BILLIARD SCATTERING ON ROUGH SETS: TWO-DIMENSIONAL CASE[*]

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Abstract. The notion of a rough two-dimensional (convex) body is introduced, and to each rough body there is assigned a measure on $\mathbb{T}^3$ describing billiard scattering on the body. The main result is characterization of the set of measures generated by rough bodies. This result can be used to solve various problems of least aerodynamical resistance.

Key words. billiards, scattering on rough surfaces, Monge–Kantorovich optimal mass transportation, problems of minimal and maximal resistance, shape optimization

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1. Definition of a rough set and statement of main theorem.

1.1. Introductory remarks and review of literature. In this paper the notion of a rough two-dimensional (convex) body is given and some properties of rough bodies are established.

Let $B \subset \mathbb{R}^2$ be a convex bounded set with nonempty interior, that is, a bounded convex body. Consider the "set" obtained from B by moving off a set of "very small" area. Such a (heuristically defined) set is called a *rough body*: from the "macroscopic" point of view, it almost coincides with B, and, from the "microscopic" point of view, it contains some "flaws." (One can imagine a detail of a mechanism that, after a period of exploitation, has some defects.) If the removed set adjoins the boundary ∂B, then one can expect that a flow of point particles incident on the rough body is reflected in another way as compared to reflection from B.

The notion of rough body arises naturally when studying Newton-like problems of the body of least resistance. The first problem of such kind was considered by Newton himself [1]. Recently there were several works made concerning the problem of least resistance in various classes of admissible bodies; see, e.g., [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [15]. The solution of a minimization problem for the case of *rotating bodies* can be naturally identified with a rough body [14]; see also the concluding remarks to this paper.

There are many papers on particle scattering by rough bodies (see, e.g., [16], [17], [18]); they describe bodies and flows of particles that occur in nature. On the contrary, we assume that a rough structure can be "manufactured," and our aim is to describe all possible rough structures.

1.2. Definition of a rough body. It is supposed that the "microscopic structure" of the boundary of a rough body can be detected from observations of particle

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scattering on the body. From this point of view, two rough bodies are considered equal if they scatter flows of particles in an identical manner. Having these observations in mind, we give the definition of a rough body.

Let B be a bounded convex body. Denote by $n(\xi)$ the unit outer normal vector to ∂B at a regular point $\xi \in \partial B$, and denote by $(\partial B \times S^1)_+$ the set of pairs $(\xi, v) \in \partial B \times S^1$ such that $\langle n(\xi), v \rangle \geq 0$. Here and in what follows, $\langle \cdot, \cdot \rangle$ means the standard scalar product in $\mathbb{R}^2$. The set $(\partial B \times S^1)_+$ is equipped with the measure μ which is defined by $d\mu(\xi, v) = \langle n(\xi), v \rangle \, d\xi \, dv$, where $d\xi$ and dv are the one-dimensional Lebesgue measures on ∂B and S^1, respectively.
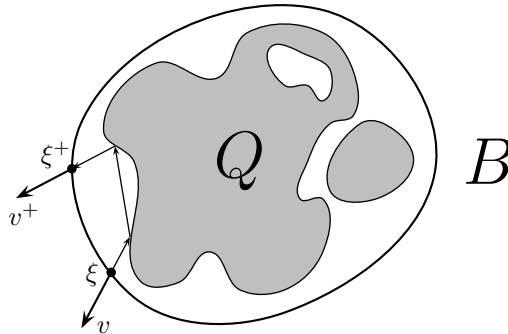

FIG. 1. *A billiard trajectory in* $\mathbb{R}^2 \setminus Q$.

Let Q be a set with piecewise smooth boundary contained in B; consider the billiard in $\mathbb{R}^2 \setminus Q$. Note that Q is not necessarily connected. For $(\xi, v) \in (\partial B \times S^1)_+$, consider a billiard particle starting at the point ξ with the velocity $-v$. After several (maybe none) reflections from $\partial Q \setminus \partial B$, the particle will intersect ∂B again, at a point $\xi^+ = \xi^+_{Q,B}(\xi, v) \in \partial B$; denote by $v^+ = v^+_{Q,B}(\xi, v)$ the velocity at this point (see Figure 1). It may happen that the initial point ξ belongs to ∂Q; in that case we have $\xi^+ = \xi$ and the vector v^+ is symmetric to v with respect to $n(\xi)$. It may also happen that at some moment the particle either gets into a singular point of ∂Q, or touches ∂Q at a regular point, or stays in $B \setminus Q$ forever and does not intersect ∂B again, or makes an infinite number of reflections in finite time. The set of corresponding points (ξ, v) has zero measure, and the corresponding values $\xi^+_{Q,B}(\xi, v)$ and $v^+_{Q,B}(\xi, v)$ are not defined.

Thus, there is defined the one-to-one mapping $T_{Q,B} : (\xi, v) \mapsto (\xi^+_{Q,B}(\xi, v), v^+_{Q,B}(\xi, v))$ of a full measure subset of $(\partial B \times S^1)_+$ onto itself. It has the following properties:

T1. $T_{Q,B}$ preserves the measure μ.

T2. $T^{-1}_{Q,B} = T_{Q,B}$.

The mapping $T_{Q,B}$ induces the measure $\nu_{Q,B}$ on $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ in the following way. Let $A \subset \mathbb{T}^3$ be a Borel set; by definition,

$$\nu_{Q,B}(A) = \mu \left(\{(\xi, v) \in (\partial B \times S^1)_+ : (v, v^+_{Q,B}(\xi, v), n(\xi)) \in A\} \right).$$

In fact, the measure $\nu_{Q,B}$ contains information about particle scattering on Q. Imagine that an observer has no means to track the trajectory of particles inside B. Instead, for each incident particle there is registered the triple of vectors: the initial and final

velocities (measured at the points of first and second intersection with ∂B), and the normal vector to ∂B at the point of first intersection with ∂B. The normal vector at the second point of intersection is not registered; as will be seen later (Lemma 1), if the area of $B \setminus Q$ is small, then the difference between the normal vectors at these two points is also small. The measure $\nu_{Q,B}$ describes the distribution of triples.

DEFINITION 1. *We say that a sequence of sets* $\{Q_m, \ m = 1, 2, \ldots\}$ *represents a* rough body *if it has the following properties:*

M1. $Q_m \subset B$ *and* $\mathrm{Area}(B \setminus Q_m) \to 0$ *as* $m \to \infty$.

M2. *The sequence of measures* $\nu_{Q_m, B}$ *weakly converges.*

Two sequences of such sets are called equivalent if the corresponding limiting measures coincide. An equivalence class is called a body obtained by roughening B, *or simply* rough body, *and denoted by* $\mathcal{B}$, *and the corresponding limiting measure is denoted by* $\nu_{\mathcal{B}}$.

Note that the sets Q_m in this definition are not necessarily connected.

Remark. Since $\mathbb{T}^3$ is compact and the full measure of $\mathbb{T}^3$ satisfies $\nu_{Q,B}(\mathbb{T}^3) \leq 2\pi |\partial B|$, one concludes that the set of measures $\{\nu_{Q,B}\}$, with fixed B, is weakly precompact. That is, any sequence of measures $\{\nu_{Q_m,B}\}$ contains a weakly converging subsequence. In this sense one can say that a sequence, satisfying only condition M1, can represent more than one rough body.

We would also like to mention that, first, two rough bodies obtained one from another by translation are identified, according to our definition. Second, particle scattering on $\mathcal{B}$ in a small neighborhood of $\xi \in \partial B$ can be detected if ξ is an extreme point of B, and cannot otherwise. Indeed, if ξ is an extreme point of B, then the scattering is described by the restriction of $\nu_{\mathcal{B}}$ on $\mathbb{T}^2 \times \mathcal{N}_{n(\xi)}$, with $\mathcal{N}_{n(\xi)}$ being a small neighborhood of $n(\xi)$ in S^1. If, otherwise, ξ is not an extreme point of B, that is, belongs to an open linear segment contained in ∂B, the scattering can be determined only on the whole segment.

Actually, from the viewpoint of applications to the problems of optimal resistance in *homogeneous* and *rarefied* media (see section 4, containing concluding remarks and applications), these drawbacks are not so serious. Indeed, resistance of a body is invariant under translations (due to homogeneity). Besides, if the boundary of a body contains a linear segment, then one does not need to know scattering at each point of the segment; it suffices to know it on the whole segment (due to homogeneity and rarefaction).

The definition of a rough body could be made in a slightly different way, basing it on measures defined on $S^1 \times S^1 \times \partial B$. In that case the triple (v, v^+, ξ) should be registered, with ξ being the point of first intersection with ∂B. That definition would allow one to register particle scattering at each point of ∂B and to distinguish between bodies obtained by translation one from another. However, we prefer to adopt the former definition, since it seems to us mathematically more transparent and makes the arguments a bit easier.

1.3. Examples. Sometimes it is convenient to use another representation of the measure $\nu_{\mathcal{B}}$. Namely, consider the change of coordinates $(v, v^+, n) \mapsto (\varphi, \varphi^+, n)$, where $\varphi = \mathrm{Arg}\, v - \mathrm{Arg}\, n$, $\varphi^+ = \mathrm{Arg}\, v^+ - \mathrm{Arg}\, n$. Here $\mathrm{Arg}\, v$ is the angle between a fixed vector and v measured, say, clockwise from this vector to v. If $(v, v^+, n) \in$ spt $\nu_{\mathcal{B}}$, then φ and φ^+ belong to $[-\pi/2, \pi/2]$ modulo 2π. Introduce the shorthand notation $\square := [-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ and define the mapping $\varpi : \square \times S^1 \to \mathbb{T}^3$ by $\varpi(\varphi, \varphi^+, n) = (v, v^+, n)$. One has spt $\nu_{\mathcal{B}} \subset \varpi(\square \times S^1)$. Denote $\check{\nu}_{\mathcal{B}} := (\varpi^{-1})^{\#} \nu_{\mathcal{B}}$. Sometimes this measure can be factorized: $\check{\nu}_{\mathcal{B}} = \eta_{\mathcal{B}} \otimes \tau_B$, where $\eta_{\mathcal{B}}$ is defined on $\square$

and τ_B is the surface measure on B; so to say, the "roughness" is "homogeneous" along the body's boundary. Consider several examples.

Example 1 ("smooth body"). The rough body represented by the sequence $Q_m = B$ is identified with B itself. The corresponding measure is $\check{\nu}_B = \eta_0 \otimes \tau_B$, where the measure η_0 has the density $\cos\varphi \cdot \delta(\varphi + \varphi^+)$; the support of η_0 is shown in Figure 2(b). In Figure 2(a), B is taken to be an ellipse.
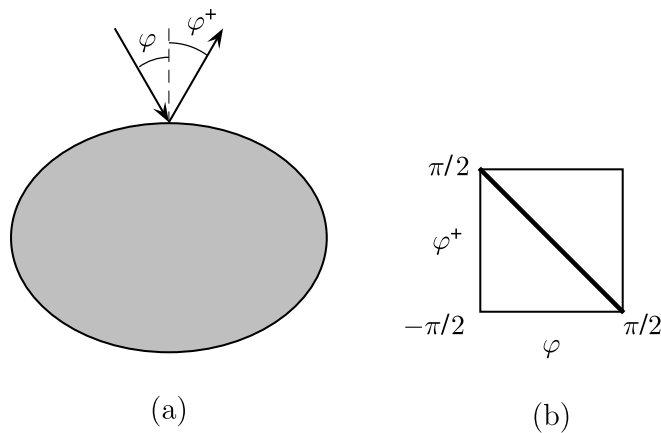

(a)

(b)

FIG. 2. *A smooth body* (a) *and the support of the corresponding measure* η_0 (b).

Example 2 (roughness formed by triangular hollows). Q_m is a $2m$-polygon; the $270°$ angles alternate with the angles that are slightly smaller than $90°$. All vertices corresponding to the angles smaller than $90°$ belong to ∂B. Any two sides that form a $270°$ angle are equal. The largest side length tends to zero as $m \to \infty$. Thus, the set Q_m is obtained by moving off m "hollows" from its convex hull, each of the hollows being an isosceles right triangle.
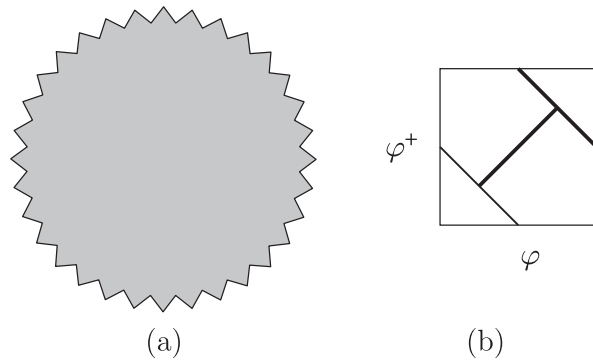

(a)   (b)

FIG. 3. *A rough body with hollows being isosceles right triangles* (a) *and the support of the corresponding measure* η_∇ (b).

