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Mathisson's helical motions for a spinning particle: Are they unphysical?

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It has been asserted in the literature that Mathisson's helical motions are unphysical, with the argument that their radius can be arbitrarily large. We revisit Mathisson's helical motions of a free spinning particle, and observe that such statement is unfounded. Their radius is finite and confined to the disk of centroids. We argue that the helical motions are perfectly valid and physically equivalent descriptions of the motion of a spinning body, the difference between them being the choice of the representative point of the particle, thus a *gauge* choice. We discuss the kinematical explanation of these motions, and we dynamically interpret them through the concept of hidden momentum. We also show that, contrary to previous claims, the frequency of the helical motions coincides, even in the relativistic limit, with the *zitterbewegung* frequency of the Dirac equation for the electron.

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I. INTRODUCTION

The equations of motion for spinning pole-dipole particles were first derived by Mathisson [1] in the context of general relativity, though similar equations, for the case of *flat* spacetime, had been derived earlier by Frenkel [2] (see also [3]) in a special relativistic treatment applying to a classical model of an electron. These equations have then been further worked out and rederived most notably by Weyssenhoff-Raabe [4,5], Möller [6], Bhabha-Corben [7–9], Dixon [10] and Gralla *et al.* [11], in the framework of special relativity; and in general relativity by Papapetrou [12], who carried out an exact derivation for pole-dipole particles, Tulczyjew [13], Taub [14], Dixon [15,16] and Souriau [17,18], who made derivations covariant *at each step*, and more recently Natário [19] and Gralla *et al.* [20]. To form a determined system, these equations require a supplementary condition, which amounts to specify the reference worldline relative to which the moments of the particle are taken. The natural choice is to require it to be the center of mass; however by contrast with Newtonian mechanics, in relativity the center of mass/energy of a spinning particle is an observer-dependent point. Thus, in order to use the concept of center of mass to fix a worldline of reference, a particular observer must be specified. This, as shall be explained in detail below, is done through a spin condition $S^{\alpha\beta}u_\beta = 0$ (for some unit timelike vector field u^α), stating that the reference worldline is the center of mass as measured by some observer of 4-velocity u^α. Its choice can be regarded as a *gauge fixing*.

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Mathisson's helical solutions [21] arise when one uses the supplementary condition $S^{\alpha\beta}U_\alpha = 0$, where U^α is the center of mass 4-velocity, thus stating that the center of mass is measured in its proper frame (i.e., the frame where it is at rest). This condition was first used by Frenkel [2], and later embodied in the derivation by Mathisson [1], and also employed by Pirani [22] as a mean of closing the Papapetrou equations. It will be hereafter dubbed as the "Mathisson-Pirani" supplementary condition, as it is best known. The helical motions have been studied since by many authors (see, e.g., [4–9,23–25]).

These helical motions exist even for a free particle in flat spacetime, and are still rather mysterious today. They were first interpreted in [21], for the case of the electron, as the classical counterpart of the "*zitterbewegung*" observed in Dirac's equation, based on the coincidence of frequencies obtained in the nonrelativistic limit; this point of view was then supported by other authors, e.g., [8,9,12,23,26,27]. Möller [6] provided a kinematical interpretation of the helices as arising from the motions of what he called the "pseudocenters of mass" (see equivalent treatment in Sec. V below). In spite of this, they have been deemed unphysical [15,28] with the argument that the radius of the helices can be arbitrarily large [4,5,15,28,29], which would be contradicted by experiment (this was actually what initially motivated Dixon's multipole approach to extended bodies [15], embodying the alternative condition $S^{\alpha\beta}P_\beta = 0$, proposed by Tulczyjew). This would make them also inconsistent with Möller's scheme. Also, it was argued [4,5] that the coincidence with the frequency of Dirac's equation *zitterbewegung* motions holds only in the nonrelativistic limit.

Herein we will show that the assessments regarding the unphysical nature of the helical motions are unfounded and originate from a mistake in the treatment in [4,5]. We argue otherwise: that they are physically acceptable, being actually alternative and equivalent (albeit more complicated) descriptions of the motion of a spinning body; these different descriptions are a matter of choice, resulting from the incompleteness of the gauge fixing provided by the Mathisson-Pirani supplementary condition, which leaves a *residual gauge freedom*. Their radius is shown to be contained within the disk of centroids, whose size is actually the *minimum* size a classical spinning particle can have if it is to have finite angular momentum and positive mass without violating special relativity. We kinematically explain these solutions, showing they are consistent with Möller's interpretation. And we show also that they are dynamically consistent descriptions of the motion of the body, which can be understood through the same concept of "hidden momentum" recently proposed in [20] as an explanation for the bobbings observed in numerical simulations of binary systems.

Finally, regarding the correspondence with the quantum problem, we point out that the assertions in [4,5] that the frequencies only coincide in the nonrelativistic limit originate from the same mistake that leads to the arbitrarily large radius; indeed the frequencies coincide exactly.

A. Notation and conventions

We summarize here the conventions that we use for the various quantities necessary to describe the motion of a spinning particle:

(1) $U^\alpha \equiv dz^\alpha/d\tau$ is the tangent vector to the worldline of reference $z^\alpha(\tau)$ (in this work it amounts to the 4-velocity of some suitably defined center of mass);

(2) u^α denotes a generic unit timelike vector defined along the worldline of reference; it can be thought as the instantaneous 4-velocity of an observer $\mathcal{O}(u)$;

(3) $\Sigma(z, u)$ is the hypersurface generated by all geodesics orthogonal to u^α at a point z^α; in flat spacetime, it is simply the 3-space orthogonal to u^α (it can be thought of as the instantaneous rest space of $\mathcal{O}(u)$);

(4) $x^\alpha_{\rm CM}(u)$ is the center of mass as measured in the instantaneous rest space of $\mathcal{O}(u)$;

(5) *Centroids*: following [30], we dub the centers of mass $x^\alpha_{\rm CM}(u)$ as measured by arbitrary observers as centroids; these divide in two subclasses: 1) *proper center of mass* $x^\alpha_{\rm CM}(U)$—center of mass as measured in its own rest frame; 2) *nonproper center of mass*—center of mass measured by an observer not comoving with it. Sometimes we shall use the abbreviation CM for center of mass.

(6) *Masses:* $m(u) \equiv -P^\alpha u_\alpha$ denotes the mass as measured by $\mathcal{O}(u)$; by $m \equiv m(U) = -P^\alpha U_\alpha$ we denote the proper mass (i.e., the mass measured in the CM frame); and $M \equiv \sqrt{-P^\alpha P_\alpha}$ is the mass as measured in the zero 3-momentum frame.

(7) We denote by $S_\star^{\alpha\beta}$ the angular momentum tensor about the centroid $x^\alpha_{\rm CM}(P)$ measured in the zero 3-momentum frame (i.e., $S_\star^{\alpha\beta}P_\beta = 0$), and by $S_\star^\alpha$ the corresponding spin vector, obeying $S_\star^{\alpha\beta} = \epsilon^{\alpha\beta}{}_{\mu\nu}S_\star^\mu P^\nu/M$.

(8) $\epsilon_{\alpha\beta\gamma\delta}$ denotes the Levi-Civita tensor; we choose $\epsilon_{0123} = -1$ (for flat spacetime). We denote by $\vec{A}\times_U\vec{B}$ the spatial part of the vector $\epsilon^\alpha{}_{\beta\gamma\delta}A^\beta B^\gamma U^\delta$ with respect to a given frame $\mathcal{O}(u)$; and $\vec{A}\times\vec{B}\equiv\vec{A}\times_u\vec{B}$.

