Asymptotic Growth in Nonlinear Stochastic and Deterministic Functional Differential Equations

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Declaration

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To My Parents

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Notation and Abbreviations

Basic Sets and Spaces

$\mathbb{R}$	$(-\infty, \infty)$
$\mathbb{R}^+$	$[0, \infty)$
$\mathbb{Z}$	$\{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{Z}^+$	$\{0, 1, 2, \dots\}$
$\mathbb{C}$	Complex numbers

Function and Sequence Spaces

$V(X;Y)$	Functions of type V with domain X and range contained in Y.
$V_{loc}(X;Y)$	Functions of type V with domain X that belong to $V(K;Y)$ for every compact subset K of X.
C	Continuous functions.
AC	Absolutely continuous functions.
C^k	k–times differentiable functions with continuous k–th derivative.
L^p	Measurable functions with finite norm $\{\int \lvert\phi(t)\rvert^p \, dt\}^{1/p}$ with $1 \le p < \infty$.
$l^p(X)$	Real–valued sequences on X with finite norm $\{\sum_X \lvert x_j\rvert^p\}^{1/p}$ with $1 \le p < \infty$.
AP	Almost periodic.

Probabilistic/Measure Theoretic Notation

$a.s.$	Almost sure (on an event of probability 1).
$\mathcal{B}(X)$	Borel sigma algebra taken on X.
$M(X;Y)$	Finite measures on $(X, \mathcal{B}(X))$ with range contained in Y and total variation norm.
$M_{loc}(X;Y)$	Finite measures on X that belong to $M(K;Y)$ for every compact subset K of X.

Dini Derivatives

$$D^+u(t) = \limsup_{h \to 0^+} \frac{u(t+h) - u(t)}{h}, \quad D_+u(t) = \liminf_{h \to 0^+} \frac{u(t+h) - u(t)}{h},$$
$$D^-u(t) = \limsup_{h \to 0^-} \frac{u(t+h) - u(t)}{h}, \quad D_-u(t) = \liminf_{h \to 0^-} \frac{u(t+h) - u(t)}{h}.$$

Operators and other Symbols

$\lvert \cdot \rvert$	Absolute value of a scalar or norm of a vector.
$\mathscr{R}$	Real part of a complex number.
$\mathscr{F}$	Imaginary part of a complex number.

Abstract

Asymptotic Growth in Nonlinear Stochastic and Deterministic Functional Differential Equations

Denis D. Patterson

This thesis concerns the asymptotic growth of solutions to nonlinear functional differential equations, both random and deterministic. How quickly do solutions grow? How do growth rates of solutions depend on the memory and the nonlinearity of the system? What is the effect of randomness on the growth rates of solutions? We address these questions for classes of nonlinear functional differential equations, principally convolution Volterra equations of the second kind.

We first study deterministic equations with sublinear nonlinearity and integrable kernels. For such systems, we prove that the growth rates of solutions are *independent* of the distribution of the memory. Hence we conjecture that stronger memory dependence is needed to generate growth rates which depend meaningfully on the delay structure. Using the theory of regular variation, we then demonstrate that solutions to a class of sublinear Volterra equations with *non–integrable kernels* grow at a memory dependent rate.

We complete our treatment of sublinear equations by examining the impact of stochastic perturbations on our previous results; we consider the illustrative and important cases of Brownian and α–stable Lévy noise. In summary, if an appropriate functional of the forcing term has a limit L at infinity, solutions behave asymptotically like the underlying unforced equation when $L = 0$ and like the forcing term when $L = +\infty$. Solutions inherit properties of both the forcing term and underlying unforced equation for $L \in (0, \infty)$. Similarly, we prove linear discrete Volterra equations with summable kernels inherit the behaviour of unbounded perturbations, random or deterministic.

Finally, we consider Volterra integro–differential equations with superlinear nonlinearity and non-singular kernels. We provide sharp estimates on the rate of blow–up if solutions are explosive, or unbounded growth if solutions are global. We also recover well–known necessary and sufficient conditions for finite–time blow–up via new methods.

Chapter 1

Introduction and Overview

1.1 Motivation and Goals

Differential equations are ubiquitous in the modelling of complex systems in both the natural and social sciences. Linear models are commonplace, and mathematically well–understood, but many real–world phenomena demand sophisticated nonlinear models. Nonlinearity alone introduces substantial mathematical complexity but, in many applications, we must also model the influence of past events on the dynamics. The system possesses a "memory" of its previous states, or delays and time lags may cause the recent past to affect the evolution of the process significantly. In this situation, the dynamical system is said to be a functional differential equation (FDE). If all former states are important the equation is called a Volterra equation. Finally, external chance events or uncertainty intrinsic to the problem demand that randomness is accounted for: in this case, the evolution studied is a stochastic functional differential equation (SFDE). The goal of this thesis is to provide a comprehensive theory regarding the growth of solutions to highly nonlinear deterministic and stochastic FDEs. In particular, we wish to understand how rapidly solutions of such equations can grow and to determine the relationship between the memory, the strength of dependence on the state (nonlinearity), the system's initial configuration, and the long–term growth rate.

According to Hale [57], FDEs have been studied in some form for over 200 years but systematic investigations only began in earnest in the 20th century. Much of this development was application–driven; Picard investigated hereditary effects in physics in 1908 and Volterra constructed integro–differential models for viscoelastic problems as early as 1909. Volterra later became more focused on hereditary effects in the interaction of species (circa 1931) and to this day population dynamics remains an active area of application for FDEs (see e.g. Cushing [42] or Kuang [71]). From the 1940's onward there was a rapid advancement in the theory of FDEs with control systems and engineering applications in mind (see Bellman and Cooke [22]). The Soviet school was particularly active during this era with Myshkis [95] and Krasovskii [24] (among others) making important contributions. In the 1960's delay and functional differential equations were widely studied in the context of economic modelling (see e.g. Samuelson [110]), but the first work in this area actually dates back to Haldane [56] and Kalecki [65, 66] in the mid 1930's. More modern applications in this area include the modelling of stock and commodity prices by FDEs and SFDEs (see e.g. [4, 40, 41, 82]). Beside the applications mentioned above, FDEs also arise naturally in models of nuclear reactors (see Levin and Nohel [76]), heat flow, lasers, advertising policies, financial management and many more diverse areas besides; Kolmanovskii and Myshkis [70] provide a more thorough description of precisely how FDEs arise in each of these applied areas.

Naturally, many researchers have devoted considerable attention and effort to answering questions concerning the existence and uniqueness of solutions to FDEs (see Gripenberg et al. [50] for a modern and comprehensive account). Broadly speaking, this theory is mature and more than fit for purpose with regard to applications. Of course, knowing that solutions exist and are unique is certainly necessary from a modelling perspective, but it is clearly not sufficient. We are typically interested in much more detailed information in practical situations – this is usually referred to as the *qualitative behaviour of solutions*. For example, does the population tend to an equilibrium level, at what temperature will a metal combust, or how does the growth rate of our economy depend on past activity? For particular models, these are all properties which can be determined analytically. Furthermore, analytical results regarding qualitative properties can serve to guide and validate numerical investigations which often seek to answer even more refined questions.

In this thesis, we aim to provide detailed results regarding the qualitative behaviour of growing solutions to general classes of FDEs under minimal hypotheses. In particular, we aim for practically useful results which allow the behaviour of solutions to be determined *explicitly in terms of the problem data*. To illustrate this point, define the spaces

$$C_h = \left\{ x \in C([0,T];\mathbb{R}) : \sup_{t\in[0,T)} \frac{|x(t)|}{h(t)} < \infty \right\}, \quad h \in C([0,T);\mathbb{R}^+), \quad T > 0,$$

and recall that a pair of spaces (A, B) is admissible with respect to the operator T if $T(A) \subset B$. Now consider the following well–known result from the admissibility theory of nonlinear Volterra equations.

Theorem 1.1.1 (Corduneanu [39, Theorem 4.1.2]). *Consider the integral equation*

$$x(t) = h(t) + \int_0^t k(t,s) f(s, x(s))\, ds, \quad t \in [0,T), \tag{1.1.1}$$

for some $T > 0$. Suppose (C_g, C_G) is admissible with respect to the linear Volterra operator $(Kx)(t) = \int_0^t k(t,s)x(s)\, ds$ for $t \in [0,T)$, $f : C_G \mapsto C_g$ is Lipschitz continuous in its second argument, and $h \in C_G$. Then (1.1.1) has a unique solution $x \in C_G$.

The result stated above is very general and *potentially* very powerful. If we can identify a suitable pair of functions (g, G), then there exists a positive constant L such that $|x(t)| \le L\, G(t)$ for $t \in [0,T)$, i.e. the growth or decay of the solution to (1.1.1) can be bounded in terms of G. However, our task in reaching such a conclusion is considerable. Clearly, (g, G) must be related in some way to the problem data, i.e. the functions h, k and f, but we are given no hint whatsoever as to how to determine the pair (g, G) from the given data. By contrast, in this thesis, we concentrate on less general classes of equations with features which appear to us to arise most frequently in applications. To wit, our focus is primarily on scalar equations of convolution–type — convolution structure is a typical feature in models of systems which are stationary in time. This judicious specialisation allows us to formulate sharp conclusions depending solely, and in an explicit fashion, on the problem data. Furthermore, addressing the scalar case is clearly a necessary first step in tackling higher dimensional equations and we believe this case is likely to contain the key steps needed for generalisation to higher dimensions. Perhaps our most innocuous decision is that we choose to work in continuous–time. However, we do not believe that this has a significant impact on our results and it is likely that much of our programme has a direct analogue in discrete–time (and perhaps even on other time scales [122]). Indeed, these claims are supported by some investigations already undertaken; in Chapter 5 we venture briefly into discrete–time in order to investigate discrete analogues to some of our nonlinear results for linear equations and we have also proven discrete analogues to many of our sublinear results [14]. Finally, we choose to primarily study equations of integro–differential type, as opposed to pure integral equations such as (1.1.1) (cf. equations (1.2.1), (1.2.2) and (1.2.3)). This choice is primarily motivated by our desire to generalise our results to stochastic equations, although our integro–differential equations are readily put in the form of (1.1.1) by integration.

To paraphrase the conclusion of Theorem 1.1.1: Under appropriate conditions, if $h \in C_G$, then $x \in C_G$. In other words, the solution to (1.1.1) inherits its asymptotic behaviour entirely from the exogenous forcing term h; the dynamics of the solution are ultimately ambivalent to both the kernel k and the nonlinearity f. This striking feature is generic in the literature on admissibility for Volterra operators, particularly linear operators [37, 50]. This thesis contains new results of a similar flavour in a nonlinear setting – we typically prove that when the forcing term h is sufficiently large relative to the nonlinearity and kernel (in an appropriate sense), then the long–term behaviour of the solution is the same as that of h to leading–order. However, this immediately begs the question: what happens when h is not "sufficiently large" to dominate the dynamics? In this case, we hope to find some function, now depending explicitly on k and f, which describes the behaviour of the system in an appropriate sense. In particular, we often prove that there exists a function A, depending only on k and f, such that $|x(t)|/A(t) \to 1$ as $t \to \infty$; in other words, the leading–order behaviour of the solution is given by A as $t \to \infty$. The bulk of this thesis will concentrate on results of the two types outlined above. Furthermore, in certain advantageous situations, we can interpolate between these two cases and describe precisely the transition from when the state–dependent term governs the dynamics to when the exogenous perturbation dominates.

The remainder of this chapter outlines some of our main results in more detail and prepares some necessary mathematical preliminaries.

1.2 Overview

In Chapter 2 we analyse growth rates of positive solutions to scalar nonlinear functional and Volterra differential equations of the form

$$x'(t) = \int_{[0,\tau]} \mu_1(ds) f(x(t-s)) + \int_{[0,t]} \mu_2(ds) f(x(t-s)), \quad t > 0; \tag{1.2.1}$$
$$x(t) = \psi(t), \quad t \in [-\tau, 0], \quad \tau > 0,$$

where $\mu_1 \in M([0,\tau]; \mathbb{R}^+)$ and $\mu_2 \in M(\mathbb{R}^+; \mathbb{R}^+)$ are finite measures. The nonlinear dependence in (1.2.1) is assumed to be sublinear, in the sense that $\lim_{x\to\infty} f(x)/x = 0$, and we impose extra regularity properties on a function asymptotic to the nonlinear function, f, rather than on the nonlinearity itself. The main result of Chapter 2 reduces the analysis of growing solutions to (1.2.1) to determining the rate of growth of solutions to the *autonomous* ODE obtained by concentrating all of the mass of the measures at zero, i.e.

$$y'(t) = \{\mu_1(\mathbb{R}^+) + \mu_2(\mathbb{R}^+)\} f(y(t));$$

this contrasts markedly with the theory for linear equations in which such estimates are (in general) not sharp. However, the aforementioned estimates on the asymptotic growth rate of solutions are implicit. In particular, we prove that $A(x(t), t) \to 1$ as $t \to \infty$ for some appropriately chosen function A; in general, one cannot hope to do better (see [16] and Section 2.3). However, under additional conditions on the nonlinearity, we supply more direct asymptotic information by showing that $x(t) \sim B(t)$ as $t \to \infty$ for some function B which is constructed explicitly from the problem data.

Chapter 3 details our investigation into *memory–dependent* growth in scalar Volterra equations of the form

$$x'(t) = \int_{[0,t]} \mu(ds) f(x(t-s)), \quad t \geq 0; \quad x(0) = \psi > 0, \tag{1.2.2}$$

where $\mu \in M_{loc}(\mathbb{R}^+; \mathbb{R}^+)$. However, in contrast to Chapter 2, we now consider the case in which $\mu(\mathbb{R}^+) = \infty$, while retaining the hypothesis that f is sublinear. A regularly varying function ϕ obeys the defining asymptotic relation $\lim_{x\to\infty}\phi(\lambda x)/\phi(x) = \lambda^\rho$ for all $\lambda > 0$ and some $\rho \in \mathbb{R}$; this is a natural generalisation of the class of power functions (see Section 1.3.3). To obtain precise results and mitigate the technical difficulties which arise because $\mu(\mathbb{R}^+) = \infty$, we assume that both M and f are regularly varying functions, where $M(t) := \int_{[0,t]}\mu(ds)$. By computing the growth rate in terms of a related *nonautonomous* ODE, we show that the growth rate of solutions depends explicitly on the memory of the system through the index of regular variation of M. In particular, the stronger the memory of the system, the slower the rate of growth of the solution. Finally, we employ a fixed point argument to determine analogous results for a perturbed Volterra equation of the form

$$x'(t) = \int_{[0,t]}\mu(ds)f(x(t-s)) + h(t), \quad t \geq 0; \quad x(0) = \psi > 0. \tag{1.2.3}$$

As one would expect, when h is small (in an appropriate sense), solutions to (1.2.3) inherit the asymptotic behaviour of the unforced equation (i.e. (1.2.2)). However, for a sufficiently large perturbation, the solution tracks the perturbation asymptotically, even when the forcing term is potentially highly non–monotone. Moreover, the transition between these asymptotic regimes can be precisely understood in terms of the indices of regular variation of f and M.

In Chapter 4 we develop bounds on the growth rates and fluctuation sizes of unbounded solutions to perturbed nonlinear Volterra equations such as (1.2.3). We now allow the forcing term h to be random or deterministic but reimpose the restriction that $\mu \in M(\mathbb{R}^+;\mathbb{R}^+)$, as in Chapter 2. We continue to assume that the nonlinearity is sublinear but, in contrast to Chapter 3, we no longer need to impose regular variation in order to proceed. Our main results show that if an appropriate functional of the forcing term and nonlinearity has a limit L at infinity, the solution to (1.2.3) behaves asymptotically like the underlying unforced equation

$$y'(t) = \mu(\mathbb{R}^+)\, f(y(t)) \tag{1.2.4}$$

when $L = 0$, like the forcing term when $L = +\infty$, and inherits properties of both the forcing term and underlying differential equation (1.2.4) for values of $L \in (0,\infty)$. The class of regularly varying functions with index less than or equal to one provides a rich class of admissible nonlinearities; by exploiting the enhanced structure of this class we provide sharp and comprehensive asymptotic results which further illuminate our analysis for general nonlinearities. Our approach carries over in a natural way to stochastic equations with additive noise and we treat the illustrative cases of Brownian and α–stable Lévy noise.

Chapter 5 concerns the asymptotic behaviour of solutions of a linear convolution Volterra summation equation with an unbounded forcing term, namely

$$x(n+1) = \sum_{j=0}^{n} k(n-j)x(j) + H(n+1), \quad n \geq 0; \quad x(0) = \xi \in \mathbb{R}. \tag{1.2.5}$$

In both Chapters 3 and 4, we address how large additive perturbations influence (even dominate) the long–term dynamics of nonlinear systems with memory. In spite of the voluminous literature regarding linear memory–dependent systems, it appears that this question has not been studied in extensive detail in a linear setting; the more usual case when perturbations are "mild" (e.g. forcing terms which lie in L^p/l^p spaces) has been the subject of intensive efforts (see [50, Theorem 7.2.3] for a representative result in continuous–time). These "mild" perturbations are evidently of greatest interest in applications such as engineering or systems control and have naturally attracted the most attention but larger, even dominant, forcing is of particular interest in economic applications. Hence it is perhaps most natural to study this problem in discrete time since (1.2.5) has the structure of classical time series models — models which are ubiquitous in economics and finance. We believe appropriate continuous–time analogues to the results of Chapter 5 can also be proven, just as our continuous–time sublinear results have discrete–time analogues [14], but our conclusions are unlikely to change materially with the time scale. Finally, the linearity of equation (1.2.5) in the state–variable is more mathematically tractable than the nonlinear equations studied in this thesis and we can consequently aim for more refined conclusions. For example, in Chapters 3 and 4, proving that x/H tends to a finite or infinite limit is considered a success, but in Chapter 5 we hope to prove that x/H (or related quantities) have illuminating asymptotic representations or inherit other significant properties from H.

In our study of equation (1.2.5) the kernel sequence $(k(n))_{n\geq 1}$ is summable and we ascribe growth bounds to the exogenous perturbation sequence $(H(n))_{n\geq 1}$. If the forcing term grows at a geometric rate asymptotically or is bounded by a geometric sequence, then the solution (appropriately scaled) admits a convenient asymptotic representation. Moreover, we use this representation to show that additional growth properties of the perturbation are preserved in the solution. If the forcing term fluctuates asymptotically, then fluctuations of the same magnitude will be present in the solution. We also connect the finiteness of time averages of the solution to (1.2.5) with those of the perturbation. Our results, and corollaries thereof, apply to stochastic as well as deterministic equations and we demonstrate this by studying some representative classes of examples. We conclude by extending our theory to cover a class of nonlinear equations via a straightforward linearisation argument.

In Chapter 6 we consider the finite–time blow–up and asymptotic behaviour of solutions to the nonlinear Volterra integro–differential equation (1.2.2). Our main contribution is to determine sharp estimates on the growth rates of both explosive and nonexplosive solutions for a class of equations with nonsingular kernels under relatively weak hypotheses on the nonlinearity. In this superlinear setting we must be content with estimates of the form $A(x(t), t) \to 1$ as $t \to \tau$, where τ is the blow–up time if solutions are explosive or $\tau = \infty$ if solutions are global. Our estimates improve on the sharpness of results in the literature and we also recover well–known blow–up criteria via new methods in the course of our analysis. Furthermore, in the nonexplosive case, we characterise the additive perturbations which preserve the growth rates from the unforced case, and detail precisely how nonautonomous forcing terms can impact the growth rates of solutions. In the presence of a blow–up, our conclusions are unaffected by continuous perturbation terms.

1.3 Mathematical Preliminaries

1.3.1 Existence and Uniqueness Theory

This section briefly describes the relevant existence and uniqueness theory for the nonlinear FDEs considered in this thesis. Since the existence and uniqueness theory for these equations is mature, we prefer to tackle this issue presently and focus on the asymptotic behaviour of solutions in the main body of the thesis. The only exceptions to this policy are in Chapter 4 when we briefly address the existence of solutions to SFDEs driven by semimartingales, and in Chapter 6 when we investigate whether or not solutions are global in the presence of superlinear nonlinearities.

Consider the nonlinear Volterra integro–differential equation

$$x'(t) = \int_{[0,t]} \mu(ds) f(x(t-s)) + h(t), \quad t \geq 0; \quad x(0) = \xi. \tag{1.3.1}$$

Definition 1.3.1. *A function solves the initial value problem* (1.3.1) *if it obeys* (1.3.1) *almost everywhere on an interval containing zero and is absolutely continuous on that interval. A solution which obeys* (1.3.1) *for almost every* $t \geq 0$ *is called a* global solution*. A solution which obeys* (1.3.1) *for* *almost every* $t \in [0,T]$ *for some* $T > 0$ *is called a* local solution*.*

The following result is well–known and representative (see Gripenberg et al. [50, Corollary 12.3.2]).

Theorem 1.3.1 (Local Existence Theorem)**.** *Suppose* $\mu \in M_{loc}(\mathbb{R}^+;\mathbb{R}^+)$*,* $h \in L^1_{loc}(\mathbb{R}^+;\mathbb{R})$*, and* $f \in C(\mathbb{R};\mathbb{R})$*. Then, for each* $\xi \in \mathbb{R}$*, there exists a locally absolutely continuous solution* x *to* (1.3.1) *on an interval* $[0,T]$ *for some* $T > 0$*.*

Moreover, every solution to (1.3.1)*, defined on some interval* $[0,T]$*, can be continued to a noncontinuable solution on* $[0,T_{max})$ *for some* $T_{max} > T$*. If* $T_{max} < \infty$*, then* $\limsup_{t\to T_{max}^-} |x(t)| = \infty$*.*

The notion of a solution described in Definition 1.3.1 is strictly weaker than the solution concept typically used for ordinary differential equations. In particular, absolute continuity only guarantees that solutions are differentiable almost everywhere (with respect to the Lebesgue measure). This fact has implications for the comparison principles used throughout this work and indeed necessitates the introduction of Dini derivatives in various arguments. One might ask: could we introduce a stronger solution concept to circumvent these issues? This cannot be done without asking too much regularity on the problem data. For ODEs, one usually first establishes that there is a continuous solution to the equivalent integral equation by a fixed point argument and this continuous solution is subsequently shown to be continuously differentiable due to the smoothing property of the solution map. By contrast, the solution map generated by equation (1.3.1) does not allow us to conclude such additional smoothness of solutions in general. The integrated form of (1.3.1) is given by

$$x(t) = x(0) + \int_0^t \left(\int_{[0,u]} \mu(ds) f(x(u-s)) \right) du + \int_0^t h(u)\,du, \quad t \in [0,T]. \tag{1.3.2}$$

Under the hypotheses of Theorem 1.3.1, if x is a continuous function obeying (1.3.2), then the first integrand on the right–hand side of (1.3.2) is Lebesgue integrable and locally bounded. Hence x is an absolutely continuous function on $[0,T]$. If we make the stronger hypothesis that μ admits a continuous kernel, i.e. $\mu(ds) = w(s)ds$, then we can conclude from (1.3.2) that $x \in C^1([0,T];\mathbb{R})$.

Theorem 1.3.1 provides an adequate local existence theorem for all nonlinear FDEs considered in this work. Furthermore, we will generally have global solutions via Theorem 1.3.1 and a trivial comparison argument. The first half of this thesis deals with FDEs which are sublinear in the state variable, or in other words, the nonlinear function f additionally obeys

$$\lim_{|x|\to\infty} \frac{f(x)}{x} = 0.$$

Hence f obeys a global linear bound of the form

$$|f(x)| \le A + B|x|, \text{ for all } x \in \mathbb{R}, \text{ for some positive constants } A \text{ and } B.$$

Therefore, whenever the initial value problem (1.3.1) has a local solution and f is sublinear, the solution to (1.3.1) is in fact global by comparison with the appropriate linear Volterra equation.

In Chapter 6 we consider the case when the nonlinearity is positive and obeys

$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty.$$

In this situation, Theorem 1.3.1 guarantees a positive solution to (1.3.1) on some interval $[0,T_{max})$, where T_{max} depends on ξ, f and μ in general. Given a local solution to (1.3.1) via Theorem 1.3.1 we proceed to establish whether or not the solution is global. If not, the solution exhibits finite–time blow–up, i.e. $T_{max} < \infty$ and $\lim_{t\to T_{max}^-} x(t) = \infty$, due to the positivity hypothesis.

Uniqueness of solutions is not a major concern in any of our results as our arguments will apply to all solutions of the equations under consideration (if indeed multiple solutions exist). However, in many applications it is natural to insist on uniqueness of solutions and this can be guaranteed for (1.3.1) by imposing Lipschitz type hypotheses on the nonlinearity. Uniqueness results for (1.3.1) can be proven by specialising the theorems of Gripenberg et al. [50, Ch. 12/13]; the following result is representative.

Theorem 1.3.2 (Uniqueness Theorem). *Suppose $\mu \in M_{loc}(\mathbb{R}^+;\mathbb{R}^+)$, $h \in L^1_{loc}(\mathbb{R}^+;\mathbb{R})$, and $f \in C(\mathbb{R};\mathbb{R})$. Suppose further that f is locally Lipschitz continuous on $\mathbb{R}$, i.e. for each $d \in \mathbb{R}^+$*

$$|f(x) - f(y)| \le K_d\,|x-y|, \text{ for all } x, y \in [-d, d],$$

for some positive constant K_d. Then, for each $\xi \in \mathbb{R}$, there exists a unique absolutely continuous solution x to (1.3.1) on an interval $[0,T]$ for some $T > 0$.

1.3.2 Asymptotics

We begin by defining a useful equivalence relation on the space of positive continuous functions; in essence, we consider two functions to be equivalent if they have the same leading order asymptotic behaviour.

Definition 1.3.2. *Suppose $x, y \in C(\mathbb{R}^+;(0,\infty))$. x and y are said to be asymptotically equivalent if $\lim_{t\to\infty} x(t)/y(t) = 1$. We write $x(t) \sim y(t)$ as $t \to \infty$, or sometimes $x \sim y$ for extra brevity.*

Note that $x(t) \sim y(t)$ implies $1/x(t) \sim 1/y(t)$ as $t \to \infty$. The following elementary lemma will be used frequently throughout this thesis, so we record it now for future reference.

Lemma 1.3.1. *If f, $\phi \in C(\mathbb{R}^+;(0,\infty))$ are asymptotically equivalent and obey*

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\phi(x)}{x} = 0, \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \phi(x) = \infty,$$

then $F(x) \sim \Phi(x)$ as $x \to \infty$, where F and Φ are defined by

$$F(x) = \int_1^x \frac{1}{f(u)}du, \quad \Phi(x) = \int_1^x \frac{1}{\phi(u)}du, \quad x > 0. \tag{1.3.3}$$

Occasionally, we employ the standard Landau "O" and "o" notation. If x and y are as above, we write $x(t) = O(y(t))$ or x is $O(y)$, if $|x(t)| \le K y(t)$ for some $K \in (0,\infty)$ and t sufficiently large. Similarly, we write $x(t) = o(y(t))$ or x is $o(y)$, if $x(t)/y(t) \to 0$ as $t \to \infty$.

1.3.3 Regular Variation

Definition 1.3.3. *Suppose a measurable function $\phi : \mathbb{R} \to (0,\infty)$ obeys*

$$\lim_{x\to\infty} \frac{\phi(\lambda x)}{\phi(x)} = \lambda^\rho, \text{ for all } \lambda > 0, \text{ some } \rho \in \mathbb{R},$$

then ϕ is regularly varying at infinity *with index ρ, or $\phi \in RV_\infty(\rho)$. We say that a function ϕ is* slowly varying at infinity *if $\phi \in RV_\infty(0)$.*

Regular variation provides a natural generalisation of the class of power functions and has found applications in a diverse array of mathematical fields, including probability theory, complex analysis, and number theory [27]. More recently, the application of the theory of regular variation to the study of qualitative properties of differential equations has become an active area of investigation. There is a burgeoning literature regarding the application of the theory of regular variation to the asymptotic behaviour of ordinary and functional differential equations (see for example the monographs of Marić [87], and Řehák [103], and recent representative papers such as those of Chatzarakis et al, [34], Matucci and Řehák [89, 90], and Takasi and Manojlović [117]). Regular variation has also been successfully utilised in the analysis of problems in partial differential equations (see Cîrstea and Rădulescu [36], and Rădulescu [102]).

We record some useful properties of regularly varying functions which will be called upon frequently throughout this thesis.

(i.) Composition and reciprocals: If $\phi \in \mathrm{RV}_\infty(\rho)$ and $\gamma \in \mathrm{RV}_\infty(\xi)$, then $\phi \circ \gamma \in \mathrm{RV}_\infty(\rho\xi)$. Hence, $1/\phi \in \mathrm{RV}_\infty(-\rho)$.

(ii.) Inverses and integration: If $\phi \in \mathrm{RV}_\infty(\rho)$ is invertible, then $\phi^{-1} \in \mathrm{RV}_\infty(1/\rho)$ for $\rho \neq 0$. Furthermore, if the function $t \mapsto \bar{\phi}$ is given by $\bar{\phi}(t) = \int_0^t \phi(s)\,ds$, then $\bar{\phi} \in \mathrm{RV}_\infty(\rho+1)$.

(iii.) Preservation of asymptotic order: Suppose $x, y \in C(\mathbb{R}^+; (0,\infty))$ obey $\lim_{t\to\infty} x(t) = +\infty$, $\lim_{t\to\infty} y(t) = +\infty$, and $x(t) \sim y(t)$ as $t \to \infty$. If $\phi \in \mathrm{RV}_\infty(\rho)$ for any $\rho \in \mathbb{R}$, then

$$\lim_{t\to\infty} \frac{\phi(x(t))}{\phi(y(t))} = 1.$$

(iv.) Smooth approximation: If $\phi \in \mathrm{RV}_\infty(\rho)$ for $\rho > 0$, then there exists $\varphi \in C^1(\mathbb{R}^+; (0,\infty)) \cap \mathrm{RV}_\infty(\rho)$ such that $\varphi'(x) > 0$ for all $x > 0$ and

$$\lim_{x\to\infty} \frac{\phi(x)}{\varphi(x)} = 1, \quad \lim_{x\to\infty} \frac{x\varphi'(x)}{\varphi(x)} = \rho.$$

Similarly, if h is in $\mathrm{RV}_\infty(-\theta)$ for $\theta > 0$, then there exists $j \in C^1(\mathbb{R}^+; (0,\infty))$ which is also in $\mathrm{RV}_\infty(-\theta)$ such that $j'(t) < 0$ for all $t > 0$ and

$$\lim_{t\to\infty} \frac{h(t)}{j(t)} = 1, \quad \lim_{t\to\infty} \frac{tj'(t)}{j(t)} = -\theta.$$

A slightly weaker result holds for slowly varying functions at ∞: if h is in $\mathrm{RV}_\infty(0)$, then there exists $j \in C^1(\mathbb{R}^+; (0,\infty)) \cap \mathrm{RV}_\infty(0)$ such that

$$\lim_{t\to\infty} \frac{h(t)}{j(t)} = 1, \quad \lim_{t\to\infty} \frac{tj'(t)}{j(t)} = 0.$$

This result is part of Theorem 1.3.3 in [27].

The following result is fundamental in the theory of regular variation and, in the present context, serves as a key tool in the precise asymptotic analysis of growth rates of nonlinear FDEs (see Chapter 3 in particular).

Theorem 1.3.3 (Karamata's Theorem [67]). *If $\phi \in RV_\infty(\rho)$ is locally bounded on $[X, \infty)$ for some $X \in \mathbb{R}^+$, then*

$$\lim_{x\to\infty} \frac{x^{\sigma+1}\phi(x)}{\int_X^x t^\sigma \phi(t)\,dt} = \sigma + \rho + 1, \quad \text{for each } \sigma \geq -(1+\rho).$$

We occasionally employ the theory of rapid variation and we now recall the definition of a rapidly varying function.

Definition 1.3.4. *Suppose a measurable function* $h : \mathbb{R} \to (0,\infty)$ *obeys for* $\lambda > 0$*:*

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = \begin{cases} 0, & \lambda < 1, \\ 1, & \lambda = 1, \\ +\infty, & \lambda > 1. \end{cases}$$

Then h *is* rapidly varying at infinity*, or* $h \in RV_\infty(\infty)$*. If on the other hand,* $h : \mathbb{R} \to (0,\infty)$ *obeys for* $\lambda > 0$*:*

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = \begin{cases} +\infty, & \lambda < 1, \\ 1, & \lambda = 1, \\ 0, & \lambda > 1. \end{cases}$$

Then we write $h \in RV_\infty(-\infty)$*.*

1.3.4 Representative Results

The following section is intended to give the reader a representative snapshot of some of the main results of this thesis without undue emphasis on technical details.

We first consider FDEs with sublinear nonlinearity, i.e. those for which $\lim_{x\to\infty} f(x)/x = 0$. The following theorem is a specialisation of the main result of Chapter 2.

Theorem 1.3.4 (Theorem 2.3.2)**.** *If* $\mu_1 \in M([0,\tau];\mathbb{R}^+)$*,* $\mu_2 \in M(\mathbb{R}^+;\mathbb{R}^+)$*,* $f \in C((0,\infty);(0,\infty))$ *is increasing and obeys* $\lim_{x\to\infty} f'(x) = 0$*, then solutions of*

$$\begin{aligned} x'(t) &= \int_{[0,\tau]} \mu_1(ds) f(x(t-s)) + \int_{[0,t]} \mu_2(ds) f(x(t-s)), \quad t \geq 0; \\ x(t) &= \psi(t), \quad t \in [-\tau, 0], \end{aligned}$$

obey

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} \frac{F(x(t))}{t} = M,$$

where $\psi \in C([-\tau,0];(0,\infty))$ *and* $M = \mu_1([0,\tau]) + \mu_2(\mathbb{R}^+)$*.*

The function F is defined in terms of f by $F : x \mapsto F(x) = \int_1^x du/f(u)$ and in the linear case we would have $F(x) = \log(x)$. Thus the quantity $F(x(t))/t$ can be thought of as a nonlinear generalisation of the classical Lyapunov exponent and captures the leading order growth rate of the solution. Of course, we prefer conclusions of the form $x(t) \sim A(t)$ as $t \to \infty$ for some specified function A. However, even for a linear ODE, $\log(x(t))/t = M$ does not imply that $x(t) \sim \exp(Mt)$ as $t \to \infty$ in general. As the following result shows, it is possible to provide simple sufficient conditions for the equivalence of these statements in the nonlinear case.

Corollary 1.3.1 (Theorem 2.3.3)**.** *Suppose that the hypotheses of Theorem 1.3.4 hold. If* $\limsup_{x\to\infty} f(x)F(x)/x < \infty$*, then*

$$x(t) \sim F^{-1}(Mt) \quad \textit{as } t \to \infty.$$

It is natural to ask whether the conclusion of Theorem 1.3.4 is robust to the addition of forcing terms, both deterministic and random. To this end, we study equation (1.3.1). Furthermore, since we aim to study stochastic perturbations, it is best to formulate this problem in integral form from the outset. Integration of (1.3.1) yields

$$x(t) = x(0) + \int_0^t M(t-s) f(x(s))\, ds + H(t), \quad t \geq 0, \tag{1.3.4}$$

where $M(t) = \mu([0,t])$ and $H(t) = \int_0^t h(s)\, ds$. One would expect to essentially retain the asymptotic rate of growth from (1.3.4) when H is small in an appropriate sense. The following quantity turns out to provide the appropriate notion of "size" of the forcing term:

$$L_f(H) := \lim_{t\to\infty} \frac{H(t)}{M \int_0^t f(H(s))\, ds} \in [0,\infty], \quad \text{where } M = \mu(\mathbb{R}^+).$$

The following theorem combines and specialises several of the main results from Chapter 4.

Theorem 1.3.5 (Theorems 4.3.2, 4.3.3, 4.3.6)**.** *Suppose* $\lim_{x\to\infty} f'(x) \downarrow 0$, $H \in C((0,\infty);(0,\infty))$ *and* $\mu \in M(\mathbb{R}^+;\mathbb{R}^+)$. *If x denotes a solution to* (1.3.1)*, then the following hold true:*

$$L_f(H) = 0: \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1 \quad \textit{and} \quad \lim_{t\to\infty} \frac{x(t)}{H(t)} = \infty$$

$$L_f(H) \in (0,\infty): \quad 1 \leq \liminf_{t\to\infty} \frac{F(x(t))}{Mt} \leq \limsup_{t\to\infty} \frac{F(x(t))}{Mt} \leq 1 + L_f(H) \tag{1.3.5}$$

$$L_f(H) \in (1,\infty): \quad 1 \leq \liminf_{t\to\infty} \frac{x(t)}{H(t)} \leq \limsup_{t\to\infty} \frac{x(t)}{H(t)} \leq \frac{L_f(H)}{L_f(H)-1} \tag{1.3.6}$$

$$L_f(H) = \infty: \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = \infty \quad \textit{and} \quad \lim_{t\to\infty} \frac{x(t)}{H(t)} = 1.$$

When $L_f(H) = 0$ we retain the conclusion of Theorem 1.3.4 but when $L_f(H) = \infty$ the dynamics of the solution are inherited completely from the forcing term. In the intermediate cases when $L_f(H) \in (0,\infty)$ and $L_f(H) \in (1,\infty)$ the solution appears to inherit characteristics of both the unforced system and the perturbation term. Indeed, these results can be thought of as lying on a continuous spectrum in the following sense: sending $L_f(H) \downarrow 0$ in (1.3.5) correctly predicts the result in the case that $L_f(H) = 0$ and likewise sending $L_f(H) \uparrow \infty$ in (1.3.6) correctly predicts the conclusion when $L_f(H) = \infty$. When $L_f(H) = 1$ there is an asymptotic balance between the competing forces of the unperturbed system and the forcing term; this critical case can be illuminated further and resolved satisfactorily using the theory of regular variation (see Theorem 4.3.7). Examples show that the upper and lower bonds given in (1.3.5) and (1.3.6) can all be achieved.

When the forcing term is a Brownian driven martingale, i.e. $H(t) = \int_0^t \sigma(s) dB(s)$, the key quantity for capturing perturbation size is

$$\Sigma(t) = \sqrt{2\left(\int_0^t \sigma^2(s)\, ds\right) \log\log\left(\int_0^t \sigma^2(s)\, ds\right)}.$$

Solutions are no longer positive but real–valued so we require some "symmetry at infinity" in order to make the problem more tractable and stop the proliferation of cases. Thus we ask that

$$\lim_{|x|\to\infty} \frac{|f(x)|}{\phi(|x|)} = 1, \quad \phi'(x) \downarrow 0 \text{ as } x \to \infty. \tag{1.3.7}$$

The following result is a compendium of the results regarding Brownian driven noise from Chapter 4 and is the stochastic counterpart to Theorem 1.3.5.

Theorem 1.3.6 (Theorems 4.4.2, 4.4.3, 4.4.4 and 4.4.5)**.** *Suppose* $f \in C((0,\infty);(0,\infty))$ *obeys* (1.3.7), $H : t \mapsto \int_0^t \sigma(s) dB(s)$ *where B is a Brownian motion on an appropriate probability space and* $\sigma \in$

$C(\mathbb{R}^+;\mathbb{R})$, *and* $\mu \in M(\mathbb{R}^+;\mathbb{R}^+)$. *If* X *denotes a solution to* (1.3.1), *then the following hold true:*

$$L_f(\Sigma) = 0 \quad (\textit{includes } \sigma \in L^2): \qquad \limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 \quad a.s.$$

$$L_f(\Sigma) \in (0,\infty) \textit{ and } \sigma \notin L^2: \quad \limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 + L_f(\Sigma) \quad a.s.$$

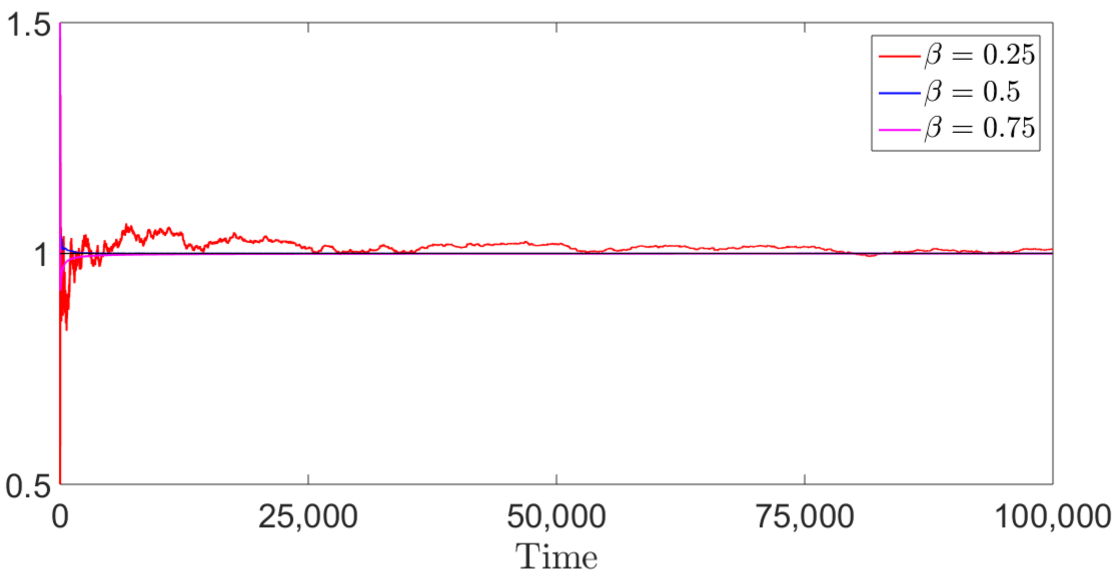
$$L_f(\Sigma) \in (1,\infty) \textit{ and } \sigma \notin L^2: \quad \frac{-L_f(H)}{L_f(H)-1} \le \liminf_{t\to\infty} \frac{X(t)}{\Sigma(t)} \le \limsup_{t\to\infty} \frac{X(t)}{\Sigma(t)} \le \frac{L_f(H)}{L_f(H)-1} \quad a.s.$$

$$L_f(\Sigma) = \infty \textit{ and } \sigma \notin L^2: \quad \liminf_{t\to\infty} \frac{X(t)}{\Sigma(t)} = -1, \; \limsup_{t\to\infty} \frac{X(t)}{\Sigma(t)} = 1 \; a.s.$$

While the conclusions of Theorem 1.3.6 mirror those of Theorem 1.3.5, we are left to wonder whether or not the conclusions are sharp (as they can be shown to be by examples in the deterministic case). In order to resolve this question we consider a simplified instance of (1.3.8) which has a power–type nonlinearity and an exponential kernel; this allows us to write the equation as a second–order system which will be approximated well by a standard Euler–Maruyama discretisation scheme.

Figure 1.3.1 shows the result of a series of numerical experiments intended to investigate the quality of Theorem 1.3.6 when $L_f(\Sigma) = 0$. In this case, extensive experimentation confirms that the upper bound of 1 on the quantity $F(|X(t)|)/Mt$ is in fact possible to achieve and hence sharp.

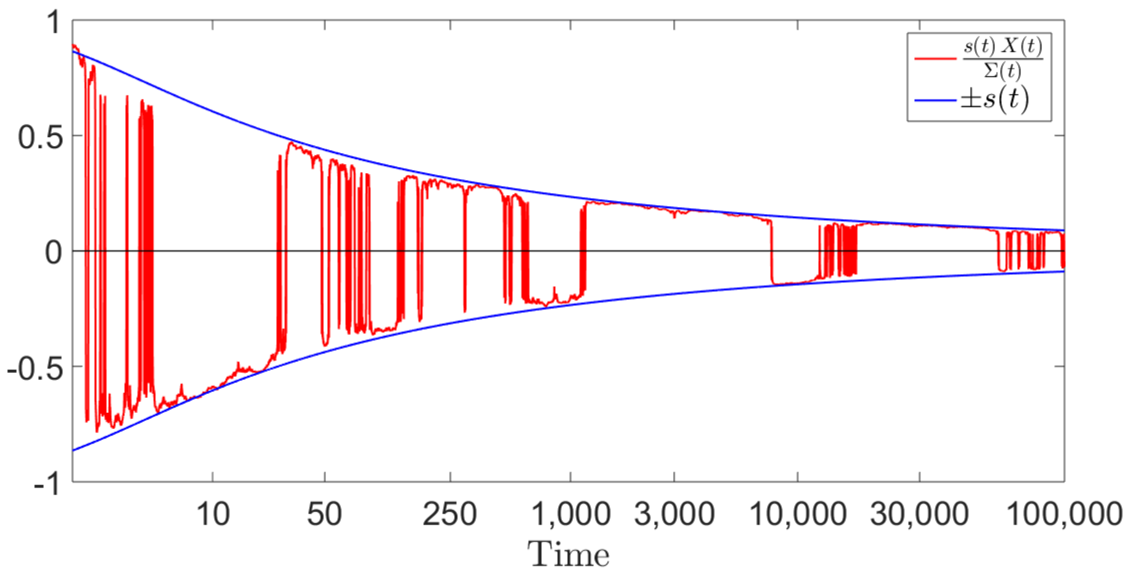
Fig. 1.3.1: Sample paths of $F(|X(t)|)/Mt$ with $L_f(\Sigma) = 0$.



$f(x) = \mathrm{sgn}(x)|x|^\beta$, $\mu(ds) = e^{-s}\,ds$ and $\sigma(t) = \sqrt{t}$.

When $L_f(\Sigma) = \infty$ we expect a decisive and positive conclusion from our numerical experiments since Theorem 1.3.6 predicts exact upper and lower fluctuation sizes in this case. We consider the quantity $X(t)/\Sigma(t)$ multiplied by a slowly decaying scaling factor; this has the advantage of generating convergent bounds which are more amenable to visualisation. In figure 1.3.2 we observe that the scaled solution does indeed cleave closely to the predicted bounds.

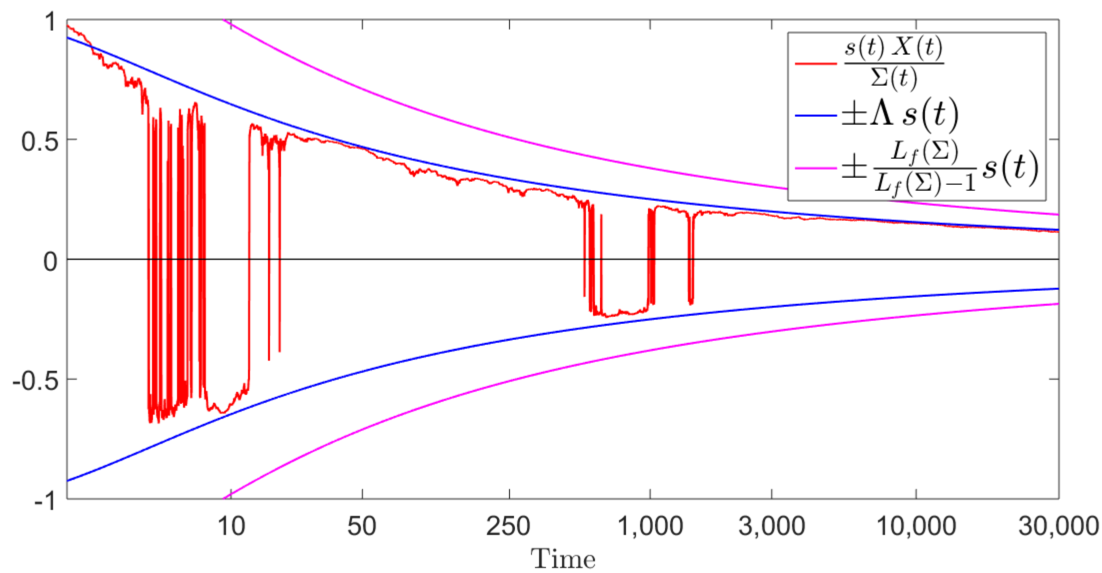
Fig. 1.3.2: Sample path of $X(t)/\Sigma(t)$ with $L_f(\Sigma) = \infty$.



$f(x) = \mathrm{sgn}(x)\sqrt{|x|}$ and $\sigma(t) \sim t^2$ as $t \to \infty$.

When $L_f(\Sigma) \in (1, \infty)$ numerical simulations indicate that the bounds given in Theorem 1.3.6 are not sharp in general (compare the pink line and the scaled solution in figure 1.3.3). However, in light of our refined deterministic results which utilise the powerful theory of regular variation, we conjecture an improved bound (the blue line in figure 1.3.3). This improved bounds depends on the index of regular variation and is given by finding the unique positive solution to a particular nonlinear algebraic equation. Figure 1.3.3 suggests that this conjecture may well be the optimal result but a definitive answer to this question is a matter for future work.

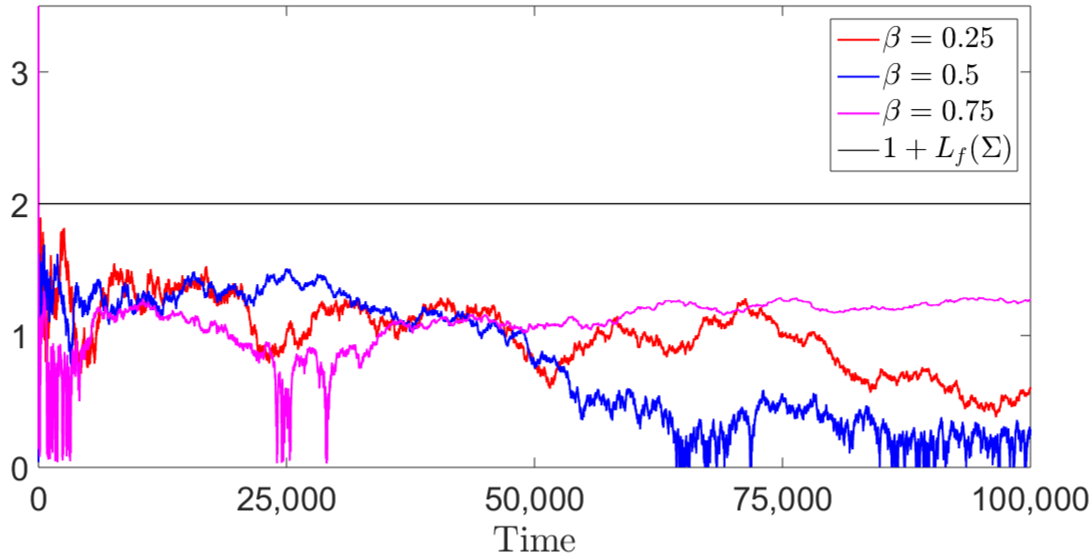
Fig. 1.3.3: Sample path of $X(t)/\Sigma(t)$ with $L_f(\Sigma) = 1.1$.



Λ is the positive solution to $\Lambda = \Lambda^\beta / L_f(\Sigma) + 1$.

Finally, we study the bounds Theorem 1.3.6 provides on the quantity $F(|X(t)|)/Mt$ when $L_f(\Sigma) \in (0, \infty)$. Figure 1.3.4 indicates that while our result appears to have the correct order of magnitude of the solution, the linear bound $1 + L_f(\Sigma)$ is not tight in general. In a similar spirit to figure 1.3.2, in the special case of a regularly varying nonlinearity, it is once more possible to conjecture a tighter bound based on our deterministic results (see Theorem 4.3.7 case (ii.)).

Fig. 1.3.4: Sample paths of $F(|X(t)|)/Mt$ with $L_f(\Sigma) = 1$.



$f(x) = \mathrm{sgn}(x)|x|^\beta$.

In sharp contrast to sublinear FDEs, solutions to superlinear FDEs ($\lim_{x\to\infty} f(x)/x = +\infty$) exhibit qualitatively different behaviour even for measures which are a.e. equal. In particular, the behaviour of the kernel near zero is known to have a strong impact on the asymptotics of solutions and thus our study of (1.3.2) in the superlinear regime requires the imposition of stronger hypotheses on μ to yield meaningful conclusions. Hence we suppose that μ is an absolutely continuous measure so that (1.3.1) becomes

$$x'(t) = \int_0^t w(t-s)f(x(s))\,ds + h(t), \quad t \ge 0. \tag{1.3.8}$$

In order to describe the asymptotics of both global and explosive solutions to (1.3.8) define the functions

$$F_U(x) = \int_1^x \frac{du}{\sqrt{\int_0^u f(s)\,ds}}, \quad x > 0,$$

and

$$F_B(x) = \int_x^\infty \frac{du}{\sqrt{\int_0^u f(s)\,ds}}, \quad x > 0.$$

F_U and F_B depend on f in an explicit fashion and hence can be computed or estimated from the problem data. The following theorem is the main result of Chapter 6.

Theorem 1.3.7. *Suppose $f \in C((0,\infty);(0,\infty))$ is increasing with $\lim_{x\to\infty} f(x)/x = \infty$, $w \in C([0,\infty);[0,\infty))$ with $w(0) > 0$ and $H \in C^1([0,\infty);[0,\infty))$. Each positive solution x to (1.3.8) blows up in finite–time for each $x(0) > 0$ if and only if*

$$\lim_{x\to\infty} F_U(x) < \infty. \tag{1.3.9}$$

If x blows up and $u \mapsto f(u)/u$ is eventually increasing, then

$$\lim_{t \uparrow T} \frac{F_B(x(t))}{T-t} = \sqrt{2w(0)}. \tag{1.3.10}$$

If x does not blow–up, $u \mapsto f(u)/u$ is eventually increasing and $w \in L^1$, then the following are equivalent:

$$(i.) \quad \lim_{t\to\infty} \frac{F_U(x(t))}{t} = \sqrt{2w(0)}, \quad (ii.) \quad \limsup_{t\to\infty} \frac{F_U(H(t))}{t} \leq \sqrt{2w(0)}.$$

Furthermore,

$$\lim_{t\to\infty} \frac{F_U(H(t))}{t} = K > \sqrt{2w(0)} \quad \textit{implies} \quad \lim_{t\to\infty} \frac{F_U(x(t))}{t} = K. \tag{1.3.11}$$

Equation (1.3.9) provides a necessary and sufficient condition for solutions to blow–up in finite time; this result is not new but our method of proof is different from previous work which established this condition. The asymptotic relation (1.3.10) gives the asymptotic rate of growth at blow–up for explosive solutions and the value of the kernel at zero appears explicitly in this identity, justifying our earlier claim that the asymptotics are sensitive to the kernel close to zero. It is also notable that the forcing term H does not feature in (1.3.10) whatsoever and this indicates that it has no affect on the blow–up rate. The asymptotic relation $(i.)$ captures the asymptotic growth rate of global solutions when $H \equiv 0$ and hence the equivalence of $(i.)$ and $(ii.)$ serves to characterise the class of forcing terms which leave the unperturbed growth rate intact. Finally, (1.3.11) suggests that when the perturbation reaches a critical size, the solution to (1.3.8) tracks the forcing term asymptotically in the sense that x and H share the same nonlinear Lyapunov exponent.

Chapter 2

Growth Rates of Sublinear Functional Differential Equations

2.1 Introduction

We study the asymptotic behaviour of unbounded solutions of the following FDE

$$x'(t) = \int_{[0,\tau]} \mu_1(ds) f(x(t-s)) + \int_{[0,t]} \mu_2(ds) f(x(t-s)), \quad t > 0; \tag{2.1.1}$$
$$x(t) = \psi(t), \quad t \in [-\tau, 0], \quad \tau \in (0,\infty),$$

where μ_1, μ_2 are nonnegative finite measures, f is a positive continuous function, and ψ is a positive continuous function. We will assume f is sublinear, in the sense that $f(x)/x \to 0$ as $x \to \infty$. Under this hypothesis, solutions of (2.1.1) will grow but will not exhibit finite time blow up; more precisely $x \in C([-\tau,\infty);(0,\infty))$ but $\lim_{t\to\infty} x(t) = \infty$. We may then ask whether the asymptotic growth rates of solutions to (2.1.1) can be captured in a meaningful way. Our main results provide sufficient conditions under which the solutions of (2.1.1) have essentially the same asymptotic behaviour as the related autonomous ordinary differential equation

$$y'(t) = Mf(y(t)), \quad t > 0; \quad y(0) = \psi, \quad M := \int_{[0,\tau]} \mu_1(ds) + \int_{[0,\infty)} \mu_2(ds). \tag{2.1.2}$$

Indeed, defining F by

$$F(x) = \int_1^x \frac{1}{f(u)}\, du, \quad x > 0, \tag{2.1.3}$$

the sublinearity of f implies that $F(x) \to \infty$ and our most general results show that

$$\lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1. \tag{2.1.4}$$

Under strengthened conditions, (2.1.4) can be improved to give direct asymptotic information in the form

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} = 1. \tag{2.1.5}$$

Chapter 2 is based on the paper [19].

If $f \in C^1(\mathbb{R}^+;(0,\infty))$, the general solution to (2.1.2) is given by

$$y(t) = F^{-1}(Mt + F(\psi)), \quad t \geq 0.$$

Hence $\lim_{t\to\infty} F(y(t))/Mt = 1$ and furthermore, if f is *sublinear*,

$$\lim_{t\to\infty} \frac{y(t)}{F^{-1}(Mt)} = 1. \tag{2.1.6}$$

Sublinearity implies $y'(t)/y(t) \to 0$ as $t \to \infty$ and this in turn implies that $y(t+c)/y(t) \to 1$ as $t \to \infty$ for any $c \in \mathbb{R}$. The limit in (2.1.6) then follows by choosing $c = -F(\psi)/M$. The asymptotic results for x mirror those for y, and if (2.1.5) holds the solution to (2.1.1) is actually asymptotic to the solution of the related ODE (2.1.2).

In order to determine explicit first order representations for the asymptotic behaviour of $y(t)$ as $t \to \infty$, it is only necessary to determine the large time behaviour of F and F^{-1}, and this is precisely what is needed to determine explicit first order representations for the asymptotic behaviour of $x(t)$ as $t \to \infty$. Thus deducing the asymptotic behaviour of (2.1.1) *reduces exactly to the related problem for the ODE.*

Growth estimates on the solutions of nonlinear convolution–type equations such as (2.1.1) have attracted attention from a wide variety of investigators (see, for example, Lipovan [79, 80], and Schneider [111]). In particular, the asymptotic theory of equations such as (2.1.1) is intimately related to the upper bound estimates furnished by inequalities of the Bellman-Bihari-Gronwall type (cf. [26]) and in some sense this chapter addresses the question of when these estimates are asymptotically sharp. The literature on such inequalities is vast, and important results are given in several monographs (e.g., Lakshmikantham and Leela [73], and Pachpatte [98]). In [98], Pachpatte provides myriad examples of differential inequalities and their applications to the qualitative theory of differential equations, including to linear integro-differential equations similar to (2.1.1). However, these results rely on the nonlinear function being monotonically increasing and generally only provide upper estimates on the size of solutions. Indeed much of the literature on growth bounds and estimates involves monotone hypotheses; these are of course natural when trying to establish uniqueness of solutions of dynamical systems since the nonlinearity can be interpreted as a modulus of continuity, which enjoys natural monotonicity properties.

If f is *increasing*, and $F(x) \to \infty$ as $x \to \infty$, immediate integration of (2.1.1) leads directly to an *upper* inequality of Bihari–type, and an estimate of the form

$$F(x(t)) \leq C + Mt, \quad t \geq 0$$

for some $C > 0$. This immediately yields

$$\limsup_{t\to\infty} \frac{F(x(t))}{Mt} \leq 1. \tag{2.1.7}$$

However, this result does not indicate whether the estimate is in any sense sharp for non–trivial functional differential equations; although it is certainly so for sublinear ODEs, since $\lim_{t\to\infty} F(y(t))/Mt = 1$. It is well–known that the estimate from the Bihari–Gronwall–Bellman approach cannot be sharp when f is linear, as exact Liapunov exponents – which do not coincide with those resulting from the Gronwall inequality – are given by solving the characteristic equation (see e.g., [50]). Therefore, it is of evident interest to develop corresponding *lower* inequalities on the solution of (2.1.1) with a view to investigating the quality of the upper bounds generated by the standard theory. An excellent paper which addresses such lower bounds in Volterra integral equations is that of Lipovan [78]; in the present work we develop suitable lower inequalities for the solutions of our equations which in fact highlight the sharp character of the bounds achieved in [78]. We are assisted in this task by the integro–differential character of (2.1.1). Furthermore, the standard approach does not seem to address the situation in which f is *decreasing*. To a certain degree, the main new contributions of this work are to furnish *sharp* lower estimates, to relax the hypothesis that f increases, and to obtain simplified but precise limiting behaviour, rather than explicit and global bounds on the solution. Indeed, as we are in any case studying differential systems, we find it sometimes reasonable not to integrate (2.1.1), in part to prevent the destruction of useful information about the solution, and this leads to a different line of attack from the aforementioned integral equation theory.

In the linear case, as noted above, the exact asymptotic behaviour is known, and the upper estimate in (2.1.7) is not sharp in general. If f grows sufficiently more rapidly than linearly, finite time blow–up of solutions is possible – this class of problems is the subject of Chapter 6. Presently we confine our attention to the case when f is sublinear. A characterisation of sublinearity which seems mild is that f is asymptotic to a function whose derivative vanishes at infinity. Sublinear equations of this type were studied by Appleby et al. [11] with a single delay term but this analysis relies on the theory of regular variation. Other works which give growth estimates for sublinear functional differential equations include Graef [49], and Kusano and Onose [72] but these articles focus on equations of arbitrary order with oscillatory solutions, leading to less general results. We impose the more general hypothesis of asymptotic monotonicity on the nonlinear function f; both the increasing and decreasing cases are addressed. This generality allows us to easily recover the results for the case of regular variation. However, our analysis reveals that relaxing either the increasing or decreasing hypothesis completely makes estimation of a sharp growth bound difficult. In the case when f is slowly varying at infinity we can still achieve results but the unbounded delay case is challenging. Only under additional hypotheses on f can we obtain exact asymptotics in this case.

