} \quad \mathcal{A}(x) := \sum_{i=1}^{n} A_i x_i + A_0 \preceq 0.$$

Let matrices M and N be defined as in section 3.1 with $Q = \mathbb{R}^n$. Then Theorem 2 can be reformulated for problem (P*) as follows:

Theorem 4 *A feasible solution x^0 is optimal for the linear SDP problem (P*) if and only if there exist matrices $\Pi \in \mathcal{P}(p_*)$ and $\Gamma \in \mathbb{R}^{s \times s_*}$ such that*

$$\tilde{\Omega} \bullet A_j + c_j = 0,\ j = 1, \ldots, n,\ \ \tilde{\Omega} \bullet \mathcal{A}(x^0) = 0 \ \textit{with}\ \tilde{\Omega} = N \Pi N^T + M \Gamma^T. \tag{35}$$

Here we take into account that $\tilde{\Omega} \bullet \mathcal{A}(x^0) = (N\Pi N^T + M\Gamma^T) \bullet \mathcal{A}(x^0) = (N\Pi N^T) \bullet \mathcal{A}(x^0)$.

Notice that for any $A \in \mathcal{S}(s)$ and any convenient matrices Π and Γ , it holds

$$(N\Pi N^T + M\Gamma^T) \bullet A = (N\Pi N^T + M\Gamma^T/2 + \Gamma M^T/2) \bullet A.$$

As any matrix $\Gamma \in \mathbb{R}^{s \times s_*}$ can be represented in the form $\Gamma = M\bar{W} + N\bar{V}$, where $\bar{W} \in \mathbb{R}^{s_* \times s_*}$, $\bar{V} \in \mathbb{R}^{p_* \times s_*}$, then $(N\Pi N^T + M\Gamma^T) \bullet A = (M, N)U(M, N)^T \bullet A$ with some $U \in \mathcal{D}$,

$$\mathcal{D} := \Big\{ U \in \mathcal{S}(s) : U = \begin{pmatrix} W & V \\ V^T & \Pi \end{pmatrix},\ \Pi \in \mathcal{P}(p_*) \Big\}. \tag{36}$$

Hence a feasible solution x^0 is optimal for problem (P*) if and only if there exists a matrix $U \in \mathcal{D}$ such that

$$U \bullet \bar{A}_j + c_j = 0, \; j = 1, \ldots, n, \;\; U \bullet \bar{\mathcal{A}}(x^0) = 0, \tag{37}$$

where $\bar{A}_j = (M,N)^T A_j (M,N), j = 0,1,...,n, \; \bar{\mathcal{A}}(x) = \sum\limits_{i=1}^{n} \bar{A}_i x_i + \bar{A}_0$.

Taking into account conditions (37), we can formulate the following dual for (P*):

$$\text{(D*)} \qquad \max U \bullet \bar{A}_0 \quad \text{s.t. } U \bullet \bar{A}_j + c_j = 0, \; j = 1, \ldots, n, \; U \in \mathcal{D}.$$

In [8], it was shown that problem (P*) is equivalent to the *regularized primal program*:

$$\text{(RP)} \qquad \min_{x \in \mathbb{R}^n} c^T x \quad \text{s.t. } -\mathcal{A}(x) \in \mathcal{P}^f,$$

where $\mathcal{P}^f$ is *the minimal cone* of problem (P*) defined as

$$\mathcal{P}^f := \cap \{K \triangleleft \mathcal{P}(s) : K \supset \mathcal{F}\}, \;\; \mathcal{F} := \{F : F = -\mathcal{A}(x), x \in \mathcal{X}\}. \tag{38}$$

A cone $K \subset \mathcal{P}$ is a face of a cone $\mathcal{P}$ denoted as $K \triangleleft \mathcal{P}$ if for all $A, B \in \mathcal{P}$, the following implication holds: $A + B \in K \Rightarrow \; A, B \in K$.

The dual program (*regularized dual*) for (RP) has the form

$$\text{(DRP):} \qquad \max U \bullet A_0 \quad \text{s.t. } U \bullet A_i + c_i = 0, \; i = 1, \ldots, n, \; U \in (\mathcal{P}^f)^+.$$

Here $(\mathcal{P}^f)^+$ is the *polar cone* defined as

$$(\mathcal{P}^f)^+ := \{U \in \mathcal{S}(s) : U \bullet P \geq 0 \;\; \forall \; P \in \mathcal{P}^f\}. \tag{39}$$

In [8, 18], it is shown that the strong duality holds for the pair of regularized dual problems (RP) and (DRP). The following theorem is the immediate consequence of these duality results.