II. EQUATIONS OF MOTION FOR FREE SPINNING PARTICLES IN FLAT SPACETIME

In a multipole expansion, a body is represented by a set of moments of $T^{\alpha\beta}$, called "inertial" or "gravitational" moments (forming the so called [1] "gravitational skeleton"). The moments are taken about a reference worldline $z^\alpha(\tau)$, which will be chosen as a suitably defined center of mass to be discussed below. Truncating the expansion at dipole order, the equations of motion involve only two moments of $T^{\alpha\beta}$ [10,15,29], the momentum P^α and the angular momentum $S^{\alpha\beta}$:

$$P^\alpha \equiv \int_{\Sigma(\tau,u)} T^{\alpha\beta}d\Sigma_\beta, \tag{1}$$

$$S^{\alpha\beta} \equiv 2\int_{\Sigma(\tau,u)} r^{[\alpha}T^{\beta]\gamma}d\Sigma_\gamma. \tag{2}$$

Here $P^\alpha(\tau)$ is the 4-momentum of the body; $S^{\alpha\beta}(\tau)$ is the angular momentum about a point $z^\alpha(\tau)$ of the reference worldline; $\Sigma(\tau, u) \equiv \Sigma(z(\tau), u)$; $r^\alpha \equiv x^\alpha - z^\alpha(\tau)$, where $\{x^\alpha\}$ is a chart on spacetime; and finally $d\Sigma_\gamma \equiv n_\gamma d\Sigma$, where n_γ is the (past-pointing) unit normal to $\Sigma(\tau, u)$ and $d\Sigma$ is the 3-volume element on $\Sigma(\tau, u)$.

For simplicity, we will consider the background to be Minkowski spacetime without any further fields. In this case P^α, $S^{\alpha\beta}$ are independent of Σ, and the equations of motion that follow from the conservation law $T^{\alpha\beta}{}_{;\beta} = 0$ are [5,10,12,15,16,29]:

$$\frac{DP^\alpha}{d\tau} = 0, \tag{3a}$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]}. \tag{3b}$$

Contracting (3b) with U^α we obtain an expression for the momentum

$$P^\alpha = mU^\alpha - \frac{DS^{\alpha\beta}}{d\tau}U_\beta, \tag{4}$$

where $m \equiv -P^\alpha U_\alpha$. Equations (3a) and (3b) form an indeterminate system. Indeed, there are 13 unknowns (P^α, three independent components of U^α, and six independent components of $S^{\alpha\beta}$) for only 10 equations.[1] To close the system we need to specify the representative point of the body (i.e., the worldline of reference relative to which $S^{\alpha\beta}$ is taken). That can be done through a supplementary spin condition of the type $S^{\alpha\beta}u_\beta = 0$, where $u^\alpha(\tau)$ is some appropriately chosen unit timelike vector, which effectively kills off three components of the angular momentum; this condition, as we shall see in the next section, means that the reference worldline is the center of mass as measured in the rest frame of the observer of velocity u^α. Hence U^α is the center of mass 4-velocity and m denotes the *proper mass*, i.e., the energy of the body as measured in the center of mass frame.

We note from Eq. (4) that the momentum of a spinning particle is not, in general, parallel to its 4-velocity; it is said to possess "hidden momentum" [20,31], which will play a key role in this discussion.

III. CENTER OF MASS AND THE SIGNIFICANCE OF THE SPIN SUPPLEMENTARY CONDITION

In relativistic physics, the center of mass of a spinning particle is observer-dependent. This is illustrated in Fig. 1.

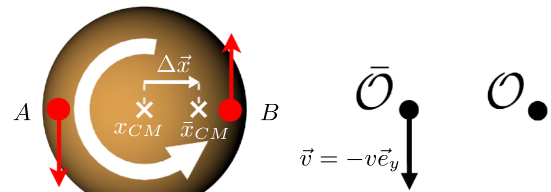

FIG. 1 (color online). Center of mass of a free spinning particle ($\vec{S} = S\vec{e}_z$, orthogonal to the page) as evaluated by two different observers. Observer $\mathcal{O}$, of 4-velocity $u^\alpha = P^\alpha/M$, is *at rest* with respect to center of mass $x^i_{\rm CM} \equiv x^i_{\rm CM}(u)$ it measures (i.e., $x^i_{\rm CM}$ is a *proper* center of mass). Observer $\bar{\mathcal{O}}$, moving with velocity $\vec{v} = -v\vec{e}_y$ relative to $\mathcal{O}$, sees the points on the right hemisphere (e.g., point B) moving faster than the points in the left hemisphere (e.g., point A), and, therefore, for $\bar{\mathcal{O}}$, the right hemisphere will be more massive than the left one. This means that the center of mass $\bar{x}^i_{\rm CM} \equiv x^i_{\rm CM}(\bar{u})$ as evaluated in the moving frame of $\bar{\mathcal{O}}$ is shifted to the right (relative to $x_{\rm CM}$). The shift is exactly $\Delta\vec{x} = \vec{S}_\star \times \vec{v}/M$.

Thus one needs to specify the frame in which the center of mass is to be evaluated. That can be done through a spin condition of the type $S^{\alpha\beta}u_\beta = 0$, as we will show next. The vector $(d^u_G)^\alpha \equiv -S^{\alpha\beta}u_\beta$ yields the "mass dipole moment" as measured in the rest frame of the observer $\mathcal{O}$ of 4-velocity u^α. This is easily seen in this frame, where $u^i = 0$ and $S^{\alpha\beta}u_\beta = S^{\alpha 0}u_0$. Thus, from Eq. (2),

$$S^{i0} = 2\int_{\Sigma(\tau,u)} r^{[i}T^{0]\gamma}d\Sigma_\gamma = \int x^i T^{00}d^3x - m(u)z^i, \quad (5)$$

where, as before, $r^\alpha \equiv x^\alpha - z^\alpha$ [note that $r^0 = 0$, since the integration is performed in the hypersurface $\Sigma(u)$ orthogonal to u^α], and $m(u) \equiv -P^\alpha u_\alpha$ denotes the mass as measured in the frame $\mathcal{O}$. The first term of (5) is by definition $m(u)x^i_{\rm CM}(u)$, where $x^i_{\rm CM}(u)$ are the coordinates of the center of mass as measured by $\mathcal{O}$, and so

$$x^i_{\rm CM}(u) - z^i = \frac{S^{i0}}{m(u)} \Leftrightarrow x^\alpha_{\rm CM}(u) - z^\alpha = -\frac{S^{\alpha\beta}u_\beta}{m(u)}. \quad (6)$$

Thus we see that the condition $S^{\alpha\beta}u_\beta = 0$ is precisely the condition that the reference worldline $z^\alpha(\tau)$ is the center of mass as measured in this frame. It allows us to write $S^{\alpha\beta} = \epsilon^{\alpha\beta}{}_{\gamma\delta}S^\gamma u^\delta$, where $S^\alpha(u)$ is the spin 4-vector, defined as being the 4-vector with components $(0, \vec{S})$ in the rest frame of $\mathcal{O}$, that is, $S^\alpha = \frac{1}{2}\epsilon^\alpha{}_{\beta\gamma\delta}u^\beta S^{\gamma\delta}$. In order to see how the center of mass position changes in a change of observer, consider now another observer $\bar{\mathcal{O}}$ moving relative to $\mathcal{O}$ with 4-velocity $\bar{u}^\alpha = \bar{u}^0(1, \vec{v})$; for this observer the center of mass will be at a different position, as depicted in Fig. 1. The "mass dipole moment" as measured by $\bar{\mathcal{O}}$ is[2] $(d^{\bar{u}}_G)^\alpha = -S^{\alpha\beta}\bar{u}_\beta$; thus, the center of mass as measured by $\bar{\mathcal{O}}$ is displaced by a vector $\Delta x^\alpha = -S^{\alpha\beta}\bar{u}_\beta/m(\bar{u})$ relative to the reference worldline z^α, where $m(\bar{u}) \equiv -P^\gamma\bar{u}_\gamma$ denotes the mass of the particle as measured by $\bar{\mathcal{O}}$. Hence we get

$$\Delta x^i = \frac{1}{P^\gamma\bar{u}_\gamma}(S^{i0}\bar{u}_0 + S^{ij}\bar{u}_j) = \frac{(\vec{S}\times\vec{v})^i}{P^0 - P^i v_i}, \quad (7)$$