The corresponding measure is $\check{\nu}_B = \eta_\nabla \otimes \tau_B$, where the measure η_∇ has the density $\cos\varphi \cdot [\chi_{[-\pi/2,-\pi/4]}(\varphi)\,\delta(\varphi+\varphi^+ + \frac{\pi}{2}) + \chi_{[-\pi/4,\pi/4]}(\varphi)\,\delta(\varphi-\varphi^+) + \chi_{[\pi/4,\pi/2]}(\varphi)\,\delta(\varphi+\varphi^+ - \frac{\pi}{2})] + |\sin\varphi| \cdot [\chi_{[-\pi/4,0]}(\varphi)\,\delta(\varphi+\varphi^+ + \frac{\pi}{2}) - \chi_{[-\pi/4,\pi/4]}(\varphi)\,\delta(\varphi-\varphi^+) + \chi_{[0,\pi/4]}(\varphi)\,\delta(\varphi+\varphi^+ - \frac{\pi}{2})]$. Thus, the support of η_∇ is the union of three segments; see Figure 3(b). The

middle segment $\varphi^+ = \varphi$ corresponds to double reflections, and the lateral segments, $\varphi^+ = -\varphi - \pi/2$ and $\varphi^+ = -\varphi + \pi/2$, correspond to single reflections, from the right or from the left side of a triangular hollow. In Figure 3(a), B is a circle.

Example 3 (roughness formed by rectangular hollows). The sets Q_m are obtained by removing a finite number of "rectangular hollows" from B. In other words, one has $Q_m = B \setminus (\cup_n \Omega_{m,n})$, where the removed sets $\Omega_{m,n}$ do not mutually intersect and each set $\partial\Omega_{m,n} \setminus \partial B$ is the union of three sides of a rectangle. The ratio (width)/(depth) of a hollow depends only on m and is denoted by h_m. Denote by $l_m = |\partial B \setminus \cup_n (\partial\Omega_{m,n})|/|\partial B|$ the relative length of the part of boundary ∂B not covered by hollows. We assume that $\lim_{m\to\infty} h_m = 0 = \lim_{m\to\infty} l_m$. In Figure 4(a), B is a square.
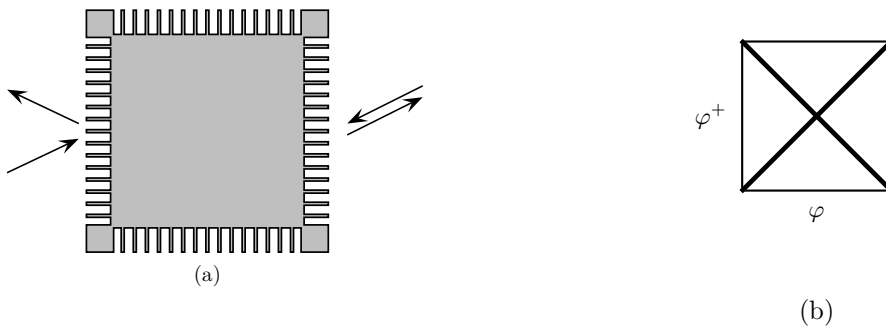

(a)

(b)

FIG. 4. *A rough body where hollows are "thin rectangles"* (a)*; the support of the corresponding measure* $\eta_\square$ (b)*.*

The measure $\check{\nu}_{\mathcal{B}}$ equals $\check{\nu}_{\mathcal{B}} = \eta_\square \otimes \tau_B$. The density of the measure $\eta_\square$ equals $\frac{1}{2}\cos\varphi \cdot (\delta(\varphi + \varphi^+) + \delta(\varphi - \varphi^+))$, and the support is the union of two diagonals, $\varphi^+ = \varphi$ and $\varphi^+ = -\varphi$; see Figure 4(b). The particles with an even (odd) number of reflections contribute to the first (second) diagonal.

1.4. Main theorem. According to Definition 1, each rough body is identified with a measure on $\mathbb{T}^3$. The question is, What is the set of these measures? The following definition and theorem give the answer.

Let us first introduce some notation: $\pi_{v,n} : \mathbb{T}^3 \to \mathbb{T}^2$, $\pi_n : \mathbb{T}^3 \to S^1$, etc., are projections onto the corresponding subspaces: $\pi_{v,n}(v, v^+, n) = (v, n)$, $\pi_n(v, v^+, n) = n$, etc.; $\pi_d : \mathbb{T}^3 \to \mathbb{T}^3$ is the symmetry with respect to the plane $v = v^+$, that is, $\pi_d(v, v^+, n) = (v^+, v, n)$; $z_+ = \max\{0, z\}$ is the positive part of $z \in \mathbb{R}$; and u means Lebesgue measure on S^1. Recall that τ_B is the surface measure on B and is defined on S^1.

DEFINITION 2. *We denote by* $\mathcal{M}_B$ *the set of measures* ν *on* $\mathbb{T}^3$ *such that the following properties hold:*

A1. *The marginal measures* $\pi_{v,n}^{\#}\nu$ *and* $\pi_{v^+,n}^{\#}\nu$ *are*

$$\pi_{v,n}^{\#}\nu = \langle v, n \rangle_+ \cdot u \otimes \tau_B, \qquad \pi_{v^+,n}^{\#}\nu = \langle v^+, n \rangle_+ \cdot u \otimes \tau_B.$$

A2. $\pi_d^{\#}\nu = \nu$.

Denote also $\mathcal{M} = \cup_B \mathcal{M}_B$, *the union being taken over all bounded convex bodies* B.

Taking into account the Alexandrov theorem on characterization of surface measures, one concludes that $\mathcal{M}$ is the set of measures ν on $\mathbb{T}^3$ such that

(1) the marginal measure $\pi_n^\# \nu =: \tau$ satisfies the conditions

    1a. $\int_{S^1} n \, d\tau(n) = 0$;

    1b. for any $v \in S^1$ it holds that $\int_{S^1} \langle n, v \rangle^2 \, d\tau(n) \neq 0$;

(2) the marginal measures $\pi_{v,n}^\# \nu$ and $\pi_{v^+,n}^\# \nu$ satisfy the conditions

    2a. $\pi_{v,n}^\# \nu = \langle v, n \rangle_+ \cdot u \otimes \tau$;

    2b. $\pi_{v^+,n}^\# \nu = \langle v^+, n \rangle_+ \cdot u \otimes \tau$.

Thus, these marginal measures coincide; the only difference is in the notation for the variables: v, n in case 2a and v^+, n in case 2b.

Now we can state the main theorem.

THEOREM. *The set of measures* $\{\nu_\mathcal{B}\}$, *with* $\mathcal{B}$ *being all possible bodies obtained by roughening* B, *coincides with* $\mathcal{M}_B$. *Therefore,* $\{\nu_\mathcal{B}, \mathcal{B}$ *is a rough body*$\} = \mathcal{M}$.

In section 2, we formulate two auxiliary lemmas and using them prove the theorem. In section 3, the lemmas are proved. Section 4 contains concluding remarks and applications of the theorem to problems of optimal aerodynamic resistance. Appendices A and B contain proofs of some auxiliary technical results.

2. Statement of auxiliary lemmas and proof of theorem.

2.1. Statement of Lemma 1. Fix a bounded convex body B. Two points $\xi_1, \xi_2 \in \partial B$, $\xi_1 \neq \xi_2$, divide the curve ∂B into two arcs. Denote by $l(\xi_1, \xi_2)$ the length of the smallest arc and denote

$$c = c_B := \inf_{\substack{\xi_1, \xi_2 \in \partial B \\ \xi_1 \neq \xi_2}} \frac{|\xi_1 - \xi_2|}{l(\xi_1, \xi_2)};$$

one obviously has $0 < c < 1$.

Let $Q \subset B$; denote

$$\overline{|\xi - \xi^+|}_{Q,B} := \iint_{(\partial B \times S^1)_+} |\xi - \xi_{Q,B}^+(\xi, v)| \, d\mu(\xi, v)$$

and

$$\overline{|n - n^+|}_{Q,B} := \iint_{(\partial B \times S^1)_+} |n(\xi) - n(\xi_{Q,B}^+(\xi, v))| \, d\mu(\xi, v).$$

LEMMA 1. (a) *The following holds true:*

$$\overline{|\xi - \xi^+|}_{Q,B} \leq 2\pi \cdot \mathrm{Area}(B \setminus Q).$$

(b) *For sufficiently small* $\mathrm{Area}(B \setminus Q)$,[1] *one has*

$$\overline{|n - n^+|}_{Q,B} \leq \frac{2\pi\sqrt{8\pi}}{\sqrt{c}} \sqrt{\mathrm{Area}(B \setminus Q)}.$$

2.2. Statement of Lemma 2. Let us first introduce the notion of a hollow.

DEFINITION 3. *Let* $\Omega \subset \mathbb{R}^2$ *be a closed bounded set with piecewise smooth boundary and* $I \subset \partial\Omega$, *where the following hold:*

(i) I *is an interval contained in a straight line* $\langle x, n \rangle = a$.

(ii) $\Omega \setminus I$ *is contained in the open half-plane* $\langle x, n \rangle < a$. *Here* n *is a fixed unit vector.*

Then the pair (Ω, I) *is called a hollow oriented by* n, *or just an* n-*hollow.*

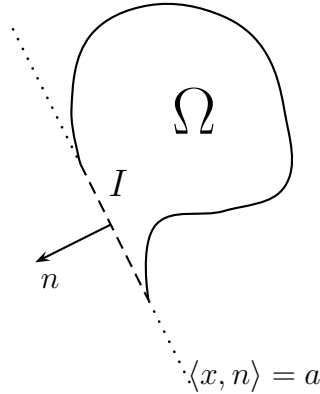[1] That is, it is smaller than a positive value depending only on B.

FIG. 5. *A hollow.*

In Figure 5 and in what follows, I is shown by a dashed line, and $\partial\Omega \setminus I$ is shown by a solid line.

Define the measure $\tilde{\mu}_I$ on $I \times S^1$ by $d\tilde{\mu}_I(\xi, v) = \frac{\langle n, v \rangle_+}{|I|} d\xi\, dv$, where $|I|$ means the length of I. Obviously, $\tilde{\mu}_I$ is supported on the set $(I \times S^1)_+ := \{(\xi, v) \in I \times S^1 : \langle n, v \rangle \geq 0\}$. Define the one-to-one mapping $(\xi, v) \mapsto (\Xi^+_{\Omega, I}(\xi, v), V^+_{\Omega, I}(\xi, v))$ of a full measure subset of $(I \times S^1)_+$ onto itself. Namely, consider the billiard in Ω. Let $(\xi, v) \in (I \times S^1)_+$; consider the billiard particle starting at the point ξ with the velocity $-v$. It makes several reflections from $\partial\Omega \setminus I$ and then reflects from I again, at a point $\Xi^+ = \Xi^+_{\Omega, I}(\xi, v)$. The velocity immediately before this reflection is denoted by $V^+ = V^+_{\Omega, I}(\xi, v)$. The mapping so defined preserves the measure $\tilde{\mu}_I$ and is an involution, that is, coincides with its inverse.

One can give an equivalent definition based on the mapping $\xi^+_{Q, B}(\xi, v), v^+_{Q, B}(\xi, v)$ just defined in section 1.2. Take a set Q such that Ω is a connected component of $\operatorname{conv} Q \setminus Q$ and I is a connected component of $\partial(\operatorname{conv} Q) \setminus \partial Q$. For $(\xi, v) \in (I \times S^1)_+$, let by definition $(\Xi^+_{\Omega, I}(\xi, v), V^+_{\Omega, I}(\xi, v)) := (\xi^+_{Q, \operatorname{conv} Q}(\xi, v), v^+_{Q, \operatorname{conv} Q}(\xi, v))$. This definition does not depend on the choice of Q.

DEFINITION 4. *Let (Ω, I) be a hollow. The measure $\eta_{\Omega, I}$ on $\mathbb{T}^2 = S^1 \times S^1$ is defined as follows. For a Borel set $A \subset \mathbb{T}^2$, put*

$$\eta_{\Omega, I}(A) := \tilde{\mu}_I(\{(\xi, v) \in (I \times S^1)_+ : (v, V^+_{\Omega, I}(\xi, v)) \in A\}).$$

We shall say that $\eta_{\Omega, I}$ is the measure generated by the hollow (Ω, I).

Here we use the notation $\pi_v, \pi_{v+} : \mathbb{T}^2 \to S^1$ for the projections onto the subspaces $\{v\}$ and $\{v^+\}$, respectively; $\pi_v(v, v^+) = v$, $\pi_{v+}(v, v^+) = v^+$. We also denote by π_d the symmetry with respect to the diagonal $v = v^+$; $\pi_d(v, v^+) = (v^+, v)$.