Theorem 5 *A feasible solution x^0 is optimal for the linear SDP problem (P*) if and only if there exists a matrix $U \in (\mathcal{P}^f)^+$ such that*

$$U \bullet A_j + c_j = 0, \; j = 1, \ldots, n, \;\; U \bullet \mathcal{A}(x^0) = 0. \tag{40}$$

We are going now to show the equivalence of the optimality conditions (35) and (40) formulated in Theorems 4 and 5. To do this, let us show that the cones $\mathcal{P}^f$ and $(\mathcal{P}^f)^+$ admit representations in terms of the subspace of immobile indices.

Lemma 3 *The sets $\mathcal{P}^f$ and $(\mathcal{P}^f)^+$ defined in (39) and (40) can be represented as follows:*

$$\mathcal{P}^f = \mathcal{P}(s) \cap \{MM^T\}^\perp = \{P : P = NYN^T, \; Y \in \mathcal{P}(p_*)\}, \tag{41}$$

$$(\mathcal{P}^f)^+ = (M,N)\mathcal{D}(M,N)^T := \{P^+ : P^+ = (M,N)U(M,N)^T, U \in \mathcal{D}\}, \tag{42}$$

where the set $\mathcal{D}$ is given in (36).

Here and in what follows, $\{A\}^\perp$ denotes the orthogonal complement of an element $A \in \mathcal{S}(s)$ in the space $\mathcal{S}(s)$.

Proof. Here we will use the following matrix properties (see [1]):

1. Given $A \in \mathbb{R}^{s\times p}$, $B \in \mathcal{P}(s)$, if $trace(A^TBA) = 0$ then $A^TB = 0$.
2. Let $A \in \mathbb{R}^{s\times p}$, rank$A = p$, and $B \in \mathbb{R}^{s\times(s-p)}$, rank$B = s-p$. If $A^TB = 0$, then for any $G \in \mathbb{R}^{s\times s}$ satisfying the equality $A^TG = 0$ there exists a matrix $W \in \mathbb{R}^{(s-p)\times s}$ such that $G = BW$.
3. Matrix $A \in \mathcal{S}(s)$ is positive semidefinite if and only if $A \bullet B \geq 0$ for all $B \in \mathcal{P}(s)$.

Now let us show that for any $A \in \mathbb{R}^{s\times p}$, rank$A = p$, and $B \in \mathbb{R}^{s\times(s-p)}$, rank$B = s-p$, such that $A^TB = 0$, the following equality takes place:

$$\mathcal{P}(s) \cap \{AA^T\}^\perp = \{P : P = BYB^T,\ Y \in \mathcal{P}(s-p)\}. \tag{43}$$

Suppose that $\tilde{P} \in \mathcal{P}(s) \cap \{AA^T\}^\perp$. Then $\tilde{P} \in \mathcal{P}(s)$ and $\tilde{P} \bullet AA^T = trace(A^T\tilde{P}A) = 0$. Hence $A^T\tilde{P} = 0$ (see matrix property **1** mentioned above). As $\tilde{P} \in \mathcal{P}(s)$, then $\tilde{P} = GG^T$ with $G \in \mathbb{R}^{s\times s}$. Consequently we get $A^TGG^T = 0$ giving $A^TG = 0$. It follows from the latest equality and property **2** that there exists a matrix $W \in \mathbb{R}^{(s-p)\times s}$ such that $G = BW$. Hence $\tilde{P} = GG^T = BWW^TB^T = BYB^T$ with $Y = WW^T \in \mathcal{P}(s-p)$ and $\tilde{P} \in \{P : P = BYB^T,\ Y \in \mathcal{P}(s-p)\}$.
Now suppose that $P \in \{P : P = BYB^T,\ Y \in \mathcal{P}(s-p)\}$. Then, by construction, $P \in \mathcal{P}(s)$ and $P \bullet AA^T = 0$. Hence $P \in \mathcal{P}(s) \cap \{AA^T\}^\perp$ and (43) is proved.

