On the max-semistable limit of maxima of stationary sequences with missing values

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Abstract

Let $\{X_n\}$ be a stationary sequence with marginal distribution in the domain of attraction of a max-semistable distribution. This includes all distributions in the domain of attraction of any max-stable distribution and also other distributions like some integer-valued distributions with exponential type tails such as the Negative Binomial case. We consider the effect of missing values on the distribution of the maximum term. The pattern of occurrence of the missing values must be either iid or strongly mixing. We obtain the expression of the extremal index for the resulting sequence.

The results generalize and extend the ones obtained for the max-stable domain of attraction.

Keywords: Extreme value theory; Integer-valued models; Max-semistable laws.

1 Introduction

Integer-valued time series have received increasing attention in the probabilistic and statistical literature over the past two decades because of its applicability in many different areas such as the natural sciences, the social sciences, international tourism demand and economy. We refer to Hall and Scotto (2006) for several references on the subject and to McKenzie (2003) for an overview of the early work in this area.

Within the integer-valued models proposed in the literature, little is known about its extremal properties. In part, this is due to the fact that many integer-valued distributions do not belong to the domain of attraction on any extreme-value distribution. Anderson (1970)

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gave an important contribution to this limitation by obtaining upper and lower bounds for the limiting distribution of the maximum term of independent and identically distributed (iid) sequences with distributions exhibiting an exponentially decaying tail. He proved that an integer-valued distribution function F, with infinite right endpoint, satisfies

$$\lim_{n\to+\infty}\frac{1-F(n-1)}{1-F(n)}=r,\; r\in]1,+\infty[, \tag{1}$$

if and only if

$$\begin{cases}\limsup_{n\to\infty}F^n(x+b_n)\leq\exp(-r^{-x})\\ \liminf_{n\to\infty}F^n(x+b_n)\geq\exp(-r^{-(x-1)})\end{cases}$$

for any real x and b_n appropriately chosen. We shall say that a distribution belongs to Anderson's class if it satisfies (1). An example of a well known member of this class is the Negative Binomial distribution.

Based on Anderson's work several stationary models have been studied with respect to the extremes and similar limiting bounds were obtained for the limiting distribution of the maximum term in presence of different dependence structures. For instance McCormick and Park (1992) considered a first-order autoregressive sequence with Negative Binomial marginal distribution, Hall (1996) considered a class of integer-valued max-autoregressive models, Hall (2003) considered a general class of infinite moving average models, and Hall and Moreira (2006) considered a particular type of moving average models with geometric marginal distribution introduced by McKenzie (1986).

In an attempt to overcome the presence of limiting bounds instead of a well defined limiting distribution, Temido (2000) proved that (1) is necessary and sufficient for the existence of a nondecreasing positive integer sequence $\{k_n\}$ satisfying

$$\lim_{n\to+\infty}\frac{k_{n+1}}{k_n}=r,\quad r\in]1,+\infty[, \tag{2}$$

and of a real sequence $\{u_n\}$ such that $k_n(1-F(u_n))\to\tau>0$, as $n\to\infty$, for some $\tau>0$. So, if instead of looking at the maximum term of the first n observations we look at the maximum term of the first k_n observations, where $\{k_n\}$ satisfies (2), we can obtain a well defined limiting distribution for the maximum term. The limiting distribution is not a max-stable (MS) distribution but belongs to a larger class of distributions known as max-semistable (MSS) distributions. This class was introduced by Pancheva (1992) and will be described below.

Using the MSS class Hall and Temido (2007) have studied the limiting distribution of the maximum term of several stationary models with margins in Anderson's class.

Over the last years attention has been given to the effect of sub-sampling on the extremes of stationary sequences. This is important for the analysis of environmental and financial processes. We refer to Hall and Scotto (2006) for an overview of the work in this area. Many of the results consider deterministic sub-sampling of the sequences. However, there are several real data applications where sub-sampling is the result of a (random) occurrence of missing values. One reason for the interest in extremes observed at random sampling rates comes from the need to compare schemes for monitoring systems with breakdowns or systems with automatic replacement of devices in case of failures. Examples are encountered, for instance, in ocean engineering and environmental studies. In these areas, missing observations appear when the measuring equipment is not working properly or is out of service. The effect of missing values on the extremes of stationary sequences has been considered by Hall and Hüsler (2006) and by Hall and Scotto (2008). The former consider models with marginal distribution in the domain of attraction of any MS distribution and also distributions in Anderson's class. The later considers models with marginal distribution in the domain of attraction of a Fréchet distribution and with a moving average structure. In this work we consider the setup given in Hall and Hüsler and study the limiting distribution of the maximum term using MSS distributions.

What happens when a missing value occurs? In the majority of situations, one of two things may happen: either the observation is replaced by a fixed value (for instance a code), or the observation is completely lost and the data sample will be sub-sampled resulting in a smaller (and random) sample size. Occasionally, it may be of interest to avoid the occurrence of missing values and an automatic replacement of a device or machine may be available. In this case the resulting sample will be a mixture of two original samples.

In this paper we consider three different models which were motivated by the situations described above.

Let $\{X_n\}$ be a strictly stationary sequence of random variables (rv's) with marginal distribution function (df) F and $M_n = \max\{X_1, \ldots, X_n\}$. Without loss of generality let the upper endpoint x_F of F be positive. Assume that $\{U_n\}$ is another stationary sequence, independent of $\{X_n\}$, having Bernoulli marginal distribution with parameter β, $0 \leq \beta \leq 1$. Based on the sequences $\{X_n\}$ and $\{U_n\}$ we define the following models:

M1 - Model with missing values: $Y_n = U_n X_n$. In this case the marginal distribution of $\{Y_n\}$ is $1 - \beta + \beta F$ for non-negative argument values. Missing values are replaced by zeros.

M2 - Model with sub-sampling: $\{Z_n\}_{n\geq 1}$ with $Z_n = X_{i_n}$, where i_n represents the indices of the sequence $\{X_n\}$ for which $U_n = 1$. Let $N_n = \sup\{j : i_j \leq n\} = \sum_{j=1}^{n} U_j$ and let us only consider the first N_n values of $\{Z_n\}$. The marginal distribution of $\{Z_n\}$ is F. In this model missing values are lost.

M3 - Model with replaced missing values: $W_n = U_n X_n + (1 - U_n) X_n^{(1)}$ where $\{X_n^{(1)}\}$ is an independent replica of $\{X_n\}$. In this case the marginal distribution of $\{W_n\}$ is also F. Missing values are replaced by a substituting sequence.

In this paper we only consider situations where $\{X_n\}$ and $\{U_n\}$ are independent.

We will be interested in the limiting distribution of the maximum term of these models. We define $M_n(Y) = \max\{Y_1, \ldots, Y_n\}$ and $M_n(W) = \max\{W_1, \ldots, W_n\}$. For convenience of notation we define also $M_n(Z) = \max\{Z_1, \ldots, Z_{N_n}\}$ because for the sequence Z_n we are only interested in the first N_n variables.

The sequences $\{U_n\}$ considered in this paper will be considered stationary and strongly mixing. Furthermore, we assume that some kind of dependence conditions hold for $\{X_n\}$. We consider an asymptotic independence condition similar to Leadbetter's $D(u_n)$ condition (Leadbetter et al. (1983)). We also consider two types of local dependence conditions similar to conditions $D^{(2)}(u_n)$ and $D^{(3)}(u_n)$ of Chernick et al. (1991). Each of these conditions will ensure that the clusters of exceedances formed in the stationary sequence $\{X_n\}$ have a particular type of pattern. Throughout this work $\{u_n\}$ represents a sequence of linearly normalized levels $u_n = a_n x + b_n,\ a_n > 0,\ b_n \in \mathbb{R}$ such that $u_n \to x_F$.

We obtain the expression of the extremal index and hence the limiting distribution of the maximum term of the transformed sequences (models M1, M2 and M3). The results generalize and extend the ones obtained for the MS domain of attraction given by Hall and Hüsler (2006).

2 The MSS class and stationary sequences

If instead of considering n rv's we consider k_n rv's, where $\{k_n\}$ satisfies

$$\lim_{n\to+\infty} \frac{k_{n+1}}{k_n} = r, \text{ with } r \text{ in } [1, +\infty[, \tag{3}$$

then we obtain a larger class of possible limiting distributions for the maxima, known as the MSS class. This class, introduced by Pancheva (1992), includes the MS distributions and also non-degenerate limiting df's for the maxima of iid rv's with either discrete or multi-modal continuous df's which are not max-stable. Following Pancheva (1992) we will say that a real df G is MSS if there are reals $r > 1$, $a = a(r) > 0$ and $b = b(r)$ such that $G(x) = G^r(ax + b),\ x \in \mathbb{R}$, or equivalently, if there exist a sequence of iid rv's with df F and two real sequences $\{a_n > 0\}$ and $\{b_n\}$ for which $\lim_{n\to+\infty} F^{k_n}(a_n x + b_n) = G(x)$, for each continuity point of G, with $\{k_n\}$ satisfying (3). In this case we will say that F belongs to the domain of attraction of G.

Note the subtle difference between the definitions of MS and MSS distributions. While in the definition of the MS distributions, G is MS if for all reals $r > 1$, there exist $a = a(r) > 0$ and $b = b(r)$ such that $G(x) = G^r(ax + b),\ x \in \mathbb{R}$, in the definition of the MSS distribution the equality may hold only for some values of $r > 1$. More precisely, if G is MSS but not MS, the equality holds only for some r and all its integer powers.

