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Convex Semi-Infinite Programming: Implicit Optimality Criterion Based on the Concept of Immobile Points

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Abstract

The paper deals with convex Semi-Infinite Programming (SIP) problems. A new concept of *immobility order* is introduced and an algorithm of determination of the immobility orders (*DIO algorithm*) and so called *immobile points* is suggested. It is shown that in the presence of the immobile points SIP problems do not satisfy the Slater condition. Given convex SIP problem, we determine all its immobile points and use them to formulate a Nonlinear Programming (NLP) problem in a special form. It is proved that optimality conditions for the (infinite) SIP problem can be formulated in terms of the analogous conditions for the corresponding (finite) NLP problem. The main result of the paper is the *Implicit Optimality Criterion* that permits to obtain new efficient optimality conditions for the convex SIP problems (even not satisfying the Slater condition) using the known results of the optimality theory of NLP.

Key words: semi-infinite programming, nonlinear programming, the Slater condition, optimality criterion

1 Introduction

Semi-Infinite Programming (SIP) models appear in mathematics, engineering, physics, social and other sciences when some processes or systems depend on a finite dimensional variable and are described with the help of an infinite number of constraints. In the last decades, semi-infinite optimization has become a topic of a special interest due to a number of practical applications and the relationship with other mathematical fields (for the references see [4], [11], [12]).

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In general, a SIP problem searches for a minimum of some function $c(x)$ (objective function) subject to an infinite system of constraints expressed as $f(x,t) \leq 0$ for all $t \in T$, where T is some set. Sometimes a solution of the SIP problem is found using *discretization approach* when the set T is replaced by some finite subset of its points (grid) and the initial infinite problem is reduced to a finitely constrained problem (or a sequence of such problems) where only the constraints corresponding to the points of the grid (or a sequence of grids respectively) are considered. Another approach is so called *reduction approach* when the infinite set of constraints of the initial problem is replaced by a finite set of constraints in the form $f(x,t^l(x)) \leq 0, l = 1,\dots,p,\ p \in \mathbb{N}$, where $t^l(x), l = 1,\dots,p$, are some functions on x. As a result one gets a so called *reduced problem*. Under certain assumptions the reduced problem is locally equivalent to the initial SIP problem. In both, discretization and reduction approaches, optimality conditions for SIP problems are formulated in terms of optimality conditions for the finite problems constructed and SIP problems are solved using the solutions of these finite problems. For the references see the survey paper [4].

The present paper deals with convex SIP problems where the objective function and the constraint function are convex w.r.t. x and the set $T \subset \mathbb{R}$ is compact. All the functions are assumed to be sufficiently smooth in their domains. The main purpose of the paper is to introduce a new approach to optimality conditions for the convex SIP problems that, in general, do not satisfy the Slater condition. For the first time the similar approach was described in [9] for linear SIP problems. The new concepts of *immobility order*[1] and of *immobile point* make it possible to formulate necessary and sufficient optimality conditions for some convex SIP problem in terms of optimality conditions for a certain finite nonlinear programming (NLP) problem. In general, the approach suggested in the paper differs from the discretization and reduction approaches (see [10]) and can be applied to the convex SIP problems without additional assumptions such as, for example, the Slater condition. Authors believe that the results obtained are useful for further investigations in the theory of infinite problems as well as for development of new SIP algorithms.

The paper is organized as follows. In Section 2 we formulate the problem and introduce the definitions of immobility order and immobile point. In Section 3 an algorithm (we call it DIO algorithm) of determination of the immobile points and the immobility orders is described and justified; the example of its application is provided. Theorem 3.1 is of a special interest as it shows how one can construct the feasible solution satisfying the definition of immobility order for all points of the set T simultaneously. Remark 3.2 indicates that in the presence of the immobile points the SIP problem does not satisfy the Slater condition. The Implicit Optimality Criterion based on the concept of immobile points is formulated and proved in Section 4. Example 4.1 illustrates the application of

[1] In [8] this term is translated from Russian as *motionless degree.*

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the Implicit Optimality Criterion to the convex SIP problems not satisfying the Slater condition. The final Section 5 contains conclusions.

2 Immobility orders and immobile points

Consider a Semi-Infinite Programming (SIP) problem

$$\begin{aligned} & c(x) \longrightarrow \min, \\ \text{s.t.}\quad & f(x,t) \le 0, \quad t \in T = [t_*, t^*],\ t_*, t^* \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$. Suppose that the functions $c(x)$ and $f(x,t)$ in (2.1) are analytically defined, sufficiently smooth in $\mathbb{R}^n$ and $\mathbb{R}^n \times T$ respectively. We also assume that $c(x)$ and $f(x,t)$ are convex w.r.t. x, i.e. for each $x_1,\ x_2 \in \mathbb{R}^n$ and for all $\alpha \in [0,1]$ the inequalities

$$c(\alpha x_1 + (1-\alpha)x_2) \le \alpha c(x_1) + (1-\alpha)c(x_2),$$

$$f(\alpha x_1 + (1-\alpha)x_2, t) \le \alpha f(x_1,t) + (1-\alpha) f(x_2,t),\ t \in T,$$

are satisfied.
Denote by $X \subset \mathbb{R}^n$ the feasible set of problem (2.1)

$$X = \{x \in \mathbb{R}^n :\ f(x,t) \le 0, t \in T\}. \tag{2.2}$$

Assumption 2.1 *Suppose, $X \neq \emptyset$ and there exists $\bar{x} \in X$ such that $f(\bar{x}, t) \not\equiv 0, t \in T$.*

For any $x \in X$, we denote by $T_a(x) = \{t \in T :\ f(x,t) = 0\}$ the corresponding set of the active points of T. Taking into account Assumption 2.1 and the sufficient smoothness of the function $f(x,t)$ in $\mathbb{R}^n \times T$, we can conclude that there exists $\bar{x} \in X$ such that $|T_a(\bar{x})|$ is finite.

In the sequel we will use the following notations

$$\begin{aligned} & f^{(0)}(x,\ t) = f(x,t),\ f^{(s)}(x,\ t) = \partial^s f(x,t)/\partial t^s,\ s \in \mathbb{N}; \\ & N(q) = \emptyset,\ \text{if } q < 0,\ N(q) = \{0,\ 1,\ \dots,\ q\}\ \text{ if } q \ge 0,\ q \in \mathbb{Z}. \end{aligned}$$

Given $t \in T,\ x \in X$, let $\rho = \rho(x,t) \in \{-1,\ 0,\ 1,\ \dots\}$ be a number such that

$$f^{(s)}(x,t) = 0,\ s \in N(\rho), \quad f^{(\rho+1)}(x,t) \neq 0. \tag{2.3}$$

Definition 2.1 *Let $t \in T$. A number $q(t) \in \{-1, 0, 1, \dots\}$ is called the order of immobility (immobility order) of t in SIP problem (2.1) if*

1. *for each $x \in X$ it is satisfied*

$$f^{(r)}(x,t) = 0,\ r \in N(q(t)), \tag{2.4}$$

2. there exists $x(t) \in X$ *such that*

$$f^{(q(t)+1)}(x(t), t) \neq 0. \tag{2.5}$$

From the definition above and from the constraints of problem (2.1) it follows that

1. if $t \in \text{int}\, T$, then $q(t) + 1$ is even and $f^{(q(t)+1)}(x(t), t) < 0$;
2. $q(t_*) \in \{-1, 0, 1, \dots\}$ and $f^{(q(t_*)+1)}(x(t_*), t_*) < 0$ for the correspondent $x(t_*) \in X$;
3. $q(t^*) \in \{-1, 0, 1, \dots\}$ and for the correspondent $x(t^*) \in X$ we have

   3.a) $f^{(q(t^*)+1)}(x(t^*), t^*) < 0$ whenever $q(t^*) + 1$ is even;

   3.b) $f^{(q(t^*)+1)}(x(t^*), t^*) > 0$ whenever $q(t^*) + 1$ is odd.

