On the Algorithm of Determination of Immobile Indices for Convex SIP problems

Kostyukova O.I.,* Tchemisova T.V. ,† Yermalinskaya S.A.‡

Abstract

We consider convex Semi-Infinite Programming (SIP) problems with a continuum of constraints. For these problems we introduce new concepts of *immobility orders* and *immobile indices*. These concepts are objective and important characteristics of the feasible sets of the convex SIP problems since they make it possible to formulate optimality conditions for these problems in terms of optimality conditions for some NLP problems (with a finite number of constraints). In the paper we describe a finite algorithm (DIO algorithm) of determination of immobile indices together with their immobility orders, study some important properties of this algorithm, and formulate the Implicit Optimality Criterion for convex SIP without any constraint qualification conditions (CQC). An example illustrating the application of the DIO algorithm is provided.

Keywords: Convex Semi-Infinite Programming, Non-Linear Programming, optimality criterion, constraint qualification condition, immobile index, immobility order.

1 Introduction

Semi-Infinite Programming (SIP) models appear in mathematics, engineering, physics, social and other sciences when some processes or systems depend on finite dimensional variables and are described with the help of an infinite number of constraints (for the references see (Hettich and Kortanek, 1993), (Polak, 1983), (Weber, 2002)). In recent years many papers have appeared that are dedicated to the theory of SIP ((Rückmann and Shapiro, 1999)-(Stein and Still, 2000),(Stein and Still, 2002),(Still, 1999) etc.), in general, and to SIP optimality conditions, in particular. New constructive

*Chief Researcher, Institute of Mathematics, National Academy of Sciences of Belarus. This work was partially supported by the state program "Mathematical Models 13" of fundamental research in Republic of Belarus

†Assistant Professor, Mathematical Department, University of Aveiro, Portugal. This work was partially supported by FCT – Fundacao para Ciencia e Tecnologia, Portugal.

‡PhD Student, Belarus ian State University of Informatics and Radio-electronics, Belarus

algorithms for solution of SIP problems have been suggested in (Gustafson, 1983), (Hettich, 1986),(Stein and Still, 2003),(Tanaka, Fukushima and Ibaraki, 1988), etc.
The main issues arising in study of semi-infinite optimization problems are the following:
1) obtaining of linear representations of convex inequality systems (Fajardo and Lopez, 2006; Goberna and Lopez, 1998; Li, Nahak and Singer, 2000);
2) obtaining of formulas for distances to a feasible set (Canovas, Dontchev, Lopes and Parra, 2005; Gugat, 2000; Li et al., 2000),
3) duality theory and stability theory (Bonnans and Shapiro, 2000; Jongen, Meer and Trieschm, 2004; Klatte, 1994; Li, Yang and Teo, 2003),
4) formulation of informative optimality conditions (Bonnans and Shapiro, 2000; Hettich and Jongen, 1978; Jongen, Rückmann and Stein, 1998; Jongen, Wetterling and Zwier, 1987; Rückmann and Shapiro, 1999; Rückmann and Shapiro, 2001; Stein and Still, 2000),
5) efficient discretizations of original semi-infinite problem (Bonnans and Shapiro, 2000; Hettich and Kortanek, 1993; Reemtsen and Gorner, 1998).

While considering the issues mentioned above the so called Constraint Qualification Conditions (CQC) play s a crucial role. Thus, in any serious research concerning these problems the first step consists in a formulation of certain conditions that the problem under consideration has to satisfy. The fact that CQC are imposed on the problem is significant since the examples show that certain results obtained can be invalid if these conditions are violated.
In (Jeyakumar, Lee and Dinh, 2003), we find the following statement: "The constraint qualifications do not always hold for finite dimensional convex programs and frequently fail for infinite dimensional convex programs. Over the years a great deal of attention has been focussed on the characterizations of optimality which avoid a constraint qualification". Therefore it is interesting and important to investigate the problems formulated in 1)-5) under possibly weaker CQC. An evidence of importance of these problems is justified by a huge number of papers (see, for example, (Auslender, 2000),(Bot and Wanka, 2006), (Canovas et al., 2005),(Gugat, 2000), (Jeyakumar, Dinh and Lee, 2004)-(Jeyakumar, Lee and Dinh, 2004),(Klatte, 1994),(Li et al., 2000),(Still, 2004), (Xin and Chong, 2007)) devoted to

• study of different CQC and their relationship;
• formulation of new CQC;
• solution of the problems from 1)-5) under weaker CQC.

It is evident that any CQC presents a certain characterization of the feasible set of the original SIP problem in a neighborhood of a given feasible point. Hence to avoid or at least to weaken CQC one has to use some specific information about the structure of the feasible set.

In general, a SIP problem consists in a search of a minimum of some function $c(x)$ (objective function) subject to an infinite system of constraints expressed as $f(x,t) \leq 0$ for all $t \in T$, where T is some compact index set.

In the present paper, we consider the *convex* SIP problems where the objective function $c(x)$ and the constraint function $f(x,t)$ are convex w.r.t. a variable x, and the index set T belongs to $\mathbb{R}$. For this problem we introduce new concepts of *immobile indices* and their *immobility orders*[1]. These concepts are the important characteristic of the feasible set of the convex SIP problems since they make it possible to understand better the nature of the optimization problem and to investigate the problems formulated in 1)-5) without or with some weaker CQC.

In the paper we describe a constructive algorithm (DIO algorithm) of the determination of the immobile indices and the correspondent immobility orders. We justify this algorithm and show that it converges in a finite number of iterations. We show that given some convex SIP problem, the DIO algorithm permits to verify whether this problem satisfies or not the Slater condition. The study of the properties of the DIO algorithm permits us to conclude that it can be easily implemented by any standard NLP solver. We show that the use of the immobile indices and their orders permits to formulate new optimality conditions for the convex SIP problem without any CQC and obtain an efficient discretization of this problem.

The paper is organized as follows. In Section 2 we formulate the problem, introduce the definitions of immobility order and immobile index. In Section 3 the algorithm of determination of the immobile indices and their immobility orders is proposed and some of its properties are described. Proposition 3.6 shows that the convex SIP problem does not satisfy the Slater condition if and only if the set of its immobile indices is not empty. In Section 4 we use the concepts of immobility orders and immobile indices to formulate the optimality conditions for convex SIP in the form of the criterion that does not use any CQC. We show also that the concepts of immobility orders and immobile indices permit to obtain for any convex SIP problem (2.1) a discretization that has the same optimal value. Section 5 contains the discussion of the obtained results.

[1] In (Kortanek and Medvedev, 2005) this term was translated from Russian as *motionless degrees*.

2 Immobility orders and immobile indices

Consider a Semi-Infinite Programming (SIP) problem of the form

$$\begin{aligned} & c(x) \longrightarrow \min, \\ \text{s.t. } & f(x,t) \leq 0, \quad \forall t \in T = [t_*, t^*],\ t_*, t^* \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$. Suppose that the functions $c(x)$ and $f(x,t)$ in (2.1) are analytically defined, sufficiently smooth in $\mathbb{R}^n$ and $\mathbb{R}^n \times T$, respectively. We assume also that $c(x)$ and $f(x,t)$ are convex w.r.t. x. Denote by $X \subset \mathbb{R}^n$ the feasible set of problem (2.1)

$$X = \{x \in \mathbb{R}^n :\ f(x,t) \leq 0, \forall\, t \in T\}. \tag{2.2}$$

Given any feasible solution $x \in X$, the set $T_a(x) = \{t \in T :\ f(x,t) = 0\}$ is called a set of the active indices of T corresponding to x.
In the sequel we will use the following notation:

$$\begin{aligned} & f^{(0)}(x,\ t) = f(x,t),\ f^{(s)}(x,\ t) = \partial^s f(x,t)/\partial t^s,\ s \in \mathbb{N}; \\ & N(q) = \emptyset, \text{ if } q < 0,\ N(q) = \{0,\ 1,\ \dots,\ q\}, \text{ if } q \geq 0,\ q \in \mathbb{Z}. \end{aligned}$$

Given $t \in T,\ x \in \mathbb{R}^n$, let $\rho = \rho(x,t) \in \{-1,\ 0,\ 1,\ \dots\}$ be a number such that

$$f^{(s)}(x,t) = 0,\ s \in N(\rho), \quad f^{(\rho+1)}(x,t) \neq 0. \tag{2.3}$$

Assumption 1. Suppose, $X \neq \emptyset$ and there exists $\bar{x} \in X$ such that $\rho(\bar{x},t) < \infty, t \in T$.

Definition 2.1. An index $t \in T$ is called an immobile index of problem (2.1) if $f(x,t) = 0$ for all feasible $x \in X$.

Let T_* be the set of the immobile indices in problem (2.1):

$$T_* = \{t \in T :\ f(x,t) = 0, \ \ \forall x \in X\}.$$

Definition 2.2. Let $t \in T$. A number $q(t) \in \{-1, 0, 1, \dots\}$ is called an order of immobility (immobility order) of t in SIP problem (2.1) if

1. for each $x \in X$ it is satisfied

$$f^{(r)}(x,t) = 0,\ r \in N(q(t)), \tag{2.4}$$

2. there exists $x(t) \in X$ such that

$$f^{(q(t)+1)}(x(t),t) \neq 0. \tag{2.5}$$

Remark 2.1. Assumption 1 guarantees that problem (2.1) has a finite number of immobile indices and their immobility orders are finite.