DEFINITION 5. *Denote by Λ_n the set of measures η on $\mathbb{T}^2$ such that*
(i) $d\pi^{\#}_v \eta(v) = \langle v, n \rangle_+\, dv$, $d\pi^{\#}_{v+} \eta(v^+) = \langle v^+, n \rangle_+\, dv^+$;
(ii) $\pi^{\#}_d \eta = \eta$.

Any measure $\eta_{\Omega, I}$ generated by an n-hollow belongs to Λ_n. Indeed, for any $A \subset S^1$ one has $\pi^{\#}_v \eta_{\Omega, I}(A) = \eta_{\Omega, I}(A \times S^1) = \tilde{\mu}_I(\{(\xi, v) \in (I \times S^1)_+ : v \in A\}) = \frac{1}{|I|} \iint_{I \times A} \langle n, v \rangle_+\, d\xi\, dv = \int_A \langle n, v \rangle_+\, dv$. This proves the first equality in (i).

Similarly, one has $\pi^{\#}_{v+} \eta_{\Omega, I}(A) = \eta_{\Omega, I}(S^1 \times A) = \tilde{\mu}_I(\{(\xi, v) \in (I \times S^1)_+ : V^+_{\Omega, I}(\xi, v) \in A\})$. Since the mapping $(\xi, v) \mapsto (\Xi^+_{\Omega, I}, V^+_{\Omega, I})$ preserves the measure, one gets the

value $\tilde{\mu}_I(\{(\xi, v) \in (I \times S^1)_+ : v \in A\})$, which in turns equals $\int_A \langle n, v \rangle_+ \, dv$. This proves the second equality in (i). Finally, the relation (ii) for $\eta_{\Omega, I}$ is a simple consequence of involutive and measure preserving properties of the mapping $(\xi, v) \mapsto (\Xi^+_{\Omega, I}, V^+_{\Omega, I})$.

LEMMA 2. *The set of measures generated by n-hollows is weakly dense in* Λ_n.

2.3. Proof of the direct statement of theorem. Here we prove that for any body $\mathcal{B}$ obtained by roughening B it holds that $\nu_{\mathcal{B}} \in \mathcal{M}_B$.

Let $Q \subset B$; define the measure $\nu'_{Q,B}$ on $\mathbb{T}^3$ by

$$\nu'_{Q,B}(A) := \mu\left(\{(\xi, v) \in (\partial B \times S^1)_+ : (v, v^+_{Q,B}(\xi, v), n(\xi^+_{Q,B}(\xi, v))) \in A\}\right),$$

where A is an arbitrary Borel subset of $\mathbb{T}^3$. Thus, the definition of both $\nu_{Q,B}$ and $\nu'_{Q,B}$ is based on observations of vector triples (v, v^+, n) and (v, v^+, n^+), respectively. Here n and n^+ are the outer normals to ∂B at the points where the particle *gets in* B and *gets out* of B. The measures $\nu_{Q,B}$ and $\nu'_{Q,B}$ have the following properties:

$$(1) \qquad\qquad \pi^{\#}_{v,n} \nu_{Q,B} = \langle v, n \rangle_+ \cdot u \otimes \tau_B,$$

$$(2) \qquad\qquad \pi^{\#}_{v^+,n^+} \nu'_{Q,B} = \langle v^+, n^+ \rangle_+ \cdot u \otimes \tau_B,$$

$$(3) \qquad\qquad \pi^{\#}_d \nu_{Q,B} = \nu'_{Q,B}.$$

Consider a sequence $\{Q_m\}$ representing $\mathcal{B}$; let us show that $\nu_{Q_m,B} - \nu'_{Q_m,B}$ weakly converges to zero as $m \to \infty$. It is enough to prove that for any continuous function f on $\mathbb{T}^3$ it holds that

$$(4) \quad \int_{\mathbb{T}^3} f(v, v^+, n) \, d\nu_{Q_m,B}(v, v^+, n) - \int_{\mathbb{T}^3} f(v, v^+, n^+) \, d\nu'_{Q_m,B}(v, v^+, n^+) \to_{m\to\infty} 0.$$

Taking into account the formulas for a change of variables

$$\int_{\mathbb{T}^3} f(v, v^+, n) \, d\nu_{Q,B}(v, v^+, n) = \int_{(\partial B \times S^1)_+} f(v, v^+_{Q,B}(\xi, v), n(\xi)) \, d\mu(\xi, v)$$

and

$$\int_{\mathbb{T}^3} f(v, v^+, n^+) \, d\nu'_{Q,B}(v, v^+, n^+) = \int_{(\partial B \times S^1)_+} f(v, v^+_{Q,B}(\xi, v), n(\xi^+_{Q,B}(\xi, v))) \, d\mu(\xi, v),$$

formula (4) takes the form

$$(5) \quad \lim_{m\to\infty} \int_{(\partial B \times S^1)_+}$$
$$\left[f(v, v^+_{Q_m,B}(\xi, v), n(\xi^+_{Q_m,B}(\xi, v))) - f(v, v^+_{Q_m,B}(\xi, v), n(\xi)) \right] d\mu(\xi, v) = 0.$$

According to Lemma 1, the difference $n(\xi^+_{Q_m,B}(\xi, v)) - n(\xi)$ converges to zero in mean, and hence it converges to zero in measure; therefore the difference

$$f(v, v^+_{Q_m,B}(\xi, v), n(\xi^+_{Q_m,B}(\xi, v))) - f(v, v^+_{Q_m,B}(\xi, v), n(\xi))$$

also converges to zero in measure. It follows that formula (5) is true.

Thus, both $\nu_{Q_m, B}$ and $\nu'_{Q_m, B}$ weakly converge to $\nu_{\mathcal{B}}$. Substituting $Q = Q_m$ into formulas (1)–(3) and passing to limit as $m \to \infty$, one gets

$$\pi^{\#}_{v,n} \nu_{\mathcal{B}} = \langle v, n \rangle_+ \cdot u \otimes \tau_B,$$

$$\pi^{\#}_{v^+, n} \nu_{\mathcal{B}} = \langle v^+, n \rangle_+ \cdot u \otimes \tau_B,$$

$$\pi^{\#}_d \nu_{\mathcal{B}} = \nu_{\mathcal{B}},$$

that is, $\nu_{\mathcal{B}} \in \mathcal{M}_B$.

2.4. Proof of the inverse statement of theorem. Here it is proved that for any $\nu \in \mathcal{M}_B$ there exists a body $\mathcal{B}$ obtained by roughening B such that $\nu_{\mathcal{B}} = \nu$. The proof is based on two statements.

STATEMENT 1. *Let B be a convex polygon. Then for any measure $\nu \in \mathcal{M}_B$ there exists a body $\mathcal{B}$ obtained by roughening B such that $\nu_{\mathcal{B}} = \nu$.*

Proof. Let us enumerate the sides of the polygon B and denote by c_i the length of the ith side and by n_i the outer unit normal to this side. By δ_n, denote the probabilistic atomic measure on S^1 concentrated at $n \in S^1$, that is, $\delta_n(n) = 1$. The surface measure of B is $\tau_B = \sum c_i \delta_{n_i}$; this implies that any measure $\nu \in \mathcal{M}_B$ has the form $\nu = \sum c_i \eta_i \otimes \delta_{n_i}$, where $\eta_i \in \Lambda_{n_i}$.

According to Lemma 2, any measure η_i is the weak limit as $m \to \infty$ of measures $\eta_{\Omega_i^m, I_i^m}$ generated by a sequence of n_i-hollows (Ω_i^m, I_i^m). Now take a sequence of sets Q_m such that $\mathrm{conv}\, Q_m = B$ and each connected component of $B \setminus Q_m$ is the image of a set Ω_i^m under the composition of a homothety with positive ratio and a translation; additionally, the image of I_i^m under this transformation belongs to $(i$th side of $B) \setminus \partial Q_m$. We also require that $\mathrm{Area}(B \setminus Q_m) \to 0$ and $|(i$th side of $B) \setminus \partial Q_m| =: c_i^m \to c_i$ as $m \to \infty$. In Appendix A it is shown how to construct such a sequence Q_m. The measure $\nu_{Q_m, B} = \tilde{\nu}_m + \sum_i \nu_i^m$ is the sum of the measure $\tilde{\nu}_m$ corresponding to reflections from $\partial B \cap \partial Q_m$ and the measures ν_i^m corresponding to particles getting into the "hollows on the ith side." One has $\tilde{\nu}_m = \sum_i (c_i - c_i^m) \cdot \eta_0 \otimes \delta_{n_i}$ and $\nu_i^m = c_i^m \cdot \eta_{\Omega_i^m, I_i^m} \otimes \delta_{n_i}$. The norm of $\tilde{\nu}_m$ goes to zero and ν_i^m weakly converges to $c_i \eta_i \otimes \delta_{n_i}$ for any i; it follows that $\nu_{Q_m, B}$ weakly converges to ν as $m \to \infty$. Therefore, the sequence Q_m represents a body $\mathcal{B}$ obtained by roughening B, and $\nu_{\mathcal{B}} = \nu$. ◻

STATEMENT 2. *For any measure $\nu \in \mathcal{M}_B$ there exist a sequence of convex polygons $B_k \subset B$ with $\mathrm{Area}(B \setminus B_k) \to 0$ and a sequence of measures $\nu_k \in \mathcal{M}_{B_k}$ weakly converging to ν as $k \to \infty$.*

Proof. Consider a partition of the circumference S^1 into a finite number of arcs, $S^1 = \cup_i \mathcal{S}^i$. It induces the partition of ∂B into arcs $\partial B^i = \{\xi \in \partial B : n(\xi) \in \mathcal{S}^i\}$. Consider the polygon $\check{B}$ inscribed into ∂B whose vertices are separation points of this partition. Denote by n_i the outer normal to the ith side of this polygon. Denote by s_{v_1, v_2} the operator of rotation on S^1 that takes v_1 to v_2, and define the mapping $\Upsilon_i : \mathbb{T}^2 \times \mathcal{S}^i \to \mathbb{T}^2$ by $\Upsilon_i(v, v^+, n) = (s_{n, n_i} v, s_{n, n_i} v^+)$. Finally, consider the measure $\check{\nu} = \sum_i |b^i| \eta_i \otimes \delta_{n_i}$, where $|b^i|$ is the length of the ith side of the polygon, and the measure η_i on $\mathbb{T}^2$ is defined by $\eta_i(A) = \frac{1}{|\partial B^i|} \nu(\Upsilon_i^{-1}(A))$ for arbitrary Borel set $A \subset \mathbb{T}^2$. Here $|\partial B^i|$ is the length of the arc ∂B^i. One easily verifies that $\check{\nu}$ belongs to $\mathcal{M}_{\check{B}}$.

Now take a sequence of partitions of S^1, $\{\mathcal{S}_k^i\}_i$, $k = 1, 2, \ldots$, where the maximum arc length of a partition goes to zero as $k \to \infty$. Denote by $\{\partial B_k^i\}_i$, $k = 1, 2, \ldots$, the sequence of induced partitions of ∂B, and take the sequence of polygons B_k generated by these partitions. One clearly has $\mathrm{Area}(B \setminus B_k) \to 0$ and

$$(6) \qquad \max_i \frac{|b_k^i|}{|\partial B_k^i|} \to 1 \qquad \text{as} \quad k \to \infty,$$

where $|b_k^i|$ is the length of the ith side of B_k. In the same way as above, one defines the mappings $\Upsilon_{ik} : \mathbb{T}^2 \times \mathcal{S}_k^i \to \mathbb{T}^2$ and the measures $\nu_k = \sum_i |b_k^i| \, \eta_{ik} \otimes \delta_{n_{ik}} \in \mathcal{M}_{B_k}$, where η_{ik} is given by $\eta_{ik}(A) := \frac{1}{|\partial B_k^i|} \nu(\Upsilon_{ik}^{-1}(A))$ and n_{ik} is the outer unit normal to the ith side of B_k.