Definition 2.2 *A point* $t \in T$ *is called the immobile point of problem (2.1), if* $q(t) > -1$.

To simplify the further laying out we make the following assumption.

Assumption 2.2 *Suppose that* $q(t_*) = q(t^*) = -1$.

In the present paper, we claim that the concept of the immobility order is an important characteristic of the constraints of SIP problem (2.1) that makes it possible to formulate optimality conditions for this problem (with an infinite number of constraints) in terms of optimality conditions for a certain NLP problem (with a finite number of constraints).

3 An algorithm of determination of the immobile points and their immobility orders

Consider the convex SIP problem in the form (2.1) with the feasible set X. Suppose that Assumptions 2.1 and 2.2 are satisfied for this problem. Choose any $\bar{x} \in X$ with a finite set of active points $T_a(\bar{x}) = \{t_i, i \in I\}$, $I = I(\bar{x}) = \{1, 2, \dots, \bar{p}\}$, $\bar{p} = p(\bar{x}) < \infty$.
The algorithm described below constructs the mapping $q : T \to \{-1, 1, 3, \dots\}$.

DIO ALGORITHM

(Determination of Immobility Orders)

Suppose, $k = 0$ and $q_i^{(0)} = -1, \ \forall i \in I$.

The <u>k-th iteration</u> starts with a set of numbers $q_i^{(k)}, i \in I$, constructed on the previous iteration of the algorithm. (It will be shown later that for any $i \in I$ either $q_i^{(k)}$ is odd or $q_i^{(k)} = -1$).

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Introduce the sets

$$X_i^{(k)} = \{z \in R^n : f^{(s)}(z, t_i) = 0,\ s \in N(q_i^{(k)}),\ f^{(q_i^{(k)}+1)}(z, t_i) \le 0\},\quad i \in I; \tag{3.1}$$

$$X^{(k)} = \bigcap_{i \in I} X_i^{(k)}. \tag{3.2}$$

For each $i \in I$, solve the nonlinear programming problem

$$f_i^{(k)}(z) = f^{(q_i^{(k)}+1)}(z,\ t_i) \longrightarrow \min_z,\quad \text{s.t. } z \in X^{(k)}. \tag{3.3}$$

It will be proved later (Lemma 3.1) that $\bar{x} \in X^{(k)}$. Therefore, $X^{(k)} \neq \emptyset$ and either problem (3.3) admits an optimal solution, or its objective function $f_i^{(k)}(z)$ is not limited from below in the feasible set $X^{(k)}$.
Denote by $x^{(i)}$ the optimal solution of problem (3.3), in the case such the solution exists. Otherwise, denote by $x^{(i)}$ any feasible solution of problem (3.3) that satisfies the inequality $f_i^{(k)}(x^{(i)}) < 0$.

Consider the set $I^{(k)} := \{i \in I :\ f_i^{(k)}(x^{(i)}) = 0\}$.

If $I^{(k)} = \emptyset$, then algorithm stops resulting with the following values of $q(t), t \in T$:

$$q(t_i) = q_i^{(k)},\ i \in I;\quad q(t) = -1,\ t \in T \backslash T_a(\bar{x}). \tag{3.4}$$

If $I^{(k)} \neq \emptyset$, then set:

$$q_i^{(k+1)} = q_i^{(k)} + 2,\ i \in I^{(k)};\quad q_i^{(k+1)} = q_i^{(k)},\ i \in I \backslash I^{(k)}, \tag{3.5}$$

and pass to the next iteration with $k := k + 1$.
The algorithm is described. □

Note that DIO algorithm is finite. Indeed, if denote by $k_* \in \mathbb{N}$, the number of its iterations, it is easy to verify that the following estimation is true: $k_* \le \sum_{i \in I} \frac{\rho(\bar{x}, t_i)+1}{2}$.

In the reminder of this section, we will demonstrate that the mapping $q(t), t \in T$, constructed by DIO algorithm, determines the immobility orders of all the points of the set T correctly.
Since k_* introduced above can be considered as a number of the last iteration of the algorithm (i.e., the number of the iteration where the algorithm has stopped), we obtain by (3.4):

$$\begin{aligned} q_i = q(t_i) &= q_i^{(k_*)} \text{ for } i \in I, \\ q(t) &= -1 \text{ for } t \in T \backslash T_a(\bar{x}). \end{aligned} \tag{3.6}$$

Lemma 3.1 *On iterations of DIO algorithm the following inclusions are satisfied:*

$$X \subset X^{(v)} = \bigcap_{i \in I} X_i^{(v)}, \tag{3.7}$$

where $v = 0, \dots, k_*$.

Proof (by induction on v). Due to (2.2), (3.1) we have $X \subset X_i^{(0)}$ for any $i \in I$. Then inclusion (3.7) is valid for $v = 0$.
Assume that (3.7) is satisfied for $v = k \geq 0,\ k < k_*$, i.e.

$$X \subset X_i^{(k)},\ i \in I. \tag{3.8}$$

Let us prove (3.7) for $v = k+1$. From (3.5) it follows that $q_i^{(k+1)} = q_i^{(k)},\ i \in I \backslash I^{(k)}$. Then (3.1) and (3.8) yield the following enclosure:

$$X \subset X_i^{(k+1)} = X_i^{(k)},\ i \in I \backslash I^{(k)}. \tag{3.9}$$

Now, suppose that for some $i_* \in I^{(k)}$, there exists $x^* \in X$ such that $f^{(q_{i_*}^{(k)}+1)}(x^*,\ t_{i_*}) \neq 0$. Then, evidently, $f^{(q_{i_*}^{(k)}+1)}(x^*,\ t_{i_*}) < 0$ and, taking into consideration (3.8), we obtain

$$x^* \in X \subset X^{(k)},\ \ f_{i_*}^{(k)}(x^*) = f^{(q_{i_*}^{(k)}+1)}(x^*,\ t_{i_*}) < 0. \tag{3.10}$$

Since $i_* \in I^{(k)}$, we have $f_{i_*}^{(k)}(x^{(i_*)}) = 0$ for the feasible solution $x^{(i_*)}$ found on k-th iteration of the algorithm. Therefore $x^{(i_*)}$ is the optimal solution of problem (3.3). The optimality of $x^{(i_*)}$ contradicts with the existence of x^* satisfying (3.10) and thus we conclude that

$$f^{(q_i^{(k)}+1)}(z,\ t_i) = 0, \quad \forall z \in X,\ \forall i \in I^{(k)}. \tag{3.11}$$