According to Definition 2.2, we have

$$q(t) = -1 \text{ for } t \in T \setminus T_* \tag{2.6}$$

and for every $t \in T_*$ there exists a vector $x(t) \in X$ such that the index t is a solution to the lower level problem

$$\max_{\tau \in [t_*,\, t^*]} \varphi_t(\tau), \tag{2.7}$$

where $\varphi_t(\tau) \equiv f(x(t), \tau),\ \tau \in [t_*, t^*]$. It follows from (2.6) and from optimality conditions for $t \in T_*$ on problem (2.7) that

1. if $t \in \text{int}\, T$, then $q(t) + 1$ is even and $f^{(q(t)+1)}(x(t), t) < 0;$
2. $q(t_*) \in \{-1, 0, 1, \dots\}$ and $f^{(q(t_*)+1)}(x(t_*), t_*) < 0$ for the correspondent $x(t_*) \in X;$
3. $q(t^*) \in \{-1, 0, 1, \dots\}$ and for the correspondent $x(t^*) \in X$ we have
   3.a) $f^{(q(t^*)+1)}(x(t^*), t^*) < 0$ whenever $q(t^*) + 1$ is even;
   3.b) $f^{(q(t^*)+1)}(x(t^*), t^*) > 0$ whenever $q(t^*) + 1$ is odd.

Besides it follows from the above definitions that the immobile index set T_* can be presented as follows: $T_* = \{t \in T : q(t) > -1\}.$
To simplify the further laying out, we make the following assumption (see (Kostyukova, 1988)).

Assumption 2. Suppose that $q(t_*) = q(t^*) = -1.$

Note that Assumption 2 is not restrictive and all the results of the paper can be applied to the problems that do not satisfy it. See, for example, (Kostyukova, 1988) for the analogous situations in linear SIP.
The set of immobile indices T_* as well as the concrete values of their immobility orders $q(t), t \in T_*$, depend only on the feasible set X of problem (2.1) and do not depend on a concrete choice of some feasible $x \in X$.
In what follows we will show that the set of immobile indices and the values of their immobility orders are objective and important characteristics of the feasible set X in problem (2.1) that make it possible to formulate optimality conditions for the convex SIP problem (with an infinite number of constraints) in terms of optimality conditions for a certain NLP problem (with a finite number of constraints) without any CQC. As it was mentioned in (Jongen et al., 1987), such the results are very important since they permit to derive sufficient conditions for local optimality.

3 The algorithm of determination of immobility orders

Consider the convex SIP problem in the form (2.1) with the feasible set X defined in (2.2). Suppose that Assumptions 1 and 2 are satisfied for this problem. Choose any $\bar{x} \in X$ satisfying Assumption 1. Consider the corresponding finite set of active indices $T_a(\bar{x}) = \{t_i, i \in I\}$, where $I = I(\bar{x}) = \{1, 2, \ldots, \bar{p}\}$, $\bar{p} = p(\bar{x}) < \infty$. The algorithm described below calculates the immobility orders $q(t)$ for all $t \in T$.

3.1 Description of the DIO algorithm

If $I = \emptyset$, then the algorithm stops resulting in $q(t) = -1,\ t \in T$. Suppose that $I \neq \emptyset$. Let $k = 0$ and $q_i^{(0)} = -1,\ \forall i \in I.$

The <u>k-th iteration</u> starts with a set of numbers $q_i^{(k)}, i \in I,$ constructed on the previous iteration of the algorithm. For any $i \in I$, either $q_i^{(k)}$ is odd or $q_i^{(k)} = -1$.

Introduce the sets

$$X_i^{(k)} = \{z \in R^n : f^{(s)}(z, t_i) = 0,\ s \in N(q_i^{(k)}),\ f^{(q_i^{(k)}+1)}(z, t_i) \le 0\}, i \in I; \tag{3.1}$$

$$X^{(k)} = \bigcap_{i \in I} X_i^{(k)}. \tag{3.2}$$

For each $i \in I$, solve the nonlinear programming problem

$$f_i^{(k)}(z) = f^{(q_i^{(k)}+1)}(z,\ t_i) \longrightarrow \min_z, \text{ s.t. } z \in X^{(k)}. \tag{3.3}$$

It will be proved later (Lemma 3.1) that $X \subset X^{(k)}$. Therefore, $X^{(k)} \neq \emptyset$ and either problem (3.3) admits an optimal solution, or its objective function $f_i^{(k)}(z)$ is not bounded below in the feasible set $X^{(k)}$.

Denote by $x^{(i)}$ an optimal solution of problem (3.3) in the case such a solution exists. Otherwise, denote by $x^{(i)}$ any feasible solution of problem (3.3) that satisfies the inequality $f_i^{(k)}(x^{(i)}) < 0$.

Consider the set $I^{(k)} := \{i \in I :\ f_i^{(k)}(x^{(i)}) = 0\}$.

If $I^{(k)} \neq \emptyset$, then set:

$$q_i^{(k+1)} = q_i^{(k)} + 2,\ i \in I^{(k)};\quad q_i^{(k+1)} = q_i^{(k)},\ i \in I \backslash I^{(k)}, \tag{3.4}$$

and pass to the next iteration with $k := k + 1$. If $I^{(k)} = \emptyset$, then the algorithm stops resulting in the following values of $q(t), t \in T$:

$$q(t_i) = q_i^{(k)},\ i \in I;\quad q(t) = -1,\ t \in T \backslash T_a(\bar{x}). \tag{3.5}$$

3.2 Justification of DIO algorithm

First of all, note that the DIO algorithm is finite if Assumption 1 is satisfied. Indeed, if we denote by $k_* \in \mathbb{N}$ the number of the iterations of the algorithm, it is easy to verify that the following estimation is true: $k_* \leq \sum\limits_{i\in I} \frac{\rho(\bar{x},t_i)+1}{2}$. Here $\bar{x} \in X$ is a vector that was chosen at the beginning of the algorithm.

Now we will demonstrate that the mapping $q(t), t \in T,$ constructed by the DIO algorithm correctly determines the immobility orders of all the indices of the set T.

Since k_* introduced above can be considered as the number of the algorithm last iteration (i.e., the number of the iteration where the algorithm has stopped), we obtain by (3.5)

$$\begin{aligned} & q_i = q(t_i) = q_i^{(k_*)} \quad \text{for } \ i \in I, \\ & q(t) = -1 \quad \text{for } \ t \in T \backslash T_a(\bar{x}). \end{aligned} \tag{3.6}$$

Lemma 3.1. *On the iterations of the DIO algorithm, the following inclusion is satisfied:*

$$X \subset X^{(v)} = \bigcap_{i\in I} X_i^{(v)}, \tag{3.7}$$

where $v = 0, \dots, k_*$.

Proof. Let us prove the lemma by induction on v. Due to (2.2), (3.1) we have $X \subset X_i^{(0)}$ for any $i \in I$. Then inclusion (3.7) is valid for $v = 0$.

Assume that (3.7) is satisfied for $v = k \geq 0, \ k < k_*$, i.e.

$$X \subset X_i^{(k)}, \ i \in I. \tag{3.8}$$

Let us prove (3.7) for $v = k + 1$. From (3.4) it follows that $q_i^{(k+1)} = q_i^{(k)}, \ i \in I \backslash I^{(k)}$. Then (3.1) and (3.8) yield

$$X \subset X_i^{(k+1)} = X_i^{(k)}, \ i \in I \backslash I^{(k)}. \tag{3.9}$$

Suppose that $f^{(q_{i_*}^{(k)}+1)}(x^*, t_{i_*}) \neq 0$ for some $i_* \in I^{(k)}$ and $x^* \in X$. Then, evidently, $f^{(q_{i_*}^{(k)}+1)}(x^*, \ t_{i_*}) < 0$ and, taking into consideration (3.8), we obtain

$$x^* \in X \subset X^{(k)}, \ \ f_{i_*}^{(k)}(x^*) = f^{(q_{i_*}^{(k)}+1)}(x^*, \ t_{i_*}) < 0. \tag{3.10}$$

Since $i_* \in I^{(k)}$, we have $f_{i_*}^{(k)}(x^{(i_*)}) = 0$ for the feasible solution $x^{(i_*)}$ found on k-th iteration of the algorithm. Therefore $x^{(i_*)}$ is the optimal solution of problem (3.3) with $i = i_*$. The optimality of $x^{(i_*)}$ contradicts with the existence of x^* satisfying (3.10) and, thus, we conclude that

$$f^{(q_i^{(k)}+1)}(z, \ t_i) = 0, \quad \forall z \in X, \ \forall i \in I^{(k)}. \tag{3.11}$$

Recall that, by construction, all the values $q_i^{(k)}+1$ are even. Then, due to the constraints of the SIP problem (2.1) and to Assumption 2, equalities (3.11) imply

$$f^{(q_i^{(k)}+2)}(z,\ t_i)=0,\ \ f^{(q_i^{(k)}+3)}(z,\ t_i)\leq 0\quad \forall z\in X,\ \forall i\in I^{(k)}.$$

From the relations above together with (3.1), (3.4), and (3.11) it follows

$$X\subset X_i^{(k+1)},\ i\in I^{(k)}. \tag{3.12}$$

From (3.9) and (3.12) we have $X\subset X_i^{(k+1)}$ for all $i\in I$. Then (3.7) is satisfied for $v=k+1$ and the proof of the lemma is complete. □

Note that the DIO algorithm is constructed in such a way that the relations

$$\begin{gathered} f^{(s)}(x^{(j)},\ t_i)=0,\ s\in N(q_i),\ \ f^{(q_i+1)}(x^{(j)},\ t_i)\leq 0,\ i,j\in I;\\ f^{(q_i+1)}(x^{(i)},\ t_i)<0,\ i\in I, \end{gathered} \tag{3.13}$$

hold true, t_i and $q_i,\ i\in I$, being the indices from $T_a(\bar{x})$ and the corresponding immobility orders of the convex SIP problem (2.1). Here and further we denote by $x^{(i)},\ i\in I$, the feasible solutions of problem (3.3) obtained on the last iteration of the DIO algorithm.