A primary motivation for the work in this chapter is to give a platform to deduce growth and fluctuation properties for deterministically and stochastically perturbed functional and Volterra equations of the form

$$dX(t) = \left(\int_{[0,\tau]} \mu_1(ds) f(X(t-s)) + \int_{[0,t]} \mu_2(ds) f(X(t-s)) + h(t) \right) dt + dZ(t), \quad t \geq 0, \tag{2.1.8}$$

where Z is a semimartingale with appropriate asymptotic properties. Indeed, the deterministic theory established presently forms the basis of our investigation of equations such as (2.1.8) in Chapter 4. Systems with the same qualitative features as those present in (2.1.8) find applications in the endogenous growth theory of mathematical economics and in particular in vintage capital models.

The inclusion of general finite measures in (2.1.8) is a key feature for applications to vintage capital as it allows both demographic and structural delay effects to be captured; the work of Benhabib and Rustichini [23] is an excellent early exemplar of how Volterra equations with general measures can be used to model non–exponential depreciation of capital, and effects such as “learning by doing” and time–to–build lags. These ideas, and variants thereof, have subsequently been developed in both the economic and mathematical literature. d’Albis et al. [43] (and the references contained therein) provide a more up to date overview of the development of such models and the associated mathematical machinery. We note that the aforementioned literature is primarily focused on models involving equations which are linear in the state variable.

The second notable qualitative feature of equation (2.1.8) is the inclusion of a sublinear nonlinearity; this arises naturally in economic models as a consequence of the so–called law of diminishing returns. In the context of endogenous growth models, Jones [62] explains the crucial need to incorporate non–unit returns to scale in order to eliminate unrealistic scale effects (see also [63]). The most common sublinearities in the economic literature are those of power type and our treatment of the case of regular variation is therefore especially pertinent in this context. The work of Lin and Shampine [77] provides a recent example of the intersection of endogenous growth theory and the theory of functional differential equations. Building on the framework of Jones and Williams [64], Lin and Shampine present a model of finite length patents in a decentralised economy which gives rise to a complex system of functional differential equations with sublinear state–dependent terms. Sublinear nonlinearities are also present in FDEs arising in population dynamics where they model overcrowding effects (see Thieme [118, 119]).

Finally, the inclusion of a deterministic state–independent term h in (2.1.8) serves to model underlying trends in the external environment, while the semimartingale term models exogenous and uncertain pertubations to the system. Interesting examples of appropriate semimartingales include

$$Z_1(t) = \int_0^t \sigma(s)\, dB(s), \quad Z_2(t) = Y_\alpha(t)$$

where B is standard Brownian motion, and Y_α is an α–stable Lévy process (see Bertoin [25] for further details). These allow us to model respectively systems subject to exogeneous, persistent time–dependent shocks, or systems perturbed by erratic, and potentially large, shocks. The state–independence of Z in our examples make (2.1.8) reminiscent of a continuous–time nonlinear time series model subject to white noise perturbations (cf. Brockwell and Lindner [30], or Marquardt and Stelzer [88]). We also note that for such stochastic systems, which are likely also subject to model uncertainty, there is less potential for global pathwise bounds to be of value. This further supports our emphasis on asymptotic results with less stringent requirements on the nonlinearity.

2.2 Mathematical Preliminaries and Hypotheses

The measures μ_1 and μ_2 will obey

$$\mu_1 \in M([0,\tau];\mathbb{R}^+), \;\; \mu_2 \in M(\mathbb{R}^+;\mathbb{R}^+), \;\; \int_{[0,\tau]} \mu_1(ds) + \int_{[0,\infty)} \mu_2(ds) =: M \in (0,\infty). \tag{2.2.1}$$

Assuming $M > 0$ guarantees that we avoid the trivial case when the right–hand side of (2.1.1) is identically zero. We suppose that

$$f \in C(\mathbb{R}^+;(0,\infty)), \tag{2.2.2}$$

so that the function $F : \mathbb{R}^+ \to \mathbb{R}$ given by (2.1.3) is well–defined, strictly increasing, and invertible. Throughout this chapter we assume that the initial function ψ is strictly positive on the initial interval $[-\tau, 0]$; this guarantees that solutions to (2.1.1) obey $x(t) \to \infty$ as $t \to \infty$. However, this assumption is not needed in proofs regarding the rate of growth and hence these arguments apply if it is known independently that solutions grow to infinity.

The non-standard mixed form of (2.1.1) owes to the fact that our methods apply equally well to both bounded delay equations and Volterra equations without a forcing term on the right–hand side; for brevity we prove results for (2.1.1) which can be immediately applied to each special class of equations as desired. We could rewrite (2.1.1) as a "pure" Volterra equation at the expense of adding an exogenous forcing term by noting that delay differential equations of the form

$$z'(t) = \int_{[0,\tau]} \mu_1(ds) f(z(t-s)), \quad t \geq 0; \quad z(t) = \psi(t), \quad t \in [-\tau, 0], \tag{2.2.3}$$

can be written as

$$z'(t) = \int_{[0,t]} \mu(ds) f(z(t-s)) + h(t), \quad t \geq 0; \quad z(0) = \psi(0), \tag{2.2.4}$$

where $\mu(E) = \mu_1(E \cap [0,\tau])$ and $h(t) = \mathbb{1}_{[0,\tau)}(t) \int_{[t,\tau]} \mu_1(ds) f(\psi(t-s))$. Indeed our results for (2.1.1) are a necessary first step in understanding the *growth* asymptotics of unbounded solutions of more general FDEs of the form (2.2.4) and such equations will be addressed in later chapters.

We say that $f \in C(\mathbb{R}^+; (0,\infty))$ is *sublinear* if it is dominated by every positive linear function at infinity, or in other words,

$$\lim_{x\to\infty} \frac{f(x)}{x} = 0. \tag{2.2.5}$$

In order to prove many of our results, we request extra properties and regularity on f that are not necessarily satisfied by sublinear functions as described by (2.2.5). However, we believe that our choice of additional hypotheses on f are not especially restrictive, relatively natural in the context of differential systems, and apply in a unified manner across a variety of situations. With $\mathcal{S}$ the class of functions given by

$$\mathcal{S} = \{\phi \in C^1(\mathbb{R}^+; (0,\infty)) : \lim_{x\to\infty} \phi'(x) = 0, \quad \phi'(x) > 0\}, \tag{2.2.6}$$

we suppose that

$$\text{There exists a } \phi \in \mathcal{S} \text{ such that } f(x) \sim \phi(x) \text{ as } x \to \infty. \tag{2.2.7}$$

Note that (2.2.7) implies the sublinear property (2.2.5) (see Lemma 2.6.1). Once f obeying (2.2.7) is fixed, we select an equivalence class representative ϕ (with respect to the relation $\sim$) from $\mathcal{S}$ and associate with this ϕ the function

$$\Phi(x) = \int_1^x \frac{du}{\phi(u)}, \quad x > 0. \tag{2.2.8}$$

By Lemma 1.3.1, $f \sim \phi$ implies $F \sim \Phi$ in the framework of this chapter.

One of the principal advantages of the strengthened hypothesis (2.2.7) is that the extra regularity requirements, i.e. monotonicity and smoothness, are *not imposed directly* on f but rather on the auxiliary function ϕ. This allows f to have a certain irregularity without any cost.

While (2.2.7) holds for large classes of sublinear functions that are commonly found in applications, it is still a strictly stronger hypothesis than sublinearity, even when f is increasing. In particular, an increasing, continuously differentiable, sublinear function f must have $\liminf_{x\to\infty} f'(x) = 0$ but there is no guarantee that $\limsup_{x\to\infty} f'(x) = 0$ and it is even possible to have

$$0 = \liminf_{x\to\infty} f'(x) < \limsup_{x\to\infty} f'(x) = \infty. \tag{2.2.9}$$

We illustrate this point more fully in Section 2.5 with some examples.

2.3 General Results

Firstly, we prove that sublinear behaviour in f implies subexponential growth in the solution, x, to (2.1.1), in the sense that x has a zero Liapunov exponent; the proof of this fact is elementary and we give it immediately below.

Theorem 2.3.1. *Suppose μ_1 and μ_2 obey (2.2.1) and $\psi \in C([-\tau, 0]; (0,\infty))$. If f obeys (2.2.2) and (2.2.5), then solutions to (2.1.1) obey $x(t) \to \infty$ as $t \to \infty$, and moreover*

$$\lim_{t\to\infty} \frac{1}{t} \log x(t) = 0. \tag{2.3.1}$$

Proof. It is shown later that (2.2.1), (2.2.2), and positivity of ψ guarantee that $\lim_{t\to\infty} x(t) = \infty$, $x'(t) \geq 0$ for a.e. $t \geq 0$ and x is nondecreasing. Since f is continuous and obeys (2.2.5), for every $\epsilon > 0$ there is $L(\epsilon) > 0$ such that $0 < f(x) < L(\epsilon) + \epsilon x$ for all $x \geq 0$. For $t > \tau$, $x(t+s) \leq x(t)$ for all $-\tau \leq s \leq 0$, and thus

$$\begin{aligned} 0 \leq x'(t) &\leq \int_{[-\tau,0]} \mu_1(ds)\,\{L(\epsilon) + \epsilon x(t+s)\} + \int_{[0,t]} \mu_2(ds)\,\{L(\epsilon) + \epsilon x(t-s)\} \\ &\leq L(\epsilon)M + \epsilon M x(t), \quad \text{for a.e. } t > \tau. \end{aligned}$$

Since $\lim_{t\to\infty} x(t) = \infty$, for each $\eta > 0$ there exists a $T(\eta, M) > 0$ such that $L(\epsilon)M/x(t) < \eta$ for all $t \geq T(\eta, M)$. Hence

$$0 \leq \frac{x'(t)}{x(t)} \leq \eta + \epsilon M, \quad \text{for a.e. } t \geq \tau + T(\eta, M) =: T_1.$$

Let $\eta = \epsilon$ and integrate the inequality above over the interval $[T_1, t]$ to show that

$$0 \leq \log x(t) - \log x(T_1) \leq \epsilon(1+M)(t - T_1), \quad \text{for each } t \geq T_1.$$

Therefore

$$0 \leq \frac{\log x(t)}{t} \leq \frac{\log x(T_1)}{t} + \frac{\epsilon(1+M)(t-T_1)}{t}, \quad \text{for each } t \geq T_1.$$

Take the limsup as $t \to \infty$, and then let $\epsilon \to 0^+$ to complete the proof. □

By strengthening the sublinearity hypothesis on f to (2.2.7), we next show that solutions to (2.1.1) grow like those of the autonomous ODE (2.1.2), in the sense that $\lim_{t\to\infty} F(x(t))/Mt = 1$.

Theorem 2.3.2. *Suppose μ_1 and μ_2 obey (2.2.1) and $\psi \in C([-\tau, 0]; (0,\infty))$. If f obeys (2.2.2) and (2.2.7), then solutions to (2.1.1) obey*

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1. \tag{2.3.2}$$

It is notable that, in contrast to linear functional differential equations with a positive measure, the rate of growth is independent of the distribution of the mass in the measures μ_1 and μ_2, but depends merely on the overall mass $M = \mu_1([0,\tau]) + \mu_2(\mathbb{R}^+)$. Therefore, the growth of solutions cannot be boosted or retarded (at least in terms of the asymptotic relation prescribed in (2.3.2)) by greater weight being allocated to more recent values of the solution.

The proof of Theorem 2.3.2 begins by establishing that $\limsup_{t\to\infty} F(x(t))/Mt \leq 1$; this is essentially a consequence of Bihari's inequality. As intimated earlier, proving the required lower bound is more challenging. Due to the hypothesis (2.2.1), our problem can effectively be reduced to the study of delay differential inequalities of the form

$$x'(t) \geq M_\epsilon\, \phi(x(t - T_\epsilon)), \quad \text{for a.e. } t \geq T_\epsilon > 0,$$

where $\phi \sim f$ and $M_\epsilon \to M$ as $\epsilon \to 0$. The *sublinearity* of ϕ is now crucial in establishing that $\lim_{t\to\infty} \phi(x(t-\theta))/\phi(x(t)) = 1$ for each fixed $\theta > 0$ – this step effectively eliminates the last remnants of the delay in our problem and a simple Bihari–type argument once more prevails.

We now state specialisations of the above result for the related delay and Volterra differential equations.

Corollary 2.3.1. *Suppose μ_1 obeys (2.2.1) with $\mu_2 \equiv 0$ and $\psi \in C([-\tau, 0]; (0,\infty))$. If f obeys (2.2.2)* *and* (2.2.7)*, then solutions to* (2.2.3) *obey*

$$\lim_{t\to\infty} z(t) = \infty, \quad \lim_{t\to\infty} \frac{F(z(t))}{Mt} = 1.$$

Corollary 2.3.2. *Suppose μ_2 obeys* (2.2.1) *with $\mu_1 \equiv 0$ and $\psi \in (0,\infty)$. If f obeys* (2.2.2) *and* (2.2.7)*, then solutions to the Volterra integro–differential equation*

$$v'(t) = \int_{[0,t]} \mu_2(ds) f(v(t-s)), \quad t \geq 0; \quad v(0) = \psi, \tag{2.3.3}$$

obey

$$\lim_{t\to\infty} v(t) = \infty, \quad \lim_{t\to\infty} \frac{F(v(t))}{Mt} = 1. \tag{2.3.4}$$

At this point it is natural to ask if we can hope to recover asymptotic behaviour similar to that of the solution of (2.1.2) if "$M = +\infty$". Using the previous result and a comparison argument, this can be immediately ruled out. We present this result for the solution of a "pure" Volterra equation since it is more natural to only consider the unbounded delay component (i.e. $\mu_1 \equiv 0$) when "$M = +\infty$".

Corollary 2.3.3. *Let μ_1 and μ_2 obey* (2.2.1) *but with*

$$\int_{\mathbb{R}^+} \mu_2(ds) = \infty, \tag{2.3.5}$$

and $\psi \in C([-\tau,0];(0,\infty))$. If f obeys (2.2.2) *and* (2.2.7)*, then solutions to* (2.3.3) *obey*

$$\lim_{t\to\infty} v(t) = \infty, \quad \lim_{t\to\infty} \frac{F(v(t))}{t} = \infty.$$

The result of Corollary 2.3.3 can be viewed as a continuous extension of Theorem 2.3.2 in the limit as $M \to \infty$. This can be seen readily by writing (2.3.4) in the form

$$\lim_{t\to\infty} \frac{F(v(t))}{t} = M,$$

and by letting $M \to \infty$ we obtain the conclusion of Corollary 2.3.3. Roughly speaking, Corollary 2.3.3 indicates that solutions to (2.1.1) now grow more rapidly than solutions to the ODE (2.1.2).

We expect that when the total mass of the measures is infinite, in the sense that (2.3.5) holds, this may well give rise to phenomena not captured by relatively crude results such as Corollary 2.3.3. Treating this issue in more detail will naturally require some additional information about the rate of growth to infinity of the function $M(t) := \int_{[0,t]} \mu_2(ds)$ ($\mu_1 \equiv 0$); this problem is the subject of Chapter 3.

The reader may view the asymptotic relation (2.3.2) as giving rather indirect information about the asymptotic behaviour of the solution x of (2.1.1), and we might naturally desire more direct information by determining a function a such that $x(t) \sim a(t)$ as $t \to \infty$. In the case of a linear equation (2.3.2) is a statement concerning the Liapunov exponent of a scalar differential equation. Therefore, the direct information we seek constitutes a type of Hartman–Wintner result (cf. Hartman [59], and Hartman and Wintner [60] for ODEs with linear leading order terms, and Pituk [100] for FDEs with linear leading order terms), in contrast to (2.3.2), which is a type of Hartman–Grobman result. A natural candidate for a in this case is $a(t) = F^{-1}(Mt)$, and the following Proposition makes this apparent.

Proposition 2.3.1. *Suppose f obeys* (2.2.7) *and let F be given by* (2.1.3)*. If $a \in C(\mathbb{R}^+;(0,\infty))$ is such that $a(t) \sim F^{-1}(Mt)$ as $t \to \infty$, then*

$$\lim_{t\to\infty} \frac{F(a(t))}{Mt} = 1.$$

Proposition 2.3.1 shows that "direct asymptotic information" regarding the solution gives stronger information than the relation (2.3.2). Consequently, it is reasonable to ask if we can impose easily–checked and natural sufficient conditions on the nonlinear function f so that this can be done. The following result gives such conditions under which Theorem 2.3.2 can be appropriately strengthened.

Theorem 2.3.3. *Suppose μ_1 and μ_2 obey (2.2.1), f obeys (2.2.2) and (2.2.7), and let $\psi \in C([-\tau, 0]; (0, \infty))$. If*

$$\limsup_{x\to\infty} \frac{f(x)\,F(x)}{x} := L < \infty, \tag{2.3.6}$$

then solutions to (2.1.1) obey (2.3.2) and moreover

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} = 1. \tag{2.3.7}$$

If f is linear we know independently that $z(t)/F^{-1}(Mt)$ does not have zero limit once $\mu_1(\{0\}) + \mu_2(\{0\}) < M$, or in other words, once (2.1.1) is a true FDE. However, (2.3.6) is merely a sufficient condition to ensure that $z(t) \sim F^{-1}(Mt)$ as $t \to \infty$ but should illustrate that the asymptotic growth of f cannot be too fast if we hope to retain results of the form (2.3.7). This begs the question: exactly how fast can the nonlinearity grow before the asymptotic relation (2.3.7) ceases to hold? A full answer to this question is outside the scope of this thesis but the interested reader can consult [16] for a more comprehensive answer.

The aforementioned caveats notwithstanding, (2.3.6) is a practically useful condition since it is relatively sharp and does not make overly stringent restrictions on the nonlinearity. For example, if

$$\begin{aligned}&\text{There exists an } \epsilon \in (0,1) \text{ such that } x \mapsto f(x)/x^{1-\epsilon}\\ &\text{is asymptotic to a decreasing function } \phi \in C(\mathbb{R}^+; (0,\infty)),\end{aligned} \tag{2.3.8}$$

then condition (2.3.6) holds. Under (2.3.8) there is $x_1 > 1$ such that $x \geq x_1$ implies $\phi(x)/2 < f(x)/x^{1-\epsilon} < 2\phi(x)$. Then, as $\phi(u) > \phi(x)$ for $u < x$, we get for $x \geq x_1$ that

$$\frac{f(x)}{x}\int_{x_1}^{x} \frac{1}{f(u)}\,du \leq \frac{2\phi(x)x^{1-\epsilon}}{x}\int_{x_1}^{x} \frac{2}{\phi(u)u^{1-\epsilon}}\,du \leq 4x^{-\epsilon}\int_{x_1}^{x} \frac{1}{u^{1-\epsilon}}\,du \leq \frac{4}{\epsilon}.$$

This gives (2.3.6), because $x \mapsto f(x)/x$ is bounded on $[1,\infty)$, and therefore so is $x \mapsto f(x)/x \cdot \int_1^{x_1} du/f(u)$.

The validity of (2.3.6) within the class of regularly varying functions also casts light on its utility. For example, for any $f \in \text{RV}_\infty(\beta)$ for $\beta \in (0,1)$, (2.3.6) holds: this is a very large class of sublinear functions satisfying (2.2.7). However, if $f \in \text{RV}_\infty(1)$, Karamata's Theorem yields

$$\lim_{x\to\infty} \frac{f(x)\,F(x)}{x} = \infty,$$

and so (2.3.6) does not hold in this case; this shows that we cannot relax (2.3.8) to allow $\epsilon = 0$.

We make one final remark concerning condition (2.3.6). Since $f(x)/x \to 0$ as $x \to \infty$, we have $F(x) \to \infty$ as $x \to \infty$: therefore the possibility arises that L in (2.3.6) could be zero. However, if f obeys (2.2.7), then $L \geq 1$ and in fact

$$\liminf_{x\to\infty} \frac{F(x)f(x)}{x} \geq 1. \tag{2.3.9}$$

This is readily seen: by (2.2.7), for every $\epsilon \in (0,1)$, there is $x_1(\epsilon) > 0$ such that $(1-\epsilon)\phi(x) < f(x) <$

$(1+\epsilon)\phi(x)$ for $x \geq x_1(\epsilon)$, where $\phi \in \mathcal{S}$ and so is increasing. Hence for $x_1(\epsilon) \leq u \leq x$ we have

$$f(u) < (1+\epsilon)\phi(u) < (1+\epsilon)\phi(x) < \frac{1+\epsilon}{1-\epsilon} f(x).$$

Therefore for $x \geq x_1(\epsilon)$

$$F(x) = F(x_1(\epsilon)) + \int_{x_1(\epsilon)}^{x} \frac{1}{f(u)}\,du \geq F(x_1(\epsilon)) + \frac{1-\epsilon}{1+\epsilon} \cdot \frac{x - x_1(\epsilon)}{f(x)}.$$

Multiplying by $f(x)/x$, using the fact that this tends to zero as $x \to \infty$, and then taking limits as $x \to \infty$, and then as $\epsilon \to 0^+$, we arrive at (2.3.9).

Our next result shows that when f is asymptotically decreasing solutions of (2.1.1) obey $x(t) \sim F^{-1}(Mt)$ as $t \to \infty$ with no additional hypotheses on f.

Theorem 2.3.4. *Suppose μ_1 and μ_2 obey (2.2.1) with $\psi \in C([-\tau, 0]; (0, \infty))$. If $f \in C(\mathbb{R}^+; (0, \infty))$ is asymptotic to a decreasing function $\phi \in C^1(\mathbb{R}^+; (0, \infty))$, then solutions to (2.1.1) obey*

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} = 1.$$

We notice that there is no restriction on how rapidly f may decrease in Theorem 2.3.4, in contrast to the restriction on sublinear increase in f in Theorem 2.3.2. Before concluding the section, we give a simple example showing an application of Theorem 2.3.4.

Example 2.3.5. *Consider the Volterra equation*

$$x'(t) = af(x(t)) + \int_0^t \frac{1}{(1+t-s)^{\theta+1}} f(x(s))\,ds, \quad t > 0; \quad x(0) = \psi > 0,$$

where $a \geq 0$, $\theta > 0$ and $f : [0, \infty) \to (0, \infty)$ is locally Lipschitz continuous with $f(x) \sim e^{-\alpha x}$ as $x \to \infty$ for $\alpha > 0$. These conditions ensure a unique positive continuous solution (see Theorem 1.3.2), and indeed, as f is asymptotic to a decreasing function, we see that all the hypotheses of Theorem 2.3.4 apply, with

$$M = a + \int_0^\infty \frac{1}{(1+u)^{1+\theta}}\,du = a + \frac{1}{\theta}, \quad F(x) \sim \int_1^x e^{\alpha u}\,du =: \Phi(x), \quad \text{as } x \to \infty.$$

It remains to determine explicitly the asymptotic behaviour of $F^{-1}(x)$ as $x \to \infty$. Since $\Phi(x) = (e^{\alpha x} - 1)/\alpha$, it follows that

$$\Phi^{-1}(x) = \frac{1}{\alpha} \log(1 + \alpha x).$$

Therefore $F^{-1}(x) \sim \Phi^{-1}(x) \sim (\log x)/\alpha$ as $x \to \infty$ (see Lemma 2.6.6), and by Theorem 2.3.4,

$$x(t) \sim F^{-1}(Mt) \sim \frac{1}{\alpha} \log(Mt) \sim \frac{1}{\alpha} \log t, \quad \text{as } t \to \infty. \tag{2.3.10}$$

If $f(x) \sim x^{-\beta}$ as $x \to \infty$ for $\beta > 0$, we can carry out similar calculations to get

$$F^{-1}(x) \sim ((\beta+1)x)^{1/(1+\beta)}, \text{ as } x \to \infty,$$

so

$$x(t) \sim F^{-1}(Mt) \sim \left\{(\beta+1)\left(a + \frac{1}{\theta}\right)\right\}^{1/(1+\beta)} t^{1/(1+\beta)}, \quad \text{as } t \to \infty. \tag{2.3.11}$$

2.4 Results with Regular Variation

We now present some auxiliary results which show that our main results can readily be applied to the case when the sublinear function f is regularly varying at infinity.

Essentially, if f is in $\mathrm{RV}_\infty(\beta)$ with $\beta \in (0,1)$, it immediately satisfies condition (2.2.7), and so Theorem 2.3.2 and all relevant corollaries can be applied. If $\beta < 0$, then the hypothesis that f is asymptotically decreasing in Theorem 2.3.4 is satisfied, and so Theorem 2.3.4 applies. If $\beta > 1$, f is not sublinear, and we are outside the scope of this chapter. The case when $\beta \in \{0,1\}$ contains subtleties which we discuss presently, but in some cases we may still apply our previous results.

Our first result is a direct application of Theorems 2.3.2 and 2.3.4, in conjunction with Theorem 2.3.3.

Theorem 2.4.1. *Suppose μ_1 and μ_2 obey (2.2.1), and $\psi \in C([-\tau,0];(0,\infty))$. If $f \in RV_\infty(\beta)$ with $\beta \in (-\infty,1)/\{0\}$, then solutions to (2.1.1) obey*

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} = 1.$$

Proof. If $\beta \in (-\infty,0)$, then f is asymptotic to a decreasing function and Theorem 2.3.4 immediately proves the claim. If $\beta \in (0,1)$ then there exists an increasing function $\phi \in C^1((0,\infty);(0,\infty)) \cap \mathrm{RV}_\infty(\beta)$ such that

$$\lim_{x\to\infty} \frac{f(x)}{\phi(x)} = 1, \quad \lim_{x\to\infty} \frac{x\,\phi'(x)}{\phi(x)} = \beta.$$

It follows that $\phi'(x) \sim \beta\,\phi(x)/x$ as $x\to\infty$ and hence that $\phi' \in \mathrm{RV}(\beta-1)$. Therefore $\lim_{x\to\infty}\phi'(x) = 0$ [27, Proposition 1.5.1]. Now apply Theorem 2.3.2 to show that $\lim_{t\to\infty} F(x(t))/Mt = 1$; we use Theorem 2.3.3 to strengthen this conclusion. By Karamata's Theorem

$$\limsup_{x\to\infty} \frac{f(x)\,F(x)}{x} = 1 - \beta < \infty.$$

Therefore applying Theorem 2.3.3 yields $x(t) \sim F^{-1}(Mt)$ as $t\to\infty$. □

Example 2.4.2. *The following is but a simple application of Theorem 2.4.1, and the reader is invited to consider others. Consider the Volterra equation*

$$x'(t) = af(x(t)) + \int_0^t \frac{1}{(1+t-s)^{\theta+1}} f(x(s))\,ds, \quad t>0; \quad x(0) = \psi > 0,$$

where $a \ge 0$, $\theta > 0$ and $f : [0,\infty) \to (0,\infty)$ is locally Lipschitz continuous with $f(x) \sim x^\beta(\log x)^\alpha$ as $x\to\infty$ for $\beta \in (0,1)$, $\alpha \in \mathbb{R}$. The conditions ensure a unique positive continuous solution, and indeed, as $f \in RV_\infty(\beta)$, we see that all the hypotheses of Theorem 2.4.1 apply, with

$$M = a + \int_0^\infty \frac{du}{(1+u)^{1+\theta}} = a + \frac{1}{\theta}, \quad F(x) \sim \int_e^x \frac{du}{u^\beta(\log u)^\alpha} =: \Phi(x), \quad \text{as } x\to\infty.$$

It remains to determine explicitly the asymptotic behaviour of $F^{-1}(x)$ as $x\to\infty$. Clearly

$$\Phi(x) = \int_1^{\log x} v^{-\alpha} e^{(1-\beta)v}\,dv.$$

Applying L'Hôpital's rule shows that

$$\int_1^y v^{-\alpha} e^{(1-\beta)v}\,dv \sim \frac{1}{1-\beta} y^{-\alpha} e^{(1-\beta)y}, \quad \text{as } y\to\infty,$$

and hence

$$F(x) \sim \frac{1}{1-\beta}(\log x)^{-\alpha}x^{1-\beta}, \quad \text{as } x \to \infty.$$

Using the asymptotic relation above, it can now readily be shown that $\log F^{-1}(y)/\log y \to 1/(1-\beta)$ *as* $y \to \infty$*. Replacing this in the asymptotic relation for* F *leads to*

$$F^{-1}(y) \sim (1-\beta)^{\frac{1-\alpha}{1-\beta}}(\log y)^{\frac{\alpha}{1-\beta}}y^{\frac{1}{1-\beta}}, \quad \text{as } y \to \infty.$$

Finally, by Theorem 2.4.1, we conclude that

$$x(t) \sim F^{-1}(Mt) \sim (1-\beta)^{\frac{1-\alpha}{1-\beta}}\left(a+\frac{1}{\theta}\right)^{\frac{1}{1-\beta}}(\log t)^{\frac{\alpha}{1-\beta}}t^{\frac{1}{1-\beta}}, \quad \text{as } t \to \infty. \tag{2.4.1}$$

Example 2.4.3. *In the last example the measure exhibited power–law decay. We consider now the same nonlinearity, but an exponentially decaying measure, i.e.*

$$x'(t) = af(x(t)) + \int_0^t e^{-\theta(t-s)}f(x(s))\,ds, \quad t>0; \quad x(0)=\psi>0,$$

where once again $a \geq 0$ *and* $\theta > 0$*. As before, there is a unique positive continuous solution and all the hypotheses of Theorem 2.4.1 apply, with*

$$M = a + \int_0^\infty e^{-\theta u}\,du = a + \frac{1}{\theta},$$

so we recover exactly the same asymptotic behaviour of the solution x *as in the last example (i.e. the asymptotic relation* (2.4.1) *holds). Therefore, even though the past behaviour of the solution is discounted much more rapidly in this example than in the previous one, there is no difference in the rate of growth of the solution (to first order) because the value of* M *is the same in each case. Indeed, if we were to consider the delay–differential equation*

$$x'(t) = af(x(t)) + \frac{1}{\theta}f(x(t-\tau)), \quad t>0; \quad x(t)=\psi(t)>0, \quad t \in [-\tau,0],$$

with the same f*, and* $\tau > 0$ *fixed, we see once again* x *obeys* (2.4.1)*. This is because the mass of the point delta measure at* τ *is* $1/\theta$*, and the mass of the point delta measure at* 0 *is* a*, so* $M = a + 1/\theta$*, just as before. In this case, even though the past behaviour of the solution makes no contribution before time* $t-\tau$*, the same growth rate eventuates.*

Example 2.4.4. *Theorem 2.4.1 does not apply when* $\beta = 1$*. If* f *is sublinear, then condition* (2.2.7) *holds, and Theorem 2.3.2 applies. However, as mentioned earlier,* f *cannot satisfy* (2.3.6)*, and so we cannot conclude directly that* x *obeys* (2.3.7)*. Indeed, it has been shown in [12, Theorem 2.2], in the case that* $f' \in RV_\infty(0)$ *(which implies* $f \in RV_\infty(1)$*) with bounded delay (*$\mu_2 \equiv 0$*), that*

$$\lim_{t\to\infty}\frac{x(t)}{F^{-1}(Mt)} = e^{-\lambda C},$$

where

$$\lim_{x\to\infty}\frac{f(x)}{x/\log x} =: \lambda \in [0,\infty], \quad C := \int_{[0,\tau]} s\,\mu_1(ds).$$

Therefore, the conclusion of Theorem 2.4.1 need not *hold if* $f(x)$ *is of larger order than* $x/\log x$ *as* $x \to \infty$*, and the delay is nontrivial, although* $x(t)/F^{-1}(Mt) \to 1$ *as* $t \to \infty$ *if* $f(x) = o(x/\log x)$ *as* $x \to \infty$*.*

The determination of the asymptotic behaviour of F *and* F^{-1} *is more delicate when* $f \in RV_\infty(1)$*,* *in large part because Karamata's theorem only shows that $1/F(x) = o(f(x)/x)$ as $x \to \infty$. However, we supply a concrete example in which the asymptotic behaviour of F can be worked out explicitly, and Theorem 2.3.2 applies.*

Consider the Volterra equation

$$x'(t) = \int_0^t \frac{1}{(1+t-s)^{\theta+1}} f(x(s))\,ds, \quad t > 0; \quad x(0) = \psi > 0,$$

where $\theta > 0$ and $f : [0, \infty) \to (0, \infty)$ is locally Lipschitz continuous with $f(x) \sim x/(\log x)^\alpha$ as $x \to \infty$ for $\alpha > 0$. We see that $f \in RV_\infty(1)$ and $f(x)/x \to 0$ as $x \to \infty$, so not only do these conditions ensure a unique positive continuous and growing solution, but moreover they ensure that Theorem 2.3.2 applies, with

$$M = \int_0^\infty \frac{1}{(1+u)^{1+\theta}}\,du = \frac{1}{\theta}, \quad F(x) \sim \int_e^x \frac{(\log u)^\alpha}{u}\,du =: \Phi(x), \quad \text{as } x \to \infty.$$

Clearly

$$\Phi(x) = \frac{(\log x)^{\alpha+1}}{\alpha+1},$$

and so

$$F(x) \sim \frac{(\log x)^{\alpha+1}}{\alpha+1}, \quad \text{as } x \to \infty.$$

Therefore, by Theorem 2.3.2,

$$\lim_{t\to\infty} \frac{1}{t} \frac{(\log x(t))^{\alpha+1}}{\alpha+1} = \frac{1}{\theta},$$

so

$$\lim_{t\to\infty} \frac{\log x(t)}{t^{1/(\alpha+1)}} = \left(\frac{\alpha+1}{\theta}\right)^{1/(\alpha+1)}.$$

Since $\alpha > 0$, the growth is slower than exponential, as expected, and we may view the limit above as a generalisation of the Liapunov exponent in this nonlinear setting.

Example 2.4.5. *To illustrate the utility of only requiring asymptotic monotonicity in our earlier results, suppose $f(x) = x^\alpha[2 + \sin(\log_2(x + 2))]$ with $\alpha \in (0, 1)$. $f \in RV_\infty(\alpha)$ and, although f is clearly non-monotone, it oscillates slowly enough that $f'(x) > 0$ for all x sufficiently large.*

Theorem 2.4.1 immediately raises the question of what happens when f is regularly varying with index zero. In this case there is no guarantee that f will be asymptotic to a monotone function and hence we cannot rely on any of our previous work. An example emphasising the extreme oscillatory behaviour possible within the class $\text{RV}_\infty(0)$ is to take

$$f(x) = \exp[\ln(2 + x)^{\frac{1}{3}} \cos(\ln(2 + x)^{\frac{1}{3}})]. \tag{2.4.2}$$

In this example, $\liminf_{x\to\infty} f(x) = 0$ and $\limsup_{x\to\infty} f(x) = \infty$.

The following pair of results partially answer the question of how our previous conclusions can be retained when $f \in \text{RV}_\infty(0)$. Our first result shows that when the delay is bounded we still have $x(t) \sim F^{-1}(Mt)$ as $t \to \infty$ without additional hypotheses.

Theorem 2.4.6. *Suppose μ_1 obeys (2.2.1) with $\mu_2 \equiv 0$ and $\psi \in C([-\tau, 0]; (0, \infty))$. If $f \in RV_\infty(0)$, then solutions to (2.2.3) obey*

$$\lim_{t\to\infty} z(t) = \infty, \quad \lim_{t\to\infty} \frac{z(t)}{F^{-1}(Mt)} = 1.$$

In the case of unbounded delay the problem is much more delicate and only under additional hypotheses have we been able to retain the asymptotic rates as before. If we assume that f is bounded away from zero we rule out highly irregular nonlinearities such as (2.4.2) and we can prove the following result.

Theorem 2.4.7. *Suppose μ_2 obeys (2.2.1) with $\mu_1 \equiv 0$ and $\psi > 0$. If $f \in RV_\infty(0)$ is bounded away from zero, then solutions to (2.3.3) obey*

$$\lim_{t\to\infty} v(t) = \infty, \quad \lim_{t\to\infty} \frac{v(t)}{F^{-1}(Mt)} = 1.$$

Of course, the hypothesis in Theorem 2.4.7 that f is bounded away from zero (by continuity of $x \mapsto f(x)$, this lower bound is meaningful in the limit as $x \to \infty$) is satisfied in the case that f is asymptotically monotone. Therefore, one can rephrase Theorem 2.4.1 to include the case that $\beta = 0$, at the small expense of assuming the asymptotic monotonicity of f (which is automatically true when $\beta > 0$).

It instructive to see how far our calculations can proceed in the case of unbounded delay, without additional hypotheses. The following lemma shows that we can obtain a sharp lower bound.

Theorem 2.4.8. *Suppose μ_2 obeys (2.2.1) with $\mu_1 \equiv 0$ and $\psi > 0$. If $f \in RV_\infty(0)$, then solutions to (2.3.3) obey*

$$\lim_{t\to\infty} v(t) = \infty, \quad \liminf_{t\to\infty} \frac{v(t)}{F^{-1}(Mt)} \geq 1.$$

Our final result demonstrates that under no additional assumptions we can at least obtain a "crude" upper bound on the growth rate of the solution to (2.3.3) which agrees with the lower bound provided by Theorem 2.4.8 up to a logarithmic factor.

Theorem 2.4.9. *Suppose μ_2 obeys (2.2.1) with $\mu_1 \equiv 0$ and $\psi > 0$. If $f \in RV_\infty(0)$ is bounded away from zero, then solutions to (2.3.3) obey*

$$\lim_{t\to\infty} v(t) = \infty, \quad \lim_{t\to\infty} \frac{\log(v(t))}{\log(t)} = 1.$$

2.5 Examples of Sublinearity

Before giving proofs of our results in Section 2.6.1, we close with the examples promised in Section 2.2 which show the scope of the strengthened sublinearity hypothesis, (2.2.7).

We find for the purposes of these examples it is more natural and instructive to construct an f with the desired properties by specifying f'. We defer the justification of the following examples to Section 2.6.4. Throughout these examples we define f', for $n \in \mathbb{N}$, as follows

$$f'(x) = \begin{cases} \eta(x), \ \ x \in (0,1] \cup (n + w_n, n+1], \\ \eta(n) + \frac{2(x-n)(h_n - \eta(n))}{w_n}, \ \ x \in (n, n + w_n/2], \\ h_n + \frac{2(x-n-w_n/2)(\eta(n+w_n)-h_n)}{w_n}, \ \ x \in (n + w_n/2, n + w_n]. \end{cases} \tag{2.5.1}$$

Choosing $\eta(x) > 0$ for all $x > 0$ and $h_n > 0$ for all $n \in \mathbb{N}$ ensures that f is strictly increasing. Define $\phi(x) := \int_0^x \eta(u)du$ and by construction we will have $\phi \sim f$. In order to have both $\liminf_{x\to\infty} f'(x) = 0$ and $\limsup_{x\to\infty} f'(x) > 0$ we want f' to largely follow the behaviour of η, which tends to zero, but to also have high, narrow spikes inherited from h_n.

Example 2.5.1. *Suppose f' is defined by (2.5.1), $\phi(x) := \int_0^x \eta(u)du$ and that $\eta(x) \downarrow 0$ as $x \to \infty$, $0 < w_n < 1$, and $h_n > \phi'(n)$ for all $n \in \mathbb{N}$. Furthermore suppose that*

$$\lim_{x\to\infty} \phi(x) = \infty, \ \lim_{n\to\infty} h_n = L \in (0,\infty], \ \lim_{n\to\infty} \sum_{j=1}^n \frac{w_j h_j}{\phi(n)} = 0, \ \lim_{n\to\infty} \sum_{j=1}^n \frac{w_j \phi'(j)}{\phi(n)} = 0.$$

Then

(i.) $\liminf_{x\to\infty} f'(x) = 0$, $\limsup_{x\to\infty} f'(x) \geq L$.

(ii.) $f(x) \sim \phi(x)$ *as* $x \to \infty$ *and hence* $\lim_{x\to\infty} f(x)/x = 0$.

The function f constructed in Example 2.5.1 has "spikes" in its derivative which can grow arbitrarily quickly but since it is asymptotic to ϕ it still obeys condition (2.2.7). When ϕ tends to a finite limit, so does f. Moreover, we do not require that ϕ grows faster than the sums of $w_j\, h_j$ and $h_j\, \phi'(j)$.

Example 2.5.2. *Suppose* f' *is defined by* (2.5.1)*,* $\phi(x) := \int_0^x \eta(u)du$ *and that* $\eta(x) \downarrow 0$ *as* $x \to \infty$*,* $0 < w_n < 1$*, and* $h_n > \phi'(n)$ *for all* $n \in \mathbb{N}$*. Furthermore, if* L^**,* L_0 *and* L_1 *are finite, suppose that*

$$\lim_{x\to\infty} \phi(x) = L^*, \ \lim_{n\to\infty} h_n = L \in (0,\infty], \ \lim_{n\to\infty} \sum_{j=1}^{n} w_j h_j = L_0, \ \lim_{n\to\infty} \sum_{j=1}^{n} w_j \phi'(j) = L_1.$$

Then

(i.) $\liminf_{x\to\infty} f'(x) = 0$, $\limsup_{x\to\infty} f'(x) \geq L$.

(ii.) $f(x) \to L' \in (0,\infty)$ *as* $x \to \infty$ *and hence* $\lim_{x\to\infty} f(x)/x = 0$.

In this case f is asymptotic to a constant so it once more obeys (2.2.7).

2.6 Proofs

2.6.1 Proofs with Increasing Nonlinearity

Before giving the proofs of our main results we state and prove some useful technical lemmata; the first makes explicit the fact that (2.2.7) implies sublinearity.

Lemma 2.6.1. *If* $f \in C(\mathbb{R}^+;(0,\infty))$ *obeys* (2.2.7)*, then* $f(x)/x \to 0$ *as* $x \to \infty$*.*

Proof. Since ϕ is increasing, either $\phi(x) \to \infty$ as $x \to \infty$ or $\phi(x) \to L \in (0,\infty)$ as $x \to \infty$. In the latter case, asymptotic equivalence of ϕ and f yields $\lim_{x\to\infty} f(x)/x = 0$. In the first case, use L'Hôpitals rule to obtain

$$\lim_{x\to\infty} \phi(x)/x = \lim_{x\to\infty} \phi'(x) = 0.$$

Thus $\lim_{x\to\infty} f(x)/x = \lim_{x\to\infty} (f(x)/\phi(x))\,(\phi(x)/x) = 0$. □

The proof of Proposition 2.3.1 requires the following preliminary lemma.

Lemma 2.6.2. *Suppose* $\phi \in C(\mathbb{R}^+;(0,\infty))$ *obeys* $\phi(x) \to \infty$ *as* $x \to \infty$*,* $\phi'(x) > 0$ *for* $x > 0$ *and* $\phi'(x)$ *is decreasing with* $\phi'(x) \to 0$ *as* $x \to \infty$*. If* $b, c \in C(\mathbb{R}^+,(0,\infty))$ *obey* $\lim_{t\to\infty} b(t) = \lim_{t\to\infty} c(t) = \infty$*, and* $b(t) \sim c(t)$ *as* $t \to \infty$*, then* $\phi(b(t)) \sim \phi(c(t))$ *as* $t \to \infty$*.*

Proof of Lemma 2.6.2. We start by showing that

$$\limsup_{x\to\infty} \frac{\phi(\Lambda x)}{\phi(x)} \leq \Lambda \quad \text{for every } \Lambda > 1. \tag{2.6.1}$$

Let $x \geq a > 0$. Then $\phi(x) - \phi(a) = \int_a^x \phi'(u)du \geq \phi'(x)(x-a)$. Thus

$$\limsup_{x\to\infty} \frac{\phi'(x)x}{\phi(x)} = \limsup_{x\to\infty} \frac{\phi'(x)(x-a)}{\phi(x)} \frac{x}{x-a} \leq \limsup_{x\to\infty} \frac{\phi(x)-\phi(a)}{\phi(x)} = 1. \tag{2.6.2}$$

To prove (2.6.1) we proceed as follows:

$$\begin{aligned}\frac{\phi(\Lambda x)}{\phi(x)} &= \frac{\int_a^{\Lambda x}\phi'(u)du+\phi(a)}{\phi(x)} = \frac{\int_a^{x}\phi'(u)du+\int_x^{\Lambda x}\phi'(u)du+\phi(a)}{\phi(x)} \\ &= 1+\frac{\int_x^{\Lambda x}\phi'(u)du}{\phi(x)} \le 1+(\Lambda-1)\frac{\phi'(x)\,x}{\phi(x)}.\end{aligned}$$

Now taking the limsup, and using (2.6.2), we have shown (2.6.1). We are now ready to prove our claim. By hypothesis, for all $\epsilon > 0$, there is a $T(\epsilon) > 0$ such that

$$(1-\epsilon)c(t) < b(t) < (1+\epsilon)c(t), \quad t \ge T(\epsilon).$$

Monotonicity of ϕ immediately yields

$$\frac{\phi((1-\epsilon)c(t))}{\phi(c(t))} < \frac{\phi(b(t))}{\phi(c(t))} < \frac{\phi((1+\epsilon)c(t))}{\phi(c(t))}, \quad t \ge T.$$

By (2.6.1), and the divergence of c, there exists $T' > T$ such that $\phi((1+\epsilon)c(t)) < (1+\epsilon)^2\phi(c(t))$ for all $t \ge T'$. Hence $\limsup_{t\to\infty}\phi(b(t))/\phi(c(t)) \le 1$. Reversing the roles of b and c in the above argument we have that

$$\limsup_{t\to\infty}\frac{\phi(c(t))}{\phi(b(t))} \le 1,$$

or equivalently, $\liminf_{t\to\infty}\phi(b(t))/\phi(c(t)) \ge 1$, completing the proof. □

Proof of Proposition 2.3.1. By (2.2.7), $\Phi(x) = \int_1^x du/\phi(u)$ obeys $\Phi(x) \sim F(x)$ as $x \to \infty$. Notice also from (2.2.7) that Φ is increasing with decreasing derivative. Now, we apply Lemma 2.6.2 with $\phi = \Phi$, so that if b and c are continuous functions with $b(t) \to \infty$ and $b(t) \sim c(t)$ as $t \to \infty$, then

$$\Phi(b(t)) \sim \Phi(c(t)) \text{ as } t \to \infty.$$

Therefore, it follows that $\Phi(b(t)) \sim F(c(t))$ as $t \to \infty$. Now take $c(t) = F^{-1}(Mt)$ and $b(t) = a(t)$, so that $\Phi(a(t))/Mt \to 1$ as $t \to \infty$. Since $\Phi(x) \sim F(x)$ as $x \to \infty$ the claim follows. □

Proof of Theorem 2.3.2. Since x is absolutely continuous, for each $t \ge 0$,

$$\begin{aligned} x(t+h)-x(t) &= \int_t^{t+h} x'(u)\,du \\ &= \int_t^{t+h}\left(\int_{[0,u]}\mu_2(ds)f(x(u-s)) + \int_{[0,\tau]}\mu_1(ds)f(x(u-s))\right)du, \end{aligned} \tag{2.6.3}$$

for each $h > 0$. In particular,

$$x(h)-x(0) = \int_0^h\left(\int_{[0,u]}\mu_2(ds)f(x(u-s)) + \int_{[0,\tau]}\mu_1(ds)f(x(u-s))\right)du. \tag{2.6.4}$$

Continuity of x and positivity of $x(0)$ mean that there exists an interval $[0, t_0)$ on which x is positive. Suppose t_0 is the minimial time at which x equals zero. Taking $h = t_0$ in (2.6.4) shows that $x(t_0) \ge x(0)$ by nonnegativity of the right–hand side, a contradiction. Thus x is a positive function and a fortiori, $x(t) \ge x(0)$ for all $t \ge 0$. The right–hand side of (2.1.1) is nonnegative for all $t \ge 0$ and hence $x'(t) \ge 0$ for a.e. $t \ge 0$. It now follows from (2.6.3) that x is nondecreasing. Therefore $\lim_{t\to\infty} x(t)$ exists.

Suppose $\lim_{t\to\infty} x(t) = L \in [x(0),\infty)$ and integrate (2.1.1) to obtain

$$x(t) = x(0) + \int_0^t \left(\int_{[0,u]} \mu_2(ds) f(x(u-s)) + \int_{[0,\tau]} \mu_1(ds) f(x(u-s)) \right) du \quad t \geq 0.$$

Since f is continuous, for each $\epsilon \in (0, f(L))$ there exists a $T_0 > \tau$ such that $f(x(t)) > f(L) - \epsilon > 0$ for all $t \geq T_0$. Now let $t > 2T_0$ and estimate as follows:

$$\begin{aligned} x(t) &\geq x(0) + \int_{2T_0}^t \left(\int_{[0,u]} \mu_2(ds) f(x(u-s)) + \int_{[0,\tau]} \mu_1(ds) f(x(u-s)) \right) du \\ &\geq x(0) + \int_{2T_0}^t \int_{[0,u/2]} \mu_2(ds) f(x(u-s))\, du + \int_{2T_0}^t \int_{[0,\tau]} \mu_1(ds) f(x(u-s))\, du. \end{aligned}$$

In the first integral above, $u - s \in [T_0, t]$ and in the second integral, since $T_0 > \tau$, $u - s \in [2T_0 - \tau, t] \subset [T_0, t]$ for each $t > 2T_0$. Hence

$$x(t) \geq x(0) + (f(L) - \epsilon) \int_{2T_0}^t \{\mu_2([0,u/2]) + \mu_1([0,\tau])\}\, du, \quad t > 2T_0. \tag{2.6.5}$$

The function $g : u \mapsto \mu_2([0,u/2]) + \mu_1([0,\tau])$ is nonnegative and measurable, and $\lim_{u\to\infty} g(u) = M > 0$. Since $f(L) - \epsilon > 0$, letting $t \to \infty$ in (2.6.5) shows that $\lim_{t\to\infty} x(t) = \infty$, a contradiction. Therefore $\lim_{t\to\infty} x(t) = \infty$, as claimed.

Step 1: First compute the upper bound on the growth rate of the solution. If $\epsilon > 0$ is arbitrary, by hypothesis, there exists $x_1(\epsilon)$ such that for all $x > x_1(\epsilon)$, $(1-\epsilon)\phi(x) < f(x) < (1+\epsilon)\phi(x)$. Since $\lim_{t\to\infty} x(t) = \infty$, there exists $T_1(\epsilon)$ such that for $t \geq T_1(\epsilon)$, $x(t) > x_1(\epsilon)$. Thus, for a.e. $t \geq T(\epsilon) := T_1(\epsilon) + \tau$,

$$x'(t) \leq (1+\epsilon) \int_{[0,\tau]} \mu_1(ds)\phi(x(t-s)) + (1+\epsilon) \int_{[0,t-T]} \mu_2(ds)\phi(x(t-s)) + R(t),$$

where $R(t) := \int_{(t-T,t]} \mu_2(ds) f(x(t-s))$. Monotonicity of $\phi \circ x$ leads to the estimate

$$\begin{aligned} x'(t) &\leq (1+\epsilon) \left(\int_{[0,\tau]} \mu_1(ds)\phi(x(t)) + \int_{[0,\infty)} \mu_2(ds)\phi(x(t)) \right) + R(t) \\ &= (1+\epsilon)\, M\, \phi(x(t)) + R(t), \quad \text{for a.e. } t \geq T(\epsilon). \end{aligned}$$

Hence

$$\frac{x'(t)}{\phi(x(t))} \leq (1+\epsilon)M + \frac{R(t)}{\phi(x(t))}, \quad \text{for a.e. } t \geq T(\epsilon). \tag{2.6.6}$$

Estimate the final term on the right–hand side of (2.6.6) as follows:

$$\frac{R(t)}{\phi(x(t))} = \frac{\int_{(t-T,t]} \mu_2(ds) f(x(t-s))}{\phi(x(t))} \leq \frac{\int_{(t-T,t]} \mu_2(ds)}{\phi(x(t))} \sup_{u\in[0,T]} f(x(u)), \quad t \geq T(\epsilon).$$

Since $f \circ x$ is a continuous function the supremum is bounded on compact intervals, and $\lim_{t\to\infty} R(t) = 0$. Also, because ϕ is nondecreasing and $x(t) \to \infty$ as $t \to \infty$, $\lim_{t\to\infty} \phi(x(t)) \in (0,\infty]$ and hence $\lim_{t\to\infty} R(t)/\phi(x(t)) = 0$. Thus there exists a $\bar{T}(\epsilon) > T(\epsilon)$ such that $R(t)/\phi(x(t)) < \epsilon$ for $t \geq \bar{T}(\epsilon)$ and (2.6.6) simplifies to

$$\frac{x'(t)}{\phi(x(t))} \leq \epsilon + (1+\epsilon)M, \quad \text{for a.e. } t \geq \bar{T}(\epsilon). \tag{2.6.7}$$

Asymptotic integration shows that

$$\frac{\Phi(x(t))}{t} \leq \frac{\Phi(x(\bar{T}))}{t} + \frac{[\epsilon + (1+\epsilon)M]\,(t-\bar{T})}{t}, \quad \text{for each } t \geq \bar{T}(\epsilon).$$

Now take the limsup in the inequality above to show that

$$\limsup_{t\to\infty} \frac{\Phi(x(t))}{t} \leq \epsilon + (1+\epsilon)M,$$

and then let $\epsilon \to 0^+$ to obtain $\limsup_{t\to\infty} \Phi(x(t))/Mt \leq 1$. The asymptotic equivalence of F and Φ then allows us to conclude that $\limsup_{t\to\infty} F(x(t))/Mt \leq 1$.

Step 2: Now compute the corresponding lower bound. Define

$$\mu_1 = \int_{[0,\tau]} \mu_1(ds), \quad \mu_2 = \int_{[0,\infty)} \mu_2(ds).$$

By (2.2.1), for an arbitrary $\epsilon \in (0,1)$, there exists $T_2(\epsilon)$ large enough that

$$(1-\epsilon)\int_{[0,\infty)} \mu_2(ds) \leq \int_{[0,T_2]} \mu_2(ds) \leq \int_{[0,\infty)} \mu_2(ds).$$

Furthermore, since $\lim_{x\to\infty} \phi'(x) = 0$ there exists $x_2(\epsilon)$ such that $x \geq x_2$ implies $\phi'(x) < \epsilon$, for all $\epsilon > 0$. Part (i) gives us the existence of a $T_3(\epsilon)$ such that $x(t) \geq x_2(\epsilon)$ whenever $t \geq T_3(\epsilon)$. Take $T_4 := \bar{T} + 2T_1(\epsilon) + 2\tau + 2T_2(\epsilon) + 2T_3(\epsilon)$ and exploit asymptotic monotonicity once more to derive the estimate

$$x'(t) \geq (1-\epsilon)\,\mu_1\,\phi(x(t-\tau)) + (1-\epsilon)^2\,\mu_2\,\phi(x(t-T_2)), \quad \text{for a.e. } t \geq T_4.$$

Thus

$$\frac{x'(t)}{\phi(x(t))} \geq (1-\epsilon)\,\mu_1\,\frac{\phi(x(t-\tau))}{\phi(x(t))} + (1-\epsilon)^2\,\mu_2\,\frac{\phi(x(t-T_2))}{\phi(x(t))}, \quad \text{for a.e. } t \geq T_4. \tag{2.6.8}$$

In a moment, we will show that

$$\lim_{t\to\infty} \frac{\phi(x(t-\theta))}{\phi(x(t))} = 1, \quad \text{for each } \theta > 0. \tag{2.6.9}$$

Let $\eta \in (0,1)$ be arbitrary. Using (2.6.8) and applying (2.6.9) twice (with $\theta = \tau$ and $\theta = T_2(\epsilon)$), we can find a $T_5(\eta,\epsilon) > T_4(\epsilon)$ such that

$$\frac{x'(t)}{\phi(x(t))} \geq (1-\eta)(1-\epsilon)\,\mu_1 + (1-\eta)(1-\epsilon)^2\,\mu_2, \quad \text{for a.e. } t \geq T_5(\eta,\epsilon).$$

Perform asymptotic integration on the inequality above to show that

$$\frac{\Phi(x(t))}{t} \geq \frac{\Phi(x(T_5))}{t} + \frac{(1-\eta)(1-\epsilon)(t-T_5)\mu_1}{t} + \frac{(1-\eta)(1-\epsilon)^2(t-T_5)\mu_2}{t},$$

for each $t \geq T_5(\eta,\epsilon)$. Take the liminf as $t \to \infty$ in the inequality above to obtain

$$\liminf_{t\to\infty} \frac{\Phi(x(t))}{t} \geq (1-\eta)(1-\epsilon)\,\mu_1 + (1-\eta)(1-\epsilon)^2\,\mu_2,$$

and then let $\eta = \epsilon$, and send $\epsilon \to 0^+$ to show that $\liminf_{t\to\infty} \Phi(x(t))/t \geq M$. The asymptotic equivalence of Φ and F yields

$$\liminf_{t\to\infty} \frac{F(x(t))}{Mt} \geq 1.$$

Combining this with the corresponding limsup from Step 1 proves the theorem.

It remains to return to the deferred proof of (2.6.9). Let $\theta > 0$ be given. Since $x(t-\theta) \leq x(t)$ for all $t \geq \theta$, and ϕ is increasing, we immediately have that

$$\limsup_{t\to\infty} \frac{\phi(x(t-\theta))}{\phi(x(t))} \leq 1.$$

To get the corresponding liminf, consider the absolutely continuous function $a : [\theta, \infty) \to \mathbb{R}$ defined by $a(t) = (\phi \circ x)(t-\theta)$ for $t \geq \theta$. Since a is absolutely continuous,

$$a(t+\theta) - a(t) = \int_t^{t+\theta} a'(s)\, ds, \quad \text{for each } t \geq \theta,$$

or equivalently,

$$\frac{\phi(x(t-\theta))}{\phi(x(t))} = 1 - \frac{1}{\phi(x(t))} \int_t^{t+\theta} \phi'(x(s-\theta))x'(s-\theta)\, ds, \quad \text{for each } t \geq \theta.$$

We claim that

$$\lim_{t\to\infty} \frac{1}{\phi(x(t))} \int_t^{t+\theta} \phi'(x(s-\theta))x'(s-\theta)\, ds = 0, \quad \text{for each } \theta > 0, \tag{2.6.10}$$

and hence that

$$\liminf_{t\to\infty} \frac{\phi(x(t-\theta))}{\phi(x(t))} \geq 1.$$

Together with the corresponding limsup, the liminf above establishes (2.6.9). To prove (2.6.10), once more fix the value of $\theta > 0$, and note that

$$\phi'(x(s-\theta)) < \epsilon, \quad \text{for each } s \geq T_3(\epsilon) + \theta.$$

From (2.6.7), if $s \in [t, t+\theta]$ for some $t \geq \bar{T}(\epsilon) + \theta$, then

$$x'(s-\theta) \leq (\epsilon + (1+\epsilon)M)\, \phi(x(s-\theta)) \leq (\epsilon + (1+\epsilon)M)\, \phi(x(t)),$$

for a.e. $t \geq \bar{T}(\epsilon) + \theta$. Thus, if $s \in [t, t+\theta]$, for a.e. $t \geq T_3(\epsilon) + \bar{T}(\epsilon) + \theta$,

$$0 \leq \phi'(x(s-\theta))x'(s-\theta) \leq \epsilon\, (\epsilon + (1+\epsilon)M)\, \phi(x(t)).$$

Hence

$$0 \leq \frac{1}{\phi(x(t))} \int_t^{t+\theta} \phi'(x(s-\theta))x'(s-\theta)\, ds \leq \theta\epsilon\, (\epsilon + (1+\epsilon)M)\,,$$

for each $t \geq T_3(\epsilon) + \bar{T}(\epsilon) + \theta$. Finally, letting $\epsilon \to 0^+$ completes the proof. □

Proof of Corollary 2.3.3. If v is a solution of (2.3.3), then $v(t) > 0$ for all $t \geq 0$, v is nondecreasing and $\lim_{t\to\infty} v(t) = \infty$ by Theorem 2.3.2 (with $\mu_1 \equiv 0$). By hypothesis, for each $N > 0$ there exists $T_1(N) > 0$ such that $\int_{[0,T_1(N)]} \mu_2(ds) > N$, for all $t \geq T_1(N)$. Similarly, by (4.2.1), for all $\epsilon \in (0,1)$ there exists $T_2(\epsilon) > 0$ such that $f(x) > (1-\epsilon)\phi(x)$ for all $x \geq T_2$. Since $\lim_{t\to\infty} v(t) = \infty$ there exists

$x(\epsilon)$ such that $v(t) > T_2(\epsilon)$ for all $t \geq x(\epsilon)$. Hence, for a.e. $t \geq T := \max(2T_1, 2T_2)$,

$$v'(t) \geq (1-\epsilon)\int_{[0,T]} \mu_2(ds)\phi(v(t-s)) \geq (1-\epsilon)N\phi(v(t-T)).$$

Hence

$$v(t) \geq v(T) + (1-\epsilon)N\int_T^t \phi(v(s-T))\,ds, \qquad \text{for each } t \geq T.$$

Define the comparison solution y_N for each fixed $\epsilon > 0$ and $N > 0$ by

$$y_N(t) = y_N(T) + \frac{N(1-\epsilon)}{2}\int_T^t \phi(y_N(s-T))\,ds, \quad \text{for each } t > T,$$

with $y_N(t) = v(t)/2$ for $t \in [0,T]$. A straightforward comparison of the integral equations in question shows that $y_N(t) < v(t)$ for all $t \geq 0$. Furthermore,

$$y_N'(t) = \tfrac{N(1-\epsilon)}{2}\phi(y_N(t-T)), \quad \text{for each } t > T.$$

Now let $u_N(t) := y_N(t+T)$ for $t \geq -T$. For $t > 0$, $t+T > T$ and hence

$$u_N'(t) = y_N'(t+T) = \frac{N(1-\epsilon)}{2}\phi(y_N(t)) = \frac{N(1-\epsilon)}{2}\phi(u_N(t-T)).$$

For $t \in [-T,0]$, $u_N(t) = y_N(t+T) = v(t+T)/2 =: \psi_N(t)$. Thus we have the following delay differential equation for u_N:

$$u_N'(t) = \frac{N(1-\epsilon)}{2}\phi(u_N(t-T)), \quad t > 0; \quad u_N(t) = \psi_N(t) > 0, \quad t \in [-T,0].$$

Applying Theorem 2.3.2 yields $\lim_{t\to\infty} F(u_N(t))/t = N(1-\epsilon)/2$. This implies that $\lim_{t\to\infty} F(y_N(t+T))/t = N(1-\epsilon)/2$. Finally, since F is increasing and v lies above our comparison solution y_N, we obtain

$$\frac{N(1-\epsilon)}{2} = \lim_{t\to\infty}\frac{F(y_N(t))}{t}\frac{t}{t-T} = \liminf_{t\to\infty}\frac{F(y_N(t))}{t} \leq \liminf_{t\to\infty}\frac{F(v(t))}{t}.$$

We can now let $\epsilon \to 0^+$ and, since N was arbitrary, we have proven that

$$\liminf_{t\to\infty}\frac{F(v(t))}{t} = \infty, \tag{2.6.11}$$

as required. □

Before the proof of Theorem 2.3.3 we establish the following useful technical result.