Recall that, by construction, all the values $q_i^{(k)} + 1$ are even. Then, due to the constraints of SIP problem (2.1) and to Assumption 2.2, we have

$$f^{(q_i^{(k)}+2)}(z,\ t_i) = 0,\ \ f^{(q_i^{(k)}+3)}(z,\ t_i) \leq 0 \quad \forall z \in X,\ \forall i \in I^{(k)}. \tag{3.12}$$

From (3.1), (3.5), (3.11) and (3.12) it follows

$$X \subset X_i^{(k+1)},\ i \in I^{(k)}. \tag{3.13}$$

From (3.9) and (3.13) we have $X \subset X_i^{(k+1)}$ for all $i \in I$. Thus (3.7) is satisfied for $v = k+1$ and the proof of lemma is complete. □

Note that DIO algorithm is constructed in such a way that it is satisfied

$$\begin{gathered} f^{(s)}(x^{(j)},\ t_i) = 0,\ s \in N(q_i),\ \ f^{(q_i+1)}(x^{(j)},\ t_i) \leq 0,\ i, j \in I; \\ f^{(q_i+1)}(x^{(i)},\ t_i) < 0,\ i \in I, \end{gathered} \tag{3.14}$$

where $t_i,\ i \in I$, are the immobile points of the convex SIP problem (2.1) and the corresponding immobility orders complied with (3.6). Here and further we denote by $x^{(i)},\ i \in I$, the feasible solutions of problem (3.3) obtained on the last iteration of DIO algorithm.

Let us prove now that relations (3.14) are valid also for convex combinations of $x^{(i)},\ i \in I$, i.e. for any vector

$$y = \sum_{i\in I} \bar{\alpha}_i\, x^{(i)} \tag{3.15}$$

such that

$$\bar{\alpha}_i > 0,\ i \in I,\ \sum_{i\in I} \bar{\alpha}_i = 1. \tag{3.16}$$

Lemma 3.2 *Let y satisfy (3.15), (3.16). Then*

$$f^{(s)}(y,\ t_i) = 0,\ s \in N(q_i),\ \ f^{(q_i+1)}(y,\ t_i) < 0,\ i \in I, \tag{3.17}$$

where $q_i,\ i \in I$, are determined by (3.6).

Proof. Let $\rho_i := \rho(y, t_i),\ i \in I$, where $\rho(y, t)$ is defined as in (2.3). Then

$$f^{(s)}(y,\ t_i) = 0,\ s \in N(\rho_i),\quad f^{(\rho_i+1)}(y,\ t_i) \neq 0,\ i \in I, \tag{3.18}$$

and the statement of the lemma will be proved if we show that for any $i \in I$ it is satisfied $\rho_i = q_i$.
1) Let us prove, first, that

$$\rho_i \leq q_i,\ i \in I. \tag{3.19}$$

Arguing by contradiction, suppose that there exists $i_0 \in I$, such that $\rho_{i_0} > q_{i_0}$. Then from (3.18) we have

$$f^{(s)}(y,\ t_{i_0}) = 0,\ s \in N(q_{i_0} + 1). \tag{3.20}$$

Since $f(x,\ t)$ is convex w.r.t. x, then for any Δt such that $t_i + \Delta t \in T$, we can write

$$f(y,\ t_i + \Delta t) \leq \sum_{j\in I} \bar{\alpha}_j\, f(x^{(j)}, t_i + \Delta t),\quad \forall i \in I, \tag{3.21}$$

where y and $\bar{\alpha}_j,\ j \in I$, satisfy (3.15) and (3.16). It is evident that for any $z \in X$ and any $i \in I$, Taylor's expansion of $f(z, t)$ in the neighborhood of t_i of the order $l,\ l \in \mathbb{N}$, can be written in the form

$$f(z, t_i + \Delta t) = f(z, t_i) + f^{(1)}(z, t_i)\Delta t + \frac{1}{2!} f^{(2)}(z, t_i)\Delta t^2 + \dots$$
$$+ \frac{1}{l!} f^{(l)}(z, t_i)\Delta t^l + \frac{1}{(l+1)!} f^{(l+1)}(z, t_i)\Delta t^{l+1} + o(\Delta t^{l+1}).$$

Given $i \in I$, applying Taylor's expansion of the order q_i to the corresponding inequality in (3.21), we get

$$
\begin{aligned}
&f(y,t_i) + f^{(1)}(y,t_i)\Delta t + \cdots + \frac{1}{(q_i+1)!} f^{(q_i+1)}(y,t_i)\Delta t^{q_i+1} + o(\Delta t^{q_i+1}) \leq \\
&\sum_{j\in I} \bar{\alpha}_j \Big(f(x^{(j)},t_i) + f^{(1)}(x^{(j)},t_i)\Delta t + \cdots + \frac{1}{(q_i+1)!} f^{(q_i+1)}(x^{(j)},t_i)\Delta t^{q_i+1} + \\
&o(\Delta t^{q_i+1})\Big).
\end{aligned} \tag{3.22}
$$

Consider (3.22) with $i = i_0$ and substitute into its right-hand side formulae (3.14) and into the left-hand side the values from (3.20), obtaining

$$
o(\Delta t^{q_{i_0}+1}) \leq \sum_{j\in I} \bar{\alpha}_j (f^{(q_{i_0}+1)}(x^{(j)}, t_{i_0}) \Delta t^{q_{i_0}+1} + o(\Delta t^{q_{i_0}+1})). \tag{3.23}
$$

Since $q_{i_0}+1$ is even, then we have $\Delta t^{q_{i_0}+1} > 0$. Divide (3.23) by $\Delta t^{q_{i_0}+1}$ and let $\Delta t \to 0$. Then, taking into account (3.14), we get a contradictory system of inequalities

$$
0 \leq \sum_{j\in I} \bar{\alpha}_j f^{(q_{i_0}+1)}(x^{(j)},\ t_{i_0}) \leq \bar{\alpha}_{i_0} f^{(q_{i_0}+1)}(x^{(i_0)},\ t_{i_0}) < 0 \tag{3.24}
$$

that proves (3.19).

2) Now, let us strengthen (3.19) and show that

$$
\rho_i = q_i, \forall i \in I. \tag{3.25}
$$

Suppose, $I_* := \{ i \in I : \rho_i < q_i \} \neq \emptyset$. To obtain a contradiction, it suffices to demonstrate that no one of the following hypotheses is true:

a) $\exists\ i_0 \in I_*$, such that ρ_{i_0} is even; b) ρ_i is odd, $\forall\ i \in I_* \neq \emptyset$.