Let us prove now that relations (3.13) are valid also for convex combinations of $x^{(i)},\ i\in I$, i.e. for any vector

$$y=\sum_{i\in I}\bar{\alpha}_i\, x^{(i)} \tag{3.14}$$

such that

$$\bar{\alpha}_i>0,\ i\in I,\ \ \sum_{i\in I}\bar{\alpha}_i=1. \tag{3.15}$$

Lemma 3.2. *Let y satisfy (3.14), (3.15). Then*

$$f^{(s)}(y,\ t_i)=0,\ s\in N(q_i),\ \ f^{(q_i+1)}(y,\ t_i)<0,\ i\in I, \tag{3.16}$$

where $q_i,\ i\in I$, are determined by (3.6).

Proof. Let $\rho_i:=\rho(y,t_i),\ i\in I$, where $\rho(y,t)$ is defined as in (2.3). Then

$$f^{(s)}(y,\ t_i)=0,\ s\in N(\rho_i),\quad f^{(\rho_i+1)}(y,\ t_i)\neq 0,\ i\in I, \tag{3.17}$$

and the statement of the lemma will be proved if we show that for any $i\in I$ it is satisfied $\rho_i=q_i$.

1) Let us prove, first, that

$$\rho_i\leq q_i,\ i\in I. \tag{3.18}$$

Arguing by contradiction, suppose that there exists $i_0 \in I$, such that $\rho_{i_0} > q_{i_0}$. Then from (3.17) we have

$$f^{(s)}(y,\ t_{i_0}) = 0,\ s \in N(q_{i_0}+1). \tag{3.19}$$

Let $\Delta t \neq 0$ be such that $t_i + \Delta t \in T,\ \forall i \in I$. Since $f(x,t)$ is convex w.r.t. x, then we can write

$$f(y,\ t_i+\Delta t) \leq \sum_{j\in I} \bar{\alpha_j}\, f(x^{(j)}, t_i + \Delta t), \quad \forall i \in I, \tag{3.20}$$

where y and $\bar{\alpha}_j,\ j \in I$, satisfy (3.14) and (3.15). It is evident that for any $z \in X$ and any $i \in I$, the Taylor expansion of the order $l,\ l \in \mathbb{N}$, of the function $f(z,t)$ in the neighborhood of t_i can be written in the form

$$f(z, t_i + \Delta t) = f(z,t_i) + f^{(1)}(z,t_i)\Delta t + \cdots + \frac{1}{(l+1)!} f^{(l+1)}(z,t_i)\Delta t^{l+1} + o(\Delta t^{l+1}).$$

Given $i \in I$, let us apply the Taylor expansions of the order q_i to the functions in (3.20). Then we get

$$\begin{aligned} f(y,t_i) + f^{(1)}(y,t_i)\Delta t + \cdots + \tfrac{1}{(q_i+1)!} f^{(q_i+1)}(y,t_i)\Delta t^{q_i+1} + o(\Delta t^{q_i+1}) \leq \\ \leq \sum_{j\in I} \bar{\alpha_j}\Big(f(x^{(j)},t_i) + f^{(1)}(x^{(j)},t_i)\Delta t + \cdots + \\ + \tfrac{1}{(q_i+1)!} f^{(q_i+1)}(x^{(j)},t_i)\Delta t^{q_i+1} + o(\Delta t^{q_i+1})\Big). \end{aligned} \tag{3.21}$$

Suppose $i = i_0$ in (3.21). If take into account (3.13) and (3.19), we obtain

$$o(\Delta t^{q_{i_0}+1}) \leq \sum_{j\in I} \bar{\alpha_j}(f^{(q_{i_0}+1)}(x^{(j)}, t_{i_0})\Delta t^{q_{i_0}+1} + o(\Delta t^{q_{i_0}+1})). \tag{3.22}$$

Since $q_{i_0}+1$ is even, then $\Delta t^{q_{i_0}+1} > 0$. Divide (3.22) by $\Delta t^{q_{i_0}+1}$ and let $\Delta t \to 0$. Then, taking into account (3.13), we get the contradictory system of the inequalities

$$0 \leq \sum_{j\in I} \bar{\alpha_j} f^{(q_{i_0}+1)}(x^{(j)},\ t_{i_0}) \leq \bar{\alpha}_{i_0} f^{(q_{i_0}+1)}(x^{(i_0)},\ t_{i_0}) < 0$$

that proves (3.18).

2) Now, let us strengthen (3.18) and show that $\rho_i = q_i, \forall i \in I$.
Suppose, $I_* := \{i \in I : \rho_i < q_i\} \neq \emptyset$. To obtain a contradiction, it suffices to demonstrate that no one of the following hypotheses is true:

a) $\exists\ i_0 \in I_*$, such that ρ_{i_0} is even; b) ρ_i is odd, $\forall\ i \in I_* \neq \emptyset$.

First of all, let us substitute each functions in (3.20) by its expansion of the order $\rho_i + 1$ in the neighborhood of t_i. With respect to (3.13), (3.17), and (3.18), we have

$$f^{(\rho_i+1)}(y,t_i)\Delta t^{\rho_i+1} + o(\Delta t^{\rho_i+1}) \leq 0,\ \forall i \in I. \tag{3.23}$$

Consider the hypothesis a). Suppose, $i = i_0$ in (3.23). Divide the inequality obtained by $\Delta t^{\rho_{i_0}+1}$ and let, first, $\Delta t \to +0$ and after, $\Delta t \to -0$. Since $\Delta t^{\rho_{i_0}+1} > 0$ for $\Delta t > 0$ and $\Delta t^{\rho_{i_0}+1} < 0$ for $\Delta t < 0$, then the values of the limits obtained can be estimated as follows:

$$\lim_{\Delta t \to +0}\big(f^{(\rho_{i_0}+1)}(y, t_{i_0}) + \frac{o(\Delta t^{\rho_{i_0}+1})}{\Delta t^{\rho_{i_0}+1}}\big) \le 0, \ \lim_{\Delta t \to -0}(f^{(\rho_{i_0}+1)}(y, t_{i_0}) + \frac{o(\Delta t^{\rho_{i_0}+1})}{\Delta t^{\rho_{i_0}+1}}) \ge 0,$$

wherefrom

$$f^{(\rho_{i_0}+1)}(y, t_{i_0}) \le 0, \quad f^{(\rho_{i_0}+1)}(y, t_{i_0}) \ge 0.$$

The latest two inequalities can be satisfied simultaneously if and only if $f^{(\rho_{i_0}+1)}(y,\ t_{i_0}) = 0$ that contradicts with (3.17). Therefore, the hypothesis a) is false.

Now, consider hypothesis b). From (3.23), taking into account $\Delta t^{\rho_i+1} > 0, i \in I_*$, we get $f^{(\rho_i+1)}(y,\ t_i) \le 0, i \in I_*$, wherefrom with respect to the inequality in (3.17) we obtain

$$f^{(\rho_i+1)}(y,\ t_i) < 0, \quad i \in I_*. \tag{3.24}$$

It was assumed above that ρ_i is odd and $\rho_i < q_i$ for $\forall i \in I_* \ne \emptyset$.
Given $i \in I_*$, let $k_i \in \{0,\ 1,\ \ldots,\ k_* - 1\}$ be an index such that $i \in I^{(k_i)},\ q_i^{(k_i)} = \rho_i$, $q_i^{(k_i+1)} > \rho_i$. Denote

$$k_0 := \min_{i \in I_*} k_i = k_{i_*}. \tag{3.25}$$

On the k_0-th iteration of the DIO algorithm the nonlinear problem (3.3) takes the form

$$f_{i_*}^{(k_0)}(z) = f^{(\rho_{i_*}+1)}(z,\ t_{i_*}) \longrightarrow \text{min}, \quad \text{s.t. } z \in X^{(k_0)}. \tag{3.26}$$

