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Convex Semi-Infinite Programming: Explicit Optimality Conditions

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Abstract

We consider the convex Semi-Infinite Programming (SIP) problem where objective function and constraint function are convex w.r.t. a finite-dimensional variable x and all of these functions are sufficiently smooth in their domains. The constraint function depends also on so called time variable t that is defined on the compact set $T \subset \mathbb{R}$. In our recent paper [15] the new concept of *immobility order* of the points of the set T was introduced and the Implicit Optimality Criterion was proved for the convex SIP problem under consideration. In this paper the Implicit Optimality Criterion is used to obtain new first and second order explicit optimality conditions. We consider separately problems that satisfy and that do not satisfy the the Slater condition. In the case of SIP problems with linear w.r.t. x constraints the optimality conditions have a form of the criterion. Comparison of the results obtained with some other known optimality conditions for SIP problems is provided as well.

Key words: semi-infinite programming, nonlinear programming, the Slater condition, optimality conditions

1 Introduction

Semi-Infinite Programming (SIP) deals with nonlinear extremal problems that involve infinitely many constraints in finite dimension. Having appeared in early sixties [14] in the works of Charnes, Cooper and Kortanek, today semi-infinite optimization continues to be a topic of a special interest due to a growing number of theoretical and practical applications (for the references see [9], [17], [27]). A number of important results concerning the topological structure of the feasible sets and the optimality conditions for semi-infinite problems have been published and different methods of solution of such problems have been developed (see, for example, [28], [2], [21], [24], [26] et al.). Generally, SIP problem

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with continuum of constraints is the problem of minimization (maximization) of some function $c(x)$ (objective function) subject to an infinite system of constraints expressed as $f(x,t) \leq 0,\ x \in \mathbb{R}^n,\ t \in T$ where T is some compact set. Sometimes (see [7],[6]) the solution of such the problem is based on the *discretization approach* when the set T is replaced by some finite number of its points (grid) and the initial infinite problem is reduced to a finitely constrained problem where only the constraints corresponding to the points of the grid are considered. Another approach is so-called *reduction approach*, when infinitely many constraints of the problem are replaced by some parameterized constraints $f(x,t_j(x)) \leq 0,\ j = 1,\dots,p$, where number p and functions $t_j(x), x \in R^n,\ j = 1,\dots,p$, are defined as in Theorem 2.1 from [10]. Under certain assumptions the reduced problem is equivalent to the initial SIP problem. In both approaches the optimality conditions for SIP problems are formulated in terms of optimality conditions for certain finite problems and the methods of solution of SIP problems are based on the methods of solution of these finite problems. For the references see, for example, the survey paper [9].

In the present paper we are concerned with *convex* SIP problem with continuum of constraints (i.e. problems, with compact index sets of constraints) where the objective function and the constraint function are supposed to be convex w.r.t. x and sufficiently smooth in their domains. In [15] a new approach to optimality conditions for convex SIP problems was suggested. Using the new concepts of immobility order and immobile point, a special class of finite nonlinear programming (NLP) problems was introduced and the optimality conditions for the convex SIP problems were formulated (Implicit Optimality Criterion) in terms of the analogous conditions for the finite NLP problems. The optimality conditions obtained do not require any additional assumption, such as, for example, the the Slater condition. Although the new approach suggested in [15] does not mean that one can simply reduce the convex SIP problem to some NLP problem, it can be useful for further developing of extremal theory as well as for creating of new algorithms of semi-infinite programming. The purpose of the paper is to deduce the new explicit optimality conditions for convex SIP problems and to compare the results obtained with other known optimality conditions.
The paper is organized as follows. In Section 2 we recall the basic notions of the immobile points, immobility orders and formulate the Implicit Optimality Criterion from [15]. In Section 3 we show how, in general case, the Implicit Optimality Criterion can be used to obtain explicit first and second order optimality conditions for convex SIP problems that satisfy and that do not satisfy the Slater condition. The case of SIP problems with linear constraints is considered separately and several examples are discussed. Section 4 is devoted to a comparison of the explicit optimality conditions obtained in Section 3 with the conditions suggested in [2, 8, 10, 12, 21, 20]. The final Section 5 contains brief discussion of the results.

2 Immobile Points, Immobility Orders and Implicit Optimality Criterion

Consider a convex Semi-Infinite Programming (SIP) problem in the form

$$\begin{array}{rl} & c(x) \longrightarrow \min, \\ \text{s.t.} & f(x,t) \leq 0, \quad t \in T = [t_*, t^*], \ t_*, t^* \in \mathbb{R}, \end{array} \tag{2.1}$$

where $x \in \mathbb{R}^n$; functions $c(x)$ and $f(x,t)$ are analytically defined, sufficiently smooth in $\mathbb{R}^n$ and $\mathbb{R}^n \times T$ respectively, convex with respect to x, i.e. for each $x_1, \ x_2 \in \mathbb{R}^n$ and for all $\alpha \in [0,1]$ it is satisfied:

$$c(\alpha x_1 + (1-\alpha)x_2) \leq \alpha c(x_1) + (1-\alpha)c(x_2),$$
$$f(\alpha x_1 + (1-\alpha)x_2, t) \leq \alpha f(x_1,t) + (1-\alpha)f(x_2,t), \ t \in T.$$

Denote by $X \subset \mathbb{R}^n$ the feasible set of problem (2.1)

$$X = \{x \in \mathbb{R}^n : \ f(x,t) \leq 0, t \in T\}.$$

Recall that it is said that the Slater condition holds in (2.1) if there exists $\bar{x} \in X$ such that $f(\bar{x},t) < 0, \ \forall t \in T$.

Assumption 2.1 *Suppose that $X \neq \emptyset$ and that there exists $\bar{x} \in X$ such that $f(\bar{x},t) \not\equiv 0, t \in T$.*

For any $x \in X$ we denote by $T_a(x) = \{t \in T : \ f(x,t) = 0\}$ the corresponding set of active points from T. Taking into account Assumption 2.1 and the sufficient smoothness of the function $f(x,t)$ in $\mathbb{R}^n \times T$, we conclude that there exists $\bar{x} \in X$ such that $|T_a(\bar{x})|$ is finite.