Lemma 2.6.3. *Suppose $f \in C(\mathbb{R}^+;(0,\infty))$ is asymptotically increasing and $\lim_{x\to\infty} F(x) = \infty$. If (2.3.6) holds, then for each $\epsilon > 0$ sufficiently small there exists a $T(\epsilon) > 0$ such that*

$$1 < \frac{F^{-1}((1+\epsilon)t)}{F^{-1}(t)} < \frac{1}{1-\frac{\epsilon(1+\epsilon)}{1-\epsilon}L}, \quad t \geq T(\epsilon).$$

Proof of Lemma 2.6.3. Consider $u'(t) = f(u(t))$, $t > 0$ with $u(0) = 1$. Then $u(t) = F^{-1}(t)$ for $t \geq 0$ and $\lim_{t\to\infty} u(t) = \infty$. Hence, for all $t \geq T_1(\epsilon)$ we have $u(t) > x_1(\epsilon)$, where $x_1(\epsilon)$ is defined by

$1-\epsilon < \frac{f(x)}{\phi(x)} < 1+\epsilon,\ x \geq x_1(\epsilon)$, and ϕ is an increasing function. Thus for $t \geq T_1(\epsilon)$,

$$\begin{aligned} 0 < F^{-1}((1+\epsilon)t) - F^{-1}(t) &= \int_t^{(1+\epsilon)t} u'(s)ds = \int_t^{(1+\epsilon)t} f(u(s))ds \\ &\leq (1+\epsilon)\int_t^{(1+\epsilon)t} \phi(u(s))ds \leq \epsilon\,(1+\epsilon)\,t\,(\phi\circ F^{-1})((1+\epsilon)t). \end{aligned}$$

Therefore

$$0 < 1 - \frac{F^{-1}(t)}{F^{-1}((1+\epsilon)t)} \leq \epsilon(1+\epsilon)t\frac{\phi(F^{-1}((1+\epsilon)t))}{F^{-1}((1+\epsilon)t)}, \quad t \geq T_1(\epsilon). \tag{2.6.12}$$

Now let $y_\epsilon(t) = F^{-1}((1+\epsilon)t)$, so $F(y_\epsilon(t)) = (1+\epsilon)t$ and $y_\epsilon(t) = F^{-1}((1+\epsilon)t) > F^{-1}(t) > x_1(\epsilon)$. Hence

$$(1+\epsilon)t\frac{\phi(F^{-1}((1+\epsilon)t))}{F^{-1}((1+\epsilon)t)} = \frac{F(y_\epsilon(t))\phi(y_\epsilon(t))}{y_\epsilon(t)} < \frac{F(y_\epsilon(t))f(y_\epsilon(t))}{(1-\epsilon)y_\epsilon(t)}.$$

Thus (2.6.12) reads

$$0 < 1 - \frac{F^{-1}(t)}{F^{-1}((1+\epsilon)t)} \leq \frac{\epsilon F(y_\epsilon(t))f(y_\epsilon(t))}{(1-\epsilon)y_\epsilon(t)}, \quad t \geq T_1(\epsilon).$$

By (2.3.6) there exists $x_2(\epsilon) > 0$ such that $f(x)F(x)/x < L(1+\epsilon)$ for all $x \geq x_2(\epsilon)$. Let $T_2(\epsilon) > 0$ be such that $F^{-1}(t) > x_2(\epsilon)$, which implies $y_\epsilon(t) > x_2(\epsilon)$ for all $t \geq T_2(\epsilon)$. Therefore, letting $T(\epsilon) = 1 + \max(T_1(\epsilon), T_2(\epsilon))$,

$$0 < 1 - \frac{F^{-1}(t)}{F^{-1}((1+\epsilon)t)} \leq \frac{\epsilon F(y_\epsilon(t))f(y_\epsilon(t))}{(1-\epsilon)y_\epsilon(t)} \leq \frac{\epsilon(1+\epsilon)L}{(1-\epsilon)}, \quad t \geq T(\epsilon).$$

Thus, choosing $\epsilon \in (0, 1/4 \vee 3L/5)$, $1 - \frac{\epsilon(1+\epsilon)}{1-\epsilon}L > 0$, we obtain

$$0 < 1 - \frac{\epsilon(1+\epsilon)}{1-\epsilon}L < \frac{F^{-1}(t)}{F^{-1}((1+\epsilon)t)}, \quad t \geq T(\epsilon).$$

Hence

$$\frac{F^{-1}((1+\epsilon)t)}{F^{-1}(t)} < \frac{1}{1 - \frac{\epsilon(1+\epsilon)}{1-\epsilon}L}, \quad t \geq T(\epsilon),$$

as claimed. □

We are now in position to give the proof of Theorem 2.3.3.

Proof of Theorem 2.3.3. By Theorem 2.3.2, $\lim_{t\to\infty} x(t) = +\infty$ and $\lim_{t\to\infty} F(x(t))/Mt = 1$. The latter limit implies that for each $\epsilon \in (0,1)$ there exists $T(\epsilon) > 0$ such that $1-\epsilon < F(x(t))/Mt < 1+\epsilon$ for all $t \geq T(\epsilon)$. Hence

$$\frac{F^{-1}((1-\epsilon)Mt)}{F^{-1}(Mt)} < \frac{x(t)}{F^{-1}(Mt)} < \frac{F^{-1}((1+\epsilon)Mt)}{F^{-1}(Mt)}, \quad t \geq T(\epsilon).$$

Since f obeys (2.2.7), $F(x) \to \infty$ as $x \to \infty$. Therefore we can apply Lemma 2.6.3 to the right–hand member of the inequality above. Doing this and then sending $\epsilon \to 0$ yields

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} \leq 1.$$

The liminf is dealt with analogously. □

2.6.2 Proofs With Decreasing Nonlinearity

This section concentrates on results in which f is asymptotic to a decreasing function, principally Theorem 2.3.4. Before proving Theorem 2.3.4 we find it useful to prepare some estimates concerning the auxiliary functions F and Φ.

Lemma 2.6.4. *If $\phi \in C^1(\mathbb{R}^+;(0,\infty))$ is decreasing and Φ is given by (2.2.8), then*

$$\lim_{t\to\infty} \frac{\Phi^{-1}(A+Bt)}{\Phi^{-1}(Bt)} = 1, \quad \textit{for each } A \in \mathbb{R} \textit{ and } B \in (0,\infty).$$

Proof of Lemma 2.6.4. By construction Φ^{-1} is a C^1, positive and strictly increasing function on $[0,\infty)$ and we can always consider it on $[0,\infty)$ by taking t sufficiently large. We begin by noting that since Φ is the integral of a nondecreasing function it is convex. Therefore Φ^{-1} is a concave function and $\Phi^{-1}(0) = 1$. This means that Φ^{-1} is subadditive and taking $A > 0$ we may write

$$\Phi^{-1}(A+Bt) \leq \Phi^{-1}(A) + \Phi^{-1}(Bt).$$

Hence $\Phi^{-1}(A+Bt)/\Phi^{-1}(Bt) \leq 1+\Phi^{-1}(A)/\Phi^{-1}(Bt)$ and since $\lim_{t\to\infty}\Phi^{-1}(t) = \infty$ taking the limsup yields

$$\limsup_{t\to\infty} \frac{\Phi^{-1}(A+Bt)}{\Phi^{-1}(Bt)} \leq 1, \quad A > 0.$$

If $A < 0$, by monotonicity, $\Phi^{-1}(A+Bt) < \Phi^{-1}(Bt)$ and we quickly obtain

$$\limsup_{t\to\infty} \frac{\Phi^{-1}(A+Bt)}{\Phi^{-1}(Bt)} \leq 1, \quad A \in \mathbb{R}.$$

Given $A > 0$, $\Phi^{-1}(A+Bt) > \Phi^{-1}(Bt)$ and we obtain

$$\liminf_{t\to\infty} \frac{\Phi^{-1}(A+Bt)}{\Phi^{-1}(Bt)} \geq 1.$$

If $A < 0$ apply the Mean Value Theorem to the C^1 function Φ^{-1} to find a $\theta_t \in [A+Bt, Bt]$ such that $\Phi^{-1}(Bt) = \Phi^{-1}(A+Bt) - A\,(\phi\circ\Phi^{-1})(\theta_t)$. Note that, for t sufficiently large, we can guarantee $\theta_t > 0$. Therefore

$$\frac{\Phi^{-1}(A+Bt)}{\Phi^{-1}(Bt)} = 1 + \frac{A\,(\phi\circ\Phi^{-1})(\theta_t)}{\Phi^{-1}(Bt)},$$

and hence by monotonicity of ϕ and Φ^{-1}

$$\frac{\Phi^{-1}(A+Bt)}{\Phi^{-1}(Bt)} \geq 1 + \frac{A\,(\phi\circ\Phi^{-1})(0)}{\Phi^{-1}(Bt)}.$$

Now we can use that $\lim_{t\to\infty}\Phi^{-1}(t) = \infty$ to obtain

$$\liminf_{t\to\infty} \frac{\Phi^{-1}(A+Bt)}{\Phi^{-1}(Bt)} \geq 1 + \lim_{t\to\infty} \frac{A\,\phi(\Phi^{-1}(0))}{\Phi^{-1}(Bt)} = 1, \quad A < 0.$$

Combining these limits gives the result for $A \in \mathbb{R}$ and any $B \in (0,\infty)$. □

Lemma 2.6.5. *If $\phi \in C^1(\mathbb{R}^+;(0,\infty))$ is strictly decreasing and Φ is given by (2.2.8), then*

$$\frac{\Phi^{-1}((1+\epsilon)t)}{\Phi^{-1}(t)} < \frac{1}{1-\epsilon}, \quad \textit{for each } \epsilon \in (0,1).$$

Proof of Lemma 2.6.5. Consider the differential equation defined by

$$w'(t) = \phi(w(t)), \quad t > 0; \quad w(0) = 1. \tag{2.6.13}$$

We have that $w(t) = \Phi^{-1}(t),\ t \geq 0$ and hence

$$\frac{\Phi^{-1}((1+\epsilon)t)}{\Phi^{-1}(t)} = \frac{w((1+\epsilon)t)}{w(t)} = \frac{w(t) + \int_t^{t+\epsilon t} w'(s)ds}{w(t)} = 1 + \frac{1}{w(t)} \int_t^{t+\epsilon t} \phi(w(s))ds.$$

Now using the monotonicity of both the solution and of ϕ we have

$$\frac{\Phi^{-1}((1+\epsilon)t)}{\Phi^{-1}(t)} \leq 1 + \frac{\epsilon t \phi(w(t))}{w(t)} = 1 + \epsilon t \frac{\phi(\Phi^{-1}(t))}{\Phi^{-1}(t)}.$$

For $t \geq 0$, by setting $y := \Phi^{-1}(t) \geq 1$, we obtain

$$\frac{t\phi(\Phi^{-1}(t))}{\Phi^{-1}(t)} = \frac{\Phi(y)\phi(y)}{y} = \frac{\phi(y)}{y} \int_1^y \frac{1}{\phi(u)} du \leq \frac{y-1}{1-\epsilon} \frac{1}{\phi(y)} \frac{\phi(y)}{y} \leq \frac{1}{1-\epsilon}.$$

Combining these estimates yields

$$\frac{\Phi^{-1}((1+\epsilon)t)}{\Phi^{-1}(t)} \leq 1 + \frac{\epsilon}{1-\epsilon} \leq \frac{1}{1-\epsilon},$$

as required. □

Lemma 2.6.6. *Suppose $f \in C(\mathbb{R}^+; (0,\infty))$ and that f is asymptotic to a decreasing function $\phi \in C^1(\mathbb{R}^+; (0,\infty))$. If F be given by (2.1.3) and Φ is given by (2.2.8), then*

$$\lim_{t\to\infty} \frac{F^{-1}(t)}{\Phi^{-1}(t)} = 1.$$

Proof of Lemma 2.6.6. The solution to the initial value problem

$$u'(t) = f(u(t)), \quad t > 0; \quad u(0) = 1 \tag{2.6.14}$$

is given by $u(t) = F^{-1}(t)$ for $t \geq 0$. For every $\epsilon \in (0, 1/2)$ there is $x_1(\epsilon) > 0$ such that $1 - \epsilon < f(x)/\phi(x) < 1 + \epsilon$ for all $x > x_1(\epsilon)$. Since $u(t) \to \infty$ as $t \to \infty$, it follows that there exists $T(\epsilon) > 0$ such that $u(t) > x_1(\epsilon)$ for all $t \geq T(\epsilon)$. Hence

$$u'(t) = f(u(t)) \in ((1-\epsilon)\phi(u(t)), (1+\epsilon)\phi(u(t))), \quad t \geq T(\epsilon).$$

Thus

$$1 - \epsilon < \frac{u'(t)}{\phi(u(t))} < 1 + \epsilon, \quad t \geq T(\epsilon).$$

and integration over $[T(\epsilon), t]$ yields, with $\Phi^* := \Phi(x(T(\epsilon)))$,

$$\Phi^* + (1-\epsilon)(t - T(\epsilon)) < \Phi(u(t)) < \Phi^* + (1+\epsilon)(t - T(\epsilon)), \quad t \geq T(\epsilon),$$

and recalling that $u(t) = F^{-1}(t)$, we have

$$\Phi^{-1}(\Phi^* + (1-\epsilon)(t - T(\epsilon))) < F^{-1}(t) < \Phi^{-1}(\Phi^* + (1+\epsilon)(t - T(\epsilon))), \quad t \geq T(\epsilon). \tag{2.6.15}$$

Applying Lemma 2.6.4 to the left and right–hand sides of (2.6.15) shows that

$$\liminf_{t\to\infty}\frac{\Phi^{-1}((1-\epsilon)t)}{\Phi^{-1}(t)} \le \liminf_{t\to\infty}\frac{F^{-1}(t)}{\Phi^{-1}(t)} \le \limsup_{t\to\infty}\frac{F^{-1}(t)}{\Phi^{-1}(t)} \le \limsup_{t\to\infty}\frac{\Phi^{-1}((1+\epsilon)t)}{\Phi^{-1}(t)}.$$

By Lemma 2.6.5, $\Phi^{-1}((1+\epsilon)t) < \Phi^{-1}(t)/1-\epsilon$, so

$$\limsup_{t\to\infty}\frac{F^{-1}(t)}{\Phi^{-1}(t)} \le \frac{1}{1-\epsilon},$$

and letting $\epsilon \to 0^+$ gives

$$\limsup_{t\to\infty}\frac{F^{-1}(t)}{\Phi^{-1}(t)} \le 1. \tag{2.6.16}$$

To deal with the liminf, write $y := (1-\epsilon)t$ and $\eta := (1-\epsilon)^{-1} - 1$. Note that $\epsilon < 1/2$ yields $\eta \in (0,1)$. Lemma 2.6.5 (with η in the role of ϵ) then yields

$$\frac{\Phi^{-1}((1-\epsilon)t)}{\Phi^{-1}(t)} = \frac{\Phi^{-1}(y)}{\Phi^{-1}((1+\eta)y)} > 1-\eta = 2 - \frac{1}{1-\epsilon}.$$

Hence

$$\liminf_{t\to\infty}\frac{F^{-1}(t)}{\Phi^{-1}(t)} \ge 2 - \frac{1}{1-\epsilon}.$$

Letting $\epsilon \to 0^+$ and combining the resulting inequality with (2.6.16) yields the desired limit. □

Proof of Theorem 2.3.4. The proof of the first claim is as in Theorem 2.3.2.

Step 1: First establish the required lower bound on the solution. If $\epsilon > 0$ is arbitrary, by hypothesis, there exists $x_1(\epsilon)$ such that for all $x > x_1(\epsilon)$, $(1-\epsilon)\phi(x) < f(x) < (1+\epsilon)\phi(x)$. Furthermore, there exists $T_1(\epsilon)$ such that for $t \ge T_1(\epsilon)$, $x(t) > x_1(\epsilon)$. Now let $T = T_1 + \tau + T_2$, where $\mu_2([0,T_2]) > (1-\epsilon)\mu_2([0,\infty))$ for $t \ge T_2$. Let $t \ge T(\epsilon)$, then $t - \tau \ge T_1$ and $x(t-s) > x_1(\epsilon)$ for $s \in [0,\tau]$. Hence $f(x(t-s)) < (1+\epsilon)\phi(x(t-s)) < (1+\epsilon)\phi(x(t-\tau))$. Therefore

$$\int_{[0,\tau]}\mu_1(ds)f(x(t-s)) < (1+\epsilon)\int_{[0,\tau]}\mu_1(ds)\phi(x(t-\tau)), \quad t \ge T. \tag{2.6.17}$$

For $t \ge T(\epsilon)$ and $s \in [0,\tau]$, $f(x(t-s)) \ge (1-\epsilon)\phi(x(t-s)) \ge (1-\epsilon)\phi(x(t))$. Thus

$$\int_{[0,\tau]}\mu_1(ds)f(x(t-s)) \ge (1-\epsilon)\int_{[0,\tau]}\mu_1(ds)\phi(x(t)). \tag{2.6.18}$$

Also, for $t \ge 2T$,

$$\int_{[0,t]}\mu_2(ds)f(x(t-s)) \ge (1-\epsilon)\int_{[0,T]}\mu_2(ds)\phi(x(t-s)) \ge (1-\epsilon)\int_{[0,T]}\mu_2(ds)\phi(x(t)).$$

These estimates give us

$$x'(t) \ge \left[(1-\epsilon)\int_{[0,\tau]}\mu_1(ds) + (1-\epsilon)\int_{[0,T]}\mu_2(ds)\right]\phi(x(t)) =: M_\epsilon\phi(x(t)),$$

for almost every $t \ge 2T$. Define Φ as before and let $\Phi_\epsilon = \Phi(x(2T))$. It can be shown by integration and rearrangement that

$$x(t) \ge \Phi^{-1}(\Phi_\epsilon + M_\epsilon(t-2T)), \quad \text{for each } t \ge 2T. \tag{2.6.19}$$

With $\Phi_{T,\epsilon} := \Phi_\epsilon - 2TM_\epsilon$, it follows readily that

$$\liminf_{t\to\infty} \frac{x(t)}{\Phi^{-1}(M_\epsilon t)} \geq \liminf_{t\to\infty} \frac{\Phi^{-1}(M_\epsilon t + \Phi_{T,\epsilon})}{\Phi^{-1}(M_\epsilon t)}.$$

For each fixed $\epsilon > 0$, apply Lemma 2.6.4 to obtain $\liminf_{t\to\infty} x(t)/\Phi^{-1}(M_\epsilon t) \geq 1$. By construction, $(1-\epsilon)^2 M < M_\epsilon < M$, and thus $\lim_{\epsilon\downarrow 0} M_\epsilon = M$. Now consider

$$\liminf_{t\to\infty} \frac{x(t)}{\Phi^{-1}(Mt)} = \liminf_{t\to\infty} \frac{x(t)}{\Phi^{-1}(M_\epsilon t)} \frac{\Phi^{-1}(M_\epsilon t)}{\Phi^{-1}(Mt)}.$$

Letting $\theta = M_\epsilon t$ and $\lambda_\epsilon = M/M_\epsilon > 1$, we have

$$\frac{\Phi^{-1}(M_\epsilon t)}{\Phi^{-1}(Mt)} = \frac{\Phi^{-1}(\theta)}{\Phi^{-1}(\lambda_\epsilon \theta)} > 2 - \lambda_\epsilon,$$

where the final inequality is obtained using Lemma 2.6.5 with $\lambda_\epsilon = 1 + \epsilon$. Since $\lim_{\epsilon\downarrow 0} \lambda_\epsilon = 1$, we conclude that $\liminf_{t\to\infty} x(t)/\Phi^{-1}(Mt) \geq 1$, and hence

$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} \geq 1.$$

Step 2: Now derive the corresponding upper bound. Use (2.6.17) to obtain

$$\begin{aligned} x'(t) &< (1+\epsilon)\int_{[0,\tau]} \mu_1(ds)\phi(x(t-\tau)) + \int_{[0,t-2T]} \mu_2(ds) f(x(t-s)) \\ &\quad + \int_{(t-2T,t]} \mu_2(ds) f(x(t-s)), \quad \text{for a.e. } t \geq 2T. \end{aligned}$$

Use the monotonicity of ϕ and (2.6.19) to show that

$$\begin{aligned} x'(t) &\leq (1+\epsilon)\int_{[0,\tau]} \mu_1(ds)(\phi\circ\Phi^{-1})(\Phi_\epsilon + M_\epsilon(t-\tau-2T)) \\ &\quad + \int_{(t-2T,t]} \mu_2(ds) f(x(t-s)) \\ &\quad\quad + (1+\epsilon)\int_{[0,t-2T]} \mu_2(ds)(\phi\circ\Phi^{-1})(\Phi_\epsilon + M_\epsilon(t-s-2T)) \\ &=: a_1(t) + a_2(t) + a_3(t), \quad \text{for a.e. } t \geq 3T. \end{aligned} \tag{2.6.20}$$

Integrate to show that $x(t) \leq x(3T) + \int_{3T}^t \{a_1(s) + a_2(s) + a_3(s)\}\, ds$ and estimate the first integral on the right–hand side as follows:

$$\int_{3T}^t a_1(s) ds \leq \frac{(1+\epsilon)\mu_1}{M_\epsilon}\left[\Phi^{-1}(\Phi_\epsilon + M_\epsilon(t-\tau-2T))\right], \quad t \geq 3T.$$

The second term can be estimated as follows

$$a_2(t) = \int_{(t-2T,t]} \mu_2(ds) f(x(t-s)) \leq \int_{(t-2T,t]} \mu_2(ds) \cdot \sup_{u\in[0,2T]} f(x(u)).$$

Integrating and changing the order of integration then yields

$$\int_{3T}^{t} a_2(s)\,ds \le \int_{3T}^{t}\int_{(s-2T,s]} \mu_2\,(dr)ds \sup_{u\in[0,2T]} f(x(u))$$
$$= \int_{[T,t]} \{t\wedge(2T+r)-(3T\vee r)\}\mu_2(dr) \sup_{u\in[0,2T]} f(x(u)).$$

We then take cases and find that this estimate can be reduced to

$$\int_{3T}^{t} a_2(s)\,ds \le 2T\mu_2 \sup_{u\in[0,2T]} f(x(u)) := A_T, \quad t\ge 3T.$$

The last term is then estimated as follows

$$\int_{3T}^{t} a_3(s)ds = (1+\epsilon)\int_{[0,T]}\mu_2(dw)\int_{T-u}^{t-2T-u}(\phi\circ\Phi^{-1})(\Phi_\epsilon+M_\epsilon s)\,ds$$
$$+(1+\epsilon)\int_{(T,t-2T]}\mu_2(dw)\int_{0}^{t-2T-u}(\phi\circ\Phi^{-1})(\Phi_\epsilon+M_\epsilon s)\,ds$$
$$\le (1+\epsilon)\mu_2\int_0^{t-2T}(\phi\circ\Phi^{-1})(\Phi_\epsilon+M_\epsilon s)\,ds.$$

Rearrange, as in the calculation of $\int_{3T}^{t} a_1(s)ds$, to simplify this estimate to

$$\int_{3T}^{t} a_3(s)ds \le \frac{(1+\epsilon)\mu_2}{M_\epsilon}\left[\Phi^{-1}(\Phi_\epsilon+M_\epsilon(t-2T))\right], \quad t\ge 3T.$$

Combining these three estimates yields

$$x(t)\le x(3T)+A_T+\frac{(1+\epsilon)(\mu_1+\mu_2)}{M_\epsilon}\Phi^{-1}(\Phi_\epsilon+M_\epsilon(t-2T)), \quad t\ge 3T.$$

Hence

$$\limsup_{t\to\infty}\frac{x(t)}{\Phi^{-1}(Mt)} \le \frac{(1+\epsilon)M}{M_\epsilon}\limsup_{t\to\infty}\frac{\Phi^{-1}(\Phi_\epsilon+M_\epsilon(t-2T))}{\Phi^{-1}(Mt)}.$$

The arguments for the limsup in Lemma 2.6.4 work for the limsup above since $M_\epsilon < M$ and hence $\limsup_{t\to\infty} x(t)/\Phi^{-1}(Mt)\le(1+\epsilon)M/M_\epsilon$. Therefore we may send $\epsilon\downarrow 0$ and the same arguments as before yield $\limsup_{t\to\infty} x(t)/F^{-1}(Mt)\le 1$. Combining this with our lower bound gives the desired conclusion. □

2.6.3 Proofs of Results with Regular Variation

Proof of Theorem 2.4.6. As before, $z(t)\to\infty$ as $t\to\infty$. From equation (2.2.3),

$$\frac{z'(t)}{z(t)} = \int_{[0,\tau]}\mu_1(ds)\frac{f(z(t-s))}{z(t-s)}\frac{z(t-s)}{z(t)} \le \int_{[0,\tau]}\mu_1(ds)\frac{f(z(t-s))}{z(t-s)}, \quad \text{for a.e. } t\ge 0,$$

because z is nondecreasing. Since $\lim_{x\to\infty} f(x)/x=0$ there exists $x_1(\epsilon)$ such that for all $x>x_1(\epsilon)$ we have $f(x)/x<\epsilon/\int_{[0,\tau]}\mu_1(ds)$, for some $\epsilon\in(0,1/2)$. Similarly, there exists $T(\epsilon)$ such that for all $t\ge T(\epsilon)$, $z(t)>x_1(\epsilon)$. Hence

$$0\le\frac{z'(t)}{z(t)}\le\frac{\epsilon}{\int_{[0,\tau]}\mu_1(ds)}\int_{[0,\tau]}\mu_1(ds)\le\epsilon, \quad \text{for a.e. } t\ge T_1(\epsilon):=T(\epsilon)+\tau. \tag{2.6.21}$$

Now let $T > 0$ be arbitrary, take $s \in [0,T]$ and suppose $t \geq T + T_1(\epsilon)$. Integrate equation (2.6.21) from $t-s$ to t to obtain

$$\int_{t-s}^{t} \frac{z'(u)}{z(u)}\,du = \log\left(\frac{z(t)}{z(t-s)}\right) \leq \epsilon s, \qquad \text{for each } t \geq T + T_1(\epsilon).$$

It follows that $z(t-s)/z(t) \geq e^{-\epsilon s}$ and hence $1 - z(t-s)/z(t) \leq 1 - e^{-\epsilon s}$ for each $t \geq T + T_1(\epsilon)$. Therefore

$$\sup_{s\in[0,T]} \left|\frac{z(t-s)}{z(t)} - 1\right| \leq 1 - e^{-\epsilon T}, \qquad \text{for each } t \geq T + T_1(\epsilon).$$

Since ϵ was arbitrary,

$$\lim_{t\to\infty} \sup_{s\in[0,T]} \left|\frac{z(t-s)}{z(t)} - 1\right| = 0, \qquad \text{for any } T > 0.$$

Consequently, for each $\eta \in (0, 1/2)$ there exists a $T_2(\eta,\epsilon) > 0$ such that

$$\sup_{s\in[0,T_1(\epsilon)]} \left|\frac{z(t-s)}{z(t)} - 1\right| < \eta, \qquad \text{for each } t \geq T_2(\eta,\epsilon).$$

Therefore $1-\eta < z(t-s)/z(t) \leq 1,\ s \in [0, T_1(\epsilon)],\ t \geq T_2(\eta,\epsilon)$. Taking $\eta = \epsilon$ yields $\lambda_{t,s} := z(t-s)/z(t) \in [1-\epsilon, 1] \subset (1/2, 1]$ for all $s \in [0, T_1(\epsilon)]$ and $t \geq T_2$. Thus

$$\begin{aligned}\sup_{s\in[0,T_1]} \left|\frac{f(z(t-s))}{f(z(t))} - 1\right| &= \sup_{s\in[0,T_1]} \left|\frac{f(\lambda_{t,s} z(t))}{f(z(t))} - 1\right| \\ &\leq \sup_{\lambda\in[1-\epsilon,1]} \left|\frac{f(\lambda z(t))}{f(z(t))} - 1\right| \leq \sup_{\lambda\in[0,1/2]} \left|\frac{f(\lambda z(t))}{f(z(t))} - 1\right|, \quad \text{for each } t \geq T_2.\end{aligned}$$

By the Uniform Convergence Theorem for slowly varying functions,

$$\lim_{t\to\infty} \sup_{\lambda\in[0,1/2]} \left|\frac{f(\lambda z(t))}{f(z(t))} - 1\right| = 0, \quad \lim_{t\to\infty} \sup_{s\in[0,T_1]} \left|\frac{f(z(t-s))}{f(z(t))} - 1\right| = 0. \tag{2.6.22}$$

Estimate $z'(t)/f(z(t)) = \int_{[0,\tau]} \mu_1(ds) f(z(t-s))/f(z(t))$ using the identities above. By (2.6.22), $1-\epsilon < f(z(t-s))/f(z(t)) < 1+\epsilon$ for $s \in [0,\tau]$ and all $t \geq T_3(\epsilon)$. Thus

$$(1-\epsilon)M < \frac{z'(t)}{f(z(t))} < (1+\epsilon)M, \qquad \text{for a.e. } t \geq T_3(\epsilon).$$

Asymptotic integration establishes that $\lim_{t\to\infty} F(z(t))/Mt = 1$. Therefore there exists a $T_4(\epsilon)$ such that $Mt(1-\epsilon) < F(z(t)) < Mt(1+\epsilon)$ for each $t \geq T_4(\epsilon)$. Hence

$$\frac{F^{-1}(Mt(1-\epsilon))}{F^{-1}(Mt)} < \frac{z(t)}{F^{-1}(Mt)} < \frac{F^{-1}(Mt(1+\epsilon))}{F^{-1}(Mt)}, \qquad t \geq T_4(\epsilon).$$

Since $F^{-1} \in \mathrm{RV}_\infty(1)$, sending $t \to \infty$ yields

$$1 - \epsilon \leq \liminf_{t\to\infty} \frac{z(t)}{F^{-1}(Mt)} \leq \limsup_{t\to\infty} \frac{z(t)}{F^{-1}(Mt)} \leq 1 + \epsilon.$$

Finally, let $\epsilon \to 0^+$ in the inequality above to obtain the claimed result. □

Proof of Theorem 2.4.7. By hypothesis there exist positive real numbers $\underline{f}$ and $\bar{f}$ such that $\underline{f} < f(x) < \bar{f}$ for all $x > 0$. Hence

$$v'(t) \leq \int_{[0,T_2]} \mu_2(ds) f(v(t-s)) + \int_{(T_2,t]} \mu_2(ds)\bar{f}, \qquad \text{for a.e. } t \geq T_2 > 0.$$

Since $f(v(t)) > \underline{f} > 0$,

$$\frac{v'(t)}{f(v(t))} \leq \frac{\bar{f}}{\underline{f}} \int_{[T_2,\infty)} \mu_2(ds) + \frac{1}{f(v(t))} \int_{[0,T_2]} \mu_2(ds) f(v(t-s)), \quad \text{for a.e. } t \geq 0.$$

Now by arguments analogous to those from the proof of Theorem 2.4.6, $\lim_{t\to\infty} v'(t)/f(v(t)) < \epsilon$ for a.e. t sufficiently large. It now follows readily that

$$\lim_{t\to\infty} \sup_{0\leq s\leq T_2} \left| \frac{f(v(t-s))}{f(v(t))} - 1 \right| = 0.$$

Hence there exists $T_3(\epsilon)$ such that for all $t \geq T_3(\epsilon)$, $f(v(t-s))/f(v(t)) < 1+\epsilon$, for all $s \in [0, T_2(\epsilon)]$. Let $T_2(\epsilon)$ be large enough that $\int_{[t,\infty)} \mu(ds) < \epsilon$ for all $t \geq T_2$ and then take $T = T_2(\epsilon) + T_3(\epsilon)$. Thus

$$\frac{v'(t)}{f(v(t))} < \epsilon \frac{\bar{f}}{\underline{f}} + (1+\epsilon) \int_{[0,T_2]} \mu_2(ds) \leq \epsilon \frac{\bar{f}}{\underline{f}} + (1+\epsilon) M, \quad \text{for a.e. } t \geq T.$$

Now use asymptotic integration to show that $\limsup_{t\to\infty} F(v(t))/Mt \leq 1$. By the usual considerations, and since $F^{-1} \in \text{RV}_\infty(1)$, we obtain the upper bound $\limsup_{t\to\infty} v(t)/F^{-1}(Mt) \leq 1$. We defer the calculation of the required lower bound to the next theorem. □

Proof of Theorem 2.4.8. If $T > 0$ is arbitrary, monotonicity of the solution implies

$$\frac{v'(t)}{v(t)} \leq \int_{[0,t-T]} \mu_2(ds) \frac{f(v(t-s))}{v(t-s)} + \int_{(t-T,t]} \mu_2(ds) \frac{f(v(t-s))}{v(t-s)}, \quad \text{for a.e. } t \geq T.$$

Estimation analogous to that performed at the start of the proof of Theorem 2.4.6 yields the existence of a $T_1(\epsilon)$ sufficiently large that $v'(t)/v(t) \leq \epsilon$ for a.e. $t \geq T_1(\epsilon)$. As in the proof of Theorem 2.4.6 it can readily be shown that

$$\lim_{t\to\infty} \sup_{s\in[0,T_1(\epsilon)]} \left| \frac{v(t-s)}{v(t)} - 1 \right| = 0, \quad \lim_{t\to\infty} \sup_{s\in[0,T(\epsilon)]} \left| \frac{f(v(t-s))}{f(v(t))} - 1 \right| = 0. \tag{2.6.23}$$

For any $\epsilon \in (0,1)$, $\int_{[0,T(\epsilon)]} \mu(ds) \geq (1-\epsilon)M$ for some $T(\epsilon) > T_1(\epsilon)$. Hence $v'(t) \geq \int_{[0,T(\epsilon)]} \mu_2(ds) f(v(t-s))$ for a.e. $t \geq T(\epsilon)$. Combine the previous a.e. inequality with (2.6.23) to find a $T^*(\epsilon)$ such that $v'(t) \geq (1-\epsilon)^2 M f(v(t))$ for a.e. $t \geq T^*(\epsilon)$. Asymptotic integration now shows that $\liminf_{t\to\infty} F(v(t))/Mt \geq 1$. Since $F^{-1} \in \text{RV}_\infty(1)$, this immediately implies that $\liminf_{t\to\infty} v(t)/F^{-1}(Mt) \geq 1$. □

Proof of Theorem 2.4.9. Let $\epsilon \in (0,1)$ be arbitrary. Since $f \in \text{RV}_\infty(0)$, $\lim_{x\to\infty} f(x)/x = 0$. Therefore there exists an $X(\epsilon)$ such that $x^{-\epsilon} < f(x) < x^\epsilon$ for all $x > X(\epsilon)$. Since $\lim_{t\to\infty} v(t) = \infty$, there exists $T(\epsilon)$ such that $v(t) > X(\epsilon)$ for all $t \geq T(\epsilon)$ and hence

$$\begin{aligned} v'(t) &= \int_{[0,t-T]} \mu_2(ds) f(v(t-s)) + \int_{(t-T,t]} \mu_2(ds) f(v(t-s)) \\ &\leq \int_{[0,t-T]} \mu_2(ds) v(t-s)^\epsilon + \int_{(t-T,t]} \mu_2(ds) f(v(t-s)), \quad \text{for a.e. } t \geq T(\epsilon). \end{aligned}$$

Letting $h(t) = \int_{(t-T,t]} \mu_2(ds) f(v(t-s))$, $h(t) \leq \int_{[t-T,t]} \mu_2(ds) \sup_{s\in[0,T]} f(v(s))$. Thus $v'(t) \leq Mv(t)^\epsilon + h(t)$ for a.e. $t \geq T(\epsilon)$. Therefore

$$\frac{v'(t)}{v(t)^\epsilon} \leq M + \frac{h(t)}{v(t)^\epsilon}, \quad \text{for a.e. } t \geq T(\epsilon).$$

Since $h(t) \to 0$ as $t \to \infty$ and $\lim_{t\to\infty} v(t) = \infty$, there exists a $T_1(\epsilon) > T(\epsilon)$ such that

$$\frac{v'(t)}{v(t)^\epsilon} \leq \epsilon + M, \qquad \text{for a.e. } t \geq T_1(\epsilon).$$

Asymptotic integration now shows that

$$v(t)^{1-\epsilon} \leq (1-\epsilon)\left[(M+\epsilon)(t-T_1) + v(T_1)^{1-\epsilon}\right], \qquad \text{for each } t > T_1$$

Take logarithms, divide by $\log t$ and send $t \to \infty$ to obtain

$$\limsup_{t\to\infty} \frac{\log v(t)}{\log t} \leq \frac{1}{1-\epsilon}$$

Finally, let $\epsilon \to 0^+$ to show that $\limsup_{t\to\infty} \log(v(t))/\log(t) \leq 1$. $f \in \mathrm{RV}_\infty(0)$ implies that $F \in \mathrm{RV}_\infty(1)$ and hence $\lim_{x\to\infty} \log(F(x))/\log(x) = 1$. Using the lower bound from Theorem 2.4.8 there exists T_2 such that for all $\epsilon \in (0,1)$ we have $v(t) > F^{-1}(M(1-\epsilon)t),\ t \geq T_2$. Similarly, $\log(v(t))/\log(t) \geq \log(F^{-1}(M(1-\epsilon)t))/\log(t)$. Taking the liminf then shows that

$$\liminf_{t\to\infty} \frac{\log(v(t))}{\log(t)} \geq \liminf_{t\to\infty} \frac{\log(F^{-1}(t))}{\log(t)} = 1.$$

Combining the upper and lower bounds gives the desired result. □

2.6.4 Justification of Examples

In this section we provide the relevant details to support the examples discussed in Section 2.5. The calculations for both examples are identical except for the final few steps where differing hypotheses are employed. We begin by stating some formulae which are derived by integrating (2.5.1). For $n \in \mathbb{N}$ and $x \in [n, n + w_n/2)$

$$f(x) = f(n) + (x-n)\eta(n) + \frac{h_n - \eta(n)}{w_n}(x-n)^2. \tag{2.6.24}$$

Hence $f(n + w_n/2) = f(n) + w_n\eta(n)/4 + (h_n w_n)/4$. For $x \in (n + w_n/2,\ w_n + n]$,

$$f(x) = f(n + \tfrac{w_n}{2}) + h_n\left(x - n - \tfrac{w_n}{2}\right) + \frac{\eta(n+w_n) - h_n}{w_n}\left(x - n - \tfrac{w_n}{2}\right)^2. \tag{2.6.25}$$

Therefore $f(n + w_n) = f(n + w_n/2) + (h_n w_n)/2 + (w_n/4)\left(\eta(n) + \eta(n+w_n)\right)$. Finally for $x \in (n + w_n, n+1)$

$$f(x) = f(n + w_n) + \int_{n+w_n}^{x} \eta(u)du. \tag{2.6.26}$$

It follows that

$$f(n+1) = f(n) + \frac{h_n w_n}{2} + \frac{w_n}{4}\left(\eta(n) + \eta(n+w_n)\right) + \int_{n+w_n}^{x} \eta(u)du, \tag{2.6.27}$$

and it can also be shown that

$$f(n+1) = f(n) + \sum_{j=1}^{n} \left\{ \frac{h_j w_j}{2} + \frac{w_j}{4}\left(\eta(j) + \eta(j+w_j)\right) + \int_{w_j+j}^{j+1} \eta(u)du \right\}. \tag{2.6.28}$$

Example 2.5.1

By hypothesis, ϕ grows more quickly than the sums of $\eta(j)w_j$ and $h_j w_j$ so we only need to study the asympotics of the final term of (2.6.28). For $n \in \mathbb{N}$,

$$S_n := \sum_{j=1}^{n} \int_{j+w_j}^{j+1} \eta(u)du \leq \sum_{j=1}^{n} \int_{j}^{j+1} \eta(u)du.$$

Thus $S_n \leq \int_1^{n+1} \eta(u)du$. (2.6.27) can be rewritten as

$$f(n+1) \leq \int_0^{n+1} \eta(u)du + T_n = \phi(n+1) + T_n, \tag{2.6.29}$$

where $T_n/\phi(n) \to 0$ as $n \to \infty$. Similarly,

$$S_n = \int_1^{n+1} \eta(u)du - \sum_{j=1}^{n} \int_{j}^{j+w_j} \eta(u)du.$$

Since η is decreasing, $\sum_{j=1}^{n} \int_j^{j+w_j} \eta(u)du \leq \sum_{j=1}^{n} w_j \eta(j)$ and

$$S_n \geq \int_1^{n+1} \eta(u)du - \sum_{j=1}^{n} w_j \eta(j).$$

Hence, from (2.6.27), we obtain the inequality

$$f(n+1) \geq \phi(n+1) - \sum_{j=1}^{n} w_j \eta(j) + T_n,$$

where $T_n/\phi(n) \to 0$ as $n \to \infty$. Combining our upper and lower estimates for $f(n+1)$ yields $\lim_{n\to\infty} f(n+1)/\phi(n+1) = 1$. Since $\lim_{x\to\infty} \eta(x) = 0$,

$$\frac{\phi(n+1) - \phi(n)}{\phi(n)} = \frac{\int_n^{n+1} \eta(u)du}{\phi(n)} \leq \frac{\eta(n)}{\phi(n)} \to 0 \text{ as } n \to \infty.$$

Hence $\lim_{n\to\infty} \phi(n+1)/\phi(n) = 1$. Thus for any $x \in [n(x), n(x)+1)$ our previous arguments show that (suppressing x–dependence in n)

$$\frac{f(x)}{\phi(x)} \leq \frac{f(n+1)}{\phi(n)} = \frac{f(n+1)}{\phi(n+1)} \frac{\phi(n+1)}{\phi(n)} \to 1 \text{ as } x \to \infty.$$

Likewise

$$\frac{f(x)}{\phi(x)} \geq \frac{f(n)}{\phi(n+1)} = \frac{f(n)}{\phi(n)} \frac{\phi(n)}{\phi(n+1)} \to 1 \text{ as } x \to \infty.$$

Note that $\lim_{x\to\infty} \phi'(x) = 0$ since $\lim_{x\to\infty} \eta(x) = 0$ by hypothesis and hence by Lemma 2.6.1 ϕ is sublinear. Therefore f is also sublinear.

We have chosen h_n so that $h_n > \eta(n)$ for each n and $f'(n + w_n/2) = h_n$. Hence

$$\limsup_{x\to\infty} f'(x) \geq \lim_{n\to\infty} f'(n + w_n/2) = \lim_{n\to\infty} h(n) = L.$$

Also $\lim_{x\to\infty} \eta(x) = 0$ implies that $\liminf_{x\to\infty} f'(x) = 0$.

Example 2.5.2

All of the arguments from Example 2.5.1 also apply here with minor changes. In (2.6.29) we now have $T_n \to \bar{L} \in (0, \infty)$ and we can proceed as before.

Chapter 3

Memory Dependent Growth in Sublinear Volterra Equations

3.1 Introduction

We now investigate explicit memory dependence in the asymptotic growth rates of positive solutions of the following scalar Volterra integro–differential equation

$$x'(t) = \int_{[0,t]} \mu(ds) f(x(t-s)), \quad t > 0; \quad x(0) = \xi > 0, \tag{3.1.1}$$

where f is a positive sublinear function (i.e. $\lim_{x\to\infty} f(x)/x = 0$) and μ is a nonnegative Borel measure. Defining

$$M(t) = \int_{[0,t]} \mu(ds), \quad t \geq 0, \tag{3.1.2}$$

and integrating (3.1.1) shows that (3.1.1) is equivalent to the integral equation

$$x(t) = x(0) + \int_0^t M(t-s) f(x(s))\, ds, \quad t \geq 0; \quad x(0) = \xi > 0. \tag{3.1.3}$$

We also study the asymptotic behaviour of the perturbed Volterra equation

$$x'(t) = \int_{[0,t]} \mu(ds) f(x(t-s)) + h(t), \quad t > 0; \quad x(0) = \xi > 0. \tag{3.1.4}$$

As with equation (3.1.1), it is useful to consider an integral form of (3.1.4), and by defining

$$H(t) = \int_0^t h(s)\, ds, \quad t \geq 0, \tag{3.1.5}$$

it follows that (3.1.4) can be written in integral form as

$$x(t) = x(0) + \int_0^t M(t-s) f(x(s))\, ds + H(t), \quad t \geq 0; \quad x(0) = \xi > 0. \tag{3.1.6}$$

Chapter 3 is based on the paper [18].

In Chapter 2, with μ a finite measure, we demonstrated that when f is sublinear and asymptotically increasing, the solution of (3.1.1) obeys $\lim_{t\to\infty} F(x(t))/t = \mu(\mathbb{R}^+) < \infty$, where

$$F(x) := \int_1^x \frac{1}{f(u)}\,du, \quad x > 0. \tag{3.1.7}$$

In other words, the structure of the memory does not affect the asymptotic growth rate of the solution to (3.1.1) when the total measure is finite: indeed, the entire mass of μ could be concentrated at 0, because the ordinary differential equation $y'(t) = \mu(\mathbb{R}^+) \cdot f(y(t))$ also obeys $F(y(t))/t \to \mu(\mathbb{R}^+)$ as $t \to \infty$. This is in contrast to the linear case where the growth rate depends crucially on the structure of the memory (cf. [50, Theorem 7.2.3]). In Chapter 2, we also showed that if $\lim_{t\to\infty} M(t) = \infty$, then $\lim_{t\to\infty} F(x(t))/t = \infty$. This result suggests that allowing the total measure to be infinite makes the long run dynamics more sensitive to the memory but that comparison with a non-autonomous ordinary differential equation may be necessary in this case.

To achieve precise asymptotic results for the solutions of (3.1.1) and (3.1.4) we employ the theory of regular variation extensively. Many of the applications of regular variation in the asymptotic theory of *linear* Volterra equations deal with the situation in which it is desired to model slow decay in the memory, as captured by a measure or kernel, or a singularity. Of course, slowly fading memory can be described in other ways, using for instance the theory of L^1 weighted spaces (see e.g. [112] and for stochastic equations, [20]). When the kernel is integrable, it is often possible to obtain precise rates of decay in L^∞ by means of a larger class of kernels (such as the subexponential class studied in [8], of which regularly varying kernels are a subclass). However, for singular equations, or equations with non–integrable kernels, the full power of the theory of regular variation is often needed: in particular, for linear equations, transform methods and the Abelian and Tauberian theorems for regular variation are exploited (see e.g. [10, 121]). It should be stressed, though, that such methods are of greatest utility for linear equations: indeed, there does not seem to be especial benefit gained in this work in applying such a transform approach. Moreover, in this case, the equation is intrinsically non–linear: $f(x)$ is not of linear order as $x \to \infty$, and regular variation arises *both* in the slow decay of μ and in the sublinear growth of f. Also, it is a general theme of the works cited above that the slow decay in the memory, combined with an appropriate type of stability, give rise to convergence at a certain rate to equilibrium. By contrast, we study growing solutions in the present work.

With a view to applications, we believe the most interesting subclass of equations will retain the property that the asymptotic contribution to the growth rate from a moving interval of any fixed duration ($\tau > 0$, say) is negligible, in the sense that

$$\lim_{t\to\infty} \int_{[t,t+\tau)} \mu(ds) = 0, \quad \text{for each } \tau > 0. \tag{3.1.8}$$

It should be noted that our proofs *do not* require this stipulation, but we mention it in order to motivate shortly a stronger hypothesis on M, as defined in (3.1.2).

With (3.1.8) still in force, if μ is absolutely continuous and admits a non–negative and continuous density k, such that $\mu(ds) = k(s)ds$, we see that $k \notin L^1(0,\infty)$ because $M(t) \to \infty$ as $t \to \infty$. In particular the property (3.1.8) is implied by $k(t) \to 0$ as $t \to \infty$. Therefore, it is perfectly possible for k to lie in another L^p space, for some $p > 1$. As an example, suppose that $k(t) \sim t^{-\theta}$ as $t \to \infty$ for $\theta \in (0,1)$: then for $p > 1/\theta > 1$, $k \in L^p(0,\infty)$, while $k \notin L^1(0,\infty)$. In this sense, our work shares concerns with existing results in the literature in which the Volterra equation does not possess an integrable kernel (see e.g. [58, 112]).

The type of fading memory property (3.1.8) we suggested was of interest motivates a stronger assumption on M. First, we see that (3.1.8) implies

$$\frac{1}{n\tau}M(n\tau) = \frac{1}{\tau}\frac{1}{n}\sum_{j=0}^{n-1}\int_{[j\tau,j\tau+\tau)}\mu(ds) \to 0 \text{ as } n \to \infty$$

and so the non–negativity of μ implies that $M(t)/t \to 0$ as $t \to \infty$. Since $M(t) \to \infty$ as $t \to \infty$, M is non–decreasing, and $M(t)/t \to 0$ as $t \to \infty$, it is reasonable to suppose that $M \in \text{RV}_\infty(\theta)$ for $\theta \in [0, 1]$. We note that the inclusion of $\theta = 1$ in the parameter range does not lead to any problems in the analysis, and indeed it transpires that our arguments are valid for all $\theta \geq 0$.

Analogously, the nonlinearity, f, is a positive and asymptotically increasing function such that $f(x) \to \infty$ and $f(x)/x \to 0$ as $x \to \infty$; hence it is natural to assume that $f \in \text{RV}_\infty(\beta)$ for $\beta \in [0, 1)$. We can rule out some choices of the parameter β rapidly: if $\beta > 1$, $f(x)/x \to \infty$ as $x \to \infty$, and if $\beta < 0$, f is asymptotic to a decreasing function. When $\beta = 0$ we append the hypotheses of asymptotic monotonicity and increase to infinity on f, as these are not necessarily satisfied by functions in $\text{RV}_\infty(0)$, but otherwise the analysis is essentially the same as when $\beta \in (0, 1)$. The exclusion of the case $\beta = 1$ is largely on technical grounds: informally, when $\beta = 1$, the inverse of the increasing function F defined by (3.1.7) is *no longer regularly varying*; F^{-1} now belongs to the class of rapidly varying functions (see Definition 1.3.4). It also can be seen from the nature of our results that the asymptotic behaviour of solutions must be of a different form from those that hold when $\beta < 1$. For $\beta < 1$, no such technical problem arises, and indeed F^{-1} is regularly varying with index $1/(1 - \beta)$.

The proof of our main result for (3.1.1), Theorem 3.2.1, relies principally upon comparison methods, properties of regularly varying functions and a time change argument for delay differential equations. We first use constructive comparison methods, similar in spirit to those employed by Appleby and Buckwar [6] for linear equations, to establish “crude” upper and lower bounds on the solution of (3.1.1). The more challenging construction is that of the lower bound and is completed by comparing solutions of (3.1.1) with those of a related nonlinear pantograph equation using time change arguments inspired by Brunner and Maset [31]. Finally, we prove a convolution lemma for regularly varying functions (cf. [5, Theorem 3.4]) which is then used, in conjunction with straightforward comparison methods, to sharpen the aforementioned “crude” upper and lower bounds, and show that they coincide. Another paper which uses similar iterative methods to sharpen estimates in the growth of solutions of nonlinear convolution Volterra equations is Schneider [111].

With $\bar{M}(t) := \int_0^t M(s)ds$, we obtain $\lim_{t\to\infty} F(x(t))/\bar{M}(t) = \Lambda(\beta, \theta)$, or that the growth rate of solutions of (3.1.1) depend explicitly on both indices of regular variation, and therefore the memory of the system (Theorem 3.2.1). The value of the parameter–dependent limit Λ can be determined explicitly in terms of the Gamma function. This result is only valid for $\beta \in [0, 1)$ and hence may not hold if f is only assumed to be sublinear (i.e. $\lim_{x\to\infty} f(x)/x = 0$). In this sense, it appears that the imposition of the hypothesis of regular variation on f and M is intrinsic to the form of the asymptotic behaviour deduced, rather than a being a purely technical contrivance.

The results and methods outlined above for (3.1.1) can also be used to yield sharp asymptotics for the perturbed equation (3.1.4). If H is positive, solutions to (3.1.4) will be positive and exhibit unbounded growth; therefore there is no need to assume pointwise positivity of h. Solutions of (3.1.4) are no longer necessarily non–decreasing and more delicate comparison techniques are required to treat this additional difficulty.

When H is of the same order of magnitude as the solution of (3.2.4), we establish non-trivial upper and lower bounds on the solution and then employ a simple fixed point iteration argument to calculate the exact asymptotic growth rate of the solution in terms of a characteristic equation (Theorem 3.3.1). Moreover, the converse also holds: growth in the solution of (3.1.4) at a rate proportional to that of the solution of (3.2.4) is possible only when H is of the same order as that solution. In these results, the parameter θ characterises the dependence of the growth rate on the degree of memory in the system. When the perturbation term grows sufficiently quickly, the solution tracks H asymptotically, in the sense that $\lim_{t\to\infty} x(t)/H(t) = 1$, even when H is allowed to be highly non-monotone. Indeed, under certain restrictions we can show that our characterisation of rapid growth in the perturbation is necessary in order for $\lim_{t\to\infty} x(t)/H(t) = 1$ to prevail.

3.2 Main Results

Throughout this chapter μ obeys

$$\mu \in M_{loc}(\mathbb{R}^+;\mathbb{R}^+), \quad \mu(\mathbb{R}^+) = \lim_{t\to\infty} M(t) = \infty, \tag{3.2.1}$$

where the function M is defined by (3.1.2). Our first result gives precise information on the asymptotic growth rate of the solution to (3.1.1); we defer the proof to Section 3.5.

Theorem 3.2.1. *Suppose μ obeys (3.2.1) with $M \in RV_\infty(\theta)$, $\theta \geq 0$ and $f \in RV_\infty(\beta)$, $\beta \in [0,1)$. When $\beta = 0$ let f be asymptotically increasing and obey $\lim_{x\to\infty} f(x) = \infty$. Then each solution, x, of (3.1.1) obeys $x \in RV_\infty((1+\theta)/(1-\beta))$ and*

$$\lim_{t\to\infty} \frac{F(x(t))}{\bar{M}(t)} = \frac{\Gamma(\theta+1)\Gamma\left(\frac{1+\beta\theta}{1-\beta}\right)}{\Gamma\left(\frac{1+\theta}{1-\beta}\right)} =: \Lambda(\beta,\theta), \tag{3.2.2}$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ and $\bar{M}(t) = \int_0^t M(s)ds$.

By Karamata's Theorem $\lim_{t\to\infty} \bar{M}(t)/tM(t) = 1/(1+\theta)$. Hence the conclusion of Theorem 3.2.1 is equivalent to

$$\lim_{t\to\infty} \frac{F(x(t))}{t\,M(t)} = (1+\theta)\frac{\Gamma(\theta+1)\Gamma\left(\frac{1+\beta\theta}{1-\beta}\right)}{\Gamma\left(\frac{1+\theta}{1-\beta}\right)} = \frac{1}{1-\beta}B\left(\theta+1, \frac{1+\theta\beta}{1-\beta}\right),$$

where B denotes the Beta function, which is defined by $B(x,y) = \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1}d\lambda$ (cf. [97, p.142]). Furthermore, since $F^{-1} \in \mathrm{RV}_\infty(1/(1-\beta))$, (3.2.2) is also equivalent to

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} = \left\{\frac{1}{1-\beta}B\left(\theta+1, \frac{1+\theta\beta}{1-\beta}\right)\right\}^{\frac{1}{1-\beta}}. \tag{3.2.3}$$

Theorem 3.2.1 expresses the leading order asymptotics of the solution in terms of the functions F and $\bar{M}$. The dependence of Λ on β and θ is known explicitly and this can be used to gain some insight into second order effects of the nonlinearity and the memory on the growth rate. The following proposition records some properties of the function $\Lambda(\beta,\theta)$ that are useful when interpreting the conclusion of Theorem 3.2.1.

Proposition 3.2.1. *Suppose $\Lambda(\beta,\theta)$ is defined by (3.2.2) with $\beta \in [0,1)$ and $\theta \in [0,\infty)$.*

(i.) $\Lambda(0,\theta) = 1$ *for fixed* $\theta \in (0,\infty)$ *and* $\Lambda(\beta,0) = 1$ *for fixed* $\beta \in (0,1)$,

(ii.) $\lim_{\beta\uparrow 1} \Lambda(\beta,\theta) = 0$ *for fixed* $\theta \in (0,\infty)$ *and* $\lim_{\theta\to\infty} \Lambda(\beta,\theta) = 0$ *for fixed* $\beta \in (0,1)$,

(iii.) $\beta \mapsto \Lambda(\beta,\theta)$ *is decreasing,* $\beta \in (0,1)$, θ *(fixed)* $\in (0,\infty)$,

(iv.) $\theta \mapsto \Lambda(\beta,\theta)$ *is decreasing,* $\theta \in (0,\infty)$, β *(fixed)* $\in (0,1)$,

(v.) $\Lambda(\beta,\theta) \in (0,1)$ *for* $\beta \in (0,1)$ *and* $\theta \in (0,\infty)$.

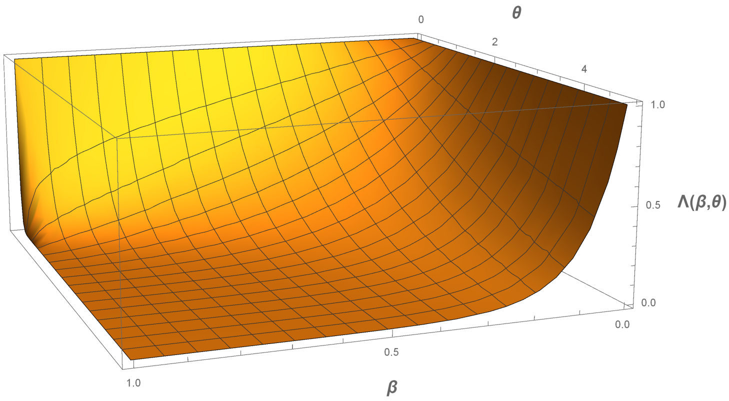


Fig. 3.2.1: Plot of the surface $\Lambda(\beta,\theta)$.

For each fixed $\beta \in (0,1)$, letting $\theta = 0$ yields $\lim_{t\to\infty} F(x(t))/\bar{M}(t) = 1$, by applying Theorem 3.2.1. In Chapter 2 we obtained this conclusion for sublinear equations of the form (3.1.3) without regular variation but with $\lim_{t\to\infty} M(t) = M \in (0,\infty)$. Therefore Theorem 3.2.1 can be thought of as a continuous extension of our previous results for (3.1.1) with sublinear nonlinearities and finite measures.

For a fixed $\beta \in (0,1)$, a decrease in the value of θ represents an increase in the rate of decay of the measure μ. This can be made precise by supposing that the measure μ is absolutely continuous, and specifically that $\mu(ds) = m(s)ds$ for continuous $m \in \mathrm{RV}_\infty(\theta - 1)$ with $\theta \in (0,1)$. Therefore, increasing the value of θ gives more weight to values of the solution in the past ("stronger memory") and we expect the growth rate of solutions of (3.1.1) to be slower than that of the related ordinary differential equation

$$y'(t) = M(t)f(y(t)), \quad t > 0; \quad y(0) = \xi > 0. \tag{3.2.4}$$

The equation (3.2.4), in contrast, places the entire weight $M(t)$ at the present time, when the solution is largest. Hence, increasing the value of θ (putting more weight further into the past) slows the growth rate and it is intuitive that $\Lambda(\beta,\theta)$ is decreasing in θ. Using this comparison with (3.2.4) once more, it is clear that Proposition 3.2.1 $(v.)$ must hold since solutions of (3.1.1) can never grow faster than those of (3.2.4) (if f is strictly increasing this can be seen by inspection).