It remains to show that ν_k weakly converges to ν. For any continuous function f on $\mathbb{T}^3$ one has

$$\iiint_{\mathbb{T}^3} f(v, v^+, n) \, d\nu_k(v, v^+, n) = \sum_i |b_k^i| \iint_{\mathbb{T}^2} f(v, v^+, n_{ik}) \, d\eta_{ik}(v, v^+)$$

$$(7) \qquad = \sum_i \frac{|b_k^i|}{|\partial B_k^i|} \iiint_{\mathbb{T}^2 \times \mathcal{S}_k^i} f(\Upsilon_{ik}(v, v^+, n), n_{ik}) \, d\nu(v, v^+, n).$$

For each k define the mapping from $\mathbb{T}^3$ to $\mathbb{T}^3$ by the relations $(v, v^+, n) \mapsto (\Upsilon_{ik}(v, v^+, n), n_{ik})$ if $n \in \mathcal{S}_k^i$. It uniformly converges to the identity mapping as $k \to \infty$; hence the function $\tilde{f}_k$, defined by the relations $\tilde{f}_k(v, v^+, n) := f(\Upsilon_{ik}(v, v^+, n), n_{ik})$ if $n \in \mathcal{S}_k^i$, uniformly converges to f as $k \to \infty$. From here and from (6) it follows that the right-hand side in (7) converges to $\iiint_{\mathbb{T}^3} f(v, v^+, n) \, d\nu(v, v^+, n)$ as $k \to \infty$. Thus, the convergence $\int f \, d\nu_k \to \int f \, d\nu$ is proved. $\square$

The inverse statement of the theorem follows from Statements 1 and 2. Indeed, let $\nu \in \mathcal{M}_B$. Using Statement 2, find a sequence of convex polygons $B_k \subset B$ and a sequence $\nu_k \in \mathcal{M}_{B_k}$ weakly converging to ν. According to Statement 1, each measure ν_k is generated by a rough body. Consider the sequence of sets $Q_{kl} \subset B_k$, $l = 1, 2, \ldots$, representing this body, and then from all of these sequences choose a diagonal sequence $\tilde{Q}_k = Q_{kl_k}$ such that the corresponding sequence of measures $\nu_{\tilde{Q}_k, B}$ weakly converges to ν and $\mathrm{Area}(B \setminus \tilde{Q}_k)$ goes to zero as $k \to \infty$. The sequence $\tilde{Q}_k$ represents a body $\mathcal{B}$ obtained by roughening B and $\nu_{\mathcal{B}} = \nu$.

3. Proof of the lemmas.

3.1. Proof of Lemma 1. Consider the billiard in $\mathbb{R}^2 \setminus Q$. For $(\xi, v) \in (\partial B \times S^1)_+$, denote by $\tau(\xi, v)$ the time the billiard trajectory with the initial data $\xi, -v$ spends in $B \setminus Q$. In particular, if $\xi \in \partial B \cap \partial Q$, then one has $\tau(\xi, v) = 0$.

Denote by D the set of points $(x, w) \in (B \setminus Q) \times S^1$ that are accessible from $(\partial B \times S^1)_+$; that is, there exists $(\xi, v) \in (\partial B \times S^1)_+$ such that the billiard particle with the data $\xi, -v$ at the zero moment of time at some moment $0 \le t \le \tau(\xi, v)$ will pass through x with the velocity w. This description defines the change of coordinates in $D : (\xi, v, t) \mapsto (x, w)$; $(\xi, v) \in (\partial B \times S^1)_+$, $t \in [0, \tau(\xi, v)]$, and the element of phase volume $d^2x \, dw$ in the new coordinates takes the form $d\mu(\xi, v) \, dt$. Hence, the phase volume of D equals $\iiint_D d^2x \, dw = \iint_{(\partial B \times S^1)_+} \tau(\xi, v) \, d\mu(\xi, v)$. Taking into account that $D \subset (B \setminus Q) \times S^1$ and the phase volume of $(B \setminus Q) \times S^1$ equals $2\pi \cdot \mathrm{Area}(B \setminus Q)$, one gets

$$(8) \qquad \iint_{(\partial B \times S^1)_+} \tau(\xi, v) \, d\mu(\xi, v) \le 2\pi \cdot \mathrm{Area}(B \setminus Q).$$

This is in fact a simple modification of the well-known *mean free path* formula (see, e.g., [19]).

One has $\tau(\xi, v) \ge |\xi - \xi_{Q,B}^+(\xi, v)|$: the time the particle spends in $B \setminus Q$ exceeds the distance between the initial and final points of the trajectory. This inequality and (8) imply (a).

The points ξ and $\xi_{Q,B}^+(\xi, v)$ divide the curve ∂B into two arcs; denote by $\gamma(\xi, v)$ the shortest one. One has $|\gamma(\xi, v)| = l(\xi, \xi_{Q,B}^+(\xi, v))$, and therefore $|\xi - \xi_{Q,B}^+(\xi, v)| \geq c\,|\gamma(\xi, v)|$. It follows that

(9)
$$c \iint_{(\partial B \times S^1)_+} |\gamma(\xi, v)|\, d\mu(\xi, v) \leq \iint_{(\partial B \times S^1)_+} |\xi - \xi_{Q,B}^+(\xi, v)|\, d\mu(\xi, v) \leq 2\pi \cdot \mathrm{Area}(B \backslash Q).$$

Let $\varrho(y)$ be a natural parametrization of the curve ∂B, $\varrho : [0, |\partial B|] \to \partial B$. By $f(y)$ denote the measure of the values (ξ, v) such that the interval $\gamma(\xi, v)$ contains the point $\varrho(y)$; that is, $f(y) := \iint_{(\partial B \times S^1)_+} \mathbb{I}(\varrho(y) \in \gamma(\xi, v))\, d\mu(\xi, v)$. Making a change of variables in the integral in the left-hand side of (9), one gets

$$\iint_{(\partial B \times S^1)_+} |\gamma(\xi, v)|\, d\mu(\xi, v) = \int_0^{|\partial B|} f(y)\, dy,$$

and therefore

(10)
$$\int_0^{|\partial B|} f(y)\, dy \leq \frac{2\pi}{c} \mathrm{Area}(B \setminus Q).$$

One easily sees that $|f(y_1) - f(y_2)| \leq 4\,|y_1 - y_2|$ for any y_1 and y_2 and $f(y) \geq 0$. From here and from (10) it follows that for sufficiently small $\mathrm{Area}(B \setminus Q)$ (namely, for $\mathrm{Area}(B \setminus Q) \leq c|\partial B|^2/(2\pi)$) it holds that $f(y) \leq \sqrt{8\pi/c}\,\sqrt{\mathrm{Area}(B \setminus Q)}$.

Recall that $\mathrm{Arg}(v)$ is the angle the vector $v \neq 0$ forms with a fixed vector v_0; the angle is measured clockwise from v_0 to v and is defined modulo 2π. Introduce the shorthand notation $\xi^+ := \xi_{Q,B}^+(\xi, v)$, and denote by $\Delta\,\mathrm{Arg}(\xi, v)$ the smallest in modulus of the values $\mathrm{Arg}(n(\xi^+)) - \mathrm{Arg}(n(\xi))$. In other words, $\Delta\,\mathrm{Arg}(\xi, v)$ equals the smallest of the values

$$\int_{\gamma(\xi, v)} |d\,\mathrm{Arg}(n_{\xi'})|, \qquad \int_{\partial B \backslash \gamma(\xi, v)} |d\,\mathrm{Arg}(n_{\xi'})|.$$

Taking into account that $|n(\xi^+) - n(\xi)| \leq |\Delta\,\mathrm{Arg}(\xi, v)|$, one gets that

$$|n(\xi^+) - n(\xi)| \leq \int_{\gamma(\xi, v)} |d\,\mathrm{Arg}(n_{\xi'})|,$$

and therefore

$$\overline{|n - n^+|}_{Q,B} \leq \iint_{(\partial B \times S^1)_+} \left(\int_{\gamma(\xi, v)} |d\,\mathrm{Arg}(n_{\xi'})| \right) d\mu(\xi, v).$$

Making a change of variables in this integral, one obtains

$$\overline{|n - n^+|}_{Q,B} \leq \int_0^{|\partial B|} f(y)\, |d\,\mathrm{Arg}(n_{\varrho(y)})| \leq 2\pi\, \sqrt{8\pi/c}\, \sqrt{\mathrm{Area}(B \setminus Q)}.$$

Thus, (b) is also proved.

3.2. Proof of Lemma 2. Fix $n \in S^1$ and $m \in \mathbb{N}$. Let σ be an involutive permutation of $\{1, \ldots, m\}$, that is, $\sigma^2 = \text{id}$. Divide the half-circumference $S_n^1 := \{v \in S^1 : \langle v, n \rangle \geq 0\}$ into m arcs $\mathcal{S}_{n,m}^1 = \mathcal{S}_n^1, \ldots, \mathcal{S}_{n,m}^m = \mathcal{S}_n^m$ numbered clockwise such that, for any i, $\int_{\mathcal{S}_n^i} \langle v, n \rangle \, dv = 2/m$. For the sake of brevity we omit the subscript m when no confusion can arise.

DEFINITION 6. *A measure η is called a (σ, n)-measure if $\eta \in \Lambda_n$ and $\text{spt}\,\eta \subset \cup_{i=1}^m \left(\mathcal{S}_n^i \times \mathcal{S}_n^{\sigma(i)} \right)$, and therefore, for any i it holds that $\eta \left(\mathcal{S}_n^i \times \mathcal{S}_n^{\sigma(i)} \right) = 2/m$.*

PROPOSITION 1. *For any measure $\eta \in \Lambda_n$ there exists a sequence of involutive permutations σ_k on $\{1, \ldots, m_k\}$, $k = 1, 2, \ldots$, such that m_k tends to infinity and any sequence of (σ_k, n)-measures weakly converges to η as $k \to \infty$.*

PROPOSITION 2. *Let σ be an involutive permutation on $\{1, \ldots, m\}$. Then the distance (in variation) between the set of measures generated by n-hollows and the set of (σ, n)-measures does not exceed $16/m$. In other words, whatever $\varepsilon > 0$, there exist a (σ, n)-measure η and an n-hollow (Ω, I) such that $\|\eta_{\Omega, I} - \eta\| < 16/m + \varepsilon$; here the norm means variation of measure.*

This distance actually equals zero, but we need only the (weaker) claim of Proposition 2.

Lemma 2 follows from Propositions 1 and 2. Indeed, let $\eta \in \Lambda_n$. First, choose the sequence of permutations σ_k, according to Proposition 1, and then, using Proposition 2, for every k choose an an n-hollow (Ω_k, I_k) such that the distance from η_{Ω_k, I_k} to the set of (σ_k, n)-measures does not exceed $17/m_k$. The sequence of chosen measures η_{Ω_k, I_k} weakly converges to η.

3.3. Proof of Proposition 1. Introduce on S_n^1 the angular coordinate $\varphi = \text{Arg}\,v - \text{Arg}\,n$; that is, φ changes between $-\pi/2$ and $\pi/2$ and increases clockwise. With this notation, to the arcs $\mathcal{S}_{n,m}^i$ correspond the segments $J_m^i = [\arcsin(-1 + 2(i-1)/m), \arcsin(-1 + 2i/m)]$. Define the measure λ on $[-\pi/2, \pi/2]$ by $d\lambda(\varphi) = \cos\varphi \, d\varphi$, and denote by Λ the set of measures η on $\square := [-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ such that (a) $\pi_\varphi^\# \eta = \lambda = \pi_{\varphi^+}^\# \eta$ and (b) $\pi_d^\# \eta = \eta$. Here π_φ, π_{φ^+}, and π_d are defined by $\pi_\varphi(\varphi, \varphi^+) = \varphi$, $\pi_{\varphi^+}(\varphi, \varphi^+) = \varphi^+$, and $\pi_d(\varphi, \varphi^+) = (\varphi^+, \varphi)$. Reformulating Definition 6, we shall say that η is a σ-measure if $\eta \in \Lambda$ and $\text{spt}\,\eta \subset \cup_{i=1}^m \left(J_m^i \times J_m^{\sigma(i)} \right)$. Notice that in the new notation the objects no longer depend on n: we write Λ instead of Λ_n, σ-measure instead of (σ, n)-measure, and hollow instead of n-hollow.

In this notation, Proposition 1 can be reformulated as follows: for any measure $\eta \in \Lambda$ there exists a sequence of involutive permutations σ_k on $\{1, \ldots, m_k\}$, $k = 1, 2, \ldots$, such that m_k tends to infinity and any sequence of σ_k-measures weakly converges to η as $k \to \infty$.

The idea of the proof is as follows. First, η is approximated by means of a rational matrix, and then this matrix is approximated by means of a larger matrix generated by a permutation.

Consider the partition of $\square$ into smaller rectangles $\square_k^{ij} = J_k^i \times J_k^j$, $i, j = 1, \ldots, k$. Choose rational nonnegative numbers c_k^{ij} such that $c_k^{ij} = c_k^{ji}$, $\sum_j c_k^{ij} = 2/k$ for any i, and $\left| \eta(\square_k^{ij}) - c_k^{ij} \right| \leq k^{-3}$ for any i and j. To do so, it suffices to take positive rational values c_k^{ij} such that $\eta(\square_k^{ij}) - k^{-4} \leq c_k^{ij} \leq \eta(\square_k^{ij})$ for $i > j$ and put $c_k^{ij} = c_k^{ji}$ for $i < j$ and $c_k^{ii} = 2/k - \sum_{j \neq i} c_k^{ij}$ for $i = j$. One has $\eta \left(J_k^i \times [-\pi/2, \pi/2] \right) = \sum_{j=1}^k \eta(\square_k^{ij}) = 2/k$; hence $c_k^{ii} - \eta(\square_k^{ii}) = \sum_{j \neq i} \left(\eta(\square_k^{ij}) - c_k^{ij} \right) \in [0, (k-1) \cdot k^{-4}] \subset [0, k^{-3}]$.