The following notations will be used in this paper. For the functions $c : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \times T \to \mathbb{R}$ denote:

$$\begin{array}{l} \nabla c(x) = c'(x); \\ f^{(0)}(x, \ t) = f(x,t), \ f^{(s)}(x, \ t) = \partial^s f(x,t)/\partial t^s, \ s \in \mathbb{N}; \\ \nabla_x f(x,t) = \partial f(x,t)/\partial x, \ \nabla_{xx} f(x,t) = \partial^2 f(x,t)/\partial x^2. \end{array}$$

In sequel we will use also the following point - set mapping $N(q)$:

$$N(q) = \emptyset, \ \text{if } q < 0, \ N(q) = \{0, \ 1, \ \ldots, \ q\} \ \text{ if } q \geq 0, \ q \in \mathbb{Z}.$$

Given $t \in T, \ x \in X$, let $\rho = \rho(x,t) \in \{-1, \ 0, \ 1, \ \ldots\}$ be such a number that

$$f^{(s)}(x,t) = 0, \ s \in N(\rho), \quad f^{(\rho+1)}(x,t) \neq 0.$$

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Definition 2.1 *Let $t \in T$. A number $q(t) \in \{-1, 0, 1, \dots\}$ is called the order of immobility (or the immobility order) of t in SIP problem (2.1) if*

1. *for each $x \in X$ the equalities*

$$f^{(r)}(x,t) = 0, \;\; r \in N(q(t)), \tag{2.2}$$

   hold;

2. *there exists $x(t) \in X$ such that*

$$f^{(q(t)+1)}(x(t),t) \neq 0. \tag{2.3}$$

From the definition above and from the constraints of problem (2.1) it follows that

1. if $t \in T$, then $q(t)+1$ is even and $f^{(q(t)+1)}(x(t),t) < 0$;
2. $q(t_*) \in \{-1, 0, 1, \dots\}$ and $f^{(q(t_*)+1)}(x(t_*),t_*) < 0$ for the correspondent $x(t_*) \in X$;
3. $q(t^*) \in \{-1, 0, 1, \dots\}$ and for the correspondent $x(t^*) \in X$ we have

   3.a) $f^{(q(t^*)+1)}(x(t^*),t^*) < 0$ for $q(t^*)+1$ even;

   3.b) $f^{(q(t^*)+1)}(x(t^*),t^*) > 0$ for $q(t^*)+1$ odd.

To simplify the further laying out, we can make the following assumption, without loss of generality.

Assumption 2.2 *Suppose that $q(t_*) = q(t^*) = -1$.*

It follows from Definition 2.1 that the constraints of problem (2.1) satisfy the the Slater condition if $q(t) = -1, \forall t \in T$.

Definition 2.2 *A point $t \in T$ is called the immobile point of problem (2.1) if $q(t) > -1$.*

In [15] the finite algorithm that finds the immobility orders of all points of the set T, was proposed. It was shown also, that the concept of the immobility order is an important characteristic of the constraints of problem (2.1) that makes it possible to formulate the optimality conditions for a given SIP problem (with an infinite number of constraints) in terms of the optimality conditions for some NLP problem (with a finite number of constraints).

Throughout we suppose that both, Assumption 2.1 and Assumption 2.2, are satisfied. Consider any feasible $x^0 \in X$ and the corresponding set of active points $T_a(x^0)$. Put $p = p(x^0) = |T_a(x^0)|$ and suppose that p is finite. Then $T_a(x^0)$ can be presented in the form

$$T_a(x^0) = \{t_j^0, j \in I(x^0)\} \tag{2.4}$$

with $I(x^0) = \{1, \dots, p\}$. Denote by $q_j = q(t_j^0), j \in I(x^0)$, the immobility orders of the active points. The following theorem is proved in [15].

Theorem 2.1 *[Implicit Optimality Criterion] The feasible solution $x^0 \in X$ with $|T_a(x^0)| < \infty$ is optimal in the convex SIP problem (2.1) if and only if it is optimal in the following Nonlinear Programming problem (NLP) problem*

$$\begin{array}{ll} & c(x) \longrightarrow \min, \\ s.t. & f^{(s)}(x, t_j^0) = 0, \; s \in N(q_j), \\ & f^{(q_j+1)}(x, t_j^0) \leq 0, \quad j \in I(x^0). \end{array} \tag{2.5}$$

3 Explicit Optimality Conditions

From the Implicit Optimality Criterion it follows that any necessary or sufficient optimality condition of the feasible solution $x^0 \in X$ in NLP problem (2.5) will be also the corresponding (necessary or sufficient) optimality condition for x^0 in convex SIP problem in the form (2.1). In this section we will obtain the new explicit optimality conditions for problem (2.1), considering separately situations when the the Slater condition is and is not satisfied.

3.1 Explicit Optimality Criterion for convex SIP problems satisfying the Slater condition

Suppose that the Slater condition is satisfied for SIP problem (2.1). Suppose also that for some feasible $x^0 \in X$ the corresponding set $T_a(x^0)$ of active points (see (2.4)) is finite. In this case $q_j = -1$ for any $j \in I(x^0)$ and NLP problem (2.5) formulated in Theorem 2.1 coincides with the locally reduced problem (SIP_D) introduced in [10]:

$$\begin{array}{ll} & c(x) \longrightarrow \min, \\ \text{s.t.} & f(x, t_i^0) \leq 0, \quad i = 1, \dots, p. \end{array} \tag{3.1}$$

If SIP problem (2.1) is convex, then problem (SIP_D) is convex too.
Since the the Slater condition is satisfied for SIP problem (2.1), it is satisfied also for problem (3.1) for which the following first order necessary and sufficient optimality conditions are true:
The feasible solution x^0 of problem (3.1) is optimal if and only if there exists a vector

$$\lambda^0 = (\lambda_0^0, \lambda_j^0, \; j = 1, \dots, p) \tag{3.2}$$

such that

$$\lambda_0^0 > 0, \; \lambda_j^0 \geq 0, j = 1, \dots, p, \tag{3.3}$$

$$\nabla_x L(x^0, \lambda^0) = 0 \tag{3.4}$$

with

$$L(x, \lambda^0) = \lambda_0^0 c(x) + \sum_{j=1}^{p} \lambda_j^0 f(x, t_j^0). \tag{3.5}$$

Let x^0 be a feasible solution of SIP problem (2.1). Then, applying the Implicit Optimality Criterion (Theorem 2.1), we can formulate the following theorem.