First of all, let us substitute each functions in (3.21) by its Taylor's expansion in the neighborhood of t_i of the order $\rho_i + 1$. With respect to (3.14), (3.18), and (3.19), we have

$$
f^{(\rho_i+1)}(y,t_i)\Delta t^{\rho_i+1} + o(\Delta t^{\rho_i+1}) \leq 0,\ \forall i \in I. \tag{3.26}
$$

Let us consider now hypothesis a). Suppose, $i = i_0$ in (3.26). Divide the inequality obtained by $\Delta t^{\rho_{i_0}+1}$ and let, first, $\Delta t \to +0$ and after, $\Delta t \to -0$. Since $\Delta t^{\rho_{i_0}+1} > 0$ whenever $\Delta t > 0$ and $\Delta t^{\rho_{i_0}+1} < 0$ whenever $\Delta t < 0$, then the values of the limits obtained can be estimated as follows:

$$
\lim_{\Delta t \to +0} \frac{f^{(\rho_{i_0}+1)}(y,t_{i_0})\Delta t^{\rho_{i_0}+1} + o(\Delta t^{\rho_{i_0}+1})}{\Delta t^{\rho_{i_0}+1}} \leq 0,
$$

$$
\lim_{\Delta t \to -0} \frac{f^{(\rho_{i_0}+1)}(y,t_{i_0})\Delta t^{\rho_{i_0}+1} + o(\Delta t^{\rho_{i_0}+1})}{\Delta t^{\rho_{i_0}+1}} \geq 0,
$$

wherefrom $\lim_{\Delta t \to +0} f^{(\rho_{i_0}+1)}(y, t_{i_0}) \leq 0$, $\lim_{\Delta t \to -0} f^{(\rho_{i_0}+1)}(y, t_{i_0}) \geq 0$. The last two inequalities can be satisfied simultaneously only if $f^{(\rho_{i_0}+1)}(y,\ t_{i_0}) = 0$, which contradicts (3.18). Therefore, hypothesis a) is false.

Now, consider hypothesis b). From (3.26), taking into account inequalities $\Delta t^{\rho_i+1} > 0, i \in I_*$, we get $f^{(\rho_i+1)}(y,\ t_i) \leq 0, i \in I_*$, wherefrom with respect to the inequality in (3.18) we obtain

$$f^{(\rho_i+1)}(y,\ t_i) < 0, \quad i \in I_*. \tag{3.27}$$

It was assumed above that ρ_i is odd and $\rho_i < q_i$ for $\forall i \in I_* \neq \emptyset$. Given $i \in I_*$, let $k_i \in \{0,\ 1,\ \ldots,\ k_* - 1\}$, be the index such that $i \in I^{(k_i)},\ q_i^{(k_i)} = \rho_i$. Denote:

$$k_0 := \min_{i \in I_*} k_i = k_{i_*}. \tag{3.28}$$

On the k_0-th iteration of DIO algorithm problem (3.3) takes the form

$$f_{i_*}^{(k_0)}(z) = f^{(\rho_{i_*}+1)}(z,\ t_{i_*}) \longrightarrow \min, \quad \text{s.t. } z \in X^{(k_0)}. \tag{3.29}$$

As $i_* \in I^{(k_0)}$, we can conclude that the problem above has an optimal solution x^* satisfying

$$f_{i_*}^{(k_0)}(x^*) = f^{(\rho_{i_*}+1)}(x^*,\ t_{i_*}) = 0. \tag{3.30}$$

From (3.28) it follows $q_i^{(k_0)} \leq \rho_i$, $i \in I$. Then, with respect to (3.18), we obtain

$$f^{(s)}(y,\ t_i) = 0,\ s \in N(q_i^{(k_0)}),\ i \in I. \tag{3.31}$$

Finally, let us show that

$$y \in X^{(k_0)}. \tag{3.32}$$

According to (3.1) and (3.31), it is sufficient to prove that the following inequalities

$$f^{(q_i^{(k_0)}+1)}(y,\ t_i) \leq 0,\ i \in I, \tag{3.33}$$

are valid. By DIO algorithm, for all $i \in I$, it is satisfied $q_i^{(k_0)} \leq \rho_i \leq q_i$. Then for any $i \in I$, substituting $q_i^{(k_0)}$ instead of q_i in (3.22) and, taking into account (3.31), we obtain

$$f^{(q_i^{(k_0)}+1)}(y, t_i)\Delta t^{q_i^{(k_0)}+1} + o(\Delta t^{q_i^{(k_0)}+1}) \leq \sum_{j \in I} \bar{\alpha}_j \Big(f^{(q_i^{(k_0)}+1)}(x^{(j)}, t_i)\Delta t^{q_i^{(k_0)}+1} + o(\Delta t^{q_i^{(k_0)}+1}) \Big).$$

Note that $q_i^{(k_0)}$ is odd here. Dividing the inequality above by $\Delta t^{q_i^{(k_0)}+1} > 0$ and taking the limit as $\Delta t \to 0$, we obtain

$$f^{(q_i^{(k_0)}+1)}(y,\ t_i) \leq \sum_{j \in I} \bar{\alpha}_j f^{(q_i^{(k_0)}+1)}(x^{(j)},\ t_i) \tag{3.34}$$

that, together with the last two groups of inequalities in (3.14), implies (3.33) and, consequently, (3.32).
From (3.27), (3.28) we have $f^{(\rho_{i_*}+1)}(y,\ t_{i_*}) < 0$ that, taking into account (3.32) and (3.30), contradicts the optimality of x^* in (3.29). Thus hypothesis b) is false too. □

Corollary 3.1 *Let y satisfy (3.15), (3.16). Then there exists $\varepsilon > 0$ such that the following inequalities are valid*

$$f(y,t) \le 0, \quad t \in [t_i - \varepsilon,\ t_i + \varepsilon],\ \forall i \in I. \tag{3.35}$$

Proof. Lemma 3.2 states that (3.17) is valid for the given y. Then $f(y,\ t_i) < 0$ for all $i \in I$, such that $q_i = -1$. Taking into account the sufficient smoothness of the function $f(y,t)$, we can extend this result to some neighborhood of t_i:

$$\text{for } \forall i \in I \text{ with } \ q_i = -1, \quad \exists \varepsilon_i > 0 : \ f(y,t) < 0, \quad t \in [t_i - \varepsilon_i,\ t_i + \varepsilon_i]. \tag{3.36}$$

If $q_i > -1$ for some $i \in I$, then q_i is odd, evidently. From (3.17) it follows that the correspondent t_i is the local maximizer of the continuous function $f(y,t)$ and that $f(y,\ t_i) = 0$. Therefore, we can state that

$$\text{for } \forall i \in I \text{ with } \ q_i > -1, \quad \exists \varepsilon_i > 0 : \ f(y,t) \le 0, \quad t \in [t_i - \varepsilon_i,\ t_i + \varepsilon_i] \tag{3.37}$$

and (3.35) follows immediately from (3.36) and (3.37) if suppose $\varepsilon := \min\limits_{i \in I} \varepsilon_i$. □

Theorem 3.1 *Given $t \in T$, the value $q(t)$ constructed by DIO algorithm satisfies Definition 2.1.*

Proof. Consider any $t \in T$. Let us prove, first, that $q(t)$ satisfies (2.4). If $q(t) = -1$, then $N(q(t)) = \emptyset$ and it is nothing to prove.
Suppose $q(t) > -1$. Then, by the algorithm, there exists $i \in I$ such that $t = t_i$. According to (2.3), for any $i \in I$ and any $z \in X$ we denote by $\rho = \rho(z, t_i) \in \{-1, 0, 1, \dots\}$ a number such that

$$f^{(s)}(z, t_i) = 0,\ s \in N(\rho),\ f^{(\rho+1)}(z, t_i) \ne 0. \tag{3.38}$$

Let us show, first, that

$$\rho(z, t_i) \ge q_i,\ \forall z \in X,\ \forall i \in I, \tag{3.39}$$

where $q_i = q(t_i)$.