Any sequence of measures η_k satisfying the conditions $\eta_k(\square_k^{ij}) = c_k^{ij}$, $1 \leq i, j \leq k$, weakly converges to η. Indeed, for any continuous function f on $\square$ it holds that

$$\int_\square f \, d\eta_k - \int_\square f \, d\eta = \sum_{i,j=1}^{k} \int_{\square_k^{ij}} f \, (d\eta_k - d\eta) \le k^{-1} \max f \to 0$$

as $k \to \infty$.

To complete the proof, it suffices to find an integer $m_k > k$ and an involutive permutation σ_k of $\{1, \ldots, m_k\}$ such that any σ_k-measure, η_k, satisfies the equalities $\eta_k(\square_k^{ij}) = c_k^{ij}$, $i, j = 1, \ldots, k$. Choose a positive integer N such that all of the values $a_{ij} := N \cdot c_k^{ij}$ are integer. The obtained matrix $A = (a_{ij})_{i,j=1}^{k}$ is symmetric, and for any i the value $\sum_{j=1}^{k} a_{ij} = 2N/k$ is a fixed positive integer. In Appendix B it is shown that there exist square matrices $B_{ij} = (b_{ij}^{\mu\nu})_{\mu,\nu}$ of size $2N/k$ such that $B_{ij}^T = B_{ji}$, the sum of elements in any matrix B_{ij} equals a_{ij}, and the block matrix $D = (B_{ij})$ composed of these matrices has exactly one unit in each row and each column, and other elements are zeros.

D is a symmetric square matrix of size $2N$; denote its elements by d_{ij}. Put $m_k = 2N$, and define the mapping σ_k on $\{1, \ldots, 2N\}$ in such a way that $d_{i\sigma_k(i)} = 1$ for any i. The so defined mapping σ_k is a permutation; it is involutive since the matrix D is symmetric. Moreover, if η_k is a σ_k-measure, then for any i and j it holds that $\eta_k(\square_k^{ij}) = N^{-1} \sum_{\mu,\nu} b_{ij}^{\mu\nu} = c_k^{ij}$. The proposition is proved.

3.4. Proof of Proposition 2.

1. Whatever the n-hollow (Ω, I), one introduces the reference system (x_1, x_2) in such a way that n coincides with $(0, -1)$, and the interval I belongs to the straight line $x_2 = 0$ and contains the origin $O = (0, 0)$. Like in the proof of Proposition 1, introduce the coordinate $\varphi = \operatorname{Arg} v - \operatorname{Arg} n$ on S_n^1. One has $v = -(\sin \varphi, \cos \varphi)$, $\varphi \in [-\pi/2, \pi/2]$. The definitions of the segments $J_m^i = J^i$, the measure λ, the set of measures Λ, and the σ-measure are seen in the beginning of the previous subsection. The mapping $(\xi, v) \mapsto V_{\Omega,I}^+(\xi, v)$ in the new coordinates ξ, φ is written as $(\xi, \varphi) \mapsto \varphi_{\Omega,I}^+(\xi, \varphi)$. Finally, define the measure μ_I on $I \times [-\pi/2, \pi/2]$ by $d\mu_I(\xi, \varphi) = \frac{\cos \varphi}{|I|} d\xi \, d\varphi$.

Denote $\square' = \left(\cup_{i=2}^{m-1} J^i\right) \times \left(\cup_{i=2}^{m-1} J^i\right)$, $\square_1 = J^1 \times [-\pi/2, \pi/2]$, $\square_2 = J^m \times [-\pi/2, \pi/2]$, $\square_3 = \left(\cup_{i=2}^{m-1} J^i\right) \times J^1$, and $\square_4 = \left(\cup_{i=2}^{m-1} J^i\right) \times J^m$. Thus, one has $\square \setminus \square' = \square_1 \cup \square_2 \cup \square_3 \cup \square_4$; see Figure 6.
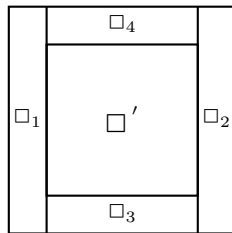

FIG. 6. *Partition of the square into rectangles.*

It suffices to construct a sequence of hollows $(\Omega_\varepsilon, I_\varepsilon)$, $\varepsilon > 0$, such that

(P) *for any $i \ne 1, m, \sigma(1), \sigma(m)$ the measure of the set of values $(\xi, \varphi) \in I_\varepsilon \times J^i$ such that $\varphi_{\Omega_\varepsilon, I_\varepsilon}^+(\xi, \varphi) \notin J^{\sigma(i)}$ goes to zero as $\varepsilon \to 0$.*

Then, speaking of restrictions of measures on the subset $\square'$, one gets that the distance from the restrictions of measures $\eta_{\Omega_\varepsilon, I_\varepsilon}$ to the set of restrictions of σ-measures goes to zero as $\varepsilon \to 0$. On the other hand, for any measure $\eta \in \Lambda$ one has $\eta(\square_1) =$

$\eta(\square_2) = 2/m$, $\eta(\square_3) \leq 2/m$, $\eta(\square_4) \leq 2/m$, and hence $\eta(\square \setminus \square') \leq 8/m$; therefore the distance between the restrictions on $\square \setminus \square'$ of any two measures η_1 and η_2 from Λ does not exceed $16/m$: $\|\eta_1\rfloor_{\square\setminus\square'} - \eta_2\rfloor_{\square\setminus\square'}\| \leq 16/m$. It follows that the upper limit of distances from $\eta_{\Omega_\varepsilon, I_\varepsilon}$ to the set of σ-measures does not exceed $16/m$, and so Proposition 2 is proved.

2. The rest of this subsection is dedicated to the detailed description of the sequence of hollows $(\Omega_\varepsilon, I_\varepsilon)$ and to the proof of property (P) for them.

First, consider an auxiliary construction (see Figure 7). Take two different points F and F' above the line $l = \{x_2 = 0\}$, with $|OF| = 2 = |OF'|$. Denote by Φ and Φ' the angles the rays OF and OF', respectively, formed with the vector $(0,1)$. The angles are counted clockwise from $(0,1)$. Thus, $F = 2(\sin\Phi, \cos\Phi)$ and F' $= 2(\sin\Phi', \cos\Phi')$. Assume, for further convenience, that F is situated on the left of F'; thus, one has $-\pi/2 < \Phi < \Phi' < \pi/2$. (The case where F is situated on the right of F' is completely similar.) Select three positive numbers λ, λ', and δ, and define two ellipses $\mathcal{E}$ and $\mathcal{E}$' and two parabolas $\mathcal{P}$ and $\mathcal{P}$'. The first ellipse has the foci O and F, the length of its large semiaxis is $\sqrt{1+\lambda}$, of the small semiaxis, $\sqrt{\lambda}$, and the focal distance equals 2. The second ellipse has the foci O and F', the lengths of its large and small semiaxes are $\sqrt{1+\lambda'}$ and $\sqrt{\lambda'}$, respectively, and the focal distance is also 2. The parabolas $\mathcal{P}$ and $\mathcal{P}$' have the foci F and F', respectively, the common axis FF', and the same focal distance δ. Thus, the parabolas are symmetric to each other with respect to the bisectrix of the triangle OFF'. The parameter δ is chosen sufficiently small, so that the point O lies in the exterior of both parabolas.
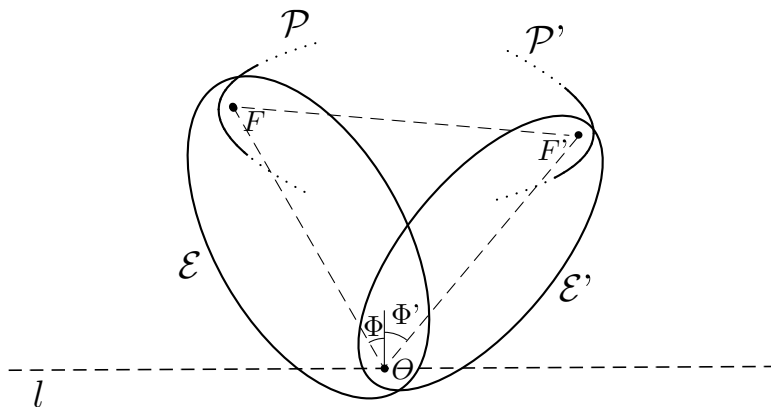

FIG. 7. *Auxiliary construction.*

In what follows, we shall distinguish between the billiard and *pseudobilliard* dynamics. The pseudobilliard dynamics is defined as follows. A particle starts at a point $(\xi, 0) \in l$ and moves with a velocity $(\sin\varphi, \cos\varphi)$ until it reflects from the interior side of $\mathcal{E}$. (Before the reflection it can intersect other curves $\mathcal{E}', \mathcal{P}, \mathcal{P}'$, or even intersect $\mathcal{E}$ from the outer side, without changing the velocity.) Then it moves again with constant velocity until it reflects from the interior side of $\mathcal{P}$. Then, in the same way, it reflects from the interior side of $\mathcal{P}$', and then from the interior side of $\mathcal{E}$', and, finally, it intersects l from above to below. Denote by $(\xi', 0)$ the point of intersection and by $-(\sin\varphi', \cos\varphi')$ the velocity at this point.

Consider the admissible set: the set of 7-tuples $(\varphi, \xi, \Phi, \Phi', \lambda, \lambda', \delta)$ such that all of the indicated reflections occur in the prescribed order. This set is open and nonempty.

Indeed, let $\delta(\Phi, \Phi')$ be the least of the values δ such that one of the parabolas (in fact, both of them simultaneously) passes through O. Put $\varphi = \Phi$, $\xi = 0$, and take arbitrary values $\lambda > 0$, $\lambda' > 0$, $-\pi/2 < \Phi < \Phi' < \pi/2$, $0 < \delta < \delta(\Phi, \Phi')$. The particle with initial data $\varphi = \Phi$, $\xi = 0$ first passes along the large semiaxis of $\mathcal{E}$, reflects from $\mathcal{E}$, returns along the same semiaxis, and reflects from $\mathcal{P}$. Then it moves with the velocity parallel to FF', reflects from $\mathcal{P}'$, moves the large semiaxis of the ellipse $\mathcal{E}'$, reflects from it, and returns to O along the same semiaxis. Thus, the admissible set is nonempty. Under a small perturbation of the parameters $\varphi, \xi, \Phi, \Phi', \lambda, \lambda', \delta$, all of the reflections are maintained and the order of reflections remains the same. This implies that the admissible set is open.

This description determines the mapping $\varphi' = \varphi'(\varphi, \xi, \Phi, \Phi', \lambda, \lambda', \delta)$, $\xi' = \xi'(\varphi, \xi,$ $\Phi, \Phi', \lambda, \lambda', \delta)^2$ from the admissible set to $\mathbb{R}^2$. This mapping is infinitely differentiable. For $\varphi = \Phi$ and $\xi = 0$ one has

$$(11) \qquad \varphi'(\Phi, 0, \Phi, \Phi', \lambda, \lambda', \delta) = \Phi'.$$

For $\xi = 0$ with arbitrary φ one has

$$(12) \qquad \xi'(\varphi, 0, \Phi, \Phi', \lambda, \lambda', \delta) = 0,$$

and

$$\varphi'(\varphi, 0, \Phi, \Phi', \lambda, \lambda', \delta) \quad \text{does not depend on} \quad \delta.$$

Indeed, a particle starting at O, after the reflection from $\mathcal{E}$ passes through F, after reflecting from $\mathcal{P}$ moves in parallel to FF', after the reflection from $\mathcal{P}'$ passes through F', and, finally, after the reflection from $\mathcal{E}'$ returns to O (see Figure 8). The initial and final velocities of the particle are, respectively, $(\sin\varphi, \cos\varphi)$ and $-(\sin\varphi', \cos\varphi')$. Denoting by α and α' the angles the second and fourth segments of the (5-segment) trajectory form, respectively, with OF and OF', one has $\alpha = \alpha'$. The angle α is a function of φ, and φ' is a function of α'; these functions depend only on the parameters of the ellipses $\mathcal{E}$ and $\mathcal{E}'$, respectively, and do not depend on the parameter δ determining the shape of parabolas.