Analytically, a df in the class MSS can be written as follows:

$$G_{\gamma,\nu}(x) = \begin{cases} \exp\left\{-(1+\gamma x)^{-1/\gamma}\nu(\log(1+\gamma x))\right\} & x \in \mathbb{R},\ 1+\gamma x > 0 \text{ and } \gamma \neq 0 \\ \mathbb{I}_{]-\infty,0[}(\gamma) & x \in \mathbb{R},\ 1+\gamma x \leq 0 \text{ and } \gamma \neq 0 \\ \exp\{-e^{-x}\nu(x)\} & x \in \mathbb{R} \text{ and } \gamma = 0 \end{cases},$$

where ν is a positive, bounded and periodic function with period $p = |\log\ a| = |\gamma| \log r$, when $\gamma \neq 0$, and $p = b = \log\ r$, when $\gamma = 0$. If the function ν is a suitable constant, we get the max-stable class.

We recall that for iid sequences with common df F, $\lim_{n\to\infty} k_n(1 - F(u_n)) = \tau$ is equivalent to $\lim_{n\to\infty} F^{k_n}(u_n) = e^{-\tau}$. Consequently, in the sequel, we shall deal with levels $u_n := u_n(\tau, k_n)$ satisfying these limits. In this context we will consider the set

$$\Gamma(F, k_n) = \{\tau > 0 : \exists\{u_n\} : \lim_{n\to+\infty} k_n(1 - F(u_n)) = \tau\},$$

introduced in Temido (2000), and note that if F is discrete and (3) holds, then $\Gamma(F, k_n)$ is not necessarily the interval $]0, +\infty[$. In fact, if for some $\tau > 0$ there exists $u_n(\tau, k_n)$, for another $\tau' > 0$ there exists $u_n(\tau', k_n)$ if and only if $\tau = r^m\,\tau'$, for some integer m. This enables us to conclude that if any discrete df belongs to the domain of attraction of G, with $\{k_n\}$ satisfying (2), then G must be discrete.

Temido (2002) proved that if F is an integer-valued df, with upper endpoint $x_F = +\infty$, and there exist sequences $\{k_n\}$ satisfying (2), $\{a_n > 0\}$ and $\{b_n\}$ such that $F^{k_n}(a_n x + b_n) \to G(x)$, $n \to +\infty$, then $G(x) = \exp(-\eta r^{-[x]})$, $x \in \mathbb{R}$, for some $\eta > 0$, if and only if (1) holds (for any real x, $[x]$ denotes the greatest integer not exceeding x). In this case, with $k'_n = [k_n/\eta]$ we get

$$\lim_{n\to+\infty} F^{k'_n}(a_n x + b_n) = \exp(-r^{-[x]}), \quad x \in \mathbb{R}\backslash\mathbb{Z}.$$

Temido and Canto e Castro (2003) consider stationary sequences $\{X_n\}$, satisfying a dependence restriction, $D_{k_n}(u_n)$, which extends Leadbetter's $D(u_n)$ condition (Leadbetter et al. (1983)).

Definition 2.1 (Temido and Canto e Castro, 2003) *Let $\{k_n\}$ be a nondecreasing sequence of positive integers. The sequence of rv's $\{X_n\}$ satisfies condition $D_{k_n}(u_n)$ if for any integers $1 \le i_1 < ... < i_p < j_1 < ... < j_q \le k_n$, for which $j_1 - i_p > \ell_n$, we have*

$$\left|P\Big(\bigcap_{s=1}^{p}\{X_{i_s} \le u_n\}, \bigcap_{m=1}^{q}\{X_{j_m} \le u_n\}\Big) - P\Big(\bigcap_{s=1}^{p}\{X_{i_s} \le u_n\}\Big)P\Big(\bigcap_{m=1}^{q}\{X_{j_m} \le u_n\}\Big)\right| \le \alpha_{n,\ell_n},$$

where $\lim_{n\to+\infty} \alpha_{n,\ell_n} = 0$ for some sequence $\ell_n = o_n(k_n)$.

Considering stationary sequences $\{X_n\}$ satisfying this long range condition, $D_{k_n}(u_n)$, Temido and Canto e Castro (2003) prove that the limiting distribution of M_{k_n} is max-semistable, whenever it exists. Namely, if there is $\{k_n\}$ as above and $\{a_n > 0\}$ and $\{b_n\}$ such that condition $D(a_n x + b_n)$ holds for the stationary sequence $\{X_n\}$, the sequence $\{k_n(1 - F(a_n x + b_n)\}$ is bounded, and $P(a_n(M_{k_n} - b_n) \le x)$ converges to $G(x)$, for each continuity point of the nondegenerate df G, then G is a max-semistable df. Furthermore, following the same authors we present the definition of extremal index.

Definition 2.2 (Temido and Canto e Castro, 2003) *We shall say that $\{X_n\}$ has an extremal index θ, with θ in $[0,1]$, if there exists a nondecreasing positive integer sequence $\{k_n\}$ satisfying (3) such that, for all $\tau \in \Gamma(F,k_n)$ and all corresponding $u_n(\tau,k_n)$, we have $\lim_{n\to+\infty} P(M_{k_n} \leq u_n(\tau,k_n)) = e^{-\theta\tau}$.*

In Hall and Hüsler (2006) an important lemma is introduced. Indeed, the authors prove that if $\{U_n\}$ is a Bernoulli strongly mixing stationary sequence and the long range condition $D(u_n)$ holds for $\{X_n\}$, then it also holds for $\{Y_n\}$ in Model M1. For the sequel and *mutatis mutandis* we have the following result.

Lemma 2.1 *Let $\{k_n\}$ be a positive integer and non decreasing sequence and suppose that $\{U_n\}$ is a Bernoulli strongly mixing stationary sequence. If condition $D_{k_n}(u_n)$ holds for $\{X_n\}$ then it also holds for $\{Y_n\}$.*

3 Independent missing values

Proposition 3.1 *Let $\{U_n\}$ be an iid Bernoulli(β) sequence, and $\{X_n\}$ a stationary sequence with extremal index θ in the sense of definition (2.2) and cluster size distribution π. Define*

$$\begin{aligned} \theta^* &= \theta(1-\Pi(1-\beta))/\beta \quad \textit{and} \quad \tau^* = \tau\beta \\ \theta^{**} &= \theta(1-\Pi(1-\beta)+1-\Pi(\beta)) \end{aligned}$$

where $\Pi(h) = \sum_{i=1}^{\infty} \pi(i)h^i$.

If $P\{M_{k_n} \leq u_n(\tau)\} \underset{n\to\infty}{\longrightarrow} e^{-\theta\tau}$, $\tau \in \Gamma(F,k_n)$, then,

$$P\{M_{k_n}(V) \leq u_n(\tau)\} \underset{n\to\infty}{\longrightarrow} e^{-\theta_V\tau_V}$$

where $\theta_V = \theta^*$, $\tau_V = \tau^*$ for the sequences $\{Y_n\}$ and $\{Z_n\}$, and $\theta_V = \theta^{**}$, $\tau_V = \tau$ for the sequence $\{W_n\}$.

Proof: Let $\{s_n\}$ be a sequence of positive integers satisfying

$$\lim_{n\to\infty} s_n^{-1} = \lim_{n\to\infty} \frac{s_n \ell_n}{k_n} = \lim_{n\to\infty} s_n \alpha_{n,\ell_n} = 0. \tag{4}$$

The idea for this proof is to divide the first k_n elements of the sequence $\{Y_n\}$ into blocks of size $r_n := [k_n/s_n]$, calculate the probability of no exceedances in those blocks and then use Lemma 4 from Temido and Canto e Castro (2003).

Let us consider the first r_n elements of the sequences $\{X_n\}$ and $\{Y_n\}$. Assuming $u_n > 0$, then,

$$\begin{aligned} P\{M_{r_n}(Y) \leq u_n\} &= P\{\{Y_1,\ldots,Y_{r_n}\} \text{ contains no exceedances of } u_n\} \\ &= P\{M_{r_n} \leq u_n\} + \sum_{j=1}^{r_n} P\{\{X_1,\ldots,X_{r_n}\} \text{ contains } j \text{ exceedances all withdrawn}\} \end{aligned}$$

From the definition of extremal index we get $P\{M_{k_n} \leq u_n\} - e^{-k_n P\{X_1 > u_n\}\theta} \underset{n\to\infty}{\longrightarrow} 0$. On the other hand, from Lemma 4 from Temido and Canto e Castro (2003) we have

$$P\{M_{k_n} \leq u_n\} - P^{s_n}\{M_{r_n} \leq u_n\} \underset{n\to\infty}{\longrightarrow} 0.$$

Combining both results we obtain

$$P\{M_{r_n} \leq u_n\} = 1 - \frac{k_n}{s_n}\theta P\{X_1 > u_n\} + o\left(\frac{1}{s_n}\right),$$

as long as $\lim_{n\to\infty} k_n P\{X_1 > u_n\} \neq \infty$.