Arguing by contradiction, suppose $\bar{\rho} = \rho(\bar{z}, t_{i_1}) < q_{i_1}$ for some $i_1 \in I$ and some $\bar{z} \in X$. Denote by $\bar{k}$, $0 \leq \bar{k} < k_*$, the number of the iteration where $q_{i_1}^{(\bar{k})} = \bar{\rho}$, $q_{i_1}^{(\bar{k}+1)} = \bar{\rho} + 2$. By DIO algorithm, there exists $x^{(i_1)} \in X^{\bar{k}}$ such that

$$0 = f_{i_1}^{(\bar{k})}(x^{(i_1)}) = f^{(\bar{\rho}+1)}(x^{(i_1)}, t_{i_1}) = \min_{x \in X^{(\bar{k})}} f^{(\bar{\rho}+1)}(x, t_{i_1}). \tag{3.40}$$

From the other hand, as $\bar{z} \in X$, from Assumption 2.2 we conclude that $\bar{\rho}$ is odd. Then from (3.38) it follows $f^{(\bar{\rho}+1)}(\bar{z}, t_{i_1}) < 0$. By Lemma 3.1, $\bar{z} \in X \subset X^{(\bar{k})}$. However, the relations obtained

$$f^{(\bar{\rho}+1)}(\bar{z}, t_{i_1}) < 0, \ \bar{z} \in X^{(\bar{k})}$$

contradict (3.40). Therefore, (3.39) is valid and together with (3.38) it yields (2.4).

Let us now show that there exists $\tilde{x} = \tilde{x}(t)$ satisfying (2.5). Recall that DIO algorithm starts with the index set I in the form $I = I(\bar{x})$ for some $\bar{x} \in X$. For any y given by (3.15), (3.16) and any $\alpha \in [0, 1]$, we consider

$$x(\alpha) = \alpha\bar{x} + (1 - \alpha)y. \tag{3.41}$$

From the convexity of the function $f(x, t)$ w.r.t. x we have

$$f(x(\alpha),\ t) \leq \alpha f(\bar{x},\ t) + (1 - \alpha)f(y,\ t) = f(y, t) + \alpha(f(\bar{x}, t) - f(y, t)), \quad \forall t \in T.$$

Let $\alpha(t)$ be a function defined in T as follows:

$$\alpha(t) = \begin{cases} 0, & \text{if } f(y,\ t) \leq 0, \\ \dfrac{f(y, t)}{f(y,\ t) - f(\bar{x},\ t)}, & \text{if } f(y,\ t) > 0. \end{cases} \tag{3.42}$$

Let us prove that $\alpha(t) < 1$, $\forall t \in T$. Indeed, from Corollary 3.1 it follows

$$\exists\, \varepsilon > 0 : \ \ \alpha(t) = 0, \ t \in [t_i - \varepsilon,\ t_i + \varepsilon], \ \ \forall i \in I. \tag{3.43}$$

Let $T_* := T \setminus \bigcup_{i \in I} [t_i - \varepsilon,\ t_i + \varepsilon]$. By construction, $f(\bar{x},\ t) < 0$, $t \in T_*$. Then

$$f(\bar{x},\ t) \leq -\delta, \ \forall t \in T_*, \tag{3.44}$$

for $\delta := \min_{t \in T_*} |f(\bar{x},\ t)| > 0$.
Consider the subset $T_{**} \subseteq T_*$ defined as follows: $T_{**} = \{t \in T_* :\ f(y, t) > 0\}$. If $T_{**} = \emptyset$, then $\alpha(t) = 0$, $\forall t \in T$, and the statement is proved.
Now, suppose $T_{**} \neq \emptyset$. By construction, for any $t \in T \setminus T_{**}$ we have $\alpha(t) = 0$. Denote $\delta_0 := \max_{t \in T_{**}} f(y, t) < +\infty$. Evidently, $\delta_0 > 0$, $\min_{t \in T_{**}} |f(\bar{x}, t)| \geq \delta > 0$. Then, taking into consideration (3.44), we obtain for $t \in T_{**}$:

$$\alpha(t) = \frac{f(y, t)}{f(y, t) - f(\bar{x}, t)} = \frac{1}{1 - \frac{f(\bar{x}, t)}{f(y, t)}} \leq \frac{1}{1 + \delta/\delta_0} < 1.$$

Let θ_* be the maximal value of the function $\alpha(t)$ constructed above:

$$\theta_* := \max_{t\in T} \alpha(t). \tag{3.45}$$

Obviously, $0 \le \theta_* < 1$. Choose some fixed parameter α_0 from the interval $]\theta_*, 1[$ and set $\tilde{x} := x(\alpha_0)$ where $x(\alpha_0)$ is calculated by (3.41). Now by the same method that was used in the proof of Lemma 3.2, we can show that $\tilde{x} \in X$ and $f^{(q(t)+1)}(\tilde{x}, t) < 0, \forall t \in T$. Here we just have to suppose $I := \{1, 2\}$, $x^{(1)} := \bar{x}$, $x^{(2)} := y$, $\bar{\alpha}_1 := \alpha_0$, $\bar{\alpha}_2 := 1 - \alpha_0$ and consider relations (3.17) and

$$\begin{gathered} f^{(s)}(\bar{x},\ t_i) = 0,\ s \in N(q_i), \quad f^{(q_i+1)}(\bar{x},\ t_i) \le 0,\ i \in I; \\ f(\bar{x},\ t) < 0,\ t \in T \backslash T_a(\bar{x}), \end{gathered} \tag{3.46}$$

instead of (3.14). This will complete the proof of the theorem. □

Remark 3.1 *Theorem 3.1 states also that there always exists a vector $\tilde{x}$ that satisfies (2.4) and (2.5)* **for all** *$t \in T$ simultaneously. Therefore, in Definition 2.1 we can always suppose $x(t) \equiv \tilde{x}$, $\forall t \in T$.*

Remark 3.2 *It follows from Definition 2.1 and Remark 3.1 that the constraints of problem (2.1) satisfy the Slater condition if and only if $q(t) = -1, \forall t \in T$.*

Example 3.1. Let us use DIO algorithm to determine the immobility orders of all points of the interval T in problem (2.1), where

$$\begin{aligned} & f(x,\ t) = 18 \cdot [(t-0.14)^6(t-0.6)^2(t-0.94)^4(x_1^2 + (x_2+\tfrac{1}{3})^2 + x_3^2 + (x_4-4)^2 - 1) + \\ & +(t-0.14)^4(1-\cos(t-0.6))\sin^4(t-0.94)((x_1+x_2+x_3+\tfrac{1}{2})^2 - 1) + \\ & +\sin^4(t-0.14)(t-0.94)^2\sin^2(t-0.94)((x_2+x_4-3)^4 + 4x_1^2x_3^2 - 1)], \\ & T = [0, 1], \quad x \in \mathbb{R}^4. \end{aligned}$$

Consider the feasible solution $\bar{x} = (0,\ 0.5,\ 0,\ 3.5)'$. Then

$$f(\bar{x},\ t) = -(t-0.14)^6(t-0.6)^2(t-0.94)^4,\ t \in [0, 1],$$

and, according to the notations used above, we have

$$\begin{aligned} & T_a(\bar{x}) = \{0.14,\ 0.6,\ 0.94\} = \{t_i,\ i \in I\}, \\ & I = \{1,\ 2,\ 3\},\ t_1 = 0.14,\ t_2 = 0.6,\ t_3 = 0.94, \\ & \rho(\bar{x},\ 0.14) = 5, \quad \rho(\bar{x},\ 0.6) = 1, \quad \rho(\bar{x},\ 0.94) = 3. \end{aligned}$$