For a fixed $\theta \in (0,\infty)$, one might expect an increase in β to lead to a faster rate of growth of the solution of (3.1.1). Therefore, it may initially be surprising that $\Lambda(\beta,\theta)$ is decreasing in β. This counter-intuitive result is best understood by explaining the error introduced in the approximation of the right-hand side of (3.1.1). From (3.1.1),

$$x'(t) = \int_{[0,t]} \mu(ds) f(x(t-s)) = \int_{[0,t]} \mu(ds) \frac{f(x(t-s))}{f(x(t))} f(x(t)), \quad t > 0.$$

The error of our upper bound on the solution is proportional to the ratio $f(x(t-s))/f(x(t))$ for $s \in (0,t)$, or $f(x(\lambda t))/f(x(t))$ for $\lambda \in (0,1)$. Since $f \circ x \in \mathrm{RV}_\infty(\beta(1+\theta)/(1-\beta))$

$$\lim_{t\to\infty} \frac{f(x(\lambda t))}{f(x(t))} = \lambda^{\frac{\beta(1+\theta)}{1-\beta}} =: \gamma(\beta).$$

When $\gamma(\beta)$ is close to one, the solution of (3.1.1) is close to that of (3.2.4) and hence our estimate is sharp. However, $\gamma(\beta)$ is decreasing and $\lim_{\beta\uparrow 1} \gamma(\beta) = 0$. Thus the zero limit as $\beta \uparrow 1$ in Proposition

3.2.1 (*ii.*) represents the fact that the solution of (3.2.4) increases much faster in β than the solution to (3.1.1), for a fixed value of θ.

3.3 Results for Perturbed Volterra Equations

We now present a result which illustrates how our precise understanding of the asymptotics of solutions to (3.1.1) can be applied to perturbed versions of the equation, such as (3.1.4). This result applies to perturbations of (3.1.1) that are of the same, or smaller, order of magnitude as solutions of the ordinary differential equation (3.2.4). Our assumptions on H guarantee that $\lim_{t\to\infty} x(t) = \infty$ but this limit is no longer necessarily achieved monotonically and this is reflected in the added complexity of certain technical aspects of the proofs. The proofs of the results in this section are largely deferred to Section 3.5.

Theorem 3.3.1. *Suppose μ obeys (3.2.1) with $M \in RV_\infty(\theta)$, $\theta \geq 0$ and $f \in RV_\infty(\beta)$, $\beta \in [0,1)$. When $\beta = 0$ let f be asymptotically increasing and obey $\lim_{x\to\infty} f(x) = \infty$. If x denotes a solution of (3.1.4) and $H \in C((0,\infty);(0,\infty))$, then the following are equivalent:*

$$(i.) \quad \lim_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} = \zeta \in [L,\infty), \quad (ii.) \quad \lim_{t\to\infty} \frac{H(t)}{F^{-1}(t\,M(t))} = \lambda \in [0,\infty), \tag{3.3.1}$$

where $L = \left\{ B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) / (1-\beta) \right\}^{1/(1-\beta)}$, and moreover

$$\zeta = \frac{\zeta^\beta}{1-\beta} B\left(1+\theta, \tfrac{1+\theta\beta}{1-\beta}\right) + \lambda. \tag{3.3.2}$$

When there is a sufficiently slowly growing forcing term H, $\lambda = 0$, and we recover from (3.3.2) exactly the asymptotic behaviour of the unperturbed equation, given by (3.2.3). Also, in the limit as $\lambda \to 0^+$, the rate of the unperturbed equation is recovered.

Condition (*ii.*) on H in Theorem 3.3.1 does not cover the case when H is of larger magnitude than the solution of the unperturbed equation (3.1.1) (or that of (3.2.4)). To deal with this case, we want to know the growth rate of the solution when $\lim_{t\to\infty} H(t)/F^{-1}(t\,M(t)) = \infty$. Insight can be gained by sending $\lambda \to \infty$ in Theorem 3.3.1. For $\lambda > 0$, from Theorem 3.3.1, we have

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = \frac{\zeta(\lambda)}{\lambda} =: \eta(\lambda),$$

where ζ depends on λ through (3.3.2). Since $\zeta = \zeta(\lambda)$ is the unique positive solution of (3.3.2), $\eta = \eta(\lambda)$ is the unique positive solution of $\eta = 1 + K\eta^\beta\lambda^{\beta-1}$ where $K > 0$ is the λ–independent positive quantity

$$K = \frac{1}{1-\beta} B\left(1+\theta, \tfrac{1+\theta\beta}{1-\beta}\right).$$

Clearly $\eta(\lambda) > 1$ and $\lambda \mapsto \eta(\lambda)$ is in C^1 by the implicit function theorem. Moreover, by implicit differentiation, $\eta'(\lambda)$ obeys

$$\eta'(\lambda)\left\{1 - \beta\frac{\eta(\lambda)-1}{\eta(\lambda)}\right\} = K(\beta-1)\eta(\lambda)^\beta\lambda^{\beta-2}.$$

Therefore, as the bracket on the left–hand side is positive, $\lambda \mapsto \eta(\lambda)$ is decreasing. Hence for $\lambda > 1$, we have $\eta(\lambda) < 1 + K\eta(1)^\beta\lambda^{\beta-1}$, so $\limsup_{\lambda\to\infty}\eta(\lambda) \leq 1$, and $\eta(\lambda) \to 1$ as $\lambda \to \infty$.

In view of this discussion, one might expect that $\lim_{t\to\infty} H(t)/F^{-1}(t\,M(t)) = \infty$ implies $x(t) \sim H(t)$ as $t \to \infty$, or less precisely that sufficiently rapid growth in H forces $x(t)$ to grow at the rate $H(t)$. Therefore, it is natural to ask under what conditions we would have $x(t) \sim H(t)$ as $t \to \infty$.

A necessary condition for $\lim_{t\to\infty} x(t)/H(t) = 1$ is $\lim_{t\to\infty} \int_0^t M(t-s)f(H(s))ds/H(t) = 0$. This motivates the hypothesis

$$\lim_{t\to\infty} \frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} = 0, \tag{3.3.3}$$

and the following result. This result requires no monotonicity in H and as such allows for H to undergo considerable fluctuation, a point we will illustrate further in Section 3.4 and Chapter 4.

Theorem 3.3.2. *Suppose the measure μ obeys (3.2.1) with $M \in RV_\infty(\theta)$, $\theta \geq 0$ and that $f \in RV_\infty(\beta)$, $\beta \in [0,1)$. When $\beta = 0$ let f be asymptotically increasing and obey $\lim_{x\to\infty} f(x) = \infty$. Let H be a function in $C((0,\infty);(0,\infty))$ satisfying (3.3.3). Then the solution, x, of (3.1.1) obeys $\lim_{t\to\infty} x(t)/H(t) = 1$.*

When H is regularly varying at infinity the hypotheses of Theorems 3.3.1 and 3.3.2 align to give a complete classification of the asymptotics (Corollary 3.3.1). However, assuming regular variation of H imposes considerable regularity constraints. In particular, H is then asymptotic to an increasing function and this restricts potential applications of Theorem 3.3.2 to SFDEs.

Corollary 3.3.1. *Let $M \in RV_\infty(\theta)$, $\theta \geq 0$ with $\lim_{t\to\infty} M(t) = \infty$. Suppose that $f \in RV_\infty(\beta)$, $\beta \in [0,1)$. When $\beta = 0$ let f be asymptotically increasing and obey $\lim_{x\to\infty} f(x) = \infty$. If $H \in RV_\infty(\alpha)$, $\alpha > 0$, then the following are equivalent:*

(i.) $\lim_{t\to\infty} M(t)\int_0^t f(H(s))\,ds/H(t) = 0$,

(ii.) $\lim_{t\to\infty} H(t)/F^{-1}(tM(t)) = \infty$,

(iii.) $\lim_{t\to\infty} \int_0^t M(t-s)f(H(s))\,ds/H(t) = 0$.

We exclude the case $\alpha = 0$, because it is covered by Theorem 3.3.1 with $\lambda = 0$.

We state without proof a partial converse to Theorem 3.3.2 with $H \in \mathrm{RV}_\infty(\alpha)$ for $\alpha > 0$. The proof follows from Corollary 3.3.1 and estimation arguments similar to those used throughout this chapter.

Theorem 3.3.3. *Suppose the measure μ obeys (3.2.1) with $M \in RV_\infty(\theta)$, $\theta \geq 0$ and that $f \in RV_\infty(\beta)$, $\beta \in [0,1)$. When $\beta = 0$ let f be asymptotically increasing and obey $\lim_{x\to\infty} f(x) = \infty$. If x denotes a solution of (3.1.4) and $H \in C((0,\infty);(0,\infty)) \cap RV_\infty(\alpha)$ with $\alpha > 0$, then the following are equivalent:*

$$(i.)\quad \lim_{t\to\infty} \frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} = 0, \qquad (ii.)\quad \lim_{t\to\infty} \frac{x(t)}{H(t)} = 1.$$

While discussing the hypothesis that $\lim_{t\to\infty} H(t)/F^{-1}(tM(t)) = \infty$ in the context of regular variation it is worth remarking that this hypothesis is also satisfied for $H \in \mathrm{RV}_\infty(\infty)$, the so-called rapidly varying functions (see Definition 1.3.4). If $H \in \mathrm{RV}_\infty(\infty)$, then (3.3.3) holds and Theorem 3.3.2 can be applied; this fact is recorded in the following corollary.

Corollary 3.3.2. *Suppose the measure μ obeys (3.2.1) with $M \in RV_\infty(\theta)$, $\theta \geq 0$ and that $f \in RV_\infty(\beta)$, $\beta \in [0,1)$. When $\beta = 0$ let f be asymptotically increasing and obey $\lim_{x\to\infty} f(x) = \infty$. If x denotes a solution of (3.1.4) and $H \in C((0,\infty);(0,\infty)) \cap RV_\infty(\infty)$ is asymptotically increasing, then $\lim_{t\to\infty} x(t)/H(t) = 1$.*

Corollary 3.3.2 will also hold if $H \in \mathrm{MR}_\infty(\infty)$, a sub-class of $\mathrm{RV}_\infty(\infty)$, because this guarantees that H is asymptotic to an increasing function (see [27, p.68 & p.83]).

3.4 Examples

3.4.1 Application of Theorem 3.2.1

The main attraction of Theorem 3.2.1 is that it largely reduces the asymptotic analysis of solutions of (3.1.1) to the computation, or asymptotic analysis, of the function F^{-1}. This is because, under appropriate hypotheses, Theorem 3.2.1 yields

$$x(t) \sim F^{-1}(t\,M(t))\left\{\frac{1}{1-\beta}B\left(\theta+1,\frac{\theta\beta+1}{1-\beta}\right)\right\}^{\frac{1}{1-\beta}}, \text{ as } t\to\infty.$$

In general, exact computation of F^{-1} in closed form is not possible. The following result provides the asymptotics of F^{-1} for a large class of $f\in \mathrm{RV}_\infty(\beta)$ for $\beta\in[0,1)$ using some classic results from the theory of regular variation. Its principal appeal is that it can be applied by calculating the limit of a readily–computed function which can be found directly in terms of f, without need for integration.

Proposition 3.4.1. *Suppose $f\in RV_\infty(\beta)$, $\beta\in[0,1)$ is continuous and $\ell(x):=\big(f(x)/x^\beta\big)^{\frac{1}{1-\beta}}$ obeys*

$$\lim_{x\to\infty}\frac{\ell(x\,\ell(x))}{\ell(x)}=1. \tag{3.4.1}$$

Then

$$F(x)\sim\frac{1}{1-\beta}\frac{x}{f(x)},\quad F^{-1}(x)\sim(1-\beta)^{\frac{1}{1-\beta}}\,\ell\left(x^{\frac{1}{1-\beta}}\right)x^{\frac{1}{1-\beta}},\ \textit{as } x\to\infty.$$

The following examples illustrate the convenience of Proposition 3.4.1 in practice.

Example 3.4.1. *Suppose $f(x)\sim a\,x^\beta\,\log\log(x^\alpha)$ as $x\to\infty$ with $\beta\in[0,1)$, $a>0$ and $\alpha>0$. In this case*

$$\ell(x)\sim\left(a\,\log\log\left(x^\alpha\right)\right)^{\frac{1}{1-\beta}},\ \textit{as } x\to\infty.$$

Hence

$$\bar{L}(x):=\frac{\ell(x\,\ell(x))}{\ell(x)}=\frac{\left(\log\log\left(x^\alpha\,a^{\frac{\alpha}{1-\beta}}\,\{\log\log(x^\alpha)\}^{\frac{\alpha}{1-\beta}}\right)\right)^{\frac{1}{1-\beta}}}{\left(\log\log\left(x^\alpha\right)\right)^{\frac{1}{1-\beta}}},\ x>0.$$

Let $\log\log(x^\alpha)=u$ to obtain

$$\begin{aligned}\bar{L}(\exp\exp(u)^{1/\alpha})&=\left(\log(e^u)/u+\log\left(1+\frac{\log(a^{\frac{\alpha}{1-\beta}})+\log(u^{\frac{\alpha}{1-\beta}})}{e^u}\right)/u\right)^{\frac{1}{1-\beta}}\\&=:(1+G(u))^{\frac{1}{1-\beta}},\end{aligned}$$

where $\lim_{u\to\infty}G(u)=0$. Therefore $\lim_{x\to\infty}\bar{L}(x):=\ell(x\,\ell(x))/\ell(x)=1$ and by Lemma 3.4.1,

$$F^{-1}(x)\sim(1-\beta)^{\frac{1}{1-\beta}}\left\{a\,\log\log\left(x^{\frac{\alpha}{1-\beta}}\right)\right\}^{\frac{1}{1-\beta}}x^{\frac{1}{1-\beta}},\ \textit{as } x\to\infty.$$

This example is valid with $\log\log(x)$ replaced by $\prod_{i=1}^n\log_{i-1}(x)$, with the convention that $\log_i(x)=\log\log_{i-1}(x)$. The proof in this case is essentially the same but the resulting formulae are rather convoluted.

Example 3.4.2. *Suppose $f(x)\sim x^\beta\,(2+\sin(\log\log(x)))$ as $x\to\infty$, with $\beta\in(0,1)$. In this case*

$$\ell(x)\sim\left(2+\sin(\log\log(x))\right)^{\frac{1}{1-\beta}},\ \textit{as } x\to\infty.$$

Hence

$$\bar{L}(x) = \left(\frac{2 + \sin(\log\log(x\,\ell(x)))}{2 + \sin(\log\log(x))} \right)^{\frac{1}{1-\beta}}.$$

Once more make the substitution $\log\log(x) = u$ *to obtain*

$$\bar{L}(\exp\exp(u)) = \left(\frac{2 + \sin\left(u + \log[1 + \log\{2 + \sin(u)\}^{\frac{1}{1-\beta}}/e^u]\right)}{2 + \sin(u)} \right)^{\frac{1}{1-\beta}}.$$

Letting $u \to \infty$ *in the above yields* $\lim_{u\to\infty} \bar{L}(\exp\exp(u)) = 1$ *and hence Proposition 3.4.1 applies. Therefore*

$$F^{-1}(x) \sim (1-\beta)^{\frac{1}{1-\beta}} \left\{2 + \sin\left(\log\log(x^{\frac{1}{1-\beta}})\right)\right\}^{\frac{1}{1-\beta}} x^{\frac{1}{1-\beta}}, \quad \text{as } x \to \infty.$$

3.4.2 Discrete Measures

It may appear that our inclusion of a general measure μ in (3.1.1) and the hypothesis that the integral of μ is regularly varying are only compatible when μ is an absolutely continuous measure. The following proposition allows us to easily construct examples to show that our results also cover a variety of equations involving discrete measures.

Proposition 3.4.2. *Let* $x \geq 0$ *and* δ_x *be the Dirac measure at* x *on* $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. *Suppose that* $\theta \in (0,1)$ *and that* $\mu_0 \in RV_\infty(\theta - 1)$. *Let* $\tau > 0$ *and*

$$\mu(ds) = \sum_{j=0}^{\lfloor t/\tau \rfloor} \mu_0(j\tau)\delta_{j\tau}(ds). \tag{3.4.2}$$

Hence

$$M(t) = \int_{[0,t]} \mu(ds) = \sum_{j=0}^{\lfloor t/\tau \rfloor} \mu_0(j\tau), \tag{3.4.3}$$

and $M \in RV_\infty(\theta)$. *Furthermore,* $M(t) \sim \tilde{M}(t) := \int_0^t \tilde{\mu}(s)ds$ *as* $t \to \infty$, *where* $\tilde{\mu} \in RV_\infty(\theta - 1)$ *is any* C^1, *decreasing function such that* $\mu_0(s) \sim \tilde{\mu}(s)$ *as* $s \to \infty$.

3.4.3 Perturbed Equations

Using a parametrized example we illustrate how the asymptotic behaviour of solutions to (3.1.4) can be classified using the range of possibilities covered by the results in Section 3.3.

Example 3.4.3. *For ease of exposition suppose that* $\beta \in (0,1)$ *and let*

$$f(x) = x^\beta, \quad x \geq 0; \quad M(t) = (1+t)^\theta - 1, \quad t \geq 0; \quad H(t) = (1+t)^\alpha\, e^{\gamma t} - 1, \quad t \geq 0,$$

with $\theta > 0$, $\alpha \in \mathbb{R}$, *and* $\gamma \geq 0$. *Hence* (3.1.4) *becomes*

$$x(t) = x(0) + \int_0^t ((1+t-s)^\theta - 1)x(s)^\beta ds + (1+t)^\alpha\, e^{\gamma t} - 1, \quad t \geq 0,$$

with $x(0) > 0$. *Therefore*

$$F^{-1}(t\,M(t)) \sim (1-\beta)^{\frac{1}{1-\beta}}\, t^{\frac{\theta+1}{1-\beta}}, \quad \text{as } t \to \infty.$$

Case *(i.) : $\gamma = 0$. In this case $H \in RV_\infty(\alpha)$ and*

$$\frac{H(t)}{F^{-1}(t\,M(t))} \sim (1-\beta)^{\frac{1}{\beta-1}} t^{\alpha - \frac{\theta+1}{1-\beta}}, \text{ as } t \to \infty.$$

If $\alpha < (\theta+1)/(1-\beta)$, then $\lim_{t\to\infty} H(t)/F^{-1}(t\,M(t)) = 0$ and Theorem 3.3.1 yields the limit

$$\lim_{t\to\infty} x(t)/F^{-1}(t\,M(t)) = L,$$

where $L = \left\{ B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right)/(1-\beta)\right\}^{1/(1-\beta)}$.

If $\alpha = (\theta+1)/(1-\beta)$, then $\lim_{t\to\infty} H(t)/F^{-1}(t\,M(t)) = (1-\beta)^{1/(\beta-1)} =: \lambda$ and Theorem 3.3.1 gives $\lim_{t\to\infty} x(t)/F^{-1}(t\,M(t)) = \zeta$, where ζ satisfies (3.3.2).

Finally, if $\alpha > (\theta+1)/(1-\beta)$, then $\lim_{t\to\infty} H(t)/F^{-1}(t\,M(t)) = \infty$. Then, by Corollary 3.3.1, (3.3.3) holds and Theorem 3.3.2 yields $\lim_{t\to\infty} x(t)/H(t) = 1$.

Case *(ii.) : $\gamma > 0$. In this case, $H \in RV_\infty(\infty)$ and by Corollary 3.3.2, $\lim_{t\to\infty} x(t)/H(t) = 1$ for all $\alpha \in \mathbb{R}$, $\beta \in (0,1)$ and $\theta > 0$.*

Particularly with a view to applications to SFDEs, it is pertinent to highlight when H is required to have some form monotonicity in Section 3.3. When $\lambda = 0$ in Theorem 3.3.1 there is no monotonicity requirement on H but $\lambda > 0$ implies that H asymptotic to the monotone increasing function F^{-1}, modulo a constant. By contrast, Theorem 3.3.2 allows for large "fluctuations", or irregular behaviour, in H; the following examples illustrate this point.

Example 3.4.4. *Suppose $f \in RV_\infty(\beta)$ with $\beta \in (0,1)$, $M \in RV_\infty(\theta)$ with $\theta \geq 0$ and $H(t) = (1+t)^\alpha\ (2+\sin(t)) - 2$, $\alpha > 0$. From Karamata's Theorem*

$$\begin{aligned}\limsup_{t\to\infty} \frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} &= \limsup_{t\to\infty} \frac{M(t)\int_0^t f((1+s)^\alpha\ (2+\sin(s)-2))\,ds}{(1+t)^\alpha\ (2+\sin(t))-2} \\ &\leq \limsup_{t\to\infty} \frac{(1+\epsilon)M(t)\int_0^t \phi(3\,s^\alpha)\,ds}{t^\alpha}.\end{aligned}$$

Since

$$\frac{M(t)\int_0^t \phi(3\,s^\alpha)\,ds}{t^\alpha} \sim \frac{M(t)\,t\,f(3\,t^\alpha)}{(1+\alpha\beta)t^\alpha}, \text{ as } t\to\infty,$$

a sufficient condition for (3.3.3) to hold, and hence for Theorem 3.3.2 to apply, is $\alpha > (1+\theta)/(1-\beta)$. Even more rapid variation in H is permitted; for example let $H(t) = e^t(2+\sin(t)) - 2$. In this case asymptotic monotonicity of f and the rapid variation of e^t yield

$$\limsup_{t\to\infty} \frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} \leq \limsup_{t\to\infty} \frac{M(t)\,t\,f(3e^t)}{e^t} = 0,$$

and once more Theorem 3.3.2 applies to yield $x(t) \sim H(t)$ as $t\to\infty$, where x is the solution to (3.1.6). By fixing $f(x) = x^\beta$, we can immediately see that it is possible to capture more general types of exponentially fast oscillation in Theorem 3.3.2. Choose $H(t) = e^{\sigma(t)\,t}$, where $\sigma(t)$ obeys $0 < \sigma_- \leq \sigma(t) \leq \sigma_+ < \infty$ for all $t \geq 0$, for some constants σ_- and σ_+. Checking condition (3.3.3) yields

$$\limsup_{t\to\infty} \frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} \leq \limsup_{t\to\infty} \frac{M(t)\,t\,e^{\beta\sigma_+ t}}{e^{\sigma_- t}}.$$

The limit of the right hand side will be zero if $\sigma_- > \beta\sigma_+$.

Finally we present an example in which condition (3.3.3) fails to hold. This example illustrates the limitations of our results by showing that when the exogenous perturbation exhibits rapid, irregular growth we are unable to capture the dynamics of the solution. This example is constructed by considering an extremely ill-behaved perturbation with periodic fluctuations of exponential order.

Example 3.4.5. *Choose* $f(x) = x^\beta$ *and* $H(t) \sim e^{t(1+\alpha p(t))} := H^*(t)$*, as* $t \to \infty$*, with* $\alpha \in (0,1)$*,* $\beta \in (0,1)$ *and* p *a continuous* $1-$*periodic function such that* $\max_{t\in[0,1]} p(t) = 1$ *and* $\min_{t\in[0,1]} p(t) = -1$*. Let* $t > 0$ *and define* $n(t) \in \mathbb{N}$ *such that* $n(t) \le t < n(t)+1$*. Then*

$$\begin{aligned} S(t) :&= \int_0^t f(H^*(s))ds = \int_0^t e^{\beta s(1+\alpha p(s))}ds \\ &= \sum_{j=0}^{n(t)-1} \int_0^1 e^{\beta(u+j)(1+\alpha p(u))}du + \int_0^{t-n(t)} e^{\beta(u+n(t))(1+\alpha p(u))}du. \end{aligned}$$

Let $I_j := \int_0^1 e^{\beta(u+j)(1+\alpha p(u))}du$ *and* $S_n := \sum_{j=0}^{n-1} I_j$*. Then*

$$S(t) = S_{n(t)} = \int_0^{t-n(t)} e^{\beta(u+n(t))(1+\alpha p(u))}du.$$

Hence $S(t) \le S_{n(t)} = \int_0^1 e^{\beta(u+n(t))(1+\alpha p(u))}du = S_{n(t)+1}$ *and* $S_{n(t)} \le S(t) \le S_{n(t)+1}$*.*

Now estimate I_j *as* $j \to \infty$ *as follows. Letting* $c(u) = e^{u(1+\alpha p(u))\beta}$ *and* $d(u) = \beta(1+\alpha p(u))$ *we have* $I_j = \int_0^1 c(u)e^{jd(u)}du$*. By hypothesis* $0 < \underline{c} = \min_{u\in[0,1]} c(u) \le \max_{u\in[0,1]} c(u) \le \overline{c} < \infty$*. Hence*

$$\underline{c}\int_0^1 e^{jd(u)}du \le I_j \le \overline{c}\int_0^1 e^{jd(u)}du,\ j \ge 0.$$

Therefore, for $j \ge 1$*,*

$$\underline{c}^{\frac{1}{j}}\left(\int_0^1 e^{jd(u)}\,du\right)^{\frac{1}{j}} \le I_j^{\frac{1}{j}} \le \overline{c}^{\frac{1}{j}}\left(\int_0^1 e^{jd(u)}\,du\right)^{\frac{1}{j}}.$$

For any continuous, non-negative function $a : [0,1] \mapsto (0,\infty)$*,*

$$\lim_{j\to\infty}\left(\int_0^1 a(u)^j du\right)^{1/j} = \max_{u\in[0,1]} a(u)$$

and thus $\lim_{j\to\infty} I_j^{1/j} = \max_{u\in[0,1]} e^{d(u)}$*. Therefore*

$$\lim_{j\to\infty}\frac{1}{j}\log I_j = \max_{u\in[0,1]} d(u) = \beta\max_{u\in[0,1]}(1+\alpha p(u)) = \beta(1+\alpha) > 0.$$

Since $S_n = \sum_{j=0}^{n-1} I_j$*, this gives us* $\lim_{n\to\infty}\log S_n/n = \beta(1+\alpha)$*. Hence*

$$\liminf_{t\to\infty}\frac{1}{t}\log S(t) \ge \liminf_{t\to\infty}\frac{1}{t}\log S_{n(t)} = \liminf_{t\to\infty}\frac{1}{n(t)}\log S_{n(t)}\frac{n(t)}{t} = \beta(1+\alpha).$$

An analogous calculation for the limsup then yields $\lim_{t\to\infty}\log S(t)/t = \lambda$*. Therefore, as* $\log H^*(t)/t = 1+\alpha p(t)$*,*

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(\frac{H^*(t)}{S(t)}\right) = \limsup_{t\to\infty}(1+\alpha p(t)) - \beta(1+\alpha) = (1+\alpha)(1-\beta) > 0.$$

Hence $\limsup_{t\to\infty} H^*(t)/\int_0^t f(H^*(s))\,ds = \infty$*, and because* $H(t) \sim H^*(t)$ *as* $t \to \infty$ *and* $f \in RV_\infty(\beta)$*,* *we have* $\limsup_{t\to\infty} H(t)/\int_0^t f(H(s))\,ds = \infty$. *Similarly*

$$\liminf_{t\to\infty} \frac{1}{t}\log\left(\frac{H^*(t)}{S(t)}\right) = \liminf_{t\to\infty}(1+\alpha p(t)) - \beta(1+\alpha) = 1-\beta-\alpha(1+\beta).$$

Choose $\alpha > (1-\beta)/(1+\beta) > 0$ *to ensure that* $1-\beta-\alpha(1+\beta) < 0$ *and*

$$\liminf_{t\to\infty} \frac{H^*(t)}{\int_0^t f(H^*(s))\,ds} = 0.$$

As above, this gives

$$\liminf_{t\to\infty} \frac{H(t)}{\int_0^t f(H(s))\,ds} = 0.$$

We remark that because the function $t \mapsto H(t)/\int_0^t f(H(s))\,ds$ *is of exponential order,* (3.3.3) *is violated for any* $M \in RV_\infty(\theta)$ *with* $\theta \in [0,\infty)$.

3.5 Proofs of Main Results

We often choose to work with a monotone function approximating f; this monotone approximation will be denoted by ϕ. If f is regularly varying with a positive index, then

$$\text{There exists } \phi \in C^1((0,\infty);(0,\infty)) \text{ such that } f(x) \sim \phi(x) \text{ and } \phi'(x) > 0 \text{ for all } x > 0. \tag{3.5.1}$$

See Section 1.3.3 for further details. The function $F(x)$ is approximated by $\Phi(x) := \int_1^x du/\phi(u)$ and Φ^{-1} is the inverse function of Φ.

The proof of Theorem 3.2.1 is decomposed into the following lemmata, the first of which provides a precise estimate on the asymptotics of the convolution of two regularly varying functions.

Lemma 3.5.1. *Suppose* $a \in RV_\infty(\rho)$ *and* $b \in RV_\infty(\sigma)$*, where* $\rho \geq 0$ *and* $\sigma \geq 0$*, and* $\lim_{t\to\infty} a(t) = \infty$*. If* $\sigma = 0$ *let* b *be asymptotically increasing and obey* $\lim_{t\to\infty} b(t) = \infty$*. Then*

$$\lim_{t\to\infty} \frac{\int_0^t a(s)b(t-s)ds}{t\,a(t)\,b(t)} = \int_0^1 \lambda^\rho(1-\lambda)^\sigma d\lambda =: B(\rho+1,\sigma+1),$$

where B *denotes the Beta function.*

Proof. Let ϵ, $\eta \in (0,\frac{1}{2})$ be arbitrary. Define

$$\begin{aligned} I(t) &= \int_0^t a(s)b(t-s)\,ds = \int_0^{\epsilon t} a(s)\,b(t-s)\,ds + \int_{\epsilon t}^{(1-\eta)t} a(s)b(t-s)\,ds + \int_{(1-\eta)t}^t a(s)b(t-s)\,ds \\ &=: I_1(t) + I_2(t) + I_3(t), \quad \text{for each } t \geq 0. \end{aligned} \tag{3.5.2}$$

By making the substitution $s = \lambda t$, we have

$$\frac{I_2(t)}{t\,a(t)\,b(t)} = \frac{\int_{\epsilon t}^{(1-\eta)t} a(s)b(t-s)\,ds}{t\,a(t)\,b(t)} = \int_\epsilon^{1-\eta} \frac{a(\lambda t)}{a(t)}\frac{b(t(1-\lambda))}{b(t)}d\lambda.$$

By the Uniform Convergence Theorem it follows that

$$\lim_{t\to\infty}\frac{I_2(t)}{t\,a(t)\,b(t)} = \int_\epsilon^{1-\eta} \lambda^\rho(1-\lambda)^\sigma d\lambda. \tag{3.5.3}$$

Since both a and b are positive functions it is clear that $I(t) \geq I_2(t)$ and hence

$$\liminf_{t\to\infty} \frac{I(t)}{t\,a(t)\,b(t)} \geq \int_\epsilon^{1-\eta} \lambda^\rho (1-\lambda)^\sigma \, d\lambda.$$

Letting η and $\epsilon \to 0^+$ then yields

$$\liminf_{t\to\infty} \frac{I(t)}{t\,a(t)\,b(t)} \geq \int_0^1 \lambda^\rho (1-\lambda)^\sigma \, d\lambda. \tag{3.5.4}$$

By hypothesis there exists an increasing, C^1 function β such that $b(t)/\beta(t) \to 1$ as $t \to \infty$. It follows that there exists $T_1 > 0$ such that $t \geq T_1$ implies $b(t)/\beta(t) \leq 2$. Therefore with $\epsilon \in (0, \frac{1}{2})$, $t \geq 2T_1$ we have that $(1-\epsilon)t \geq T_1$. Suppose $t \geq 2T_1$ and estimate as follows

$$I_1(t) = \int_0^{\epsilon t} a(s)\, b(t-s)\, ds \leq 2\beta(t) \int_0^{\epsilon t} a(s)\, ds = 2\beta(t)\, \epsilon\, t\, a(\epsilon t)\, \frac{\int_0^{\epsilon t} a(s)\, ds}{\epsilon\, t\, a(\epsilon t)}.$$

Hence, for $t \geq 2T_1$,

$$\frac{I_1(t)}{t\,a(t)\,b(t)} \leq 2\epsilon\, \frac{\beta(t)}{b(t)}\, \frac{a(\epsilon t)}{a(t)}\, \frac{\int_0^{\epsilon t} a(s)\, ds}{\epsilon\, t\, a(\epsilon t)}.$$

$a \in \mathrm{RV}_\infty(\rho)$ implies that $\lim_{t\to\infty} a(\epsilon t)/a(t) = \epsilon^\rho$ and by Karamata's Theorem,

$$\lim_{t\to\infty} \int_0^{\epsilon t} \frac{a(s)}{\epsilon\, t\, a(\epsilon t)} ds = 1/(1+\rho).$$

Thus

$$\limsup_{t\to\infty} \frac{I_1(t)}{t\,a(t)\,b(t)} \leq \frac{2\epsilon^{\rho+1}}{1+\rho}. \tag{3.5.5}$$

Finally, consider $I_3(t)$. By construction $t \geq T_1$ implies $b(t)/\beta(t) \leq 2$. Since b, β are continuous and positive, with β bounded away from zero, $\sup_{0\leq t\leq T_1} b(t)/\beta(t) = \max_{0\leq t\leq T_1} b(t)/\beta(t) = B_1 < \infty$. Thus there exists $B_2 > 0$ such that $b(t) \leq B_2\, \beta(t)$ for all $t \geq 0$. Therefore

$$I_3(t) = \int_{(1-\eta)t}^t a(s)\, b(t-s)\, ds \leq B_2 \int_{(1-\eta)t}^t a(s)\, \beta(t-s)\, ds \leq B_2 \beta(\eta t) \int_{(1-\eta)t}^t a(s)\, ds.$$

Hence

$$\limsup_{t\to\infty} \frac{I_3(t)}{t\,a(t)\,b(t)} \leq B_2 \limsup_{t\to\infty} \frac{\beta(\eta t)}{b(t)} \limsup_{t\to\infty} \frac{\int_{(1-\eta)t}^t a(s)\, ds}{t\, a(t)} = B_2\, \eta^\sigma \limsup_{t\to\infty} \frac{\int_{(1-\eta)t}^t a(s)\, ds}{t\, a(t)}. \tag{3.5.6}$$

The final limit on the right–hand side of (3.5.6) is calculated by once more calling upon the Uniform Convergence Theorem as follows:

$$\lim_{t\to\infty} \frac{\int_{(1-\eta)t}^t a(s)\, ds}{t\, a(t)} = \lim_{t\to\infty} \int_{1-\eta}^1 \frac{a(\lambda t)}{a(t)} d\lambda = \int_{1-\eta}^1 \lambda^\rho d\lambda.$$

Returning to (3.5.6) and using the identity above appropriately yields

$$\limsup_{t\to\infty} \frac{I_3(t)}{t\,a(t)\,b(t)} \leq B_2\, \eta^\sigma \int_{1-\eta}^1 \lambda^\rho d\lambda = B_2\, \eta^\sigma \left(\frac{1}{\rho+1} - \frac{(1-\eta)^{\rho+1}}{\rho+1} \right). \tag{3.5.7}$$

Therefore, combining (3.5.3), (3.5.5) and (3.5.7), we obtain

$$\limsup_{t\to\infty} \frac{I(t)}{t\,a(t)\,b(t)} \le 2\epsilon^{\rho+1}\frac{1}{1+\rho} + \int_\epsilon^{1-\eta} \lambda^\rho(1-\lambda)^\sigma d\lambda + B_2\,\eta^\sigma \int_{1-\eta}^1 \lambda^\rho d\lambda.$$

Letting η and $\epsilon \to 0^+$ in the above then yields

$$\limsup_{t\to\infty} \frac{I(t)}{t\,a(t)\,b(t)} \le \int_0^1 \lambda^\rho(1-\lambda)^\sigma d\lambda. \tag{3.5.8}$$

Combining (3.5.8) with (3.5.4) gives the desired conclusion. □

The proof of Theorem 3.2.1 now begins in earnest by proving a "rough" lower bound on the solution which we will later refine. Lemmas 3.5.2, 3.5.3 and 3.5.4 are all proven under the same set of hypotheses and are presented separately for the sake of readability.

Lemma 3.5.2. *Suppose μ obeys (3.2.1) with $M \in RV_\infty(\theta)$, $\theta \ge 0$ and $f \in RV_\infty(\beta)$, $\beta \in [0,1)$. If $\beta = 0$, let f be asymptotically increasing and obey $\lim_{x\to\infty} f(x) = \infty$. Then each solution, x, of (3.1.1) obeys*

$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} > 0.$$

Proof. Let $\epsilon \in (0,1)$ be arbitrary. By hypothesis there exists ϕ such that (3.5.1) holds and hence there exists $x_1(\epsilon) > 0$ such that $f(x) > (1-\epsilon)\phi(x)$ for all $x > x_1(\epsilon)$. Furthermore, there exists $T_0(\epsilon) > 0$ such that $t \ge T_0$ implies $x(t) > x_1(\epsilon)$. Similarly, there exists $T_1(\epsilon) > 0$ such that $M(t) > 0$ for all $t \ge T_1$. Since $M \in \text{RV}_\infty(\theta)$, there exists a C^1 function M_1 such that for all $\epsilon \in (0,1)$ there exists $T_2(\epsilon) > 0$ such that for all $t \ge T_2$, $M(t) > (1-\epsilon)M_1(t)$. Let $T_3 := T_0 + T_1 + T_2$ and estimate as follows

$$\begin{aligned} x'(t) &= \int_{[0,t-T_3]} \mu(ds) f(x(t-s)) + \int_{(t-T_3,t]} \mu(ds) f(x(t-s)) \ge (1-\epsilon)\int_{[0,t-T_3]} \mu(ds)\phi(x(t-s)) \\ &= (1-\epsilon)\int_{[0,(t-T_3)/2]} \mu(ds)\phi(x(t-s)) + (1-\epsilon)\int_{((t-T_3)/2,t-T_3]} \mu(ds)\phi(x(t-s)) \\ &\ge (1-\epsilon)\int_{[0,(t-T_3)/2]} \mu(ds)\phi(x(t-s)) \ge (1-\epsilon) M(\tfrac{1}{2}(t-T_3))\phi\left(x(\tfrac{1}{2}(t+T_3))\right), \quad \text{for a.e. } t \ge 4T_3. \end{aligned}$$

Since $M \in \text{RV}_\infty(\theta)$, $\lim_{t\to\infty} M((t-T_3)/2)/M(t-T_3) = 2^{-\theta}$. Thus there exists a positive constant C and a time $\tilde{T_3} \ge 4T_3$ such that

$$x'(t) \ge C\,M(t-T_3)\phi\left(x(\tfrac{1}{2}(t+T_3))\right), \quad \text{for a.e. } t \ge \tilde{T_3}. \tag{3.5.9}$$

Furthermore, since $t \ge \tilde{T_3}$ implies $t - T_3 > T_2$, there exists $C_0 > 0$ such that

$$x'(t) \ge C_0\,M_1(t-T_3)\phi\left(x((t+T_3)/2)\right), \quad \text{for a.e. } t \ge \tilde{T_3}. \tag{3.5.10}$$

Now define the C^2, positive, increasing function $\bar{M}_1(t) := \int_0^t M_1(s)ds$ for $t \ge 0$. Let

$$\alpha(t) = \bar{M}_1^{-1}(t) + T_3, \quad t \ge \bar{M}_1(\tilde{T_3}). \tag{3.5.11}$$

For $t \ge \bar{M}_1(\tilde{T_3})$, $\alpha(t) \ge \alpha(\bar{M}_1(\tilde{T_3})) = \tilde{T_3} + T_3 > \tilde{T_3}$ since α is increasing. Define $\tilde{x}(t) = x(\alpha(t))$ for $t \ge \bar{M}_1(\tilde{T_3})$. Note that $\tilde{x} \in AC([\bar{M}_1(\tilde{T_3}),\infty);(0,\infty))$ and $\alpha'(t) = 1/M_1(\bar{M}_1^{-1}(t))$. Now use (3.5.10) to compute as follows:

$$\begin{aligned}\tilde{x}'(t) = \alpha'(t)x'(\alpha(t)) &\geq \frac{C_0\, M_1(\alpha(t) - T_3)}{M_1(\bar{M}_1^{-1}(t))}\phi\left(x(\tfrac{1}{2}(\alpha(t)+T_3))\right) \\ &= C_0\, \phi\left(x(\tfrac{1}{2}(\alpha(t)+T_3))\right), \quad \text{for a.e. } t \geq \bar{M}_1(\tilde{T}_3). \end{aligned} \tag{3.5.12}$$

Define $\tau(t) = t - \bar{M}_1\left(\bar{M}_1^{-1}(t)/2\right) > 0$ for $t \geq \bar{M}_1(\tilde{T}_3)$ and note that $(\alpha(t)+T_3)/2 = \alpha(t-\tau(t))$. Hence

$$\tilde{x}'(t) \geq C_0\, \phi\left(x(\tfrac{1}{2}(\alpha(t)+T_3)\right) = C_0\, \phi\left(x(\alpha(t-\tau(t))\right) = C_0\, \phi\left(\tilde{x}(t-\tau(t))\right), \quad \text{for a.e. } t \geq \bar{M}_1(\tilde{T}_3). \tag{3.5.13}$$

Owing to the monotonicity of $\bar{M}_1$, $\tau(t) > 0$ for $t \geq \bar{M}_1(\tilde{T}_3)$. Furthermore, since $\bar{M}_1 \in \mathrm{RV}_\infty(\theta+1)$,

$$\lim_{t\to\infty} \frac{t-\tau(t)}{t} = \lim_{t\to\infty} \frac{\bar{M}_1\left(\frac{1}{2}\bar{M}_1^{-1}(t)\right)}{\bar{M}_1\left(\bar{M}_1^{-1}(t)\right)} = \lim_{t\to\infty} \frac{\bar{M}_1\left(\frac{1}{2}\bar{M}_1^{-1}(t)\right)}{\bar{M}_1\left(\bar{M}_1^{-1}(t)\right)} = \left(\frac{1}{2}\right)^{\theta+1}.$$

Thus there exists a $T_4 > 0$ such that for all $t \geq T_4$, $t - \tau(t) > 2^{-(\theta+2)}\, t$. If $\bar{T}_4 = \max(T_4, \bar{M}_1(\tilde{T}_3))$, then

$$\tilde{x}'(t) \geq C_0\, \phi(\tilde{x}(qt)), \quad \text{for a.e. } t \geq \bar{T}_4, \quad \text{where } q = 2^{-(\theta+2)} \in (0,1). \tag{3.5.14}$$

Let $h \in (-1, 0)$ be arbitrary, take $t \geq \bar{T}_4 + 1$ and integrate (3.5.14) to obtain

$$\int_{t+h}^{t} \tilde{x}'(s)\, ds = \tilde{x}(t) - \tilde{x}(t+h) \geq C_0 \int_{t+h}^{t} \phi(\tilde{x}(qs))\, ds, \quad \text{for each } t \geq \bar{T}_4 + 1.$$

Now rearrange the inequality above to show that

$$\frac{\tilde{x}(t+h) - \tilde{x}(t)}{h} \geq \frac{C_0}{h} \int_{t}^{t+h} \phi(\tilde{x}(qs))\, ds, \quad \text{for each } t \geq \bar{T}_4 + 1.$$

Take the liminf as $h \to 0^-$ to derive the differential inequality

$$D_-\tilde{x}(t) \geq C_0\, \phi(\tilde{x}(qt)), \quad \text{for each } t \geq T_5 := \bar{T}_4 + 1. \tag{3.5.15}$$

The following estimates are needed for a comparison argument. Since $\phi \circ \Phi^{-1} \in \mathrm{RV}_\infty\left(\beta/(1-\beta)\right)$,

$$\lim_{x\to\infty} \frac{(\phi\circ\Phi^{-1})(\frac{x}{q})}{(\phi\circ\Phi^{-1})(x)} = \left(\frac{1}{q}\right)^{\frac{\beta}{1-\beta}}.$$

Thus there exists $x_2 > 0$ such that for all $x \geq x_2$

$$\frac{(\phi\circ\Phi^{-1})(\frac{x}{q})}{(\phi\circ\Phi^{-1})(x)} < 2\left(\frac{1}{q}\right)^{\frac{\beta}{1-\beta}}.$$

Next let $T_5^{'} > 0$ be so large that $\Phi(\tilde{x}(qT_5^{'})/2) - x_2 > 0$ and let $T_6 = \max(T_5, T_5^{'}) + 1$. Then $\Phi(\tilde{x}(qT_6)/2) > \Phi(\tilde{x}(qT_5^{'})/2) > x_2$. Define

$$c = \min\left(\frac{C_0}{4} q^{\frac{\beta}{1-\beta}}, \frac{\Phi(\tilde{x}(qT_6)/2) - x_2}{2T_6(1-q)}, \frac{\Phi(\tilde{x}(qT_6)/2)}{2T_6}\right) \tag{3.5.16}$$

and

$$d = cT_6 - \Phi(\tilde{x}(qT_6)/2). \tag{3.5.17}$$

Finally, define $x_0 = cqT_6 - d = \Phi(\tilde{x}(qT_6)/2) - cT_6(1-q) > x_2$. Note that $d < 0$ due to (3.5.16). Therefore $1/q - 1 > 0$ and for any $x \geq x_0$, $x/q + (1/q - 1)\, d < x/q$. Hence

$$\frac{(\phi \circ \Phi^{-1})\left(\frac{x}{q} + \left(\frac{1}{q} - 1\right) d\right)}{(\phi \circ \Phi^{-1})(x)} \leq \frac{(\phi \circ \Phi^{-1})\left(\frac{x}{q}\right)}{(\phi \circ \Phi^{-1})(x)} < 2\left(\frac{1}{q}\right)^{\frac{\beta}{1-\beta}}, \quad x \geq x_0. \tag{3.5.18}$$

Letting $t = (x+d)/cq$ in (3.5.18) and noting that (3.5.16) implies $C_0/c \geq 4\,(1/q)^{\beta/(1-\beta)}$, we have

$$\frac{(\phi \circ \Phi^{-1})(ct - d)}{(\phi \circ \Phi^{-1})(cqt - d)} < 2\left(\frac{1}{q}\right)^{\frac{\beta}{1-\beta}} < \frac{C_0}{c}, \quad \text{for each } t \geq T_6. \tag{3.5.19}$$

Define the lower comparison solution, x_-, by

$$x_-(t) = \Phi^{-1}(ct - d), \quad t \geq qT_6. \tag{3.5.20}$$

Owing the monotonicity of Φ^{-1}, $\tilde{x}$ and x_-, and the identity (3.5.17),

$$x_-(t) \leq x_-(T_6) = \Phi^{-1}(cT_6 - d) = \frac{\tilde{x}(qT_6)}{2} < \tilde{x}(t), \quad \text{for each } t \in [qT_6, T_6].$$

Hence

$$x_-(t) < \tilde{x}(t), \quad t \in [qT_6, T_6]. \tag{3.5.21}$$

Since $\Phi(x_-(t)) = ct - d$,

$$x_-'(t) = c\,\left(\phi \circ \Phi^{-1}\right)(ct - d) = c\,\phi(x_-(t)) = \frac{c}{C_0}\frac{\phi(x_-(t))}{\phi(x_-(qt))}\, C_0\, \phi(x_-(qt)), \quad t > T_6.$$

By (3.5.19)

$$\frac{c}{C_0}\frac{\phi(x_-(t))}{\phi(x_-(qt))} = \frac{c}{C_0}\frac{(\phi \circ \Phi^{-1})(ct - d)}{(\phi \circ \Phi^{-1})(cqt - d)} < \frac{c}{C_0}\frac{C_0}{c} = 1, \quad \text{for each } t > T_6.$$

Thus

$$x_-'(t) < C_0\, \phi(x_-(qt)), \quad \text{for each } t > T_6. \tag{3.5.22}$$

Recall that $D_-\tilde{x}(t) \geq C_0\, \phi(\tilde{x}(qt))$ for each $t \geq T_6 > T_5$, by (3.5.15). We further claim that $\tilde{x}(t) > x_-(t)$ for each $t \geq T_6$. To see this, let

$$Z = \{t \geq T_6 : \tilde{x}(t) \leq x_-(t)\}.$$

If the claim if false, then Z is nonempty, and by continuity of $\tilde{x}$ and x_-

$$T_6 < \inf Z =: t_1 < \infty.$$

Furthermore, $x_-(t_1) = \tilde{x}(t_1)$ and $x_-(t) < \tilde{x}(t)$ for each $t \in [T_6, t_1)$. Thus,for each $h < 0$ sufficiently small, $x_-(t_1 + h) < \tilde{x}(t_1 + h)$ and hence

$$\frac{x_-(t_1 + h) - x_-(t_1)}{h} > \frac{\tilde{x}(t_1 + h) - \tilde{x}(t_1)}{h}.$$

Now take the liminf as $h \to 0^-$ to show that

$$x_-'(t_1) = D_- x_-(t_1) \geq D_-\tilde{x}(t_1). \tag{3.5.23}$$

However, by (3.5.15) and (3.5.22),

$$x_-'(t_1) = D_- x_-(t_1) < C_0\, \phi(x_-(qt_1)) = C_0\, \phi(\tilde{x}(qt_1)) \le D_- \tilde{x}(t_1),$$

which is in contradiction with equation (3.5.23). Therefore Z is empty and

$$\tilde{x}(t) > x_-(t) = \Phi^{-1}(ct-d), \quad \text{for each } t \ge qT_6.$$

Hence

$$x(\alpha(t)) = \tilde{x}(t) > \Phi^{-1}(ct-d), \quad \text{for each } t \ge qT_6.$$

From the definition of α, in (3.5.11), $\alpha^{-1}(t) = \bar{M}_1(t-T_3)$ and therefore

$$x(t) = \tilde{x}(\alpha^{-1}(t)) > \Phi^{-1}(c\,\alpha^{-1}(t)-d) = \Phi^{-1}(c\,\bar{M}_1(t-T_3)-d), \quad \bar{M}_1(t-T_3) > qT_6.$$

Hence, recalling that $d<0$,

$$\Phi(x(t)) > c\bar{M}_1(t-T_3) - d > c\bar{M}_1(t-T_3), \quad \bar{M}_1(t-T_3) > qT_6. \tag{3.5.24}$$

Note that for $t > 2T_3$, $t/2 < t-T_3$. Since $\bar{M}_1$ is increasing this implies that $\bar{M}_1(t/2) \le \bar{M}_1(t-T_3)$. Thus (3.5.24) implies

$$\liminf_{t\to\infty} \frac{\Phi(x(t))}{\bar{M}_1(t)} \ge \liminf_{t\to\infty} \frac{c\,\bar{M}_1\left(\frac{t}{2}\right)}{\bar{M}_1(t)} = c\,2^{-(\theta+1)} > 0.$$

By Karamata's Theorem $\lim_{t\to\infty} \bar{M}_1(t)/tM_1(t) = 1/(1+\theta)$. Therefore

$$\liminf_{t\to\infty} \frac{\Phi(x(t))}{t\,M_1(t)} \ge c\,(1+\theta)\,2^{-(\theta+1)} > 0.$$

Finally, since $\Phi^{-1} \in \mathrm{RV}_\infty(1/(1-\beta))$ and M is asymptotic to M_1, we conclude that

$$\liminf_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} > 0,$$

as required. □

Lemma 3.5.3. *If the hypotheses of Lemma 3.5.2 hold, then solutions of* (3.1.1) *obey*

$$\limsup_{t\to\infty} \frac{F(x(t))}{t\,M(t)} \le \frac{1}{1-\beta} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right).$$

Proof. Once again let ϕ satisfying (3.5.1) obey $f(x)/\phi(x) < (1+\epsilon)$ for all $x > x_1(\epsilon)$, for any $\epsilon > 0$ and for some $x_1(\epsilon) > 0$. Owing to the fact that $\lim_{t\to\infty} x(t) = \infty$ there exists $T_1(\epsilon)$ such that $t \ge T_1(\epsilon)$ implies $x(t) > x_1(\epsilon)$. Since $\lim_{t\to\infty} M(t) = \infty$ there exists $T_2(\epsilon)$ such that $M(t) > 0$ for all $t \ge T_2$. Hence, for all $t \ge 2\max(T_1, T_2)$, (3.1.3) becomes

$$\frac{x(t)}{\phi(x(t))} \le \frac{x(0)}{\phi(x(t))} + \frac{\int_0^{T_1} M(t-s) f(x(s))\,ds}{\phi(x(t))} + (1+\epsilon)\,t\,M(t), \tag{3.5.25}$$

where the upper bound on the term $\int_{T_1}^t M(t-s)\phi(x(s))\,ds$ was obtained by exploiting the fact that $t \mapsto x(t)$ and $t \mapsto M(t)$ are non-decreasing. By Karamata's Theorem and the regular variation of ϕ, it is true that $\lim_{x\to\infty}(1-\beta)\phi(x)\Phi(x)/x = 1$. Thus for all $\epsilon > 0$ there exists $x_2(\epsilon)$ such that

$$\Phi(x) < \frac{(1+\epsilon)x}{(1-\beta)\phi(x)}, \quad \text{for all } x > x_2(\epsilon).$$

Once more the divergence of $x(t)$ yields the existence of a $T_3(\epsilon)$ such that $x(t) > x_2(\epsilon)$ for all $t \geq T_3(\epsilon)$. Letting $T_4 = 2\max(T_1, T_2, T_3)$ we obtain

$$\frac{\Phi(x(t))}{t\,M(t)} < \frac{(1+\epsilon)x(t)}{(1-\beta)\phi(x(t))\,t\,M(t)}, \quad \text{for all } t \geq T_4.$$

Combining the above estimate with (3.5.25) yields

$$\frac{\Phi(x(t))}{t\,M(t)} < \frac{(1+\epsilon)x(0)}{(1-\beta)\phi(x(t))\,t\,M(t)} + \frac{(1+\epsilon)\int_0^{T_1} M(t-s)f(x(s))\,ds}{(1-\beta)\phi(x(t))\,t\,M(t)} + \frac{(1+\epsilon)^2}{(1-\beta)}, \quad t \geq T_4(\epsilon).$$

Hence, letting $t \to \infty$ and then sending $\epsilon \to 0^+$, we get

$$\limsup_{t\to\infty} \frac{\Phi(x(t))}{t\,M(t)} \leq \frac{1}{1-\beta}.$$

Since $\Phi^{-1} \in \text{RV}_\infty\left(1/(1-\beta)\right)$ the above estimate can be restated as

$$\limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} \leq (1-\beta)^{\frac{1}{\beta-1}} < \infty.$$

We now seek to refine the "crude" upper bound on the growth of the solution obtained above. From the above construction and Lemma 3.5.2 we may suppose

$$\limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} =: \eta \in (0,\infty). \tag{3.5.26}$$

From (3.5.26) it follows that for all $\epsilon > 0$ there exists $T_5(\epsilon) > 0$ such that for all $t \geq T_5(\epsilon)$, $x(t) < (\eta+\epsilon)\Phi^{-1}(t\,M(t))$. By monotonicity of ϕ it follows that

$$\frac{\phi(x(t))}{\phi\left(\Phi^{-1}(t\,M(t))\right)} < \frac{\phi\left((\eta+\epsilon)\Phi^{-1}(t\,M(t))\right)}{\phi\left(\Phi^{-1}(t\,M(t))\right)}, \quad t \geq T_5(\epsilon).$$

Since $\phi \in \text{RV}_\infty(\beta)$

$$\limsup_{t\to\infty} \frac{\phi(x(t))}{\phi\left(\Phi^{-1}(t\,M(t))\right)} \leq (\eta+\epsilon)^\beta.$$

Thus for all $\epsilon > 0$ there exists $T_6(\epsilon) > 0$ such that for all $t \geq T_6$,

$$\phi(x(t)) < (1+\epsilon)(\eta+\epsilon)^\beta \phi\left(\Phi^{-1}(t\,M(t))\right).$$

Integrating the previous estimate yields

$$\int_{T_6}^t M(t-s)\phi(x(s))\,ds \leq (1+\epsilon)(\eta+\epsilon)^\beta \int_{T_6}^t M(t-s)\phi\left(\Phi^{-1}(s\,M(s))\right)ds, \tag{3.5.27}$$

for $t \geq T_6(\epsilon)$. Since $(\phi \circ \Phi^{-1})(t\,M(t)) \in \text{RV}_\infty\left(\beta(1+\theta)/(1-\beta)\right)$ and $M \in \text{RV}_\infty(\theta)$, Lemma 3.5.1 can be applied to obtain

$$\lim_{t\to\infty} \frac{\int_0^t M(t-s)\phi\left(\Phi^{-1}(s\,M(s))\right)ds}{t\,M(t)\,\phi\left(\Phi^{-1}(t\,M(t))\right)} = B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right). \tag{3.5.28}$$

Combining (3.5.27) and (3.5.28) yields

$$\limsup_{t\to\infty} \frac{\int_{T_6}^t M(t-s)\phi(x(s))\,ds}{t\,M(t)\,\phi\left(\Phi^{-1}(t\,M(t))\right)} \le (1+\epsilon)(\eta+\epsilon)^\beta B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right).$$

Apply the estimate above to (3.1.3) as follows:

$$\begin{aligned}
\eta &= \limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} \\
&\le \limsup_{t\to\infty} \frac{\int_0^{T_6} M(t-s)f(x(s))\,ds}{\Phi^{-1}(t\,M(t))} + \limsup_{t\to\infty} \frac{(1+\epsilon)\int_{T_6}^t M(t-s)\phi(x(s))\,ds}{\Phi^{-1}(t\,M(t))} \\
&\le (1+\epsilon)^2(\eta+\epsilon)^\beta B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right) \limsup_{t\to\infty} \frac{t\,M(t)\,\phi\left(\Phi^{-1}(t\,M(t))\right)}{\Phi^{-1}(t\,M(t))} \\
&= (1+\epsilon)^2(\eta+\epsilon)^\beta B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right) \limsup_{x\to\infty} \frac{x\,\phi\left(\Phi^{-1}(x)\right)}{\Phi^{-1}(x)}.
\end{aligned}$$

Letting $\epsilon \to 0^+$ and using Karamata's Theorem to the remaining limit on the right–hand side yields

$$\eta^{1-\beta} = \limsup_{y\to\infty} \frac{\Phi(y)\phi(y)}{y} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right) = \frac{1}{1-\beta} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right),$$

with $y = \Phi^{-1}(x)$ so that $y \to \infty$ as $x \to \infty$. Thus

$$\eta = \limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} \le \left\{\frac{1}{1-\beta} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right)\right\}^{\frac{1}{1-\beta}}.$$

Since $\Phi \in \mathrm{RV}_\infty(1-\beta)$ and $\Phi(x) \sim F(x)$ as $x \to \infty$, the above upper bound can be reformulated as

$$\limsup_{t\to\infty} \frac{F(x(t))}{t\,M(t)} \le \frac{1}{1-\beta} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right),$$

which is the required estimate. $\square$

Lemma 3.5.4. *If the hypotheses of Lemma 3.5.2 hold, then solutions of (3.1.1) obey*

$$\liminf_{t\to\infty} \frac{F(x(t))}{t\,M(t)} \ge \frac{1}{1-\beta} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right).$$

Proof. By Lemma 3.5.2 and Lemma 3.5.3

$$\liminf_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} =: \eta \in (0,\infty).$$

Then for all $\epsilon \in (0,\eta)\cap(0,1)$ there exists $T_1(\epsilon) > 0$ such that for all $t \ge T_1$ $\eta - \epsilon < x(t)/\Phi^{-1}(t\,M(t))$. Since $\lim_{t\to\infty} M(t) = \infty$ there exists T_2 such that $M(t) > 0$ for all $t \ge T_2$. Hence

$$x(t) > (\eta-\epsilon)\Phi^{-1}\left(t\,M(t)\right), \quad t \ge T_3 := \max(T_1, T_2). \tag{3.5.29}$$

Using monotonicity and regular variation of ϕ it follows from (3.5.29) that

$$\liminf_{t\to\infty} \frac{\phi(x(t))}{(\phi\circ\Phi^{-1})(t\,M(t))} \ge (\eta-\epsilon)^\beta.$$

Now, because $\phi(x) \sim f(x)$ as $x \to \infty$, for all $\epsilon \in (0,\eta)\cap(0,1)$ there exists $T_4(\epsilon) > 0$ such that

$$f(x(t)) > (1-\epsilon)\phi(x(t)) > (1-\epsilon)^2(\eta-\epsilon)^\beta\,(\phi\circ\Phi^{-1})(t\,M(t)), \quad t \ge T_4(\epsilon).$$

Integration then yields

$$\int_0^t M(t-s)f(x(s))\,ds > (1-\epsilon)^2(\eta-\epsilon)^\beta \int_{T_4}^t M(t-s)(\phi\circ\Phi^{-1})(s\,M(s))\,ds.$$

Hence, as in the proof of Lemma 3.5.3, applying Lemma 3.5.1 gives

$$\liminf_{t\to\infty}\frac{\int_0^t M(t-s)f(x(s))\,ds}{t\,M(t)\,(\phi\circ\Phi^{-1})(t\,M(t))} \geq (1-\epsilon)^2(\eta-\epsilon)^\beta\, B\left(\theta+1,\frac{\theta\beta+1}{1-\beta}\right). \tag{3.5.30}$$

Now apply the estimate from (3.5.30) to (3.1.3) as follows

$$\begin{aligned}\eta &= \liminf_{t\to\infty}\frac{x(t)}{\Phi^{-1}(t\,M(t))} \geq \liminf_{t\to\infty}\frac{\int_0^t M(t-s)f(x(s))\,ds}{\Phi^{-1}(t\,M(t))}\\ &= (1-\epsilon)^2(\eta-\epsilon)^\beta\, B\left(\theta+1,\frac{\theta\beta+1}{1-\beta}\right)\liminf_{t\to\infty}\frac{t\,M(t)\,(\phi\circ\Phi^{-1})(t\,M(t))}{\Phi^{-1}(t\,M(t))}\\ &= (1-\epsilon)^2(\eta-\epsilon)^\beta\, B\left(\theta+1,\frac{\theta\beta+1}{1-\beta}\right)\liminf_{x\to\infty}\frac{x\,\phi\left(\Phi^{-1}(x)\right)}{\Phi^{-1}(x)}.\end{aligned}$$

The limit of the final term on the right-hand side is $1/(1-\beta)$ by Karamata's Theorem and sending $\epsilon\to 0^+$ yields

$$\eta = \frac{\eta^\beta}{1-\beta}B\left(\theta+1,\frac{\theta\beta+1}{1-\beta}\right).$$

Hence

$$\liminf_{t\to\infty}\frac{x(t)}{F^{-1}(t\,M(t))} \geq \left\{\frac{1}{1-\beta}B\left(\theta+1,\frac{\theta\beta+1}{1-\beta}\right)\right\}^{\frac{1}{1-\beta}}.$$

Since $F\in \mathrm{RV}_\infty(1-\beta)$, this can be rewritten in the form

$$\liminf_{t\to\infty}\frac{F(x(t))}{t\,M(t)} \geq \frac{1}{1-\beta}B\left(\theta+1,\frac{\theta\beta+1}{1-\beta}\right),$$

which is the desired bound. □

As with Theorem 3.2.1, the proof of Theorem 3.3.1 is split into a series of lemmata. A final consolidating argument then establishes the result as stated in Section 3.2.