(recall that, in this frame, $S^{i0} = x^i_{\rm CM}(u) - z^i = 0$, since we chose $x^i_{\rm CM}(u)$ as the reference worldline). Note that the coordinates of the 3-vector Δx^i are the same in the frame $\mathcal{O}$ or $\bar{\mathcal{O}}$, since $\Delta\vec{x} \perp \vec{v}$. If $u^\alpha = P^\alpha/M$, $M \equiv \sqrt{-P^\alpha P_\alpha}$, i.e., if we take as reference worldline the center of mass $x^\alpha_{\rm CM}(P)$ as measured in the zero 3-momentum frame, then:

$$\Delta x^i = \frac{(\vec{S}_\star \times \vec{v})^i}{M}, \quad (8)$$

[1]Substituting (4) in (3a) and (3b), we obtain the equations in Mathisson's representation [1,12,22,29]; in this case we would have 10 independent unknowns (m, three independent components of U^α, and six of $S^{\alpha\beta}$), for seven independent equations: four from (3a) and only three from (3b), since contracting the latter with U^α leads to an identity.

[2]We have shown in discussion above and in Eq. (5) that $(d^u_G)^\alpha \equiv -S^{\alpha\beta}u_\beta$, with $S^{\alpha\beta}$ defined integrating in an hypersurface $\Sigma(\tau, u)$ orthogonal to u^α, yields the mass dipole as measured by $\mathcal{O}$. Note that it follows from the conservation laws $T^{\alpha\beta}{}_{;\beta} = 0$ that the *2-tensor* **S** does not depend on Σ. Only its *components* $S^{\alpha\beta}$ do, since a choice of Σ amounts in this case to *choose the frame* where $S^{\alpha\beta}$ are expressed (see, e.g., [30]). Hence $(d^{\bar{u}}_G)^\alpha = -S^{\alpha\beta}\bar{u}_\beta$, with $S^{\alpha\beta}$ again defined with respect to $\Sigma(\tau, u)$, yields indeed the mass dipole measured by $\bar{\mathcal{O}}$, only written in the coordinates of $\mathcal{O}$. But since $\vec{d}_G(\bar{u}) \perp \vec{v}$, cf. Equation (7), the coordinates $(d^{\bar{u}}_G)^\alpha$ are actually the same in the systems of $\mathcal{O}$ and $\bar{\mathcal{O}}$.

where we denote by $S_{\star}^{\alpha\beta}$ the angular momentum with respect to $x^{\alpha}_{\rm CM}(P)$, cf. point 7 of Sec. IA. In general one wants the equations of motion not to depend on quantities (the center of mass) measured by a particular observer, but instead a center of mass defined only in terms of properties "intrinsic" to the particle. Two conditions accomplishing this are frequently found in the literature[3]: the Mathisson-Pirani condition

$$S^{\alpha\beta}U_{\beta} = 0 \tag{9}$$

(that is, $u^{\alpha} = U^{\alpha}$), and the Tulczyjew-Dixon condition $S^{\alpha\beta}P_{\beta} = 0$ (that is, $u^{\alpha} = P^{\alpha}/M$). The latter amounts to taking as reference worldline the center of mass as measured in the frame of zero 3-momentum, $P^i = 0$; the former, Eq. (9), comes as the most natural choice, as it amounts to compute the center of mass in its proper frame, i.e., *in the frame where it has zero 3-velocity*. Such center of mass is dubbed a "proper center of mass."

IV. MATHISSON'S HELICAL SOLUTIONS

Using the Mathisson-Pirani condition (9), implying $S^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}S_{\mu}U_{\nu}$, we can rewrite (4) as

$$P^{\alpha} = mU^{\alpha} + S^{\alpha\beta}a_{\beta} = mU^{\alpha} + \epsilon^{\alpha\beta}{}_{\gamma\delta}a_{\beta}S^{\gamma}U^{\delta}, \tag{10}$$

where $a^{\alpha} = DU^{\alpha}/d\tau$. It follows from Eqs. (3a) and (10) that the proper mass $m = -P^{\alpha}U_{\alpha}$ is a *constant* of the motion: $dm/d\tau = 0$. Eq. (3b) can be written as $DS^{\mu}/d\tau = a_{\nu}S^{\nu}U^{\mu}$, stating that the spin vector S^{α} is Fermi-Walker transported along the CM worldline. This equation, coupled with (10) and (3a), effectively means that the spin vector is parallel transported

$$\frac{DS^{\alpha}}{d\tau} = 0,$$

since, as can be seen substituting (10) in (3a) and contracting with S^{α}, the spin vector is orthogonal to the acceleration: $a_{\nu}S^{\nu} = 0$. Noting, from Eq. (10), that $P^{\alpha}S_{\alpha} = 0$, we can take, without loss of generality, the constant spin vector pointing along the z-axis

$$S^{\alpha} = (0, 0, 0, S),$$

in the global Cartesian frame of zero 3-momentum

$$P^{\alpha} = (M, 0, 0, 0) = \left(\frac{m}{\gamma}, 0, 0, 0\right).$$

Here $M \equiv \sqrt{-P^{\alpha}P_{\alpha}}$ denotes the mass/energy of the particle as measured in this frame, and $\gamma \equiv -P^{\alpha}U_{\alpha}/M$ is a constant. The equations of motion to be solved are (10). These require $U^t = \gamma$, $U^z = 0$ and

$$U^x + \frac{\gamma S}{m}\frac{dU^y}{d\tau} = 0, \qquad U^y - \frac{\gamma S}{m}\frac{dU^x}{d\tau} = 0. \tag{11}$$

The general solution for the worldline of reference describes the famous *helical motions*, which correspond to *clockwise* (i.e., *opposite* to the spin direction) circular motions with radius R and speed v on the xy plane; taking their center as the spatial origin of the frame, they read

$$z^{\alpha}(\tau) = \left(\gamma\tau, -R\cos\left(\frac{v\gamma}{R}\tau\right), R\sin\left(\frac{v\gamma}{R}\tau\right), 0\right), \tag{12}$$

(where $\tau \equiv$ proper time, $\gamma = 1/\sqrt{1 - v^2}$) with 4-velocity and acceleration

$$U^{\alpha} = \left(\gamma, v\gamma\sin\left(\frac{v\gamma}{R}\tau\right), v\gamma\cos\left(\frac{v\gamma}{R}\tau\right), 0\right), \tag{13}$$

$$a^{\alpha} = \frac{v^2\gamma^2}{R}\left(0, \cos\left(\frac{v\gamma}{R}\tau\right), -\sin\left(\frac{v\gamma}{R}\tau\right), 0\right). \tag{14}$$

The radius of the trajectory R is

$$R = \frac{v\gamma^2 S}{m}. \tag{15}$$

All these helical solutions are equivalent descriptions of the motion of a spinning body, the difference between them being the representative point they use to describe the body. Note that (this is true in flat spacetime, and in the absence of electromagnetic field) the nonhelical solution $R = 0$ corresponds to $P^{\alpha} \parallel U^{\alpha}$, i.e., to the (unique) solution defined by the Tulczyjew-Dixon condition $S^{\alpha\beta}P_{\beta} = 0$. The center of the circular motions [i.e., the spatial origin of the frame in Eq. (12)] is thus the centroid measured in the zero 3-momentum frame, $x^{\alpha}_{\rm CM}(P)$.