Using properties of ellipses, one derives the formulas connecting φ, α, and $\varphi' = \varphi'(\varphi, 0, \Phi, \Phi', \lambda, \lambda', \delta)$:

$$(13)$$
$$\sin(\varphi - \Phi) = \frac{\lambda \sin\alpha}{2 + \lambda - 2\cos\alpha\sqrt{1 + \lambda}}, \qquad \sin(\varphi' - \Phi') = -\frac{\lambda' \sin\alpha}{2 + \lambda' - 2\cos\alpha\sqrt{1 + \lambda'}}.$$

It follows that

$$(14) \qquad \left.\frac{\partial\varphi'}{\partial\varphi}\right|_{\substack{\varphi=\Phi \\ \xi=0}} = -\left(\frac{\sqrt{\lambda'}}{1 + \sqrt{\lambda'}}\frac{1 + \sqrt{\lambda}}{\sqrt{\lambda}}\right)^2.$$

With fixed Φ, Φ', λ, λ', and δ the mapping $\varphi'(\varphi, \xi)$, $\xi'(\varphi, \xi)$ preserves the measure, $\cos\varphi\, d\varphi\, d\xi = \cos\varphi'\, d\varphi'\, d\xi'$, and hence

$$\cos\varphi = \pm\cos\varphi' \begin{vmatrix} \frac{\partial\varphi'}{\partial\varphi} & \frac{\partial\varphi'}{\partial\xi} \\ \frac{\partial\xi'}{\partial\varphi} & \frac{\partial\xi'}{\partial\xi} \end{vmatrix}.$$

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[2]Note that throughout this paper the sign ' (prime) never means derivation.
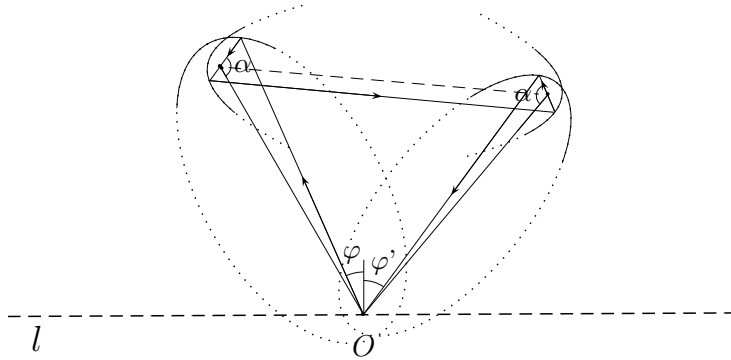
FIG. 8. *Pseudobilliard dynamics.*

Using (12), one gets that $\frac{\partial \xi'}{\partial \varphi}\big|_{\xi=0} = 0$, and hence

$$\begin{vmatrix} \frac{\partial \varphi'}{\partial \varphi} & \frac{\partial \varphi'}{\partial \xi} \\ \frac{\partial \xi'}{\partial \varphi} & \frac{\partial \xi'}{\partial \xi} \end{vmatrix}_{\xi=0} = \frac{\partial \varphi'}{\partial \varphi} \frac{\partial \xi'}{\partial \xi}\bigg]_{\xi=0};$$

therefore

$$(15) \qquad\qquad \cos \varphi = \pm \cos \varphi' \frac{\partial \varphi'}{\partial \varphi} \frac{\partial \xi'}{\partial \xi}\bigg]_{\xi=0}.$$

Putting $\varphi = \Phi$ and $\xi = 0$, and taking into account (11), (14), and (15), one gets

$$(16) \qquad\qquad \cos \Phi = \pm \cos \Phi' \left(\frac{\sqrt{\lambda'}}{1 + \sqrt{\lambda'}} \frac{1 + \sqrt{\lambda}}{\sqrt{\lambda}} \right)^2 \frac{\partial \xi'}{\partial \xi}\bigg]_{\substack{\varphi=\Phi \\ \xi=0}}.$$

Define the positive continuous functions $\lambda(\Phi')$ and $\lambda'(\Phi)$ by the relations

$$(17) \qquad\qquad \left(\frac{\sqrt{\lambda}}{1 + \sqrt{\lambda}} \right)^2 = \frac{1}{2} \cos \Phi', \qquad \left(\frac{\sqrt{\lambda'}}{1 + \sqrt{\lambda'}} \right)^2 = \frac{1}{2} \cos \Phi;$$

then one has

$$(18) \qquad\qquad \left| \frac{\partial \xi'}{\partial \xi}\bigg]_{\substack{\varphi=\Phi; \, \lambda=\lambda(\Phi') \\ \xi=0; \, \lambda'=\lambda'(\Phi)}} \right| = 1.$$

Additionally, taking into account (14) and (17), one gets

$$(19) \qquad\qquad \frac{\cos \Phi'}{\cos \Phi} \frac{\partial \varphi'}{\partial \varphi}\bigg|_{\substack{\varphi=\Phi; \, \lambda=\lambda(\Phi') \\ \xi=0; \, \lambda'=\lambda'(\Phi)}} = -1.$$

Recall that $\varphi' = \varphi'(\varphi, 0, \Phi, \Phi', \lambda, \lambda')$; that is, the restriction of the function φ' to the subspace $\xi = 0$ does not depend on δ. Hence the function $\frac{\partial \varphi'}{\partial \varphi}\big|_{\xi=0}$ and, by formula (15), the function $\frac{\partial \xi'}{\partial \xi}\big|_{\xi=0}$ also do not depend on δ. Put $\Phi_0 = \arcsin(1 - 2/m)$,

so that $J^1 = [-\pi/2, -\Phi_0]$ and $J^m = [\Phi_0, \pi/2]$, and put $\Delta\Phi = 2/m$. The set $\{(\Phi, 0, \Phi, \Phi', \lambda(\Phi'), \lambda'(\Phi)) : -\Phi_0 \leq \Phi, \Phi' \leq \Phi_0, \Phi' - \Phi \geq \Delta\Phi\}$ is compact and belongs to the (open) domain of the function φ'. Choose a sufficiently large integer value $k = k(\varepsilon)$, so that for

$$(20) \qquad |\sin\varphi - \sin\Phi| < 2/(km), \qquad \xi = 0, \qquad -\Phi_0 \leq \Phi, \Phi' \leq \Phi_0,$$
$$\Phi' - \Phi \geq \Delta\Phi, \qquad \lambda = \lambda(\Phi'), \qquad \lambda' = \lambda'(\Phi)$$

it holds true that

$$(21) \qquad -\frac{\cos\Phi'}{\cos\Phi}\frac{\partial\varphi'}{\partial\varphi} \in [(1+\varepsilon)^{-1}, 1+\varepsilon].$$

Formulas (21) and (11) mean that, under conditions (20), φ' is also close to Φ'. Increasing k if necessary, ensure (under the same conditions) that

$$(22) \qquad \frac{\cos\Phi'}{\cos\Phi}\frac{\cos\varphi}{\cos\varphi'} \in [(1+\varepsilon)^{-1}, 1+\varepsilon].$$

Taking into account (15), (21), and (22), one obtains that under conditions (20) it holds true that

$$(23) \qquad \left|\frac{\partial\xi'}{\partial\xi}\right| \in [(1+\varepsilon)^{-2}, (1+\varepsilon)^2].$$

3. Now we proceed to the description of the hollow $(\Omega_\varepsilon, I_\varepsilon)$.

(a) If $2 \leq i \neq \sigma(i) \leq m-1$, then divide the interval J^i into k subintervals $J^{i,j}$ of equal measure λ, going in increasing order: $J^i = \cup_{j=1}^k J^{i,j}$, $\lambda(J^{i,j}) = 2/(km)$ for any $j = 1, \ldots, k$. Recall that $d\lambda(\varphi) = \cos\varphi\,d\varphi$ and the value $k = k(\varepsilon)$ is defined above. Without loss of generality assume that $k(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

To each pair of intervals, $J^{i,j}$ and $J^{\sigma(i),j}$, we apply the construction described above; see Figure 9. Namely, draw arcs of ellipses $\mathcal{E}_{i,j} = \overbrace{AB}$, $\mathcal{E}'_{i,j} = \overbrace{A'B'}$ and arcs of parabolas $\mathcal{P}_{i,j}$, $\mathcal{P}'_{i,j}$. Without loss of generality suppose that $i < \sigma(i)$. The angles AOB and $A'OB'$ correspond to the angular intervals $J^{i,j}$ and $J^{\sigma(i),j}$, respectively. The foci $\bar{F} = F_{i,j}$ and $\bar{F}' = F'_{i,j}$ belong to the intervals OA and OA', respectively. The endpoints of the arcs $\mathcal{P}_{i,j}$ and $\mathcal{P}'_{i,j}$ also belong to the intervals OA and OA', respectively. The angle corresponding to the ray OA (and therefore to the left endpoint of the interval $J^{i,j}$) will be denoted by $\bar{\Phi} = \Phi_{i,j}$, and the angle corresponding to the ray OA' (and therefore to the right endpoint of the interval $J^{\sigma(i),j}$) will be denoted by $\bar{\Phi}' = \Phi'_{i,j}$. Denote $\bar{\lambda} = \lambda_{i,j} := \lambda(\bar{\Phi}')$ and $\bar{\lambda}' = \lambda'_{i,j} := \lambda'(\bar{\Phi})$, according to formula (17). Next, select a value $\bar{\delta} = \delta_{i,j}$ and draw two curves (*lateral reflectors*) in such a way that (i) each of the curves contains an arc of parabola (the first curve contains $\mathcal{P}_{i,j}$ and the second one $\mathcal{P}'_{i,j}$), an arc of circumference centered at O, and three radial segments; (ii) these curves do not intersect the intervals whose endpoints belong to the set $\{F_{\alpha,\beta}, F'_{\gamma,\delta} : (\alpha, \beta) \neq (i, j), (\gamma, \delta) \neq (\sigma(i), j)\}$: this will guarantee free passage of particles from one parabola to another; and (iii) the λ-measure of the angular interval occupied by each lateral reflector does not exceed $\varepsilon/(km)$. In Figure 9, the angular reflectors are the curves joining the points A and C, and the points A' and C'.

Notice that $-\Phi_0 \leq \bar{\Phi}, \bar{\Phi}' \leq \Phi_0$, and $\bar{\Phi}' - \bar{\Phi} \geq \Delta\Phi$. Indeed, $\bar{\Phi}$ and $\bar{\Phi}'$ do not belong to the intervals $J^1 = [-\pi/2, -\Phi_0]$ and $J^m = [\Phi_0, \pi/2]$. On the other hand, one has $\bar{\Phi}' - \bar{\Phi} \geq \sin\bar{\Phi}' - \sin\bar{\Phi} = \lambda([\bar{\Phi}, \bar{\Phi}']) \geq 2/m = \Delta\Phi$.
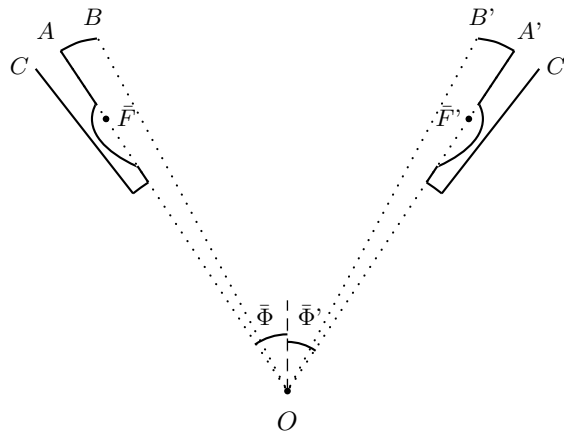
FIG. 9. *Part of the hollow corresponding to the angular intervals $J^{i,j}$ and $J^{\sigma(i),j}$.*

Introduce the shorthand notation $\varphi'(\varphi, \xi) = \varphi'(\varphi, \xi, \Phi_{i,j}, \Phi'_{i,j}, \lambda_{i,j}, \lambda'_{i,j}, \delta_{i,j})$. According to (21) and (23), for $\varphi \in J^{i,j}$ it holds true that

$$(24) \qquad -\frac{\cos \bar{\bar{\Phi}}'}{\cos \bar{\bar{\Phi}}} \frac{\partial \varphi'}{\partial \varphi}(\varphi, 0) \in [(1+\varepsilon)^{-1}, 1+\varepsilon]$$

and

$$(25) \qquad \left| \frac{\partial \xi'}{\partial \xi}(\varphi, 0) \right| \in [(1+\varepsilon)^{-2}, (1+\varepsilon)^2].$$

According to (11), one has $\varphi'(\bar{\bar{\Phi}}, 0) = \bar{\bar{\Phi}}'$; this equality and formula (24) imply that for $\varphi \in J^{i,j}$ and $\varphi' = \varphi'(\varphi, 0)$ one has

$$(26) \qquad -\frac{\cos \bar{\bar{\Phi}}'}{\cos \bar{\bar{\Phi}}} \frac{\varphi' - \bar{\bar{\Phi}}'}{\varphi - \bar{\bar{\Phi}}} \in [(1+\varepsilon)^{-1}, 1+\varepsilon].$$

On the other hand, one has

$$(27) \qquad \cos \bar{\bar{\Phi}} \, |J^{i,j}| = \frac{2}{km} (1 + o(1)),$$

$$(28) \qquad \cos \bar{\bar{\Phi}}' \, |J^{\sigma(i),j}| = \frac{2}{km} (1 + o(1)),$$

with $o(1)$ being uniformly small over all i, j as $\varepsilon \to 0$, and $|J|$ being the Lebesgue measure of J. (Recall that the parameters $\bar{\bar{\Phi}}, \bar{\bar{\Phi}}', k$ and the intervals $J^{i,j}$ implicitly depend on ε.)