Since $k_n P\{X_1 > u_n\} \underset{n\to\infty}{\longrightarrow} \tau > 0$ we have

$$P\{M_{r_n}(Y) \leq u_n\} = 1 - \frac{\theta\tau}{s_n}(1 - \sum_{j=1}^{r_n} \frac{s_n}{\theta\tau} P\{\{X_1, \ldots, X_{r_n}\} \text{ contains } j$$

$$\text{exceedances all withdrawn}\}) + o\left(\frac{1}{s_n}\right).$$

But, with $\chi_{n,i}(\cdot) := \mathbb{1}_{\{X_i > u_n\}}(\cdot)$, we also have

$$\begin{aligned}
&P\{\{X_1, \ldots, X_{r_n}\} \text{ contains } j \text{ exceedances all withdrawn}\} \\
&= \sum_{i_1 < i_2 < \ldots < i_j} P\left\{\bigcap_{m=1}^{j} \{X_{i_m} > u_n\},\ \text{all other } X_i \leq u_n,\ \bigcap_{m=1}^{j} \{U_{i_m} = 0\}\right\} \\
&= \sum_{i_1 < i_2 < \ldots < i_j} P\left\{\bigcap_{m=1}^{j} \{X_{i_m} > u_n\},\ \text{all other } X_i \leq u_n\right\} P(U_1 = 0)^j \\
&= P\left(\sum_{i=1}^{r_n} \chi_{n,i} = j\right)(1-\beta)^j
\end{aligned}$$

Now, using the above arguments again,

$$\begin{aligned}
P\left(\sum_{i=1}^{r_n} \chi_{n,i} = j\right) &= \pi(j) P\left(\sum_{i=1}^{r_n} \chi_{n,i} > 0\right)(1 + o(1)) \\
&= \pi(j)(1 - P\left(\sum_{i=1}^{r_n} \chi_{n,i} = 0\right))(1 + o(1)) \\
&= \pi(j)(1 + o(1))\tfrac{\theta\tau}{s_n}.
\end{aligned}$$

Therefore

$$P\{M_{r_n}(Y) \le u_n\} = 1 - \frac{\theta\tau}{s_n}\left(1 - \sum_{j=1}^{r_n} \pi(j)(1-\beta)^j(1+o(1))\right) + o\left(\frac{1}{s_n}\right).$$

Considering Lemma 2.1 and Lemma 4 from Temido and Canto e Castro (2003), we use dominated convergence to obtain

$$P\{M_{k_n}(Y) \le u_n\} \underset{n\to\infty}{\longrightarrow} e^{-\theta\tau(1-\Pi(1-\beta))},$$

where $\Pi(s) = \sum_{j=1}^{\infty} \pi(j)s^j$ represents the probability generating function of the cluster size distribution.

As for the sequences $\{Z_n\}$ and $\{W_n\}$ it suffices to notice that $P\{M_{k_n}(Z) \le u_n\} = P\{M_{k_n}(Y) \le u_n\}$ and that $P\{M_{k_n}(W) \le u_n\} = P\{M_{k_n}(Y) \le u_n\}P\{M_{k_n}(\bar{Y}) \le u_n\}$ where $\{\bar{Y}_n\}$ is a sequence defined by $\bar{Y}_n = (1-U_n)X_n$.

□

4 The effect of missing values under condition $D_{k_n}^{(2)}(u_n)$

We now consider a stationary sequence $\{X_n\}$ satisfying a local dependence condition, $D_{k_n}^{(2)}(u_n)$, and with marginal df F in the domain of attraction of some max-semistable df.

The natural extension of the local dependence condition $D^{(2)}(u_n)$ of Chernick et al. (1991)), which is similar to the condition $D''(u_n)$ (defined in Leadbetter and Nandagopalan (1989), in this new context is given in the following definition.

Definition 4.1 (Temido (2000)) *Let $\{k_n\}$ be a nondecreasing positive integer sequence such that $\lim_{n\to\infty} k_n = +\infty$. The stationary sequence $\{X_n\}$ satisfies the condition $D_{k_n}^{(2)}(u_n)$ if $D_{k_n}(u_n)$ holds and, for some positive integer-valued sequence $\{s_n\}$ satisfying (4), we have*

$$\lim_{n\to\infty} k_n P\{X_1 > u_n \ge X_2, M_{3,r_n} > u_n\} = 0, \tag{5}$$

where $r_n = [\frac{k_n}{s_n}]$ and $M_{ij} = \max\{X_k, k = i, ..., j\}$.

Clearly, (5) is implied by the condition

$$\lim_{n\to+\infty} k_n \sum_{j=3}^{r_n} P\{X_1 > u_n, X_{j-1} \le u_n < X_j\} = 0,$$

which in some situations is easier to check.

Recall that under condition $D^{(2)}(u_n)$ the clusters of exeedances form groups of consecutive observations. Once the sequence falls below the high threshold u_n the probability of a new upcrossing in the near future is negligible.

Under $D_{k_n}^{(2)}(u_n)$ we can compute the extremal index applying the following result.

Proposition 4.1 (Temido (2000)) *Let $\{k_n\}$ be a non decreasing positive integer sequence satisfying (1) and $\{X_n\}$ a stationary sequence under condition $D_{k_n}^{(2)}(u_n(\tau, k_n))$, for all τ in $\Gamma(F, k_n)$ and all corresponding $u_n(\tau, k_n)$. Then $\{X_n\}$ has extremal index θ if and only if*

$$\lim_{n\to+\infty} P(X_2 \le u_n(\tau, k_n)\,|X_1 > u_n(\tau, k_n)) = \theta.$$

The following proposition, our main result of this section, is a generalization of Theorem 4 in Hall and Hüsler (2006) concerning normalized levels $u_n(\tau, k_n)$. We use $\{V_n\}$ to denote any of the sequences $\{Y_n\}$ $\{Z_n\}$ and $\{W_n\}$ and recall that in the model **M2** we consider the maximum of the first N_{k_n} rv's.

Proposition 4.2 *Let $\{X_n\}$ be a stationary sequence with df F and $\{U_n\}$ be a Bernoulli $B(\beta)$ strongly mixing stationary sequence independent of $\{X_n\}$. Let $\{k_n\}$ be a nondecreasing positive integer sequence satisfying (3). Suppose that $\{X_n\}$ satisfies $D_{k_n}^{(2)}(u_n(\tau, k_n))$, for all $\tau \in \Gamma(F, k_n)$ and all corresponding $u_n(\tau, k_n)$. In addition suppose that $\{X_n\}$ has extremal index $\theta > 0$ associated with a limiting cluster size distribution π. For $i \ge 1$, take*

$$\nabla(i) = P\{U_1 = 0, U_2 = 0, ..., U_i = 0\}, \qquad \bar{\nabla}(i) = P\{U_1 = 1, U_2 = 1, \ldots, U_i = 1\},$$

$$\theta^* = \theta\left(1 - \sum_{j=1}^{\infty} \pi(j)\nabla(j)\right)/\beta \quad \text{and} \quad \theta^{**} = \beta\theta^* + \theta\left(1 - \sum_{j=1}^{\infty} \pi(j)\bar{\nabla}(j)\right).$$

Then

$$\lim_{n\to\infty} P(M_{k_n}(V) \le u_n(\tau, k_n)) = e^{-\theta_V \tau_V}$$

where $\theta_V = \theta^*$, $\tau_V = \tau\beta$ for the sequences $\{Y_n\}$ and $\{Z_n\}$, and $\theta_V = \theta^{**}$, $\tau_V = \tau$ for the sequence $\{W_n\}$, for all $\tau \in \Gamma(F, k_n)$.

The proof of this proposition follows the arguments used in Theorem 4 of Hall and Hüsler (2006) where, for F integer-valued, $e^{-\theta\tau(x-1)} \le \liminf_{n\to\infty} P\{M_n \le u_n\} \le \limsup_{n\to\infty} P\{M_n \le u_n\} \le e^{-\theta\tau(x)}$ is replaced by $\lim_{n\to\infty} P\{M_{k_n} \le u_n(\tau, k_n)\} = e^{-\theta\tau(x)}$. Observe that, as a consequence of what was said before, if F is an integer-valued df satisfying (1) and $u_n(\tau, k_n) = a_n x + b_n$, for some $a_n > 0$ and b_n, then, there exist $\{k_n\}$ satisfying (2) such that $\lim_{n\to+\infty} P(M_{k_n} \le u_n(\tau, k_n))$ $= \exp(-\theta r^{-[x]})$, $x \in \mathbb{R}\backslash\mathbb{Z}$.

5 The effect of missing values under condition $D_{k_n}^{(3)}(u_n)$

In this section we consider that the stationary sequence $\{X_n\}$ satisfies an extension of the local dependence condition $D^{(3)}(u_n)$ of Chernick et al. (1991) defined by Temido (2000).

Definition 5.1 *Let* $\{k_n\}$ *be a nondecreasing positive integer sequence such that* $\lim_{n\to\infty} k_n = +\infty$. *The stationary sequence* $\{X_n\}$ *satisfies the condition* $D^{(3)}_{k_n}(u_n)$ *if* $D_{k_n}(u_n)$ *holds and, for some positive integer-valued sequence* $\{s_n\}$ *satisfying (4), we have*

$$\lim_{n\to+\infty} k_n P(X_1 > u_n \geq M_{2,3}, M_{4,r_n} > u_n) = 0, \tag{6}$$

where $r_n = [\frac{k_n}{s_n}]$ *and* $M_{ij} = \max\{X_k, k = i, ..., j\}$.