As $i_* \in I^{(k_0)}$, we can conclude that the problem above has an optimal solution x^* satisfying

$$f_{i_*}^{(k_0)}(x^*) = f^{(\rho_{i_*}+1)}(x^*,\ t_{i_*}) = 0. \tag{3.27}$$

From (3.25) it follows $q_i^{(k_0)} \le \rho_i$, $i \in I$. Then, due to (3.17), we obtain

$$f^{(s)}(y,\ t_i) = 0,\ s \in N(q_i^{(k_0)}),\ i \in I. \tag{3.28}$$

Finally, let us show that

$$y \in X^{(k_0)}. \tag{3.29}$$

According to (3.1) and (3.28), it is sufficient to prove that the following inequalities

$$f^{(q_i^{(k_0)}+1)}(y,\ t_i) \le 0,\ i \in I, \tag{3.30}$$

are valid. By the DIO algorithm, for all $i \in I$, it is satisfied $q_i^{(k_0)} \leq \rho_i \leq q_i$. Then for any $i \in I$, substituting $q_i^{(k_0)}$ instead of q_i in (3.21) and taking into account (3.28), we obtain

$$f^{(q_i^{(k_0)}+1)}(y, t_i)\Delta t^{q_i^{(k_0)}+1} + o(\Delta t^{q_i^{(k_0)}+1}) \leq$$
$$\leq \sum_{j\in I} \bar{\alpha}_j \Big(f^{(q_i^{(k_0)}+1)}(x^{(j)}, t_i)\Delta t^{q_i^{(k_0)}+1} + o(\Delta t^{q_i^{(k_0)}+1})\Big).$$

Note that $q_i^{(k_0)}$ is odd here. Dividing the latest obtained inequality by $\Delta t^{q_i^{(k_0)}+1} > 0$ and taking the limit as $\Delta t \to 0$, we obtain

$$f^{(q_i^{(k_0)}+1)}(y,\ t_i) \leq \sum_{j\in I} \bar{\alpha}_j f^{(q_i^{(k_0)}+1)}(x^{(j)},\ t_i)$$

that, together with the last two groups of the inequalities in (3.13), implies (3.30) and, consequently, (3.29). From (3.24), (3.25) we have $f^{(\rho_{i_*}+1)}(y,\ t_{i_*}) < 0$ that, taking into account (3.29) and (3.27), contradicts the optimality of x^* in (3.26) and we can conclude that the hypothesis b) is false as well. □

Corollary 3.3. *Let y satisfy (3.14), (3.15). Then there exists $\varepsilon > 0$ such that the following inequalities are valid*

$$f(y,t) \leq 0,\ \forall t \in [t_i - \varepsilon,\ t_i + \varepsilon],\ \forall i \in I. \tag{3.31}$$

Proof. If y is feasible in problem (2.1) then inequalities (3.31) are trivially satisfied. Suppose that y is not feasible. Lemma 3.2 states that relations (3.16) are valid for the given y. Then $f(y,\ t_i) < 0$ for all $i \in I$, such that $q_i = -1$. Taking into account the sufficient smoothness of the function $f(y,t)$, we can extend this result to some neighborhood of t_i:

$$\forall i \in I \text{ with } \ q_i = -1, \quad \exists \varepsilon_i > 0: \ f(y,t) < 0, \quad t \in [t_i - \varepsilon_i,\ t_i + \varepsilon_i]. \tag{3.32}$$

If $q_i > -1$ for some $i \in I$, then q_i is odd, evidently. From (3.16) it follows that the correspondent t_i is a local maximizer of the continuous function $f(y,t)$ and that $f(y,\ t_i) = 0$. Therefore, we can state:

$$\forall i \in I \text{ with } \ q_i > -1, \quad \exists \varepsilon_i > 0: \ f(y,t) \leq 0, \quad t \in [t_i - \varepsilon_i,\ t_i + \varepsilon_i]. \tag{3.33}$$

Then (3.31) follows immediately from (3.32) and (3.33) if suppose $\varepsilon := \min\limits_{i\in I} \varepsilon_i$. □

Theorem 3.4. *Given $t \in T$, the value $q(t)$ constructed by the DIO algorithm satisfies Definition 2.2.*

Proof. Consider any $t \in T$. Let us prove, first, that $q(t)$ satisfies (2.4).
If $q(t) = -1$, then $N(q(t)) = \emptyset$ and there is nothing to prove.
If $q(t) > -1$, then, by the algorithm, there exists $i \in I$ such that $t = t_i$. According to (2.3), for any $i \in I$ and any $z \in X$ we denote by $\rho = \rho(z, t_i) \in \{-1, 0, 1, \dots\}$ a number such that

$$f^{(s)}(z, t_i) = 0,\ s \in N(\rho),\ f^{(\rho+1)}(z, t_i) \neq 0. \tag{3.34}$$

Let us show that

$$\rho(z, t_i) \geq q_i,\ \forall z \in X,\ \forall i \in I, \tag{3.35}$$

where $q_i = q(t_i)$. Arguing by contradiction, suppose $\bar{\rho} = \rho(\bar{z}, t_{i_1}) < q_{i_1}$ for some $i_1 \in I$ and some $\bar{z} \in X$. Denote by $\bar{k}$, $0 \leq \bar{k} < k_*$, the number of the iteration where $q_{i_1}^{(\bar{k})} = \bar{\rho},\ q_{i_1}^{(\bar{k}+1)} = \bar{\rho} + 2$. By the DIO algorithm, there exists $x^{(i_1)} \in X^{(\bar{k})}$ such that

$$0 = f_{i_1}^{(\bar{k})}(x^{(i_1)}) = f^{(\bar{\rho}+1)}(x^{(i_1)}, t_{i_1}) = \min_{x \in X^{(\bar{k})}} f^{(\bar{\rho}+1)}(x, t_{i_1}). \tag{3.36}$$

On the other hand, as $\bar{z} \in X$, then from Definition 2.1 and Assumption 2 we conclude that $\bar{\rho}$ is odd. Therefore, the inequality in (3.34) takes the form

$$f^{(\bar{\rho}+1)}(\bar{z}, t_{i_1}) < 0. \tag{3.37}$$

By Lemma 3.1, $\bar{z} \in X \subset X^{(\bar{k})}$. However, inequality (3.37) contradicts (3.36). Thus, (3.35) is valid and together with (3.34) it yields (2.4).
Let us show now that there exists $\tilde{x} = \tilde{x}(t)$ satisfying (2.5). Recall that the DIO algorithm starts with the index set I in the form $I = I(\bar{x})$ for some $\bar{x} \in X$. For any y given by (3.14), (3.15) and any $\alpha \in [0, 1]$, we consider

$$x(\alpha) = \alpha \bar{x} + (1 - \alpha) y. \tag{3.38}$$

From the convexity of the function $f(x, t)$ w.r.t. x we have

$$f(x(\alpha),\ t) \leq \alpha f(\bar{x},\ t) + (1 - \alpha) f(y,\ t) = f(y, t) + \alpha(f(\bar{x}, t) - f(y, t)),\ \forall t \in T.$$

Let $\alpha(t)$ be a function defined in T as follows:

$$\alpha(t) = \begin{cases} 0, & \text{if } f(y,\ t) \leq 0, \\ \dfrac{f(y, t)}{f(y,\ t) - f(\bar{x},\ t)}, & \text{if } f(y,\ t) > 0. \end{cases}$$

Let us prove that $\alpha(t) < 1,\ \forall t \in T$. Indeed, from Corollary 3.3 it follows

$$\exists\, \varepsilon > 0 :\ \ \alpha(t) = 0,\ t \in [t_i - \varepsilon,\ t_i + \varepsilon],\ \ \forall i \in I.$$

Let $T_\varepsilon := T \setminus \bigcup\limits_{i\in I}[t_i-\varepsilon,\ t_i+\varepsilon]$. By construction, $f(\bar{x},\ t)<0,\, t\in T_\varepsilon$. Then

$$f(\bar{x},\ t)\le -\delta,\ t\in T_\varepsilon, \tag{3.39}$$

for $\delta := \min\limits_{t\in T_\varepsilon}|f(\bar{x},\ t)|>0$.

Consider the subset $T_\varepsilon^+\subseteq T_\varepsilon$ defined as follows: $T_\varepsilon^+ = \{t\in T_\varepsilon:\ \ f(y,t)>0\}$. If $T_\varepsilon^+=\emptyset$, then $\alpha(t)=0,\ \forall t\in T$, and the statement is proved.
Now suppose $T_\varepsilon^+\neq\emptyset$. By construction, for any $t\in T\setminus T_\varepsilon^+$ we have $\alpha(t)=0$. Let $\delta_0 := \max\limits_{t\in\ T_\varepsilon^+} f(y,t)<+\infty$. Evidently, $\delta_0>0,\ \ \min\limits_{t\in T_\varepsilon^+}|f(\bar{x},t)|\ge\delta>0$. Then, taking into consideration (3.39), we obtain for $t\in T_\varepsilon^+$

$$\alpha(t)=\frac{f(y,t)}{f(y,t)-f(\bar{x},t)}=\frac{1}{1-\frac{f(\bar{x},t)}{f(y,t)}}\le\frac{1}{1+\delta/\delta_0}<1.$$

Let θ_* be the maximal value of the function $\alpha(t)$ constructed above

$$\theta_* := \max_{t\in T}\alpha(t). \tag{3.40}$$

Obviously, $0\le\theta_*<1$. Choose some fixed parameter α_0 from the interval $]\theta_*,1[$ and set $\tilde{x}:=x(\alpha_0)$ where $x(\alpha_0)$ is calculated by (3.38). By the same method that was used in the proof of Lemma 3.2, we can show that $\tilde{x}\in X$ and $f^{(q(t)+1)}(\tilde{x},t)<0,\forall t\in T$. Here we just have to suppose $I:=\{1,2\}$, $x^{(1)}:=\bar{x},\ x^{(2)}:=y,\ \bar{\alpha}_1:=\alpha_0,\ \bar{\alpha}_2:=1-\alpha_0$ and consider relations (3.16) together with

$$f^{(s)}(\bar{x},\ t_i)=0,\ s\in N(q_i),\quad f^{(q_i+1)}(\bar{x},\ t_i)\le 0,\ i\in I;$$
$$f(\bar{x},\ t)<0,\ t\in T\setminus T_a(\bar{x}),$$

instead of (3.13). This completes the theorem proof. □

3.3 Properties of DIO algorithm and some useful remarks

It follows from Theorem 3.4 that the following Proposition is true.