Lemma 3.5.5. *Suppose μ obeys (3.2.1) with $M\in RV_\infty(\theta)$, $\theta\geq 0$ and $f\in RV_\infty(\beta)$, $\beta\in[0,1)$. If $\beta=0$, let f be asymptotically increasing and obey $\lim_{x\to\infty}f(x)=\infty$. If x denotes a solution to (3.1.4) and $H\in C((0,\infty);(0,\infty))$, then*

$$\liminf_{t\to\infty}\frac{x(t)}{F^{-1}(t\,M(t))} \geq L := \left\{\frac{1}{1-\beta}B\left(1+\theta,\frac{1+\theta\beta}{1-\beta}\right)\right\}^{\frac{1}{1-\beta}} > 0. \tag{3.5.31}$$

Proof. With $\epsilon\in(0,1)$ arbitrary and $T_0(\epsilon)$ and $T_1(\epsilon)$ defined as in Lemma 3.5.2, (3.1.4) admits the initial lower estimate

$$x(t) > x(0) + H(t) + (1-\epsilon)\int_T^t M(t-s)\phi(x(s))\,ds, \quad t\geq T(\epsilon) := T_0(\epsilon)+T_1(\epsilon).$$

Letting $y(t) = x(t+T)$ and noting that $H(t) > 0$ for $t > 0$ we get

$$\begin{aligned} y(t) &> x(0) + (1-\epsilon)\int_T^{t+T} M(t+T-s)\phi(x(s))\,ds \\ &= x(0) + (1-\epsilon)\int_0^t M(t-u)\phi(x(u+T))\,du \\ &= x(0) + (1-\epsilon)\int_0^t M(t-u)\phi(y(u))\,du, \quad t \geq T(\epsilon). \end{aligned}$$

Now consider the comparison equation defined by

$$x_\epsilon'(t) = (1-\epsilon)\int_{[0,t]} \mu(ds)\phi(x_\epsilon(t-s)), \quad t > 0, \quad x_\epsilon(0) = x(0)/2. \tag{3.5.32}$$

In contrast to the solution to (3.1.4), the solution to (3.5.32) is nondecreasing. Integrating (3.5.32) using Fubini's Theorem yields

$$x_\epsilon(t) = \frac{x(0)}{2} + (1-\epsilon)\int_0^t M(t-u)\phi(x_\epsilon(u))du, \quad t \geq 0.$$

By construction $x_\epsilon(t) < y(t) = x(t+T)$ for all $t \geq 0$, or $x(t) > x_\epsilon(t-T)$ for all $t \geq T$. Applying Theorem 3.2.1 to x_ϵ then yields

$$\lim_{t\to\infty} \frac{F(x_\epsilon(t))}{t\,M_\epsilon(t)} = \frac{1}{1-\beta}B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right),$$

where $M_\epsilon(t) = (1-\epsilon)M(t)$. Hence

$$\lim_{t\to\infty} \frac{F(x_\epsilon(t))}{t\,M(t)} = \frac{1-\epsilon}{1-\beta}B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right).$$

Therefore

$$\begin{aligned} \liminf_{t\to\infty} \frac{F(x(t))}{t\,M(t)} &\geq \liminf_{t\to\infty} \frac{F(x_\epsilon(t-T))}{t\,M(t)} = \liminf_{t\to\infty} \frac{F(x_\epsilon(t-T))}{(t-T)M(t-T)} \frac{(t-T)M(t-T)}{t\,M(t)} \\ &= \frac{1-\epsilon}{1-\beta}B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right), \end{aligned}$$

where the final equality follows from the trivial fact that $t - T \sim t$ as $t \to \infty$ and noting that M preserves asymptotic equivalence because $M \in \text{RV}_\infty(\theta)$. Finally, letting $\epsilon \to 0^+$ and using the regular variation of F^{-1} yields

$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} \geq \left\{\frac{1}{1-\beta}B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right)\right\}^{\frac{1}{1-\beta}} = L,$$

which finishes the proof. $\square$

Lemma 3.5.6. *If the hypotheses of Lemma 3.5.5 hold and*

$$\lim_{t\to\infty} \frac{H(t)}{F^{-1}(t\,M(t))} = \lambda \in [0,\infty).$$

then solutions of (3.1.4) *obey*

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} \leq U := \left(\frac{\lambda}{L^\beta} + \frac{1}{1-\beta}\right)^{\frac{1}{1-\beta}}, \tag{3.5.33}$$

where L is defined by (3.5.31).

Proof. We begin by constructing a monotone comparison solution which will majorise the solution of (3.1.4) and to which Lemma 3.5.5 can be applied. Let $\epsilon \in (0,1)$ be arbitrary and define $T_1(\epsilon)$ and $T_2(\epsilon)$ as in the proof of Lemma 3.5.3.

By hypothesis $\lim_{t\to\infty} H(t)/F^{-1}(t\,M(t)) = \lambda \in [0,\infty)$ and so there exists a $T(\epsilon) > 0$ such that $t \geq T(\epsilon)$ implies $H(t) < (\lambda+\epsilon)\Phi^{-1}(t\,M(t))$. $M \in \text{RV}_\infty(\theta)$ implies there exists $M_1 \in C^1$ asymptotic to M and $T_0(\epsilon) > T$ such that $M(t) < (1+\epsilon)M_1(t)$ for all $t \geq T_0$. For $t \geq T_0$, because Φ^{-1} is increasing, $\Phi^{-1}(t\,M(t)) < \Phi^{-1}(t\,(1+\epsilon)\,M_1(t))$ and since $\Phi^{-1} \in \text{RV}_\infty(1/(1-\beta))$ there exists $T^* > T_0$ such that $\Phi^{-1}(t\,M(t)) < (1+\epsilon)^{(2-\beta)/(1-\beta)}\Phi^{-1}(t\,M_1(t))$ for all $t \geq T^*$.

Let $\epsilon^* = (1+\epsilon)^{(2-\beta)/(1-\beta)} - 1$ and note that $(1+\epsilon^*) \to 1$ as $\epsilon \to 0^+$. Define $T_2' = T^* + T_1 + T_2$ and estimate as follows:

$$\begin{aligned}
x(t) &< x(0) + H(t) + \int_0^{T_2'} M(t-s) f(x(s))\,ds + (1+\epsilon)\int_{T_2'}^t M(t-s)\phi(x(s))\,ds \\
&\leq x(0) + H(t) + M(t)\,T_2'\,F^* + (1+\epsilon)\int_{T_2'}^t M(t-s)\phi(x(s))\,ds \\
&< x(0) + (\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M_1(t)) + (1+\epsilon)M_1(t)\,T_2'\,F^* \\
&\quad + (1+\epsilon)\int_{T_2'}^t M(t-s)\phi(x(s))\,ds, \quad t \geq T_2',
\end{aligned} \tag{3.5.34}$$

where $F^* := \max_{0\leq s\leq T_2'} f(x(s))$. Next define the constant $x^* = \max_{0\leq s\leq T_2'} x(s)$ and the function

$$\bar{H}(t) = (\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M_1(t)) + (1+\epsilon)M_1(t)\,T_2'\,F^* - (\lambda+\epsilon)(1+\epsilon^*), \quad t \geq 0.$$

Since $\Phi^{-1}(0) = 1$ and $M_1(0) = 0$, $H(0) = 0$ and $H \in C^1((0,\infty);(0,\infty))$. The initial upper estimate (3.5.34) motivates the definition of the following upper comparison equation:

$$y_\epsilon'(t) = \bar{H}'(t) + (1+\epsilon)\int_{[0,t]} \mu(ds)\phi(y_\epsilon(t-s))\,ds,\ t \geq 0,\ y_\epsilon(0) = x(0) + x^* + (\lambda+\epsilon)(1+\epsilon^*).$$

Integration using Fubini's theorem quickly shows that

$$y_\epsilon(t) = x(0) + x^* + (\lambda+\epsilon)(1+\epsilon^*) + \bar{H}(t) + (1+\epsilon)\int_0^t M(t-s)\phi(y_\epsilon(s))\,ds, \quad t \geq 0.$$

Since $y_\epsilon(t)$ is nondecreasing it is immediately clear that $x(t) < y_\epsilon(t)$ for all $t \in [0, T_2']$. A simple time of the first breakdown argument using the estimate (3.5.34) then shows that $x(t) < y_\epsilon(t)$ for all $t \geq 0$. We now compute an explicit upper bound on $\limsup_{t\to\infty} y_\epsilon(t)/F^{-1}(t\,M(t))$. By monotonicity,

$$y_\epsilon(t) \leq x(0) + x^* + (\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M_1(t)) + (1+\epsilon)M_1(t)\,T_2'\,F^* + (1+\epsilon)M(t)\,t\,\phi(y_\epsilon(t)), \quad t \geq 0.$$

Hence, with $C(t)$ suitably defined,

$$\frac{y_\epsilon(t)}{t\,M(t)\,\phi(y_\epsilon(t))} \leq C(t) + \frac{(\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M_1(t))}{t\,M(t)\,\phi(y_\epsilon(t))} + (1+\epsilon), \quad t \geq 0.$$

A short calculation reveals that $\lim_{t\to\infty} C(t) = 0$. By Karamata's Theorem there exists a $T_3(\epsilon)$ such that

$$\frac{\Phi(y_\epsilon(t))}{t\,M(t)} < \frac{(1+\epsilon)C(t)}{1-\beta} + \frac{(1+\epsilon)(\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M_1(t))}{(1-\beta)\,t\,M(t)\,\phi(y_\epsilon(t))} + \frac{(1+\epsilon)^2}{1-\beta}, \tag{3.5.35}$$

for $t \geq T_4 := T_3 + T_2'$. By applying Lemma 3.5.5 to y_ϵ we conclude that

$$\liminf_{t\to\infty} \frac{y_\epsilon(t)}{\Phi^{-1}(t\,M(t))} =: L \in (0, \infty].$$

If $L \in (0, \infty)$ then there exists a $T_5(\epsilon)$ such that for all $t \geq T_6 := T_5 + T_4$

$$\begin{aligned} \frac{\Phi(y_\epsilon(t))}{t\,M(t)} &< \frac{(1+\epsilon)C(t)}{1-\beta} + \frac{(1+\epsilon)(\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M(t))}{(1-\beta)\,t\,M(t)\,\phi((1-\epsilon)\,L\,\Phi^{-1}(t\,M(t)))} + \frac{(1+\epsilon)^2}{1-\beta} \\ &< \frac{(1+\epsilon)C(t)}{1-\beta} + \frac{(1+\epsilon)(\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M(t))}{(1-\beta)\,t\,M(t)\,(1-\epsilon)^\beta\,L^\beta\,\phi(\Phi^{-1}(t\,M(t)))} + \frac{(1+\epsilon)^2}{1-\beta}. \end{aligned} \tag{3.5.36}$$

By Karamata's Theorem the following asymptotic equivalence holds

$$(1-\beta)\,t\,M(t)\,\phi\left(\Phi^{-1}(t\,M(t))\right) \sim \Phi^{-1}(t\,M(t)) \text{ as } t \to \infty.$$

Therefore taking the limit superior across (3.5.36) yields

$$\limsup_{t\to\infty} \frac{\Phi(y_\epsilon(t))}{t\,M(t)} \leq \frac{(1+\epsilon)(\lambda+\epsilon)(1+\epsilon^*)}{(1-\epsilon)^\beta\,L^\beta} + \frac{(1+\epsilon)^2}{1-\beta}.$$

By letting $\epsilon \to 0^+$ and using the regular variation of Φ^{-1}

$$\limsup_{t\to\infty} \frac{x(t)}{\Phi^1(t\,M(t))} \leq \left(\frac{\lambda}{L^\beta} + \frac{1}{1-\beta}\right)^{\frac{1}{1-\beta}} =: U.$$

If $L = \liminf_{t\to\infty} y_\epsilon(t)/\Phi^{-1}(t\,M(t)) = \infty$, then the construction above will yield

$$\limsup_{t\to\infty} \frac{y_\epsilon(t)}{\Phi^{-1}(t\,M(t))} < \infty,$$

a contradiction. Hence $L \in (0, \infty)$ and the claim is proven. □

Lemma 3.5.7. *Suppose $\beta \in [0, 1)$, $\lambda \in [0, \infty)$ and consider the iterative scheme defined by*

$$x_{n+1} = g(x_n) := \frac{x_n^\beta}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lambda, \quad n \geq 1; \quad x_0 \in [L, C^*], \tag{3.5.37}$$

with L defined by (3.5.31), U defined by (3.5.33) and

$$C^* := \max\left(U,\, L + \frac{\lambda}{1-\beta}\right). \tag{3.5.38}$$

Then there exists a unique $x_\infty \in [L, C^*]$ such that $\lim_{n\to\infty} x_n = x_\infty$.

Proof. By inspection, $g \in C([L, \infty); (0, \infty))$. We calculate as follows

$$g'(x) = \frac{\beta}{1-\beta} x^{\beta-1} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) > 0, \quad x > 0,$$

and similarly

$$g''(x) = -\beta x^{\beta-2} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) < 0, \quad x > 0.$$

Therefore $g'(L) = \beta > g'(x) > 0$ for all $x > L$ and $|g'(x)| \leq \beta < 1$ for all $x \in [L, \infty)$. Since g is increasing it is sufficient check that g maps $[L, C^*]$ to $[L, C^*]$ as follows. Firstly,

$$g(L) = \frac{L^\beta}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lambda = L + \lambda \in [L, C^*]. \tag{3.5.39}$$

By the Mean Value Theorem there exists $\xi \in [L, C^*]$ such that

$$\frac{g(C^*) - g(L)}{C^* - L} = g'(\xi) \le \beta.$$

Therefore $g(C^*) \le \beta(C^* - L) + g(L)$ and thus a sufficient condition for $g(C^*) \le C^*$ is $\beta(C^* - L) + g(L) \le C^*$ or $C^* \ge (g(L) - L\beta)/(1-\beta) = L + \lambda/(1-\beta)$, using (3.5.39). Thus with C^* as defined in (3.5.38), $g : [L, C^*] \to [L, C^*]$. Hence (3.5.37) has a unique fixed point in $[L, C^*]$ and the claim follows. $\square$

We now supply the proof of Theorem 3.3.1, as promised.

Proof of Theorem 3.3.1. Suppose that $(ii.)$ holds, or $\lim_{t\to\infty} H(t)/F^{-1}(t\,M(t)) = \lambda \in [0, \infty)$. The idea here is to combine the crude bounds on the solution from Lemmas 3.5.5 and 3.5.6 with a fixed point argument based on Lemma 3.5.7 to complete the proof that $(ii.)$ implies $(i.)$. We compute $\limsup_{t\to\infty} x(t)/F^{-1}(t\,M(t))$ in detail only as the calculation of the corresponding limit inferior proceeds in an analogous manner. To begin make the following induction hypothesis

$$(H_n) \qquad \limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(Mt)} \le \zeta_n, \quad \zeta_{n+1} := \frac{\zeta_n^\beta}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lambda, \quad n \ge 0,$$

and choose $\zeta_0 := U$. (H_0) is true by Lemma 3.5.6. Suppose that (H_n) holds. Thus there exists $T(\epsilon) > 0$ such that $x(t) < (\zeta_n + \epsilon)\Phi^{-1}(t\,M(t))$ for all $t \ge T$. Hence

$$\frac{\phi(x(t))}{\phi(\Phi^{-1}(t\,M(t)))} < \frac{\phi((\zeta_n + \epsilon)\Phi^{-1}(Mt))}{\phi(\Phi^{-1}(t\,M(t)))}, \ t \ge T.$$

The regular variation of ϕ thus yields $\limsup_{t\to\infty} \phi(x(t))/\phi(\Phi^{-1}(t\,M(t))) \le (\zeta_n + \epsilon)^\beta$. Therefore there exists a $T_2(\epsilon) > 0$ such that $t \ge T_2$ implies $f(x(t)) < (1+\epsilon)[(\zeta_n+\epsilon)^\beta + \epsilon]\phi(\Phi^{-1}(t\,M(t)))$. From (3.1.6)

$$\limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} = \limsup_{t\to\infty} \frac{\int_0^t M(t-s) f(x(s))ds}{\Phi^{-1}(t\,M(t))} + \lim_{t\to\infty} \frac{H(t)}{\Phi^{-1}(t\,M(t))}.$$

Using the upper bound derived from our induction hypothesis this becomes

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} \le (1+\epsilon)[(\zeta_n+\epsilon)^\beta + \epsilon] \limsup_{t\to\infty} \frac{\int_{T_2}^t M(t-s)\phi(\Phi^{-1}(s\,M(s)))}{\Phi^{-1}(t\,M(t))} + \lambda.$$

Applying Karamata's Theorem and Lemma 3.5.1

$$\begin{aligned}\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} &\le (1+\epsilon)[(\zeta_n+\epsilon)^\beta + \epsilon] \limsup_{t\to\infty} \frac{\int_{T_2}^t M(t-s)\phi(\Phi^{-1}(s\,M(s)))}{(1-\beta)t\,M(t)\,\phi(\Phi^{-1}(t\,M(t)))} \\ &\quad + \lambda \\ &= \frac{(1+\epsilon)[(\zeta_n+\epsilon)^\beta + \epsilon]}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lambda.\end{aligned}$$

Letting $\epsilon \to 0^+$ yields

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} \le \frac{\zeta^\beta}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lambda = \zeta_{n+1},$$

proving the induction hypothesis (H_{n+1}). Hence (H_n) holds for all n, or Hence

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} \leq \zeta_n, \text{ for all } n \geq 0.$$

By Lemma 3.5.7, $\lim_{n\to\infty} \zeta_n = \zeta$, where ζ is the unique solution in $[L, U]$ of the "characteristic" equation (3.3.2). Thus

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} \leq \zeta.$$

In the case of the corresponding limit inferior the only modification is to the induction hypothesis, take $\zeta_0 := L$, and the argument proceeds as above to yield $\liminf_{t\to\infty} x(t)/F^{-1}(Mt) \geq \zeta$, completing the proof.

Now suppose that $(i.)$ holds, or that $\lim_{t\to\infty} x(t)/F^{-1}(t\,M(t)) = \zeta \in [L, \infty)$. It follows that there exists $T_3(\epsilon) > 0$ such that for all $t \geq T_3$,

$$\phi((\zeta-\epsilon)\Phi^{-1}(t\,M(t))) < \phi(x(t)) < \phi((\zeta+\epsilon)\Phi^{-1}(t\,M(t))).$$

Hence for $t \geq T_3$

$$\begin{aligned}\int_{T_3}^t M(t-s)\phi((\zeta-\epsilon)\Phi^{-1}(s\,M(s)))ds &\leq \int_{T_3}^t M(t-s)\phi(x(s))ds \\ &\leq \int_{T_3}^t M(t-s)\phi((\zeta+\epsilon)\Phi^{-1}(s\,M(s)))ds.\end{aligned}$$

Using the regular variation of ϕ the above estimate can be reformulated as

$$\begin{aligned}\frac{(\zeta-\epsilon)^\beta \int_{T_3}^t M(t-s)\phi(\Phi^{-1}(s\,M(s)))\,ds}{\Phi^{-1}(t\,M(t))} &\leq \frac{\int_{T_3}^t M(t-s)\phi(x(s))\,ds}{\Phi^{-1}(t\,M(t))} \\ &\leq (\zeta+\epsilon)^\beta \frac{\int_{T_3}^t M(t-s)\phi((\zeta+\epsilon)\Phi^{-1}(s\,M(s)))\,ds}{\Phi^{-1}(t\,M(t))}, \quad t \geq T_3.\end{aligned}$$

Using Lemma 3.5.1 and letting $\epsilon \to 0^+$ thus yields

$$\lim_{t\to\infty} \frac{\int_0^t M(t-s)\phi(x(s))ds}{\Phi^{-1}(t\,M(t))} = \frac{\zeta^\beta}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right).$$

Therefore assuming $(i.)$ and taking the limit across (3.1.6) we obtain

$$\zeta = \frac{\zeta^\beta}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lim_{t\to\infty} \frac{H(t)}{\Phi^{-1}(t\,M(t))},$$

as claimed. ☐

We now give the proof of Theorem 3.3.2 in which the perturbation is large. The reader will note that this proof makes much less use of properties of regular varying functions: in fact, we establish the asymptotic result by observing that a key functional of the solution is well approximated by a linear non–autonomous differential inequality. Indeed, this line of argument will be used in a more general setting, and in the presence of stochastic perturbations, in Chapter 4.

Proof of Theorem 3.3.2. As always $\epsilon \in (0, 1)$ is arbitrary. From (3.5.1) there exists a ϕ such that

$$\lim_{x\to\infty} f(x)/\phi(x) = 1, \quad \lim_{x\to\infty} x\,\phi'(x)/\phi(x) = \beta$$

(see e.g., [27, Theorem 1.3.3]). Therefore there exists $x_1(\epsilon) > 0$ such that $f(x) < (1+\epsilon)\phi(x)$ for all

$x \geq x_1(\epsilon)$ and $x_0(\epsilon)$ such that $\phi'(x) < (\beta+\epsilon)\phi(x)/x$ for all $x \geq x_0(\epsilon)$. Similarly, since $\lim_{t\to\infty} x(t) = \infty$, there exists $T_1(\epsilon) > 0$ such that $x(t) > \max(x_0(\epsilon), x_1(\epsilon))$ for all $t \geq T_1(\epsilon)$. The regular variation of M means that there exists a non-decreasing function $M_1 \in C^1$ and $T_2(\epsilon) > 0$ such that $(1-\epsilon)M_1(t) < M(t) < (1+\epsilon)M_1(t)$ for all $t \geq T_2(\epsilon)$. Hence

$$(1-\epsilon)M_1(t) < \max_{T_2 \leq s \leq t} M(s) < (1+\epsilon)M_1(t), \quad t \geq T_2.$$

Thus for $t \geq T_2$

$$\begin{aligned}(1-\epsilon)M_1(t) < \max_{0\leq s\leq t} M(s) < \max\left(\max_{0\leq s\leq T_2} M(s), \max_{T_2\leq s\leq t} M(s)\right)\\ \leq \max\left(\max_{0\leq s\leq T_2} M(s), (1+\epsilon)M_1(t)\right).\end{aligned}$$

Therefore

$$1-\epsilon \leq \frac{\max_{0\leq s\leq t} M(s)}{M_1(t)} \leq \max\left(\frac{\max_{0\leq s\leq T_2} M(s)}{M_1(t)}, 1+\epsilon\right),$$

and because $\lim_{t\to\infty} M_1(t) = \infty$ we conclude that $\lim_{t\to\infty} \max_{0\leq s\leq t} M(s)/M_1(t) = 1$. It follows that there exists a $T_3(\epsilon) > 0$ such that $\max_{0\leq s\leq t} M(s) < (1+\epsilon)M_1(t)$ for all $t \geq T_3(\epsilon)$. Now let $T = 1 + \max(T_1, T_2, T_3)$. From (3.1.6), with $t \geq 2T$,

$$\begin{aligned}x(t) &= x(0) + H(t) + \int_0^T M(t-s)f(x(s))ds + \int_T^t M(t-s)f(x(s))\,ds\\ &< x(0) + H(t) + \int_0^T M(t-s)f(x(s))ds + (1+\epsilon)\int_T^t M(t-s)\phi(x(s))\,ds\\ &= x(0) + H(t) + \int_0^T M(t-s)f(x(s))ds + (1+\epsilon)\int_T^{t-T} M(t-s)\phi(x(s))\,ds\\ &+ (1+\epsilon)\int_{t-T}^t M(t-s)\phi(x(s))\,ds.\end{aligned}$$

If $s \in [T, t-T]$, then $t-s \geq T > T_1$, and for $t \geq 2T$

$$\begin{aligned}x(t) &< x(0) + H(t) + \int_0^T M(t-s)f(x(s))\,ds + (1+\epsilon)^2 M_1(t)\int_T^{t-T}\phi(x(s))\,ds\\ &+ (1+\epsilon)\max_{0\leq s\leq T} M(s)\int_{t-T}^t \phi(x(s))\,ds\end{aligned}$$

Now, as $T > T_3(\epsilon)$, $\max_{0\leq s\leq T} M(s) < (1+\epsilon)M_1(T) < (1+\epsilon)M_1(t)$. Hence

$$x(t) < x(0) + H(t) + \int_0^T M(t-s)f(x(s))\,ds + (1+\epsilon)^2 M_1(t)\int_T^t \phi(x(s))\,ds, \quad t \geq 2T.$$

For $t \geq 2T > T$, $\max_{0\leq s\leq T} M(t-s) = \max_{t-T\leq u\leq t} M(u) \leq \max_{0\leq u\leq t} M(u) < (1+\epsilon)M_1(t)$. Thus

$$x(t) < x(0) + H(t) + (1+\epsilon)M_1(t)\int_0^T f(x(s))\,ds + (1+\epsilon)^2 M_1(t)\int_T^t \phi(x(s))\,ds, \quad t \geq 2T. \tag{3.5.40}$$

For $t \in [T, 2T]$, $x(t) \leq \max_{s\in[0,2T]} x(s) := x_1^*(\epsilon)$. Combining this with (3.5.40)

$$x(t) < x_1^*(\epsilon) + H(t) + (1+\epsilon)M_1(t)x_2^*(\epsilon) + (1+\epsilon)^2 M_1(t)\int_T^t \phi(x(s))\,ds, \quad t \geq 2T, \tag{3.5.41}$$

where $x_2^*(\epsilon) := \int_0^T f(x(s))ds$. Define the function H_ϵ by

$$H_\epsilon(t) = x_1^*(\epsilon) + H(t) + (1+\epsilon)M_1(t)x_2^*(\epsilon), \quad t \geq 2T. \tag{3.5.42}$$

Note that by construction $\lim_{t\to\infty} H_\epsilon(t)/H(t) = 1$. Consolidating (3.5.41) and (3.5.42) yields

$$x(t) < H_\epsilon(t) + (1+\epsilon)^2 M_1(t) \int_T^t \phi(x(s))\, ds, \quad t \geq 2T. \tag{3.5.43}$$

Defining

$$I_\epsilon(t) = \int_T^t \phi(x(s))\, ds, \quad t \geq 2T.$$

we can formulate an advantageous auxiliary differential inequality as follows. Since x is continuous and $\phi \in C^1(0,\infty)$, $I'_\epsilon(t) = \phi(x(t))$, $t \geq 2T$. Moreover, $\lim_{t\to\infty} I_\epsilon(t) = \infty$. By (3.5.43),

$$I'_\epsilon(t) = \phi(x(t)) < \phi\left(H_\epsilon(t) + (1+\epsilon)^2 M_1(t) I_\epsilon(t)\right), \quad t \geq 2T. \tag{3.5.44}$$

By the Mean Value Theorem, for each $t \geq 2T$, there exists $\xi_\epsilon(t) \in [0,1]$ such that

$$\phi\left(H_\epsilon(t) + (1+\epsilon)^2 M_1(t) I_\epsilon(t)\right) = \phi(H_\epsilon) + \phi'\left(H_\epsilon(t)\xi_\epsilon(t)(1+\epsilon)^2 M_1(t) I_\epsilon(t)\right)(1+\epsilon)^2 M_1(t) I_\epsilon(t).$$

Let $a_\epsilon(t) = H_\epsilon(t) + \xi_\epsilon(t)(1+\epsilon)^2 M_1(t) I_\epsilon(t)$ for $t \geq 2T$. For $t \geq 2T$,

$$a_\epsilon(t) \geq H_\epsilon(t) > x_1^*(\epsilon) := \max_{s\in[0,2T]} x(s) > x_0(\epsilon).$$

Therefore, with $\psi \in \mathrm{RV}_\infty(\beta - 1)$ a decreasing function asymptotic to $\phi(x)/x$,

$$\phi'(a_\epsilon(t)) < (\beta+\epsilon)\frac{\phi(a_\epsilon(t))}{a_\epsilon(t} < (\beta+\epsilon)(1+\epsilon)\psi(a_\epsilon(t)) < (\beta+\epsilon)(1+\epsilon)\psi(H_\epsilon(t)), \quad t \geq 2T.$$

But since $\psi(x) \sim \phi(x)/x$ we also have $\psi(H_\epsilon(t))/(1+\epsilon) < \phi(H_\epsilon(t))/H_\epsilon(t)$ and hence

$$\phi'(a_\epsilon(t)) < (\beta+\epsilon)(1+\epsilon)^2 \frac{\phi(H_\epsilon(t))}{H_\epsilon(t)}, \quad t \geq 2T.$$

Combining this estimate with (3.5.44) yields

$$I'_\epsilon(t) < \phi(H_\epsilon(t)) + (\beta+\epsilon)(1+\epsilon)^4 \frac{\phi(H_\epsilon(t))}{H_\epsilon(t)} M_1(t) I_\epsilon(t), \quad t \geq 2T.$$

Letting $\alpha_\epsilon(t) = (\beta+\epsilon)(1+\epsilon)^4 M_1(t)\, \phi(H_\epsilon(t))/H_\epsilon(t)$, this becomes

$$I'_\epsilon(t) < \phi(H_\epsilon(t)) + \alpha_\epsilon(t)\, I_\epsilon(t), \quad t \geq 2T.$$

Applying the variation of constants formula yields

$$I_\epsilon(t) \leq e^{\int_T^t \alpha_\epsilon(s)ds} \int_T^t e^{-\int_T^s \alpha_\epsilon(u)du} \phi(H_\epsilon(s))ds, \quad t \geq 2T.$$

We reformulate this as

$$\frac{I_\epsilon(t)}{\int_T^t \phi(H_\epsilon(s))ds} \leq \frac{\int_T^t e^{-\int_T^s \alpha_\epsilon(u)du} \phi(H_\epsilon(s))ds}{e^{-\int_T^t \alpha_\epsilon(s)ds} \int_T^t \phi(H_\epsilon(s))ds} =: \frac{C_\epsilon(t)}{B_\epsilon(t)}, \quad t \geq 2T. \tag{3.5.45}$$

Since $C_\epsilon'(t) = \phi(H_\epsilon(t))e^{-\int_T^t \alpha_\epsilon(u)du} > 0$, $\lim_{t\to\infty} C_\epsilon(t) = C^*(\epsilon) \in (0,\infty)$ or $\lim_{t\to\infty} C_\epsilon(t) = \infty$. Also, for $t \geq 2T$,

$$\begin{aligned} B_\epsilon'(t) &= \phi(H_\epsilon(t))e^{-\int_T^t \alpha_\epsilon(u)du} - \alpha_\epsilon(t)e^{-\int_T^t \alpha_\epsilon(u)du} \int_T^t \phi(H_\epsilon(s))ds \\ &= C_\epsilon'(t) - \frac{\alpha_\epsilon(t)\, C_\epsilon'(t)\, \int_T^t \phi(H_\epsilon(s))ds}{\phi(H_\epsilon(t))} = C_\epsilon'(t)\left\{1 - \frac{\alpha_\epsilon(t)\, \int_T^t \phi(H_\epsilon(s))ds}{\phi(H_\epsilon(t))}\right\}. \end{aligned}$$

Therefore, recalling the definition of $\alpha_\epsilon(t)$ and rearranging,

$$\frac{B_\epsilon'(t)}{C_\epsilon'(t)} = 1 - (\beta+\epsilon)(1+\epsilon)^4 \left(\frac{M_1(t)\int_T^t \phi(H_\epsilon(s))ds}{H_\epsilon(t)}\right), \quad t \geq 2T.$$

Letting $t \to \infty$ and using the hypothesis (3.3.3), and that $H_\epsilon(t) \sim H(t)$ and $M_1(t) \sim M(t)$ as $t \to \infty$, yields $\lim_{t\to\infty} B_\epsilon'(t)/C_\epsilon'(t) = 1$, or equivalently $\lim_{t\to\infty} C_\epsilon'(t)/B_\epsilon'(t) = 1$. Hence there exists T_4 such that $B_\epsilon'(t) > 0$ for each $t \geq T_4$ and thus $\lim_{t\to\infty} B_\epsilon(t) = B^*(\epsilon) \in (0,\infty)$ or $\lim_{t\to\infty} B_\epsilon(t) = \infty$. Furthermore, asymptotic integration shows that $\lim_{t\to\infty} C_\epsilon(t) = \infty$ implies $\lim_{t\to\infty} B_\epsilon(t) = \infty$ and $\lim_{t\to\infty} C_\epsilon(t) = C^*(\epsilon)$ implies $\lim_{t\to\infty} B_\epsilon(t) = B^*(\epsilon)$. Hence,

$$\Lambda(\epsilon) := \lim_{t\to\infty} \frac{C_\epsilon(t)}{B_\epsilon(t)} = \begin{cases} 1, & \lim_{t\to\infty} C_\epsilon(t) = \infty, \\ \frac{C*(\epsilon)}{B^*(\epsilon)}, & \lim_{t\to\infty} C_\epsilon(t) = C^*, \end{cases}$$

where the first limit is calculated using L'Hôpital's rule. Taking the limit superior across equation (3.5.45) then yields

$$\limsup_{t\to\infty} \frac{\int_T^t \phi(x(s))ds}{\int_T^t \phi(H_\epsilon(s))ds} = \limsup_{t\to\infty} \frac{I_\epsilon(t)}{\int_T^t \phi(H_\epsilon(s))ds} \leq \Lambda(\epsilon) \in (0,\infty). \tag{3.5.46}$$

Since $H_\epsilon(t) \sim H(t)$ as $t \to \infty$ and ϕ is increasing we can apply L'Hôpital's rule once more to compute

$$\lim_{t\to\infty} \frac{\int_T^t \phi(H_\epsilon(s))ds}{\int_0^t \phi(H_\epsilon(s))ds} = \lim_{t\to\infty} \frac{\phi(H_\epsilon(t))}{\phi(H(t))} = 1,$$

using that $\phi \in \text{RV}_\infty(\beta)$. A similar argument relying on the divergence of $\phi(x(t))$ and L'Hôpital's rule yields $\int_T^t \phi(x(s))ds \sim \int_0^t \phi(x(s))ds$ as $t \to \infty$. Therefore (3.5.46) is equivalent to

$$\limsup_{t\to\infty} \frac{\int_0^t \phi(x(s))ds}{\int_0^t \phi(H(s))ds} \leq \Lambda(\epsilon) \in (0,\infty). \tag{3.5.47}$$

Therefore there exists a $\Lambda^* \in (0,\infty)$ such that

$$\limsup_{t\to\infty} \frac{\int_0^t \phi(x(s))ds}{\int_0^t \phi(H(s))\, ds} \leq \Lambda^*,$$

with Λ^* independent of ϵ. Thus there exists a $T_6(\epsilon)$ such that

$$\int_0^t \phi(x(s))ds < (\Lambda^* + \epsilon) \int_0^t \phi(H(s))ds, \quad t \geq T_6(\epsilon).$$

Letting $\bar{T} = 1 + \max(2T, T_6)$ we apply this estimate to (3.5.43) as follows

$$\begin{aligned}\frac{x(t)}{H(t)} &< \frac{H_\epsilon(t)}{H(t)} + \frac{(1+\epsilon)^2\, M_1(t)\, \int_T^t \phi(x(s))ds}{H(t)} \\ &< \frac{H_\epsilon(t)}{H(t)} + \frac{(1+\epsilon)^2\, M_1(t)\, (\Lambda^* + \epsilon) \int_0^t \phi(H(s))ds}{H(t)}, \quad t \geq \bar{T}.\end{aligned}$$

Now, since $H_\epsilon(t) \sim H(t)$ as $t \to \infty$ and $M_1 \sim M$, applying (3.3.3) to the above estimate yields $\limsup_{t\to\infty} x(t)/H(t) \leq 1$. By positivity (3.1.6) admits the trivial bound $x(t) > H(t)$ for all $t \geq 0$ and hence $\liminf_{t\to\infty} x(t)/H(t) \geq 1$, completing the proof. □

Chapter 4

Growth and Fluctuation in Perturbed Nonlinear Volterra Equations

4.1 Introduction

Consider the scalar Volterra integro–differential equation

$$x'(t) = \int_{[0,t]} \mu(ds) f(x(t-s)) + h(t), \quad t > 0; \quad x(0) = \psi \in \mathbb{R}. \tag{4.1.1}$$

We concentrate on the behaviour of unbounded but non-explosive solutions to (4.1.1), i.e. $x \in C(\mathbb{R}^+;\mathbb{R})$ but $\limsup_{t\to\infty} |x(t)| = \infty$. As suggested in the title we draw a distinction between when solutions grow, $\lim_{t\to\infty} x(t) = \infty$, and when solutions can be said to fluctuate asymptotically, $\liminf_{t\to\infty} x(t) = -\infty$ and $\limsup_{t\to\infty} x(t) = +\infty$. When solutions grow it is natural to ask at what rate they grow and when they fluctuate to ask if the size of these fluctuations can be captured in an appropriate sense; this chapter investigates these types of questions for equations such as (4.1.1).

Throughout μ is a measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ obeying

$$\mu(E) \geq 0 \text{ for all } E \in \mathcal{B}(\mathbb{R}^+), \quad \mu(\mathbb{R}^+) = M \in (0,\infty). \tag{4.1.2}$$

Define $M(t) = \mu([0,t])$, so that $\lim_{t\to\infty} M(t) = M$ and let $H(t) = \int_{[0,t]} h(s)ds$ for $t \geq 0$. From Chapter 2, we have the following sufficient condition for solutions of (4.1.1) to remain positive and grow:

$$f \in C(\mathbb{R}^+;(0,\infty)), \quad H \in C(\mathbb{R}^+;\mathbb{R}^+). \tag{4.1.3}$$

When we do not restrict ourselves to positive solutions we ask for a degree of symmetry in the problem to simplify the analysis. In particular, we require "asymptotic oddness" of the nonlinearity in the following sense:

$$f \in C(\mathbb{R};\mathbb{R}) \text{ and } \lim_{|x|\to\infty} \frac{|f(x)|}{\phi(|x|)} = 1 \text{ for some } \phi \in C^1(\mathbb{R}^+;(0,\infty)). \tag{4.1.4}$$

Chapter 4 is based on the working paper [13].

After developing results regarding the asymptotics of unbounded solutions of (4.1.1) we extend our deterministic analysis to consider the asymptotic behaviour of the related stochastic Volterra equation

$$dX(t) = \int_{[0,t]} \mu(ds) f(X(t-s))\, dt + dZ(t), \quad t > 0, \tag{4.1.5}$$

where Z is a semimartingale. We establish a simple existence and uniqueness theorem for equation (4.1.5) and then specialise to the cases of Brownian and Lévy noise in order to prove precise asymptotic results.

The differential equations (4.1.1) and (4.1.5) can be viewed as perturbations of the underlying deterministic Volterra integro-differential equation

$$y'(t) = \int_{[0,t]} \mu(ds) f(y(t-s)), \quad t > 0, \quad y(0) = \psi. \tag{4.1.6}$$

When f is positive and sublinear at infinity, we know from the results of Chapter 2 that the solution $y(t)$ of (4.1.6) obeys $y(t) \to \infty$ as $t \to \infty$ and grows asymptotically like the solution of the ordinary differential equation

$$z'(t) = Mf(z(t)), \quad t > 0; \quad z(0) = \psi, \tag{4.1.7}$$

in the sense that

$$\lim_{t\to\infty} \frac{F(y(t))}{Mt} = 1, \tag{4.1.8}$$

where F is the function defined by

$$F(x) = \int_1^x \frac{1}{f(u)}\, du, \quad x > 0. \tag{4.1.9}$$

It is natural to ask how large the forcing terms h in (4.1.1) and Z in (4.1.5) can become while the solutions x of (4.1.1) and X of (4.1.5) continue to grow in the manner described by (4.1.8). Furthermore, can we identify a new asymptotic regime or growth rate if the forcing terms exceed this critical rate? Our main goal is to identify such critical rates of growth on h and Z, and to determine precise estimates on the growth rate of solutions, or the rate of growth of the partial maxima when solutions fluctuate.

Much of our analysis flows from the simple matter of integrating (4.1.1) to obtain the forced Volterra integral equation

$$x(t) = x(0) + \int_0^t M(t-s) f(x(s))\, ds + H(t), \quad t \geq 0. \tag{4.1.10}$$

Since Itô stochastic "differential" equations are rigorously formulated in integral form it is perhaps even more natural to treat (4.1.5) similarly, which results in

$$X(t) = X(0) + \int_0^t M(t-s) f(X(s))\, ds + Z(t), \quad t \geq 0. \tag{4.1.11}$$

The representation (4.1.10) shows that the solution to (4.1.1) is a functional of the "aggregate" behaviour of the forcing term h purely through H and hence it is natural to formulate asymptotic results in terms of H. When studying the asymptotic behaviour of many forced differential systems it is frequently the case that the "aggregate" or "average" behaviour of the forcing terms are important, rather than more restrictive pointwise estimates. When studying stochastic equations pointwise estimates become unrealistically restrictive—or indeed impossible—and it is more natural and perhaps necessary to consider average behaviour. Another issue is whether the deterministic or stochastic character of the perturbation matters, or is it simply a question of the "size" of the perturbation. For these reasons we have found it of interest to study deterministic and stochastic equations in parallel, especially because it transpires that the general form of many results in the stochastic case can be conjectured by appealing to corresponding deterministic results.

To help the discussion we make our hypotheses more specific and outline typical results. In order for solutions of (4.1.6) to behave similarly to those of (4.1.7), it is important that f be *sublinear*: for example, we do not expect linear Volterra equations of the form (4.1.6) to share the exact exponential rate of growth of a linear ordinary differential equation in which all the mass of μ is concentrated at zero (cf. Gripenberg et al. [50, Theorem 7.2.3]). Also, as we are interested in growing solutions, it is quite natural that the function f should be in some sense *monotone*. In Chapter 2, we showed that if f is asymptotic to a C^1 function ϕ which is increasing and obeys $\phi'(x) \to 0$ as $x \to \infty$, then the solution of (4.1.6) obeys (4.1.8). We retain this hypothesis and occasionally strengthen it so that $\phi'(x)$ decays monotonically to 0 as $x \to \infty$; the implications and technical motivations for such hypotheses are discussed in Section 4.2.

Before stating our main results precisely we give a heuristic argument as to their likely validity. In this discussion we consider the simple (deterministic) case in which both the solution and the perturbation are positive. If the unperturbed equation (4.1.6) is integrated as above, $H \equiv 0$. In this case, the solution of the integral equation is, roughly, of order $F^{-1}(Mt)$. This leads to the naive idea that if H is of smaller order than y (i.e., than $F^{-1}(Mt)$), then H on the right–hand side of (4.1.10) could be absorbed into x on the left–hand side, without changing the leading order asymptotic behaviour of x. However, if H dominates y, or is of comparable order, such an outcome is improbable and the asymptotic behaviour of x is unlikely to be determined by y. Since the asymptotic behaviour of (4.1.6) is described well by $F(y(t))/Mt \to 1$ as $t \to \infty$, and F^{-1} is increasing, it is natural to seek to characterise the forcing term as "small" or "large" according as to whether $F(H(t))/Mt$ tends to a small or large limit as $t \to \infty$ (if such a limit exists). Define the dimensionless parameter $L \in [0, \infty]$ by

$$\lim_{t\to\infty} \frac{F(H(t))}{Mt} = L. \tag{4.1.12}$$

In some sense $L = 1$ is critical; for $L < 1$, H is dominated by the solution of (4.1.6). But for $L > 1$, H dominates the solution of (4.1.6). The cases $L = 0$ and $L = +\infty$ are especially decisive; in these cases it is very clear whether the solution of the unperturbed equation or the perturbation dominates. A condition which implies (4.1.12), and turns out to be very useful in classifying asymptotic behaviour, is

$$\lim_{t\to\infty} \frac{H(t)}{M\int_0^t f(H(s))ds} = L. \tag{4.1.13}$$

If $L = 0$ in (4.1.13), then

$$\lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1, \quad \lim_{t\to\infty} \frac{x(t)}{H(t)} = +\infty,$$

so small perturbations give rise to asymptotic behaviour as in (4.1.6), and the solution dominates the perturbation. If $L = +\infty$, then

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = 1, \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = +\infty,$$

so large perturbations cause the solution to grow at exactly the same rate as H, and the solution grows much faster than the original unperturbed Volterra equation. When the perturbation is of a scale comparable to the solution of (4.1.6), in the sense that $L \in (0, \infty)$,

$$1 \le \liminf_{t\to\infty} \frac{F(x(t))}{Mt} \le \limsup_{t\to\infty} \frac{F(x(t))}{Mt} \le 1 + L, \quad \liminf_{t\to\infty} \frac{x(t)}{H(t)} \ge 1 + \frac{1}{L}. \tag{4.1.14}$$

Examples show that the limits in the first part of (4.1.14) are not, in general, equal to 1 or $1 + L$.

Further investigation for finite and positive L leads to better estimates, especially when $L > 1$. The critical character of the case when $L = 1$ is demonstrated by the following result: if $L \in (1, \infty)$ then

$$1 \leq \liminf_{t\to\infty} \frac{x(t)}{H(t)} \leq \limsup_{t\to\infty} \frac{x(t)}{H(t)} \leq \frac{L}{L-1}. \tag{4.1.15}$$

We notice that this provides sharper estimates for large L than the asymptotic bounds given for $L \in (0, \infty)$ above and identifies that x is of order H. We also show by means of examples that when $L \in (0, 1]$, the limit

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = +\infty$$

can result, so that x can only be expected to be exactly of the order of H for $L > 1$ (see example 4.3.9). However, if $L \in (0, 1]$, it is not necessarily the case that $x(t)/H(t) \to \infty$ as $t \to \infty$ (see Section 2.4). Notice finally that as $L \to \infty$, equation (4.1.15) correctly anticipates that $x(t)/H(t) \to 1$ as $t \to \infty$, which is what pertains when $L = +\infty$. To generalise the analysis above to stochastic equations, and for notational convenience, we define the following functional

$$L_f(\gamma) = \lim_{t\to\infty} \frac{\gamma(t)}{M \int_0^t f(\gamma(s))ds}, \text{ where } M = \mu(\mathbb{R}^+) \in (0, \infty), \tag{4.1.16}$$

for all functions f and $\gamma \in C(\mathbb{R}^+; (0, \infty))$ such that the above limit is well defined.

4.2 Discussion of Hypotheses

We impose the following sublinearity hypothesis on the nonlinear function f:

$$\begin{aligned} f \sim \phi \in C^1 \text{ such that } &\lim_{|x|\to\infty} \phi(x) = \infty, \ \phi'(x) > 0 \text{ for all } x \in \mathbb{R} \\ &\text{and } \phi'(x) \to 0 \text{ as } |x| \to \infty. \end{aligned} \tag{4.2.1}$$

In some cases the following slightly stronger hypothesis is necessary:

$$\begin{aligned} f \sim \phi \in C^1 \text{ such that } &\lim_{|x|\to\infty} \phi(x) = \infty, \ \phi'(x) > 0 \text{ for all } x \in \mathbb{R} \\ &\text{and } \phi'(x) \downarrow 0 \text{ as } |x| \to \infty. \end{aligned} \tag{4.2.2}$$

If f is an increasing, sublinear function, then $\liminf_{x\to\infty} f'(x) = 0$ but it is still possible that $\limsup_{x\to\infty} f'(x) = \infty$ in the "worst" case. In Chapter 2, we provided an example of such a pathological f but such nonlinearities are unlikely to arise naturally in applications so condition (4.2.1) is a relatively mild strengthening of sublinearity in this context. Assuming further that ϕ' tends to zero monotonically, as in (4.2.2), one can establish the following lemmata which prove crucial in the asymptotic analysis of (4.1.1) and (4.1.5).

Lemma 4.2.1. *If (4.2.2) holds, then ϕ obeys*

$$\limsup_{x\to\infty} \frac{x\,\phi'(x)}{\phi(x)} \leq 1, \quad \limsup_{x\to\infty} \frac{\phi(\Lambda x)}{\phi(x)} \leq \Lambda, \quad \Lambda \in [1, \infty). \tag{4.2.3}$$

The conclusions of Lemma 4.2.3 are remarkably close to some of the key properties enjoyed by the class of regularly varying functions with unit index (denoted $RV_\infty(1)$). Namely, $\phi \in RV_\infty(1)$ implies $\lim_{x\to\infty} \phi(\Lambda x)/\phi(x) = \Lambda$ for all $\Lambda > 0$ and $\lim_{x\to\infty} x\,\phi'(x)/\phi(x) = 1$. The next lemma shows that the auxiliary function ϕ preserves asymptotic equivalence. Hence $L_f(\gamma) = L_\phi(\gamma)$, if the limit exists.

The connection between the "natural" size hypothesis on H, (4.1.12), and the functional condition, (4.1.16), is supplied by the following result.

Proposition 4.2.1. *Suppose $\phi \in C(\mathbb{R}^+;(0,\infty))$ is increasing and continuous with Φ defined by (1.3.3). Let $\gamma \in C(\mathbb{R}^+;(0,\infty))$. If $L_\phi(\gamma)$ from (4.1.16) is well defined, then*

$$\lim_{t\to\infty} \frac{\Phi(\gamma(t))}{Mt} = L_\phi(\gamma).$$

4.3 Deterministic Volterra Equations

4.3.1 Growth Results

Throughout this section we suppose that (4.1.3) holds so that $0 < x(t) \to \infty$ as $t \to \infty$, subject to a positive initial condition. Our first result provides an easy to check sufficient condition on H which guarantees solutions of (4.1.1) retain the rate of growth of solutions to the ordinary differential equation (4.1.7). This sufficient condition is of a different character to conditions involving the functional $L_f(\cdot)$ and expresses more explicitly the idea that the perturbation term, H, should be small relative to the solution of (4.1.7).

Theorem 4.3.1. *Suppose (4.1.2), (4.1.3), and (4.2.1) hold and $\psi > 0$. If*

$$\lim_{t\to\infty} \frac{H(t)}{F^{-1}\left(M(1+\epsilon)t\right)} = 0 \text{ for each } \epsilon \in (0,1), \tag{4.3.1}$$

then solutions of (4.1.1) obey

$$\lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1, \quad \lim_{t\to\infty} \frac{x(t)}{H(t)} = \infty. \tag{4.3.2}$$

Now we formulate a sufficient condition for $\lim_{t\to\infty} F(x(t))/Mt = 1$ to hold in terms of $L_f(\cdot)$. Compared to condition (4.3.1) such functional based conditions have much better scope for generalization. We also prove that when the solution of (4.1.1) retains the growth rate of solutions of (4.1.7) it is of a strictly larger order of magnitude than the perturbation term, H.

Theorem 4.3.2. *Suppose (4.1.2), (4.1.3), and (4.2.1) hold and $\psi > 0$. If $L_f(H) = 0$, then solutions of (4.1.1) obey*

$$\lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1, \quad \lim_{t\to\infty} \frac{x(t)}{H(t)} = \infty. \tag{4.3.3}$$

Note that we do not assume in Theorem 4.3.2 that $H(t) \to \infty$ as $t \to \infty$; this is in the case where $L_f(H) = 0$. However, if $L_f(H) \in (0,\infty]$, then $\lim_{t\to\infty} H(t) = \infty$. The rationale is as follows in the case $L_f(H) \in (0,\infty)$, with the case of $L_f(H) = \infty$ being similar. By hypothesis $H(t) > 0$ for $t > 0$ and as f is a positive function, $t \mapsto \int_0^t f(H(s))ds$ is increasing. Therefore, H either tends to ∞ or to a finite limit. In the former case, $H(t) \to \infty$ as $t \to \infty$ automatically. If, to the contrary, the limit is finite, then $H(t)$ tends to a finite positive limit as $t \to \infty$. But this forces $\int_0^t f(H(s))\,ds \to \infty$ as $t \to \infty$, a contradiction.

When $L_f(H)$ is nonzero but finite we expect the solution of (4.1.1) to inherit properties of both the underlying ordinary differential equation and the perturbation term. Our next theorem investigates results of the type (4.1.8) when $L_f(H) \in (0,\infty)$; we show that the growth of solutions to (4.1.1) is at least as fast as that of solutions to the underlying ordinary differential equation and we prove an upper bound on the growth rate. The resulting upper bound is linear in $L_f(H)$ and this is intuitively appealing as a "larger" H should speed up growth. However, this upper estimate on the growth rate is not sharp in general. Without additional hypotheses this upper bound is hard to improve but can be shown to be suboptimal for specific classes of nonlinearity, for example when f is regularly varying with less than unit index (see Section 4.3.3).

Theorem 4.3.3. *Suppose (4.1.2), (4.1.3), and (4.2.1) hold and $\psi > 0$. If $L_f(H) \in (0,\infty)$, then solutions of (4.1.1) obey*

$$1 \leq \liminf_{t\to\infty} \frac{F(x(t))}{Mt} \leq \limsup_{t\to\infty} \frac{F(x(t))}{Mt} \leq 1 + L_f(H).$$

If (4.2.1) is strengthened to (4.2.2), solutions of (4.1.1) also obey

$$\liminf_{t\to\infty} \frac{x(t)}{H(t)} \geq 1 + \frac{1}{L_f(H)}.$$

We note that the asymptotic lower bound on the quantity $x(t)/H(t)$ in the result above agrees with Theorem 4.3.2 as $L_f(H)$ tends to zero, in the sense that it correctly predicts $\lim_{t\to\infty} x(t)/H(t) = \infty$ when $L_f(H) = 0$.

The results of this section can all be restated with positivity assumptions on f and H replaced by (4.1.4) and

$$H \in C(\mathbb{R}^+;\mathbb{R}). \tag{4.3.4}$$

In this case one obtains upper bounds on the rate of growth of solutions of (4.1.1) in terms of the related ODE, i.e. results of the type $\limsup_{t\to\infty} F(|x(t)|)/Mt < \infty$.

The main results of this section are all proven by comparison arguments and the careful asymptotic analysis of the resulting differential inequalities. Since we assume positivity of H to ensure asymptotic growth of solutions, it is straightforward to establish that $\liminf_{t\to\infty} F(x(t))/Mt \geq 1$; this is proven by a translation argument and appealing to Corollary 2.3.2. The proof of the corresponding upper bound, $\limsup_{t\to\infty} F(x(t))/Mt < \infty$, is more involved but can be roughly summarized as follows:

Step 1: Use monotonicity and finiteness of the measure to construct the crude upper inequality

$$x(t) < H_\epsilon(t) + (1+\epsilon)M\int_T^t \phi(x(s))\,ds, \quad t \geq T, \tag{4.3.5}$$

where H_ϵ includes constants and lower order terms, ϕ is a monotone function asymptotic to f and we define $I_\epsilon(t) = \int_T^t \phi(x(s))\,ds$ for $t \geq T$.

Step 2: Using hypotheses on the size of the perturbation term try to show that H_ϵ is $o(I_\epsilon)$ (or $O(I_\epsilon)$ respectively).

Step 3: Conclude the argument via a variation on Bihari's inequality.

4.3.2 Fluctuation Results

The existence of the limit $L_f(H)$ (even when it takes the value $+\infty$) is too strong a condition if we hope to apply our deterministic arguments to related equations with stochastic perturbations. We seek to weaken the hypothesis $L_f(H) \in (0,\infty)$ as follows: assume that there exists a function γ such that

$$\gamma \in C((0,\infty);(0,\infty)) \text{ is increasing and } \lim_{t\to\infty}\gamma(t) = \infty \text{ and } \limsup_{t\to\infty}\frac{|H(t)|}{\gamma(t)} = 1. \tag{4.3.6}$$

We now make hypotheses on $L_f(\gamma)$, as opposed to $L_f(H)$. We take $\limsup_{t\to\infty} |H(t)|/\gamma(t) = 1$, rather than positive and finite since we can always normalise this quantity while keeping the properties of γ unchanged. Since $L_f(\gamma) \in (0,\infty)$ forces γ to be eventually increasing, we simply suppose that γ is always increasing for ease of exposition but there is strictly no need to make this assumption. Under (4.3.6) we can permit highly irregular behaviour in H as long as we can capture some underlying regularity in the asymptotics of H via a well-behaved auxiliary function, γ. For example, in applications to stochastic equations, H could be a stochastic process whose partial maxima are described in terms of a deterministic function; this is the case for classes of processes obeying so-called iterated logarithm laws for instance. The following result illustrates the immediate utility of the hypothesis (4.3.6) for deterministic equations and furthermore details how this hypothesis carries over to the case when $L_f(\gamma) = \infty$.

Theorem 4.3.4. *Suppose* (4.1.2)*,* (4.1.4)*,* (4.3.4)*,* (4.2.2) *and* (4.3.6) *hold. Let x denote a solution of* (4.1.1)*.*

(a.) If $L_f(\gamma) \in (1, \infty)$, then

$$\limsup_{t\to\infty} \frac{|x(t)|}{\gamma(t)} \in \left[0, \frac{L_f(H)}{L_f(H) - 1}\right).$$

(b.) If $L_f(\gamma) = \infty$, then

$$\limsup_{t\to\infty} \frac{|x(t)|}{\gamma(t)} = 1, \quad \lim_{t\to\infty} \frac{x(t) - H(t)}{\gamma(t)} = 0.$$

Case $(a.)$ of the result above indicates that when the perturbation is of intermediate size, in the sense that $L_f(\gamma) \in (1, \infty)$, solutions of (4.1.1) are at most the same order of magnitude as H, modulo a multiplier. In case $(b.)$, when the perturbation is so large that $L_f(\gamma) = \infty$, solutions of (4.1.1) have partial maxima of exactly the same order as those of H. This conclusion is strongly hinted at in case $(a.)$ of Theorem 4.3.4 if one lets $L_f(\gamma) \to \infty$ in that result.

The restriction $L_f(\gamma) > 1$ is in fact crucial to the proof of Theorem 4.3.6 and cannot be relaxed within the framework of the current argument. We make this comment precise at the relevant moment during the proof itself (see remark 4.5.2). In fact, $L_f(\gamma) > 1$ is not a purely technical contrivance but is also essential to the validity of our result. In example 4.3.9 we demonstrate that when $L_f(\gamma) \in (0, 1]$ it is possible to have $\lim_{t\to\infty} |x(t)|/\gamma(t) = \infty$.

If $\limsup_{t\to\infty} |H(t)|/\gamma(t) = 0$ in (4.3.6) we can use the following hypothesis and the arguments from Theorem 4.3.4 to extend the scope of the result above.

$$\limsup_{t\to\infty} \frac{|H(t)|}{\gamma_+(t)} = 0, \quad \limsup_{t\to\infty} \frac{|H(t)|}{\gamma_-(t)} = \infty. \tag{4.3.7}$$

Theorem 4.3.5. *Suppose* (4.1.2)*,* (4.1.4)*,* (4.3.4) *and* (4.2.2) *hold. Furthermore suppose there exist increasing functions $\gamma_\pm \in C((0, \infty); (0, \infty))$ obeying*
$\lim_{t\to\infty} \gamma_\pm(t) = \infty$ such that (4.3.7) *holds and let x denote a solution of* (4.1.1)*.*

(a.) If $L_f(\gamma_\pm) \in (1, \infty]$, then

$$\limsup_{t\to\infty} \frac{|x(t)|}{\gamma_+(t)} \in \left[0, \frac{1}{L_f(\gamma_+)}\right], \quad \limsup_{t\to\infty} \frac{|x(t)|}{\gamma_-(t)} = \infty.$$

(b.) If $L_f(\gamma_\pm) = \infty$, then

$$\lim_{t\to\infty} \frac{|x(t)|}{\gamma_+(t)} = 0, \quad \limsup_{t\to\infty} \frac{|x(t)|}{\gamma_-(t)} = \infty, \tag{4.3.8}$$

where we interpret $1/L_f(\gamma_+) = 0$ if $L_f(\gamma_+) = \infty$.

In the presence of limited information about the behaviour of H, in the sense that (4.3.7) holds, the result above tells us that the solution of (4.1.1) is roughly the same order of magnitude as H, in the sense that x also obeys (4.3.7), when $L_f(\gamma_\pm) = \infty$. When $L_f(\gamma_\pm) \in (1, \infty]$ we are still left with a weak conclusion and we are tempted to ask if this is an artifact of the argument used to establish it. Example 4.3.11 shows that we cannot expect to conclude that $\limsup_{t\to\infty} |x(t)|/\gamma_+(t) = 0$ in general. However, in attempting to apply this Theorem 4.3.5, one would likely seek to refine their choice of γ_+ in order to obtain a γ_+ obeying $L_f(\gamma_+) = \infty$ and hence make the stronger conclusion that x is $o(\gamma_+)$.