Theorem 3.1 *Let convex SIP problem (2.1) satisfy the Slater condition. Then feasible $x^0 \in X$ with $|T_a(x^0)| < \infty$ is optimal in problem (2.1) if and only if there exists a vector λ^0 in the form (3.2) such that conditions (3.3) and (3.4) hold.*

3.2 Explicit Optimality Conditions for convex SIP problems not satisfying the Slater condition

Now we consider the convex SIP problem (2.1) that does not satisfy the Slater condition. Let x^0 be feasible in (2.1). According to the Implicit Optimality Criterion, instead of verifying the optimality of x^0 in SIP problem (2.1) we can verify its optimality in NLP problem (2.5). It is evident that here the last problem (2.5) differs from (SIP_D) problem in the form (3.1) and both, equality and inequality constraints, are presented in it. It will be shown later that problem (2.5) is always degenerated in the sense that the "classical" first order optimality conditions (see [16]) for it are always satisfied with vanishing Lagrange multiplier that corresponds to the objective function. It is evident that such conditions are not efficient and, therefore, they cannot produce efficient optimality conditions for SIP problem. At the moment a number of results concerning the optimality conditions for the degenerated NLP problems (see, for example, [1], [4], [5]) is known. In the sequel we apply some of these results for the convex SIP problem under consideration.

3.2.1 Necessary Optimality Conditions

Let x^0 be some feasible solution of (2.1) with the corresponding active points set $T_a(x^0)$ in the form (2.4), $|T_a(x^0)| < \infty$. Denote:

$$h_{js}(x) = f^{(s)}(x, t_j^0),\ s \in N(q_j);\ \ g_j(x) = f^{(q_j+1)}(x, t_j^0),\ j \in I(x^0). \tag{3.6}$$

Then problem (2.5) can be written in the form

$$\begin{aligned} c(x) &\longrightarrow \min, \\ \text{s.t.}\ \ h_{js}(x) = 0,& \quad s \in N(q_j), \\ g_j(x) \leq 0,&\ j \in I(x^0). \end{aligned} \tag{3.7}$$

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Let us formulate , first, necessary optimality conditions of the first and the second order for the NLP problem in the form (2.5) (3.7) based on results from [4].

The Lagrange function for problem (3.7) has the form

$$\mathcal{L}(x,\lambda) = \lambda_0 c(x) + \sum_{j\in I(x^0)} \Big(\sum_{s\in N(q_j)} \lambda_{js} h_{js}(x) + \mu_j g_j(x) \Big) \tag{3.8}$$

with Lagrange multipliers vector

$$\lambda = (\lambda_0, \lambda_{js}, s \in N(q_j),\ \mu_j, j \in I(x^0)). \tag{3.9}$$

Denote

$$J_A(x^0) = \{j \in I(x^0) : g_j(x^0) = 0\} \tag{3.10}$$

and consider the cone of critical directions for problem (3.7) in the point x^0

$$\begin{aligned} \mathcal{K}(x^0) :=& \{\xi \in \mathbb{R}^n :\ \xi'\nabla c(x^0) \le 0,\ \xi'\nabla h_{js}(x^0) = 0,\ s \in N(q_j), j \in I(x^0), \\ & \xi'\nabla g_j(x^0) \le 0,\ j \in J_A(x^0)\}. \end{aligned} \tag{3.11}$$

According to [] we can formulate the necessary optimality conditions for problem (3.7) in the form of the following theorem.

Theorem 3.2 *Let x^0 be a local minimizer in problem (3.7). Then the following conditions are satisfied*
1^*. *(the first order necessary condition) The set*

$$\Lambda(x^0) = \{\lambda :\ \lambda \ne 0,\ \nabla_x \mathcal{L}(x^0,\lambda) = 0, \lambda_0 \ge 0, \mu_j \ge 0,\ \mu_j g_j(x^0) = 0, j \in I(x^0)\} \tag{3.12}$$

is nonempty;

2^*. *(the second order necessary condition)*

$$\max_{\lambda\in\Lambda(X^0), \|\lambda\|=1} \xi'\nabla^2_{xx}\mathcal{L}(x^0,\lambda)\xi \ge 0,\ \forall \xi \in \mathcal{K}(x^0). \tag{3.13}$$

It is evident that conditions 1^* and 2^* of Theorem 3.2 can be considered as necessary optimality conditions for feasible x^0 in SIP too. But, unfortunately, for SIP problem that does not satisfy the Slater condition, the correspondent NLP problem (2.5)(or (3.7)) is always degenerated that means that the necessary optimality conditions above are fulfilled for every feasible $x^0 \in X$ with $p = |T_a(x^0)| < \infty$. Indeed, consider any $x^0 \in X$ with finite set $T_a(x^0)$ of active points and apply to problem (2.1) DIO algorithm from [15]. As the Slater condition is not satisfied for (2.1), then, according to the algorithm, we have

$I^{(0)} \neq \emptyset$ and $\{t^0_j, j \in I^{(0)}\} \subset \{t^0_j, j = 1, \dots, p\}$. Choose any $j_* \in I^{(0)}$ and consider the following problem

$$\begin{aligned} & f(x, t^0_{j_*}) \to \min \\ \text{s.t.} \ \ & f(x, t^0_j) \leq 0, \ j \in I(x^0) \setminus \{j_*\}. \end{aligned} \tag{3.14}$$

From DIO algorithm we have that our vector x^0 is the optimal solution of the last problem. Applying the classical Lagrange multipliers rule to problem (3.14), we obtain that the following conditions

$$\Gamma = \{\gamma = (\gamma_j \geq 0, \ j \in I(x^0)) : \ \gamma \neq 0, \ \sum_{j \in I(x^0)} \gamma_j \nabla_x f(x^0, t^0_j) = 0\} \neq \emptyset, \tag{3.15}$$

$$\max_{\gamma \in \Gamma, \|\gamma\|=1} \sum_{j \in I(x^0)} \gamma_j \xi' \nabla_{xx} f(x^0, t^0_j) \xi \geq 0, \forall \xi \in S(x^0).$$

hold with $S(x^0) := \{\xi \in \mathbb{R}^n : \ \xi' \nabla_x f(x^0, t^0_j) \leq 0, \ j \in I(x^0)\}$. It follows from the conditions above that both statements of Theorem 3.2 are fulfilled with

$$\lambda_0 = \gamma_0, \ \lambda_{0j} = \gamma_j, \lambda_{sj} = 0, s = 1, \dots, N(q_j);$$

$$\mu_j = \begin{cases} 0, & \text{if } q_j > -1, \\ \gamma_j, & \text{if } q_j = -1, \quad j = 1, \dots, p. \end{cases}$$

Thus, if the Slater condition is not satisfied in (2.1), the necessary optimality conditions 1^* and 2^* are satisfied for any $x^0 \in X$ with $|T_a(x^0)| < \infty$ and , therefore, are not informative.