Proposition 3.5. *There always exists a vector $\tilde{x}\in X$ such that*

$$f^{(q(t)+1)}(\tilde{x},t)<0,\forall t\in T.$$

From the proof of Theorem 3.4 one can easily see that the vector $\tilde{x}$ can be constructed in the form of some linear combination of the vectors $x^{(i)}$, $i\in I$, obtained on the final iteration of the DIO algorithm, and of the vector $\bar{x}$, that was the starting vector on the iterations of the algorithm (see (3.38), (3.14)). The Proposition 3.5 shows that in Definition 2.2 we can always suppose $x(t)=\tilde{x},\ \forall t\in T.$
The following proposition follows from Definition 2.1 and Proposition 3.5.

Proposition 3.6. *The constraints of problem (2.1) satisfy the Slater condition if and only if the set of immobile indices T_* is empty.*

Remind that constraints of problem (2.1) are said to satisfy the Slater condition if there exists a vector $\hat{x} \in X$ such that $f(\hat{x}, t) < 0,\ t \in T$.
Suppose that the DIO algorithm has stopped on the iteration with the number k^*. Then $q(t_i) = q_i^{(k_*)}, i \in I = \{1, ..., p(\bar{x})\}$. The following proposition states that the auxiliary NLP problems (3.3) that are solved on the iterations of the algorithm are convex.

Proposition 3.7. *For any $k \in \{0, \dots, k_*\}$, the set $X^{(k)}$ constructed on the correspondent iteration of the DIO algorithm, is convex and the functions $f^{(q_i^{(k)}+1)}(x, t_i),\ \ i \in I$, are convex w.r.t. x in $X^{(k)}$.*

Proof. From the DIO algorithm we have $q_i^{(k+1)} \geq q_i^{(k)}$, hence the following inclusions are valid:

$$X_i^{(k+1)} \subseteq X_i^{(k)},\ i \in I,\ k \in \{0, \dots, k_* - 1\}. \tag{3.41}$$

Taking into account (3.2), we obtain

$$X^{(k+1)} \subseteq X^{(k)},\ k \in \{0, \dots, k_* - 1\}. \tag{3.42}$$

We will prove the statement of the proposition by induction. First, suppose that $k = 0$. All the sets

$$X_i^{(0)} = \{x \in \mathbb{R}^n :\ f(x, t_i) \leq 0\},\ i \in I,$$

are convex as $f(x, t)$ is convex w.r.t. x. Then from (3.2) we conclude that $X^{(0)}$ is convex too, being the intersection of the convex sets. By construction, $q_i^{(0)} = -1$ for any $i \in I$. Then the functions $f^{(0)}(x, t_i) = f(x, t_i)$ are convex w.r.t. x in $\mathbb{R}^n$ and, therefore they are convex in the set $X^{(0)}$. Thus proposition is valid for $k = 0$.
Suppose now that the statement of the proposition is true for all $k < k_*$. Let us prove the theorem for $k + 1$. By the algorithm, $q_i^{(k+1)} = q_i^{(k)}$ for all $i \in I \backslash I^{(k)}$. Consequently,

$$X_i^{(k+1)} = X_i^{(k)},\ \forall i \in I \backslash I^{(k)}. \tag{3.43}$$

Consider now any $i \in I^{(k)}$. Then

$$\min_{x \in X^{(k)}} f^{(q_i^{(k)}+1)}(x, t_i) = 0,\ \forall i \in I^{(k)}. \tag{3.44}$$

Let us show that the set

$$S_i := \{x \in X^{(k)} : f^{(q_i^{(k)}+1)}(x, t_i) = 0, f^{(q_i^{(k)}+2)}(x, t_i) = 0, f^{(q_i^{(k)}+3)}(x, t_i) \leq 0\}$$

is convex. If $x, y \in S_i$, then $x, y \in X^{(k)}$, and

$$f^{(q_i^{(k)}+1)}(x,t_i) = 0, f^{(q_i^{(k)}+1)}(y,t_i) = 0, f^{(q_i^{(k)}+2)}(x,t_i) = 0, f^{(q_i^{(k)}+2)}(y,t_i) = 0, \quad (3.45)$$

$$f^{(q_i^{(k)}+3)}(x,t_i) \le 0, \ f^{(q_i^{(k)}+3)}(y,t_i) \le 0. \quad (3.46)$$

Denote $x(\alpha) = \alpha x + (1-\alpha)y$. By the assumption of induction, the set $X^{(k)}$ is convex and the functions $f^{(q_i^{(k)}+1)}(x,t_i)$ are convex in $X^{(k)}$ for any $i \in I$. Consequently,

$$x(\alpha) \in X^{(k)}, \alpha \in [0,1], \quad (3.47)$$

and

$$f^{(q_i^{(k)}+1)}(x(\alpha),t_i) \le \alpha f^{(q_i^{(k)}+1)}(x,t_i) + (1-\alpha) f^{(q_i^{(k)}+1)}(y,t_i), \quad \alpha \in [0,1], \ i \in I^{(k)},$$

wherefrom, taking into account (3.45) we obtain

$$f^{(q_i^{(k)}+1)}(x(\alpha),t_i) \le 0, \quad i \in I^{(k)}, \ \alpha \in [0,1].$$

The latest inequalities together with (3.44) and (3.47) imply

$$f^{(q_i^{(k)}+1)}(x(\alpha),t_i) = 0, \quad i \in I^{(k)}, \ \alpha \in [0,1]. \quad (3.48)$$

Furthermore, as $x, \ x(\alpha)$, and y belong to $X^{(k)}$, then the following equalities hold true:

$$f^{(s)}(x,t_i) = f^{(s)}(x(\alpha),t_i) = f^{(s)}(y,t_i) = 0, \ s \in \mathcal{N}(q_i^{(k)}), \ i \in I, \ \alpha \in [0,1]. \quad (3.49)$$

From convexity of $f(x,t)$ w.r.t. x and from its sufficient smoothness, we conclude that

$$f(x(\alpha), t_i + \Delta t) \le \alpha f(x, t_i + \Delta t) + (1-\alpha) f(y, t_i + \Delta t), \ i \in I, \ \alpha \in [0,1], \quad (3.50)$$

and that for any $m > 0$, $z \in \mathbb{R}^n$, the following Taylor expansion is valid:

$$f(z, t_i + \Delta t) = \sum_{l=0}^{m} \frac{1}{l!} f^{(l)}(z,t_i) \Delta t^l + o(\Delta t^m), \ i \in I^{(k)}. \quad (3.51)$$

Substituting (3.51) in (3.50) with $z = x(\alpha), \ z = x, \ z = y$, and $m = q_i^{(k)} + 2$ and taking into consideration (3.45), (3.48), (3.49), we obtain

$$f^{(q_i^{(k)}+2)}(x(\alpha),t_i)\Delta t^{(q_i^{(k)}+2)} + o(\Delta t^{(q_i^{(k)}+2)}) \le \alpha f^{(q_i^{(k)}+2)}(x,t_i)\Delta t^{(q_i^{(k)}+2)} +$$
$$+(1-\alpha) f^{(q_i^{(k)}+2)}(y,t_i)\Delta t^{(q_i^{(k)}+2)} + o(\Delta t^{(q_i^{(k)}+2)}), \ i \in I^{(k)}, \ \alpha \in [0,1].$$

Divide the latest inequality by $\Delta t^{(q_i^{(k)}+2)}$ and pass to limit, taking into account that all $q_i^{(k)}$ are odd:

$$\lim_{\Delta t \to +0} \frac{f^{(q_i^{(k)}+2)}(x(\alpha),t_i)\Delta t^{(q_i^{(k)}+2)} + o(\Delta t^{(q_i^{(k)}+2)})}{\Delta t^{(q_i^{(k)}+2)}} \le \lim_{\Delta t \to +0} \frac{\alpha f^{(q_i^{(k)}+2)}(x,t_i)\Delta t^{(q_i^{(k)}+2)}}{\Delta t^{(q_i^{(k)}+2)}} +$$

$$+\lim_{\Delta t\to +0}\frac{(1-\alpha)f^{(q_i^{(k)}+2)}(y,t_i)\Delta t^{(q_i^{(k)}+2)}+o(\Delta t^{(q_i^{(k)}+2)})}{\Delta t^{(q_i^{(k)}+2)}},$$

$$\lim_{\Delta t\to -0}\frac{f^{(q_i^{(k)}+2)}(x(\alpha),t_i)\Delta t^{(q_i^{(k)}+2)}+o(\Delta t^{(q_i^{(k)}+2)})}{\Delta t^{(q_i^{(k)}+2)}}\geq \lim_{\Delta t\to -0}\frac{\alpha f^{(q_i^{(k)}+2)}(x,t_i)\Delta t^{(q_i^{(k)}+2)}}{\Delta t^{(q_i^{(k)}+2)}}+$$

$$+\lim_{\Delta t\to -0}\frac{(1-\alpha)f^{(q_i^{(k)}+2)}(y,t_i)\Delta t^{(q_i^{(k)}+2)}+o(\Delta t^{(q_i^{(k)}+2)})}{\Delta t^{(q_i^{(k)}+2)}}.$$