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The results of the proceeding of DIO algorithm are presented in the following table

k	$q_1^{(k)}$	$f^{(q_1^{(k)}+1)}(x^{(1)},t_1)$	$q_2^{(k)}$	$f^{(q_2^{(k)}+1)}(x^{(2)},t_2)$	$q_3^{(k)}$	$f^{(q_3^{(k)}+1)}(x^{(3)},t_3)$	$I^{(k)}$
0	-1	0	-1	-0.009	-1	0	$\{1,\ 3\}$
1	1	0	-1	-0.009	1	0	$\{1,\ 3\}$
2	3	-154.1681	-1	-0.009	3	-136.043	$\emptyset$

The feasible solutions $x^{(i)}, i \in I$, obtained at the last iteration are

$$x^{(1)} = (-0.0056,\ -0.4888,\ -0.0056,\ 3.4832)', \quad x^{(2)} = (0,\ 0,\ 0,\ 3)',$$
$$x^{(3)} = (0.0131,\ -0.5431,\ 0.0131,\ 3.7769)'.$$

The immobility orders of the points $t_i,\ i \in I$, are equal to the values $q_i^{(k_*)},\ i \in I$, from the last iteration of the algorithm. In our case $k_* = 2$ and

$$q_1 = q(0.14) = q_1^{(2)} = 3,\ \ q_2 = q(0.6) = q_2^{(2)} = -1,\ \ q_3 = q(0.94) = q_3^{(2)} = 3.$$

Thus the algorithm results in the function $q(t),\ t \in [0,\ 1]$, such that

$$q(t) = -1,\ t \in [0,\ 0.14) \cup (0.14,\ 0.94) \cup (0.94,\ 1]; \quad q(0.14) = q(0.94) = 3.$$

Let us now find $\tilde{x}$ that satisfies Definition 2.1. According to the proceeding described in Theorem 3.1, we have to construct, first, some vector y in the form (3.15), (3.16). If, for example, we assume that $\bar{\alpha}_i = 1/3,\ i \in I$, then (approximately)

$$y = \sum_{i \in I} \frac{1}{3} x^{(i)} = (0.0025,\ -0.344,\ 0.0025,\ 3.42)'.$$

The functions $f(\bar{x},\ t)$ (see Fig. 1) and $f(y,\ t)$ (see Fig. 2) are not positive in $T = [0,\ 1]$ and, consequently, the value θ_*, defined in (3.45), is zero.

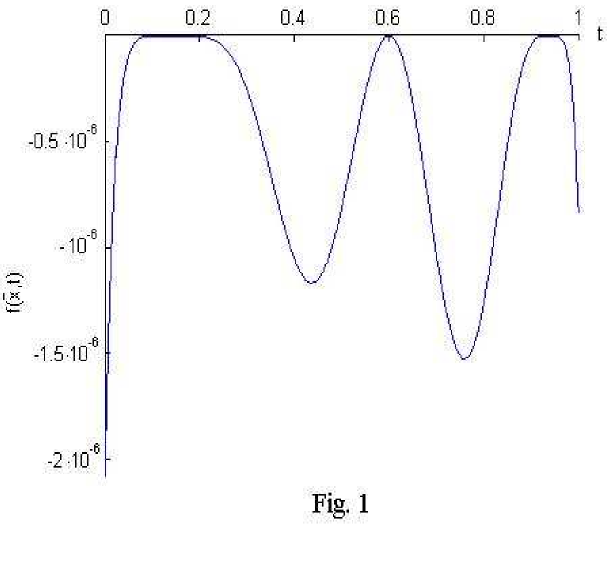

Fig. 1

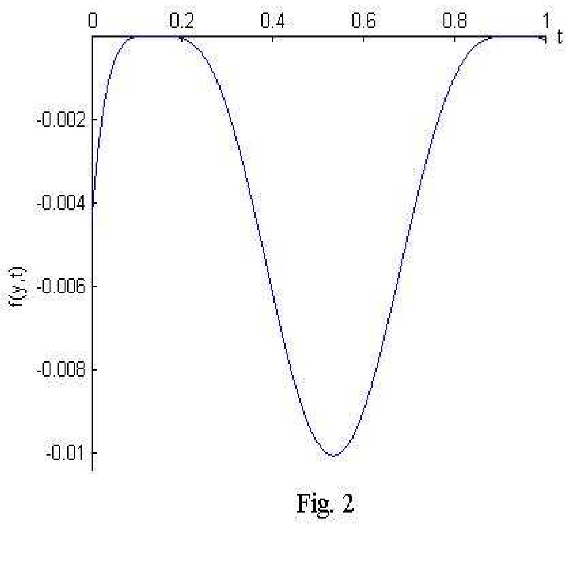

Fig. 2

According to Theorem 3.1 we have to choose, now, some α_0 from the interval $]\theta_*, 1[=]0, 1[$. Suppose, for example, that $\alpha_0 = 0.5$. Then

$$\tilde{x} = \bar{x}(\alpha_0) = \bar{x}(0.5) = (0.00125,\ 0.078,\ 0.00125,\ 3.46)'.$$

Fig. 3 shows (in two different scales) the graphic of the function $f(\tilde{x},\ t)$, $t \in [0,\ 1]$. Here

$$\begin{aligned}
&f(\tilde{x},\ t) < 0,\ t \in [0,\ 1] \setminus \{0.14,\ 0.94\},\\
&f^{(s)}(\tilde{x},\ t_i) = 0,\ s \in N(q_i),\ i \in \{1,\ 3\},\ t_1 = 0.14,\ t_3 = 0.94,\\
&f^{(4)}(\tilde{x},\ 0.14) = -138.2412275,\\
&f^{(4)}(\tilde{x},\ 0.94) = -118.5898889,\\
&f(\tilde{x},\ 0.6) = f^{(0)}(\tilde{x},\ 0.6) = -0.008236
\end{aligned}$$

and, evidently, conditions (2.4) and (2.5) are satisfied in $\tilde{x}$ for all $t \in T$. Thus we have confirmed that DIO Algorithm has correctly determined two immobile points $t_1 = 0.14$, $t_3 = 0.94$ and their immobility orders $q_1 = q_3 = 3$.