Now let us prove (41). Denote $\mathcal{P}^f_M := \mathcal{P}(s) \cap \{MM^T\}^\perp$. It is evident that $\mathcal{P}^f_M$ is a face of $\mathcal{P}(s)$. Let us show that for the set $\mathcal{F}$ defined in (38), it holds

$$\mathcal{F} \subset \mathcal{P}^f_M. \tag{44}$$

In fact, for any $x \in \mathcal{X}$, we have $-\mathcal{A}(x) \in \mathcal{P}(s)$ and $\mathcal{A}(x)MM^T = 0$. Hence (44) is true.

Let K be a face of $\mathcal{P}(s)$ such that $\mathcal{F} \subset K$. According to [8], there exists a matrix $W = (w_1, \dots, w_k) \in \mathbb{R}^{s\times k}$, $w_i \in \mathbb{R}^s, i = 1, \dots, k$, such that $K = \mathcal{P}(s) \cap \{WW^T\}^\perp$. Hence, for any $x \in \mathcal{X}$, we have $-\mathcal{A}(x) \in \mathcal{P}(s)$ and $\mathcal{A}(x) \bullet WW^T = 0$. It follows from the latest equality and property **1** that $\mathcal{A}(x)w_i = 0,\ i = 1, \dots, k$, for all $x \in \mathcal{X}$. By definition this means that $w_i \in \mathcal{M},\ i = 1, \dots, k$, and consequently, there exists $A \in \mathbb{R}^{s_*\times k}$ such that $W = MA$. The last equality implies $\mathcal{P}(s) \cap \{MM^T\}^\perp \subset \mathcal{P}(s) \cap \{WW^T\}^\perp$. Thus we have proved that the face $\mathcal{P}^f_M$ is the minimal one containing the set $\mathcal{F}$. Hence by definition, $\mathcal{P}^f = \mathcal{P}^f_M$ and relations (41) are proved.

Finally, let us prove (42). Consider matrices $U \in \mathcal{D}$ and $P = NYN^T \in \mathcal{P}^f$. Since for these matrices the equality $(M,N)U(M,N)^T \bullet P = N^TN\Pi N^TN \bullet Y$ takes place, then taking into account property **3**, we conclude that $(M,N)U(M,N)^T \bullet P \geq 0$ for all $U \in \mathcal{D}$ and all $P \in \mathcal{P}^f$. Therefore $(\mathcal{P}^f)^+ \supset (M,N)\mathcal{D}(M,N)^T$.
For any $U \in (\mathcal{P}^f)^+$ there exists a matrix $\tilde{U} = \begin{pmatrix} \tilde{W} & \tilde{V} \\ \tilde{V}^T & \tilde{\Pi} \end{pmatrix} \in \mathcal{S}(s)$ such that $U = (M,N)\tilde{U}(M,N)^T$. The condition $U \in (\mathcal{P}^f)^+$ implies $U \bullet P = (M,N)\tilde{U}(M,N)^T \bullet NYN^T = N^T(M,N)\tilde{U}(M,N)^TN \bullet Y \geq 0$ for all $Y \in \mathcal{P}(p_*)$. Then it follows from property **3** that $N^T(M,N)\tilde{U}(M,N)^TN \succeq 0$ or equivalently $(0, N^TN)\tilde{U}(0, N^TN)^T \succeq 0$.

The latest condition together with inequality $det(N^TN) \neq 0$ imply $\tilde{\Pi} \in \mathcal{P}(p_*)$. Hence $\tilde{U} \in \mathcal{D}$ and $(\mathcal{P}^f)^+ \subset (M,N)\mathcal{D}(M,N)^T$. The Lemma is proved. ■

It follows from Lemma 3 that the dual problems (D*) and (DRP) are equivalent. Hence the optimality conditions (35) and (40) formulated in Theorems 4 and 5, are equivalent as well. At the same time notice that these conditions were obtained using different approaches: the dual problem (DRP) and the optimality conditions (40) obtained on the base of this dual use the minimal cone of problem (P*) and its polar cone defined implicitly in (38) and (39), while the the optimality conditions (35) use the subspace of immobile indices. Recall that to check the conditions of Theorem 4, one just has to find matrices $\Pi \in \mathcal{P}(p_*)$, $W \in \mathcal{S}(s_*)$, and $V \in \mathbb{R}^{s_* \times p_*}$ satisfying conditions (37), i.e. these optimality conditions do not need any additional constructions except a base of the space of immobile index set.