Hence,

$$f^{(q_i^{(k)}+2)}(x(\alpha),t_i)=\alpha f^{(q_i^{(k)}+2)}(x,t_i)+(1-\alpha)f^{(q_i^{(k)}+2)}(y,t_i), i\in I^{(k)},\alpha\in[0,1]. \quad (3.52)$$

Substituting (3.45) in (3.52), we get

$$f^{(q_i^{(k)}+2)}(x(\alpha),t_i)=0,\ i\in I^{(k)},\ \alpha\in[0,1]. \quad (3.53)$$

Now, let us substitute in (3.50) the expansion (3.51) with $m=q_i^{(k)}+3$ for $x=x(\alpha),\ x=x,\ x=y$. Then, considering (3.45), (3.48), and (3.53), we obtain

$$f^{(q_i^{(k)}+3)}(x(\alpha),t_i)\Delta t^{(q_i^{(k)}+3)}+o(\Delta t^{(q_i^{(k)}+3)})\leq \alpha f^{(q_i^{(k)}+3)}(x,t_i)\Delta t^{(q_i^{(k)}+3)}+$$

$$+(1-\alpha)f^{(q_i^{(k)}+3)}(y,t_i)\Delta t^{(q_i^{(k)}+3)}+o(\Delta t^{(q_i^{(k)}+3)}),\ i\in I^{(k)},\ \alpha\in[0,1].$$

Divide the inequality above by $\Delta t^{(q_i^{(k)}+3)}$ and let $\Delta t\longrightarrow 0$. Since $q_i^{(k)}, i\in I^{(k)}$, are odd, then

$$f^{(q_i^{(k)}+3)}(x(\alpha),t_i)\leq \alpha f^{(q_i^{(k)}+3)}(x,t_i)+(1-\alpha)f^{(q_i^{(k)}+3)}(y,t_i),\ i\in I^{(k)},\ \alpha\in[0,1]. \quad (3.54)$$

From (3.46) and (3.54) we obtain

$$f^{(q_i^{(k)}+3)}(x(\alpha),t_i)\leq 0,\ i\in I^{(k)},\ \alpha\in[0,1]. \quad (3.55)$$

Relations (3.47), (3.48), (3.53), and (3.55) imply $x(\alpha)\in S_i$ for all $\alpha\in[0,1]$. Therefore, the set S_i is convex for any $i\in I^{(k)}$. With respect to (3.2) and (3.41)-(3.43), we obtain

$$X^{(k+1)}=X^{(k)}\bigcap\left(\bigcap_{i\in I^{(k)}}X_i^{(k+1)}\right)=\bigcap_{i\in I^{(k)}}\left(X^{(k)}\bigcap X_i^{(k+1)}\right)=\bigcap_{i\in I^{(k)}}S_i, \quad (3.56)$$

Taking into account convexity of sets $S_i,\ i\in I^{(k)}$, we conclude that the set $X^{(k+1)}$ is convex. By the DIO algorithm, $q_i^{(k+1)}=q_i^{(k)},\ i\in I\backslash I^{(k)}$. Then the functions

$$f^{(q_i^{(k+1)}+1)}(x,t_i)=f^{(q_i^{(k)}+1)}(x,t_i),\ i\in I\backslash I^{(k)},$$

are convex w.r.t. x in $X^{(k+1)}$ since they are convex in $X^{(k)}$ and (3.42) is true.

Table 1: Iterations of the DIO algorithm in Example 3.4.

k	$q_1^{(k)}$	$f^{(q_1^{(k)}+1)}(x^{(1)},t_1)$	$q_2^{(k)}$	$f^{(q_2^{(k)}+1)}(x^{(2)},t_2)$	$q_3^{(k)}$	$f^{(q_3^{(k)}+1)}(x^{(3)},t_3)$	$I^{(k)}$
0	-1	0	-1	-0.009	-1	0	$\{1,3\}$
1	1	0	-1	-0.009	1	0	$\{1,3\}$
2	3	-154.1681	-1	-0.009	3	-136.043	$\emptyset$

For $i \in I^{(k)}$ we have $q_i^{(k+1)} = q_i^{(k)} + 2$. Hence

$$f^{(q_i^{(k+1)}+1)}(x,t_i) = f^{(q_i^{(k)}+3)}(x,t_i),\ i \in I^{(k)},$$

all these functions being convex w.r.t. x in S_i that follows from inequalities (3.54). Taking into account (3.56), we can state also that these functions are also convex w.r.t. x in $X^{(k+1)}$.
Therefore, we have proved that the statement of the proposition is valid for $k+1$, that concludes the proof. □

Remark 3.1. It follows from the description of the DIO algorithm that it is not necessary to solve problem (3.3) completely. It is enough to show that the initially chosen feasible $\bar{x} \in X$ is optimal in problem (3.3) or (otherwise) to find a feasible $x^{(i)} \in X^{(k)}$ such that $f_i^{(k)}(x^{(i)}) < 0$.

3.4 An example of application of the DIO algorithm

Let us use the DIO algorithm to determine the immobility orders of all indices of the interval T in problem (2.1), where

$$\begin{aligned}
&f(x,t) = 18\big[(t-0.14)^6(t-0.6)^2(t-0.94)^4\big(x_1^2 + (x_2+\tfrac{1}{3})^2 + x_3^2 + (x_4-4)^2 - 1\big)+\\
&+(t-0.14)^4(1-\cos(t-0.6))\sin^4(t-0.94)\big((x_1+x_2+x_3+\tfrac{1}{2})^2 - 1\big)+\\
&+\sin^4(t-0.14)(t-0.94)^2\sin^2(t-0.94)\big((x_2+x_4-3)^4 + 4x_1^2x_3^2 - 1\big)\big],\\
&T = [0,1], \quad x \in \mathbb{R}^4.
\end{aligned}$$

Consider the feasible solution $\bar{x} = (0, 0.5, 0, 3.5)'$. Then $f(\bar{x},\ t) = -(t-0.14)^6(t-0.6)^2\times$ $\times(t-0.94)^4,\ t \in [0,1]$, and, according to the notations used above, we obtain

$$T_a(\bar{x}) = \{0.14,\ 0.6,\ 0.94\},\ I = \{1,\ 2,\ 3\},\ t_1 = 0.14,\ t_2 = 0.6,\ t_3 = 0.94,$$

$$\rho(\bar{x},\ 0.14) = 5,\ \rho(\bar{x},\ 0.6) = 1,\ \rho(\bar{x},\ 0.94) = 3.$$

The results of the iterations of the DIO algorithm are presented in Table 1.

The feasible solutions $x^{(i)}, i \in I$, to problems (3.3) obtained at the latest iteration are

$$x^{(1)} = (-0.0056, -0.4888 - 0.0056, 3.4832)',\ x^{(2)} = (0, 0,\ 0, 3)',$$

$$x^{(3)} = (0.0131, -0.5431,\ 0.0131, 3.7769)'.$$

The immobility orders of the indices $t_i,\ i \in I$, are equal to the values $q_i^{(k_*)},\ i \in I$, from the latest iteration of the algorithm. In our case $k_* = 2$ and $q_1 = q(0.14) = q_1^{(2)} = 3,\ q_2 = q(0.6) = q_2^{(2)} = -1,\ \ q_3 = q(0.94) = q_3^{(2)} = 3$. Thus, the algorithm results in the function $q(t),\ t \in [0,\ 1]$, such that $q(t) = -1,\ t \in [0,\ 0.14) \cup (0.14,\ 0.94) \cup (0.94,\ 1]$; $q(0.14) = q(0.94) = 3$.

Let us now find $\tilde{x}$ that satisfies Definition 2.2. According to the rules described in Theorem 3.4, we have to construct, first, some vector y in the form (3.14), (3.15). If assume, for example, that $\bar{\alpha}_i = 1/3,\ i \in I$, then (approximately)

$$y = \sum_{i \in I} \frac{1}{3} x^{(i)} = (0.0025,\ -0.344,\ 0.0025,\ 3.42)'.$$

The functions $f(\bar{x},\ t)$ (see Fig. 1) and $f(y,\ t)$ (see Fig. 2) are not positive in $T = [0,\ 1]$ and, consequently, the value θ_*, defined in (3.40), is zero.