As in the previous paragraph, (6) is implied by the condition

$$\lim_{n\to+\infty} k_n \sum_{i=4}^{r_n} P(X_1 > u_n, M_{i-2,i-1} \leq u_n < X_i) = 0$$

which in some situations is easier to deal with.

We also suppose that

$$\lim_{n\to\infty} k_n P(X_1 > u_n, X_2 > u_n) = 0, \tag{7}$$

which together with the previous condition implies that the clusters have an alternating pattern above and below the threshold u_n.

As mentioned before, under condition $D^{(2)}_{k_n}(u_n)$ the clusters of exceedances are formed by runs of consecutive observations over u_n. However, under $D^{(3)}_{k_n}(u_n)$ the clusters may exhibit a wide variety of patterns which have in common the following property: within a cluster, the sequence does not stay below u_n for more than one time instance. One particular pattern which may occur under $D^{(3)}_{k_n}(u_n)$ is the purely oscillating pattern which is characteristic of several time series models such as first-order autoregressive models with negative coefficient. We shall only consider such type of patterns and for that we must impose the additional condition (7).

Following Chernick et al. (1991) and O'Brien (1987), Temido (2000) proved that under condition $D^{(3)}_{k_n}(u_n)$, $\lim_{n\to\infty} P\left(M_{k_n} \leq u_n\right) - \exp\left(-k_n P\left(X_1 > u_n, X_2 \leq u_n, X_3 \leq u_n\right)\right) = 0$ and that the extremal index is obtained as

$$\theta = \lim_{n\to\infty} P\left(X_2 \leq u_n(\tau, k_n), X_3 \leq u_n(\tau, k_n) | X_1 > u_n(\tau, k_n)\right).$$

However, considering in addition condition (7) we can compute the extremal index in a different way. Indeed, like in the case when $D^{(2)}_{k_n}(u_n)$ holds, a bivariate tail distribution suffices to determine the extremal index.

Lemma 5.1 *Let* $\{k_n\}$ *be a nondecreasing positive integer sequence satisfying (3) and* $\{X_n\}$ *a stationary sequence satisfying* $D_{k_n}(u_n)$, *for some real sequence* $\{u_n\}$ *such that* $\limsup k_n(1 - F(u_n)) < +\infty$.

1. *If $\{X_n\}$ satisfies $D^{(3)}_{k_n}(u_n)$ and (7) then*
$$\lim_{n\to\infty} P\left(M_{k_n} \le u_n\right) - \exp\left(-k_n P\left(X_1 > u_n, X_3 \le u_n\right)\right) = 0. \tag{8}$$

2. *If $\{X_n\}$ satisfies $D^{(3)}_{k_n}(u_n(\tau,k_n))$ and $\lim_{n\to\infty} k_n P(X_1 > u_n(\tau,k_n), X_2 > u_n(\tau,k_n)) = 0$, for all $\tau \in \Gamma(F,k_n)$ and all corresponding $u_n(\tau,k_n)$, then $\{X_n\}$ has extremal index θ if and only if*
$$\lim_{n\to\infty} P\left(X_3 \le u_n(\tau,k_n) | X_1 > u_n(\tau,k_n)\right) = \theta. \tag{9}$$

Proof: 1. Using the arguments of O'Brien (1987) we prove that, under $D_{k_n}(u_n)$,

$$P\left(M_{k_n} \le u_n\right) - \exp\left(-k_n P\left(X_1 > u_n, M_{2,r_n} \le u_n\right)\right) \to 0,\ n \to \infty.$$

But, under the assumptions of the lemma, we have

$$\begin{aligned}
P\left(X_1 > u_n, M_{2,r_n} \le u_n\right) &= P\left(X_1 > u_n \ge M_{2,3}, M_{4,r_n} \le u_n\right) \\
&= P\left(X_1 > u_n \ge M_{2,3}\right) - P\left(X_1 > u_n \ge M_{2,3}, M_{4,r_n} > u_n\right) \\
&= P\left(X_1 > u_n\right) - P\left(X_1 > u_n, X_2 > u_n\right) - P\left(X_1 > u_n, X_3 > u_n\right) \\
&\qquad + P\left(X_1 > u_n, X_2 > u_n, X_3 > u_n\right) + o_n\left(\tfrac{1}{k_n}\right) \\
&= P\left(X_1 > u_n\right) - P\left(X_1 > u_n, X_3 > u_n\right) + o_n\left(\tfrac{1}{k_n}\right) \\
&= P\left(X_1 > u_n, X_3 \le u_n\right) + o_n\left(\tfrac{1}{k_n}\right)
\end{aligned}$$

and thus (8) holds.

2. If $\{X_n\}$ has extremal index θ we have

$$\lim_{n\to\infty} P(M_{k_n} \le u_n(\tau,k_n)) = \exp(-\theta\tau), \tag{10}$$

for all $\tau \in \Gamma(F,k_n)$. Hence, by the first part of the lemma we get

$$\lim_{n\to\infty} k_n P\left(X_1 > u_n(\tau,k_n), X_3 \le u_n(\tau,k_n)\right) = \theta\tau,$$

or, equivalently, the limit (9) occurs.

The proof of the converse is similar. □

After this result we can establish the limit in distribution of $M_{k_n}(V)$ for all the models considered in this work.

In what follows we write $A^{(n)}_{t,h} := \{X_t > u_n, X_{t+1} \le u_n < X_{t+2}, \cdots, X_{h-1} \le u_n < X_h\}$ and $B^{(n)}_{t,h} := \{X_t > u_n, X_{t+2} > u_n, \cdots, X_{h-2} > u_n, X_h > u_n\}$ where $h = t + 2m$ for some nonnegative integer m. We also consider $M_{l,k} = -\infty$ if $k < l$.

Proposition 5.1 *Let $\{X_n\}$ be a stationary sequence with df F and $\{U_n\}$ be a Bernoulli $B(\beta)$ strongly mixing stationary sequence independent of $\{X_n\}$. Let $\{k_n\}$ be a nondecreasing positive integer sequence satisfying (3). Suppose that $\{X_n\}$ satisfies $D_{k_n}^{(3)}(u_n(\tau, k_n))$ and $\lim_{n\to\infty} k_n P(X_1 > u_n(\tau, k_n), X_2 > u_n(\tau, k_n)) = 0$, for all $\tau \in \Gamma(F, k_n)$ and all corresponding $u_n(\tau, k_n)$. In addition suppose that $\{X_n\}$ has extremal index $\theta > 0$ associated with a limiting cluster size distribution π. For $i \geq 1$, take*

$$\nabla_{even}(i) = P\{U_2 = 0, U_4 = 0, ..., U_{2i} = 0\}, \quad \bar{\nabla}_{even}(i) = P\{U_2 = 1, U_4 = 1, \ldots, U_{2i} = 1\},$$

$$\theta^* = \theta \left(1 - \sum_{j=1}^{\infty} \pi(j) \nabla_{even}(j)\right) / \beta \quad \textit{and} \quad \theta^{**} = \beta\theta^* + \theta \left(1 - \sum_{j=1}^{\infty} \pi(j) \bar{\nabla}_{even}(j)\right).$$

Then

$$\lim_{n\to\infty} P(M_{k_n}(V) \leq u_n(\tau, k_n)) = e^{-\theta_V \tau_V},$$

where $\theta_V = \theta^*$, $\tau_V = \tau\beta$ for the sequences $\{Y_n\}$ and $\{Z_n\}$, and $\theta_V = \theta^{**}$, $\tau_V = \tau$ for the sequence $\{W_n\}$, for all $\tau \in \Gamma(F, k_n)$.

Proof: Consider the first r_n elements of the sequences $\{X_n\}$ and $\{Y_n\}$. Take $u_n := u_n(\tau, k_n) > 0$.

As in the proof of proposition 3.1 we obtain

$$P\{M_{r_n}(Y) \leq u_n\} = 1 - \frac{\theta\tau}{s_n}\big(1 - \sum_{j=1}^{r_n} \frac{s_n}{\theta\tau} P\{\{X_1, \ldots, X_{r_n}\} \text{ contains } j$$

$$\text{exceedances all withdrawn}\}\big) + o\left(\frac{1}{s_n}\right).$$

But,

$$
\begin{aligned}
&P\{\{X_1,\ldots,X_{r_n}\} \text{ contains } j \text{ exceedances all withdrawn}\}\\
&\quad = P\{A^{(n)}_{1,2j-1}, M_{2j,r_n} \le u_n\} P\{\bigcap_{i=1}^{j}\{U_{2i}=0\}\}\\
&\qquad + P\{X_1 > u_n, A^{(n)}_{2,2j-2}, M_{2j-1,r_n} \le u_n\} P\{U_1 = 0, \bigcap_{i=1}^{j-1}\{U_{2i}=0\}\}\\
&\qquad + P\{M_{1,r_n-2j+1} \le u_n, A^{(n)}_{r_n-2j+2,r_n}\} P\{\bigcap_{i=1}^{j}\{U_{2i}=0\}\}\\
&\qquad + P\{M_{1,r_n-2j+2} \le u_n, A^{(n)}_{r_n-2j+3,r_n-1}, X_{r_n} > u_n\} P\{\bigcap_{i=1}^{j-1}\{U_{2i-1}=0\}, U_{2j-2}=0\}\\
&\qquad + \sum_{i=2}^{r_n-2j-1} P\{M_{1,i} \le u_n, A^{(n)}_{i+1,i+2j-1}, M_{i+2j,r_n} \le u_n\} P\{\bigcap_{i=1}^{j}\{U_{2i}=0\}\}\\
&\qquad + \sum_{\substack{\text{remaining}\\ \text{terms}}} P\{\{X_1,\ldots,X_{r_n}\} \text{ contains } j \text{ exceedances all withdrawn}\} \times\\
&\qquad\qquad \times P\{U_1,\ldots,U_{r_n} \text{ equals zero where the exceedances occur}\}.
\end{aligned}
$$