Let us deduce now other necessary optimality conditions for SIP problem (2.1), applying to NLP problem (3.7) (or, equally, (2.5)) the results from [1] .

Suppose that problem (3.7) has m_0 equality constraints: $m_0 = \sum_{j=1}^{p} |N(q_j)|$. Let x^0 be its feasible solution with $|J_A(x^0)| = m_1$. Consider the symmetric matrix $\mathcal{P}$ that determines the orthogonal projection of the space $\mathbb{R}^n$ on its subspace

$$\{\xi \in \mathbb{R}^n : \xi' \nabla h_{js}(x^0) = 0, \quad s \in N(q_j), j \in I(x^0), \quad \xi' \nabla g_j(x^0) = 0, \ j \in J_A(x^0)\}.$$

Let $r := m_0 + m_1 - m_2$, where

$$m_2 := rank(\nabla h_{js}(x^0), \ s \in N(q_j), j \in I(x^0), \quad \nabla g_j(x^0), j \in J_A(x^0)).$$

It can be shown (see [1]), that $r \leq m_0 + p$. Let $\Lambda_r \subseteq \Lambda(x^0)$ be a set of Lagrange multipliers λ from (3.9) such that the matrix $\mathcal{P} \nabla^2_{xx} \mathcal{L}(x^0, \lambda) \mathcal{P}$ has no more than r negative eigenvalues (i.e., the nonnegative index of this matrix is less or equal to r). On the base of the results obtained in [1], using the Implicit Optimality Criterion, we can now formulate the following theorem.

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Theorem 3.3 *Let $x^0 \in X$ with $|T_a(x^0)| < \infty$ be a local minimizer in SIP problem (2.1). Then*

1. $\Lambda_r \neq 0$,

2. $\max\limits_{\lambda \in \Lambda_r, \|\lambda\|=1} \xi' \nabla^2_{xx} \mathcal{L}(x^0, \lambda)\xi \geq 0, \ \forall \xi \in \mathcal{K}(x^0).$

Statement 1 of Theorem (3.3) gives the first order necessary optimality condition and statement 2 gives the second order necessary optimality condition for SIP problem (2.1).

Remark 3.1 *Application of the necessary optimality conditions formulated in Theorem 3.3 is not trivial as it involves a study of the topological structure of the set Λ_r. That's why the search for the another necessary optimality conditions for degenerated nonlinear problems and obtaining on the base of them the new necessary optimality conditions for SIP problems is the object of the further investigation of the authors.*

3.2.2 Sufficient Optimality Conditions

I. The first order sufficient optimality conditions. Denote by X_D and X_N the sets of the feasible solutions in finite problems (SIP_D) (see (3.1)) and (3.7), respectively. It is not difficult to verify that

$$X \subset X_N \subset X_D. \tag{3.16}$$

Therefore, if $x^0 \in X$ is optimal in (3.1) or in (3.7), then it is optimal in SIP too.
Taking into account the convexity of problem (3.1), we can formulate for it the following first order sufficient optimality conditions

Statement I. *Let $x^0 \in X_D$. If there exists a vector λ^0 defined by (3.2) such that conditions (3.3), (3.4) are fulfilled, then x^0 is optimal in problem (3.1).*

From (3.16) it follows that the statement above remains true if we substitute (3.1) by (3.7) or by (2.1).

II. The second order sufficient optimality conditions. Here we formulate the second order sufficient optimality conditions for SIP problem (2.1), applying the sufficient conditions from [] (classical results for NLP) to NLP problem (3.7).
Given a feasible solution x^0 of problem (3.7), construct the corresponding set $J_A(x^0)$, the cone of critical directions $\mathcal{K}(x^0)$ and the set $\Lambda(x^0)$ of Lagrange multipliers that satisfy the first order necessary optimality conditions according to (3.10), (3.11) and (3.12) respectively. Then the following sufficient optimality conditions can be formulated for NLP problem (3.7) (see [3, 10]):

Statement II. *Given x^0 feasible solution of (3.7), suppose that $\Lambda(x^0) \neq \emptyset$ and*

$$\max_{\lambda \in \Lambda(x^0), ||\lambda||=1} \xi' \nabla_{xx} \mathcal{L}(x^0, \lambda)\xi > 0, \ \forall \xi \in \mathcal{K}(x^0) \setminus \{0\}. \tag{3.17}$$

Then x^0 is a strict local minimizer in NLP problem (3.7).

Now suppose that x^0 is a feasible solution of SIP problem (2.1). Then by (3.16) $x^0 \in X_N$ and from the Implicit Optimality Criterion we conclude that Statement II supplies the second order sufficient optimality conditions for SIP problem (2.1) as well.
Consequently, we have proved the following theorem.

Theorem 3.4 *Let x^0 be a feasible solution of the convex SIP problem (2.1). Suppose that $|T_a(x^0)| < \infty$. If one of the following two conditions is hold*
A) there exists a vector λ^0 (3.2) satisfying (3.3) and (3.4) (the first order sufficient optimality condition);
B) $\Lambda(x^0) \neq \emptyset$ and condition (3.17) is satisfied (the second order sufficient optimality conditions),
then x^0 is optimal in SIP problem (2.1).