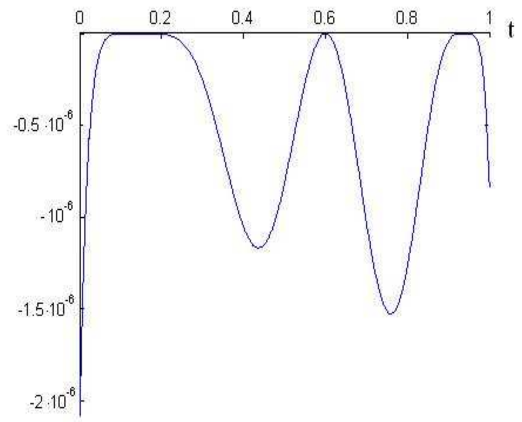

Fig.1. Function $f(\bar{x},t)$, $t \in T$

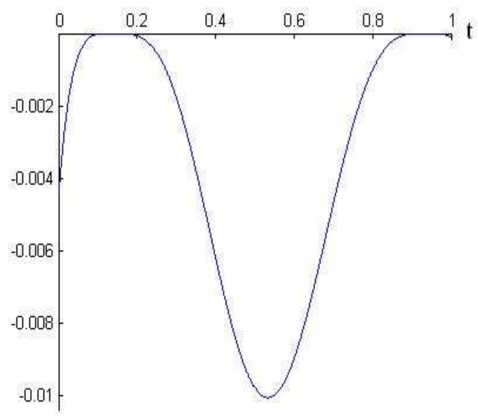

Fig. 2. Function $f(y,t)$, $t \in T$

Now, according to Theorem 3.1, we have to choose some α_0 from the interval $]\theta_*, 1[=]0, 1[$. Suppose, for example, that $\alpha_0 = 0.5$. Then

$$\tilde{x} = \bar{x}(\alpha_0) = \bar{x}(0.5) = (0.00125,\ 0.078,\ 0.00125,\ 3.46)'.$$

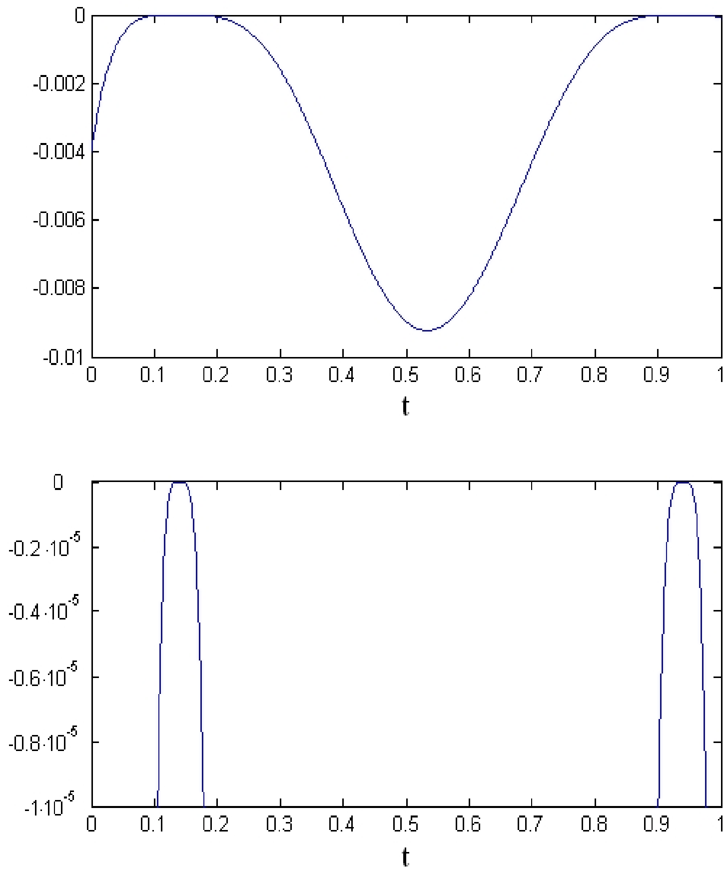

Fig. 3. Function $f(\tilde{x},t)$, $t \in T$

Fig. 3 shows (in two different scales) the graphic of the function $f(\tilde{x},t)$, $t \in [0,1]$. Here $f(\tilde{x},t) < 0,\ t \in [0,\ 1] \setminus \{0.14,\ 0.94\},\ t_1 = 0.14,\ t_3 = 0.94,\ f^{(s)}(\tilde{x},t_i) = 0,\ s \in N(q_i),\ i \in \{1,\ 3\},\ f^{(4)}(\tilde{x},\ 0.14) = -138.2412275,\ \ f^{(4)}(\tilde{x},\ 0.94) = -118.5898889,\ \ f(\tilde{x},\ 0.6) = f^{(0)}(\tilde{x},\ 0.6) = -0.008236$ and, evidently, conditions (2.4) and (2.5) are satisfied in $\tilde{x}$ for all $t \in T$. Thus we have confirmed that the DIO algorithm has correctly determined two immobile indices $t_1 = 0.14$, $t_3 = 0.94$ and their immobility orders $q_1 = q_3 = 3$. Note that the index $t_2 = 0.6$ is not immobile, nevertheless $f(\bar{x},t_2) = 0$.

4 Use of the concepts of immobility orders and immobile indices in a study of convex SIP problems

It was stated in Introduction that the concepts of immobility orders and the immobile indices play an important role in a study of SIP problems. These concepts permit to generalize a lot of classical results in the sense that the analogs of some known results of convex SIP can be obtained without any CQC. In this section we consider two examples of such generalizations.

4.1 Implicit Optimality Criterion

Consider the convex SIP problem (2.1). Suppose that Assumptions 1 and 2 are satisfied. Let $x^0 \in X$, X being the feasible set in (2.1). Consider the corresponding active index set $T_a(x^0)$ and suppose that $p := |T_a(x^0)| < \infty$. Then the set $T_a(x^0)$ can be written in the form $T_a(x^0) = \{t_j^0, j = 1, \ldots, p\}$. Denote $q_j = q(t_j^0),\ j = 1, \ldots, p$.
Using the notations above, form the following nonlinear programming (NLP) problem:

$$\begin{gathered} c(x) \longrightarrow \min, \\ \text{s.t.} \quad f^{(s)}(x, t_j^0) = 0,\ s \in N(q_j), \\ f^{(q_j+1)}(x, t_j^0) \leq 0, \quad j = 1, \ldots, p. \end{gathered} \tag{4.1}$$

The following theorem is proved in (Kostyukova, Tchemisova and Yermalinskaya, 2005).

Theorem 4.1. *[Implicit Optimality Criterion] The feasible solution $x^0 \in X$ such that $|T_a(x^0)| < \infty$ is optimal in the convex SIP problem (2.1) if and only if it is optimal in the NLP problem (4.1).*

The Theorem 4.1 generalizes the following well known (see (Hettich and Kortanek, 1993; Hettich and Still, 1995) et al.) result for convex SIP.

Theorem 4.2. *Let the convex SIP problem (2.1) satisfy the Slater condition. Then a feasible solution x^0 is optimal to problem (2.1) if and only if it is optimal to the following problem:*

$$\begin{gathered} c(x) \longrightarrow \min, \\ \textit{s.t.} \quad f(x, t_j^0) \leq 0, \quad j = 1, \ldots, p. \end{gathered} \tag{4.2}$$

It is known that the statement of Theorem 4.2 in not true if the Slater condition is not satisfied for (2.1). Note that problems (4.1) and (4.2) coincide if the convex SIP problem (2.1) satisfies the Slater condition.
It is easy to see that in our Criterion (Theorem 4.1) none of CQC are assumed for the SIP problem (2.1) and the test of optimality for the SIP problem is reduced to such a test for the corresponding nonlinear problem. It is evident that this result certainly (and valuably) contributes to solution of issue 4) (see Introduction) for convex problems.
The following quotation from (Jongen et al., 1987) supports the importance of this reduction: "In the case that ..." $T_a(x^0)$ "... in non-empty, one needs some specific information about the structure of the feasible set in a neighborhood of ..." x^0 "... in order to derive sufficient conditions for local optimality. From this point of view it is quite natural to find conditions which can be used to reduce the ... (SIP) problem locally to a nonlinear programming problem, i.e. such that the feasible set can be locally described by a finite number of constraints."

4.2 Discretized Problem

It is evident that the results of the present paper contribute also to solution of the issue 5) (see Introduction) for the convex SIP problems. Let us consider some NLP problem

$$\begin{gathered} c(x) \longrightarrow \min, \\ \text{s.t.} \quad h_j(x) = 0,\ j = 1, \dots, s, \ \ h_j(x) \leq 0,\ j = s+1, \dots, s_*, \end{gathered} \tag{4.3}$$

and denote by X_D the feasible set:

$$X_D = \{x \in R^n : h_j(x) = 0, \ \ j = 1, \dots, s, \ \ h_j(x) \leq 0, \ j = s+1, \dots, s_*\}.$$

Problem (4.3) is said to be a *discretization* of the original SIP problem if $X \subset X_D$.
The main question studied when the discretization approach is applied is the following (see (Hettich and Kortanek, 1993)): *If there exists a discretization (4.3) of the SIP problem (2.1) with the same optimal value, i.e. such that* $\min\limits_{x \in X} c(x) = \min\limits_{x \in X_D} c(x)$?
The answer to this question plays an important role in construction of numerical methods as well as in theoretical investigations (in the duality theory of SIP, for example).
The following statement is known (see(Bonnans and Shapiro, 2000) et al.).