The first and third terms of the right-hand side of the equality are clearly $o\,(1/s_n)$ and the second and fourth are $o(1/k_n)$ by (7). The last term is bounded by

$$
\begin{aligned}
&\sum_{\substack{\text{remaining}\\ \text{terms}}} P\{\{X_1,\ldots,X_{r_n}\} \text{ contains } j \text{ exceedances all withdrawn}\}\\
&\quad \le \; r_n P(X_1 > u_n, X_2 > u_n) + r_n P(X_1 > u_n \ge M_{2,3}, M_{4,r_n} > u_n)\\
&\quad = \; o\left(\frac{1}{s_n}\right),
\end{aligned}
$$

by condition $D^{(3)}_{k_n}(u_n)$ and (7). On the other hand, by the same condition,

$$
\sum_{i=2}^{r_n-2j-1} P\{M_{1,i} \le u_n, A^{(n)}_{i+1,i+2j-1}, M_{i+2j,r_n} \le u_n\} = \pi(j)(1+o(1))\frac{\theta\tau}{s_n}.
$$

Therefore

$$
\frac{s_n}{\theta\tau} P\{\{X_1,\ldots,X_{r_n}\} \text{ contains } j \text{ exceedances all withdrawn}\} =
$$

$$= \nabla_{even}(j)\pi(j)(1+o(1))$$

and

$$P\{M_{r_n}(Y) \leq u_n\} = 1 - \frac{\theta\tau}{s_n}\left(1 - \sum_{j=1}^{r_n} \pi(j)\nabla_{even}(j)(1+o(1))\right) + o\left(\frac{1}{s_n}\right).$$

Since the sequence $\{Y_n\}$ satisfies condition $D_{k_n}(u_n)$, once again by Lemma 4 of Temido and Canto e Castro (2003) and dominated convergence we obtain

$$P\{M_{k_n}(Y) \leq u_n\} \underset{n\to\infty}{\longrightarrow} e^{-\theta\tau\left(1-\sum_{j=1}^{\infty}\pi(j)\nabla_{even}(j)\right)}.$$

As for the sequences $\{Z_n\}$ and $\{W_n\}$ it suffices to notice that $P\{M_{k_n}(Z) \leq u_n\} = P\{M_{k_n}(Y) \leq u_n\}$ and that $P\{M_{k_n}(W) \leq u_n\} = P\{M_{k_n}(Y) \leq u_n\}P\{M_{k_n}(\bar{Y}) \leq u_n\}$ where $\{\bar{Y}_n\}$ is a sequence defined by $\bar{Y}_n = (1-U_n)X_n$.

□

We now establish a result which enables us to compute the limiting cluster size distribution $\pi(\cdot)$ for stationary sequences satisfying (7) and $D^{(3)}_{k_n}(u_n)$.

Lemma 5.2 *Let $\{X_n\}$ be a stationary sequence that satisfies condition $D^{(3)}_{k_n}(u_n(\tau,k_n))$ and $\lim_{n\to\infty} k_n P(X_1 > u_n(\tau,k_n), X_2 > u_n(\tau,k_n)) = 0$, for all $\tau \in \Gamma(F,k_n)$ and all corresponding $u_n(\tau,k_n)$. In addition suppose that $\{X_n\}$ has extremal index $\theta > 0$ associated with a limiting cluster size distribution π. Then, for $j \geq 1$,*

$$\pi(j) = \frac{\lim_{n\to\infty} k_n P(X_1 \leq u_n, X_3 > u_n, X_5 > u_n, \cdots, X_{2j+1} > u_n, X_{2j+3} \leq u_n)}{\theta\tau}.$$

Proof: Observe first that

$$\pi_n(j) = P\left(\sum_{i=1}^{r_n} \chi_{n,i} = j | \sum_{i=1}^{r_n} \chi_{n,i} > 0\right) = \frac{P\left(\sum_{i=1}^{r_n} \chi_{n,i} = j\right)}{P\left(M_{r_n} > u_n\right)}.$$

Thus, attending to the proof of proposition 5.1 we have

$$\begin{aligned}
\pi_n(j) &= \frac{P\left(\sum_{i=1}^{r_n} \chi_{n,i} = j\right)}{\theta\tau\,/s_n + o(1/s_n)} \\
&= \frac{\sum_{i=2}^{r_n-2j-1} P\Big(M_{1,i} \leq u_n, A^{(n)}_{i+1,i+2j-1}, M_{i+2j,r_n} \leq u_n\Big) + o(1/s_n)}{\theta\tau\,/s_n + o(1/s_n)}.
\end{aligned}$$

If $i \geq 3$ we have

$$P(X_{i-1} \leq u_n, X_i \leq u_n, A^{(n)}_{i+1,i+2j-1}, M_{i+2j,r_n} \leq u_n)$$
$$- P(M_{1,i-2} \leq u_n, X_{i-1} \leq u_n, X_i \leq u_n, A^{(n)}_{i+1,i+2j-1}, M_{i+2j,r_n} \leq u_n)$$
$$\leq \sum_{l=1}^{i-2} P(X_l > u_n, X_{i-1} \leq u_n, X_i \leq u_n, A^{(n)}_{i+1,i+2j-1}) = o(1/k_n),$$

by condition $D^{(3)}_{k_n}(u_n)$ and stationarity. Moreover, for $i \geq 2$, we get

$$P(X_{i-1} \leq u_n, A^{(n)}_{i+1,i+2j-1}, M_{i+2j,r_n} \leq u_n)$$
$$- P(X_{i-1} \leq u_n, X_i \leq u_n, A^{(n)}_{i+1,i+2j-1}, M_{i+2j,r_n} \leq u_n)$$
$$\leq P(X_{i-1} \leq u_n, X_i > u_n, X_{i+1} > u_n) = o(1/k_n),$$

by (7). Then, for $i \geq 2$,

$$P(M_{1,i-2} \leq u_n, X_{i-1} \leq u_n, X_i \leq u_n, A^{(n)}_{i+1,i+2j-1}, M_{i+2j,r_n} \leq u_n)$$
$$= P(X_{i-1} \leq u_n, A^{(n)}_{i+1,i+2j-1}, M_{i+2j,r_n} \leq u_n) + o(1/k_n).$$

On the other hand, for $i \leq r_n - 2j - 2$, we have

$$P(X_{i-1} \leq u_n, A^{(n)}_{i+1,i+2j-1}, X_{i+2j} \leq u_n, X_{i+2j+1} \leq u_n)$$
$$- P(X_{i-1} \leq u_n, A^{(n)}_{i+1,i+2j-1}, X_{i+2j} \leq u_n, X_{i+2j+1} \leq u_n, M_{i+2j+2,r_n} \leq u_n)$$
$$\leq \sum_{l=i+2j+2}^{r_n} P(X_{i-1} \leq u_n, A^{(n)}_{i+1,i+2j-1}, X_{i+2j} \leq u_n, X_{i+2j+1} \leq u_n, X_l > u_n)$$
$$= o(1/k_n),$$

once again by $D^{(3)}_{k_n}(u_n)$ and stationarity. In the same way we deduce

$$P(X_{i-1} \leq u_n, A^{(n)}_{i+1,i+2j-1}, X_{i+2j} \leq u_n, X_{i+2j+1} \leq u_n)$$
$$= P(X_{i-1} \leq u_n, A^{(n)}_{i+1,i+2j-1}, X_{i+2j+1} \leq u_n) + o(1/k_n)$$

and

$$P(X_{i-1} \leq u_n, B^{(n)}_{i+1,i+2j-1}, X_{i+2j+1} \leq u_n)$$
$$= P(X_{i-1} \leq u_n, A^{(n)}_{i+1,i+2j-1}, X_{i+2j+1} \leq u_n) + o(1/k_n).$$

Thus, the stationarity of the process enables us to conclude that

$$\begin{aligned}
\pi_n(j) &= \frac{\sum_{i=2}^{r_n-2j-1}\left(P\left(X_{i-1}\le u_n, B^{(n)}_{i+1,i+2j-1}, X_{i+2j+1}\le u_n\right)+o(1/k_n)\right)+o(1/s_n)}{\theta\tau/s_n+o(1/s_n)}\\
&= \frac{(r_n-2j-2)\left(P\left(X_1\le u_n, B^{(n)}_{3,2j+1}, X_{2j+3}\le u_n\right)+o(1/k_n)\right)+o(1/s_n)}{\theta\tau/s_n+o(1/s_n)}\\
&\to \frac{\lim_{n\to\infty} k_n\, P\left(X_1\le u_n, B^{(n)}_{3,2j+1}, X_{2j+3}\le u_n\right)}{\theta\tau} := \pi(j).
\end{aligned}$$

□

6 Examples

In this section we give several examples of application of the the results of the previous sections. We shall consider five types of stationary sequences with extremal index $\theta<1$. For the first two examples condition $D^{(2)}_{k_n}(u_n)$ holds, while for the last three condition $D^{(2)}_{k_n}(u_n)$ fails but condition $D^{(3)}_{k_n}(u_n)$ holds.