Theorem 4.3.5 could equally well be stated as follows: If $L_f(\gamma_+) \in (1,\infty]$, then $\limsup_{t\to\infty} |x(t)|/\gamma_+(t) \le 1/L_f(\gamma_+)$ and if $L_f(\gamma_-) \in (1,\infty]$, then $\limsup_{t\to\infty} |x(t)|/\gamma_+(t) = \infty$. These two statements are proved independently of one another but we chose to present them as part of a single result as we feel this is the manner in which they would prove most useful in practice; choosing γ_+ and γ_- "close together" can give useful bounds on the size of the solution but using either bound in isolation only gives very crude information (see Example 4.4.8).

If we consider the case of positive, growing solutions once more we can impose hypotheses regarding the functional $L_f(\cdot)$ directly on H in Theorem 4.3.4 and quickly establish the following result:

Theorem 4.3.6. *Suppose* (4.1.2), (4.1.3), (4.2.2) *hold and that* $\psi > 0$. *Let* x *denote a solution of* (4.1.1).

(a.) If $L_f(H) \in (1,\infty)$, *then*

$$G_L := 1 + \frac{1}{L_f(H)} \le \liminf_{t\to\infty} \frac{x(t)}{H(t)} \le \limsup_{t\to\infty} \frac{x(t)}{H(t)} \le \frac{L_f(H)}{L_f(H)-1} =: G_U.$$

(b.) If $L_f(H) = \infty$, *then*

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = 1, \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = \infty. \tag{4.3.9}$$

Under stronger hypotheses the result above provides more refined conclusions than Theorems 4.3.4 and 4.3.5. In particular, case $(a.)$ establishes bounds which demonstrate that x will closely track the asymptotic behaviour of H and case $(b.)$ establishes that when the noise term, H, is sufficiently large $x(t) \sim H(t)$ as $t \to \infty$. Furthermore, when $x(t) \sim H(t)$ as $t \to \infty$, x is of a strictly larger order of magnitude than the solution of the corresponding ordinary differential equation. We also note that this result allows us to pick up fluctuations in the solution even when H is nonnegative. Even though the solution grows to infinity it may not do so monotonically and the conclusion of Theorem 4.3.6 identifies upper and lower rates of growth of the solution ($G_L H(t)$ and $G_U H(t)$, respectively, when $L_f(\gamma) \in (1,\infty)$ and when $L_f(\gamma) = \infty$ the fluctuations are entirely determined by H).

The results of this section are proven via the usual machinery of comparison and asymptotic analysis but also rely crucially on the construction of a *linear* differential inequality. The key steps in the argument can be understood as follows:

Step 1: Using (4.3.5), derive the nonlinear differential inequality

$$I_\epsilon'(t) < \phi(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t), \quad t \ge T,$$

where $I_\epsilon(t) = \int_T^t \phi(x(s))\,ds$.

Step 2: Use (4.2.2) to derive the *linear* differential inequality

$$I_\epsilon'(t) < \phi(H_\epsilon(t)) + \frac{\phi(H_\epsilon(t))}{H_\epsilon(t)} M(1+\epsilon)^2 I_\epsilon(t), \quad t \ge T_1 > T. \tag{4.3.10}$$

Since we can solve this inequality directly, there is no additional loss of sharpness here.

Step 3: Careful asymptotic analysis of the solution to the inequality (4.3.10) using hypotheses on $L_f(H)$ yield upper bounds on the size of the solution to (4.1.1).

Step 4: The upper bounds achieved in Step 3 are recycled and further simple estimation yields the conclusions shown in the results above.

The steps outlined above are also very successful in the presence of random forcing, as we will demonstrate presently.

4.3.3 Refinements using Regular Variation

An important class of nonlinear functions obeying the hypotheses of this chapter are the class $\mathrm{RV}_\infty(\beta)$ for $\beta \in (0,1]$. Hence, in this section, we outline refinements of the results presented in Sections 4.3.1 and 4.3.2 using the theory of regular variation.

Since the case of $\mathrm{RV}_\infty(1)$ must be dealt with separately, we require the following pair of hypotheses on the nonlinearity:

$$f \in \mathrm{RV}_\infty(\beta) \text{ with } \beta \in (0,1), \tag{4.3.11}$$

and

$$f \in \mathrm{RV}_\infty(1), \quad \lim_{x\to\infty} \frac{f(x)}{x} = 0. \tag{4.3.12}$$

When (4.3.11) holds, f obeys (4.2.2) but $f \in \mathrm{RV}_\infty(1)$ does not necessarily imply that $f(x)/x \to 0$ as $x \to \infty$.

The following theorem not only illustrates the power of regular variation in providing sharp asymptotic results, but also shows that hypotheses on $L_f(\cdot)$ give a comprehensive taxonomy of the behaviour of solutions to (4.1.1).

Theorem 4.3.7. *Suppose (4.1.2) and (4.1.3) hold, and let x be the unique continuous solution to (4.1.1). If (4.3.11) holds, then*

(i.) $L_f(H) = 0$ *implies*

$$\lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1, \quad \lim_{t\to\infty} \frac{x(t)}{H(t)} = \infty.$$

(ii.) $L_f(H) \in (0,\infty)$ *implies*

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = \frac{1}{1-\zeta^{\beta-1}}, \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = \zeta^{1-\beta},$$

where $\zeta \in (1,\infty)$ is the unique solution of $\zeta = \zeta^\beta + L_f(H)^{1/(1-\beta)}$.

(iii.) $L_f(H) = \infty$ *implies*

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = 1, \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = \infty.$$

If (4.3.12) holds, then

(I.) $L_f(H) \in [0,1]$ *implies*

$$\lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1, \quad \lim_{t\to\infty} \frac{x(t)}{H(t)} = \infty.$$

(II.) $L_f(H) \in (1,\infty)$ *and $f(x)/x \downarrow 0$ as $x \to \infty$ imply*

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = \frac{L_f(H)}{L_f(H)-1}, \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = L_f(H).$$

(III.) $L_f(H) = \infty$ *and* $f(x)/x \downarrow 0$ *as* $x \to \infty$ *imply*

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = 1, \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = \infty.$$

In cases $(i.)$ and $(I.)$ the asymptotic behaviour of the underlying ODE is preserved, as in Theorem 4.3.2. When $f \in \mathrm{RV}_\infty(1)$ this behaviour is preserved until $L_f(H) > 1$, but if (4.3.11) holds this behaviour is destroyed once $L_f(H) > 0$ and a new asymptotic growth rate pertains – see case $(ii.)$.

Cases $(ii.)$, $(I.)$, and $(II.)$ illustrate the limitations of Theorem 4.3.3. In particular, case $(I.)$ shows that it is possible to have $\lim_{t\to\infty} F(x(t))/Mt = 1$, for a large class of nonlinearities when $L_f(H) > 0$ and hence the upper bound on the limsup in Theorem 4.3.3 is not sharp in general. Indeed, cases $(ii.)$ and $(II.)$ show that it is not possible to sharpen this bound without further hypotheses on the nonlinearity since the true limits depend on the index of regular variation. Case $(II.)$ essentially shows that Theorem 4.3.6 part $(a.)$ is exact for nonlinear functions obeying (4.3.12) (but we show via examples that this is not true in general).

Cases $(iii.)$ and $(III.)$ above are special cases of Theorem 4.3.6 part $(b.)$. However, with the additional hypothesis of regular variation we can also prove the following converse (whose proof follows from arguments very similar to those used in Theorems 4.3.5 and 4.3.7).

Theorem 4.3.8. *Suppose* (4.1.2) *and* (4.1.3) *hold. Let x be the unique continuous solution to* (4.1.1) *and suppose there is an increasing continuous function γ with $\gamma(t) \to \infty$ as $t \to \infty$ such that*

$$0 < \liminf_{t\to\infty} \frac{H(t)}{\gamma(t)} \leq \limsup_{t\to\infty} \frac{H(t)}{\gamma(t)} < +\infty. \tag{4.3.13}$$

If either (4.3.11) *or* (4.3.12) *with $f(x)/x \downarrow 0$ as $x \to \infty$ hold, then $L_f(H) = \infty$ is equivalent to $x(t) \sim H(t)$, as $t \to \infty$.*

4.3.4 Examples

Consider the Volterra integro–differential equation given by

$$x'(t) = \int_0^t e^{-(t-s)} f(x(s))\, ds + h(t), \quad t > 0; \quad x(0) = \psi > 0.$$

In the notation of (4.1.10), $M(t) = \int_0^t e^{-s} ds = 1 - e^{-t}$ and hence

$$H(t) = x(t) - x(0) - \int_0^t f(x(s))\, ds + \int_0^t e^{-(t-s)} f(x(s))\, ds, \quad t \geq 0. \tag{4.3.14}$$

We construct examples by choosing a solution, up to asymptotic equivalence, and then using (4.3.14) to figure out how large the perturbation term, H, must have been to generate a solution of this size. The supporting calculations are elementary and hence omitted. For simplicity we forego any mention of hypotheses of the form (4.3.6) in this section and concentrate on the special case $\gamma = H$ with H positive.

Example 4.3.9. *This example highlights the potential problems that emerge when one attempts to address the case $L_f(H) \in (0, 1]$ (resp. $L_f(\gamma)$) in the context of Theorem 4.3.4. In particular, one cannot extend the conclusion of Theorem 4.3.4 to cover $L_f(H) \in (0, 1]$ without additional hypotheses because when $L_f(H) \in (0, 1]$ it is possible to have $\lim_{t\to\infty} x(t)/H(t) = \infty$.*

Let $f(x) = (x + e)/\log(x + e)$, so

$$F(x) \sim \frac{1}{2}\log^2(x + e) \text{ and } F^{-1}(x) \sim e^{\sqrt{2x}}, \text{ as } x \to \infty. \tag{4.3.15}$$

Choose $x(t) = \exp\left(\lambda(t) + \sqrt{2(t+1)}\right) - e = \exp(P(t)) - e$ *for* $t \geq 0$ *and let* $\lambda(t) = (1+t)^\alpha$ *for some* $\alpha \in (0, 1/2)$. *In this case* $H(t) \sim K\, P(t)^{2\alpha-1} \exp(P(t))$. *Furthermore,* H *obeys* $L_f(H) = 1$ *and by construction* $\lim_{t\to\infty} x(t)/H(t) = \infty$. *However, we still have* $\lim_{t\to\infty} F(x(t))/Mt = 1$.

Example 4.3.10. *We now show that the bounds on* $\lim_{t\to\infty} x(t)/H(t)$ *and* $\liminf_{t\to\infty} F(x(t))/Mt$ *obtained in Theorems 4.3.3 and 4.3.6 can actually be attained. Once more suppose that* $f(x) = (x+e)/\log(x+e)$.

Suppose $L_f(H) \in (1, \infty)$ *and choose* $x(t) = \exp\left(\sqrt{2L_f(H)(t+1)}\right) - e$ *for* $t \geq 0$. *This gives* $H(t) \sim ((L_f(H)-1)/L_f(H)) \exp\left(\sqrt{2L_f(H)(t+1)}\right)$ *as* $t \to \infty$ *and*

$$\lim_{t\to\infty} \frac{H(t)}{M\int_0^t f(H(s))ds} = L_f(H) \in (1, \infty).$$

Hence $\lim_{t\to\infty} x(t)/H(t) = L_f(H)/(L_f(H)-1)$, *achieving the upper bound predicted by Theorem 4.3.6. Furthermore,* $\lim_{t\to\infty} F(x(t))/Mt = 1$, *achieving the lower bound from Theorem 4.3.3.*

Example 4.3.11. *In Theorem 4.3.5 (a.),* $\limsup_{t\to\infty} H(t)/\gamma_+(t) = 0$ *but if* $L_f(\gamma_+) \in (1, \infty)$, *then* $\limsup_{t\to\infty} x(t)/\gamma_+(t) > 0$ *is possible. Hence there is no straightforward improvement of the conclusion of Theorem 4.3.5 when* $L_f(\gamma_+) \in (1, \infty)$.

Let $f(x) = x^\beta$ *with* $\beta \in (0, 1)$, $H = 0$, *and* $\gamma_+(t) = F^{-1}(\alpha M t)$ *with* $\alpha \in (1, \infty)$. *This implies that* $x(t) \sim F^{-1}(Mt)$ *as* $t \to \infty$ *and hence* $\lim_{t\to\infty} x(t)/\gamma_+(t) = \alpha^{-1/(1-\beta)} > 0$, *as required. It is straightforward to verify that* $L_f(\gamma_+) = \alpha \in (1, \infty)$.

4.4 Stochastic Volterra Equations

We now study the pathwise asymptotic behaviour of solutions to (4.1.5). Our approach is to treat (4.1.5) as a perturbed version of (4.1.1) where the forcing term is now stochastic and hence to leverage our deterministic results as much as possible. We first establish existence of unique strong solutions to (4.1.5). We then use the pathwise asymptotic theory for continuous *Brownian martingales* and *α–stable Lévy processes* to show that the main results from the previous section are sufficiently general that we can extend them to provide asymptotic estimates on the path-wise growth and fluctuation of solutions to (4.1.5).

We work on a given probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ which is complete and has a right continuous filtration. We ask that the nonlinear function $f : \mathbb{R} \mapsto \mathbb{R}$ obeys the following local Lipschitz condition: for each $d > 0$ there exists $K_d > 0$ such that

$$|f(x) - f(y)| \leq K_d|x - y|, \text{ for each } x \text{ and } y \in [-d, d], \tag{4.4.1}$$

and that f obeys a global linear bound of the form

$$|f(x)| \leq K + \eta|x|, \text{ for each } x \in \mathbb{R}, \text{ where } K \text{ and } \eta \text{ are positive constants.} \tag{4.4.2}$$

In order to leverage the framework of Métivier and Pellaumail [91] we make a slight modification to the formulation of (4.1.5): consider the stochastic integral equation

$$X(t) = X(0) + \int_0^t \left(\int_{(0,s]} \mu(du) f(X(s-u)) + \mu(\{0\}) f(X(s-))\right) ds + Z(t), \quad t \geq 0. \tag{4.4.3}$$

By applying Fubini's theorem and making a suitable change of variable (4.4.3) can be written as

$$X(t) = X(0) + \mu(\{0\}) \int_{[0,t]} f(X(s-))ds + \int_{[0,t)} M_-(t-s) f(X(s))\, ds + Z(t), \quad t \geq 0, \tag{4.4.4}$$

where $X(t-) = X(\lim_{s\uparrow t})$ and $M_-(t) = \int_{(0,t]} \mu(du)$. This adjustment is necessary for the functional

$$a(s, \omega, X) = \int_{(0,s]} \mu(du) f(X(s-u)) + \mu(\{0\}) f(X(s-)), \quad s \geq 0, \tag{4.4.5}$$

to define a predictable process (measurable with respect to the filtration generated by adapted, left continuous processes) and hence be integrable with respect to general semimartingales (see Protter [101] for details).

In order to define the notion of a strong solution for SFDEs such as (4.4.4), we recall some standard terminology from the theory of stochastic processes: a regular process is one which is adapted and has right continuous paths with left hand limits (RCLL). A process X is called $\mathbb{P}$–null if almost surely the paths $t \mapsto X(t)$ are identically zero functions.

Definition 4.4.1. *A process X defined on $[0, \tau)$ is said to be a strong solution to (4.4.4) on $[0, \tau)$ with initial value $X(0)$ if the process*

$$\mu(\{0\}) \int_{[0,t]} f(X(s-))ds + \int_{[0,t)} M_-(t-s) f(X(s))\, ds + Z(t)$$

is well–defined on $[0, \tau)$ as a regular process and differs from $X(t) - X(0)$ by a $\mathbb{P}$–null process.

We say that the solution to (4.4.4) is unique if for any two processes X and Y obeying Definition 4.4.1, $X - Y$ is a $\mathbb{P}$–null process.

Theorem 4.4.1. *Let (4.1.2) hold and let Z be a cádlag semimartingale. If $f : \mathbb{R} \mapsto \mathbb{R}$ is measurable and obeys (4.4.1), and (4.4.2), then there exists a unique, strong solution to (4.4.4).*

Proof. This theorem is a natural specialisation of a result of Métivier and Pellaumail [91, Theorem 5]. In order to apply the aforementioned result we must check that the functional from (4.4.5) and also the constant functional $a(s, \omega, X) = 1$ obey the following pair of conditions: firstly for any regular processes X and Y, for each $d > 0$ there exists a constant $L_d > 0$ such that

$$|a(t, \omega, X) - a(t, \omega, Y)| \leq L_d \sup_{0 \leq s < t} |X(s) - Y(s)| \tag{4.4.6}$$

for each $t \in \mathbb{R}^+$, $\sup_{0\leq s<t} |X(s)| \leq d$ and $\sup_{0\leq s<t} |Y(s)| \leq d$. Secondly, for any regular process X there exists $C > 0$ such that

$$|a(t, \omega, X)| \leq C \sup_{0 \leq s < t} (|X(s)| + 1) \tag{4.4.7}$$

for each $t \in \mathbb{R}^+$. When the functional a is constant the conditions above are trivially satisfied so suppose now that a is given by (4.4.5) and proceed to verify condition (4.4.6). Let X and Y be any two regular processes satisfying $\sup_{0\leq s<t} |X(s)| \leq d$ (resp. Y), fix $t \in \mathbb{R}^+$ and estimate as follows:

$$\begin{aligned} |a(t, \omega, X) - a(t, \omega, Y)| &\leq \mu(\{0\}) |f(X(t-)) - f(Y(t-))| \\ &\qquad + \int_{(0,t]} \mu(ds) |f(X(t-s)) - f(Y(t-s))| \\ &\leq M\, K_d \sup_{0 \leq s < t} |X(s) - Y(s)|, \end{aligned}$$

where we have used both (4.1.2) and (4.4.1). Now check (4.4.7); assume X is a regular process and fix $t \in \mathbb{R}^+$. The following inequality is a straightforward consequence of (4.1.2) and (4.4.2):

$$|a(t,\omega,X)| \le \mu(\{0\})|f(X(t-))| + \int_{(0,t]} \mu(ds)|f(X(t-s))| \le C^* \sup_{0\le s<t} (|X(s)|+1),$$

where $C^* = M\,K$. □

Remark 4.4.1. *Note that the condition (4.4.2) will always be satisfied in this section since the hypotheses (4.1.4) and (4.2.2) will be imposed throughout. The assumption (4.1.2) is also present throughout so the only additional hypothesis imposed by Theorem 4.4.1 is that of local Lipschitz continuity on the nonlinear function f.*

We pause now to consider the method by which the results of this section are proven and to illustrate that this presents a framework for generating similar pathwise asymptotic results for a wide range of suitable stochastic forcing terms. Our method of proof relies principally on building appropriate comparison equations, so we are not concerned about the pathwise regularity of the solution to (4.1.5); this allows us to treat quite irregular forcing processes. To summarise, if (4.1.5) is driven by a stochastic process Z, our general approach is as follows:

(i.) Establish the existence and uniqueness of *strong* solutions.

(ii.) Prove pathwise bounds on the size of the process Z in terms of a well-behaved deterministic function, γ, on which we can formulate functional hypotheses in terms of $L_f(\cdot)$; these bounds should be in the spirit of (4.3.6) or (4.3.7).

(iii.) Construct an upper comparison solution (pathwise) in terms of γ which majorizes the solution to the (4.1.5); this essentially reduces the stochastic problem to a deterministic one.

(iv.) Conclude using suitable hypotheses on $L_f(\gamma)$ and the results of Section 4.3.

4.4.1 Brownian Noise

For the remainder of the chapter let X denote the unique, strong solution to (4.1.5). Furthermore, suppose

$$Z(t) = \int_0^t \sigma(s)dB(s), \text{ B standard Brownian motion, } \sigma \in C(\mathbb{R}^+,\mathbb{R}), \tag{4.4.8}$$

and define

$$\Sigma(t) = \sqrt{2\left(\int_0^t \sigma^2(s)ds\right)\log\log\left(\int_0^t \sigma^2(s)ds\right)}.$$

Analogously to the deterministic case, we classify the behaviour of solutions to (4.1.5) according to whether the number $L_f(\Sigma)$ is zero, finite or infinite.

The existence and uniqueness of solutions of (4.1.5) is naturally simpler in the case of Brownian noise. In particular, there is a unique, continuous (strong) solution to (4.1.5) with Brownian noise if (4.1.2) holds and the nonlinearity is locally Lipschitz continuous with a global linear bound (see Mao [86, Ch. 5]).

When formulating functional conditions on (4.1.5) to preserve growth of the type (4.1.8) it is necessary to distinguish between the cases $\sigma \in L^2(0,\infty)$ and $\sigma \notin L^2(0,\infty)$. When $\sigma \in L^2(0,\infty)$ the martingale term in (4.1.11), $\int_0^t \sigma(s)dB(s)$, will tend to an a.s. finite random variable and in this case we clearly expect to retain the growth rate of solutions of (4.1.7). However, when $\sigma \notin L^2(0,\infty)$ the martingale term is recurrent on $\mathbb{R}$ and has large fluctuations of order $\Sigma(t)$ (see Revuz and Yor [104, Ch. V, Ex. 1.15]). Our first result shows that when $\sigma \notin L^2(0,\infty)$ and $L_f(\Sigma) = 0$, the solution to (4.1.5) cannot grow faster than that of the ordinary differential equation (4.1.7).

Theorem 4.4.2. *Let (4.1.2), (4.1.4), (4.2.1), and (4.4.8) hold with* $\lim_{x\to\infty} f(x) = \infty$.

(a) If $\sigma \notin L^2(0,\infty)$ *and* $L_f(\Sigma) = 0$, *then*

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 \ a.s..$$

(b) If $\sigma \in L^2(0,\infty)$, *then* $L_f(\Sigma) = 0$ *and*

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 \ a.s..$$

An interesting special case of Theorem 4.4.2, which is likely to be important in applications, is when the function σ is a nonzero constant. In this case we can additionally show that the size of solution to (4.1.5) becomes unbounded with probability one.

Corollary 4.4.1. *Let (4.1.2), (4.1.4), (4.2.1), and (4.4.8) hold with* $\lim_{x\to\infty} f(x) = \infty$. *If* $\sigma(t) = \sigma \in \mathbb{R}/\{0\}$ *for all* $t \ge 0$, *then*

$$\limsup_{t\to\infty} |X(t)| = \infty \ a.s. \ and \ \limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 \ a.s..$$

As in the deterministic case, when the perturbation is of intermediate or critical magnitude, i.e. $L_f(\Sigma) \in (0,\infty)$, we expect the solution to inherit characteristics of both the perturbation and the ordinary differential equation (4.1.7). Indeed, our next result demonstrates that if the solution to (4.1.5) grows then its growth rate is at most of the same order of size as that of the solution to (4.1.7), possibly with a different multiplier which we can bound in terms of $L_f(\Sigma)$.

Theorem 4.4.3. *Let (4.1.2), (4.1.4), (4.2.2) and (4.4.8) hold with* $\lim_{x\to\infty} f(x) = \infty$ *and* $\sigma \notin L^2(0,\infty)$. *If* $L_f(\Sigma) \in (0,\infty)$, *then*

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 + L_f(\Sigma) \ a.s..$$

When $L_f(\Sigma) \in (1,\infty)$ we show that if the the solution to (4.1.5) fluctuates, then these fluctuations are at most of order $\Sigma(t)$ times a multiplier which we can bound in terms of $L_f(\Sigma)$.

The nonnegativity of the measure μ no longer plays an important role in the results above; primarily because we are reduced to proving upper bounds on the growth rate of solutions once solutions are no longer necessarily of one sign. For ease of exposition we have left the hypothesis (4.1.2) in place but it could equally well be replaced by the hypothesis that μ is a Borel measure with finite total variation norm equal to M with the results above unchanged.

Theorem 4.4.4. *Let (4.1.2), (4.1.4), (4.2.2) and (4.4.8) hold with* $\lim_{x\to\infty} f(x) = \infty$ *and* $\sigma \notin L^2(0,\infty)$. *If* $L_f(\Sigma) \in (1,\infty)$, *then*

$$\frac{-L_f(\Sigma)}{L_f(\Sigma)-1} \le \liminf_{t\to\infty} \frac{X(t)}{\Sigma(t)} \le \limsup_{t\to\infty} \frac{X(t)}{\Sigma(t)} \le \frac{L_f(\Sigma)}{L_f(\Sigma)-1} \ a.s.$$

Remark 4.4.2. *Under the hypotheses of Theorem 4.4.4, we can also prove that*

$$\liminf_{t\to\infty} \frac{X(t)}{\Sigma(t)} \le \frac{2-L_f(\Sigma)}{L_f(\Sigma)-1} \ a.s., \quad \limsup_{t\to\infty} \frac{X(t)}{\Sigma(t)} \ge \frac{L_f(\Sigma)-2}{L_f(\Sigma)-1} \ a.s..$$

Hence, when $L_f(\Sigma) > 2$, X is recurrent on $\mathbb{R}$. This leaves open the question of recurrence, or in other words, whether or not the process actually fluctuates, for $L_f(\Sigma) \in (1,2)$.

Finally, when the perturbation term is so large that $L_f(\Sigma) = \infty$ we expect this exogenous force to dominate the system; our intuition is confirmed by our next result. In particular, we prove that the solution to (4.1.5) is recurrent on $\mathbb{R}$ and that its fluctuations are precisely of order Σ.

Theorem 4.4.5. *Let (4.1.2), (4.1.4), (4.2.2) and (4.4.8) hold with $\lim_{x\to\infty} f(x) = \infty$ and $\sigma \notin L^2(0,\infty)$. If $L_f(\Sigma) = \infty$, then*

$$\liminf_{t\to\infty} \frac{X(t)}{\Sigma(t)} = -1 \ a.s. \ and \ \limsup_{t\to\infty} \frac{X(t)}{\Sigma(t)} = 1 \ a.s.,$$

and furthermore

$$\lim_{t\to\infty} \frac{X(t) - \int_0^t \sigma(s)dB(s)}{\Sigma(t)} = 0 \ a.s.. \tag{4.4.9}$$

4.4.2 Lévy Noise

We now assume that the semimartingale Z in (4.1.5) is an α–stable Lévy process; the results which follow further emphasize the fact that our methods do not rely on the path continuity of the process in any essential way. For the readers convenience we recall the relevant definitions from the theory of Lévy processes.

Definition 4.4.2. *If $Z = (Z)_{t\geq 0}$ is a Lévy process, then it's characteristic function $\mathscr{F}_Z$ is given by*

$$\mathscr{F}_Z(\lambda)(t) = \mathbb{E}\left[e^{i\lambda Z_t}\right] = e^{-t\Psi(\lambda)}, \qquad t \in \mathbb{R}^+, \quad \lambda \in \mathbb{R},$$

where $\Psi : \mathbb{R} \mapsto \mathbb{C}$ is of the form

$$\Psi(\lambda) = ia\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \left(1 - e^{ix\lambda} + ix\lambda \mathbb{1}_{\{|x|<1\}}\right) \Pi(dx), \tag{4.4.10}$$

with $a \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and Π a measure on $\mathbb{R}/\{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge |x|^2)\Pi(dx) < \infty$.

The number a in (4.4.10) corresponds to the linear "drift" coefficient of the Lévy process in question, σ is called the Gaussian coefficient and corresponds to the Brownian or continuous random component; Π is called the Lévy measure and represents the pure jump part of the process. A Lévy process is uniquely specified by the triple (a, σ, Π). Ψ is called the *characteristic exponent* of the process Z.

Definition 4.4.3. *For each $\alpha \in (0,2]$, a Lévy process with characteristic exponent Ψ is called a stable process with index α (α–stable for short) if $\Psi(k\lambda) = k^\alpha \Psi(\lambda)$ for each $k > 0$, $\lambda \in \mathbb{R}^d$.*

Stable processes are closely related to the class of stable distributions which gain their importance as "attractors" for normalised sums of independent and identically distributed random variables. In particular, a sum of random variables with power law decay in the tails, proportional to $|x|^{-1-\alpha}$, will tend to a stable distribution if $0 < \alpha < 2$ and to a normal distribution if $\alpha \geq 2$. Integrability of the Lévy measure forces us to consider $\alpha \in (0,2]$ and in this section we also ignore the case $\alpha = 2$ since this corresponds to the case of Brownian noise (which was considered in detail in Section 4.4.1). We tacitly exclude the degenerate case when Z is a pure drift process (i.e. σ, Π trivial) and assume for the remainder of this section that

$$Z \text{ is an } \alpha\text{–stable process with } \alpha \in (0,2). \tag{4.4.11}$$

The interested reader can consult Bertoin [25, Ch. VIII] for further details of stable processes, including the asympotic properties employed in the proofs of our results.

Our first result is a stochastic analogue of Theorem 4.3.2 and provides a sufficient condition to retain growth to infinity no faster than the solution of (4.1.7) in the presence of α–stable noise.

Theorem 4.4.6. *Let (4.1.2), (4.1.4), (4.2.1) and (4.4.11) hold. If $\lim_{x\to\infty} f(x) = \infty$ and there exists an increasing function $\gamma \in C((0,\infty);(0,\infty))$ such that $L_f(\gamma) = 0$ and $\int_0^\infty \gamma(s)^{-\alpha} ds < \infty$, then*

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \leq 1 \ a.s.$$

The next results provides a direct stochastic analogue of Theorem 4.3.5.

Theorem 4.4.7. *Let (4.1.2), (4.1.4), (4.2.2) and (4.4.11) hold with $\lim_{x\to\infty} f(x) = \infty$, and $\gamma \in C((0,\infty);(0,\infty))$ an increasing function such that $L_f(\gamma) \in (1,\infty]$. If $\int_0^\infty \gamma(s)^{-\alpha} ds < \infty$, then*

$$\limsup_{t\to\infty} \frac{|X(t)|}{\gamma(t)} \leq \frac{1}{L_f(\gamma)} \ a.s.,$$

where we interpret $1/L_f(\gamma) = 0$ if $L_f(\gamma) = \infty$. If $\int_0^\infty \gamma(s)^{-\alpha} ds = \infty$, then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\gamma(t)} = \infty \ a.s.$$

4.4.3 Stochastic Examples

Example 4.4.8. *To illustrate the practical utility of the results in Section 4.4.1, we present an example with power type nonlinearity and Brownian noise, i.e. $Z(t) = \int_0^t \sigma(s) dB(s)$. Suppose*

$$f(x) = sgn(x)|x|^\beta, \quad x \in \mathbb{R}, \quad \beta \in (0,1),$$

$\sigma(t) = t^\alpha$, $t \geq 0$, for some $\alpha > 0$, and μ is a measure obeying (4.1.2). Thus

$$\Sigma(t) \sim t^{\alpha+1/2} A(t,\alpha) \text{ as } t \to \infty, \text{ where } A(t,\alpha) = \sqrt{\frac{2\log\log t}{2\alpha+1}}, \tag{4.4.12}$$

and

$$F(x) \sim \frac{1}{1-\beta} x^{1-\beta} \text{ as } x \to \infty.$$

Clearly, $\Sigma(t) \to \infty$ as $t \to \infty$ and therefore $L_f(\Sigma) = \lim_{t\to\infty} \Sigma'(t)/Mf(\Sigma(t))$. It is straightforward to show that

$$\Sigma'(t) = t^{\alpha-1/2} \left(\frac{2}{2\alpha+1}\right)^{-1/2} \left(\log\log\left(\frac{t^{2\alpha+1}}{2\alpha+1}\right)\right)^{1/2} + \\ t^{\alpha-1/2} \left(\frac{2}{2\alpha+1}\right)^{-1/2} \left(\log\log\left(\frac{t^{2\alpha+1}}{2\alpha+1}\right)\right)^{1/2} \left(\log\left(\frac{t^{2\alpha+1}}{2\alpha+1}\right) \log\log\left(\frac{t^{2\alpha+1}}{2\alpha+1}\right)\right)^{-1},$$

for $t \geq 0$ and hence

$$L_f(\Sigma) = \begin{cases} 0, & 0 < \alpha < (1+\beta)/2(1-\beta), \\ \infty, & \alpha \geq (1+\beta)/2(1-\beta). \end{cases}$$

Now, by Theorem 4.4.2, we can conclude that the unique, strong solution of (4.1.5) obeys

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} = \limsup_{t\to\infty} \frac{|X(t)|^{1-\beta}}{M(1-\beta)t} \leq 1 \ a.s., \ 0 < \alpha < \frac{1+\beta}{2(1-\beta)}.$$

Similarly, by Theorem 4.4.5,

$$\liminf_{t\to\infty} \frac{X(t)}{A(t,\alpha)t^{\alpha+1/2}} = -1 \text{ a.s. and } \limsup_{t\to\infty} \frac{X(t)}{A(t,\alpha)t^{\alpha+1/2}} = 1 \text{ a.s., } \alpha \geq \frac{1+\beta}{2(1-\beta)},$$

where the function $A(t,\alpha)$ is given by (4.4.12).

Example 4.4.9. *Let Z be an α–stable process with index $\alpha \in (0,2)$ and, as in the previous example, suppose we have a power–type nonlinearity given by*

$$f(x) = sgn(x)|x|^{\beta}, \quad x \in \mathbb{R}, \quad \beta \in (0,1).$$

Let μ be a measure obeying (4.1.2) and let the function γ_+ be given by

$$\gamma_+(t) = (1+t)^{\epsilon}, \quad t \geq 0, \quad \epsilon > \frac{1}{\alpha} > 0.$$

By construction, γ_+ is increasing, positive and satisfies $\int_0^\infty \gamma_+(t)^{-\alpha}\,dt < \infty$. Furthermore,

$$L_f(\gamma_+) = \begin{cases} 0, & 1/\alpha < \epsilon < 1/(1-\beta), \\ \epsilon/M, & 1/\alpha < \epsilon = 1/(1-\beta), \\ \infty, & \epsilon > \max\left(1/\alpha, 1/(1-\beta)\right). \end{cases}$$

If the interval $(1/\alpha,\ 1/(1-\beta))$ is nonempty, then we can take γ in the statement of Theorem 4.4.6 to be γ_+ with $\epsilon \in (1/\alpha,\ 1/(1-\beta))$. Hence the solution of (4.1.5) obeys

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \leq 1 \text{ a.s., when } \beta > 1-\alpha.$$

This essentially means that if the nonlinearity is sufficiently strong we cannot experience growth in the solution of (4.1.5) faster than that seen in (4.1.7) with positive probability. The restriction $\beta > 1-\alpha$ is intuitive in the following sense: the smaller α is the more mass there is in the tail of the Lévy measure of Z and hence the partial maxima of Z will tend to grow faster the smaller the value of α; when α is small we require a stronger nonlinearity (larger value of β) to retain the unperturbed growth rate. When $\alpha \geq 1$, we retain the growth rate of the unperturbed equation.

If we take $\epsilon = 1/(1-\beta)$, then $L_f(\gamma_+) = 1/M(1-\beta)$ and we can apply Theorem 4.4.7 to yield

$$\limsup_{t\to\infty} \frac{|X(t)|}{t^{1/(1-\beta)}} \leq M(1-\beta) \text{ a.s., when } \beta > \max\left(1-\alpha,\ \frac{M-1}{M}\right), \tag{4.4.13}$$

where we require $\beta > (M-1)/M$ to ensure that $L_f(\gamma_+) > 1$. By Theorem 4.4.7,

$$\limsup_{t\to\infty} \frac{|X(t)|}{t^{\epsilon}} = 0 \text{ a.s., for each } \epsilon > \max\left(\frac{1}{\alpha},\ \frac{1}{1-\beta}\right).$$

In other words, the solution of (4.1.5) is $o(t^\epsilon)$ with probability one for ϵ sufficiently large (in terms of both the noise and nonlinearity). Next define the function γ_- by

$$\gamma_-(t) = (1+t)^{\delta}, \quad t \geq 0, \quad 0 < \delta \leq \frac{1}{\alpha}.$$

Note that γ_- is positive, increasing and obeys $\int_0^\infty \gamma_-(t)^{-\alpha}\,dt = \infty$. Since we aim to apply Theorem

4.4.7 we are only interested in the case $L_f(\gamma_-) \in (1,\infty]$. It is straightforward to show that

$$L_f(\gamma_-) = \begin{cases} \delta/M, & \delta = 1/(1-\beta) \le 1/\alpha, \\ \infty, & 1/(1-\beta) < \delta \le 1/\alpha. \end{cases}$$

Hence Theorem 4.4.7 yields

$$\limsup_{t\to\infty} \frac{|X(t)|}{t^{1/(1-\beta)}} = \infty \text{ a.s., when } \frac{M-1}{M} < \beta \le 1-\alpha, \quad \text{i.e. } \frac{1}{1-\beta} = \delta \le \frac{1}{\alpha},$$

and

$$\limsup_{t\to\infty} \frac{|X(t)|}{t^{\delta}} = \infty \text{ a.s. for each } \delta \text{ such that } \frac{1}{1-\beta} < \delta \le \frac{1}{\alpha}.$$

4.5 Proofs

4.5.1 Proofs of Miscellaneous Results

Proof of Lemma 4.2.1. Suppose that $x \ge a > 0$. $\phi(x) - \phi(a) = \int_a^x \phi'(u)du \ge \phi'(x)(x-a)$. Thus

$$\limsup_{x\to\infty} \frac{\phi'(x)x}{\phi(x)} = \limsup_{x\to\infty} \frac{\phi'(x)(x-a)}{\phi(x)} \frac{x}{x-a} \le \limsup_{x\to\infty} \frac{\phi(x)-\phi(a)}{\phi(x)} = 1, \tag{4.5.1}$$

establishing the first part of (4.2.3). The second claim is part of Lemma 2.6.2. □

Proof of Proposition 4.2.1. Define $J(t) = \int_0^t \phi(\gamma(s))ds$, $t \ge 0$. Then, because ϕ is increasing and invertible, $J'(t) = \phi(\gamma(t))$ and $\gamma(t) = \phi^{-1}(J'(t))$. We begin by considering the case $L_\phi(\gamma) \in (0,\infty)$, so

$$\lim_{t\to\infty} \frac{\phi^{-1}(J'(t))}{J(t)} = L_\phi(\gamma)M.$$

Thus for any $\epsilon \in (0,1)$ there exists $T(\epsilon) > 0$ such that for all $t \ge T$, $L_\phi(\gamma)M(1-\epsilon) < \phi^{-1}(J'(t))/J(t) < L_\phi(\gamma)M(1+\epsilon)$. Now since ϕ is increasing

$$\phi\left(L_\phi(\gamma)M(1-\epsilon)J(t)\right) < J'(t) < \phi\left(L_\phi(\gamma)M(1+\epsilon)J(t)\right), \tag{4.5.2a}$$

$$L_\phi(\gamma)M(1-\epsilon)J(t) < \gamma(t) < L_\phi(\gamma)M(1+\epsilon)J(t), \tag{4.5.2b}$$

for all $t \ge T(\epsilon)$. From integrating (4.5.2a) we obtain

$$\int_T^t \frac{J'(s)ds}{\phi\left(L_\phi(\gamma)M(1-\epsilon)J(s)\right)} \ge t-T; \quad \int_T^t \frac{J'(s)ds}{\phi\left(L_\phi(\gamma)M(1+\epsilon)J(s)\right)} \le t-T,$$

for all $t \ge T(\epsilon)$. If a is a positive constant, then

$$\int_T^t \frac{J'(s)ds}{\phi(aJ(s))} = \int_{aJ(T)}^{aJ(t)} \frac{du}{a\phi(u)} = \frac{1}{a}\left\{\Phi(aJ(t)) - \Phi(aJ(T))\right\}.$$

With $a = L_\phi(\gamma)M(1\pm\epsilon)$, we have

$$\frac{1}{L_\phi(\gamma)M(1-\epsilon)}\left\{\Phi(L_\phi(\gamma)M(1-\epsilon)J(t)) - \Phi(L_\phi(\gamma)M(1-\epsilon)J(T))\right\} \ge t-T,$$

$$\frac{1}{L_\phi(\gamma)M(1+\epsilon)}\left\{\Phi(L_\phi(\gamma)M(1+\epsilon)J(t)) - \Phi(L_\phi(\gamma)M(1+\epsilon)J(T))\right\} \le t-T.$$

Thus

$$\Phi(L_\phi(\gamma)M(1-\epsilon)J(t)) \geq L_\phi(\gamma)M(1-\epsilon)(t-T) + \Phi(L_\phi(\gamma)M(1-\epsilon)J(T)),$$
$$\Phi(L_\phi(\gamma)M(1+\epsilon)J(t)) \leq L_\phi(\gamma)M(1+\epsilon)(t-T) + \Phi(L_\phi(\gamma)M(1+\epsilon)J(T)), \quad t \geq T.$$

Applying the monotone function Φ to (4.5.2b), for $t \geq T$, we have

$$\Phi(\gamma(t)) > L_\phi(\gamma)M(1-\epsilon)(t-T) + \Phi(L_\phi(\gamma)M(1-\epsilon)J(T)),$$
$$\Phi(\gamma(t)) < L_\phi(\gamma)M(1+\epsilon)(t-T) + \Phi(L_\phi(\gamma)M(1+\epsilon)J(T)).$$

Taking limits across the final two sets of inequalities above we obtain

$$\liminf_{t\to\infty} \frac{\Phi(\gamma(t))}{t} \geq ML_\phi(\gamma)(1-\epsilon); \ \limsup_{t\to\infty} \frac{\Phi(\gamma(t))}{t} \leq L_\phi(\gamma)M(1+\epsilon).$$

Letting $\epsilon \to 0^+$ gives the desired result. When $L_\phi(\gamma) = 0$ we will have

$$\gamma(t) = \phi^{-1}(J'(t)) < \epsilon J(t), \ t \geq T_1(\epsilon).$$

Thus $J'(t) < \phi(\epsilon J(t))$ for all $t \geq T_1(\epsilon)$. Integrating we obtain

$$\Phi(\epsilon J(t)) < \epsilon(t - T_1) + \Phi(\epsilon J(T_1)), \ t \geq T_1.$$

Hence

$$\limsup_{t\to\infty} \frac{\Phi(\gamma(t))}{t} \leq \limsup_{t\to\infty} \frac{\Phi(\epsilon J(t))}{t} \leq \epsilon.$$

It follows immediately that $\lim_{t\to\infty} \Phi(\gamma(t))/t = 0$. When $L_\phi(\gamma) = \infty$, we have

$$\gamma(t) = \phi^{-1}(J'(t)) > NJ(t), \ t \geq T_2(N), \ N \in \mathbb{R}^+.$$

Integrating by substitution yields $\Phi(NJ(t)) \geq N(t - T_1) - \Phi(NJ(T_1)), \ t \geq T_1$. Hence

$$\liminf_{t\to\infty} \frac{\Phi(\gamma(t))}{t} \geq \liminf_{t\to\infty} \frac{\Phi(NJ(t)}{t} \geq N,$$

and letting $N \to \infty$ completes the proof that $\lim_{t\to\infty} \Phi(\gamma(t))/t = \infty$. □

4.5.2 Proofs of Results for Deterministic Volterra Equations

Proof of Theorem 4.3.1. With Φ defined by (1.3.3), condition (4.2.1) and Lemma 1.3.1 imply $F(x) \sim \Phi(x)$ as $x \to \infty$. Therefore, for every $\epsilon \in (0,1)$, there exists $x_1(\epsilon)$ such that

$$\frac{1}{1+\epsilon}\Phi(x) < F(x) < (1+\epsilon)\Phi(x), \quad x > x_1(\epsilon).$$

Thus $F^{-1}(x) > x_1(\epsilon)$ implies $\frac{1}{1+\epsilon}\Phi(F^{-1}(x)) < x$ or $x > F(x_1(\epsilon)) = x_2(\epsilon)$ implies $F^{-1}(x) < \Phi^{-1}((1+\epsilon)x)$. By hypothesis, for every $\epsilon \in (0,1)$ and $\eta \in (0,1)$, there is $T(\epsilon,\eta)$ such that

$$H(t) < \eta F^{-1}(M(1+\epsilon)t), \quad t \geq T(\epsilon,\eta).$$

Define $T_1(\epsilon) = T(\epsilon,\epsilon)$. For $t \geq T_1(\epsilon)$, $H(t) < \epsilon F^{-1}(M(1+\epsilon)t)$. Now let $T_2(\epsilon) = x_2(\epsilon)/(M(1+\epsilon))$ and $T_3 = T_1 + T_2$. Hence

$$F^{-1}(M(1+\epsilon)t) < \Phi^{-1}(M(1+\epsilon)^2 t), \quad t \geq T_3.$$

But since $t \geq T_3 \geq T_1$, we also have $H(t) < \epsilon\Phi^{-1}(M(1+\epsilon)^2 t) < \epsilon\Phi^{-1}(M(1+3\epsilon)t)$. Next, because $f(x) \sim \phi(x)$ as $x \to \infty$, there exists $x_3(\epsilon) > 0$ such that

$$\frac{1}{1+4\epsilon} < \frac{f(x)}{\phi(x)} < 1+4\epsilon, \quad x > x_4(\epsilon).$$

Since $\lim_{t\to\infty} x(t) = \infty$, there is $T_4(\epsilon) > 0$, so $x(t) > x_3(\epsilon)$ for $t \geq T_4$. Let $T^* = T_4 + T_3$ and for $t \geq T^*$ make the upper estimate

$$x(t) \leq x(0) + \epsilon\Phi^{-1}(M(1+3\epsilon)t) + x_*(\epsilon) + (1+4\epsilon)M\int_{T^*}^t \phi(x(s))ds, \tag{4.5.3}$$

where $x_*(\epsilon) = M\int_0^{T^*} f(x(s))ds$. For $t \geq T^*$, define the function z_ϵ by

$$z_\epsilon(t) = 1 + x_*(\epsilon) + \epsilon\Phi^{-1}(M(1+3\epsilon)t) + (1+4\epsilon)M\int_{T^*}^t \phi(z_\epsilon(s))ds.$$

By construction $x(t) < z_\epsilon(t)$ for all $t \geq T^*$. Since z_ϵ is differentiable we have

$$\begin{aligned} z_\epsilon'(t) &= \epsilon M(1+3\epsilon)\phi(\Phi^{-1}(M(1+3\epsilon)t)) + (1+4\epsilon)M\phi(z_\epsilon(t)), \quad \text{for each } t \geq T^*, \\ z_\epsilon(T^*) &= 1 + x_*(\epsilon) + \epsilon\Phi^{-1}(M(1+3\epsilon)T^*) = z_*(\epsilon). \end{aligned}$$

Define

$$z_+(t) = \Phi^{-1}(A(\epsilon) + M(1+8\epsilon)(t-T^*)), \; t \geq T^*,$$

where $A(\epsilon) > \Phi(z_*(\epsilon)) + M(1+8\epsilon)T^*$. Then $z_+'(t) = M(1+8\epsilon)\phi(z_+(t))$ for $t \geq T^*$ or $z_+'(t) = M(1+4\epsilon)\phi(z_+(t)) + 4M\epsilon\phi(z_+(t))$. Since $\epsilon \in (0,1)$, we have

$$4M\epsilon\phi(z_+(t)) > 4M\epsilon\phi(\Phi^{-1}(M(1+7\epsilon)t)) > \epsilon M(1+3\epsilon)\phi(\Phi^{-1}(M(1+3\epsilon)t)).$$

Hence

$$z_+'(t) > M(1+4\epsilon)\phi(z_+(t)) + \epsilon M(1+3\epsilon)\phi(\Phi^{-1}(M(1+3\epsilon)t)), \quad t \geq T^*,$$

and $z_+(T^*) = \Phi^{-1}(A(\epsilon)) > z_*(\epsilon) = z(T^*)$. From the preceding construction it follows that $z_+(t) > z_\epsilon(t) > x(t)$ for all $t \geq T^*$. Hence, from the definition of z_+,

$$\Phi(x(t)) < A(\epsilon) + M(1+8\epsilon)(t-T^*), \quad t \geq T^*.$$

It follows that $\limsup_{t\to\infty} \Phi(x(t))/t \leq M(1+8\epsilon)$ and letting $\epsilon \to 0^+$

$$\limsup_{t\to\infty} \frac{\Phi(x(t))}{Mt} \leq 1.$$

The lower bound is proved similarly and we refer the reader to Theorem 4.3.2. Since $F \sim \Phi$, we will have $\lim_{t\to\infty} F(x(t))/Mt = 1$, as claimed.

We now establish the second part of (4.3.2), namely that $\lim_{t\to\infty} x(t)/H(t) = \infty$. By hypothesis and the first part of (4.3.2), for an arbitrary $\epsilon \in (0,1)$ (chosen so small that $M(1-\epsilon)/\epsilon > 1$), there exists $T_0(\epsilon) > 0$ such that

$$F(x(t)) > M(1-\epsilon)t, \; F(H(t)) < \epsilon t, \quad t \geq T_0(\epsilon).$$

Therefore, for $t \geq T_0(\epsilon)$,

$$\frac{x(t)}{H(t)} > \frac{F^{-1}(M(1-\epsilon)t)}{F^{-1}(\epsilon t)}.$$

Hence with $K = K(\epsilon) = M(1-\epsilon)/\epsilon > 1$, and with y defined by $y'(t) = f(y(t))$ for $t > 0$ and $y(0) = 1$, we get

$$\liminf_{t\to\infty} \frac{x(t)}{H(t)} \geq \liminf_{t\to\infty} \frac{F^{-1}(M(1-\epsilon)t)}{F^{-1}(\epsilon t)} = \liminf_{\tau\to\infty} \frac{F^{-1}(K\tau)}{F^{-1}(\tau)} = \liminf_{\tau\to\infty} \frac{y(K\tau)}{y(\tau)}.$$

We show momentarily that

$$\liminf_{\tau\to\infty} \frac{y(N\tau)}{y(\tau)} \geq N, \text{ for any } N \geq 1. \tag{4.5.4}$$

Using (4.5.4) yields

$$\liminf_{t\to\infty} \frac{x(t)}{H(t)} \geq \liminf_{\tau\to\infty} \frac{y(K\tau)}{y(\tau)} \geq K = \frac{M(1-\epsilon)}{\epsilon}.$$

Letting $\epsilon \to 0^+$ yields $\liminf_{t\to\infty} x(t)/H(t) = +\infty$, as required.

Now we return to the proof of (4.5.4). Clearly, $\lim_{t\to\infty} y(t) = \infty$ and therefore there exists $T_1(\epsilon) > 0$ such that $f(y(t)) > (1-\epsilon)\phi(y(t))$ for all $t \geq T_1(\epsilon)$. Let $t \geq T_1(\epsilon)$ and $N > 1$. Using the monotonicity of ϕ, we get

$$\begin{aligned} y(Nt) &= y(t) + \int_t^{Nt} f(y(s))\, ds \\ &\geq y(t) + \int_t^{Nt} (1-\epsilon)\phi(y(s))\, ds \geq y(t) + (N-1)t(1-\epsilon)\phi(y(t)). \end{aligned}$$

Since $y(t) = F^{-1}(t)$ for $t \geq 0$, we have for $t \geq T_1(\epsilon)$

$$\frac{y(Nt)}{y(t)} \geq 1 + (1-\epsilon)(N-1)\frac{t\,\phi(F^{-1}(t))}{F^{-1}(t)}.$$

Letting $t \to \infty$ yields

$$\liminf_{t\to\infty} \frac{y(Nt)}{y(t)} \geq 1 + (1-\epsilon)(N-1) \liminf_{x\to\infty} \frac{\Phi(x)\phi(x)}{x},$$

since $\Phi(x) \sim F(x)$ as $x \to \infty$. Finally, as ϕ is increasing

$$\Phi(x) = \int_1^x \frac{1}{\phi(u)}\, du \geq \frac{x-1}{\phi(x)},$$

so

$$\liminf_{t\to\infty} \frac{y(Nt)}{y(t)} \geq 1 + (1-\epsilon)(N-1).$$

Letting $\epsilon \to 0^+$ in the inequality above gives the desired bound (4.5.4). $\square$

Proof of Theorem 4.3.2. Firstly, with $\epsilon \in (0,1)$ arbitrary, rewrite (4.1.1) as follows:

$$\begin{aligned} x(t) &\leq x(0) + H(t) + M\int_0^T f(x(s))ds + M\int_T^t f(x(s))ds \\ &\leq H_\epsilon(t) + (1+\epsilon)M\int_T^t \phi(x(s))ds, \quad t \geq T, \end{aligned}$$

where $H_\epsilon(t) = x(0) + H(t) + M\int_0^T f(x(s))ds$. Define $I_\epsilon(t) = \int_T^t \phi(x(s))ds$ for $t \geq T$, so that

$$x(t) \leq H_\epsilon(t) + (1+\epsilon)MI_\epsilon(t), \quad t \geq T. \tag{4.5.5}$$

Hence

$$I'_\epsilon(t) = \phi(x(t)) < \phi\left(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)\right), \quad t \geq T. \tag{4.5.6}$$

Note that $\lim_{t\to\infty} I_\epsilon(t) = \infty$. We claim that

$$\lim_{t\to\infty} \frac{H_\epsilon(t)}{I_\epsilon(t)} = 0. \tag{4.5.7}$$

Suppose $\limsup_{t\to\infty} H(t) < \infty$. In this case, $\limsup_{t\to\infty} H_\epsilon(t) < \infty$, but $\lim_{t\to\infty} I_\epsilon(t) = \infty$ and (4.5.7) holds.

Suppose next that $\limsup_{t\to\infty} H(t) = +\infty$. Since $f(x) \sim \phi(x)$ as $x \to \infty$, there is $x_1(\epsilon) > 0$ such that $f(x) < (1+\epsilon)\phi(x)$ for all $x \ge x_1(\epsilon)$. By the continuity of f and ϕ the number $K = K_0(\epsilon)$ given by

$$K_0(\epsilon) = \inf_{x\in(0,x_1(\epsilon))} \frac{\phi(x)}{f(x)}$$

is well–defined, and in $(0,\infty)$, even when $f(0) = 0$. Therefore, with $K_1(\epsilon) = \min(K_0(\epsilon), 1/(1+\epsilon))$, $\phi(x) \ge K_1(\epsilon) f(x)$ for all $x > 0$. Since $H(t) > 0$ for $t > 0$, the estimate

$$\int_T^t \phi(H(s))\,ds \ge K_1(\epsilon) \int_T^t f(H(s))\,ds$$

holds for $t \ge T$. Therefore,

$$\frac{H(t)}{\int_T^t \phi(H(s))\,ds} \le \frac{1}{K_1(\epsilon)} \cdot \frac{H(t)}{\int_0^t f(H(s))\,ds} \cdot \frac{\int_0^t f(H(s))\,ds}{\int_T^t f(H(s))\,ds}, \quad t \ge T. \tag{4.5.8}$$

Since f and H are positive, $t \mapsto \int_0^t f(H(s))\,ds$ tends to some $L \in (0,\infty)$ or infinity as $t \to \infty$. Suppose the former pertains. Then, because $L_f(H) = 0$, $H(t) \to 0$ as $t \to \infty$, contradicting the hypothesis that $\limsup_{t\to\infty} H(t) = \infty$. Thus, $\int_0^t f(H(s))\,ds \to \infty$ as $t \to \infty$, and the last quotient on the righthand side of (4.5.8) is an indeterminate limit as $t \to \infty$. But by l'Hôpital's rule, and because $L_f(H) = 0$,

$$\lim_{t\to\infty} \frac{H(t)}{\int_T^t \phi(H(s))\,ds} = 0.$$

To complete the proof of (4.5.7) note that positivity of H implies $\phi(x(t)) > \phi(x(0) + H(t)) > \phi(H(t))$. Thus $I_\epsilon(t) = \int_T^t \phi(x(s))ds \ge \int_T^t \phi(H(s))ds$. Hence, because $I_\epsilon(t) \to \infty$ as $t \to \infty$,

$$\begin{aligned}
\limsup_{t\to\infty} \frac{H_\epsilon(t)}{I_\epsilon(t)} &= \limsup_{t\to\infty} \left\{ \frac{x(0) + M\int_0^T f(x(s))ds}{I_\epsilon(t)} + \frac{H(t)}{I_\epsilon(t)} \right\} \\
&\le \limsup_{t\to\infty} \frac{H(t)}{\int_T^t \phi(H(s))ds} = 0,
\end{aligned}$$

and (4.5.7) holds.

Equation (4.5.7) implies that for every $\eta \in (0,1)$ there is $T'(\eta,\epsilon) > 0$ such that $H_\epsilon(t) < \eta I_\epsilon(t)$ for all $t \ge T'(\eta,\epsilon)$. Hence for $t \ge T'(\epsilon,\epsilon)$, $H_\epsilon(t) < M\epsilon I_\epsilon(t)$. Then for $t \ge T_2 = T + T'$,

$$I_\epsilon'(t) < \phi(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)) < \phi(M(1+2\epsilon)I_\epsilon(t)).$$

Integrating we obtain

$$\int_{T_2}^t \frac{I_\epsilon'(s)ds}{\phi(M(1+2\epsilon)I_\epsilon(t))} \le t - T_2, \quad t \ge T_2.$$

Integrating by substitution with $u = M(1+2\epsilon)I_\epsilon(s)$ yields

$$\Phi\left(M(1+2\epsilon)I_\epsilon(t)\right) - \Phi\left(M(1+2\epsilon)I_\epsilon(T_2)\right) \le M(1+2\epsilon)(t - T_2), \quad t \ge T_2.$$

Letting $\Phi_\epsilon = \Phi\left(M(1+2\epsilon)I_\epsilon(T_2)\right)$

$$I_\epsilon(t) \le \frac{1}{M(1+2\epsilon)}\Phi^{-1}(\Phi_\epsilon + M(1+2\epsilon)(t-T_2)), \quad t \ge T_2.$$

From (4.5.5) we have $x(t) \le H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)$ for $t \ge T$ and for $t \ge T'$ we have $H_\epsilon(t) < M\epsilon I_\epsilon(t)$. Hence for $t \ge T_2$

$$x(t) \le M\epsilon I_\epsilon(t) + M(1+\epsilon)I_\epsilon(t) = M(1+2\epsilon)I_\epsilon(t) \le \Phi^{-1}(\Phi_\epsilon + M(1+2\epsilon)(t-T_2)).$$

Therefore $\Phi(x(t)) < \Phi_\epsilon + M(1+2\epsilon)(t-T_2)$ and $\limsup_{t\to\infty}\Phi(x(t))/t \le M(1+2\epsilon)$. Letting $\epsilon \to 0^+$ we have $\Phi(x(t))/Mt \le 1$ and, since $F(x) \sim \Phi(x)$ as $x \to \infty$ by Lemma 1.3.1, this implies

$$\limsup_{t\to\infty}\frac{F(x(t))}{Mt} \le 1.$$

We now proceed to compute the corresponding lower bound. Since $\lim_{t\to\infty} M(t) = M < \infty$, there exists $T_3 > 0$ such that $M(t) > M(1-\epsilon)$, for all $t \ge T_3$, with $\epsilon \in (0,1)$ arbitrary. For $t \ge 2T_3$,

$$\begin{aligned} x(t) &\ge x(0) + \int_0^{T_3} M(t-s)f(x(s))ds + \int_{T_3}^t M(t-s)f(x(s))ds \\ &\ge x(0) + (1-\epsilon)\int_{T_3}^t M(t-s)\phi(x(s))ds \ge x(0) + (1-\epsilon)^2 M\int_{T_3}^t \phi(x(s))ds. \end{aligned}$$

Letting $y(t) = x(t+T)$ for $t \ge 2T_3$, it is straightforward to show that

$$y(t) \ge x(0) + M(1-\epsilon)^2\int_0^{t-T_3}\phi(y(u))du, \quad t \ge T_3.$$

Now define the lower comparison solution

$$z(t) = z^* + M(1-\epsilon)^2\int_0^{t-T_3}\phi(z(u))du, \quad t \ge T_3,$$

and $z(t) = z^* = \frac{1}{2}\min_{t\in[0,2T_3]} x(t)$, $t \in [0,T_3]$. Thus for $t \in [0,T_3]$, $y(t) = x(t+T_3) > z^* = z(t)$ and $z^* < x(0)$. Now suppose that $y(t) > z(t)$ for $t \in [0,\bar{T})$, $\bar{T} > T_3$, but $y(\bar{T}) = z(\bar{T})$. Then $s \in [0,\bar{T}-T_3]$ implies $\phi(y(s)) > \phi(z(s))$ and $\int_0^{\bar{T}-T_3}\phi(y(s))ds \ge \int_0^{\bar{T}-T_3}\phi(z(s))ds$. Therefore

$$\begin{aligned} y(\bar{T}) &\ge x(0) + M(1-\epsilon)^2\int_0^{\bar{T}-T_3}\phi(y(s))ds \ge x(0) + M(1-\epsilon)^2\int_0^{\bar{T}-T_3}\phi(z(s))ds \\ &> z^* + M(1-\epsilon)^2\int_0^{\bar{T}-T_3}\phi(z(s))ds = z(\bar{T}) = y(\bar{T}), \end{aligned}$$

a contradiction. Hence $x(t+T_3) = y(t) > z(t)$ for all $t \ge 0$. For $t \ge T_3$, $z'(t) = M(1-\epsilon)^2\phi(z(t-T_3))$ and thus by [19, Corollary 2], $\lim_{t\to\infty}\Phi(z(t))/t = M(1-\epsilon)^2$, under (4.2.1). Hence

$$\liminf_{t\to\infty}\frac{\Phi(x(t+T_3))}{t} \ge \liminf_{t\to\infty}\frac{\Phi(z(t))}{t} \ge M(1-\epsilon)^2.$$

Thus

$$M(1-\epsilon)^2 \le \liminf_{t\to\infty}\frac{\Phi(x(t))}{t-T_3} = \liminf_{t\to\infty}\frac{\Phi(x(t))}{t}.$$

Recall Lemma 1.3.1 and let $\epsilon \to 0^+$ to obtain $\liminf_{t\to\infty} F(x(t))/Mt \ge 1$, proving the first limit in (4.3.3). The proof of the second limit in (4.3.3) is identical to the proof of the same statement in

Theorem 4.3.1. □

Proof of Theorem 4.3.3. The required lower bound, $\liminf_{t\to\infty} F(x(t))/Mt \ge 1$, can be derived exactly as in Theorem 4.3.2. For the upper bound begin by recalling the estimate (4.5.6) from the proof of Theorem 4.3.2:

$$I_\epsilon'(t) < \phi\left(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)\right), \quad t \ge T,$$

where $I_\epsilon(t) = \int_T^t \phi(x(s))\,ds$ for $t \ge T$ and $H_\epsilon(t) = x(0) + H(t) + M\int_0^T f(x(s))\,ds$.