Proposition 4.3. *Let the SIP problem (2.1) be convex and suppose that it satisfies the Slater condition. Then there exists a discretization of the problem (2.1) with the same optimal value. This discretization has the form*

$$\begin{gathered} c(x) \longrightarrow \min, \\ \textit{s.t.} \quad f(x, t_j) \leq 0, \ \ j = 1, \dots, s_*. \end{gathered} \tag{4.4}$$

where t_j, $j = 1, ..., s_*$, *are some fixed indices from the index set* T.

It is also well known that without the Slater condition this statement is not true even for linear semi-infinite problems.
The use of the concepts of immobility orders and immobile indices permits to strengthen the statement of Proposition 4.3. Let $T_* = \{t^0, j = 1, ..., p_*\}$ be the set of immobile indices in SIP problem (2.1).

Proposition 4.4. *For any convex SIP problem (2.1) there exists a discretization with the same optimal value. This discrete problem has the form*

$$\begin{gathered} c(x) \longrightarrow \min, \\ \textit{s.t.} \quad f^{(s)}(x, t_j^0) = 0, \ s \in N(q_j), j = 1, \dots, p_*, \\ f^{(q_j+1)}(x, t_j^0) \leq 0, \ \ j = 1, \dots, p_*, \ f(x, t_j) \leq 0, \ \ j = p_*, \dots, s_*, \end{gathered} \tag{4.5}$$

where $s_* \leq \max\{0, n - p_*\}$, *and* t_j, $j = p_* + 1, ..., s_*$, *are some fixed indices from the index set* T.

The proof of Proposition 4.4 follows from Theorem 4.1.
It is evident that if the Slater condition is satisfied, then $T_* = \emptyset$ and problems (4.4) and (4.5) are the same.
Finally, we would like to note that the concepts of immobile indices and their immobility orders can valuably contribute to solution of the issues 1)-3) as well. However, the size limits of this publication do not permit us to comment this statement.

5 Discussion

The main results of the paper, the algorithm of determination of the immobile indices for the convex SIP problems and the Implicit Optimality Criterion, can be used either in the optimality theory of SIP or for constructing new SIP algorithms. The Criterion is based on the concepts of immobility indices and immobility orders that themselves are important characteristics of the index set T and the admissible set X of the SIP problem (2.1). The DIO Algorithm determines the immobile indices and their immobility orders in a finite number of iterations and permits to verify whether the Slater condition is satisfied for the convex SIP problem. The important properties of the Implicit Optimality Criterion are that it works without any special CQC (for example, the Slater condition is not necessarily satisfied), and that it reduces the optimality conditions for the convex SIP problems to optimality conditions for some related NLP problem. Application of the Implicit Optimality Criterion gives the possibility to develop new efficient optimality conditions for SIP problems. As a matter of fact, different optimality conditions for the NLP problem mentioned above generate the corresponding explicit optimality conditions for the original convex SIP problem. The study of such explicit optimality conditions is the subject of a special investigation (see (Kostyukova et al., 2005) et al.)

References

Auslender, A. 2000. Existence of optimal solutions and duality results under weak conditions, *Math.Program.* **88**: 45–59.

Bonnans, J. and Shapiro, A. 2000. *Perturbation analysis of optimization problems*, Springer.

Bot, R. and Wanka, G. 2006. An alternative formulation for a new closed cone constraint qualification, *Nonlinear Analysis* **64**: 1367–1381.

Canovas, M., Dontchev, A., Lopes, M. and Parra, J. 2005. Metric regularity of semi-infinite constraint systems, *Math.Program.* **104**: 329–346.

Fajardo, M. and Lopez, M. 2006. Some results about the facial geometry of convex semi-infinite systems, *Optimization* **55**(5-6): 661–684.

Goberna, M. A. and Lopez, M. 1998. *Linear Semi-Infinite Optimization*, John Wiley and Sons.

Gugat, M. 2000. Error bounds for infinite systems of convex inequalities without Slater's condition, *Math.Program.* **88**: 255–275.

Gustafson, S. 1983. A three-phase algorithm for semi-infinite programs, in Semi-infinite programming and applications, *Lecture Notes in Econom. and Math.Systems. A.V.Fiacco and K.O.Kortanek, eds.* **215**: 138–157.

Hettich, R. 1986. An implementation of a discretization method for semi-infinite programming, *Math. Programming* **34**(3): 354–361.

Hettich, R. and Jongen, H. 1978. Semi-infinite programming: conditions of optimality and applications (in J.Stoer editor, Optimization Techniques, Part 2), *Lecture Notes in Control and Informations Sciences* **7**: 1–11.

Hettich, R. and Kortanek, K. 1993. Semi-infinite programming: theory, methods and applications, *SIAM Review* **35**: 380–429.

Hettich, R. and Still, G. 1995. Second order optimality conditions for generalized semi-infinite programming problems, *Optimization* **34**: 195–211.

Jeyakumar, V., Dinh, N. and Lee, G. M. 2004. A new closed cone constrained qualification for convex optimization, *Appl.Math.Report* **AMR04/8**.

Jeyakumar, V., Lee, G. M. and Dinh, N. 2003. New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs, *SIAM J. Optim.* **14**(2): 534–547.

Jeyakumar, V., Lee, G. M. and Dinh, N. 2004. Liberating the subgradient optimality conditions from constraint qualifications, *Appl.Math.Report* **AMR04/7**.

Jongen, H., Meer, K. and Trieschm, E. 2004. *Optimization Theory*, Lecture Notes in Control and Informations Sciences, Kluwer Academic Publishers.

Jongen, H., Rückmann, J. and Stein, O. 1998. Generalized semi-infinite optimization: a first order optimality condition and examples, *Math. Progr.* **83**: 145–158.

Jongen, H., Wetterling, W. and Zwier, G. 1987. On sufficient conditions for local optimality in semi-infinite programming, *Optimization* **18**(2): 165–178.

Klatte, D. 1994. Stable local minimizers in semi-infinite optimization: regularity and second-order conditions, *J. of Comput. and Appl. Math.* **56**(1-2): 137–157.

Kortanek, K. and Medvedev, V. 2005. *Semi-Infinite Programming and new applications in finance(C.Floudas and Panos Pardalos, Editors)*, Encyclopedia of Optimization, Kluwer Academic.

Kostyukova, O. 1988. Investigation of the linear extremal problems with continuum constraints, *Technical Report. Institute of Mathematics of Belarusian Academy of Sciences* **26/336**.

Kostyukova, O., Tchemisova, T. and Yermalinskaya, S. 2005. Convex semi-infinite programming: Implicit Optimality Criterion based on the concept of immobile points, *Technical Report. Aveiro University* **05/I-38**.

Li, S., Yang, X. and Teo, K. 2003. Duality for semi-definite and semi-infinite programming, *Optimization* **52**: 507–528.

Li, W., Nahak, C. and Singer, I. 2000. Constraint qualifications for semi-infinite systems of convex inequalities, *SIAM J. Optim* **11**(1): 31–52.

Polak, E. 1983. *Semi-Infinite Programming and Applications (A.V.Fiacco and K.O.Kortanek, Editors)*, Lecture Notes in Econom. and Math.Systems, Springer-Verlad.

Reemtsen, R. and Gorner, S. 1998. *Numerical methods for Semi-Infinite Programming: A survey. In "Semi-inifinte Programming", Reemtsen/Ruckmann (eds.)*, Kluwer.

Rückmann, J. J. and Shapiro, A. 1999. First-order optimality conditions in generalized semi-infinite programming, *J. of Optimization Theory and Appl.* **101**(3): 677–691.

Rückmann, J. J. and Shapiro, A. 2001. Second-order optimality conditions in generalized semi-infinite programming, *Set-Valued Analysis* **9**(1-2): 169–186.

Stein, O. and Still, G. 2000. On optimality conditions for generalized semi-infinite programming problems, *JOTA* **104**(2): 443–458.

Stein, O. and Still, G. 2002. On generalized semi-infinite optimization and bi-level optimization, *EJOR* **142**(3): 444–462.

Stein, O. and Still, G. 2003. Solving semi-infinite optimization problems with interior point techniques, *SIAM J. on Control and Optimization* **42**: 769–788.

Still, G. 1999. Generalized semi-infinite programming: theory and methods, *EJOR* **119**: 301–313.

Still, G. 2004. On constraint qualifications in nonsmooth optimization, *JOTA* **121**(3): 647–671.

Tanaka, Y., Fukushima, M. and Ibaraki, T. 1988. A globally convergent method for semi-infinite nonlinear optimization, *J. of Comp. and Appl. Math.* **23**: 141–153.

Weber, G.-W. 2002. Generalized semi-infinite optimization: theory and applications in optimal control and discrete optimization. Special issue ”Optimality conditions, generalized convexity and duality in vector optimization”, A. Cambini, L. Martein (eds), *J.Statistic and Management Systems* **5**: 359–388.

Xin, T. Y. and Chong, L. 2007. On basic constraint qualifications for infinite system of convex inequalities in banach spaces, *Acta Mathematica Sinica, English Series* **23**(1): 65–76.