Choose closed intervals $\tilde{J}^{i,j} \subset J^{i,j}$ and $\tilde{J}^{\sigma(i),j} \subset J^{\sigma(i),j}$ satisfying the following conditions: (i) $\varphi'(\tilde{J}^{i,j} \times \{0\}) = \tilde{J}^{\sigma(i),j}$; (ii) some neighborhoods of $\tilde{J}^{i,j}$ and $\tilde{J}^{\sigma(i),j}$ belong to $J^{i,j}$ and $J^{\sigma(i),j}$, respectively; and (iii) the pseudobilliard trajectory with the initial data $(\varphi, 0)$, $\varphi \in \tilde{J}^{i,j}$ does not intersect the neighbor lateral reflectors (that is, the lateral reflectors corresponding to the intervals $\tilde{J}^{i,j+1}$ and $\tilde{J}^{\sigma(i),j-1}$ if $j \neq 1, k$; if $j = 1$, then $\tilde{J}^{\sigma(i),j-1}$ should be replaced with $\tilde{J}^{\sigma(i)-1,k}$, and if $j = k$, then $\tilde{J}^{i,j+1}$ should be replaced with $\tilde{J}^{i+1,1}$). Note in this regard that the neighbor lateral reflectors occupy a small part of the angular intervals $J^{i,j}$ and $J^{\sigma(i),j}$ (represented in the figure

by the arcs AB and $B'A'$). Other lateral reflectors will not be intersected by the choice of lateral reflectors.

By virtue of (26), (27), (28) and because of smallness of the angular intervals occupied by the lateral reflectors, $\tilde{J}^{i,j}$ and $\tilde{J}^{\sigma(i),j}$ may be chosen in such a way that the ratios $\lambda(\tilde{J}^{i,j})/\lambda(J^{i,j})$ and $\lambda(\tilde{J}^{\sigma(i),j})/\lambda(J^{\sigma(i),j})$ uniformly (with respect to i, j) tend to 1 as $\varepsilon \to 0$. Thus, a billiard particle going from O in a direction $\varphi \in \tilde{J}^{i,j}$ makes the same reflections and in the same order as under the pseudobilliard dynamics: first, reflection from $\mathcal{E}_{i,j}$, from $\mathcal{P}_{i,j}$, from $\mathcal{P}'_{i,j}$, and from $\mathcal{E}'_{i,j}$; finally, the particle goes back to O in the direction $\varphi'(\varphi, 0) \in \tilde{J}^{\sigma(i),j}$.

Choose $a_{i,j}$ in such a way that the following conditions are fulfilled: if $(\xi, \varphi) \in [-a_{i,j}, a_{i,j}] \times \tilde{J}^{i,j}$, then (i) the corresponding billiard trajectory does not intersect the lateral reflectors and the indicated order of reflections is preserved; (ii) $\varphi'(\varphi, \xi) \in J^{\sigma(i),j}$; (iii) $\left|\frac{\partial \xi'}{\partial \xi}(\varphi, \xi)\right| \in [(1+\varepsilon)^{-3}, (1+\varepsilon)^3]$. Analogously, choose $a_{\sigma(i),j}$ in such a way that the conditions are fulfilled: if $(\xi, \varphi) \in [-a_{\sigma(i),j}, a_{\sigma(i),j}] \times \tilde{J}^{\sigma(i),j}$, then (i) the billiard trajectory does not intersect the lateral reflectors and the order of its reflections is reversed; (ii) $\varphi'(\varphi, \xi) \in J^{i,j}$; (iii) $\left|\frac{\partial \xi'}{\partial \xi}(\varphi, \xi)\right| \in [(1+\varepsilon)^{-3}, (1+\varepsilon)^3]$. Note that the values $a_{i,j}$ and $a_{\sigma(i),j}$ implicitly depend on ε.

Select $a_\varepsilon \le \min_{i,j} a_{ij}$ in such a way that $a_\varepsilon \to 0$ as $\varepsilon \to 0$, and denote $I_\varepsilon = (-a_\varepsilon, a_\varepsilon) \times \{0\}$, $\tilde{I}_\varepsilon = (-a_\varepsilon(1+\varepsilon)^{-3}, a_\varepsilon(1+\varepsilon)^{-3}) \times \{0\}$, and $\tilde{J}^i_\varepsilon = \tilde{J}^i := \cup_j \tilde{J}^{i,j}_\varepsilon$. The part of the boundary of Ω_ε related to the angular intervals $J^{i,j}$ and $J^{\sigma(i),j}$ under consideration is formed by the arcs of ellipses $\mathcal{E}_{i,j}$, $\mathcal{E}'_{i,j}$ and the corresponding lateral reflectors. Then a billiard particle with initial conditions $(\xi, \varphi) \in \tilde{I}_\varepsilon \times \tilde{J}^{i,j}$ after making four reflections will intersect l at a point $(\xi', 0) \in I_\varepsilon$, and the angle at the point of intersection will be $\varphi^+_{\Omega_\varepsilon, I_\varepsilon}(\xi, \varphi) = \varphi'(\varphi, \xi) \in J^{\sigma(i),j} \subset J^{\sigma(i)}$. Thus, the set of values $(\xi, \varphi) \in I_\varepsilon \times J^i$ such that $\varphi^+_{\Omega_\varepsilon, I_\varepsilon}(\xi, \varphi) \notin J^{\sigma(i)}$ is contained in the set $\left(I_\varepsilon \times J^i\right) \setminus \left(\tilde{I}_\varepsilon \times \tilde{J}^i_\varepsilon\right)$, whose measure is vanishing as $\varepsilon \to 0$.

(b) If $2 \le i = \sigma(i) \le m - 1$, then the corresponding part of the boundary is the arc of circumference of radius 2 with the center at O occupying the angular interval J^i, that is, the set $\{2(\sin \varphi, \cos \varphi), \varphi \in J^i\}$. Next, we will show that for all values $(\xi, 0) \in I_\varepsilon$, $\varphi \in J^i$, except for a portion of order $o(1)$, the corresponding billiard particle makes one reflection from the arc and then goes back to I_ε in the direction $\varphi' \in J^i$.

For all values $\varphi \in J^i$, except for the union of two intervals of vanishing length (each of the intervals is contained in J^i, has the length $2 \arctan(a_\varepsilon/4)$, and contains an endpoint of J^i), the particle starting at $(\xi, 0) \in I_\varepsilon$ in the direction φ will reflect from the indicated arc of circumference. Let $\psi \in J^i$ be the angular coordinate of the reflection point. By $(\xi', 0)$ denote the point at which the reflected particle intersects the straight line l. One easily verifies that

$$(29) \qquad \frac{1}{\xi} + \frac{1}{\xi'} = \cos \psi.$$

One has

$$(30) \qquad |\xi| < a_\varepsilon,$$

and hence

$$(31) \qquad \frac{1}{|\xi'|} = \left|\cos \psi - \frac{1}{\xi}\right| > \frac{1}{a_\varepsilon} - 1.$$

From (29) it follows that $|\xi + \xi'|/|\xi\xi'| = |\cos\psi| \leq 1$, and, taking into account (30) and (31), one finds that $|\xi + \xi'| < a_\varepsilon^2/(1 - a_\varepsilon)$. This implies that for all values $(\xi, 0) \in I_\varepsilon$, except for a set of measure $O(a_\varepsilon^2)$, the second point of intersection of the billiard trajectory belongs to I_ε; moreover, the velocity at this point, $\varphi_{\Omega_\varepsilon, I_\varepsilon}^+(\xi, \varphi)$, belongs to $\mathcal{N}_{2\arctan(a_\varepsilon/4)}(J^i)$, the neighborhood of J^i of radius $2\arctan(a_\varepsilon/4)$. This finally implies that for all $(\xi, \varphi) \in I_\varepsilon \times J^i$, except for a portion of order $O(a_\varepsilon)$, it holds that $\varphi_{\Omega_\varepsilon, I_\varepsilon}^+(\xi, \varphi) \in J^i$.

(c) The parts of the hollow's boundary, corresponding to J^1 and J^m, are formed by smooth curves joining the corresponding endpoints of I_ε and the points $2(\sin\Phi_0, -\cos\Phi_0)$ and $2(\sin\Phi_0, \cos\Phi_0)$, respectively. The unique condition on these curves is that they can be parametrized by the monotonically increasing angular coordinate. For those values $\sigma(1), \sigma(m)$ that coincide with neither 1 nor m take just the arcs of circumference of radius 2 corresponding to the angular intervals $J^{\sigma(1)}, J^{\sigma(m)}$.

Consider the union of all of the elliptic arcs $\mathcal{E}_{i,j}, \mathcal{E}'_{i,j}$ introduced in item (a), all of the arcs of circumference defined in items (a) and (b), and the two curves introduced in this item (c). Let us call this union the *main element*. Each lateral reflector is a curve; select it in such a way that both its endpoints belong to the main element. Finally, the curve $\partial\Omega_\varepsilon \setminus I_\varepsilon$ is the union of all of the lateral reflectors and the part of the main element visible from O (that is, which is not shielded by the adjacent lateral reflectors). Thus, the definition of the hollow $(\Omega_\varepsilon, I_\varepsilon)$ is complete.

In Figure 10, there is shown a particular hollow $(\Omega_\varepsilon, I_\varepsilon)$ corresponding to the permutation $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{smallmatrix}\right)$. The angular intervals $J^1, \ldots, J^5$ are separated by dotted lines. The family of hollows $(\Omega_\varepsilon, I_\varepsilon)$, with vanishingly small ε, has the following property: for almost all particles with the initial direction from J^2 (resp. J^3, J^4), the final direction will belong to J^4 (resp. J^3, J^2). In the figure, there is shown the trajectory of a particle with the initial direction $\varphi \in J^2$ and the final direction $\varphi^+ \in J^4$. The particle makes a reflection from an elliptic arc, then two reflections from (very small) parabolic arcs, and, finally, again from an elliptic arc. According to our notation, these arcs are $\mathcal{E}_{2,2}, \mathcal{P}_{2,2}, \mathcal{P}'_{2,2}$, and $\mathcal{E}'_{2,2}$.
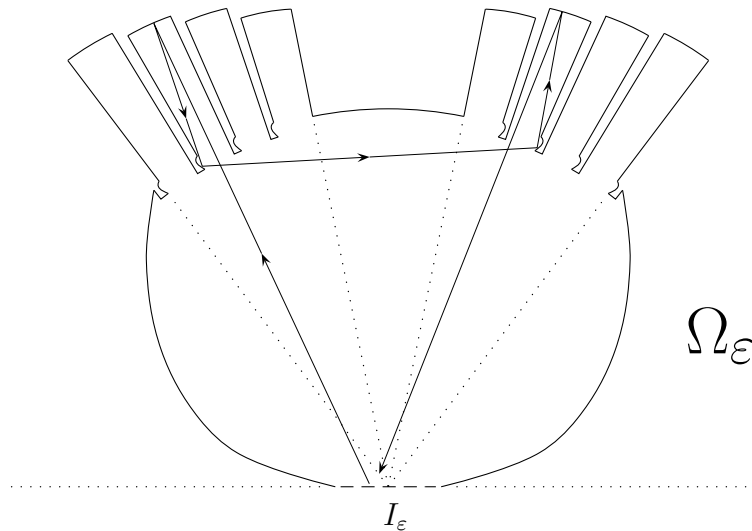

FIG. 10. *A hollow $(\Omega_\varepsilon, I_\varepsilon)$ approximating a σ-measure.*

4. Concluding remarks and applications. Physical bodies in the real world have atomic structure and therefore are disconnected. This is a reason for using (generally) disconnected sets Q_m in the definition of a rough body. In the future we intend to turn to propose and study the notion of a three-dimensional rough body, where the connectivity assumption is absolutely useless; this is another reason. By removing this assumption, the consideration in two dimensions (namely, the proof of Lemma 1) is made somewhat more difficult, but at the same time prerequisites for passing to the three-dimensional case are created.

In fact, the notions of "disconnected" (as everywhere in this paper) and "connected" rough bodies are equivalent. There is a natural one-to-one correspondence between the equivalence classes in the connected and disconnected cases,[3] the former classes being subclasses of the latter ones under this correspondence.

Let us now consider applications of the main theorem to problems of the body of minimal or maximal aerodynamic resistance. A two-dimensional convex body B moves, at constant velocity, through a rarefied homogeneous medium in $\mathbb{R}^2$, and at the same time it slowly rotates. The rotation is generally nonuniform; we assume that, during a sufficiently long observation period, in a reference system connected with the body the body's velocity is distributed in S^1 according to a given density function ρ, with $\int_{S^1} \rho(v)\,dv = 1$. The medium particles do not mutually interact, and collisions of the particles with the body are absolutely elastic. The resistance of the medium to the motion of the body is a vector-valued function of time. After averaging it over a sufficiently long period of time, one gets a vector. We are interested in the projection of this vector onto the direction of motion; for the sake of brevity, it will be called mean resistance, or just resistance. The problem is as follows: given B, determine the roughness on it in such a way that the main resistance of the resulting rough body is minimal or maximal.