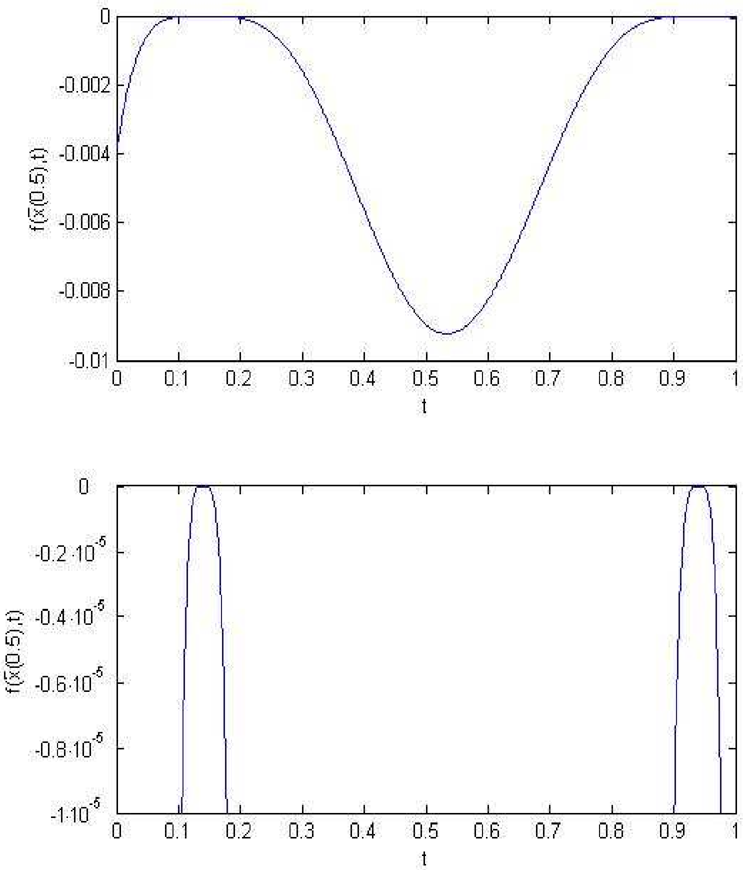

Fig. 3

Note that the point $t_2 = 0.6$ is not immobile, nevertheless $f(\bar{x}, t_2) = 0$.

4 Implicit Optimality Criterion

In this section, we consider optimality conditions for the convex SIP problem (2.1). Suppose that Assumptions 2.1 and 2.2 are satisfied. Let $x^0 \in X$, X being the feasible set in (2.1). Consider the corresponding set of active points $T_a(x^0)$ and suppose that $p := |T_a(x^0)| < \infty$. Then the set $T_a(x^0)$ can be written in the form

$$T_a(x^0) = \{t^0_j, j = 1, \dots, p\}. \tag{4.1}$$

Denote: $q_j = q(t^0_j),\ j = 1, \dots, p$. Using the notations above, form the following nonlinear programming problem (NLP problem):

$$\begin{aligned} & c(x) \longrightarrow \min, \\ \text{s.t.} \quad & f^{(s)}(x, t^0_j) = 0,\ s \in N(q_j), \\ & f^{(q_j+1)}(x, t^0_j) \leq 0, \quad j = 1, \dots, p. \end{aligned} \tag{4.2}$$

Let $Y \subset \mathbb{R}^n$ be the feasible set of (4.2). It is evident that $X \subset Y$.

Theorem 4.1 *[Implicit Optimality Criterion] The feasible solution $x^0 \in X$ with $|T_a(x^0)| < \infty$ is optimal in the convex SIP problem (2.1) if and only if it is optimal in the NLP problem (4.2).*

Proof. $\Leftarrow$) As $X \subset Y$, we can state that the optimality of the feasible solution $x^0 \in X$ in (4.2) immediately implies its optimality of x^0 in SIP problem (2.1).

$\Rightarrow$) (By contradiction). Suppose that x^0 is optimal for (2.1) but there exists $y \in Y$ such that $c(y) < c(x^0)$. It is evident that $y \notin X$. Let $\tilde{x} \in X$ be the feasible solution of problem (2.1) constructed in the proof of Theorem 3.1 and, therefore, satisfying (2.5). Consider the vector

$$x(\alpha) = \alpha_1 \tilde{x} + \alpha_2 y, \tag{4.3}$$

where $\alpha = (\alpha_1, \alpha_2)$ such that

$$\alpha_i \geq 0,\ i = 1, 2;\ \alpha_1 + \alpha_2 = 1. \tag{4.4}$$

Since $c(x)$ is a convex function, we have

$$c(x(\alpha)) \leq \alpha_1 c(\tilde{x}) + \alpha_2 c(y). \tag{4.5}$$

By construction, $c(\tilde{x}) \geq c(x^0) > c(y)$. Then for $\alpha^0_2 := \dfrac{c(x^0) - c(\tilde{x})}{c(y) - c(\tilde{x})}$ it is verified: $0 \leq \alpha^0_2 < 1$. Having assumed that α_2 defined in (4.4) satisfies, additionally, the inequality

$$\alpha_2 > \alpha^0_2, \tag{4.6}$$

we obtain in (4.5)

$$c(x(\alpha)) \leq \alpha_1 c(\tilde{x}) + \alpha_2 c(y) < c(x^0). \tag{4.7}$$

Since $f(x,t)$ is convex w.r.t. x, we have

$$f(x(\alpha),t) \leq \alpha_1 f(\tilde{x},t) + \alpha_2 f(y,t). \tag{4.8}$$

Let us show that there exists $\Delta = \Delta(\alpha) > 0$ such that for α from (4.4) with $\alpha_1 > 0$ and for any $j = 1, \ldots, p$ it is satisfied

$$f(x(\alpha),t) \leq 0,\ t \in [t_j^0 - \Delta, t_j^0 + \Delta]. \tag{4.9}$$

For $t = t_j^0, j = 1, \ldots, p$, the last formula is evident.
Now consider $t \in [t_j^0 - \Delta, t_j^0 + \Delta],\ t \neq t_j^0$ for some $j = 1, \ldots, p$. We can write t in the form: $t = t_j^0 + \Delta t_j$, where $0 < |\Delta t_j| \leq \Delta$. Then (4.8) takes the form

$$f(x(\alpha),t) = f(x(\alpha), t_j^0 + \Delta t_j) \leq \alpha_1 f(\tilde{x}, t_j^0 + \Delta t_j) + \alpha_2 f(y, t_j^0 + \Delta t_j). \tag{4.10}$$

It is easy to verify that

$$f(\tilde{x}, t_j^0 + \Delta t_j) = \frac{1}{(q_j+1)!} f^{(q_j+1)}(\tilde{x}, t_j^0) \Delta t_j^{q_j+1} + o(\Delta t_j^{q_j+1}),$$

$$f(y, t_j^0 + \Delta t_j) = \frac{1}{(q_j+1)!} f^{(q_j+1)}(y, t_j^0) \Delta t_j^{q_j+1} + o(\Delta t_j^{q_j+1})$$

where, by construction, $\Delta t_j^{q_j+1} > 0,\ f^{(q_j+1)}(\tilde{x}, t_j^0) < 0,\ f^{(q_j+1)}(y, t_j^0) \leq 0$. Therefore, from (4.10) we conclude that there exists a number $\Delta > 0$ such that (4.9) is satisfied for α from (4.4) with $\alpha_1 > 0$.
Consider now vector $\alpha^* = (\alpha_1^*, \alpha_2^*)$ where

$$\alpha_1^* + \alpha_2^* = 1,\ \alpha_1^* > 0,\ \alpha_2^* > \alpha_2^0.$$

It is easy to verify that $x(\alpha^*)$, calculated by (4.3), satisfies both (4.7) and (4.9) for some $\Delta^* = \Delta(\alpha^*) > 0$, i.e.