A dual program in another form is proposed in [17]. It is called an extended Lagrange–Slater dual and denoted as (ELSD). Strong duality holds for this dual as well and one can formulate optimality conditions for SDP in terms of this problem. In [8], it was shown that the problem (ELSD) is equivalent to the regularized dual problem (DRP). This equivalence means here that the constraints and the set of Lagrange multipliers are the same in both problems. The difference between these dual problems consists in the fact that in (ELSD) the feasible set of Lagrange multipliers, denoted as $(\mathcal{P}^f)^+$, is expressed implicitly in the form of solutions of n recursive systems of additional constraints, whereas in (DRP) it is defined implicitly according to (38), (39). From the equivalence of dual problems (ELSD) and (DRP), taking into account the equivalence of (D*) and (DRP), we can conclude that the new optimality conditions proved in Theorem 4, are equivalent to the conditions from [17] as well.
It is noticed in [8] that the equivalence of two dual problems (ELSD) and (DRP) found using different techniques is more than a coincidence. Analyzing the results of this section, we can continue this statement and say that the equivalence of the problems (ELSD) and (DRP) (obtained using the minimal cone and dual cone approaches) to the dual problem (D*) (obtained using the immobile indices approach), is not just a coincidence as well. This equivalence, on the one hand, indicates that all these approaches are valid and faithful, and, on the other hand, supports the conclusion about the important role that the subspace of immobile indices plays in characterizing of the properties of infinite extremal problems.

5 DIIS algorithm of determination of the immobile index subspace for SDP problem (11)

Given a feasible solution of SDP problem (11), to verify if it satisfies the optimality conditions formulated in Theorems 2 and 3, we need to know a basis $m_i, i = 1, \dots, s_*$, of the subspace of immobile indices $\mathcal{M}$ (this subspace is defined by formula (20)). In this section, we describe and justify an algorithm that finds such basis. We call this algorithm the *DIIS algorithm* since it will be used for determination of the immobile index subspace.

5.1 Description of the DIIS algorithm

Consider the linear SDP problem (11). Suppose that its feasible set $\mathcal{X}$ is nonempty.

Set $I^1 = \emptyset$ and $k = 1$.

General iteration. At the beginning of the k-th iteration of the algorithm a set of linearly independent vectors $m_i \in \mathbb{R}^s,\ i \in I^k$, is known.

Put $p_k = s - |I^k|$ and find a solution

$$l_i \in \mathbb{R}^s, i = 1, ..., p_k;\ \gamma_i \in \mathbb{R}^s, i \in I^k;\ \rho_i, i \in I_1 \cup I_2 \tag{45}$$

to the system

$$\begin{aligned} &\sum_{i=1}^{p_k} l_i^T A_j l_i + \sum_{i \in I^k} \gamma_i^T A_j m_i + \sum_{i \in I_1 \cup I_2} \rho_i h_{ij} = 0, j = 0, 1, ..., n, \\ &\sum_{i=1}^{p^k} ||l_i||^2 = 1,\ \rho_i \geq 0, i \in I_2;\ l_i^T m_j = 0,\ j \in I^k,\ i = 1, ..., p_k, \end{aligned} \tag{46}$$

where $h_{i0} := -b_i,\ i \in I_1 \cup I_2$.

If the system does not admit any solution then STOP: it will be proved later that in this case the vectors $m_j, j \in I^k$, form a basis of the subspace $\mathcal{M}$.

Otherwise consider a solution (45) to the system (46). Let $\{m_j,\ j \in \Delta I^k\}$, $\Delta I^k = \{|I^k| + 1, ..., |I^k| + s_k\}$, $s_k \geq 1$, be a maximal subset of linearly independent vectors from the set $\{l_i, i = 1, ..., p_k\}$. Notice that $\sum\limits_{i=1}^{p_k} ||l_i||^2 = 1$ and hence $\Delta I^k \neq \emptyset$.
Put $I^{k+1} := I^k \cup \Delta I^k$, $k := k + 1$ and repeat the iteration.