Remark 4.5.1. *The stronger hypothesis* (4.2.2) *can be used to improve the estimate above. We state this improvement here for convenience. Using the mean value theorem,* (4.2.2)*, and the first part of Lemma 4.2.1, estimate as follows:*

$$\begin{aligned} I_\epsilon'(t) &\le \phi(H_\epsilon(t)) + \phi'(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)\theta_t)M(1+\epsilon)I_\epsilon(t) \\ &\le \phi(H_\epsilon(t)) + \phi'(H_\epsilon(t))M(1+\epsilon)I_\epsilon(t) \le \phi(H_\epsilon(t)) + \frac{\phi(H_\epsilon(t))}{H_\epsilon(t)}M(1+\epsilon)^2 I_\epsilon(t), \end{aligned} \tag{4.5.9}$$

where $\theta_t \in [0,1]$ results from using the mean value theorem. The differential inequality above is now linear in $I_\epsilon(t)$ and can be solved explicitly.

Next, since $x(t) > H(t)$, $\phi(x(t)) > \phi(H(t))$ and

$$\frac{H_\epsilon(t)}{MI_\epsilon(t)} = \frac{H_\epsilon(t)}{H(t)}\frac{H(t)}{M\int_T^t \phi(x(s))ds} \le \frac{H_\epsilon(t)}{H(t)}\frac{H(t)}{M\int_0^t \phi(H(s))ds}\frac{\int_0^t \phi(H(s))ds}{\int_T^t \phi(H(s))ds}, \quad t \ge T.$$

Hence

$$\limsup_{t\to\infty}\frac{H_\epsilon(t)}{MI_\epsilon(t)} \le L_\phi(H)\limsup_{t\to\infty}\left\{\frac{H_\epsilon(t)}{H(t)}\frac{\int_0^t \phi(H(s))ds}{\int_T^t \phi(H(s))ds}\right\} = L_\phi(H).$$

Thus $H_\epsilon(t) < ML_\phi(H)(1+\epsilon)I_\epsilon(t)$ for $t \ge T' > T$. Combine this estimate with (4.5.6) to obtain

$$I_\epsilon'(t) \le \phi(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)) \le \phi((M + ML_\phi(H))(1+\epsilon)I_\epsilon(t)), \quad t \ge T'.$$

Integrated the inequality above reads

$$\int_{T'}^t \frac{I_\epsilon'(s)ds}{\phi((M+ML_\phi(H))(1+\epsilon)I_\epsilon(s))} \le t - T', \quad t \ge T'.$$

Make the substitution $u = (M + ML_\phi(H))(1+\epsilon)I_\epsilon(s)$ to obtain

$$\Phi((M+ML_\phi(H))(1+\epsilon)I_\epsilon(t)) - \Phi((M+ML_\phi(H))(1+\epsilon)I_\epsilon(T')) \le \\ (M+ML_\phi(H))(1+\epsilon)(t-T').$$

Define $\Phi_\epsilon = (M + ML_\phi(H))(1+\epsilon)I_\epsilon(T')$, so

$$M(1+L_\phi(H))(1+\epsilon)I_\epsilon(t) \le \Phi^{-1}(\Phi_\epsilon + (M+ML_\phi(H))(1+\epsilon)(t-T')).$$

Now combine equation (4.5.5) with the inequality above as follows:

$$\begin{aligned} x(t) &\le H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t) < M(1+\epsilon)(1+L_\phi(H))I_\epsilon(t) \\ &< \Phi^{-1}(\Phi_\epsilon + M(1+L_\phi(H))(1+\epsilon)(t-T')), \end{aligned}$$

for all $t \geq T'$. Thus

$$\Phi(x(t)) < \Phi_\epsilon + M(1 + L_\phi(H))(1+\epsilon)(t - T'), \quad t \geq T',$$

and letting $t \to \infty$ yields $\limsup_{t\to\infty} \Phi(x(t))/Mt \leq (1 + L_\phi(H))(1+\epsilon)$. Recall Lemma 1.3.1 and let $\epsilon \to 0^+$ to obtain

$$\limsup_{t\to\infty} \frac{F(x(t))}{Mt} \leq 1 + L_f(H).$$

Now assume that (4.2.2) holds and show that $\liminf_{t\to\infty} x(t)/H(t) \geq 1 + 1/L_f(H)$. Since $t \mapsto M(t)$ is increasing there exists $T_2(\epsilon) > 0$ such that $M(t) > (1-\epsilon)M$ for all $t \geq T_2(\epsilon)$. Also, $f(x) > (1-\epsilon)\phi(x)$ for all $x \geq x_1(\epsilon)$ and owing to the divergence of $x(t)$ there exists $T_1(\epsilon)$ such that $x(t) > x_1(\epsilon)$ for all $t \geq T_1(\epsilon)$. Therefore

$$x(t) > H(t) + \int_{T_1}^{t-T_2} M(t-s) f(x(s))\, ds > H(t) + M(1-\epsilon)^2 \int_{T_1}^{t-T_2} \phi(x(s))\, ds, \quad t > T_1 + T_2.$$

Then, since $x(t) > H(t)$ for all $t \geq 0$,

$$x(t) > H(t) + M(1-\epsilon)^2 \int_{T_1}^{t-T_2} \phi(H(s)) ds, \quad t \geq T_1 + T_2,$$

and it follows immediately that

$$\frac{x(t)}{H(t)} > 1 + \frac{1}{L_f(H)} \frac{L_f(H)\, M(1-\epsilon)^2 \int_{T_1}^{t-T_2} \phi(H(s)) ds}{H(t)}, \quad t \geq T_1 + T_2. \tag{4.5.10}$$

By hypothesis $H(t) \sim L_f(H)\, M \int_0^t \phi(H(s)) ds$ as $t \to \infty$ and consequently

$$\max_{t-T_2 \leq s \leq t} H(s) \sim \max_{t-T_2 \leq s \leq t} L_f(H)\, M \int_0^s \phi(H(u)) du = L_f(H)\, M \int_0^t \phi(H(s)) ds.$$

Furthermore, because ϕ preserves asymptotic equivalence (see Lemma 2.6.2),

$$\phi\left(\max_{t-T_2 \leq s \leq t} H(s)\right) \sim \phi\left(L_f(H)\, M \int_0^t \phi(H(s)) ds\right) \sim \phi(H(t)) \text{ as } t \to \infty.$$

Hence

$$\limsup_{t\to\infty} \frac{\int_{t-T_2}^t \phi(H(s)) ds}{\phi(H(t))} \leq \limsup_{t\to\infty} \frac{T_2\, \phi\left(\max_{t-T_2 \leq s \leq t} H(s)\right)}{\phi(H(t))} = T_2.$$

Using the facts collected above compute as follows:

$$\limsup_{t\to\infty} \frac{\int_{t-T_2}^t \phi(H(s))}{H(t)} = \limsup_{t\to\infty} \frac{\int_{t-T_2}^t \phi(H(s)) ds}{\phi(H(t))} \frac{\phi(H(t))}{H(t)} \leq T_2 \limsup_{t\to\infty} \frac{\phi(H(t))}{H(t)} = 0.$$

Similarly, because $\lim_{t\to\infty} H(t) = \infty$, $\lim_{t\to\infty} \int_0^{T_1} \phi(H(s)) ds / H(t) = 0$. Thus

$$\lim_{t\to\infty} \frac{L_f(H)\, M \int_{T_1}^{t-T_2} \phi(H(s)) ds}{H(t)} = 1.$$

Returning to (4.5.10) and using the limit above yields

$$\liminf_{t\to\infty} \frac{x(t)}{H(t)} \geq 1 + \frac{(1-\epsilon)^2}{L_f(H)}.$$

Finally, let $\epsilon \to 0^+$ to give the desired conclusion. $\square$

Proof of Theorem 4.3.4 (a.) Hypotheses (4.1.4) and (4.2.2) imply that there exists a function $\phi \in C^1(\mathbb{R}^+;\mathbb{R}^+)$ and $K(\epsilon) > 0$ such that

$$|f(x)| < K(\epsilon) + (1+\epsilon)\phi(|x|), \text{ for all } x \in \mathbb{R}. \tag{4.5.11}$$

Now use equation (4.5.11) to derive the following preliminary upper estimate on the size of the solution:

$$|x(t)| < |x(0)| + |H(t)| + M\,K(\epsilon)\,t + M(1+\epsilon)\int_0^t \phi(|x(s)|)ds, \quad t \geq 0.$$

By L'Hôpital's rule, $\lim_{x\to\infty} \Phi(x)/x = \lim_{x\to\infty} 1/\phi(x) = 0$ and hence $\lim_{t\to\infty} \Phi(\gamma(t))/\gamma(t) = 0$. By Proposition 4.2.3, and since $L_f(\gamma) \in (1,\infty)$ by hypothesis,

$$\lim_{t\to\infty} \frac{A+Bt}{\gamma(t)} = \lim_{t\to\infty} \frac{A+Bt}{\Phi(\gamma(t))}\frac{\Phi(\gamma(t))}{\gamma(t)} = 0, \tag{4.5.12}$$

for any nonnegative constants A and B. Thus there exists $T(\epsilon) > 0$ such that for all $t \geq T(\epsilon)$ we have $|x(0)| + M\,(K(\epsilon)\,t < \epsilon\,\gamma(t)$. By (4.3.6), and the previous estimate, there exists $T_2(\epsilon) > T(\epsilon)$ such that for all $t \geq T_2(\epsilon)$, $|x(0)| + M\,K(\epsilon)\,t + |H(t)| < (1+\epsilon)\gamma(t)$. Combining this with our initial estimate we obtain

$$|x(t)| < (1+\epsilon)\gamma(t) + M(1+\epsilon)\int_0^t \phi(|x(s)|)ds, \quad t \geq T_2(\epsilon).$$

Choose $x^* = \max_{0\leq s\leq T_2} \phi(|x(s)|)$, so that $\int_0^{T_2} \phi(|x(s)|)ds \leq T_2\,x^*$. Hence

$$|x(t)| < T_2\,x^* + (1+\epsilon)\gamma(t) + M(1+\epsilon)\int_{T_2}^t \phi(|x(s)|)ds, \quad t \geq T_2.$$

Define the upper comparison solution, x_+, as follows:

$$\begin{aligned} x_+(t) &= 1 + T_2\,x^* + (1+\epsilon)\gamma(t) + M(1+\epsilon)\int_{T_2}^t \phi(x_+(s))\,ds \\ &= \gamma_\epsilon(t) + M(1+\epsilon)I_\epsilon(t), \quad t \geq T_2, \end{aligned} \tag{4.5.13}$$

where $\gamma_\epsilon(t) = 1 + T_2\,x^* + (1+\epsilon)\gamma(t)$ and $I_\epsilon(t) = \int_{T_2}^t \phi(x_+(s))\,ds$. By construction, $|x(t)| < x_+(t)$ for all $t \geq T_2$ (this follows immediately via a "time of the first breakdown" argument). Applying the same estimation procedures as in Theorems 4.3.2 and 4.3.3 to x_+, and in particular to the quantity $I_\epsilon(t)$, we obtain an estimate analogous to (4.5.9):

$$I_\epsilon'(t) < \phi(\gamma_\epsilon(t)) + a_\epsilon(t)I_\epsilon(t), \quad t \geq T_3(\epsilon), \tag{4.5.14}$$

where $a_\epsilon(t) = M(1+\epsilon)^2\phi(\gamma_\epsilon(t))/\gamma_\epsilon(t)$. Note once more that the hypothesis (4.2.2) is needed to obtain the differential inequality (4.5.14). Before proceeding further with the line of argument from Theorem 4.3.3 we need to refine the estimate above. $L_f(\gamma) \in (0,\infty)$ implies that $\lim_{t\to\infty}\gamma(t) = \infty$ and thus $\limsup_{t\to\infty} \phi(\gamma_\epsilon(t))/\phi(\gamma(t)) \leq (1+\epsilon)$ by Lemma 4.2.1. Therefore there exists a $T_4(\epsilon) > T_3(\epsilon)$ such that for all $t \geq T_4$ we have $\phi(\gamma_\epsilon(t)) < (1+\epsilon)^2\phi(\gamma(t))$. Hence

$$I_\epsilon'(t) < (1+\epsilon)^2\phi(\gamma(t)) + M(1+\epsilon)^4\frac{\phi(\gamma(t))}{\phi(\gamma_\epsilon(t))}I_\epsilon(t), \quad t \geq T_4.$$

$\gamma_\epsilon(t) \sim (1+\epsilon)\gamma(t)$ as $t \to \infty$ implies that there exists $T_5(\epsilon) > T_4(\epsilon)$ such that $\gamma_\epsilon(t) > (1-\epsilon)(1+\epsilon)\gamma(t)$

for all $t \geq T_5$. Taking reciprocals of the previous inequality and apply it to the previous estimate of $I_\epsilon'(t)$ to obtain

$$I_\epsilon'(t) < (1+\epsilon)^2\phi(\gamma(t)) + M(1+\epsilon)^3\frac{\phi(\gamma(t))}{(1-\epsilon)\phi(\gamma(t))}I_\epsilon(t), \quad t \geq T_5.$$

Now let

$$\alpha_\epsilon = (1+\epsilon)^2, \quad a_\epsilon(t) = M(1+\epsilon)^3\frac{\phi(\gamma(t))}{(1-\epsilon)\phi(\gamma(t))},$$

to obtain the consolidated estimate

$$I_\epsilon'(t) \leq \alpha_\epsilon\,\phi(\gamma(t)) + a_\epsilon(t)\,I_\epsilon(t), \quad t \geq T_5. \tag{4.5.15}$$

Let $T' > T_5$ and solve the differential inequality above as follows

$$\begin{aligned}\frac{d}{dt}\left(I_\epsilon(t)e^{-\int_{T'}^t a_\epsilon(s)ds}\right) &= I_\epsilon'(t)e^{-\int_{T'}^t a_\epsilon(s)ds} - a_\epsilon(t)I_\epsilon(t)e^{-\int_{T'}^t a_\epsilon(s)ds}\\ &= e^{-\int_{T'}^t a_\epsilon(s)ds}\{I_\epsilon'(t) - a_\epsilon(t)I_\epsilon(t)\} \leq \alpha_\epsilon\,\phi(\gamma(t))e^{-\int_{T'}^t a_\epsilon(s)ds}, \quad t \geq T'.\end{aligned}$$

Integration yields

$$I_\epsilon(t)e^{-\int_{T'}^t a_\epsilon(s)ds} \leq I_\epsilon(T') + \alpha_\epsilon\int_{T'}^t \phi(\gamma(s))e^{-\int_{T'}^s a_\epsilon(u)du}ds, \quad t \geq T'.$$

Hence

$$\frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \leq \frac{I_\epsilon(T')}{\int_{T'}^t \phi(\gamma(s))ds\,e^{-\int_{T'}^t a_\epsilon(s)ds}} + \frac{\alpha_\epsilon\int_{T'}^t \phi(\gamma(s))e^{-\int_{T'}^s a_\epsilon(u)du}ds}{\int_{T'}^t \phi(\gamma(s))ds\,e^{-\int_{T'}^t a_\epsilon(s)ds}}, \quad t \geq T'. \tag{4.5.16}$$

In the analysis which is required to show that the second term on the right-hand side of (4.5.16) is bounded it emerges that the first term on the right-hand side is also bounded so we immediately focus on the second term. Define

$$C_\epsilon(t) = \alpha_\epsilon\int_{T'}^t \phi(\gamma(s))e^{-\int_{T'}^s a_\epsilon(u)du}\,ds, \quad B_\epsilon(t) = \int_{T'}^t \phi(\gamma(s))\,ds\,e^{-\int_{T'}^t a_\epsilon(s)ds},$$

and restate (4.5.16) as

$$\frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \leq \frac{I_\epsilon(T')}{B_\epsilon(t)} + \frac{C_\epsilon(t)}{B_\epsilon(t)}, \quad t \geq T'.$$

By inspection $C_\epsilon'(t) > 0$, so either $\lim_{t\to\infty} C_\epsilon(t) = \infty$ or $\lim_{t\to\infty} C_\epsilon(t) = C(\epsilon) \in (0,\infty)$. Differentiating B_ϵ we obtain

$$\begin{aligned}B_\epsilon'(t) &= \phi(\gamma(t))e^{-\int_{T'}^t a_\epsilon(s)ds} - a_\epsilon(t)e^{-\int_{T'}^t a_\epsilon(s)ds}\int_{T'}^t \phi(\gamma(s))\,ds\\ &= e^{-\int_{T'}^t a_\epsilon(s)ds}\left\{\phi(\gamma(t)) - a_\epsilon(t)\int_{T'}^t \phi(\gamma(s))\,ds\right\}\\ &= C_\epsilon'(t)\left\{\frac{1}{\alpha_\epsilon} - a_\epsilon(t)\frac{\int_{T'}^t \phi(\gamma(s))\,ds}{\alpha_\epsilon\,\phi(\gamma(t))}\right\} = C_\epsilon'(t)\left\{\frac{1}{\alpha_\epsilon} - M\frac{(1+\epsilon)^4}{(1-\epsilon)}\frac{\int_{T'}^t \phi(\gamma(s))\,ds}{\alpha_\epsilon\,\gamma(t)}\right\}.\end{aligned}$$

Hence

$$\frac{B_\epsilon'(t)}{C_\epsilon'(t)} = \frac{1}{\alpha_\epsilon} - \frac{(1+\epsilon)^3}{(1-\epsilon)}\frac{M\int_T^t \phi(\gamma(s))ds}{\alpha_\epsilon\,\gamma(t)}, \quad t \geq T'. \tag{4.5.17}$$

Therefore, for ϵ sufficiently small,

$$\lim_{t\to\infty} \frac{B_\epsilon'(t)}{C_\epsilon'(t)} = \frac{1}{\alpha_\epsilon} - \frac{(1+\epsilon)^3}{(1-\epsilon)\,\alpha_\epsilon\, L_\phi(\gamma)} > 0. \tag{4.5.18}$$

Remark 4.5.2. *Note that the hypothesis $L_\phi(\gamma) > 1$ implies that $B_\epsilon(t)$ is eventually increasing and hence has a limit $B(\epsilon) \in (0,\infty]$ at infinity. If $\lim_{t\to\infty} C_\epsilon(t) = \infty$ and $L_\phi(\gamma) \in (0,1]$, $B_\epsilon(t)$ is eventually decreasing and $\lim_{t\to\infty} B_\epsilon(t) \in [0,\infty)$. In this case $\lim_{t\to\infty} B_\epsilon(t) = 0$ for all $\epsilon \in (0,1)$ and we will be unable to obtain the required estimates to continue the proof.*

From (4.5.18), by asymptotic integration, the convergence and divergence of B_ϵ and C_ϵ are equivalent. Hence

$$\lim_{t\to\infty} \frac{C_\epsilon(t)}{B_\epsilon(t)} = \begin{cases} \left(1/\alpha_\epsilon - (1+\epsilon)^3/(1-\epsilon)\,\alpha_\epsilon\, L_\phi(\gamma)\right)^{-1}, & \lim_{t\to\infty} C_\epsilon(t) = \infty, \\ C_\epsilon / B_\epsilon, & \lim_{t\to\infty} C_\epsilon(t) = C(\epsilon). \end{cases}$$

In both cases

$$\limsup_{t\to\infty} \frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} = K(\epsilon) < \infty.$$

Therefore there exists $\bar{T} > T'$ such that $I_\epsilon(t) < K(\epsilon)(1+\epsilon)\int_{T'}^t \phi(\gamma(s))ds$ for all $t \geq \bar{T}$. Thus, recalling (4.5.13),

$$x_+(t) = \gamma_\epsilon(t) + M(1+\epsilon)I_\epsilon(t) \leq (1+2\epsilon)\gamma(t) + M(1+\epsilon)^2 K(\epsilon) \int_{T'}^t \phi(\gamma(s))\,ds,$$

for $t \geq \bar{T}$. Hence

$$\begin{aligned} \limsup_{t\to\infty} \frac{x_+(t)}{\gamma(t)} &\leq 1 + 2\epsilon + M(1+\epsilon)^2 K(\epsilon) \limsup_{t\to\infty} \frac{\int_{T'}^t \phi(\gamma(s))ds}{\gamma(t)} \\ &= 1 + 2\epsilon + \frac{(1+\epsilon)^2 K(\epsilon)}{L_\phi(\gamma)} < \infty. \end{aligned}$$

Therefore, since $|x(t)| < x_+(t)$ for all $t \geq T_2$, $\limsup_{t\to\infty} |x(t)|/\gamma(t) < \infty$. Now let

$$\limsup_{t\to\infty} \frac{|x(t)|}{\gamma(t)} = \lambda \in [0,\infty), \tag{4.5.19}$$

One can compute a definite upper bound on λ in terms of the problem parameters as follows. Define $J(t) = \int_0^t M(t-s)f(x(s))ds$ and estimate as above

$$\begin{aligned} |J(t)| &\leq M \int_0^t K(\epsilon) + (1+\epsilon)\phi(|x(s)|)ds \\ &\leq M\,K(\epsilon)\,t + M\,T_2\,(1+\epsilon) \sup_{s\in[0,T_2]} \phi(|x(s)|) + M(1+\epsilon)\int_{T_2}^t \phi(|x(s)|)ds, \end{aligned} \tag{4.5.20}$$

for $t \geq T_2$. Using (4.5.19) there exists a $\bar{T}(\epsilon) > T_2$ such that

$$\limsup_{t\to\infty} \frac{|J(t)|}{\gamma(t)} \leq M(1+\epsilon) \limsup_{t\to\infty} \frac{\int_{\bar{T}}^t \phi((\lambda+\epsilon)\gamma(s))ds}{\gamma(t)} \leq \frac{\max(1,\,\lambda+\epsilon)}{L_\phi(\gamma)}.$$

Return to (4.1.1), take absolute values and apply the estimates above as follows

$$\lambda = \limsup_{t\to\infty} \frac{|x(t)|}{\gamma(t)} \leq \limsup_{t\to\infty} \frac{|x(0)|}{\gamma(t)} + \limsup_{t\to\infty} \frac{|H(t)|}{\gamma(t)} + \limsup_{t\to\infty} \frac{|J(t)|}{\gamma(t)} \leq 1 + \frac{\max(1,\,\lambda)}{L_f(\gamma)}. \tag{4.5.21}$$

Solving the inequalities above yields

$$\lambda \leq \max\left((1+L_f(\gamma))/L_f(\gamma),\, L_f(\gamma)/(L_f(\gamma)-1)\right).$$

In fact the second quantity is always larger, so

$$\limsup_{t\to\infty} \frac{|x(t)|}{\gamma(t)} \leq \frac{L_f(\gamma)}{L_f(\gamma)-1}.$$

□

Proof of Theorem 4.3.4 (b.) Follow the argument of Theorem 4.3.4 (*a.*) exactly to equation (4.5.17), which we recall below.

$$\frac{B_\epsilon'(t)}{C_\epsilon'(t)} = \frac{1}{\alpha_\epsilon} - \frac{(1+\epsilon)^3}{(1-\epsilon)} \frac{M\int_T^t \phi(\gamma(s))ds}{\alpha_\epsilon\,\gamma(t)}, \quad t \geq T'.$$

Now $L_f(\gamma) = \infty$ implies $\lim_{t\to\infty} B_\epsilon'(t)/C_\epsilon'(t) = 1/\alpha_\epsilon$. Thus $0 < C_\epsilon'(t) \sim \alpha_\epsilon B_\epsilon'(t)$ as $t \to \infty$. Recall equation (4.5.16)

$$\frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \leq \frac{I_\epsilon(T')}{B_\epsilon(t)} + \frac{C_\epsilon(t)}{B_\epsilon(t)}, \quad t \geq T'.$$

If $\lim_{t\to\infty} C_\epsilon(t) = \infty$, then $\lim_{t\to\infty} B_\epsilon(t) = \infty$ and $C_\epsilon(t) \sim \alpha_\epsilon B_\epsilon(t)$ as $t \to \infty$. Thus, when $C_\epsilon(t) \to \infty$ as $t \to \infty$,

$$\limsup_{t\to\infty} \frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \leq \alpha_\epsilon.$$

Alternatively, if $\lim_{t\to\infty} C_\epsilon(t) = C(\epsilon)$, $\lim_{t\to\infty} B_\epsilon(t) = B(\epsilon) \in (0,\infty)$, then

$$\limsup_{t\to\infty} \frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \leq \frac{I_\epsilon(T') + C(\epsilon)}{B(\epsilon)}.$$

In both cases

$$\limsup_{t\to\infty} \frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \leq K(\epsilon) < \infty.$$

Therefore $\limsup_{t\to\infty} x_+(t)/\gamma(t) < \infty$ and hence $\limsup_{t\to\infty} |x(t)|/\gamma(t) < \infty$. By an argument exactly analogous to that which completes the proof of Theorem 4.3.4 case (*a.*) we can show that $\lim_{t\to\infty} |J(t)|/\gamma(t) = 0$. Now write

$$\frac{x(t)}{\gamma(t)} = \frac{x(0)}{\gamma(t)} + \frac{J(t)}{\gamma(t)} + \frac{H(t)}{\gamma(t)}, \quad t \geq 0. \tag{4.5.22}$$

Because $\limsup_{t\to\infty} |H(t)|/\gamma(t) = 1$, either $\limsup_{t\to\infty} H(t)/\gamma(t) = 1$, or $\liminf_{t\to\infty} H(t)/\gamma(t) = -1$. Since $\lim_{t\to\infty} J(t)/\gamma(t) = 0$, taking the limsup and liminf across (4.5.22) gives $\limsup_{t\to\infty} x(t)/\gamma(t) = 1$ or $\liminf_{t\to\infty} x(t)/\gamma(t) = -1$. In both cases, $\limsup_{t\to\infty} |x(t)|/\gamma(t) = 1$. Noting that $J(t)/\gamma(t) \sim (x(t) - H(t))/\gamma(t)$ as $t \to \infty$ yields the second part of the conclusion. □

Proof of Theorem 4.3.5 (*a.*). The argument of Theorem 4.3.4 (*a.*) implies that $\limsup_{t\to\infty} |x(t)|/\gamma_+(t) < \infty$. Let $\lambda_+ = \limsup_{t\to\infty} |x(t)|/\gamma_+(t) \in [0,\infty)$ and estimate as before to obtain $\limsup_{t\to\infty} |J(t)|/\gamma_+(t) \leq \max(1,\, \lambda_+)/L_f(\gamma_+)$. Calculating as in (4.5.21) then yields $\lambda_+ \leq \max(1,\, \lambda_+)/L_f(\gamma_+)$. In fact, in all cases, $\limsup_{t\to\infty} |x(t)|/\gamma(t) \in [0, 1/L_f(\gamma_+)]$.

For the second part of the claim, suppose to the contrary that

$$\limsup_{t\to\infty} |x(t)|/\gamma_-(t) = \lambda_- < \infty.$$

Now argue, as in Theorem 4.3.4, that $\limsup_{t\to\infty} |J(t)|/\gamma_-(t) < \max(1,\lambda_-)/L_\phi(\gamma_-)$, where $J(t) = \int_0^t M(t-s)f(x(s))\,ds$. However, by rearranging (4.1.1) and taking absolute values

$$|H(t)| \le |x(0)| + |x(t)| + |J(t)|, \quad t \ge 0.$$

Dividing across by γ_- and taking the limsup yields $\limsup_{t\to\infty} |H(t)|/\gamma_-(t) < \infty$, in contradiction to (4.3.7). Hence $\lambda_- = \infty$, as claimed. □

Proof of Theorem 4.3.5 (*b.*). As in case (*a.*), the proof is a consequence of Theorem 4.3.4. The stronger conclusion, $\lim_{t\to\infty} |x(t)|\gamma_+(t) = 0$, holds because in (4.5.21) we now have $\limsup_{t\to\infty} |H(t)|/\gamma_+(t) = 0$ and $\limsup_{t\to\infty} |J(t)|/\gamma_+(t) = 0$. The argument to show that $\limsup_{t\to\infty} |x(t)|/\gamma_-(t) = \infty$ is essentially unchanged from the proof above. □

4.5.3 Proofs with Regular Variation

We divide the proof of Theorem 4.3.7 into several separate results in order to make it more manageable for the reader. We also choose, in some instances, to state and prove results in a slightly different form to that which appear in Theorem 4.3.7 so that the proofs may proceed more naturally.

Theorem 4.5.1. *If* (4.3.11), (4.1.2), *and* (4.1.3) *hold, then the following are equivalent:*

$$(i.) \quad \lim_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} = \zeta \in [1,\infty), \quad (ii.) \quad \lim_{t\to\infty} \frac{H(t)}{F^{-1}(Mt)} = \lambda \in [0,\infty).$$

Moreover,

$$\zeta = \zeta^\beta + \lambda. \tag{4.5.23}$$

Proof of Theorem 4.5.1. Suppose that (*i.*) holds and rewrite (4.1.10) in the following form

$$\frac{H(t)}{F^{-1}(Mt)} = \frac{x(t)}{F^{-1}(Mt)} - \frac{x(0)}{F^{-1}(Mt)} - \frac{\int_0^t M(t-s)f(x(s))ds}{F^{-1}(Mt)}, \quad t \ge 0. \tag{4.5.24}$$

If we can show that

$$\lim_{t\to\infty} \frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} = \zeta^\beta$$

we will have proven that (*i.*) implies (*ii.*). Let $\epsilon \in (0,1)$ be arbitrary. There exists a C^1, monotone function φ such that $f(x) < (1+\epsilon)\varphi(x)$, for all $x > x_1(\epsilon)$. Owing to the divergence of $x(t)$, there exists $T_1(\epsilon) > 0$ such that $x(t) > x_1(\epsilon)$ for all $t \ge T_1(\epsilon)$. Similarly, from (*i.*), there exists $T_2(\epsilon) > 0$ such that $x(t) < (\zeta+\epsilon)\Phi^{-1}(Mt)$ for all $t \ge T_2(\epsilon)$. Letting $T_3 := T_1 + T_2/M$ and combining these estimates yields

$$\frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} \le \frac{M\int_0^{T_3} f(x(s))ds}{\Phi^{-1}(Mt)} + \frac{(1+\epsilon)M\int_{T_3}^t \varphi((\zeta+\epsilon)\Phi^{-1}(Ms))ds}{\Phi^{-1}(Mt)}, \quad t \ge T_3$$

Hence

$$\limsup_{t\to\infty} \frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} \le \limsup_{t\to\infty} \frac{(1+\epsilon)M\int_{T_3}^t \varphi((\zeta+\epsilon)\Phi^{-1}(Ms))ds}{\Phi^{-1}(Mt)}. \tag{4.5.25}$$

To estimate the right–hand side of (4.5.25), make the substitution $u = \Phi^{-1}(Ms)$

$$\frac{(1+\epsilon)M\int_{T_3}^t \varphi((1+\epsilon)\Phi^{-1}(Ms))ds}{\Phi^{-1}(Mt)} = \frac{(1+\epsilon)\int_{\Phi^{-1}(MT_3)}^{\Phi^{-1}(Mt)} \frac{\varphi((\zeta+\epsilon)u)}{\varphi(u)}du}{\Phi^{-1}(Mt)}, \quad t \geq T_3. \tag{4.5.26}$$

Since $\varphi \in \mathrm{RV}_\infty(\beta)$, there exists $T_4(\epsilon)$ such that for all $u \geq T_4$, $\varphi((\zeta+\epsilon)u)/\varphi(u) < \epsilon + (\zeta+\epsilon)^\beta$. Choose $T_5 := T_3 + \varphi(T_4)/M$; this estimate and equation (4.5.26) yield

$$\frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} \leq \frac{(1+\epsilon)(\epsilon+(\zeta+\epsilon)^\beta)\left\{\Phi^{-1}(Mt) - \Phi^{-1}(MT_5)\right\}}{\Phi^{-1}(Mt)}, \quad t \geq T_5.$$

Taking the limsup and letting $\epsilon \to 0^+$ we obtain

$$\limsup_{t\to\infty} \frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} \leq \zeta^\beta.$$

We now obtain a corresponding bound for the limit inferior. Since $\lim_{t\to\infty} M(t) = M$ is finite there exists T^* so large that for any $\epsilon \in (0,1)$, $(1-\epsilon)M < M(T^*) < M$. Using this fact, and that $M(t)$ is non-decreasing,

$$\frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} \geq \frac{\int_0^{t-T^*} M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} \geq \frac{(1-\epsilon)M\int_0^{t-T^*} f(x(s))ds}{\Phi^{-1}(Mt)},$$

for $t \geq T^*$. From $(i.)$ there exists a $T_6(\epsilon) > 0$ such that $(\zeta-\epsilon)\Phi^{-1}(Mt) < x(t)$ for all $t \geq T_6(\epsilon)$. As before, there exists a $T_7(\epsilon) > 0$ so that $f(x(t)) > (1-\epsilon)\varphi(x(t))$ for $t \geq T_7(\epsilon)$. Take $\bar{T} = \max(T_6, T_7)$. Hence

$$\frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} \geq \frac{(1-\epsilon)^2 M\int_{\bar{T}}^{t-T^*} \varphi((\zeta-\epsilon)\Phi^{-1}(Ms))ds}{\Phi^{-1}(Mt)}, \quad t \geq T^* + \bar{T}.$$

The right–hand side of the above equation is estimated as before and it follows that

$$\liminf_{t\to\infty} \frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} \geq \zeta^\beta,$$

and hence

$$\lim_{t\to\infty} \frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} = \zeta^\beta.$$

For the converse result we first note that by the divergence of x and the regular variation of f there exists $T > 0$ and a monotone increasing function φ such that for any $\epsilon \in (0,1)$ we have

$$x(t) \geq x(0) + (1-\epsilon)\int_T^t M(t-s)\varphi(x(s))ds + H(t), \quad t \geq T. \tag{4.5.27}$$

Let $y(t) = x(t+T)$ and estimate as follows:

$$y(t) \geq x(0) + (1-\epsilon)\int_T^{t+T} M(t+T-s)\varphi(x(s))\,d = x(0) + (1-\epsilon)\int_0^t M(t-u)\varphi(y(u))\,du, \quad t \geq T.$$

There exists a function x_ϵ which satisfies the equation

$$x'_\epsilon(t) = (1-\epsilon)\int_{[0,t]} \mu(ds)\varphi(x_\epsilon(t-s))ds, \quad t > 0; \quad x_\epsilon(0) = x(0)/2.$$

By the above construction $x_\epsilon(t) = x(0)/2 + (1-\epsilon)\int_0^t M(t-u)\varphi(x_\epsilon(u))du$ for $t \geq 0$ and $x_\epsilon(t) < y(t) =$

$x(t+T)$ for $t \geq 0$. Applying Corollary 2.3.2 to x_ϵ yields

$$\lim_{t\to\infty} \frac{x_\epsilon(t)}{F^{-1}(Mt)} = 1-\epsilon.$$

Therefore,

$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} \geq \liminf_{t\to\infty} \frac{x_\epsilon(t-T)}{F^{-1}(Mt)} = \liminf_{t\to\infty} \frac{x_\epsilon(t-T)}{F^{-1}(M(t-T))} \frac{F^{-1}(M(t-T))}{F^{-1}(Mt)} = 1-\epsilon.$$

Letting $\epsilon \to 0^+$ we have proven that

$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} \geq 1. \tag{4.5.28}$$

We now rule out the possibility that $\limsup_{t\to\infty} x(t)/F^{-1}(Mt) = \infty$. By (*ii.*) there exists $T_8(\epsilon) > 0$ such that for all $t \geq T_8$, $H(t) < (\lambda+\epsilon)F^{-1}(Mt)$. With $T_2(\epsilon)$ defined as before let $\tilde{T} = T_2 + T_8$ and estimate (4.1.10) as follows:

$$\begin{aligned} x(t) &\leq x(0) + H(t) + \int_0^{\tilde{T}} M(t-s)f(x(s))ds + (1+\epsilon)\int_{\tilde{T}}^t M(t-s)\varphi(x(s))ds \\ &\leq x(0) + (\lambda+\epsilon)\Phi^{-1}(Mt) + M\,\tilde{T}\,F_{\max} + (1+\epsilon)M\int_{\tilde{T}}^t \varphi(x(s))ds, \quad t \geq \tilde{T}, \end{aligned} \tag{4.5.29}$$

where $F_{\max} = \sup_{0\leq t\leq \tilde{T}} f(x(t))$. The preceding estimate motivates the definition of the upper comparison equation

$$y_\epsilon(t) = 1 + x(0) + (\lambda+\epsilon)\Phi^{-1}(Mt) + M\,\tilde{T}\,F_{\max} + (1+\epsilon)M\int_{\tilde{T}}^t \varphi(y_\epsilon(s))ds, \quad t \geq \tilde{T}.$$

By construction $y_\epsilon(t)$ is differentiable and strictly increasing and obeys $x(t) < y_\epsilon(t)$ for all $t \geq \tilde{T}$. Now compute an explicit upper bound on $\limsup_{t\to\infty} y_\epsilon(t)/F^{-1}(Mt)$. Using the monotonicity of y_ϵ and dividing by $M\,t\,\varphi(y_\epsilon(t))$ yields

$$\frac{y_\epsilon(t)}{M\,t\,\varphi(y_\epsilon(t))} \leq \frac{1+x(0)}{M\,t\,\varphi(y_\epsilon(t))} + \frac{(\lambda+\epsilon)\Phi^{-1}(Mt)}{M\,t\,\varphi(y_\epsilon(t))} + \frac{M\,\tilde{T}\,F_{\max}}{M\,t\,\varphi(y_\epsilon(t))} + 1 + \epsilon, \quad t \geq \tilde{T}. \tag{4.5.30}$$

By Karamata's Theorem there exists $x_2(\epsilon) > 0$ such that $\Phi(x) < (1+\epsilon)x/(1-\beta)\varphi(x)$, for all $x > x_2(\epsilon)$. If $y_\epsilon(t) > x_2(\epsilon)$ for all $t \geq T_9(\epsilon)$, then (4.5.30) can be improved to

$$\frac{\Phi(y_\epsilon(t))}{Mt} < \frac{(1+\epsilon)\left\{1+x(0)+M\,\tilde{T}\,F_{\max}\right\}}{M\,t\,(1-\beta)\,\varphi(y_\epsilon(t))} + \frac{(1+\epsilon)(\lambda+\epsilon)\Phi^{-1}(Mt)}{(1-\beta)M\,t\,\varphi(y_\epsilon(t))} + \frac{(1+\epsilon)^2}{(1-\beta)},$$

for $t \geq \tilde{T} + T_9$. Therefore

$$\limsup_{t\to\infty} \frac{\Phi(y_\epsilon(t))}{Mt} \leq \frac{(1+\epsilon)^2}{(1-\beta)} + \limsup_{t\to\infty} \frac{(1+\epsilon)(\lambda+\epsilon)\Phi^{-1}(Mt)}{(1-\beta)M\,t\,\varphi(y_\epsilon(t))}.$$

Now use (*ii.*) and (4.5.28) to estimate the final term on the right–hand side. (4.5.28) implies that there exists $T_{10}(\epsilon) > 0$ such that $y_\epsilon(t) > (1-\epsilon)\Phi^{-1}(Mt)$ for $t \geq T_{10}(\epsilon)$. Hence

$$\frac{(1+\epsilon)(\lambda+\epsilon)\Phi^{-1}(Mt)}{(1-\beta)M\,t\,\varphi(y_\epsilon(t))} < \frac{(1+\epsilon)(\lambda+\epsilon)\Phi^{-1}(Mt)}{(1-\beta)M\,t\,\varphi((1-\epsilon)\Phi^{-1}(Mt))}, \quad t \geq T_{10}.$$

It immediately follows that

$$\limsup_{t\to\infty} \frac{(1+\epsilon)(\lambda+\epsilon)\Phi^{-1}(Mt)}{(1-\beta)M\,t\,\varphi(y_\epsilon(t))} \le \limsup_{t\to\infty} \frac{(1+\epsilon)(\lambda+\epsilon)\Phi^{-1}(Mt)}{(1-\beta)M\,t\,(1-\epsilon)^\beta\,\varphi(\Phi^{-1}(Mt))}.$$

Karamata's Theorem implies that

$$\Phi^{-1}(Mt) \sim Mt\,(1-\beta)\,\varphi(\Phi^{-1}(Mt)) \text{ as } t\to\infty,$$

and thus

$$\limsup_{t\to\infty} \frac{(1+\epsilon)(\lambda+\epsilon)\Phi^{-1}(Mt)}{(1-\beta)M\,t\,\varphi(y_\epsilon(t))} \le \frac{(1+\epsilon)(\lambda+\epsilon)}{(1-\epsilon)^\beta}.$$

Hence

$$\limsup_{t\to\infty} \frac{\Phi(y_\epsilon(t))}{Mt} \le \frac{(1+\epsilon)^2}{(1-\beta)} + \frac{(1+\epsilon)(\lambda+\epsilon)}{(1-\epsilon)^\beta}.$$

A final application of Karamata then yields

$$\limsup_{t\to\infty} \frac{y_\epsilon(t)}{\Phi^{-1}(Mt)} \le \left(\frac{(1+\epsilon)^2}{(1-\beta)} + \frac{(1+\epsilon)(\lambda+\epsilon)}{(1-\epsilon)^\beta}\right)^{\frac{1}{1-\beta}}.$$

Now since $x(t) < y_\epsilon(t)$ for all $t \ge 0$ we may let $\epsilon \to 0^+$ to obtain

$$\limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(Mt)} \le \left(\frac{1}{1-\beta} + \lambda\right)^{\frac{1}{1-\beta}} < \infty. \tag{4.5.31}$$

Finally, to show that $\zeta = \zeta^\beta + \lambda$, we make the following induction hypothesis

$$(H_n) \qquad \limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(Mt)} \le \zeta_n, \;\; \zeta_{n+1} := \zeta_n^\beta + \lambda,$$

where $\zeta_0 := \left(\frac{1}{1-\beta} + \lambda\right)^{\frac{1}{1-\beta}}$. (H_0) is simply (4.5.31); suppose now that (H_n) holds. Thus there exists $T_{10}(\epsilon) > 0$ such that $x(t) < (\zeta_n + \epsilon)\Phi^{-1}(Mt)$ for all $t \ge T_{10}$. Taking the limsup across (4.5.24) yields

$$\limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(Mt)} = \limsup_{t\to\infty} \frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} + \lambda.$$

The term on the right–hand side is estimated as before so we only record the key asymptotic estimate.

$$\limsup_{t\to\infty} \frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(Mt)} \le (1+\epsilon)\limsup_{t\to\infty} \frac{\int_1^{\Phi^{-1}(Mt)} \frac{\varphi((\zeta_n+\epsilon)u)}{\varphi(u)}du}{\Phi^{-1}(Mt)} = (1+\epsilon)(\epsilon + (\zeta_n+\epsilon)^\beta).$$

Therefore,

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} \le \zeta_n^\beta + \lambda = \zeta_{n+1},$$

or (H_n) implies (H_{n+1}). Hence

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} \le \zeta_n, \;\text{ for all } n \ge 0.$$

By Lemma 3.5.7, $\lim_{n\to\infty} \zeta_n = \zeta$, where ζ is the unique solution in $[1,\infty)$ of $\zeta = \zeta^\beta + \lambda$. Thus

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} \le \zeta.$$

The calculation to prove that

$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} \geq \zeta,$$

is analogous to that for the limit superior. In this case the iterative scheme is started at $\zeta_0 = 1$, the estimation is performed as before and once more Lemma 3.5.7 yields the desired conclusion. $\square$

Theorem 4.5.2. *If* (4.3.12), (4.1.2), *and* (4.1.3) *hold, then the following are equivalent:*

$$(i.) \quad \limsup_{t\to\infty} \frac{F(H(t))}{Mt} \leq 1, \quad (ii.) \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1.$$

Proof of Theorem 4.5.2 (converse and upper bound). We first prove that $(ii.)$ implies $(i.)$ By positivity, $x(t) > H(t)$ for all $t \geq 0$. Hence $F(x(t))/Mt \geq F(H(t))/Mt$ and

$$\limsup_{t\to\infty} \frac{F(H(t))}{Mt} \leq \limsup_{t\to\infty} \frac{F(x(t))}{Mt} = 1.$$

Now consider the claim that $(i.)$ implies $(ii.)$. The function $t \mapsto M(t) = \int_{[0,t]} \mu(ds)$ is non-decreasing and tends to M as $t \to \infty$. Hence

$$x(t) < x(0) + H(t) + M \int_0^t f(x(s))ds, \quad t \geq 0.$$

By hypothesis there exists $T(\epsilon) > 0$ such that for all $t \geq T$ we have $F(H(t)) < M(1+\epsilon/2)t$, or $H(t) < F^{-1}(M(1+\epsilon/2)t)$. However, since $\Phi^{-1}(x) \sim F^{-1}(x)$ as $x \to \infty$, there exists $T_1(\epsilon) > T(\epsilon)$ such that for all $t \geq T_1$ we have $F^{-1}(M(1+\epsilon/2)t) < (1+\epsilon)\Phi^{-1}(M(1+\epsilon/2)t)$. Therefore for all $t \geq T_1$

$$x(t) < x(0) + (1+\epsilon)\Phi^{-1}(M(1+\epsilon/2)t) + M \int_0^t f(x(s))ds, \quad t \geq T_1(\epsilon).$$

As in the proof of Theorem 4.5.1, we use the monotone, C^1 approximation of f, which we denote by ϕ, to improve our estimate to the following:

$$x(t) < x^* + (1+\epsilon)\Phi^{-1}(M(1+\epsilon/2)t) + M(1+\epsilon) \int_{T_2}^t \phi(x(s))ds, \quad t \geq T_2(\epsilon) > T_1,$$

where $x^* = x(0) + M\,T_2 \max_{0\leq s\leq T_2} f(x(s))$. Define the upper comparison solution x_+ as follows:

$$x_+(t) = 1 + x^* + (1+\epsilon)\Phi^{-1}(M(1+\epsilon/2)t) + M(1+\epsilon) \int_{T_2}^t \phi(x_+(s))ds, \quad t \geq T_2.$$

Note that $x_+(T_2) > x(T_2)$ by construction and $x(t) < x_+(t)$ for all $t \geq T_2$. Differentiating we obtain

$$x_+'(t) = M(1+\epsilon)(1+\epsilon/2)\left(\phi \circ \Phi^{-1}\right)(M(1+\epsilon/2)t) + M(1+\epsilon)\phi(x_+(t)), \quad t \geq T_2.$$

By inspection, $x_+'(t) > M(1+\epsilon)\phi(x_+(t))$ and asymptotic integration will yield

$$x_+(t) > \Phi^{-1}\left(C + M(1+\epsilon)t\right), \quad t \geq T_2,$$

where $C = \Phi(x_+(T_2))$. Hence $\phi(x_+(t)) > \left(\phi \circ \Phi^{-1}\right)(C + M(1+\epsilon)t)$, $t \geq T_2$. Of course, for every $\epsilon \in (0,1)$ there is a $T_3(\epsilon) > 0$ such that $C > -\epsilon t/4$ for all $t \geq T_3(\epsilon)$ Therefore, for $t \geq T_4 := \max(T_2, T_3)$,

$$\frac{x_+'(t)}{\phi(x_+(t))} < M(1+\epsilon) + M(1+\epsilon)(1+\epsilon/2)\frac{\left(\phi \circ \Phi^{-1}\right)(M(1+\epsilon/2)t)}{\left(\phi \circ \Phi^{-1}\right)(M(1+3\epsilon/4)t)}.$$

Now, because $f \in \mathrm{RV}_\infty(1)$, $\Phi^{-1} \in \mathrm{RV}_\infty(\infty)$ and hence

$$\lim_{x\to\infty} \frac{\Phi^{-1}(\lambda x)}{\Phi^{-1}(x)} = 0, \text{ for all } \lambda \in (0,1).$$

Thus, because $\phi \in \mathrm{RV}_\infty(1)$,

$$\lim_{t\to\infty} \frac{\left(\phi \circ \Phi^{-1}\right)(M(1+\epsilon/2)t)}{\left(\phi \circ \Phi^{-1}\right)(M(1+3\epsilon/4)t)} = 0.$$

It follows immediately that $\limsup_{t\to\infty} x_+'(t)/\phi(x_+(t)) \le M(1+\epsilon)$ and asymptotic integration yields $\limsup_{t\to\infty} \Phi(x_+(t))/Mt \le 1$. Now we note that $x_+(t) \ge x(t)$, for all $t \ge T_2$, and monotonicity of Φ implies $\limsup_{t\to\infty} \Phi(x(t))/Mt \le 1$. Since Φ and F are asymptotically equivalent, we have $\limsup_{t\to\infty} F(x(t))/Mt \le 1$. □

Proof of Theorem 4.5.2 (lower bound). Since $\lim_{t\to\infty} M(t) = M < \infty$, $x(t) \to \infty$ as $t \to \infty$ and $f(x) \sim \phi(x)$ as $x \to \infty$, there exists $T_3 > 0$ such that $M(t) > M(1-\epsilon)$ and $f(x(t)) > (1-\epsilon)\phi(x(t))$ for all $t \ge T_3$, with $\epsilon \in (0,1)$ arbitrary. For $t \ge 2T_3$,

$$x(t) \ge x(0) + (1-\epsilon)\int_{T_3}^t M(t-s)\phi(x(s))ds \ge x(0) + (1-\epsilon)^2 M \int_{T_3}^t \phi(x(s))ds.$$

Letting $y(t) = x(t+T_3)$ for $t \ge 2T_3$, it is straightforward to show that

$$y(t) \ge x(0) + M(1-\epsilon)^2 \int_0^{t-T_3} \phi(y(u))du, \ t \ge T_3.$$

Now define the lower comparison solution

$$z(t) = z^* + M(1-\epsilon)^2 \int_0^{t-T_3} \phi(z(u))du, \ t \ge T_3,$$

and $z(t) = z^* := \frac{1}{2}\min_{t\in[0,2T_3]} x(t)$, $t \in [0,T_3]$. Thus for $t \in [0,T_3]$, $y(t) = x(t+T_3) > z^* = z(t)$ and $z^* < x(0)$. Now suppose that $y(t) > z(t)$ for $t \in [0,\bar{T})$, $\bar{T} > T_3$, but $y(\bar{T}) = z(\bar{T})$. Then $s \in [0,\bar{T}-T_3]$ implies $\phi(y(s)) > \phi(z(s))$ and hence $\int_0^{\bar{T}-T_3} \phi(y(s))ds \ge \int_0^{\bar{T}-T_3} \phi(z(s))ds$. Therefore

$$\begin{aligned} y(\bar{T}) &\ge x(0) + M(1-\epsilon)^2 \int_0^{\bar{T}-T_3} \phi(y(s))ds \ge x(0) + M(1-\epsilon)^2 \int_0^{\bar{T}-T_3} \phi(z(s))ds \\ &> z^* + M(1-\epsilon)^2 \int_0^{\bar{T}-T_3} \phi(z(s))ds = z(\bar{T}) = y(\bar{T}), \end{aligned}$$

a contradiction. Hence $x(t+T_3) = y(t) > z(t)$ for all $t \ge 0$. For $t \ge T_3$, $z'(t) = M(1-\epsilon)^2\phi(z(t-T_3))$ and thus by Theorem 2.3.1, $\lim_{t\to\infty} \Phi(z(t))/t = M(1-\epsilon)^2$. Hence

$$\liminf_{t\to\infty} \frac{\Phi(x(t+T_3))}{t} \ge \liminf_{t\to\infty} \frac{\Phi(z(t))}{t} \ge M(1-\epsilon)^2.$$

Thus

$$M(1-\epsilon)^2 \le \liminf_{t\to\infty} \frac{\Phi(x(t))}{t-T_3} = \liminf_{t\to\infty} \frac{\Phi(x(t))}{t}.$$

Since $F \sim \Phi$, letting $\epsilon \to 0^+$ yields $\liminf_{t\to\infty} F(x(t))/Mt \ge 1$. □

Theorem 4.5.3. *Let (4.1.2), (4.1.3), and (4.3.12) hold with $f(x)/x \downarrow 0$ as $x \to \infty$. If $L_f(H) \in (1,\infty)$, then*

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = \frac{L_f(H)}{L_f(H)-1}, \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = L_f(H). \tag{4.5.32}$$

Proof of Theorem 4.5.3. The upper bound

$$\limsup_{t\to\infty} \frac{x(t)}{H(t)} \le \frac{L_f(H)}{L_f(H)-1},$$

is furnished by Theorem 4.3.6. To apply this result, note that $f(x)/x \downarrow 0$ as $x \to \infty$ implies that ϕ' is asymptotically decreasing since $\phi'(x) \sim \phi(x)/x \sim f(x)/x$ as $x \to \infty$, where ϕ is a C^1, monotone approximation of f (see Section 1.3.3).

We now establish the companion lower bound

$$\liminf_{t\to\infty} \frac{x(t)}{H(t)} \ge \frac{L_f(H)}{L_f(H)-1}$$

when $L_f(H) \in (1,\infty)$, which will settle the result for these values of $L_f(H)$. We start by defining the auxiliary parameter

$$\alpha = \lim_{t\to\infty} \frac{M\int_0^t f(H(s))\,ds}{H(t)} = \frac{1}{L_f(H)} \in (0,1).$$

Now define the sequence $\{\lambda_n\}_{n\ge 0}$ as follows:

$$\lambda_{n+1} = 1 + \alpha\lambda_n, \quad n \ge 0; \quad \lambda_0 = 1,$$

and make the induction hypothesis

$$(H_n) \quad \liminf_{t\to\infty} \frac{x(t)}{H(t)} \ge \lambda_n.$$

From equation (4.1.10), positivity implies $x(t) > H(t)$ and hence $\liminf_{t\to\infty} x(t)/H(t) \ge 1$, i.e. (H_0) holds. Suppose that (H_n) holds for some $n \ge 1$. With $\epsilon \in (0,1)$ arbitrary, there exists $T(\epsilon) > 0$ such that $t \ge T(\epsilon)$ implies

$$x(t) > (1-\epsilon)\lambda_n H(t). \tag{4.5.33}$$

Rewrite (4.1.10) as follows

$$x(t) = x(0) + H(t) + \int_0^t M(s) f(x(t-s))ds, \quad t \ge 0. \tag{4.5.34}$$

Before estimating the right hand side of (4.5.34) we need some preliminary estimates. $f \in \text{RV}_\infty(1)$ guarantees the existence of a monotone increasing function $\phi \in \text{RV}_\infty(1) \cap C^1$ such that f is asymptotic to ϕ at infinity. Hence there exists $x_1(\epsilon) > 0$ such that for all $x > x_1(\epsilon)$ we have $f(x) > (1-\epsilon)\phi(x)$. Similarly, since $\lim_{t\to\infty} x(t) = \infty$, there exists $T_1(\epsilon) > 0$ such that $x(t) > x_1(\epsilon)$ for all $t \ge T_1(\epsilon)$. Finally, $\lim_{t\to\infty} M(t) = M \in (0,\infty)$ implies that $M(t) > (1-\epsilon)M$ for all $t \ge T_2(\epsilon)$, for some $T_2(\epsilon) > 0$. Now, from (4.5.34), we have

$$x(t) > H(t) + (1-\epsilon)M \int_{T_2}^t f(x(t-s))\,ds = H(t) + (1-\epsilon)M \int_0^{t-T_2} f(x(u))\,du,$$

for $t \ge T_2(\epsilon)$. Furthermore,

$$x(t) > H(t) + (1-\epsilon)^2 M \int_{T_1}^{t-T_2} \phi(x(u))\,du, \quad t \ge T_1 + T_2.$$

Define $T_3(\epsilon) = T(\epsilon) + T_1(\epsilon)$ and use (4.5.33) to obtain

$$\frac{x(t)}{H(t)} > 1 + (1-\epsilon)^2 M \frac{\int_{T_3}^{t-T_2} \phi((1-\epsilon)\lambda_n H(u))du}{H(t)}, \quad t \geq T_3(\epsilon) + T_2(\epsilon).$$

Since $\phi \in \text{RV}_\infty(1)$ and $\lim_{t\to\infty} H(t) = \infty$, $\phi((1-\epsilon)\lambda_n H(u)) \sim (1-\epsilon)\lambda_n \phi(H(u))$ as $u \to \infty$. Hence there exists $T_4(\epsilon) > 0$ such that for all $u \geq T_4(\epsilon)$, $\phi((1-\epsilon)\lambda_n H(u)) > (1-\epsilon)^2 \lambda_n \phi(H(u))$. Thus, letting $T_5(\epsilon) = T_3 + T_4$,

$$\frac{x(t)}{H(t)} > 1 + (1-\epsilon)^4 M \lambda_n \frac{\int_{T_5}^{t-T_2} \phi(H(u))du}{H(t)}, \quad t \geq T_2 + T_5.$$

Since $L_f(H) \in (1,\infty)$, it follows that H is asymptotic to the increasing, continuous function $\gamma(t) = L_f(H) M \int_0^t f(H(s))\, ds$. Therefore, using the ideas at the end of the proof of Theorem 4.3.8, we can show that

$$\int_{t-T_2}^{t} \phi(H(u))\, du = o(H(t)), \qquad \text{as } t \to \infty.$$

Hence, $\liminf_{t\to\infty} x(t)/H(t) \geq 1 + (1-\epsilon)^4 \alpha \lambda_n$ and letting $\epsilon \to 0^+$,

$$\liminf_{t\to\infty} \frac{x(t)}{H(t)} \geq 1 + \alpha\lambda_n = \lambda_{n+1}.$$

We have now proven that (H_n) implies (H_{n+1}). Since $\alpha \in (0,1)$,

$$\lim_{n\to\infty} \lambda_n = \frac{1}{1-\alpha}.$$

Thus we conclude that $\liminf_{t\to\infty} x(t)/H(t) \geq 1/(1-\alpha)$ and, rewriting this in terms of $L_f(H)$, $\liminf_{t\to\infty} x(t)/H(t) \geq L_f(H)/(L_f(H)-1)$. Combining this with the corresponding upper bound completes the proof of the first limit in (4.5.32).

Now we establish the second limit in (4.5.32) when $L_f(H) \in (1,\infty)$. Proposition 4.2.1 implies that $\lim_{t\to\infty} F(H(t))/Mt = L_f(H)$ and it was proven above that

$$x(t) \sim \frac{L_f(H)}{L_f(H)-1} H(t), \quad \text{as} \quad t \to \infty.$$

Since $F \in RV_\infty(0)$,

$$L_f(H) = \lim_{t\to\infty} \frac{F(H(t))}{Mt} = \lim_{t\to\infty} F\left(\frac{(L_f(H)-1)\, x(t)}{L_f(H)}\right)/Mt = \lim_{t\to\infty} \frac{F(x(t))}{Mt},$$

as required. □

Proof of Theorem 4.3.7. Before starting the proof in earnest, we make a preliminary observation which holds throughout the proof. Note that $\lim_{t\to\infty} F(H(t))/Mt = L_f(H)$, when $L_f(H) \in [0,\infty)$, by Proposition 4.2.1. If $f \in \text{RV}_\infty(\beta)$, $\beta \in (0,1)$, then $F^{-1} \in \text{RV}_\infty(1/(1-\beta))$. Hence $\lim_{t\to\infty} F(H(t))/Mt = L_f(H)$ implies $\lambda := \lim_{t\to\infty} H(t)/F^{-1}(Mt) = L_f(H)^{1/(1-\beta)}$, in the notation of Theorem 4.5.1.

(i.) $L_f(H) = 0$: Invoking our preliminary observation we have $\lambda = 0$ in Theorem 4.5.1. Hence the only solution in $[1,\infty)$ of (4.5.23) is $\zeta = 1$ and $\lim_{t\to\infty} x(t)/F^{-1}(Mt) = 1$, or equivalently $\lim_{t\to\infty} F(x(t))/Mt = 1$. The second claimed limit, i.e. $x(t)/H(t) \to \infty$ as $t \to \infty$, can be read off from Theorem 4.3.1.

(ii.) $L_f(H) \in (0,\infty)$: Invoking our preliminary observation, $\lambda = L_f(H)^{1/(1-\beta)}$ in Theorem 4.5.1 and (4.5.23) becomes $\zeta = \zeta^\beta + L_f(H)^{1/(1-\beta)}$.By Theorem 4.5.1, $\lim_{t\to\infty} x(t)/F^{-1}(Mt) = \zeta$ becomes

$\lim_{t\to\infty} F(x(t))/Mt = \zeta^{1-\beta}$ upon applying F. Similarly,

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = \lim_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} \frac{F^{-1}(Mt)}{H(t)} = \frac{\zeta}{L_f(H)^{1/(1-\beta)}} = \frac{\zeta}{\zeta - \zeta^\beta} = \frac{1}{1-\zeta^{1-\beta}}.$$

(iii.) This follows directly from Theorem 4.3.8.