The fact that γ in Eq. (15) can be arbitrarily large has led some authors [4,5,15,28,29] to believe that the same extended body may be represented by circular trajectories with any radius. That would be inconsistent with the results in Sec. V below, with Möller's treatment in Ref. [6], and with the results in, e.g., [24]. This is not the case, however; as we shall now show; indeed the radius is finite (and confined to the disk of centroids, cf. Figure 2), and the misunderstanding originates from the fact that keeping the parameters m and S fixed *does not* correspond to considering the same extended body.

In a multipole expansion, an extended body is characterized by its multipole moments. In the pole-dipole approximation, that amounts to specifying its momentum P^{α} and its spin tensor $S^{\alpha\beta}$. These, cf. Equations (1) and (2), are defined with respect to an hypersurface of integration Σ (which is interpreted as the rest space of the observer, see also Footnote 2), and, in the case of $S^{\alpha\beta}$, also with respect to a reference worldline z^{α}. Different representations of the same extended body must yield the same

[3]A review (with a comprehensive list of references) on the literature regarding this subject may be found in [32].

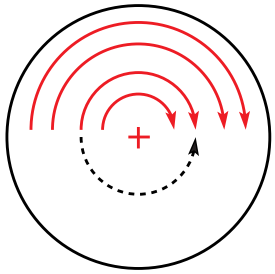

FIG. 2 (color online). Kinematical explanation of the helical motions allowed by $S^{\alpha\beta}U_\beta = 0$: every point within a disk of radius $S_\star/M$ is a *centroid* corresponding to some observer; and it is also a *proper* center of mass *if it rotates with angular velocity* $\omega = M/S_\star$ *in the opposite sense of the spinning body* (solid red lines).

moments with respect to *the same observer* and the *same reference worldline*. So instead of $m = -U^\alpha P_\alpha$ (which depends, via U^α, on the particular helix chosen), we must in fact fix

$$M = \sqrt{-P_\alpha P^\alpha} = \frac{m}{\gamma}. \tag{16}$$

Similarly, it is *not* the spin vector S^α (nor the spin tensor $S^{\alpha\beta}$ obeying $S^{\alpha\beta}U_\beta = 0$) that we must fix for different trajectories representing the same extended body. By choosing the Mathisson-Pirani condition, the tensor $S^{\alpha\beta}$ showing up in Eqs. (3) is always orthogonal to U^α; as explained in Sec. III, that means that $S^{\alpha\beta}$ is the angular momentum evaluated with respect to $x^\alpha_{\rm CM}(U)$, i.e., the center of mass as measured by the observer of 4-velocity U^α. Let U^α and $\bar{U}^\alpha$ denote the 4-velocity vectors, Eq. (13), of two different helical representations. The tensor $S^{\alpha\beta}$, obeying $S^{\alpha\beta}U_\beta = 0$, must be, in general, different from the tensor $\bar{S}^{\alpha\beta}$, obeying $\bar{S}^{\alpha\beta}\bar{U}_\beta = 0$, *if* $S^{\alpha\beta}$ *and* $\bar{S}^{\alpha\beta}$ *are to represent the same body*, since the former is the angular momentum about the point $x^\alpha_{\rm CM}(U)$, and the latter about the point $x^\alpha_{\rm CM}(\bar{U})$.

Let $S_\star^{\alpha\beta}$ denote the spin tensor for the *nonhelical* trajectory (corresponding to $R = 0$, $\tau = t$, $z^\alpha(\tau) = \tau\delta^\alpha_0$ and $U^\alpha = P^\alpha/M$),

$$S_\star^{\alpha\beta} = 2\int_{\Sigma(\tau,P)} r^{[\alpha}T^{\beta]\gamma}d\Sigma_\gamma \tag{17}$$

with $r^\alpha = x^\alpha - z^\alpha(\tau)$. This corresponds to a spin vector

$$S_\star^\alpha = (0, 0, 0, S_\star), \tag{18}$$

and so

$$S_\star^{\alpha\beta} = \epsilon^{\alpha\beta}{}_{\mu\nu}S_\star^\mu U^\nu = \epsilon^{\alpha\beta}{}_{30}S_\star = \epsilon^{0\alpha\beta 3}S_\star. \tag{19}$$

Therefore the nonvanishing components of $S_\star^{\alpha\beta}$ are $S_\star^{12} = -S_\star^{21} = S_\star$. The spin tensor for a helical trajectory, however, is (in the same global Cartesian frame of zero 3-momentum[4]):

$$S^{\alpha\beta} = 2\int_{\Sigma(\tau,P)} \bar{r}^{[\alpha}T^{\beta]\gamma}d\Sigma_\gamma, \tag{20}$$

where $\bar{r}^\alpha = x^\alpha - \bar{z}^\alpha(\tau)$, $\bar{z}^\alpha(\tau)$ being given by Eq. (12) with $R \neq 0$. Therefore

$$S^{\alpha\beta} = S_\star^{\alpha\beta} - 2\bar{z}^{[\alpha}(\tau)P^{\beta]}. \tag{21}$$

The condition $S^{\alpha\beta}U_\beta = 0$ yields

$$S_\star^{\alpha\beta}U_\beta + m\bar{z}^\alpha(\tau) - \gamma^2\tau P^\alpha = 0, \tag{22}$$

which when written out in components reduces to

$$mR = v\gamma S_\star, \tag{23}$$

and so

$$S_\star = \gamma S. \tag{24}$$

The fixed quantities for a given body are then $\gamma S = S_\star$ and $m/\gamma = M$, not m and S. Accordingly, the same extended body will be represented by helical trajectories whose radius R satisfies

$$R = \frac{vS_\star}{M}, \tag{25}$$

and thus must be smaller than $S_\star/M$. The angular frequency of the helices is, from Eqs. (12) and (25),

$$\omega = \frac{v}{R} = \frac{M}{S_\star} = \frac{m}{\gamma^2 S}, \tag{26}$$

which is thus the same for all helical solutions representing the same extended body. As we shall see in the next section, this is entirely consistent with Möller's picture of the disk, rotating *rigidly* with frequency ω, formed by the many proper centers of mass.

V. KINEMATICAL EXPLANATION OF THE HELICAL MOTIONS

In this section we will provide a kinematical explanation for the helical motions. Although stated and derived in a different form, it is equivalent to Möller's treatment in [6], which does not seem to be well understood in the literature.

The origin of the helical motions is the fact that the condition $S^{\alpha\beta}U_\beta = 0$ does not determine the reference worldline uniquely (i.e., it *does not fix completely the gauge freedom*). In other words, there is not a unique

[4]Note that integrating in the hypersurface $\Sigma(\tau, P)$, orthogonal to P^α, amounts to write $S^{\alpha\beta}$ in the frame $P^i = 0$, see Footnote 2.