Theorem 3.4 is formulated under the assumption that $|T_a(x^0)| < \infty$. Let us show that this assumption is not restrictive.
Consider any subset

$$\{t^0_j, j \in I_*(x^0)\} \subset T_a(x^0), \quad I_*(x^0) = \{1, \dots, p\}, \ p < \infty, \tag{3.18}$$

and form a nonlinear programming problem

$$\begin{aligned} & c(x) \longrightarrow \min, \\ \text{s.t. } \ & h_{js}(x) = 0, \quad s \in N(q_j), \\ & g_j(x) \leq 0, \ j \in I_*(x^0). \end{aligned} \tag{3.19}$$

Here $q_j := q(t^0_j)$ is the immobility order of point $t^0_j \in T_a(x^0)$, functions $h_{js}(x)$ and $g_j(x)$ are defined in (3.6).
Denote by X_* the set of feasible solutions of problem (3.19). It is easy to show that $X \subset X_*$. Therefore, any sufficient optimality conditions for $x^0 \in X$ and, consequently, for $x^0 \in X_*$ in problem (3.19) give the sufficient optimality conditions for $x^0 \in X$ in SIP problem (2.1). Hence, we can formulate the following result.

Theorem 3.5 *Let x^0 be feasible in the convex SIP problem (2.1). Suppose that there exists a subset (3.18) of active points $T_a(x^0)$ such that for the sets $J_A(x^0)$, $\mathcal{K}(x^0)$, $\Lambda(x^0)$ constructed by (3.10), (3.11), (3.12) with $I(x^0)$ replaced by $I_*(x^0)$ at least one of two conditions (A or B), of Theorem 3.4 are fulfilled. Then x^0 is optimal in SIP problem (2.1).*

Example 3.1. Consider the following convex SIP problem in the form (2.1)

$$\begin{aligned} & -4x_1 + x_2 + 3x_3 + \tfrac{x_1^2}{2} + \tfrac{x_2^2}{2} + \tfrac{x_3^2}{2} + \tfrac{x_4^2}{2} \longrightarrow \min, \\ \text{s.t.} \quad & -t^2x_1 + tx_2 + \sin(t)x_3 + x_4^2 \le 0, \quad t \in T = [-1,\ 2], \end{aligned} \tag{3.20}$$

where $x \in \mathbb{R}^4$.

It was shown in [15] that this problem has a unique immobile point $t_1 = 0$ with $q_1 = q(t_1) = 1$. As $q_1 > -1$ we can conclude that the Slater condition is not satisfied here. Let us apply Theorem 3.4 to check if the feasible $x^0 = (4, 1, -1, 0)' \in X$ is optimal in (3.20). It is easy to verify that condition A) of the theorem is not fulfilled.
Now consider condition B). In our example, evidently, $p = 1$, $I(x^0) = \{1\}$, $t_1^0 = 0$ $q_1 = 1$. In terms of (3.6) we have $h_{10}(x) = x_4^2$, $\quad h_{11}(x) = x_2 + x_3$, $\quad g_1(x) = -2x_1$, and the Lagrange function for (3.20) takes a form

$$\mathcal{L}(x,\lambda) = \lambda_0\Big(-4x_1 + x_2 + 3x_3 + \frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_3^2}{2} + \frac{x_4^2}{2}\Big) + \lambda_1 x_4^2 + \lambda_2(x_2 + x_3) + \mu_1(-2x_1)$$

with Lagrange multipliers vector $\lambda = (\lambda_0, \lambda_1, \lambda_2, \mu_1)$. Since

$$\nabla_x \mathcal{L}(x,\lambda) = \lambda_0 \begin{pmatrix} -4 + x_1 \\ 1 + x_2 \\ 3 + x_3 \\ x_4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2x_4 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

then condition $\nabla_x \mathcal{L}(x,\lambda) = 0$ takes in x^0 the form

$$\lambda_0 \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0.$$

From the system above we obtain

$$\mu_1 = 0,\ 2\lambda_1 + \lambda_2 = 0.$$

Then the set $\Lambda(x^0)$ defined in (3.12) takes the form

$$\Lambda(x^0) = \{\lambda \in \mathbb{R}^4 :\ \lambda = (\lambda_0, \lambda_1, -2\lambda_1, 0)',\ \lambda_0 \ge 0\}.$$

It is evident that $\Lambda(x^0) \neq \emptyset$. To verify (3.17), calculate

$$\nabla_{xx}\mathcal{L}(x^0,\lambda) = \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_0 + 2\lambda_1 \end{pmatrix}$$

and find $\mathcal{K}(x^0)$ from (3.11) :

$$\mathcal{K}(x^0) = \{\xi = (\xi_1, \xi_2, \xi_3, \xi_4)' \in \mathbb{R}^4 :, \ \xi_2 + \xi_3 = 0, -2\xi_1 \leq 0\}$$

$$= \{\xi \in \mathbb{R}^4 : \ \xi = (\xi_1, \xi_2, -\xi_2, \xi_4)', \xi_1 \geq 0\}.$$

Note here that $g_1(x^0) < 0$, hence $J_A(x^0) = \emptyset$. Finally, for $\xi \in \mathcal{K}(x^0)$ we have

$$\xi' \nabla_{xx} \mathcal{L}(x, \lambda) \xi = \lambda_0 \xi_1^2 + 2\lambda_0 \xi_2^2 + (\lambda_0 + 2\lambda_1) \xi_4^2.$$

From the last formula we can conclude that for any $\lambda \in \Lambda(x^0)$ such that $||\lambda|| = 1, \ \lambda_0 > 0, \ \lambda_1 \geq 0$ it is satisfied $\xi' \nabla_{xx} \mathcal{L}(x, \lambda) \xi > 0, \ \forall \xi \in \mathcal{K}(x^0) \setminus \{0\}$ and, hence, condition (3.17) is fulfilled. Therefore, condition B of Theorem 3.4 is satisfied and, hence, vector $x^0 = (4, 1, -1, 0)'$ is the optimal solution of SIP problem (3.20).

Remark 3.2 *Example 3.1 illustrates that fact that conditions (3.3) and (3.4) can not be used as necessary optimality condition for SIP problems that do not satisfy the Slater condition.*

Remark 3.3 *One can use another known results of extremal theory (for example, from [5]) to obtain the new explicit optimality conditions by the same scheme.*

Remark 3.4 *Using the results of this section, one can easily deduce the optimality conditions for convex SIP problems obtained after addition to the constraints of problem (2.1) of a finite number of convex inequality constraints.*

We would like to close this section with the following observation. All optimality conditions presented above for convex SIP problems with continuum of constraints were obtained without consideration of the specific features of the NLP problem (2.5). Due to the way the last problem was constructed, it has a special structure and we believe that it is possible to get more effective optimality conditions for SIP problems if take this structure into account.