Remark 4 *It is evident that the algorithm will stop after no more than s iterations, s being the dimension of the matrix space in problem (11).*

Remark 5 *The system (46) can be written in the equivalent matrix form*

$$\begin{gathered} (\Omega + M_k \Gamma^T) \bullet A_j + \sum_{i \in I_1 \cup I_2} \rho_i h_{ij} = 0, j = 0, 1, ..., n, \\ \Omega M_k = 0,\ trace\ \Omega = 1,\ \Omega \succeq 0,\ \rho_i \geq 0, i \in I_2. \end{gathered}$$

Here $M_k = (m_i, i \in I^k)$.

5.2 Justification of the DIIS algorithm

Suppose that the DIIS algorithm has stopped at k_*−th iteration, $k_* \leq s$ and consider the vectors

$$m_i, i \in I^{k_*}. \tag{47}$$

By construction, these vectors are linearly independent. Denote by $\tilde{\mathcal{M}} \subset \mathbb{R}^s$ the subspace generated by the vectors (47).

Theorem 6 *Given SDP problem (11) with $\mathcal{X} \neq \emptyset$, the set $\tilde{\mathcal{M}}$ constructed by the DIIS algorithm, is the subspace of immobile indices in problem (11).*

Proof. Suppose, first, that the constraints of problem (11) satisfy a Slater type condition. Then, evidently, the algorithm will stop with $k_* = 1$, $\tilde{\mathcal{M}} = 0 \in \mathbb{R}^s$ and $I^{k_*} = \emptyset$. On the other hand, from Remark 2, we have that in this case $\mathcal{M} = 0 \in \mathbb{R}^s$ and the theorem is proved. Notice that here $s_* = 0$.

Suppose now that the Slater type CQs are not satisfied. Let us prove that $\tilde{\mathcal{M}} = \mathcal{M}$ as well. To prove the inclusion $\tilde{\mathcal{M}} \subset \mathcal{M}$, let us show that

$$m_j \in \mathcal{M}, j \in I^{k_*}. \tag{48}$$

For $k = 1$, the relations $m_j \in \mathcal{M}, j \in I^k$, are trivially satisfied. Suppose that the inclusions $m_j \in \mathcal{M}, j \in I^k$, are true for some k, $1 \leq k < k_*$. Let us show that

$$m_j \in \mathcal{M}, j \in I^{k+1}.$$

For this purpose we will prove that for any solution (45) of system (46) the following relations take place

$$l_i^T \mathcal{A}(x) l_i = 0 \;\; \forall\, x \in \mathcal{X}, \;\; i = 1, ..., p_k. \tag{49}$$

It follows from (46) that for all $x = (x_1, ..., x_n) \in \mathcal{X}$ and $x_0 = 1$, it is satisfied:

$$\begin{aligned} 0 = &\sum_{j=0}^{n} x_j \Big(\sum_{i=1}^{p_k} l_i^T A_j l_i + \sum_{i \in I^k} \gamma_i^T A_j m_i + \sum_{i \in I_1 \cup I_2} \rho_i h_{ij}\Big) = \\ &\sum_{i=1}^{p_k} l_i^T \mathcal{A}(x) l_i + \sum_{i \in I^k} \gamma_i^T \mathcal{A}(x) m_i + \sum_{i \in I_1 \cup I_2} \rho_i (h_i^T x - b_i) = \\ &\sum_{i=1}^{p_k} l_i^T \mathcal{A}(x) l_i + \sum_{i \in I_2} \rho_i (h_i^T x - b_i). \end{aligned} \tag{50}$$

Here we have taken into account that by assumption $m_i \in \mathcal{M}, i \in I^k$, and $h_{i0} = -b_i$, $h_i^T x = b_i, i \in I_1$, for $x \in \mathcal{X}$. Moreover from the conditions $x \in \mathcal{X}$, and $\rho_i \geq 0, i \in I_2$, we have

$$l_i^T \mathcal{A}(x) l_i \leq 0, \;\; i = 1, ..., p_k; \quad \rho_i (h_i^T x - b_i) \leq 0, i \in I_2.$$