answer to the question: which is the center of mass such that it is at rest relative to the frame where it is evaluated? In order to see this, consider for simplicity a free particle in flat spacetime. Clearly, for this case, *one of the solutions* of Eqs. (3a) and (3b) supplemented by (9) is straight line motion, with $U^\alpha = P^\alpha/M$ constant. Let $\mathcal{O}$ be the observer of 4-velocity $u^\alpha = P^\alpha/M$ (i.e., its rest frame is the zero 3-momentum frame). The center of mass as measured by this observer is the point $x^\alpha_{\rm CM}(P)$ in Fig. 1. This point is at rest relative to $\mathcal{O}$, so that it is clearly a proper center of mass. But now let again $\bar{\mathcal{O}}$ be an observer moving relative to $\mathcal{O}$ with 3-velocity $\vec{v}$. The 4-velocity of $\bar{\mathcal{O}}$ is $\bar{u}^\alpha = \gamma(u^\alpha + v^\alpha)$, where $\gamma \equiv -u_\alpha \bar{u}^\alpha$ and v^α is the relative velocity vector which is spatial with respect to u^α. Observer $\bar{\mathcal{O}}$ measures the center of mass $x^\alpha_{\rm CM}(\bar{u})$ (i.e., its centroid) to be at a different position, as shown by Eq. (8); and in general that point will *not* be a proper center of mass, since it will be moving relative to $\bar{\mathcal{O}}$. Choosing $x^\alpha_{\rm CM}(P)$ as our reference worldline ($z^\alpha = x^\alpha_{\rm CM}(P)$), let τ_P be the proper time along it. The relative position $\Delta x^\alpha = x^\alpha_{\rm CM}(\bar{u}) - x^\alpha_{\rm CM}(P)$ is the spatial (with respect to u^α) vector $\Delta x^\alpha = S_\star^{\alpha\beta}\bar{u}_\beta/P^\gamma u_\gamma = -S_\star^{\alpha\beta}v_\beta/M$. Noting, from Eqs. (3a) and (3b), that $DS_\star^{\alpha\beta}/d\tau_P = 0$, it evolves along $z^\alpha(\tau_P)$ as

$$\frac{D\Delta x^\alpha}{d\tau_P} = -\frac{S_\star^{\alpha\beta}}{M}\frac{Dv_\beta}{d\tau_P} \Leftrightarrow \frac{d\vec{\Delta x}}{dt} = \frac{\vec{S}_\star \times \vec{a}_c}{M}. \tag{27}$$

The second equation holds in the rest frame of $\mathcal{O}$ (the frame $u^i = 0 = P^i$), where the time coordinate is $t = \tau_P$, and $\vec{a}_c \equiv d\vec{v}/dt$ denotes the *coordinate* acceleration of observer $\bar{\mathcal{O}}$ in the frame of $\mathcal{O}$. Note that it can be directly obtained from (8) by simply differentiating with respect to the coordinate t. Thus we see that if v^α is parallel transported along $z^\alpha(\tau_P)$, which in flat spacetime is ensured by taking $\bar{\mathcal{O}}$ inertial, then $D\Delta x^\alpha/d\tau_P = 0$, implying that $x^\alpha_{\rm CM}(\bar{u})$ is fixed relative to $x^\alpha_{\rm CM}(P)$; and thus moves relative to $\bar{\mathcal{O}}$ at a speed $-\vec{v}$. The set of centroids measured by all the possible *inertial* observers forms a disk of points *all at rest* with respect to each other and (again, for a *free particle in flat spacetime*) with respect to $x^\alpha_{\rm CM}(P)$, around which the disk is centered. However if we consider $\bar{\mathcal{O}}$ to be accelerating, then the velocity $\vec{v}_{\rm CM}(\bar{u}) = d\Delta\vec{x}/dt$ of the centroid he measures changes in a nontrivial way, as shown by Eqs. (27). Now if we take the case that $\bar{\mathcal{O}}$ itself also moves with 3-velocity

$$\vec{v} = \frac{\vec{S}_\star \times \vec{a}_c}{M} \equiv \frac{1}{M}\left(\vec{S}_\star \times \frac{d\vec{v}}{dt}\right), \tag{28}$$

then $x^\alpha_{\rm CM}(\bar{u})$ is at rest relative to $\bar{\mathcal{O}}$, i.e., it is a *proper* center of mass. Equation (28) is equivalent to Eqs. (11), its solutions being circular motions in the plane orthogonal to $\vec{S}_\star$, with radius $R = \Delta x = |\vec{v} \times \vec{S}_\star|/M$, and angular velocity $\vec{\omega} = -M\vec{S}_\star/S_\star^2$. Note that the angular velocity is *constant* (does not depend on R) and is in *opposite sense* to the rotation of the body. Hence the set of all possible *proper* centers of mass fills a disk of radius $\Delta x_{\rm max} = S_\star/M$ (i.e., of the same size of the disk of centroids) in the plane orthogonal to $\vec{S}_\star$, *counter-rotating rigidly* with angular velocity $\vec{\omega}$. In other words: from Eq. (8) we see that the possible centroids measured by the different observers fill a disk of radius $R_{\rm max} = S_\star/M$ about the point $x^\alpha_{\rm CM}(P)$; every point of such disk could also be a proper center of mass, provided that it rotates with angular velocity $\vec{\omega}$. This is the result found by Möller [6]. In a frame where $P^i \neq 0$ (i.e., moving relative to $\mathcal{O}$) this leads to helical motions, as depicted in Fig. 2, which are precisely the ones explicitly derived in the previous section. We emphasize that the angular velocity $\vec{\omega}$ of the disk of proper centers of mass is *not* the same as the angular velocity the body; indeed it is *opposite* to the sense of rotation of the body; and note also that the points of the circle $R_{\rm max} = S_\star/M$ move at the speed of light.

Finally, it is clear, from the analysis above, that *all the helical solutions are contained within a tube of radius* $R_{\rm max} = S_\star/M$, which is actually the *minimum* size a classical spinning particle can have if it is to have finite $S_\star$ and positive mass without violating the laws of special relativity (see also [6,33]) .

VI. DYNAMICAL INTERPRETATION OF THE HELICAL MOTIONS

The concept of hidden momentum is central to the understanding of dynamics of the helical solutions; namely, it explains how the motion of a free spinning particle can be consistently described by helical solutions without violating any conservation principle, and that they are a phenomenon which can be cast as analogous to the bobbing of a magnetic dipole in an external electric field studied in [20]. As we have seen in the previous sections, for a *free* spinning particle in flat spacetime, Eqs. (3a) and (3b), supplemented with (9), yield, as possible solutions, straight line motion plus a set of helical motions contained within a tube of radius $S_\star/M$. This seems odd at first glance: how can a solution where the center of mass of the particle is accelerating without any external force be physically acceptable? The answer is that the acceleration results from an interchange between kinetic momentum mU^α and hidden momentum $S^{\alpha\beta}a_\beta$ (we dub it "inertial" hidden momentum, the reason for such denomination being explained below) which occurs in a way such that their variations cancel out at every instant, keeping the total momentum constant, as illustrated in Fig. 3. That is what we are going to show next. Consider a generic spin condition $S^{\alpha\beta}u_\beta = 0$, where u^α denotes the 4-velocity of an arbitrary observer $\mathcal{O}(u)$. As discussed in Sec. III, this

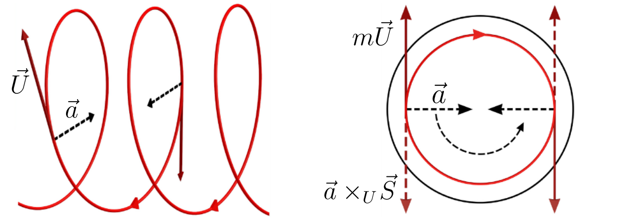

FIG. 3 (color online). Hidden momentum provides dynamical interpretation for the helical motions (left panel): the acceleration results from an interchange between kinetic $P^{\alpha}_{\rm kin} = mU^{\alpha}$ and hidden "inertial" momentum $P^{\alpha}_{\rm hid} = S^{\alpha\beta}a_{\beta}$, which occurs in a way that their variations cancel out at every instant, keeping the total momentum constant. This is made manifest in the right panel, representing the $\vec{P} = 0$ frame, wherein $\vec{P}_{\rm hid} = \vec{a}\times_U\vec{S} = -m\vec{U} = -\vec{P}_{\rm kin}$. The description is formally analogous to the bobbing [20] of a magnetic dipole orbiting a cylindrical charge.