3.3 Optimality Criterion for convex SIP problems with linear constraints

Consider now the particular case of problem (2.1), when the objective function $c(x)$ is convex and the constraints function $f(x, t)$ is linear w.r.t. x. In this case, on the base of the Implicit Optimality Criterion (Theorem 2.1) we can formulate the following (explicit) first order optimality criterion.

Theorem 3.6 *[Explicit Optimality Criterion] Consider the convex SIP problem (2.1), where $f(x,t)$ is linear w.r.t. x. Then the feasible solution x^0 is optimal in (2.1) if and only if there exists a finite subset $\{t_j^0, j=1,\dots,p\} \subset T_a(x^0)$ and numbers $\lambda_{sj}, s \in N(q_j),\ \mu_j \geq 0,\ j=1,\dots,p$ such that*

$$\nabla c(x^0) + \sum_{j=1}^{p}\Big(\sum_{s\in N(q_j)} \lambda_{sj}\nabla_x f^{(s)}(x^0,t_j^0) + \mu_j \nabla_x f^{(q_j+1)}(x^0,t_j^0)\Big) = 0, \tag{3.21}$$

$$\mu_j f^{(q_j+1)}(x^0,t_j^0) = 0,\ j=1,\dots,p, \quad \sum_{j=1}^{p}\Big(\sum_{s\in N(q_j)} |\lambda_{sj}| + \mu_j\Big) > 0.$$

Here $q_j = q(t_j^0),\ j=1,\dots,p$.

Note that for the linear case we have formulated the explicit optimality **criterion** without any additional assumption or restriction.

Example 3.2. To illustrate the application of Theorem 3.6, consider the following SIP problem

$$\begin{array}{c} \min\ (-4x_1 + x_2 + 3x_3 + \frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_3^2}{2}) \\ \text{s.t.}\ -t^2x_1 + tx_2 + \sin(t)x_3 \leq 0, \quad t \in T = [-1,\ 2], \end{array} \tag{3.22}$$

where $x \in \mathbb{R}^3$. Here we have $c(x) = -4x_1+x_2+3x_3+\frac{x_1^2}{2}+\frac{x_2^2}{2}+\frac{x_3^2}{2},\ f(x,t) = -t^2x_1+tx_2+\sin(t)x_3,\ f^{(1)}(x,t) = -2tx_1+x_2+\cos(t)x_3,\ f^{(2)}(x,t) = -2x_1-\sin(t)x_3$. It is easy to verify that $c(x)$ is convex, $f(x,t)$ is linear w.r.t. x and $x^0 = (4,1,-1)'$ is the feasible solution of (3.22) with $c(x^0) = -9$. The unique immobile point in the given problem is (one can verify it using DIO algorithm) $t_1^0 = 0$ such that $q_1 = q(t_1^0) = 1$. Then, according to Theorem 3.6, x^0 is optimal if and only if there exists some nonzero vector $\lambda = (\lambda_{01}, \lambda_{11}, \mu_1),\ \mu_1 \geq 0$ such that

$$\nabla c(x^0) + \lambda_{01}\nabla_x f(x^0,t_1^0) + \lambda_{11}\nabla_x f^{(1)}(x^0,t_1^0) + \mu_1 \nabla_x f^{(2)}(x^0,t_1^0) = 0$$

or, equivalently, if there exists a solution of the system

$$\begin{cases} -2\cdot\mu_1^0 = 0, \\ 2 + \lambda_{11}^0 = 0. \end{cases}$$

It is easy to verify that the system above has a solution $\lambda^0 = (0,-2,0)'$ and, therefore, x^0 is the optimal solution in problem (3.22). Note that if we choose any another feasible solution, for example, $x^1 = (5,9,-9)$, it will not be optimal as system (3.21) with x^0 replaced by x^1 has no feasible solutions in this case. Compare: $c(x^0) = -9$, and $c(x^1) = 54.5$.

4 Comparison with other optimality conditions for SIP problems

There are many papers devoted to optimality conditions for SIP problems (see, for example, [2, 8, 12, 13, 18, 21]). Summarizing these results, we can formulate the necessary and sufficient optimality conditions for the convex SIP problem (2.1) in the form of the following Theorems 4.1, 4.2, 4.3 (see [4]).

Theorem 4.1 *Let x^0 be an optimal solution of (2.1). Then there exists vector $\lambda^0 \neq 0$, $\lambda^0 \geq 0$, in the form (3.2) and a finite set $\{t^0_j, j = 1, \dots, p\} \subset T_a(x^0)$ such that condition (3.4) is verified.*

Let x^0 be feasible in SIP problem (2.1) with finite set $T_a(x^0) = \{t^0_j, j = 1, \dots, p\}$. Denote

$$K(x^0) = \{\xi \in \mathbb{R}^n : \xi'\nabla_x c(x^0) \leq 0,\ \xi'\nabla_x f(x^0, t^0_j) \leq 0,\ j = 1, \dots, p\}. \tag{4.1}$$

For every $j = 1, \dots, p$, and for every $\xi \in K(x^0)$, denote by $\eta_j(\xi)$ an optimal solution of the following quadratic problem

$$\frac{1}{2} f^{(2)}(x^0, t^0_j)\eta^2 + \xi'\nabla_x f^{(1)}(x^0, t^0_j)\eta \to \max_{\eta} \tag{4.2}$$

if such a solution exists.

Theorem 4.2 *Let x^0 be an optimal solution of (2.1) with finite set $T_a(x^0)$ (2.4). Suppose that $f^{(2)}(x^0, t^0_j) \neq 0, j = 1, \dots, p$.*
Then(second order necessary conditions) for every $\xi \in K(x^0)$ there exists $\lambda^0 \neq 0$ satisfying the conditions of Theorem 4.1 and such that

$$\xi'\nabla_{xx} L(x^0, \lambda^0)\xi - \sum_{j=1}^{p} \lambda^0_j (\eta_j(\xi))^2 f^{(2)}(x^0, t^0_j) \geq 0. \tag{4.3}$$

Let us formulate the known sufficient optimality conditions.