We consider two particular types of sequences $\{U_n\}$.

1. The first is the simplest case where $\{U_n\}$ is iid with $P(U_n=1)=\beta$, $\beta\in]0,1[$.

2. The second case is a homogeneous Markov chain with one-step transition probabilities

$$\begin{cases} P\{U_n=1|U_{n-1}=1\}=\eta\\ P\{U_n=1|U_{n-1}=0\}=\mu \end{cases}$$

   In this model $\{U_n\}$ defines a sequence where the probability of failure ($U_n=0$) depends only on whether a failure has just occurred. $\{U_n\}$ forms geometric blocks of consecutive zeros followed by geometric blocks of consecutive ones. In order to have a stationary chain the initial distribution is such that $P\{U_0=1\}=\frac{1-\eta}{1-\eta+\mu}$.

   Hence, given any values of $\eta,\mu\in[0,1]$

$$\beta=\frac{\mu}{1-\eta+\mu}. \tag{11}$$

   Hall and Hüsler (2006) proved that $\{U_n\}$ is strongly mixing.

For the $\{U_n\}$ Markov chain Hall and Hüsler (2006) have determined the expressions for $\nabla(j)$ and $\bar{\nabla}(j)$:

$$\begin{aligned}\nabla(j) &= \frac{(1-\mu)^{j-1}(1-\eta)}{1-\eta+\mu},\ j \geq 1,\\ \bar{\nabla}(j) &= P\{U_1 = 1, \ldots, U_j = 1\} = \frac{\mu\eta^{j-1}}{1-\eta+\mu},\ j \geq 1.\end{aligned}$$

Using the properties of homogeneous Markov chains we obtain

$$\begin{aligned}\nabla_{even}(j) &= (1-\mu(1-\mu+\eta))^{j-1}\frac{1-\eta}{1-\eta+\mu},\ j \geq 1\\ \bar{\nabla}_{even}(j) &= (\eta^2+\mu(1-\eta))^{j-1}\frac{\mu}{1-\eta+\mu},\ j \geq 1.\end{aligned}$$

We now consider different types of stationary sequences and compute the limiting distribution of the maximum term of models M1, M2 and M3, whenever these sequences are transformed through any of the sequences $\{U_n\}$ described above.

6.1 First order max-autoregressive model (multiplicative)

We first consider a multiplicative max-autoregressive model

$$X_n = k \max\{X_{n-1}, \epsilon_n\},\ n \geq 1,$$

where $\{\epsilon_n\}$ is a sequence of iid rv's with df F, $k \in]0,1[$ and X_0 is independent of $\{\epsilon_n\}$. Alpuim (1988) has proved that $\{X_n\}$ is strong mixing and has stationary df H if and only if $w_F > 0$ and, for some $x_0 > 0$, $F(x_0/k) > 0$ and $\sum_{j=1}^{+\infty} -\ln F(x_0/k^j) < +\infty$. In this case $H(x) = \prod_{j=1}^{+\infty} F(x/k^j)$. Considering $\{\epsilon_n\}$ with a max-semistable df

$$F(x) = \exp\{-x^{-\alpha}\nu(\log x)\},\ \alpha > 0, x > 0,$$

and $r = k^{-\alpha/m}$, where m is a positive integer, Temido and Canto e Castro (2003) proved that $\{X_n\}$ has extremal index $\theta = 1 - r^{-m}$ and $H(x) = F^{\gamma}(x)$, with $\gamma = r^{-m}/(1-r^{-m})$, which is again max-semistable. Due to $H^{[r^n]}(r^{n/\alpha}x) \to H(x), n \to +\infty$, we get

$$P(M_{[r^n]} \leq r^{n/\alpha}x) \to H^{1-r^{-m}}(x), n \to +\infty.$$

Moreover, since

$$P(X_1 > u_n, X_{j-1} \leq u_n < X_j) \leq P(X_1 > u_n, X_{j-1} \leq u_n, \epsilon_j > u_n/k)$$

$$\leq (1 - H(u_n))(1 - F(u_n/k))$$

condition $D^{(2)}_{k_n}(r^{n/\alpha}x))$ holds. Following Alpuim (1988) the cluster size distribution is geometric given by $\pi(j) = (1 - r^{-m})(r^{-m})^{j-1}$, $j \geq 1$.

Considering the two different types of sequences $\{U_n\}$ described in the beginning of the section we obtain the following results:

1. **IID missing values**

   Applying proposition 3.1 we obtain

$$\theta^* = \frac{1 - r^{-m}}{1 - (1-\beta)r^{-m}} \quad \text{and} \quad \theta^{**} = (1 - r^{-m})\frac{1 - r^{-m}(1 - 2\beta(1-\beta))}{(1 - r^{-m}(1-\beta))(1 - r^{-m}\beta)}.$$

2. **Missing values through a Markov chain**

   Since $\{X_n\}$ satisfies condition $D^{(2)}_{k_n}(u_n)$ we may apply proposition 4.2 and obtain

$$\theta^* = \frac{(1 - r^{-m})\,(1 - r^{-m}(\eta - \mu))}{1 - r^{-m}(1-\mu)}$$

$$\theta^{**} = \frac{(1 - r^{-m})\,(1 - r^{-m}(\eta - \mu))}{1 - \eta + \mu}\left(\frac{\mu}{1 - r^{-m}(1-\mu)} + \frac{1-\eta}{1 - r^{-m}\eta}\right).$$

In either case

$$\lim_{n\to\infty} P(M_{[r^n]}(V) \le r^{n/\alpha}x) = e^{-\theta_V \tau_V}$$

where $\theta_V = \theta^*$, $\tau_V = \tau\beta$ for the sequences $\{Y_n\}$ and $\{Z_n\}$, and $\theta_V = \theta^{**}$, $\tau_V = \tau$ for the sequence $\{W_n\}$, $\tau = x^{-\alpha}\gamma\nu(\log x)$.

6.2 First order max-autoregressive model (additive)

Now suppose that $\{X_n\}$ is an integer-valued stationary additive max-autoregressive sequence with marginal df H satisfying (1) (Anderson's class). More precisely,

$$X_n = \max\{X_{n-1}, \epsilon_n\} - c,\; n \ge 1,$$

where $c \in \mathbb{N}$, $\{\epsilon_n\}$ is an iid sequence with df F and X_0 is independent of $\{\epsilon_n\}$. Hall (1996) proved that $\{X_n\}$ is strong mixing and has stationary df $H(x) = \prod_{j=1}^{+\infty} F(x + jc)$. She also proved that condition $D^{(2)}(u_n)$ holds for $\{X_n\}$. Thus $D^{(2)}_{k_n}(u_n(\tau, k_n))$ also holds. Moreover, with the arguments of Hall (1996), we prove that $\{X_n\}$ has extremal index $\theta = 1 - r^{-c}$. It is also known, due to Alpuim (1988), that the cluster size distribution is geometric, $\pi(j) = (1 - r^{-c})r^{-(j-1)c}$, $j \ge 1$.

For instance, if H is the df of the Negative Binomial $NB(m,p)$ distribution, we have

$$P(M_{[p^{-n}]} \le x + b_n) \to \exp(-(1-p^c)p^{[x]}), n \to +\infty, x \in \mathbb{R} \setminus \mathbb{Z},$$

with $b_n = n - 1 - \frac{1}{\log p}\{(m-1)\log n + \log(\frac{(1-p)^{m-1}}{(m-1)!})\}$ and $\pi(j) = (1-p^c)p^{(j-1)c}$, $j \ge 1$.

Again, considering the two different types of sequences $\{U_n\}$ described in the beginning of the section we obtain the following results:

1. **IID missing values**

   Applying proposition 3.1 we obtain

$$\theta^* = \frac{(1-r^{-c})}{1-(1-\beta)r^{-c}} \quad \text{and} \quad \theta^{**} = (1-r^{-c})\frac{1-r^{-c}(1-2\beta(1-\beta))}{(1-r^{-c}(1-\beta))(1-r^{-c}\beta)}.$$

2. **Missing values through a Markov chain**

   Since $\{X_n\}$ satisfies condition $D^{(2)}_{k_n}(u_n)$ we may apply proposition 4.2 and obtain

$$\theta^* = \frac{(1-r^{-c})\,(1-r^{-c}(\eta-\mu))}{(1-r^{-c}(1-\mu))}$$

$$\theta^{**} = \frac{(1-r^{-c})\,(1-r^{-c}(\eta-\mu))}{1-\eta+\mu}\left(\frac{\mu}{1-r^{-c}(1-\mu)}+\frac{1-\eta}{1-r^{-c}\eta}\right).$$

In either case

$$\lim_{n\to\infty} P(M_{[r^n]}(V) \le x+b_n) = e^{-\theta_V\tau_V}, \quad x\in\mathbb{R}\setminus\mathbb{Z},$$

where $\theta_V = \theta^*$, $\tau_V = \tau\beta$ for the sequences $\{Y_n\}$ and $\{Z_n\}$, and $\theta_V = \theta^{**}$, $\tau_V = \tau$ for the sequence $\{W_n\}$, $\tau = r^{-[x]}$ and $\{b_n\}$ is an appropriate sequence of constants satisfying $\lim_{n\to\infty} b_n = +\infty$.