condition means that we take as reference worldline the center of mass as measured by $\mathcal{O}(u)$. Contracting (3b) with u_{β}, and using $S^{\alpha\beta}u_{\beta} = 0 \Rightarrow u_{\beta}DS^{\alpha\beta}/d\tau = -S^{\alpha\beta}Du_{\beta}/d\tau$, leads to

$$S^{\alpha\beta}\frac{Du_{\beta}}{d\tau} = \gamma(u, U)P^{\alpha} - m(u)U^{\alpha}, \qquad (29)$$

where $\gamma(u, U) \equiv -U^{\beta}u_{\beta}$ and $m(u) \equiv -P^{\beta}u_{\beta}$. Hence, if $Du_{\beta}/d\tau \neq 0$, then in general the momentum is not parallel to the 4-velocity: $P^{[\alpha}U^{\beta]} \neq 0$, and the particle is said to have *hidden momentum* [20]. The momentum of the particle may be split in its projections parallel and orthogonal to the CM 4-velocity U^{α}:

$$P^{\alpha} = P^{\alpha}_{\rm kin} + P^{\alpha}_{\rm hid}; \quad P^{\alpha}_{\rm kin} \equiv mU^{\alpha}, \quad P^{\alpha}_{\rm hid} \equiv (h^U)^{\alpha}{}_{\beta}P^{\beta}, \quad (30)$$

where $(h^U)^{\alpha}{}_{\beta} \equiv U^{\alpha}U_{\beta} + \delta^{\alpha}{}_{\beta}$ denotes the space projector with respect to U^{α}. We dub the time projection $P^{\alpha}_{\rm kin} = mU^{\alpha}$ "kinetic momentum" associated with the motion of the centroid; and the component $P^{\alpha}_{\rm hid}$ orthogonal to U^{α} is what we dub "hidden momentum." The reason for the latter denomination is easily seen taking the perspective of an observer $\mathcal{O}(U)$ comoving with the particle: in the frame of $\mathcal{O}(U)$ (i.e., the frame $U^i = 0$) the 3-momentum is in general not zero: $\vec{P} = \vec{P}_{\rm hid} \neq 0$; however, by definition, the particle's CM is at rest in that frame; such momentum must thus be hidden somehow.

Now, if $Du_{\beta}/d\tau = 0$, that is, if we take as reference worldline the center of mass as measured by an observer $\mathcal{O}(u)$ *such that* u^{α} *is parallel transported along it*, then from Eq. (29) we have $P^{\alpha} \parallel U^{\alpha}$, and there is no hidden momentum. This is actually cast in [24] as one of the possible spin supplementary conditions. Thus indeed this form of hidden momentum is pure gauge; it also means that the motion effects induced by it (such as the bobbings studied in [20]) must be confined to the worldtube of centroids, so that they can be made to vanish by a suitable choice of reference worldline. For this reason it is dubbed in [20] "kinematical hidden momentum" (by contrast with gauge-invariant hidden momentum present in electromagnetic systems, dubbed "dynamic" therein). In flat spacetime, we can say that if the observer is inertial (which implies that its 4-velocity is parallel transported along the particle's worldline), then there is no hidden momentum. (The "laboratory" observer considered in p. 9 of [20], for the case of flat spacetime, is an example of an inertial observer, more precisely the static observer with u^{α} tangent to the time Killing vector).

This hidden momentum is, of course, related to the relativity of the center of mass (its shift in different frames, discussed in Sec. III), and taking this perspective makes quite clear why U^{α} decouples from P^{α} if $Du^{\alpha}/d\tau \neq 0$, and the hidden momentum arises. But first let us make some remarks:

Remark.—Whereas in the previous sections we dealt essentially with flat spacetime, Eq. (29) above is general. In the previous sections we illustrated the nonuniqueness of the center of mass by its relativity with respect to what we called "observers" and/or "frames," and that their "acceleration" was the underlying reason behind the nontrivial velocity the centroid has in some cases. But what we are implicitly doing (and what actually holds in a more general formulation e.g., [15,16,20,31]), is to assume a continuous field of timelike unit vectors u^{α} along $z^{\alpha}(\tau)$; at each event, u^{α} provides the hypersurface $\Sigma(u, \tau)$ over which the integrals defining the moments P^{α}, $S^{\alpha\beta}$ (as well as the center of mass) are performed. $\Sigma(u, \tau)$ is generically defined as the hypersurface formed by all geodesics orthogonal to u^{α} at the point $z^{\alpha}(\tau)$. Thus in this construction, the vectors u^{α} (which we can always think about as the instantaneous 4-velocity of some *local* observer) are all that matter; the concept of an observer $\mathcal{O}(u)$, in the traditional sense of a worldline to which u^{α} is tangent, has no place; except for the case $u^{\alpha} = U^{\alpha} \equiv dz^{\alpha}/d\tau$, there is no worldline tangent to the field u^{α} (and therefore no acceleration is defined for it). The field u^{α} only has to exist along the reference worldline, and $Du^{\alpha}/d\tau$ is the only derivative defined for it. In the special case of flat spacetime (but not in general curved spacetime!), where vectors at different points can be compared, we can indeed think of the field u^{α} as the tangent to the worldline of some distant (as such worldline in general will not coincide with z^{α}) observer, and $Dv^{\alpha}/d\tau$ as its coordinate acceleration with respect to the CM frame $U^i = 0$. This is what was implicitly done in Sec. V.

As we have seen in Sec. III, the position of the centroid of a spinning body depends on the vector u^{α} relative to which it is computed. If that vector varies along the reference worldline it is clear that this is reflected in the velocity U^{α} of the centroid, which in general will accelerate even without the action of any force. Also U^{α} will in general no longer be parallel to P^{α}, i.e., $\vec{U} \neq 0$, and thus the centroid is not at rest in the frame $P^i = 0$. Let us show explicitly that the decoupling of U^α from P^α, manifest in Eq. (29), indeed comes from the shift of the centroid, given in Eq. (6). As we have seen in Sec. III, if we choose the reference line to be the center of momentum centroid, $z^\alpha = x^\alpha_{\rm CM}(P)$, then the shift of the centroid measured by the observer u^α is $\Delta x^\alpha = -S_\star^{\alpha\beta}u_\beta/m(u)$, with $m(u) \equiv -P^\alpha u_\alpha$. Therefore

$$\frac{D\Delta x^\alpha}{d\tau} = -\frac{S_\star^{\alpha\beta}}{m(u)}\frac{Du_\beta}{d\tau} + \frac{dm(u)}{d\tau}\frac{S_\star^{\alpha\beta}u_\beta}{m(u)^2}, \qquad (31)$$

as by (3), $DS_\star^{\alpha\beta}/d\tau = 0$. By (21) we know that the spin tensor $S^{\alpha\beta}$ computed by u^α satisfies

$$S^{\alpha\beta} = S_\star^{\alpha\beta} - \Delta x^\alpha P^\beta + P^\alpha \Delta x^\beta. \qquad (32)$$

Substituting in (31), using $S^{\alpha\beta}u_\beta = 0$ and $\Delta x^\beta u_\beta = 0$, we obtain

$$m(u)\frac{D\Delta x^\alpha}{d\tau} = -S^{\alpha\beta}\frac{Du_\beta}{d\tau} - \Delta x^\alpha P^\beta \frac{Du_\beta}{d\tau} - P^\alpha \frac{D\Delta x^\beta}{d\tau}u_\beta - \frac{dm(u)}{d\tau}\Delta x^\alpha,$$

or, noticing that $dm(u)/d\tau = -P^\beta Du_\beta/d\tau$, we have

$$m(u)\frac{D\Delta x^\alpha}{d\tau} = -S^{\alpha\beta}\frac{Du_\beta}{d\tau} - P^\alpha \frac{D\Delta x^\beta}{d\tau}u_\beta. \qquad (33)$$

In flat spacetime and Cartesian coordinates, we may always write

$$U^\alpha = \frac{dx^\alpha_{\rm CM}(u)}{d\tau} = \frac{dx^\alpha_{\rm CM}(P)}{d\tau} + \frac{d\Delta x^\alpha}{d\tau} = fP^\alpha + \frac{D\Delta x^\alpha}{d\tau}, \quad (34)$$

where f is a function to be determined. Since $\Delta x^\alpha P_\alpha = 0$ and $DP^\alpha/d\tau = 0$, it follows that $P_\alpha D\Delta x^\alpha/d\tau = 0$; thus contracting (34) with P_α we obtain $f = m/M^2$. Finally, substituting Eq. (34) for $D\Delta x^\alpha/d\tau$ in (33), we obtain (29) exactly.