Theorem 4.3 *Let x^0 be feasible in problem (2.1). Suppose that for each $\xi \in K(x^0)$ there exists a finite set in the form (3.18) and a nonzero vector $\lambda^0(\xi) = (\lambda^0_j(\xi), j = 0, 1, \dots, p)$, $\lambda^0_j(\xi) \geq 0, j = 0, 1, \dots, p$, such that*

(i) $\nabla_x L(x^0, \lambda^0(\xi)) = 0$;
(ii) one of the following two conditions is verified:

1. $\xi'\nabla_{xx} L(x^0, \lambda^0(\xi))\xi - \sum\limits_{j=1}^{p} \lambda^0_j(\xi)(\eta_j(\xi))^2 f^{(2)}(x^0, t^0_j) > 0$, *where $\eta_j(\xi)$ is an optimal solution of problem (4.2), case such a solution exists;*

2. there exists $1 \leq j_0 \leq p$ *such that* $\lambda^0_{j_0}(\xi) > 0$ *and problem (4.2) has no solution.*

Then x^0 *is optimal in SIP.*

Remark 4.1 *In [11, 10, 20, 19] the necessary and sufficient optimality conditions are provided for GSIP problems. Evidently, the conditions obtained for GSIP can be reformulated for SIP problems. It can be shown that being applied to SIP problem these conditions turn to be the same as in Theorems 4.1, 4.2, 4.3.*

Based on the fact that any optimal vector x^0 of problem (2.1) is also optimal in problem (3.14), it is easy to show that the necessary optimality conditions of Theorems 4.1 and 4.2 are satisfied with $\lambda^0_0 = 0,\ \lambda^0_j = \gamma_j \geq 0,\ j \in I(x^0),\ \lambda^0 \neq 0$, where vector $\gamma = (\gamma_j \geq 0,\ i \in I(x^0))$ belongs to the set Γ defined in (3.15) for any $x^0 \in X$ with $|T_a(x^0)| < \infty$. Therefore, these conditions are not informative as well as the conditions of Theorem 3.2.

Now let us compare the sufficient optimality conditions from Theorem 4.3 with the conditions of Theorem 3.4 (or Theorem 3.5).
Given convex SIP problem (2.1), let X^0_{suf} and X^*_{suf} be the sets of its optimal solutions that satisfy the sufficient optimality conditions of Theorem 3.4 and Theorem 4.3 respectively. Let us show that the following two statements are true.

α) $X^*_{suf} \subset X^0_{suf}$ and
β) in general, there exists x^0 such that $x^0 \in X^0_{suf}$, but $x^0 \notin X^*_{suf}$.

To prove statement α), we will show that $x^0 \in X^*_{suf} \subset X$ implies $x^0 \in X^0_{suf}$.
Suppose that $x^0 \in X$ satisfy conditions of Theorem 4.3. It is evident that $\mathcal{K}(x^0) \subset K(x^0)$. Then any $\xi \in \mathcal{K}(x^0)$ belongs also to $K(x^0)$ and, by assumption, there exists a vector $\lambda^0(\xi) = (\lambda^0_j(\xi), j = 0, 1, \ldots, p)$, $\lambda^0_j(\xi) \geq 0, j = 0, 1, \ldots, p$, such that conditions (i) and (ii) are verified.
If $\lambda^0_0(\xi) > 0$, then, taking into account (i), we conclude that condition A) of Theorem 3.4 is verified with $\lambda^0 = \lambda^0(\xi)$. Therefore, $x^0 \in X^0_{suf}$ and statement α) is proved.
Now suppose that $\lambda^0_0(\xi) = 0$. Then condition (i) of Theorem 4.3 takes the form

$$\sum_{j=1}^{p} \lambda^0_j(\xi) \nabla_x f(x^0, t^0_j) = 0. \tag{4.4}$$

Consider $\xi \in \mathcal{K}(x^0)$. By (4.1) and by Definition 2.1 we can conclude that for $j \in \{1, \ldots, p\}$ with $q_j := q(t^0_j) > -1$, the following relations are satisfied:

$$f^{(2)}(x^0, t^0_j) \leq 0,\ \xi' \nabla_x f^{(1)}(x^0, t^0_j) = 0.$$

Therefore, $\eta_j(\xi) = 0$ is the solution of (4.2), wherefrom

$$\lambda^0_j(\xi)(\eta_j(\xi))^2 f^{(2)}(x^0, t^0_j) = 0 \text{ if } q_j > -1. \tag{4.5}$$

Suppose now that for some $j_* \in \{1, \dots, p\}$ it is satisfied

$$\lambda^0_{j_*}(\xi) > 0, \;\; q_{j_*} = -1. \tag{4.6}$$

Then, taking into account (4.4), we can conclude that vector x^0 is optimal in the problem (3.14). However, this contradicts our assumption that $q_{j_*} = -1$. Consequently, there is no j_* satisfying (4.6) or, in other words, $\lambda^0_j(\xi) = 0$ if $q_j = -1$. Then for any $\eta_j(\xi)$ we have

$$\lambda^0_j(\xi)(\eta_j(\xi))^2 f^{(2)}(x^0, t^0_j) = 0 \text{ if } q_j = -1. \tag{4.7}$$

Due to (4.5) and (4.7), condition (ii) of Theorem 4.3 takes the form

$$\xi' \nabla_{xx} L(x^0, \lambda^0(\xi))\xi > 0. \tag{4.8}$$

It follows from (4.8) and from condition (i) of Theorem 4.3 that condition B) of Theorem 3.4 is satisfied with

$$\lambda_0 = \lambda^0_0(\xi), \;\; \lambda_{0j} = \lambda^0_j(\xi), \;\; \lambda_{sj} = 0, s = 1, \dots, N(q_j),$$

$$\mu_j = \begin{cases} 0, & \text{if } q_j > -1, \\ \lambda^0_j(\xi) = 0 & \text{if } q_j = -1, \quad j = 1, \dots, p. \end{cases}$$

Hence, $x^0 \in X^0_{suf}$ and statement α) is completely proved.