6.3 Second order max-autoregressive model (additive)

Now suppose that $\{X_n\}$ is an integer-valued stationary additive second order max-autoregressive sequence with marginal df H satisfying (1) (Anderson's class). More precisely,

$$X_n = \max\{X_{n-2},\epsilon_n\} - c, \quad n\ge 2,$$

where $c\in\mathbb{N}$ and $\{\epsilon_n\}$ is an iid sequence with df F satisfying $\frac{1-F(n-1)}{1-F(n)} = r$, with r in $]1,+\infty[$. Assuming that X_0, X_1 and $\{\epsilon_n\}$ are independent Hall (1998) proved that $\{X_n\}$ is strong mixing and has a stationary df given by $H(x)=\prod_{j=1}^{+\infty}F(x/k^j)$. Moreover, Hall and Temido (2007) proved that condition $D^{(3)}_{k_n}(u_n)$ holds and $\{X_n\}$ has extremal index $\theta = 1-r^{-c}$. Consequently

$$P(M_{[r^n]} \le x+b_n) \to \exp\{-(1-r^{-c})r^{-[x]}\}, \quad n\to+\infty, \quad x\in\mathbb{R}\setminus\mathbb{Z}.$$

Furthermore, since X_0 and X_1 are independent, condition (7) holds for $\{X_n\}$. Then, applying Lemma 5.2, by induction, we prove that the cluster size distribution is geometric given by $\pi(j) = (1-r^{-c})(r^{-c})^{j-1}$, $j\ge 1$.

Considering the two different types of sequences $\{U_n\}$ described in the beginning of the section we obtain the following results:

1. **IID missing values**

   Applying proposition 3.1 we obtain

$$\theta^* = \frac{1 - r^{-c}}{1 - (1-\beta)r^{-c}} \quad \text{and} \quad \theta^{**} = (1 - r^{-c})\frac{1 - r^{-c}(1 - 2\beta(1-\beta))}{(1 - r^{-c}(1-\beta))(1 - r^{-c}\beta)}.$$

2. **Missing values through a Markov chain**

   Since $\{X_n\}$ satisfies condition $D^{(3)}_{k_n}(u_n)$ and (7) we may apply proposition 5.1 and obtain

$$\begin{aligned}
\theta^* &= \tfrac{(1-r^{-c})}{\beta}\left(1 - \tfrac{\left(1-r^{-c}\right)(1-\beta)}{1-r^{-c}(1-\mu(1-\mu+\eta))}\right) \\
\theta^{**} &= (1 - r^{-c})\left(1 - \tfrac{\left(1-r^{-c}\right)(1-\beta)}{1-r^{-c}(1-\mu(1-\mu+\eta))} + 1 - \tfrac{\left(1-r^{-c}\right)\beta}{1-r^{-c}(\eta^2+\mu(1-\eta))}\right),
\end{aligned}$$

   where $\beta = \frac{\mu}{1-\eta+\mu}$.

In either case

$$\lim_{n\to\infty} P(M_{[r^n]}(V) \le x + b_n) = e^{-\theta_V \tau_V}, \quad x \in \mathbb{R} \setminus \mathbb{Z},$$

where $\theta_V = \theta^*$, $\tau_V = \tau\beta$ for the sequences $\{Y_n\}$ and $\{Z_n\}$, and $\theta_V = \theta^{**}$, $\tau_V = \tau$ for the sequence $\{W_n\}$, $\tau = r^{-[x]}$ and $\{b_n\}$ is an appropriate real sequence satisfying $\lim_{n\to\infty} b_n = +\infty$.

The last two examples consist of sequences with marginal distribution in the domain of attraction of a max-stable distribution. Hence, it suffices to consider $k_n = n$ in these cases. Although max-semistability is not required to study the effect of missing values on their extremes, we include them in the present work because the condition $D^{(2)}_{k_n}(u_n)$ does not hold and the results of Hall and Hüsler (2006) are not enough to obtain the limiting distribution of the maximum. Due to their alternating nature, conditions $D^{(3)}_{k_n}(u_n)$ and (7) hold, and hence the present results allow us to obtain the desired MS limiting distributions.

6.4 Negative AR(1) model with uniform margins

This example concerns the negatively correlated uniform AR(1) defined by

$$X_n = -\frac{1}{\varrho}X_{n-1} + \varepsilon_n,\, n \ge 1,$$

where $\varrho > 1$, $X_0 \sim \mathcal{U}(0,1)$, $\{\varepsilon_n\}$ is a sequence of iid rv's with $P(\varepsilon_1 = j/\varrho) = 1/\varrho$, for $j \in \{1, 2, ..., \varrho\}$ and X_0 is independent of $\{\varepsilon_n\}$. Asymptotic results for the extremes from this model were originally obtained in Chernick and Davis (1982). Chernick et al. (1991)

proved that $\{X_n\}$ satisfies condition $D^{(3)}(1-x/n)$ and that the extremal index is given by $\theta = 1 - 1/\varrho^2$.

Take $u_n = 1 - x/n$ and observe that $\tau := \tau(x) = x$. In view of

$$\begin{aligned} nP\left(X_1 > u_n, X_2 > u_n\right) &= nP\left(X_1 > u_n, \varepsilon_2 > u_n\left(1+\tfrac{1}{\varrho}\right)\right) \\ &= nP\left(X_1 > u_n\right)P\left(\varepsilon_2 > u_n\left(1+\tfrac{1}{\varrho}\right)\right) \to 0, n \to +\infty, \end{aligned}$$

and hence condition (7) holds. Using this result we can compute the extremal index in a simpler way. Indeed due to (9) and attending that

$$\lim_{n\to\infty} P(-\frac{1}{\varrho}\varepsilon_j + \varepsilon_{j+1} > u_n(1-\varrho^2) = P(\varepsilon_{j+1} = 1, \varepsilon_j = 1/\varrho) = 1/\varrho^2,\ j \geq 1,$$

we deduce

$$\begin{aligned} \theta &= \lim_{n\to\infty} \frac{P(X_1 > u_n, \frac{1}{\varrho^2}X_1 - \frac{1}{\varrho}\varepsilon_2 + \varepsilon_3 \leq u_n)}{P(X_1 > u_n)} \\ &= \lim_{n\to\infty} P(-\frac{1}{\varrho}\varepsilon_2 + \varepsilon_3 \leq u_n(1 - 1/\varrho^2)) \\ &= 1 - 1/\varrho^2. \end{aligned}$$

Using the previous results we obtain

$$P(M_n \leq 1 - x/n) \to \exp(-(1-\varrho^{-2})x), n \to +\infty,\ x > 0.$$

Now, in order to compute the limit cluster size $\pi(j)$ we first prove, by induction, that

$$P(B^{(n)}_{1,2j+1}) = (\frac{1}{\varrho^2})^j \frac{x}{n}(1 + o_n(1)),\ j \geq 1.$$

In fact

$$\begin{aligned} P(X_1 > u_n, X_3 > u_n) &= P(X_1 > u_n) - P(X_1 > u_n, X_3 \leq u_n) \\ &= \tfrac{x}{n}(1-\theta)(1 + o_n(1)) = \tfrac{x}{n}\tfrac{1}{\varrho^2}(1 + o_n(1)) \end{aligned}$$

and

$$P(B^{(n)}_{1,2j-1}, X_{2j+1} > u_n) = P(B^{(n)}_{1,2j-1}, \tfrac{1}{\varrho^2}X_{2j-1} - \tfrac{1}{\varrho}\varepsilon_{2j} + \varepsilon_{2j+1} > u_n)$$

$$= P(B^{(n)}_{1,2j-1})P(-\tfrac{1}{\varrho}\varepsilon_{2j} + \varepsilon_{2j+1} > u_n(1 - \tfrac{1}{\varrho^2})) = (\tfrac{1}{\varrho^2})^{j-1}\tfrac{x}{n}\tfrac{1}{\varrho^2}(1 + o_n(1)).$$

Thus, by stationarity we obtain

$$\begin{aligned} P(X_1 \leq u_n, B^{(n)}_{3,2j+1}, X_{2j+3} > u_n) &= P(B^{(n)}_{3,2j+1}) - 2P(B^{(n)}_{1,2j+1}) + P(B^{(n)}_{1,2j+3}) \\ &= \left((\tfrac{1}{\varrho^2})^{j-1}\tfrac{x}{n} - 2(\tfrac{1}{\varrho^2})^j\tfrac{x}{n} + (\tfrac{1}{\varrho^2})^{j+2}\tfrac{x}{n}\right)(1 + o_n(1)) \\ &= \tfrac{x}{n}(\tfrac{1}{\varrho^2})^{j-1}(1 - \tfrac{1}{\varrho^2})^2(1 + o_n(1)) \end{aligned}$$

and so

$$\pi(j) = \frac{\lim_{n\to\infty} \frac{x}{n} (\frac{1}{\varrho^2})^{j-1} (1 - \frac{1}{\varrho^2})^2 (1 + o_n(1))}{x(1 - \frac{1}{\varrho^2})} = (\frac{1}{\varrho^2})^{j-1} (1 - \frac{1}{\varrho^2}), \text{ for } j \geq 1,$$

which corresponds to a geometric distribution.