Hence we have different, and equivalent, descriptions for the same motion (of a free particle in flat spacetime). The most simple ones are the centroids measured by every possible inertial observers, whose trajectories are straight lines parallel to each other, and to P^α. In the frame $P^i = 0$, all these centroids are at rest. But if we take the centroid with respect to an u^α not constant along the curve, which, as discussed above, may be thought as the point of view of some accelerated observer $\mathcal{O}(u)$, then the centroid will have in general a different velocity, and also accelerate, see Eqs. (27). However P^α is always the same (it does not depend on the choice of centroid)! This makes evident the role of $P^\alpha_{\rm hid}$ in a consistent dynamical description: when one describes the body through the centroid measured by an accelerated observer, *there must be a hidden momentum* $\vec{P}_{\rm hid}$ that cancels out the kinetic momentum $\vec{P}_{\rm kin} = m\vec{U}$ the moving centroid $x^\alpha_{\rm CM}(u)$ has in the frame $P^i = 0$.

If the observer's acceleration itself changes in a way such that the signal in Eqs. (27) oscillates, we may have a bobbing; or if it is such that $\mathcal{O}(u)$ sees its centroid to be at rest (i.e., if $\mathcal{O}(u)$ moves with 3-velocity (28) in the frame $P^i = 0$), then we have a helical solution. In this case ($u^\alpha = U^\alpha$), decomposition (30) takes a simple form, cf. Equation (10):

$$P^\alpha_{\rm kin} = mU^\alpha, \quad P^\alpha_{\rm hid} = S^{\alpha\beta}a_\beta = \epsilon^{\alpha\beta}{}_{\gamma\delta}a_\beta S^\gamma U^\delta; \qquad (35)$$

which in the frame $U^i = 0$ reads $\vec{P} = \vec{P}_{\rm hid} = \vec{a}\times\vec{S}$. Since $\vec{G} = -\vec{a}$ is the "gravitoelectric" field [34,35] as measured in that frame (which is a field of "inertial forces"), $\vec{P}_{\rm hid}$ is cast in [31] as the inertial analogue of the hidden momentum $\vec{\mu}\times\vec{E}$ of electromagnetic systems (see, e.g., [20,36]), and its origin explained therein by an analogous model. In this spirit, the dynamics of the helical representations may be cast as analogous to the bobbing of a magnetic dipole orbiting a cylindrical charge, discussed in Sec. III.B.1 of [20]. Let the line charge be along the z axis, and $\vec{E}$ the electric field it produces; and consider an oppositely charged test particle, with magnetic dipole moment $\vec{\mu} = (\mu^x, \mu^y, 0)$, orbiting it. The z component of the force vanishes for this setup; hence $P^z = 0 =$ constant. But the particle will possess a hidden momentum, which for slow motion [20] reads $\vec{P}_{\rm hid} = \vec{\mu}\times\vec{E}$; as it orbits the line charge, $\vec{P}_{\rm hid}$ oscillates between positive and negative values along the z-axis, implying the particle to bob up and down in order to keep the total momentum along z constant: $P^z = P^z_{\rm kin} + P^z_{\rm hid} = 0$. Thus, just like in the case of the helical motions, the bobbing arises not through the action of a force, but from an interchange between kinetic $\vec{P}_{\rm kin} = m\vec{U}$ and hidden momentum. The difference being that the hidden momentum $\vec{S}\times\vec{a}$ is *pure gauge* (which indeed allows a helical solution to be a consistent description of the motion even in the case of a free particle in flat spacetime, but can always be made to vanish by choosing the nonhelical representation), whereas, by contrast, the electromagnetic effect mentioned above is physical and *gauge-independent*.

Hence again we see that the straight line and helical solutions are *alternative* and *physically consistent* descriptions of the motion of a free spinning body: in the first case, we have no acceleration and no hidden momentum; in the second case we have a helix, but also inertial hidden momentum.

VII. CONCLUSION. MISCONCEPTIONS ABOUT THE HELICAL SOLUTIONS AND QUANTUM *ZITTERBEWEGUNG*

Mathisson's helical solutions have been deemed unphysical in a number of treatments, e.g., [15,28], due to the wrong idea that Eq. (15), which is equivalent to Eq. (25), allows the helical motions of (the representative point of) a given particle to have an arbitrarily large radius [4,5,15,28,29]. This would also imply that these solutions were not equivalent to the ones derived in Sec. V, and found in Möller's treatment [6]. As we have seen, this is just a misconception, based on the failure to notice that in order to have a set of helical solutions representing the same physical body, we must fix $\gamma S = S_\star$ and $m/\gamma = M$, not m and S; i.e., we must require that, regardless of the different possible representations, it has the *same moments* as measured with respect to the *same observer* and *reference worldline*.

There is nothing unphysical with Mathisson's helical solutions; they are all perfectly valid and equivalent descriptions of the motion of a classical spinning body. The helices are, as we have seen, all contained within a worldtube of radius $R_{\max} = S_\star/M$ centered at $x^\alpha(P)$ (i.e., the center of mass as measured in the zero 3-momentum frame). And that should be a natural result from the analysis in Sec. III: the radius of a helical motion of 4-velocity U^α corresponds to the displacement of the center of mass measured in the frame $U^i = 0$ relative to $x^\alpha(P)$; the maximum shift is $\Delta x_{\max} = S_\star/M$, corresponding to the case that the relative velocity between the two observers is the speed of light. Now one also has to note that $R_{\max} = S_\star/M$ is also the *minimum* size that a classical spinning particle can have if it is to have finite $S_\star$ and positive mass without violating special relativity (i.e., without containing points moving faster than the speed of light; see also [6,33]). This means that not only R is not arbitrarily large, but also it can never exceed the minimum size of the particle; i.e., the helical trajectories always fall *within* the convex hull of the body. Furthermore, they have a clear kinematical explanation as shown in Sec. V; and their dynamics may be interpreted in analogy with the hidden momentum of electromagnetic systems [31].

The helical solutions were interpreted by some authors [8,9,12,21,23] as the classical limit of the quantum *zitterbewegung*, due to the similarity between the *zitterbewegung* frequency of the Dirac equation for the electron and the frequency of the corresponding classical helical motions. Indeed, indentifying $S_\star = \hbar/2$ and $M = M_e$, we obtain from Eq. (26) that all classical helical representations for the free electron have the frequency $\omega = 2M_e/\hbar$, which is precisely Dirac's *zitterbewegung* frequency (this extends Mathisson's observation in [21] to the relativistic limit). Other authors [4,5,15,28] have rejected this correspondence, based on two arguments: 1) that Mathisson's helical solutions for the electron might have arbitrarily large radius which would make them macroscopically measurable [4,5,15,28]; 2) that the coincidence between the frequencies holds only in the nonrelativistic limit [4,5], based on the expression (26) in the form $\omega = m/\gamma^2 S$, and identifying instead $m = M_e$, $S = \hbar/2$. That is, repeating the same misunderstanding that led to the arbitrary radius, it would imply that different helical solutions corresponding to the electron would have different frequencies, which would only match Dirac's frequency in the nonrelativistic limit $\gamma \approx 1$. A deeper analysis of this problem will be presented elsewhere. Herein we would just like to point out that, as made clear by the analysis above, both arguments put forward against this correspondence between classical and quantum solutions arise from misconceptions.