To prove statement β), let us once again consider the convex SIP problem (3.22) from Example 3.1. It was showed above the feasible solution of problem (3.22), vector $x^0 = (4, -1, -1, 0)'$, satisfy condition B) of Theorem 3.4 and, hence, $x^0 \in X^0_{suf}$. Let us show now that x^0 does not satisfy conditions of Theorem 4.3.
Recall that in our example we have $T_a(x^0) = \{t^0_1 = 0\}, \; p = 1$. From formulae (4.1) and (3.5), respectively, we have

$$K(x^0) = \{\xi = (\xi_i, i = 1, \dots, 4) : \xi_2 + \xi_3 \le 0\}, \quad L(x, \lambda^0) = \lambda^0_0 c(x) + \lambda^0_1 f(x, 0).$$

Then

$$\nabla_x L(x, \lambda^0) = \lambda^0_0 \begin{pmatrix} -4 + x_1 \\ 1 + x_2 \\ 3 + x_3 \\ x_4 \end{pmatrix} + \lambda^0_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2x_4 \end{pmatrix},$$

$$\nabla_{xx} L(x, \lambda^0) = \lambda^0_0 \text{diag } (1, 1, 1, 1) + \lambda^0_1 \text{diag } (0, 0, 0, 2).$$

The set of Lagrange multipliers that satisfy the first order optimality conditions (3.5) is

$$\Lambda(x^0) = \{\lambda^0 = (\lambda^0_0, \lambda^0_1) : \nabla_x L(x^0, \lambda^0) = 0\} = \{(0, \lambda^0_1) : \;\; \lambda^0_1 \ge 0\}.$$

It is easy to verify that for $\bar{\xi} = (0, 1, -1, 0)' \in K(x^0)$, and for $t_1^0 = 0$, the value $\eta_1(\bar{\xi}) = 0$ is optimal in the quadratic problem (4.2). Then we get

$$\bar{\xi}'\nabla_{xx}L(x^0, \lambda^0)\bar{\xi} - \lambda_1^0(\eta_1(\bar{\xi}))^2 f^{(2)}(x^0, t_1^0) = \bar{\xi}'\nabla_{xx}L(x^0, \lambda^0)\bar{\xi} = 0 \text{ for } \forall\lambda^0 \in \Lambda(x^0)$$

that means that condition (ii) of Theorem 4.2 is not satisfied. Hence, the optimal solution x^0 ($x^0 \in X^*_{suf}$) does not satisfy sufficient optimality conditions of Theorem 4.3, i.e. $x^0 \notin X^0_{suf}$. This proves statement β).

Statements α) and β) mean that in the case of convex semi-infinite problem in the form (2.1) our sufficient optimality conditions are better than such the conditions from [2, 8, 12, 13, 18, 21].

5 Conclusion

The main result of the paper consists in the explicit formulation of the first and second order optimality conditions for convex SIP problem (2.1) with continuum of constraints. These optimality conditions are obtained with the help of Implicit Optimality Criterion proved in [15]. The Criterion is based on the concepts of immobility points and immobility orders which are the important characteristics of the points of the index set T and of the feasible set X of problem (2.1). Application of the Implicit Optimality Criterion makes it possible to develop new efficient optimality conditions for SIP problems without special assumptions (for example, the Slater condition has not be necessarily satisfied).

In recent years many papers dedicated to the theory of SIP ([18, 19, 20, 22, 24] etc.), in general, and to SIP optimality conditions, in particular, have appeared. We refer to papers [2, 8] for the first order optimality conditions and to papers [8, 12, 13, 18, 21] for second order optimality conditions. New constructive algorithms for solving SIP problems have been suggested in [6, 7, 23, 26] etc.

Usually, to obtain the optimality conditions for SIP problems the reduction approach is used. Note that there exist different types of reduction (for the references see [9]). Sometimes, given some feasible solution x^0, the active points $T_a(x^0) \subset T$ are found and the following finite problem

$$\min_x c(x), \quad \text{s.t.} \quad f(x,t) \leq 0, \ t \in T_a(x^0) \qquad (SIP_D)$$

is associated with the original SIP problem ([9]). If SIP problem (2.1) is convex and satisfies the Slater condition, then every optimal solution x^0 in SIP is optimal in (SIP_D) as well. Here it is supposed that $|T_a(x^0)| < \infty$.

In [10] another reduction approach is proposed. Being applied to a SIP problem in the form (2.1), this approach consists in local reducing (under certain assumptions) of this problem to the following finite problem (the reduced problem):

$$\min_x c(x), \text{ s.t. } v^l(x) = f(x, t^l(x)) \leq 0, \; l = 1, \dots, p. \qquad (SIP_{red})$$

Here the number p and the functions $t^l(x), \; x \in \mathbb{R}^n, \; l = 1, \dots, p$, are defined in Theorem 2.1 from [10].

Using the reduction approaches mentioned above, either necessary or sufficient optimality conditions for SIP problem can be formulated in terms of the correspondent optimality conditions for some finite nonlinear programming problem ((SIP_D) or (SIP_{red})). In the case when the initial SIP problem does not satisfy the Slater condition, the first order necessary optimality conditions for (SIP_D) [10] and (SIP_{red}) [19] are the same and are trivially satisfied (any feasible solution satisfies this conditions) whereas the first order sufficient optimality conditions are not satisfied.

In the present paper to obtain the necessary and sufficient optimality conditions for the convex SIP problem we used the finite NLP problem in the form (2.5). Generally (in the case when the Slater condition is not satisfied), this NLP problem differs from problems (SIP_D) and (SIP_{red}) and permits to obtain the new optimality conditions in situations when the traditional approach is not efficient.To illustrate this we have applied the results from [1] to the convex SIP problem that do not satisfy the Slater conditions, having obtained the new necessary optimality conditions that are not trivially satisfied and the sufficient optimality condition that are not so strong as the sufficient optimality conditions from [8, 12, 13, 18, 21].

We would like to finish this paper with the following observation. It was not the aim of the paper to consider the numerical methods for SIP problems' solutions. At the same time it is evident to us that according to Implicit Optimality Criterion, the theory and methods of NLP can be used to obtain the new results for convex and not convex semi-infinite programming.

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