$$c(x(\alpha^*)) < c(x^0);\ f(x(\alpha^*),t) \leq 0, t \in [t_j^0 - \Delta^*, t_j^0 + \Delta^*],\ j = 1, \ldots, p. \tag{4.11}$$

Let $\varepsilon = \min |f(x^0,t)|,\ t \in T \backslash \bigcup_{j=1}^{p} [t_j^0 - \Delta^*, t_j^0 + \Delta^*]$. Evidently, $\varepsilon > 0$ and

$$f(x^0,t) \leq -\varepsilon,\ t \in T \backslash \bigcup_{j=1}^{p} [t_j^0 - \Delta^*, t_j^0 + \Delta^*]. \tag{4.12}$$

Consider a convex combination of x^0 and $x(\alpha^*)$ in the form

$$x(\lambda) = \lambda x^0 + (1-\lambda) x(\alpha^*),\ 0 \leq \lambda \leq 1.$$

Then, taking into account convexity of $f(x,t)$ and formulae (4.11), (4.12), it is not difficult to prove that there exists $\lambda^*: \ 0 \leq \lambda^* < 1$ such that

$$f(x(\lambda^*),t) \leq 0, \ t \in T. \tag{4.13}$$

The value of λ^* can be chosen arbitrary from the interval $[\lambda^0, 1[\subseteq [0,1[$ where

$$\lambda^0 = \begin{cases} 0, & \text{if } \ T^+ := \{t \in T : f(x(\alpha^*),t) > 0\} = \emptyset, \\ \max\limits_{t \in T^+} \dfrac{f(x(\alpha^*),t)}{f(x(\alpha^*),t) - f(x^0,t)} < \max\limits_{t \in T^+} \dfrac{f(x(\alpha^*),t)}{f(x(\alpha^*),t)+\varepsilon} < 1, & \text{otherwise.} \end{cases}$$

From (4.13) we conclude that $x(\lambda^*)$ is feasible for problem (2.1). Taking into account convexity of $c(x)$ and the first inequality in (4.11), we get

$$c(x(\lambda^*)) \leq \lambda^* c(x^0) + (1-\lambda^*)c(x(\alpha^*)) < \lambda^* c(x^0) + (1-\lambda^*)c(x^0) = c(x^0)$$

that contradicts with the optimality of x^0 in (2.1) and Theorem 4.1 is proved. $\square$

The Implicit Optimality Criterion permits to verify optimality conditions for the NLP problem (4.2) instead of such the conditions for the convex SIP problem (2.1).

Remark 4.1 *Note that in the case when the convex SIP problem (2.1) satisfies the Slater condition, the correspondent NLP problem (4.2) introduced in this paper coincides with the nonlinear programming problem (SIP_D) formulated in [5] and we can replace in Theorem 4.1 the NLP problem (4.2) by (SIP_D) problem. In the case when the Slater condition is not satisfied for (2.1) such the replacement is not possible.*

Example 4.1. Consider the following SIP problem

$$\begin{array}{c} -4x_1 + x_2 + 3x_3 + \frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_3^2}{2} + \frac{x_4^2}{2} \longrightarrow \min, \\ \text{s.t.} \ \ -t^2 x_1 + t x_2 + \sin(t) x_3 + x_4^2 \leq 0, \quad t \in [-1, \ 2], \end{array} \tag{4.14}$$

where $x \in \mathbb{R}^4$.

There is a unique immobile point $t_1 = 0$ with $q_1 = 1$ (one can easily check it using DIO algorithm). As $q_1 > -1$, we can conclude that problem (4.14) does not satisfy the Slater condition.
Consider the feasible solution $x^0 = (4, 1, -1, 0)'$ of problem (4.14). To verify the optimality of x^0 using the Implicit Optimality Criterion, we have to construct the corresponding NLP problem in the form (4.2). In our example this problem takes the form

$$\begin{array}{ll} & -4x_1 + x_2 + 3x_3 + \dfrac{x_1^2}{2} + \dfrac{x_2^2}{2} + \dfrac{x_3^2}{2} + \dfrac{x_4^2}{2} \longrightarrow \min, \\ \text{s.t.} & x_4^2 = 0, \quad x_2 + x_3 = 0, \quad -2x_1 \leq 0. \end{array} \tag{4.15}$$

One can easily confirm that problem (4.15) is equivalent to some quadratic problem and that vector x^0 is optimal in this quadratic problem. Then, from Theorem 4.1 we can conclude that vector x^0 is optimal in problem (4.14) too.

To illustrate Remark 4.1, let us show that x^0 is not optimal in problem (SIP_D). According to [5], (SIP_D) has the form

$$\begin{aligned} & c(x) \longrightarrow \min, \\ \text{s.t.} \quad & f(x, t_i^0) \leq 0, \quad i = 1, \ldots, p, \end{aligned} \tag{4.16}$$

where $T_a(x^0) = \{t_i^0, \ i = 1, \ldots, p\}$ is the set of active points of T corresponding to x^0. In the case of SIP problem (4.14) with the feasible solution $x^0 = (4, 1, -1, 0)'$, problem (4.16) has the form

$$\begin{aligned} & -4x_1 + x_2 + 3x_3 + \frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_3^2}{2} + \frac{x_4^2}{2} \longrightarrow \min, \\ \text{s.t.} \quad & x_4^2 \leq 0. \end{aligned} \tag{4.17}$$

It is evident that vector x^0 is feasible but not optimal for (4.17) (this problem has better feasible solutions, for instance, $x^* = (4, -1, -3, 0)'$). Therefore, the statement of Theorem 4.1 will not be true if replace (4.15) with (4.16).

We would like to finish this section with two remarks.

Remark 4.2 *The results presented in the paper can be easily reformulated for the case when problem (2.1) has, additionally, a finite number of inequality constraints* $g_j(x) \leq 0, j = 1, \ldots, m, \ m \in \mathbb{N}$, *where functions* $g_j(x), \ j = 1, \ldots, m$, *are convex w.r.t.* $x, \ x \in \mathbb{R}^n$.

Remark 4.3 *Theorem 4.1 was formulated for convex SIP problems under the condition that* $|T_a(x^0)| < \infty$ *(Assumption 2.1). Authors believe that it is possible to formulate and prove the similar theorem without such the assumption and are going to do it in a separate publication.*

5 Conclusion

The main result of the paper is the Implicit Optimality Criterion that can be used for further investigations in the optimality theory of SIP as well as for constructing the new SIP algorithms. The Criterion is based on the concepts of immobility points and immobility orders that themselves are the important characteristics of the points of the index set T. That is why a special attention in the paper is given to description and substantiation of the algorithm (DIO algorithm) that determines the immobile points and their immobility orders in a finite number of iterations. The important properties of the Implicit Optimality Criterion are that it works without special assumptions (for example, the Slater condition has not be necessarily satisfied) and that it reduces optimality conditions for SIP problems to optimality conditions for some NLP problem. The last property gives the possibility to develop new efficient optimality conditions for SIP problems. As the matter of fact, each of the known optimality conditions (for example, from [1], [3] or others) being formulated for NLP problem in the form (4.2), can generate different (and not known yet) optimality conditions for the convex SIP problem (2.1). A study of such (explicit) optimality conditions as well as a comparison of these conditions with the known optimality conditions of SIP (see [2],[5],[8]) is the subject of a separate investigation [10].

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