Finally, an aspect that has drawn skepticism (see, e.g., [11,20]) into the equations of motion supplemented with the Mathisson-Pirani condition, Eqs. (3) and (10), is the fact that they are of third order in time, meaning that, in order for the motion to be determined,[5] not only one must prescribe the initial position and velocity (for known m and S^α), but also the initial acceleration. This might seem odd as in Newtonian mechanics (where the center of mass is an invariant) the motion of the CM of an extended body is fully determined by the force laws given its initial position and velocity. But in relativity that is only true for a monopole particle; for a general extended (spinning) body, due to the relativity of the center of mass, in addition to those two initial conditions, one needs also to determine the field of unit timelike vectors u^α relative to which the CM is computed. It is important to note that, as explained in Sec. VI, the acceleration of the CM does not originate solely from the force, but also from the variation of the field u^α along the CM worldline (leading to the hidden momentum). Some spin conditions, such as the Tulczyjew-Dixon [13,15,16] or the Corinaldesi-Papapetrou [38] conditions, fully fix the reference worldline and the vector field u^α along it; the Mathisson-Pirani condition, as explained in Sec. V, does not, and the higher order of the equations merely reflects that incompleteness of gauge fixing.

[5]It has been recently pointed out [37] that Eqs. (3) and (9) uniquely determine the solution *if* one prescribes, as initial conditions, $\{S^{\alpha\beta}, x^\alpha_{\rm CM}(U), P^\alpha\}$. This does not clash with the statements above, and is indeed correct: knowing P^α and $S^{\alpha\beta}$, we immediately determine the shift $\Delta x^\alpha = x^\alpha_{\rm CM}(U) - x^\alpha_{\rm CM}(P)$ from the expression $\Delta x^\alpha = S^{\alpha\beta}P_\beta/M^2$; since $x^\alpha_{\rm CM}(U)$ is given, then we know also $x^\alpha_{\rm CM}(P)$. $S^{\alpha\beta}_\star$ follows using (32). This tells us everything about the motion: it is a superposition of a circular motion of radius $R = \Delta x$, centered at $x^\alpha_{\rm CM}(P)$, and of angular velocity $\vec{\omega} = -M\vec{S}_\star/S_\star^2$, with a boost of 4-velocity P^α/M. The set of initial data $\{S^{\alpha\beta}, x^\alpha_{\rm CM}(U), P^\alpha\}$ is thus equivalent to $\{m, S^{\alpha\beta}, x^\alpha_{\rm CM}(U), U^\alpha, a^\alpha\}$.

Why does it matter?—In addition to the physical clarification of the helical motions, an important point made in this work is to prove the physical validity of the Mathisson-Pirani condition. We have shown it is as valid as any other of the infinite number of possible spin conditions. Indeed, a condition of the type $S^{\alpha\beta}u_\beta = 0$, for some unit timelike vector u^α, amounts to choosing as the representative point of the body the center of mass as measured by some observer of 4-velocity u^α. But u^α is arbitrary; whether it is itself the 4-velocity of the center of mass U^α (Mathisson-Pirani condition), or it is parallel to P^α (Tulczyjew-Dixon condition [13,15,16]), or it corresponds to the static observers in Schwarzschild spacetime (Papapetrou-Corinaldesi condition [38]), or any other type of observers (there is an infinite number of possibilities), the choice should be based on convenience.[6] So the question might be posed, why worry about Mathisson-Pirani's condition, which leads to degenerate solutions, while the Tulczyjew-Dixon condition yields a unique definition of CM? The point is that there are also situations where it is the Mathisson-Pirani condition that gives the simplest and more enlightening solution. For a free particle in flat spacetime, indeed the Tulczyjew-Dixon condition $S^{\alpha\beta}P_\beta = 0$ provides the simplest description for the center of mass motion, which is uniform straight line motion (and coincides *in this special case* with Mathisson's nonhelical solution), whereas the Mathisson-Pirani condition includes also the helical solutions, which are more complicated descriptions. However in the presence of gravitational and electromagnetic fields, the Tulczyjew-Dixon solution no longer coincides with any of Mathisson's solutions,[7] and it turns out, as exemplified in several applications in [31], that in some more complex setups it is the Mathisson-Pirani condition that provides the simplest and clearest description. This condition arises also in a natural fashion in a number of treatments [14,19] (see also [25]); for massless particles, it has been argued in [39,40] that it is actually the only one that can be applied. And for the case of the equation for the spin evolution (3b), it is *always* the Mathisson-Pirani condition that yields the simplest and physically more sound description: in the absence of electromagnetic field (or other external torques), S^α is Fermi-Walker transported; i.e., the gyroscope's axis is fixed relative to a nonrotating frame, which is the natural, expected result. The Tulczyjew-Dixon condition yields a different equation (see Eq. (7.11) of [16]), meaning that S^α undergoes transport orthogonal to P^α (dubbed therein ''M-transport''). There is no conflict because the transport is along a different worldline. But the equation for the evolution of S^α puts a strong emphasis on the relevance of acknowledging the physical validity of the Mathisson-Pirani condition, which has to do with the deepest notions of inertia and rotation in general relativity: a Fermi-Walker transported frame is by definition a frame that does not rotate relative to the local spacetime (i.e., to the ''local compass of inertia'', as described in some literature, e.g., [41]); in order for this law to be more than a mere mathematical abstraction, and for the rotation to be something absolute and locally measurable, this has to have a correspondence to a physical object, which is the torque-free gyroscope (gyroscopes are objects that oppose to changes in direction of their axis of rotation!). In a gravitational field, a spinning particle which has only pole-dipole moments (no quadrupole or higher moments) is the idealization corresponding to the torque-free gyroscope. *Only* if the Mathisson-Pirani condition holds is it Fermi-Walker transported. Hence, deeming the Mathisson-Pirani condition as physically unacceptable (as many authors do), amounts to saying that the whole concept of Fermi-Walker transport makes no sense from the physical point of view (or is at best an approximation).

Finally, probably one of the most interesting features of the Mathisson-Pirani condition is the fact that it makes explicit two *exact* gravito-electromagnetic analogies: the tidal tensor analogy relating the gravitational force on a spinning particle with the electromagnetic force exerted on a magnetic dipole, discussed in [31,42], and the analogy based on the ''gravitoelectromagnetic fields'' from the $1 + 3$ formalism discussed in, e.g., [31,34,35], the latter with the following realizations: one again relating the two forces [35], other relating the evolution of the spin of a gyroscope with the precession of a magnetic dipole under the action of a magnetic field [31,34,35], and a third one relating the hidden inertial momentum with the hidden momentum of electromagnetic systems [31]. These analogies provide valuable insight, and a familiar formalism to treat otherwise exotic gravitational effects, as well as a means to contrast them with their electromagnetic counterparts. Such comparison allows one to notice some fundamental aspects of both interactions, as explained in detail in [31].

[6]In this work we dealt with free particles in flat spacetime, where it was clear that all the centroids (including those corresponding to the helical motions) remain inside the worldtube of radius $S_\star/M$ forever. However, in the presence of strongly inhomogeneous external fields, the point we chose to represent the particle makes a difference. The trajectories are seen to diverge (outside any such worldtube) in [24] for a Kerr background. This actually signals the breakdown of the pole-dipole approximations (not that the spin conditions are not pure gauge after all!). The approximation is only acceptable when the choice of centroid (and the spin condition) does not matter; i.e., when the scale of variation of the external field is much larger than $S_\star/M$, cf. [24].

[7]When an electromagnetic and/or gravitational field (or any other external force) is present, $x^\alpha_{\rm CM}(P)$ is *not*, in general, a proper center of mass; i.e., the 4-velocity of the centroid defined by $S^{\alpha\beta}P_\beta = 0$ is not parallel to P^α; that can be seen from, e.g., Eq. (35) of [20] (see also discussion in [31]). In other words, the centroid measured in the $P^i = 0$ frame is not at rest in that frame.

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