A prototype of such a mechanical system is an artificial satellite of the Earth on relatively low altitudes ($100 \div 200$ km), with restricted capacity of rotation angle control. The satellite's motion is slowing down by the rest of the atmosphere; the problem is to minimize or maximize the effect of slowing down. The problems of resistance *maximization* may also arise when considering solar sail: a spacecraft driven by the pressure of solar photons.

The initial velocity of an incident particle (in the reference system connected with the body) is $-v$, and the final velocity is v^+; therefore, the momentum transmitted by the particle to the body is $v + v^+$. The projection of the transmitted momentum onto the direction of motion of the body equals $1 + \langle v, v^+ \rangle$. Averaging this value over all particles incident on the body within a sufficiently long time interval, one gets the mean resistance. The averaging amounts to integration over $\rho(v)\,d\nu_{\mathcal{B}}(v, v^+, n)$; that is, the mean resistance of the rough body equals

$$R(\nu_{\mathcal{B}}) = \iiint_{\mathbb{T}^3} \left(1 + \langle v, v^+ \rangle\right) \rho(v)\,d\nu_{\mathcal{B}}(v, v^+, n).$$

Using the main theorem and Fubini's theorem, one rewrites this formula in the form

$$(32) \qquad R(\nu_{\mathcal{B}}) = \int_{S^1} d\tau_B(n) \iint_{\mathbb{T}^2} \left(1 + \langle v, v^+ \rangle\right) \rho(v)\,d\eta_{\mathcal{B},n}(v, v^+),$$

---

[3]More precisely, we mean equivalence classes formed by sequences of connected/disconnected sets.

where $\eta_{\mathcal{B},n} \in \Lambda_n$. Thus, the minimization problem for $R(\nu_{\mathcal{B}})$ reduces to minimization, for any n, of the functional $\iint_{\mathbb{T}^2} (1 + \langle v, v^+ \rangle) \rho(v) \, d\eta(v, v^+)$ over all $\eta \in \Lambda_n$. Using the notation introduced in section 3.3, one comes to the problem:

$$(33) \qquad \inf_{\eta \in \Lambda} \iint_{\square} (1 + \cos(\varphi - \varphi^+)) \varrho(\varphi) \, d\eta(\varphi, \varphi^+),$$

where $\varrho(\varphi) = \rho(v)$ for $\varphi = \operatorname{Arg} v - \operatorname{Arg} n$. This problem, in turn, by symmetrization of the cost function reduces to a particular Monge–Kantorovich problem:

$$(34) \qquad \inf_{\eta \in \Lambda_{\lambda,\lambda}} \mathcal{F}(\eta), \quad \text{where} \quad \mathcal{F}(\eta) = \iint_{\square} c(\varphi, \varphi^+) \, d\eta(\varphi, \varphi^+),$$

where $c(\varphi, \varphi^+) = (1 + \cos(\varphi - \varphi^+)) \frac{\varrho(\varphi) + \varrho(\varphi^+)}{2}$ and $\Lambda_{\lambda,\lambda}$ is the set of measures η on $\square$ having both marginal measures equal to λ: $\pi_\varphi^\# \eta = \lambda = \pi_{\varphi^+}^\# \eta$. Recall that λ is defined by $d\lambda(v) = \cos\varphi \, d\varphi$.

Problem (34) can be exactly solved in several particular cases. Consider the case of uniform motion, where the function ρ, and therefore ϱ, is constant, and thus one can take $c(\varphi, \varphi^+) = \frac{3}{8}(1 + \cos(\varphi - \varphi^+))$.[4] Note that $\mathcal{F}(\eta_0) = 1$ and therefore resistance of the smooth body is equal to its perimeter: $R(\nu_B) = \int_{S^1} d\tau_B(n) \mathcal{F}(\eta_0) = |\partial B|$. (Recall that the measure η_0 belongs to Λ and is supported on the diagonal $\varphi^+ = -\varphi$.) The minimization problem (34) for constant ϱ was solved in [14]: one has $\inf_{\mathcal{B}} R(\nu_{\mathcal{B}}) = 0.9878\ldots \cdot |\partial B|$, the infimum being taken over all rougherings of B.

Note that the corresponding *maximization* problem for (34) has the trivial solution, which does not depend on the function ϱ: $\eta = \eta_\star$, the measure $\eta_\star \in \Lambda$ being supported on the diagonal $\varphi^+ = \varphi$. One has $\sup_{\mathcal{B}} R(\nu_{\mathcal{B}}) = \kappa |\partial B|$, where $\kappa = \left(\int_{-\pi/2}^{\pi/2} \varrho(\varphi) \cos\varphi \, d\varphi \right) / \left(\int_{-\pi/2}^{\pi/2} \varrho(\varphi) \cos^3\varphi \, d\varphi \right) > 1$; in the case of uniform rotation one has $\kappa = 1.5$. The maximization problem was studied in more detail in [20].

Appendix A. The construction below is simple (see Figure 11), but its description is a bit cumbersome.

Take a point in the interior of B and connect it by segments with all vertices. The polygon is thus divided into several triangles; fix i and m and consider the triangle with the base $\mathfrak{b}_i$, the ith side of B. Denote by $d(\Omega_i^m)$ the diameter of the orthogonal projection of Ω_i^m onto the straight line containing I_i^m; one obviously has $d(\Omega_i^m) \geq |I_i^m|$. Fix a positive number $\kappa < |I_i^m|/d(\Omega_i^m)$.

Take a rectangle Π^1 contained in the triangle and such that one side of Π^1 belongs to $\mathfrak{b}_i$. By δ_1 denote the total length of the part of $\mathfrak{b}_i$ which is not occupied by this side.

For the sake of brevity, the image of a set under the composition of a homothety with positive ratio and a translation will be called a copy of this set. Take several copies of Ω_i^m (copies of first order) that do not mutually interact, that belong to Π^1, whose corresponding copies of I_i^m belong to $\mathfrak{b}_i$, and whose portion of the side of Π^1 occupied by them is more than κ.

Next, take several rectangles that do not mutually intersect and do not intersect with the chosen copies of Ω_i^m, belong to Π^1, and have one side contained in $\mathfrak{b}_i$. Denote by Π^2 the union of these rectangles and by δ_2 the total length of the part of the side of Π^1 which is not occupied by the rectangles from Π^2 and by the copies of I_i^m. Next,

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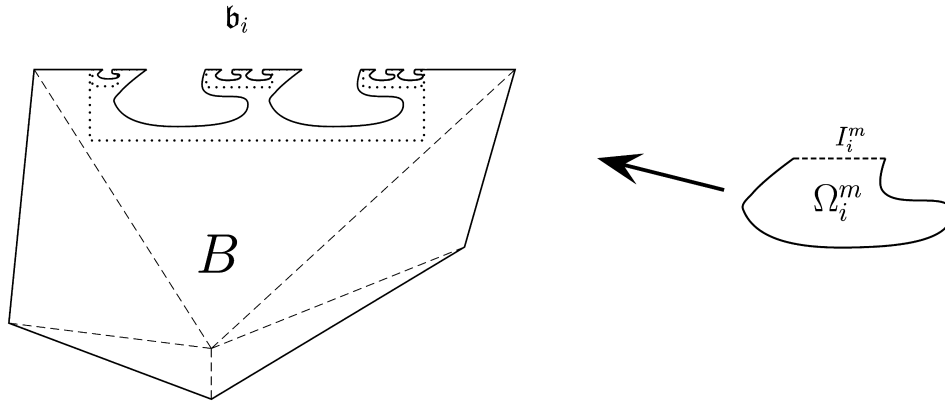[4] The normalization constant 3/8 is taken for further convenience.

FIG. 11. *Making hollows on the ith side of a polygon.*

for each rectangle from Π^2 choose several copies of Ω_i^m (copies of second order) in the way completely similar to the described above (see Figure 11).

Continuing this process, one obtains a sequence $\Pi^1, \Pi^2, \ldots$ of unions of rectangles and collections of copies of Ω_i^m of first, second, $\ldots$ order. Choose the rectangles in such a way that $\delta_1 + \delta_2 + \cdots < 1/m$ and $\mathrm{Area}(\Pi^1) < 1/m$. Finally, choose k such that the total length of sides of rectangles from Π^{k+1} contained in $\mathfrak{b}_i$ is less than $1/m$, and take the collection of copies of Ω_i^m of order $1, 2, \ldots, k$ (we shall call it full collection). The total length of the part of $\mathfrak{b}_i$ not occupied by the corresponding copies of I_i^m is less than $2/m$, and therefore it goes to zero as $m \to \infty$.

By definition, the desired set Q_m is B minus the union of full collections of copies of Ω_i^m over all i.

Appendix B. We prove here slightly more than needed.

STATEMENT 3. *Let $A = (a_{ij})_{i,j=1}^k$ be a symmetric matrix, with a_{ij} being non-negative integers. Denote $n_i = \sum_{j=1}^k a_{ij}$. Then there exist matrices $B_{ij} = (b_{ij}^{\mu\nu})_{\mu,\nu}$ of size $n_i \times n_j$ such that $b_{ij}^{\mu\nu} \in \{0,1\}$, $B_{ij}^T = B_{ji}$, the sum of elements in B_{ij} equals a_{ij}, and the block matrix $D = (B_{ij})$ contains exactly one unit in each row and each column.*

Note that for some values $i = i_1, i_2, \ldots$ it may happen that $n_i = 0$, that is, $a_{ij} = 0$ for all $j = 1, \ldots, k$. Then the corresponding matrices B_{ij} have the size $0 \times n_j$, that is, are empty. In this case D coincides with the block matrix $D' = (B_{ij})$ having the rows $i_1, i_2, \ldots$ and columns $i_1, i_2, \ldots$ crossed out.

Proof. The proof is by induction on k. Let the statement be true for $k-1$; prove it for k. Take the matrix $\tilde{A} = (a_{ij})_{i,j=2}^k$; there exists a block matrix $\tilde{B} = (\tilde{B}_{ij})_{i,j=2}^k$ satisfying the statement. Note that the order of $\tilde{B}_{ij}$ is $\tilde{n}_i \times \tilde{n}_j$, where $\tilde{n}_i = \sum_{j=2}^k a_{ij} = n_i - a_{i1}$. Define the matrices B_{ij} as follows.

(a) Put $B_{11} = \mathrm{diag}\{\underbrace{1, \ldots, 1}_{a_{11}}, 0, \ldots, 0\}$.

(b) Put $b_{12}^{a_{11}+1,1} = \cdots = b_{12}^{a_{11}+a_{12},a_{12}} = 1$; $b_{13}^{a_{11}+a_{12}+1,1} = \cdots = b_{13}^{a_{11}+a_{12}+a_{13},a_{13}} = 1$; $\cdots$; $b_{1k}^{a_{11}+\cdots+a_{1,k-1}+1,1} = \cdots = b_{1k}^{a_{11}+\cdots+a_{1k},a_{1k}} = 1$; the other elements of the matrices B_{1j}, $j = 2, \ldots, k$, are zeros. Thus, on the diagonal of B_{1j} starting from the element at the first column and the $(a_{11} + a_{12} + \cdots + a_{1,j-1} + 1)$th row, the first a_{1j} elements equal 1, and the remaining elements on this diagonal and all of the elements

off the diagonal are zeros. This defines the matrices B_{1j}, $j = 2, \ldots, k$. The matrices B_{i1}, $i = 2, \ldots, k$, are determined by the condition $B_{i1} = B_{1i}^T$.

(c) For $i \geq 2$, $j \geq 2$ define the matrix B_{ij} as follows. For $\mu \leq a_{1i}$ or $\nu \leq a_{1j}$, put $b_{ij}^{\mu\nu} = 0$, and for $\mu \geq a_{1i} + 1$, $\nu \geq a_{1j} + 1$, put $b_{ij}^{\mu\nu} = \tilde{b}_{ij}^{\mu - a_{1i}, \nu - a_{1j}}$. Thus, in the obtained matrix B_{ij}, the right lower corner coincides with the matrix $\tilde{B}_{ij}$, and all of the remaining elements are equal to zero. The number of rows of this matrix equals $a_{1i} + \tilde{n}_i = n_i$, and the number of columns equals $a_{1j} + \tilde{n}_j = n_j$. One obviously has $B_{ij}^T = B_{ji}$.

One easily verifies that $\sum_{\mu\nu} b_{ij}^{\mu\nu} = a_{ij}$ and that each row and each column of the obtained block matrix $D = (B_{ij})_{i,j=1}^k$ contains precisely one unit. $\square$

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