We now obtain the limiting distribution of the maximum term of models M1, M2 and M3, whenever the negative uniform AR(1) model is transformed through either the sequences $\{U_n\}$ described in the beginning of the section.

1. **IID missing values** Applying proposition 3.1 we obtain

$$\theta^* = \frac{(1 - \varrho^{-2})}{1 - (1 - \beta)\varrho^{-2}} \quad \text{and} \quad \theta^{**} = (1 - \varrho^{-2}) \frac{1 - \varrho^{-2}(1 - 2\beta(1 - \beta))}{(1 - \varrho^{-2}(1 - \beta))(1 - \varrho^{-2}\beta)}.$$

2. **Missing values through a Markov chain**
Since $\{X_n\}$ satisfies condition $D^{(3)}_{k_n}(u_n)$ and (7) we may apply proposition 5.1 and obtain

$$\theta^* = \frac{(1-\varrho^{-2})}{\beta} \left(1 - \frac{(1-\varrho^{-2})(1-\beta)}{1-\varrho^{-2}(1-\mu(1-\mu+\eta))}\right)$$

$$\theta^{**} = (1 - \varrho^{-2}) \left(1 - \frac{(1-\varrho^{-2})(1-\beta)}{1-\varrho^{-2}(1-\mu(1-\mu+\eta))} + 1 - \frac{(1-\varrho^{-2})\beta}{1-\varrho^{-2}(\eta^2+\mu(1-\eta))}\right),$$

where $\beta = \frac{\mu}{1-\eta+\mu}$.

In either case

$$\lim_{n\to\infty} P(M_n(V) \leq 1 - x/n) = e^{-\theta_V \tau_V},$$

where $\theta_V = \theta^*$, $\tau_V = \tau\beta$ for the sequences $\{Y_n\}$ and $\{Z_n\}$, and $\theta_V = \theta^{**}$, $\tau_V = \tau$ for the sequence $\{W_n\}$, with $\tau = x > 0$.

6.5 Negative AR(1) model with regularly varying tails

Our last example concerns again an AR(1) stationary process

$$X_n = -\psi X_{n-1} + Z_n,\ n \geq 1,$$

where $\psi \in]0, 1[$ and $\{Z_n\}$ is a sequence of iid rv's independent of X_0. Following Scotto et al. (2003) we consider that the margins of $\{Z_n\}$ possesses regularly varying balanced tails

$$\lim_{t\to\infty} \frac{P(|Z_1| > tx)}{P(|Z_1| > t)} = x^{-\alpha},$$

for $\alpha > 0$ and $x > 0$, and satisfies the tail balancing conditions

$$\lim_{x\to\infty} \frac{P(Z_1 > x)}{P(|Z_1| > x)} = p \in [0, 1],$$

and

$$\lim_{x\to\infty} \frac{P(Z_1 < -x)}{P(|Z_1| > x)} = 1 - p.$$

We now prove that this sequence satisfies (7) and $D^{(3)}(u_n(\tau, n))$. In fact

$$\begin{aligned} nP\left(X_1 > u_n, X_2 > u_n\right) &= nP\left(X_1 > u_n, Z_2 > u_n + \psi X_1\right) \\ &= nP\left(Z_2 > u_n(1+\psi)\right) P\left(X_1 > u_n\right) \to 0, n \to +\infty, \end{aligned}$$

because $\psi > 0$. On the other hand

$$\begin{aligned} &P\left(X_1 > u_n, X_{j-2} \le u_n, X_{j-1} \le u_n, X_j > u_n\right) \\ &\quad = P\left(X_1 > u_n, X_{j-2} \le u_n, X_{j-1} \le u_n, \psi^2 X_{j-2} - \psi Z_{j-1} + Z_j > u_n\right) \\ &\quad = P\left(X_1 > u_n, X_{j-2} \le u_n, X_{j-1} \le u_n, -\psi Z_{j-1} + Z_j > u_n(1-\psi^2)\right) \\ &\quad \le P\left(X_1 > u_n, X_{j-2} \le u_n, X_{j-1} \le u_n, |-\psi Z_{j-1} + Z_j| > u_n(1-\psi^2)\right) \\ &\quad \le P\left(X_1 > u_n, X_{j-2} \le u_n, X_{j-1} \le u_n, \psi|Z_{j-1}| + |Z_j| > u_n(1-\psi^2)\right) \\ &\quad \le P\left(X_1 > u_n, X_{j-2} \le u_n, X_{j-1} \le u_n, 2\max\{\psi|Z_{j-1}|, |Z_j|\} > u_n(1-\psi^2)\right) \\ &\quad \le P\left(X_1 > u_n, X_{j-2} \le u_n, X_{j-1} \le u_n, \psi|Z_{j-1}| > \tfrac{u_n(1-\psi^2)}{2}\right) \\ &\qquad\quad + P\left(X_1 > u_n, X_{j-2} \le u_n, X_{j-1} \le u_n, |Z_j| > \tfrac{u_n(1-\psi^2)}{2}\right). \end{aligned}$$

Thus

$$\begin{aligned} &n\sum_{j=2}^{r_n} P\left(X_1 > u_n, X_{j-2} \le u_n, X_{j-1} \le u_n, X_j > u_n\right) \\ &\quad \le nr_nP\left(X_1 > u_n\right) P\left(|Z_2| > u_n\tfrac{1-\psi^2}{2\psi}\right) + nr_nP\left(X_1 > u_n\right) P\left(|Z_3| > u_n\tfrac{(1-\psi^2)}{2}\right) \\ &\quad = r_nP\left(X_1 > u_n\right) nP\left(|Z_2| > u_n\right) \frac{P\left(|Z_2| > u_n\frac{1-\psi^2}{2\psi}\right)}{P\left(|Z_2| > u_n\right)} \\ &\qquad\qquad + r_nP\left(X_1 > u_n\right) nP\left(|Z_3| > u_n\right) \frac{P\left(|Z_3| > u_n\frac{1-\psi^2}{2}\right)}{P\left(|Z_3| > u_n\right)} \\ &\quad \to 0 \times \tau \times (\tfrac{1-\psi^2}{2\psi})^{-\alpha} + 0 \times \tau \times (\tfrac{1-\psi^2}{2})^{-\alpha} = 0, n \to +\infty. \end{aligned}$$

The extremal index of $\{X_n\}$ is given by $\theta = 1 - \psi^{2\alpha}$ and $\tau := \tau(x) = \frac{p+q\psi^\alpha}{1-\psi^{2\alpha}} x^{-\alpha}$ (Scotto et al. (2003)) and

$$P(M_n \le a_n x) \to \exp\{-(p + q\psi^\alpha)x^{-\alpha}\}, \quad n \to +\infty, \quad x > 0,$$

with $\{a_n\}$ satisfying $nP(|Z_1| > a_n) \to 1$, $n \to +\infty$.

Furthermore, with the arguments used in the last example we deduce that

$$\pi(j) = \psi^{2\alpha(j-1)}(1 - \psi^{2\alpha}), \text{ for } j \ge 1.$$

We observe that this coincides with the result of Scotto et al. (2003) for $\pi_1(j)$, taking into account the differences of the definitions of $\pi(j)$ used in this work (given by Leadbetter and Nandagopalan (1989)) and of $\pi_1(j)$ used by those authors (which is the same as in Embrechts et al. (1997), pag 273).

We now obtain the limiting distribution of the maximum term of models M1, M2 and M3, whenever the negative heavy tailed AR(1) model is transformed through either the sequences $\{U_n\}$ described in the beginning of the section.

1. **IID missing values**

   Applying proposition 3.1 we obtain

$$\theta^* = \frac{1 - \psi^{2\alpha}}{1 - (1-\beta)\psi^{2\alpha}} \quad \text{and} \quad \theta^{**} = (1 - \psi^{2\alpha})\frac{1 - \psi^{2\alpha}(1 - 2\beta(1-\beta))}{(1 - \psi^{2\alpha}(1-\beta))(1 - \psi^{2\alpha}\beta)}.$$

2. **Missing values through a Markov chain**

   Since $\{X_n\}$ satisfies condition $D^{(3)}_{k_n}(u_n)$ and (7) we may apply proposition 5.1 and obtain

$$\begin{aligned}
\theta^* &= \frac{(1-\psi^{2\alpha})}{\beta}\left(1 - \frac{\left(1-\psi^{2\alpha}\right)(1-\beta)}{1-\psi^{2\alpha}(1-\mu(1-\mu+\eta))}\right)\\
\theta^{**} &= (1 - \psi^{2\alpha})\left(1 - \frac{\left(1-\psi^{2\alpha}\right)(1-\beta)}{1-\psi^{2\alpha}(1-\mu(1-\mu+\eta))} + 1 - \frac{\left(1-\psi^{2\alpha}\right)\beta}{1-\psi^{2\alpha}(\eta^2+\mu(1-\eta))}\right),
\end{aligned}$$

   where $\beta = \frac{\mu}{1-\eta+\mu}$.

In either case

$$\lim_{n\to\infty} P(M_n(V) \le a_n x) = e^{-\theta_V \tau_V}, \quad x > 0,$$

where $\theta_V = \theta^*$, $\tau_V = \tau\beta$ for the sequences $\{Y_n\}$ and $\{Z_n\}$, and $\theta_V = \theta^{**}$, $\tau_V = \tau$ for the sequence $\{W_n\}$, with τ as above.

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