From the latest relations and equality (50) we get (49).
It is evident that (49) implies $m_j \in \mathcal{M}, j \in \Delta I^k$, and hence $m_j \in \mathcal{M}, j \in I^{k+1}$. Repeating the above reasoning for $k = 1, 2, ..., k_* - 1$, we find that (48) is true, hence $\tilde{\mathcal{M}} \subset \mathcal{M}$.
Now let us prove that $\mathcal{M} \subset \tilde{\mathcal{M}}$. Let N_* be an orthogonal basis of $\tilde{\mathcal{M}}^\perp$ and $M_* = (m_i, i \in I^{k_*})$. Consider the auxiliary SDP problem

$$J_* := \min_{\xi \in \mathbb{R},\; x \in Q} \xi \tag{51}$$

$$\text{s.t. } \mathcal{A}(x) M_* = 0, \;\; N_*^T \mathcal{A}(x) N_* - E_* \xi \preceq 0, \; -\xi \leq 1.$$

Here E_* is the identity $(s - |I^{k_*}|) \times (s - |I^{k_*}|)$-matrix. In problem (51), the constraints satisfy the Slater type conditions and the objective function is bounded. Hence the correspondent dual problem admits an optimal solution.
Suppose that $J_* = 0$. Then by duality theory, for $k = k_*$ system (46) has a solution. But this contradicts the assumption that k_* is the number of the iteration where the system has no

solutions. Hence $J_* < 0$ and there exists a vector $\bar{x} \in Q$ such that $\mathcal{A}(\bar{x})M_* = 0$, $N_*^T\mathcal{A}(\bar{x})N_* \prec 0$ or, equivalently,

$$l^T\mathcal{A}(\bar{x})l = 0 \;\text{ for all } l \in \tilde{\mathcal{M}}; \quad l^T\mathcal{A}(\bar{x})l < 0 \;\text{ for all } l \in \tilde{\mathcal{M}}^{\perp},\; l \neq 0. \tag{52}$$

Taking into account the last relations, it is not difficult to prove that $\bar{x} \in \mathcal{X}$.
Let $\bar{l} \in \mathcal{M}$. Since $\mathbb{R}^s = \tilde{\mathcal{M}} \oplus \tilde{\mathcal{M}}^{\perp}$, it holds: $\bar{l} = l^{(1)} + l^{(2)}$, $l^{(1)} \in \tilde{\mathcal{M}}$, $l^{(2)} \in \tilde{\mathcal{M}}^{\perp}$.
Let us show that $l^{(2)} = 0$. Suppose that on the contrary, $l^{(2)} \neq 0$. For $\bar{x} \in \mathcal{X}$ and $\bar{l} \in \mathcal{M}$ the equality

$$\bar{l}^T\mathcal{A}(\bar{x})\bar{l} = 0$$

takes place by definition of $\mathcal{M}$. On the other hand, taking into account relations (52), we have

$$\bar{l}^T\mathcal{A}(\bar{x})\bar{l} = (l^{(1)} + l^{(2)})^T\mathcal{A}(\bar{x})(l^{(1)} + l^{(2)}) = (l^{(2)})^T\mathcal{A}(\bar{x})l^{(2)} < 0.$$

The contradiction obtained proves that the inclusion $\bar{l} \in \mathcal{M}$ implies $l^{(2)} = 0$ and hence $\bar{l} \in \tilde{\mathcal{M}}$. Therefore $\mathcal{M} \subset \tilde{\mathcal{M}}$.

■

6 Conclusions

The main result of this paper consists in the proof of new optimality conditions for linear SDP. These conditions do not use any CQ, and have the form of criteria. We compare these criteria with some other gap-free optimality conditions.
Thus, in [8, 17], and [18] dual representations of the SDP problem are considered. These representations use the *minimal cone* $\mathcal{P}^f$ and its *polar cone* $(\mathcal{P}^f)^+$ defined (implicitly) in (38), (39), and permit to obtain optimality conditions (in the form of criterion) that do not use CQ. We analyze these conditions and conclude that they are equivalent to the optimality criteria formulated in this paper in spite of the fact that they differ in form and are obtained using different techniques. The proof of the equivalence of different optimality conditions is one more important result of this paper.
Moreover, the explicit representations of the minimal and polar cones in terms of the subspace of immobile indices are obtained here. These representations permit one to understand more deeply the structure of the feasible set of the linear SDP problem and to conclude that the notion of immobile indices is the important characteristics of this problem.
In the future, we plan to generalize the results obtained for linear SDP problems to convex SIP problems with multidimensional index set.

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