(I.) $L_f(H) \in [0,1]$: By Proposition 4.2.1, $\lim_{t\to\infty} F(H(t))/Mt = L_f(H) \in [0,1]$ and $\lim_{t\to\infty} F(x(t))/Mt = 1$ by Theorem 4.5.2. To prove that $x \sim H$ it is necessary to take cases. Firstly, if $L_f(H) = 0$, applying Theorem 4.3.1 yields the desired conclusion. If $L_f(H) \in (0,1)$, then the result can be deduced using the same argument as in Theorem 4.3.1 by now exploiting the fact that F^{-1} is rapidly varying at infinity. Finally, if $L_f(H) = 1$, we can prove that $x(t)/H(t) \to \infty$ as $t \to \infty$ using the argument in Theorem 4.5.3: we prove the induction hypothesis

$$\liminf_{t\to\infty} \frac{x(t)}{H(t)} \geq \lambda_n,$$

where $\lambda_{n+1} = 1+\lambda_n$ for $n \geq 0$ and $\lambda_0 = 1$. Since $\lambda_n \to \infty$ as $n \to \infty$, the claim follows directly from the proven induction hypothesis.

(II.) $L_f(H) \in (1,\infty)$: See Theorem 4.5.3.

(III.) $L_f(H) = +\infty$: See Theorem 4.3.6 part (b.) and note that the relevant hypotheses on f are satisfied (see Section 1.3.3 for the requisite properties of regularly varying functions).

□

4.5.4 Proofs of Results with Brownian Noise

Proof of Theorem 4.4.2. We start with part (a), which covers the case when $\sigma \notin L^2(0,\infty)$. Let ϵ, $\eta \in (0,1)$ be arbitrary. Rewrite (4.1.5) in integral form as follows:

$$X(t) = X(0) + \int_0^t M(t-s)f(X(s))ds + \int_0^t \sigma(s)dB(s), \quad t \geq 0.$$

Hence

$$|X(t)| \leq |X(0)| + \int_0^t M(t-s)|f(X(s))|ds + \left|\int_0^t \sigma(s)dB(s)\right|, \quad t \geq 0.$$

Denote by Ω_1 the a.s. event on which $t \mapsto X(t)(\omega)$ is continuous. We now recall the law of the iterated logarithm for continuous local martingales (see Revuz and Yor [104, Ch. V, Ex. 1.15]) which states that if $N = \{N_t, t \geq 0\}$ is a continuous local martingale with $\langle N,N\rangle_\infty = \infty$, then

$$\limsup_{t\to\infty} \frac{N_t}{\sqrt{2\langle N,N\rangle_t \log\log\langle N,N\rangle_t}} = 1 \ a.s.,$$

where $\langle N,N\rangle = \{\langle N,N\rangle_t, t \geq 0\}$ is the quadratic variation process. In our case,

$$\left\langle \int_0^\cdot \sigma(s)dB(s), \int_0^\cdot \sigma(s)dB(s) \right\rangle_t = \int_0^t \sigma^2(s)ds$$

and thus $\sigma \notin L^2(0,\infty)$ implies $\limsup_{t\to\infty} \left|\int_0^t \sigma(s)dB(s)\right| / \Sigma(t) = 1$ a.s..

Let $\eta > 0$ be arbitrary. By hypothesis there exists $\phi \in C^1$ such that

$$|f(x)| \leq K(\eta) + (1+\eta)\phi(|x|), \ x \in \mathbb{R}. \tag{4.5.35}$$

Define $H_\eta(t) = MK(\eta)t + (1+\eta)\Sigma(t)$ for $t \geq 0$. With $L_f(\Sigma) = 0$, Proposition 4.2.1 implies $\lim_{t\to\infty} \Phi(\Sigma(t))/t = 0$. Therefore, for every $\epsilon \in (0,1)$ there exists $T_2(\epsilon) > 0$ such that

$$\Sigma(t) < \Phi^{-1}(\epsilon t), \quad t \geq T_2(\epsilon). \tag{4.5.36}$$

Similarly, by L'Hôpital's rule,

$$\lim_{t\to\infty} \frac{MK(\eta)t}{\int_0^t \phi(MK(\eta)s)ds} = \lim_{t\to\infty} \frac{MK(\eta)}{\phi(MK(\eta)t)} = 0.$$

Thus, again appealing to L'Hôpital's rule, $\lim_{t\to\infty} \Phi(MK(\eta)t)/t = 0$ and moreover, for any $\eta \in (0,1)$, $\lim_{t\to\infty} \Phi\left(MK(\eta)t/\eta\right)/t = 0$. Hence for every $\epsilon \in (0,1)$ there exists $T_3(\epsilon,\eta)$ such that

$$MK(\eta)t < \eta\,\Phi^{-1}(\epsilon t), \quad t \geq T_3(\epsilon,\eta). \tag{4.5.37}$$

Combining (4.5.36) and (4.5.37) yields

$$H_\eta(t) = MK(\eta)t + (1+\eta)\Sigma(t) < (1+2\eta)\Phi^{-1}(\epsilon t), \quad t \geq T_4(\epsilon,\eta) = T_2 + T_3.$$

Rearrange this inequality, let $t \to \infty$, and then let $\epsilon \to 0^+$ to obtain

$$\lim_{t\to\infty} \frac{\Phi(H_\eta(t)/(1+2\eta))}{Mt} = 0.$$

Thus, by proceeding as above, for every $\epsilon \in (0,1)$ there is $T_4'(\epsilon,\eta) > 0$ such that

$$H_\eta(t) < (1+2\eta)\Phi^{-1}(\epsilon Mt), \quad t \geq T_4'(\epsilon,\eta). \tag{4.5.38}$$

Since Φ is concave, Φ^{-1} is convex and $\Phi^{-1}(\epsilon Mt) \leq \epsilon\Phi^{-1}(Mt) + (1-\epsilon)\Phi^{-1}(0)$. Therefore

$$\limsup_{t\to\infty} \Phi^{-1}(\epsilon Mt)/\Phi^{-1}(Mt) \leq \epsilon.$$

Take limits in (4.5.38) to give

$$\limsup_{t\to\infty} \frac{H_\eta(t)}{\Phi^{-1}(Mt)} \leq (1+2\eta)\epsilon,$$

and then let $\epsilon \to 0^+$ to yield $\lim_{t\to\infty} H_\eta(t)/\Phi^{-1}(Mt) = 0$. Therefore, for every $\epsilon \in (0,1)$, there exists $T_5'(\epsilon,\eta) > 0$ such that

$$H_\eta(t) < \epsilon\,\Phi^{-1}(Mt), \quad t \geq T_5'(\epsilon,\eta).$$

Now, let $T_5(\eta) = T_5'(\eta,\eta)$, so that

$$H_\eta(t) < \eta\,\Phi^{-1}(Mt), \quad t \geq T_5(\eta). \tag{4.5.39}$$

On the other hand, because $\limsup_{t\to\infty} \left|\int_0^t \sigma(s)dB(s)\right|/\Sigma(t) = 1$ a.s., there exists an almost sure event Ω_2 such that for all $\omega \in \Omega_2$

$$\left|\int_0^t \sigma(s)dB(s)(\omega)\right| \leq (1+\eta)\Sigma(t),\ t \geq T_1(\eta,\omega).$$

Now let $T(\eta,\omega) = \max(T_1(\eta,\omega), T_5(\eta))$. Thus, for all $\omega \in \Omega^* = \Omega_1 \cap \Omega_2$,

$$|X(t)| \leq |X(0)| + \int_0^t M(t-s)|f(X(s))|ds + (1+\eta)\Sigma(t), \quad t \geq T(\eta,\omega).$$

Using the estimate (4.5.35) on f and the finiteness of $\lim_{t\to\infty} M(t)$ we have

$$\begin{aligned}|X(t)| &\leq |X(0)| + MK(\eta)t + M(1+\eta)\int_0^t \phi(|X(s)|)ds + (1+\eta)\Sigma(t)\\ &\leq X_0^* + H_\eta(t) + M(1+\eta)\int_T^t \phi(|X(s)|)ds, \quad t \geq T(\eta,\omega), \quad \omega\in\Omega^*,\end{aligned} \tag{4.5.40}$$

where $X(0)^* = |X(0)| + MT\sup_{s\in[0,T]}\phi(|X(s)|)$. Since $t \geq T(\eta,\omega) \geq T_5(\eta)$, we have from (4.5.39) that for all $\omega\in\Omega^*$

$$|X(t)| \leq X(0)^* + \eta\,\Phi^{-1}(Mt) + M(1+\eta)\int_T^t \phi(|X(s)|)ds, \quad t \geq T(\eta,\omega). \tag{4.5.41}$$

At this point we note that we are in the same position as in the proof of Theorem 4.3.1 at equation (4.5.3). From here a calculation exactly analogous to that which completes the proof of Theorem 4.3.1 will yield

$$\limsup_{t\to\infty}\frac{F(|X(t)|)}{Mt} \leq 1 \ a.s..$$

To prove part (b), let ϵ, $\eta\in(0,1)$ be arbitrary and rewrite (4.1.5) in integral form as before. Take absolute values to obtain

$$|X(t)| \leq |X(0)| + \int_0^t M(t-s)|f(X(s))|ds + \left|\int_0^t \sigma(s)dB(s)\right|, \quad t\geq 0.$$

Let Ω_1 be as before. By the Martingale Convergence Theorem (see Revuz and Yor [104, Ch. V, Prop. 1.8]), if $N = \{N_t,\, t\geq 0\}$ is a continuous local martingale with $\langle N,N\rangle_\infty < +\infty$, then

$$\lim_{t\to\infty} N_t \in (-\infty,\infty), \quad \text{a.s.}.$$

In our case,

$$\left\langle \int_0^\cdot \sigma(s)dB(s), \int_0^\cdot \sigma(s)dB(s)\right\rangle_t = \int_0^t \sigma^2(s)ds$$

and thus $\sigma\in L^2(0,\infty)$ implies that $\lim_{t\to\infty} N_t$ exists and is finite a.s. Therefore, as $t\mapsto N_t$ is a.s. continuous, there exists an a.s. event Ω_2 such that for all $\omega\in\Omega_2$

$$\sup_{t\geq 0}\left|\int_0^t \sigma(s)dB(s)(\omega)\right| \leq N^*(\omega) < +\infty.$$

Thus for all $\omega\in\Omega^* = \Omega_1\cap\Omega_2$ and $t\geq 0$,

$$|X(t)| \leq |X(0)| + N^* + \int_0^t M(t-s)|f(X(s))|ds.$$

Using the estimate (4.5.35) on f and the finiteness of $\lim_{t\to\infty} M(t)$, we have

$$|X(t)| \leq |X(0)| + N^* + MK(\eta)t + M(1+\eta)\int_0^t \phi(|X(s)|))ds, \quad t\geq 0.$$

Lastly, define $X(0)^* = |X(0)| + N^*$ and $H_\eta(t) = MK(\eta)t$. Then we have

$$|X(t)| \leq X(0)^* + H_\eta(t) + M(1+\eta)\int_0^t \phi(|X(s)|))ds, \quad t\geq 0.$$

Note that this estimate is in precisely the form of (4.5.40). It is easy to show that $H_\eta(t) = MK(\eta)t$ obeys an estimate of the form (4.5.39) for all $t\geq T_5(\eta)$. Hence for all $t\geq T(\eta) = T_5(\eta)$ and for all

$\omega \in \Omega^*$, the estimate

$$|X(t)| \le X(0)^* + \eta\, \Phi^{-1}(Mt) + M(1+\eta) \int_T^t \phi(|X(s)|)ds, \quad t \ge T(\eta), \tag{4.5.42}$$

holds. At this point we note that we are in the same position as in the proof of part (a) after (4.5.41), and exactly analogous calculations yield

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 \ a.s..$$

□

Proof of Corollary 4.4.1. We first prove that $\limsup_{t\to\infty} |X(t)| = \infty$ a.s. by showing that X cannot be bounded with positive probability. Suppose there exists an event A, with positive probability, such that $|X(t)| \le N < \infty$ for all $t \ge 0$ on A. Now consider the linear SDE

$$dY(t) = -Y(t)dt + \sigma dB(t), \quad t > 0, \quad Y(0) = 0.$$

The solution to the SDE above is given by $Y(t) = \sigma \int_0^t e^{-(t-s)} dB(s)$. Furthermore, it can be shown that Y obeys $\limsup_{t\to\infty} |Y(t)| = \infty$ a.s. and $\liminf_{t\to\infty} |Y(t)| = 0$ a.s. (see Appleby et al. [7, Theorem 4.1]). Write (4.1.5) as

$$dX(t) = -X(t)dt + \{X(s) + \int_0^t \mu(ds) f(X_{t-s})\}dt + \sigma dB(t), \quad t > 0.$$

Applying the variation of constants formula we obtain

$$\begin{aligned} X(t) &= e^{-t}X(0) + \int_0^t e^{-(t-s)} \left\{ X(s) + \int_0^s \mu(du) f(X_{s-u}) \right\} ds + \sigma \int_0^t e^{-(t-s)} dB(s) \\ &= e^{-t}X(0) + \int_0^t e^{-(t-s)} \left\{ X(s) + \int_0^s \mu(du) f(X_{s-u}) \right\} ds + Y(t), \quad t \ge 0. \end{aligned}$$

With some simple estimation it follows that, on A, $\limsup_{t\to\infty} X(t) = \infty$, a contradiction. To show that $\limsup_{t\to\infty} F(|X(t)|)/Mt \le 1$ a.s. we check $\sigma(t) \equiv \sigma \in \mathbb{R}/\{0\}$ obeys $L_f(\Sigma) = 0$, so we can apply Theorem 4.4.2. By L'Hôpital's rule

$$\lim_{t\to\infty} \frac{\Sigma(t)}{\int_0^t f(\Sigma(s))ds} = \lim_{t\to\infty} \frac{\Sigma'(t)}{f(\Sigma(t))},$$

assuming the limit on the right–hand side exists. In fact

$$\Sigma'(t) = \frac{\sigma^2}{\log(t\sigma^2)\sqrt{2t\sigma^2 \log\log(t\sigma^2)}} + \frac{\sigma^2 \log\log(t\sigma^2)}{\sqrt{2t\sigma^2 \log\log(t\sigma^2)}}.$$

Hence $\lim_{t\to\infty} \Sigma'(t) = 0$ and $L_f(\Sigma) = 0$, as required. □

Proof of Theorem 4.4.3. Let $\epsilon \in (0,1)$ be arbitrary and follow the line of argument from the proof of Theorem 4.4.4 to obtain

$$|X(t)| \le A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon) \int_T^t \phi(|X(s)|)ds, \quad t \ge T, \quad \omega \in \Omega,$$

where $A_\epsilon = M\,T\,\sup_{s\in[0,T_1]}|X(s)|$. Define X_ϵ, as in (4.5.47), by

$$X_\epsilon(t) = 1 + A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)\int_T^t \phi(X_\epsilon(s))ds, \quad t \geq T.$$

Now by (4.5.45) there exists $T_1(\epsilon) > T$ such that

$$X_\epsilon(t) \leq (1+3\epsilon)\Sigma(t) + M(1+\epsilon)\int_T^t \phi(X_\epsilon(s))ds, \quad t \geq T_1(\epsilon). \tag{4.5.43}$$

Let $I_\epsilon(t) = \int_T^t \phi(X_\epsilon(s))ds$; monotonicity yields

$$\lim_{t\to\infty}\frac{\Sigma(t)}{M\,I_\epsilon(t)} \leq \lim_{t\to\infty}\frac{\Sigma(t)}{M\,\int_T^t \phi(\Sigma(s))ds} = L_\phi(\Sigma) \in (0,\infty).$$

Hence there exists $T_2(\epsilon) > T_1$ such that

$$\Sigma(t) \leq L_\phi(\Sigma)M(1+\epsilon)I_\epsilon(t), \quad t \geq T_2. \tag{4.5.44}$$

For $t \geq T_2$, using (4.5.44), calculate as follows

$$\begin{aligned} I'_\epsilon(t) = \phi(X_\epsilon(t)) &\leq \phi\left((1+3\epsilon)\Sigma(t) + M(1+\epsilon)I_\epsilon(t)\right) \\ &\leq \phi\left(L_\phi(\Sigma)M(1+3\epsilon)(1+\epsilon)I_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)\right) \\ &\leq \phi\left((1+7\epsilon)(M + L_\phi(\Sigma)M)I_\epsilon(t)\right). \end{aligned}$$

Integrating the previous inequality we obtain

$$\int_{T_2}^t \frac{I'_\epsilon(s)ds}{\phi\left((1+7\epsilon)(M+L_\phi(\Sigma)M)I_\epsilon(s)\right)} \leq t - T_2, \quad t \geq T_2.$$

Hence making the substitution $u = (1+7\epsilon)(M+L_\phi(\Sigma)M)I_\epsilon(s)$ yields

$$\Phi\left((1+7\epsilon)(M+L_\phi(\Sigma)M)I_\epsilon(t)\right) \leq (t-T_2)(1+7\epsilon)(M+L_\phi(\Sigma)M) + \Phi_\epsilon, \quad t \geq T_2,$$

where $\Phi_\epsilon = \Phi\left((1+7\epsilon)(M+L_\phi(\Sigma)M)I_\epsilon(T_2)\right)$. Thus

$$(1+7\epsilon)(M+L_\phi(\Sigma)M)I_\epsilon(t) \leq \Phi^{-1}\left((t-T_2)(1+7\epsilon)(M+L_\phi(\Sigma)M) + \Phi_\epsilon\right), \quad t \geq T_2.$$

Returning to (4.5.43) and using the estimate above, we obtain,

$$\begin{aligned} X_\epsilon(t) &\leq (1+3\epsilon)L_\phi(\Sigma)M(1+\epsilon)I_\epsilon(t) + M(1+\epsilon)I_\epsilon(t) \leq (1+7\epsilon)(M+L_\phi(\Sigma)M)I_\epsilon(t) \\ &\leq \Phi^{-1}\left((t-T_2)(1+7\epsilon)(M+L_\phi(\Sigma)M) + \Phi_\epsilon\right), \quad t \geq T_2. \end{aligned}$$

It immediately follows that

$$\limsup_{t\to\infty}\frac{\Phi(X_\epsilon(t))}{Mt} \leq (1+L_\phi(\Sigma))(1+7\epsilon).$$

Let $\epsilon \to 0^+$ and note that by construction $|X(t)| \leq X_\epsilon(t)$ for all $t \geq T$. Therefore,

$$\limsup_{t\to\infty}\frac{\Phi(|X(t)|)}{Mt} \leq 1 + L_\phi(\Sigma) \; a.s.,$$

as required. □

Proof of Theorem 4.4.4. By L'Hôpital's rule, $\lim_{x\to\infty} \Phi(x)/x = \lim_{x\to\infty} 1/\phi(x) = 0$ and hence $\lim_{t\to\infty} \Phi(\Sigma(t))/\Sigma(t) = 0$. Therefore, using Proposition 4.2.1,

$$\lim_{t\to\infty} \frac{A+Bt}{\Sigma(t)} = \lim_{t\to\infty} \frac{A+Bt}{\Phi(\Sigma(t))} \frac{\Phi(\Sigma(t))}{\Sigma(t)} = 0, \tag{4.5.45}$$

for any nonnegative constants A and B. Arguing as in the proof of Theorem 4.4.2, with T and Ω defined analogously, we have the initial estimate

$$|X(t)| \le |X(0)| + MK(\epsilon)t + (1+\epsilon)\Sigma(t) + M(1+\epsilon)\int_0^t \phi(|X(s)|)ds,$$

for $t \ge T(\epsilon,\omega)$, $\omega \in \Omega$, where $\mathbb{P}[\Omega] = 1$. By (4.5.45) there is $T_1(\epsilon,\omega) > T(\epsilon,\omega)$ such that for all $t \ge T_1(\epsilon,\omega)$ $|X(0)| + MK(\epsilon)t < \epsilon\Sigma(t)$. Hence

$$|X(t)| \le A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)\int_{T_1}^t \phi(|X(s)|)ds, \quad t \ge T_1, \quad \omega \in \Omega, \tag{4.5.46}$$

where $A_\epsilon = M\,T_1 \sup_{s\in[0,T_1]} \phi(|X(s)|)$. Now define the function $X_\epsilon(t)$ for $t \ge T_1$ by

$$X_\epsilon(t) = 1 + A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)\int_{T_1}^t \phi(X_\epsilon(s))ds. \tag{4.5.47}$$

By construction $|X(t)| \le X_\epsilon(t)$ for all $t \ge T_1(\epsilon)$. Let $I_\epsilon(t) = \int_{T_1}^t \phi(X_\epsilon(s))ds$, so

$$I_\epsilon'(t) = \phi(X_\epsilon(t)) = \phi(1 + A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)I_\epsilon(t)), \quad t \ge T_1(\epsilon).$$

Since ϕ is increasing and there exists $T_2(\epsilon) > T_1(\epsilon)$ such that $1 + A_\epsilon < \epsilon\Sigma(t)$,

$$I_\epsilon'(t) \le \phi((1+3\epsilon)\Sigma(t) + M(1+\epsilon)I_\epsilon(t)), \quad t \ge T_2.$$

By the mean value theorem there exists $\theta_t \in [0,1]$ such that

$$\begin{aligned} I_\epsilon'(t) &\le \phi((1+3\epsilon)\Sigma(t)) + \phi'((1+3\epsilon)\Sigma(t))\, M(1+\epsilon)I_\epsilon(t) \\ &\le \phi((1+3\epsilon)\Sigma(t)) + M(1+\epsilon)^2 \frac{\phi((1+3\epsilon)\Sigma(t))}{(1+3\epsilon)\Sigma(t)} I_\epsilon(t), \quad t \ge T_2, \end{aligned} \tag{4.5.48}$$

where the final inequality follows from Lemma 4.2.1. Once more we exploit the mean value theorem and the first part of Lemma 4.2.1 as follows:

$$\begin{aligned} \phi((1+3\epsilon)\Sigma(t)) &= \phi(\Sigma(t)) + \phi'(\Sigma(t) + \rho_t 3\epsilon\Sigma(t))\, 3\epsilon\Sigma(t), \quad \rho_t \in [0,1] \\ &\le \phi(\Sigma(t)) + \phi'(\Sigma(t))\, 3\epsilon\Sigma(t) = \phi(\Sigma(t)) \left\{ 1 + 3\epsilon \frac{\phi'(\Sigma(t))\Sigma(t)}{\phi(\Sigma(t))} \right\} \\ &\le \phi(\Sigma(t))(1+4\epsilon), \quad t \ge T^* > T_2. \end{aligned} \tag{4.5.49}$$

Hence (4.5.48) becomes

$$I_\epsilon'(t) \le (1+4\epsilon)\phi(\Sigma(t)) + M(1+\epsilon)^2 \frac{(1+4\epsilon)}{(1+3\epsilon)} \frac{\phi(\Sigma(t))}{\Sigma(t)} I_\epsilon(t), \quad t \ge T^*.$$

Let

$$a_\epsilon(t) = M(1+\epsilon)^2 \frac{(1+4\epsilon)}{(1+3\epsilon)} \frac{\phi(\Sigma(t))}{\Sigma(t)} \quad \text{and} \quad H_\epsilon(t) = \Sigma(t).$$

Now apply the argument from the proof of Theorem 4.3.6 beginning at (4.5.14). Following this line of argument shows that

$$\limsup_{t\to\infty} \frac{I_\epsilon(t)}{\int_{T_1}^t \phi(\Sigma(s))ds} \le N(\epsilon) < \infty.$$

Returning to (4.5.47) this yields

$$X_\epsilon(t) < 1 + A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)^2 N(\epsilon)\int_{T_1}^t \phi(\Sigma(s))ds, \quad t \ge T^*.$$

Therefore

$$\frac{X_\epsilon(t)}{\Sigma(t)} < 1 + 2\epsilon + \frac{1+A_\epsilon}{\Sigma(t)} + \frac{M(1+\epsilon)^2 N(\epsilon)\int_{T_1}^t \phi(\Sigma(s))ds}{\Sigma(t)}, \quad t \ge T^*.$$

Thus

$$\limsup_{t\to\infty} \frac{X_\epsilon(t)}{\Sigma(t)} \le 1 + 2\epsilon + \frac{M(1+\epsilon)^2 N(\epsilon)}{L_\phi(\Sigma)} < \infty.$$

Hence we have that $\limsup_{t\to\infty} |X(t)|/\Sigma(t) < \infty$ a.s..

Suppose that $\limsup_{t\to\infty} |X(t)|/\Sigma(t) = 0$ on an event Ω_p of positive probability, then there exists $\bar{T}(\epsilon) > 0$ such that $|X(t)| < \epsilon\Sigma(t)$ for all $t \ge \bar{T}$, $\omega \in \Omega_p$. Let $J(t) = \int_0^t M(t-s)f(X(s))ds$ and estimate as before. For all $\omega \in \Omega_p$, we obtain

$$|J(t)| \le M\,C(\epsilon)\,t + M\,\bar{T}\,(1+\epsilon)\sup_{s\in[0,\bar{T}]}\phi(|X(s)|) + M(1+\epsilon)\int_{\bar{T}}^t \phi(|X(s)|)ds, \tag{4.5.50}$$

for $t \ge \bar{T}$. Hence

$$\limsup_{t\to\infty} \frac{|J(t)|}{\Sigma(t)} \le M(1+\epsilon)\limsup_{t\to\infty} \frac{\int_{\bar{T}}^t \phi(\epsilon\Sigma(s))ds}{\Sigma(t)} \le \frac{1+\epsilon}{L_\phi(\Sigma)},$$

for all $\omega \in \Omega_p$ and $\epsilon \in (0,1)$. Therefore, since $L_f(\Sigma) > 1$, $\limsup_{t\to\infty} |J(t)|/\Sigma(t) = \lambda \in [0,1)$ on Ω_p. It follows that there exists $T' > \bar{T}$ such that $J(t)/\Sigma(t) > -\lambda - \epsilon$ for all $t \ge T'$. Consider the stochastic integral equation

$$X(t) = X(0) + \int_0^t M(t-s)f(X(s))ds + \int_0^t \sigma(s)dB(s), \quad t \ge 0.$$

For all $t \ge T'$ and $\omega \in \Omega_p$,

$$\frac{X(t)}{\Sigma(t)} = \frac{X(0)}{\Sigma(t)} + \frac{J(t)}{\Sigma(t)} + \frac{\int_0^t \sigma(s)dB(s)}{\Sigma(t)} \ge \frac{X(0)}{\Sigma(t)} + \frac{\int_0^t \sigma(s)dB(s)}{\Sigma(t)} - \lambda - \epsilon.$$

This implies that $\limsup_{t\to\infty} X(t)/\Sigma(t) \ge 1 - \lambda - \epsilon$ for all $\omega \in \Omega_p$ and for all $\epsilon \in (0,1)$. Hence $\limsup_{t\to\infty} X(t)/\Sigma(t) \ge 1 - 1/L_\phi(\Sigma)$ on Ω_p and similarly
$\liminf_{t\to\infty} X(t)/\Sigma(t) \le -1 + 1/L_\phi(\Sigma)$ on Ω_p, a contradiction. Hence $\mathbb{P}[\Omega_p] = 0$ and

$$\limsup_{t\to\infty} \frac{|X(t)|}{\Sigma(t)} = \Lambda \in (0,\infty)\, a.s..$$

From (4.5.50) we obtain the following a.s. estimate

$$|J(t)| \le M\,C(\epsilon)\,t + M\,\bar{T}\,(1+\epsilon)\sup_{s\in[0,\bar{T}]}\phi(|X(s)|) + M(1+\epsilon)\int_{\bar{T}}^t \phi((\Lambda+\epsilon)\Sigma(s))ds,$$

for $t \ge \bar{T}$. If we have $\Lambda \in (0,1)$, then we can choose $\epsilon > 0$ sufficiently small that $\Lambda + \epsilon < 1$ and monotonicity of ϕ and Σ will yield $\limsup_{t\to\infty} |J(t)|/\Sigma(t) \le \Lambda/L_\phi(\Sigma)$, as before. If $\Lambda \in [1,\infty)$, we can estimate via the second part of Lemma 4.2.1. Suppose $\Lambda \in [1,\infty)$, then

$$\limsup_{t\to\infty} \frac{|J(t)|}{\Sigma(t)} \le M(1+\epsilon)(\Lambda+\epsilon)\frac{\int_{\bar T}^t \phi(\Sigma(s))ds}{\Sigma(t)} = (1+\epsilon)\frac{\Lambda+\epsilon}{L_\phi(\Sigma)},$$

and letting $\epsilon \to 0^+$ we obtain $\limsup_{t\to\infty} |J(t)|/\Sigma(t) \le \Lambda/L_\phi(\Sigma)$ a.s.. Therefore

$$\begin{aligned}\limsup_{t\to\infty} \frac{X(t)}{\Sigma(t)} \le \Lambda &\le \limsup_{t\to\infty} \frac{|X(0)|}{\Sigma(t)} + \limsup_{t\to\infty} \frac{|J(t)|}{\Sigma(t)} + \limsup_{t\to\infty} \frac{|\int_0^t \sigma(s)dB(s)|}{\Sigma(t)} \\ &\le \frac{\Lambda}{L_\phi(\Sigma)} + 1 \; a.s..\end{aligned}$$

Finally $\Lambda \le L_f(\Sigma)/(L_f(\Sigma)-1)$. Thus, $\limsup_{t\to\infty} X(t)/\Sigma(t) \le L_f(\Sigma)/(L_f(\Sigma)-1)$ a.s. and similarly $\liminf_{t\to\infty} X(t)/\Sigma(t) \ge -L_f(\Sigma)/(L_f(\Sigma)-1)$ a.s.. □

Proof of Theorem 4.4.5. We follow closely the line of argument from the proof of Theorem 4.4.4. First we establish the required analogue of (4.5.45). $L_f(\Sigma) = \infty$, so Proposition 4.2.1 implies that $\lim_{t\to\infty} \Phi(\Sigma(t))/\Sigma(t) = \infty$. Hence

$$\lim_{t\to\infty} \frac{A+Bt}{\Sigma(t)} = \lim_{t\to\infty} \frac{A+Bt}{\int_0^t f(\Sigma(s))ds}\frac{\int_0^t f(\Sigma(s))ds}{\Sigma(t)} = 0, \quad \text{for all } A, B > 0.$$

Now proceed with the argument from Theorem 4.4.4 to obtain

$$|X(t)| \le A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)\int_{T_1}^t \phi(|X(s)|)ds, \quad t \ge T_1, \quad \omega \in \Omega,$$

where $A_\epsilon = MT_1 \sup_{s\in[0,T_1]} |X(s)|$. Define $X_\epsilon(t)$ as in (4.5.47) and estimate as before to obtain $\limsup_{t\to\infty} \int_T^t \phi(X_\epsilon(s))ds / \int_T^t \phi(\Sigma(s))ds < N(\epsilon) < \infty$. Therefore, since $L_f(\Sigma) = \infty$,

$$\limsup_{t\to\infty} \frac{X_\epsilon(t)}{\Sigma(t)} \le 1 + 2\epsilon + M(1+\epsilon)^2 N(\epsilon) \limsup_{t\to\infty} \frac{\int_T^t \phi(\Sigma(s))ds}{\Sigma(t)} = 1 + 2\epsilon.$$

Note that $|X(t)| \le X_\epsilon(t)$ a.s for all $t \ge T$ and let $\epsilon \to 0^+$ to conclude that

$$\limsup_{t\to\infty} \frac{|X(t)|}{\Sigma(t)} \le 1 \; a.s..$$

The event on which $\limsup_{t\to\infty} |X(t)|/\Sigma(t) = 0$ has probability zero by exactly the line of argument which concludes the proof of Theorem 4.4.4. Hence $\limsup_{t\to\infty} |X(t)|/\Sigma(t) = \lambda \in (0,1]$ a.s. and $|X(t)| \le (\lambda+\epsilon)\Sigma(t)$ for all $t \ge T(\epsilon)$ on an event of probability one. Using the notation $J(t) = \int_T^t M(t-s)f(X(s))\,ds$, we recall the a.s. estimate (4.5.50)

$$|J(t)| \le M\,C(\epsilon)\,t + M\,\bar T\,(1+\epsilon) \sup_{s\in[0,T]} \phi(|X(s)|) + M(1+\epsilon)\int_T^t \phi(|X(s)|)ds, \;\; t \ge T.$$

Using the monotonicity of ϕ, an estimate of the form (4.5.49), and that $L_\phi(\Sigma) = \infty$,

$$\begin{aligned}\limsup_{t\to\infty} \frac{|J(t)|}{\Sigma(t)} &\le M(1+\epsilon) \limsup_{t\to\infty} \frac{\int_T^t \phi((\lambda+\epsilon)\Sigma(s))ds}{\Sigma(t)} \\ &\le M(1+\epsilon)(1+2\epsilon) \limsup_{t\to\infty} \frac{\int_T^t \phi(\Sigma(s))ds}{\Sigma(t)} = 0 \; a.s..\end{aligned}$$

Hence $\lim_{t\to\infty} J(t)/\Sigma(t) = 0$ a.s. and the claim (4.4.9) is proven. Now we compute the value of

$\limsup_{t\to\infty} X(t)/\Sigma(t)$ as follows:

$$\limsup_{t\to\infty}\frac{X(t)}{\Sigma(t)} = \limsup_{t\to\infty}\left\{\frac{X(0)}{\Sigma(t)} + \frac{J(t)}{\Sigma(t)} + \frac{\int_0^t \sigma(s)dB(s)}{\Sigma(t)}\right\} = 1 \ a.s..$$

Taking the liminf above yields $\liminf_{t\to\infty} X(t)/\Sigma(t) = -1$ a.s., concluding the proof. □

4.5.5 Proofs of Results with Stable Lévy Noise

Proof of Theorem 4.4.6. $\int_0^\infty \gamma(s)^{-\alpha}ds < \infty$ implies $\limsup_{t\to\infty}|Z(t)|/\gamma(t) = 0$ a.s. (see Bertoin [25, Theorem 5, Ch. VIII]). This proof follows by applying the argument used to establish Theorem 4.4.2 with Σ replaced by γ as appropriate. □

Proof of Theorem 4.4.7. Suppose γ_+ and γ_- both satisfy the hypotheses on γ with $\int_0^\infty \gamma_+(s)^{-\alpha}ds < \infty$ and $\int_0^\infty \gamma_-(s)^{-\alpha}ds = \infty$. It follows that

$$\limsup_{t\to\infty}\frac{|Z(t)|}{\gamma_+(t)} = 0 \text{ a.s. and } \limsup_{t\to\infty}\frac{|Z(t)|}{\gamma_-(t)} = \infty \text{ a.s..} \tag{4.5.51}$$

We first deal with the claim that $\limsup_{t\to\infty}|X(t)|/\gamma_+(t) \le 1/L_f(\gamma_+)$ a.s. when $L_f(\gamma_+) \in (1,\infty)$. As in the proof of Theorem 4.4.4, use Proposition 4.2.1 to show that

$$\lim_{t\to\infty}\frac{A+Bt}{\gamma_+(t)} = \lim_{t\to\infty}\frac{A+Bt}{\Phi(\gamma_+(t))}\frac{\Phi(\gamma_+(t))}{\gamma_+(t)} = 0,$$

for any nonnegative constants A and B. With the estimate above in hand and the proof proceeds as in that of Theorem 4.4.4 but we arrive at a slightly different initial upper estimate to that derived in equation (4.5.46) since we employ (4.5.51) for the asymptotics of Z. In this case

$$|X(t)| \le A_\epsilon + 3\epsilon\,\gamma_+(t) + M(1+\epsilon)\int_{T_1}^t \phi(|X(s)|)ds, \quad t \ge T_1, \quad \omega \in \Omega_1, \tag{4.5.52}$$

where $A_\epsilon = M\,T_1 \sup_{s\in[0,T_1]}\phi(|X(s)|)$. Define the comparison solution

$$X_\epsilon(t) = 1 + A_\epsilon + 3\epsilon\,\gamma_+(t) + M(1+\epsilon)\int_{T_1}^t \phi(X_\epsilon(s))ds, \quad t \ge T_1. \tag{4.5.53}$$

By following exactly the steps from the proof of Theorem 4.4.4 we obtain $\limsup_{t\to\infty}|X_\epsilon(t)|/\gamma_+(t) < \infty$ with probability one and hence

$$\limsup_{t\to\infty}\frac{|X(t)|}{\gamma_+(t)} < \infty \text{ a.s.} \tag{4.5.54}$$

With the usual notation that $J(t) = \int_0^t M(t-s)f(X(s))\,ds$ write

$$\frac{|X(t)|}{\gamma_+(t)} \le \frac{|X(0)|}{\gamma_+(t)} + \frac{|J(t)|}{\gamma_+(t)} + \frac{|Z(t)|}{\gamma_+(t)}.$$

To finally derive the required bound on $\limsup_{t\to\infty}|X(t)|/\gamma_+(t)$ estimate $|J(t)|$ using (4.5.54) (as was done in the proof of Theorem 4.4.4, for example); conclude by plugging in this estimate above and using (4.5.51).

The proof is essentially the same when $L_f(\gamma_+) = \infty$. To show that $\limsup_{t\to\infty}|X(t)|/\gamma_+(t) = 0$ a.s. proceed as before in applying the argument of Theorem 4.4.4 but note now that this will give $\limsup_{t\to\infty} X_\epsilon(t)/\gamma_+(t) \le 3\epsilon$ for the comparison solution. The conclusion now follows readily.

It remains to show that $\limsup_{t\to\infty}|X(t)|/\gamma_-(t)=\infty$ a.s.. First assume to the contrary that there exists an event Ω_2 with positive probability on which $\limsup_{t\to\infty}|X(t)|/\gamma_-(t) =: L \in [0,\infty)$. First show $\limsup_{t\to\infty}|J(t)|/\gamma_-(t)<\infty$ on an event of positive probability; work on Ω_2 and estimate as follows:

$$\begin{aligned}|J(t)| &\le M\int_0^t \{K+(1+\epsilon)\phi(|X(s))|\}\,ds \\ &\le MKt + M(1+\epsilon)T\sup_{s\in[0,T]}\phi(|X(s)|) + M(1+\epsilon)^2\max(1,L)\int_T^t \phi(\gamma_-(s))\,ds, \end{aligned} \tag{4.5.55}$$

for T sufficiently large and $t\ge T$ (the last inequality uses Lemma 4.2.1). Divide by γ_- and take the limsup across (4.5.55); the final term on the right–hand side can be dealt with using the hypothesis $L_f(\gamma_-)\in(1,\infty]$, the first two terms are $o(\gamma_-)$ and thus $\limsup_{t\to\infty}|J(t)|/\gamma_-(t)<\infty$ with positive probability. Therefore the following holds on an event of positive probability

$$\limsup_{t\to\infty}\frac{|Z(t)|}{\gamma_-(t)} \le \limsup_{t\to\infty}\left\{\frac{|X(0)|}{\gamma_-(t)}+\frac{|X(t)|}{\gamma_-(t)}+\frac{|J(t)|}{\gamma_-(t)}\right\}<\infty,$$

in contradiction of the fact that $\limsup_{t\to\infty}|Z(t)|/\gamma_-(t)=\infty$ a.s.. □

Chapter 5

Growth and Fluctuation in Linear Volterra Summation Equations

5.1 Introduction

In contrast to the focus on nonlinear differential equations in the rest of this thesis, we now determine conditions under which the *linear* Volterra summation equation

$$x(n+1) = \sum_{j=0}^{n} k(n-j)x(j) + H(n+1), \quad n \geq 0; \quad x(0) = \xi \in \mathbb{R} \tag{5.1.1}$$

has unbounded solutions with additional growth properties. Volterra equations, both discrete and continuous, have found myriad applications in the field of economics and it is this area of application we have in mind throughout this chapter. In the context of economic growth models it is especially pertinent to analyze qualitative features of unbounded solutions, such as growth rates and fluctuation sizes. Moreover, in applications, it is important to understand the impact of random perturbations and hence we show how our deterministic results extend naturally to handle stochastic forcing terms.

We assume henceforth that

$$k \in \ell^1(\mathbb{Z}^+) \tag{5.1.2}$$

and that the unperturbed, or resolvent, equation

$$r(n+1) = \sum_{j=0}^{n} k(n-j)r(j), \quad n \geq 0; \quad r(0) = 1. \tag{5.1.3}$$

has a summable solution, i.e.

$$r \in \ell^1(\mathbb{Z}^+). \tag{5.1.4}$$

The summability of r can be characterised entirely in terms of the kernel k; in particular, $r \in \ell^1(\mathbb{Z}^+)$ is equivalent to the characteristic equation condition

$$1 - \sum_{l=0}^{\infty} k(l) z^{-(l+1)} \neq 0, \quad \text{for all } z \in \mathbb{C} \text{ with } |z| \geq 1. \tag{5.1.5}$$

In the important and special case that $k(n) \geq 0$ for all $n \geq 0$, the condition (5.1.5) is equivalent to

Chapter 5 is based on the paper [17].

$\sum_{j=0}^{\infty} k(j) < 1$. Hence, a useful and sharp sufficient condition for r to be summable, which does not require sign conditions on k, is $\sum_{j=0}^{\infty} |k(j)| < 1$.

The summability of r is intimately related to the boundedness of the solution to (5.1.1) under bounded perturbations. In fact, it is true that:

(a.) If r is summable, then x is bounded if and only if H is bounded,

(b.) If, for every bounded sequence H, x is bounded, then r is summable,

(see Corduneanu [37] and Perron [99]). From (a.) it is clear that if H is unbounded, then so is x. We seek to understand how more refined properties of unbounded forcing sequences H give rise to corresponding unboundedness properties of x and in this sense our results are related to classic admissibility theory for Volterra equations. The right–hand side of (5.1.1) defines a linear Volterra operator and if this operator maps a space S onto itself then S is said to be admissible with respect to the operator (cf. [51, 52, 53, 54, 55, 115, 116]). Typical admissibility results for operators of the type considered in this paper assert that for every $H \in S$, we will have $x \in S$ (cf. Gripenberg et al. [50, Theorem 2.4.5]), where S is a "standard" space (such as the space of bounded, convergent, periodic, or ℓ^p sequences). In this context, the main contribution of the present work is to expand the collection of admissible spaces for the discrete Volterra operator defined by (5.1.1) to spaces more germane to economic applications (as opposed to the classic theory which studies spaces more appropriate for applications in engineering and related areas). In particular, we show that if the unbounded sequence H has an interesting property A which characterises its growth or fluctuation, then x possesses the property A as well; in many situations the converse also holds (cf. Appleby and Patterson [14]).

We investigate both bounds on the fluctuations of solutions, and on exact rates of growth. H is assumed to be unbounded, but its growth bounds are characterised, in the sense that there is an increasing sequence $(a(n))_{n\geq 0}$ with $a(n) \to \infty$ as $n \to \infty$ such that

$$\limsup_{n\to\infty} \frac{|H(n)|}{a(n)} =: \Lambda_a|H|. \tag{5.1.6}$$

It is already known in the case when $a(n) \to +\infty$ as $n \to \infty$ (and is increasing) that $\Lambda_a|H|$ finite implies $\Lambda_a|x|$ finite (see, for example, Gol'dengerŝhel' [48]). We show that the existence of finite, zero, or infinite values of $\Lambda_a|x|$ and $\Lambda_a|H|$ are closely linked. Specifically, let a be a monotone sequence with $a(n) \to \infty$ as $n \to \infty$, and define the sequence spaces

$$\begin{aligned}
B_a &= \left\{ (g(n))_{n\geq 0} : \limsup_{n\to\infty} |g(n)|/a(n) < +\infty \right\}, \\
B_a(0) &= \left\{ (g(n))_{n\geq 0} : \limsup_{n\to\infty} |g(n)|/a(n) = 0 \right\}, \\
B_a(+) &= \left\{ (g(n))_{n\geq 0} : \limsup_{n\to\infty} |g(n)|/a(n) \in (0,\infty) \right\}, \\
B_a(\infty) &= \left\{ (g(n))_{n\geq 0} : \limsup_{n\to\infty} |g(n)|/a(n) = +\infty \right\}.
\end{aligned}$$

We show that $x \in V$ if and only if $H \in V$, where V is one of the spaces listed above (Theorem 5.3.1). The aforementioned analysis is relatively straightforward and we expand upon it to establish more refined properties of solutions by asking more of the forcing sequence H and the normalising sequence a.

Results of the type described above are especially interesting if H is a stochastic process, since they show that the large fluctuations in H determine those in x: we cannot get "smaller" fluctuations in x than those of H, but neither can we get larger ones. The limits zero and infinity in equation (5.1.6) are of special interest in probability theory. We provide examples when H is a sequence of independent and identically distributed random variables in which a sequence a cannot be found so that $\Lambda_a|H|$ is nontrivial (but that bounding sequences $a_\pm$ can be found so that $\Lambda_{a_+}|H| = 0$ and $\Lambda_{a_-}|H| = +\infty$).

Moreover, we characterise an important class of asymptotic growth. Define

$$G_\lambda = \left\{ (g(n))_{n\geq 0} : |g(n)| \to \infty \text{ as } n \to \infty,\ \lambda := \lim_{n\to\infty} \frac{g(n-1)}{g(n)} \in [0,1] \right\}, \tag{5.1.7}$$

and the following equivalence relation on the space of real–valued sequences:

Definition 5.1.1. *$(y(n))_{n\geq 0}$ and $(z(n))_{n\geq 0}$ are asymptotically equivalent if $y(n)-z(n) \to 0$ as $n \to \infty$. We write $y(n) \approx z(n)$ as $n \to \infty$, or $y \approx z$, for short.*

The space G_λ is related to a class of weight functions first introduced by Chover et al. [35] and later employed by Appleby et al. [9] to calculate rates of convergence to zero of solutions to linear Volterra convolution problems. Intuitively, we use sequences in G_λ to scale unbounded quantities of interest in much the same way time–series or economic growth models are detrended; we are particularly interested in conditions under which economically relevant growth properties are preserved by this scaling process.

In fact, $x \in G_\lambda$ if and only if $H \in G_\lambda$, and both imply $\lim_{n\to\infty} x(n)/H(n)$ is finite, nontrivial, and can be computed explicitly in terms of k and λ (Theorem 5.2.1).

We also consider a larger class of sequences which are bounded by sequences in G_λ. Define, for $a \in G_\lambda$, the space

$$BG_{a,\lambda} = \left\{ (g(n))_{n\geq 0} : \frac{g(n)}{a(n)} \approx (\lambda_a g)(n) \text{ is bounded} \right\}. \tag{5.1.8}$$

The sequence $\lambda_a g$ in (5.1.8) is only defined up to asymptotic equivalence but one could of course choose $(\lambda_a g)(n) = g(n)/a(n)$ for definiteness. With the notation outlined above, $x \in BG_{a,\lambda}$ if and only if $H \in BG_{a,\lambda}$ and the sequence $\lambda_a x$ can be taken as

$$(\lambda_a x)(n) \approx (\lambda_a H)(n) + \sum_{j=1}^{n} r(j)\lambda^j (\lambda_a H)(n-j), \quad \text{as } n \to \infty. \tag{5.1.9}$$

The result stated above (Theorem 5.2.2) is of particular interest if the forcing term grows in a reasonably regular manner, but has proportional fluctuations around a growth path given by an increasing sequence. This allows, for example, for cyclic growth in H around an exponential trend, leading to similar cyclic growth in x (Proposition 5.2.1).

On the other hand, if H is experiencing fluctuations, and the large fluctuations of H can be crudely bounded by an increasing sequence a, the asymptotic relation (5.1.9) shows how large fluctuations in x arise as a "weighted average" of the fluctuations in H. In a sense, if the decay in k is relatively slow, then r tends to experience a slower decay to zero (see Appleby et al. [9]), so the weight attached to big values of $\lambda_a H$ in the past tends to be larger, and so the present impact of fluctuations in H in the past lingers longer in the fluctuations in x. This mechanism also explains how fluctuations around a growth trend in H propagate through to those in x.

We also prove nonlinear variants of each of the growth and fluctuation results outlined above. If the linear term $x(j)$ in the convolution in (5.1.1) is replaced by a nonlinear term $f(x(j))$ such that $f(x)/x \to 1$ as $|x| \to \infty$, then the nonlinear equation thus formed inherits all the growth properties of the underlying linearised equation.

Our proofs primarily rely on the following variation of constants formula for the *convolution* problem (5.1.1) which allows x to be written directly in terms of r and H (see, for example, Elaydi [46]):

$$x(n) = r(n)x(0) + \sum_{j=1}^{n} r(n-j)H(j), \quad n \geq 1. \tag{5.1.10}$$

Since our results frequently involve the quantity x/a, where a is a sequence which captures the growth of the unbounded forcing term, another natural line of attack (following Appleby et al. [9] and Reynolds [105]) would be to rewrite (5.1.1) as follows:

$$\frac{x(n+1)}{a(n+1)} = \sum_{j=0}^{n} k(n-j)\frac{x(j)}{a(n+1)} + \frac{H(n+1)}{a(n+1)}, \quad n \geq 0.$$

We can now let $\tilde{x}(n) = x(n)/a(n)$ for each $n \geq 0$ and consider the *nonconvolution* problem given by

$$\tilde{x}(n+1) = \sum_{j=0}^{n} \tilde{k}(n,j)\tilde{x}(j) + \tilde{H}(n+1), \quad n \geq 0,$$

where $\tilde{k}(n,j) = k(n-j)a(j)/a(n+1)$ for each $n \geq 0$ and $j \in \{0,\dots,n\}$. While this alternative approach would likely lead to more general results, it would also offer weaker, or less precise, conclusions. Hence we prefer to exploit the convolution structure of (5.1.1) and use the formula (5.1.10), as opposed to its nonconvolution analogue (see Vecchio [120]); we believe this approach leads to results more likely to be of use in economic applications where it is of interest to freeze the asymptotically autonomous structure of the equation for the purposes of fitting to data and parsimonious modelling.

5.2 Growth Rates

Before stating and discussing our main results we first provide a brief motivation for our interest in equations such as (5.1.1) and outline connections to applications.

When $H \equiv 0$, the right–hand side of (5.1.1) defines an asymptotically autonomous convolution operator. This is particularly desirable in the context of applications where we generally wish to keep the structure of models time independent, not least because we prefer models with time–invariant properties amenable to statistical inference. Another feature of (5.1.1) worth remarking upon is our decision to write "$H(n+1)$" as opposed to "$H(n)$" when denoting the forcing term. Both formulations are common in the literature but we prefer the former because we are expressly interested in applications to random forcing sequences. In particular, if H is a stochastic process and each random variable $H(n)$ is $\mathcal{F}(n)$–measurable (for each $n \geq 0$), then $x(n)$ is also $\mathcal{F}(n)$–measurable. Hence the value of $x(n)$ is not known with certainty by observers of the system until time n; this is the natural setup for problems arising in an economic context (and indeed in most other applications).

One well–known class of economic models which has a formulation closely related to (5.1.1) is the classic linear multidimensional Leontief input–output model (see Leontief [74, 75]). In the Leontief model, H represents final demand, x is output or production, and the contribution of the convolution term is known as intermediate demand. Time lags reflect the fact that there is a delay between production and satisfaction of final demand. Due to its linear structure, and the exogenous nature of the forcing term, (5.1.1) is also reminiscent of classic time series models. For example, with appropriate choice of H, (5.1.1) is particularly closely related to ARMA(p,q) and AR(∞) models (see, for example, Brockwell and Davis [29]). Such models are often used to capture so–called long range dependence or long memory phenomena, which have been shown to arise in a variety of applied contexts (see Baillie [21], and Ding and Grainger [45]). In contrast to classical analysis of time series models, we focus not on studying the autocovariance function of solutions but on pathwise properties of solutions inherited from the exogenous forcing term.

Our first result below characterises the rate of growth of solutions to (5.1.1) in terms of the sequence space G_λ, defined by (5.1.7).

Theorem 5.2.1. *If $k \in \ell^1(\mathbb{Z}^+)$ obeys (5.1.5) and x is the solution to (5.1.1), then the following are equivalent:*

(a.) $H \in G_\lambda$;

(b.) $x \in G_\lambda$.

Moreover, both imply that

$$\lim_{n\to\infty} \frac{x(n)}{H(n)} = L, \tag{5.2.1}$$

where

$$L = \frac{1}{1 - \sum_{l=0}^{\infty} \lambda^{l+1} k(l)}, \quad \lambda \in [0, 1]. \tag{5.2.2}$$

When $\lambda = 0$ we see that the sum in (5.2.2) collapses to zero, so that $L = 1$. The quantity L in (5.2.1) is always nontrivial (and finite) because the summability of r implies from (5.1.5) that $1 - \sum_{j=0}^{\infty} k(j)\lambda^{j+1} \neq 0$ for $\lambda \in [0, 1]$. When k is positive, there is a "multiplier effect" from the input sequence k to the output x, because $L > 1$ when $\lambda > 0$. Equally, there is no multiplier effect if $\lambda = 0$. In Section 5.4.1 we show how Theorem 5.2.1 can be used to deal with random forcing sequences which have appropriate structure.

Our next result is similar in spirit to Theorem 5.2.1 but deals with more general growth of the type characterised by the space $BG_{a,\lambda}$, defined by (5.1.8).

Theorem 5.2.2. *If $k \in \ell^1(\mathbb{Z}^+)$ obeys (5.1.5) and x is the solution to (5.1.1), then the following are equivalent:*

(a.) $H \in BG_{a,\lambda}$;

(b.) $x \in BG_{a,\lambda}$.

Moreover, when $H \in BG_{a,\lambda}$,

$$\frac{x(n)}{a(n)} \approx (\lambda_a H)(n) + \sum_{j=1}^{n} r(j)\lambda^j (\lambda_a H)(n-j), \text{ as } n \to \infty, \tag{5.2.3}$$

and similarly, when $x \in BG_{a,\lambda}$,

$$\frac{H(n)}{a(n)} \approx (\lambda_a x)(n) - \sum_{j=0}^{n-1} k(j)\lambda^{j+1} (\lambda_a x)(n-j-1), \text{ as } n \to \infty. \tag{5.2.4}$$

One feature of the asymptotic representations in Theorem 5.2.2 is particularly noteworthy. In both (5.2.3) and (5.2.4) the summands decay rapidly to zero if $\lambda \in (0, 1)$ and hence the first few terms in the resolvent/kernel sequence are most important. The kernel is in principle known (or could be approximated by time series techniques in the case that H is a stationary process), so there is no difficulty with regard to (5.2.4). However, there seems to be limited theory for small time or transient behaviour of the resolvent, although some global results regarding the differential resolvent are available (see, for example, Gripenberg et al. [50, Theorem 5.4.1]). Of course, in practice, the first few terms of $(r(n))_{n\geq 0}$ can be easily calculated by hand and this will likely provide sufficient insight in many instances.

As we will see momentarily, the main strengths of Theorem 5.2.2 are its generality and the convenient asymptotic representations (5.2.3) and (5.2.4). We now extend Theorem 5.2.2 by showing that $H \in U \subset BG_{a,\lambda}$ if and only if $x \in U \subset BG_{a,\lambda}$, where the set U is endowed with additional growth properties.

We first address a type of "periodic–growth" which can be thought of as modeling the effect of economic cycles on the long term growth rate of an economy. The following definition of an almost periodic sequence is standard in the literature (see, for example, Agarwal [2], or, for a more detailed exposition, Corduneanu [38]).

Definition 5.2.1. *A sequence $\pi = (\pi(n))_{n\in\mathbb{Z}}$ is almost periodic if for each $\epsilon > 0$ there exists an integer $X(\epsilon)$ such that in any set of X consecutive integers there exists an integer N such that*

$$|\pi(n+N) - \pi(n)| < \epsilon, \text{ for each } n \in \mathbb{Z}.$$

We write $\pi \in AP(\mathbb{Z})$ for short.

The following definition of an asymptotically almost periodic sequence is also standard (see, for example, Henríquez [61] or Song [114]).

Definition 5.2.2. *A sequence $\pi = (\pi(n))_{n\geq 0}$ is asymptotically almost periodic if there exists sequences $\psi \in AP(\mathbb{Z})$ and $(\phi(n))_{n\geq 0}$ obeying $\phi(n) \to 0$ as $n \to \infty$ such that $\pi(n) = \psi(n) + \phi(n)$ for each $n \geq 0$. We write $\pi \in AAP(\mathbb{Z}^+)$ for short.*

For $a \in G_\lambda$, define the space of sequences

$$PG_{a,\lambda} = \left\{ (g(n))_{n\geq 0} : (g(n)/a(n))_{n\geq 0} \in \mathrm{AAP}(\mathbb{Z}^+) \right\}. \tag{5.2.5}$$

We are now in a position to state the following result which characterises a type of asymptotic growth incorporating almost periodic cycles.

Proposition 5.2.1. *If $k \in \ell^1(\mathbb{Z}^+)$ obeys (5.1.5) and x is the solution to (5.1.1), then the following are equivalent:*

(a.) $H \in PG_{a,\lambda}$;

(b.) $x \in PG_{a,\lambda}$.

Moreover, when $H \in PG_{a,\lambda}$, the almost periodic part of x/a is given by

$$\pi_x(n) = \pi_H(n) + \sum_{j=1}^{\infty} r(j)\lambda^j \pi_H(n-j), \quad n \in \mathbb{Z}, \tag{5.2.6}$$

where $H/a \approx \pi_H \in AP(\mathbb{Z})$. Similarly, when $x \in PG_{a,\lambda}$, the almost periodic part of H/a is given by

$$\pi_H(n) = \pi_x(n) - \sum_{j=0}^{\infty} k(j)\lambda^{j+1}\pi_x(n-j-1), \quad n \in \mathbb{Z}, \tag{5.2.7}$$

where $x/a \approx \pi_x \in AP(\mathbb{Z})$.

The result above is similar to the work of Diblík et al. [44] in which the authors prove sufficient conditions for the solution of a linear nonconvolution Volterra equation to have an asymptotically periodic solution, when scaled by an appropriate weight sequence. Moreover, they show that the solution omits an asymptotic representation which identifies the periodic component. In the aforementioned work, and usually in the extant literature (see, for example, Győri and Reynolds [55] and the references therein), the solution essentially inherits asymptotic periodicity as a perturbation of an underlying non–delay equation. By contrast, in our result the periodicity of the weighted solution sequence is inherited purely from the periodic behaviour of the exogenous forcing sequence.

Proposition 5.2.1 also holds if almost periodicity is replaced simply by standard periodicity; in other words, if the almost periodic part of H/a is periodic, then the almost periodic part of x/a is periodic with the same period (and vice versa).

In the same spirit as the previous result, we consider the case when the forcing sequence H in (5.1.1) has a stable time average when appropriately scaled by a sequence in G_λ. Given a sequence $a \in G_\lambda$ and another sequence g, define the weighted average sequence $(\mu_a g(n))_{n\geq 1}$ by

$$(\mu_a g)(n) = \frac{1}{n}\sum_{j=1}^{n}\frac{g(j)}{a(j)}, \quad \text{for each } n \geq 1.$$

Now, for $a \in G_\lambda$, define the space of sequences

$$AG_{a,\lambda} = \left\{(g(n))_{n\geq 0} : \frac{g(n)}{a(n)} \approx (\lambda_a g)(n) \text{ is bounded and } \lim_{n\to\infty}(\mu_a g)(n) \text{ exists}\right\}.$$

Proposition 5.2.2. *If k and the solution r of (5.1.3) are summable, and x is the solution to (5.1.1), then the following are equivalent:*

(a.) $H \in AG_{a,\lambda}$;

(b.) $x \in AG_{a,\lambda}$.

Moreover, if $H \in AG_{a,\lambda}$ and $\lim_{n\to\infty}(\mu_a H)(n) =: \mu_a H^*$, then

$$\lim_{n\to\infty}(\mu_a x)(n) = \frac{\mu_a H^*}{1-\sum_{j=0}^{\infty}.k(j)\lambda^{j+1}},$$

Similarly, if $x \in AG_{a,\lambda}$ and $\lim_{n\to\infty}(\mu_a x)(n) =: \mu_a x^*$, then

$$\lim_{n\to\infty}(\mu_a H)(n) = \mu_a x^*\left(1-\sum_{j=0}^{\infty}k(j)\lambda^{j+1}\right).$$

The most natural context for the result above is when the forcing sequence H is random and an ergodic theorem can be applied to conclude that H has stable time averages; we consider an elementary example of this type and invite the interested reader to consider others of similar character.

Example 5.2.3. *Suppose $(\Omega, \mathcal{B}(\mathbb{R}), \mathbb{P})$ is a probability space and $(h(n))_{n\geq 0}$ is a stationary sequence of random variables (not necessarily independent). In other words,*

$$\mathbb{P}[h(n_1) \in B_1, \ldots, h(n_r) \in B_r] = \mathbb{P}[h(n_1+k) \in B_1, \ldots, h(n_r+k) \in B_r]$$

for any nonnegative integers $n_1 < n_2 < \ldots < n_r$, Borel sets $B_1, \ldots, B_r$, and positive integer k. Suppose further that the $h(n)$'s are bounded on the a.s. event Ω_1. Let $H(n) = h(n)\alpha^n$ for each $n \geq 0$ and some $\alpha \in (1,\infty)$. Hence we may choose $(a(n))_{n\geq 0} = (\alpha^n)_{n\geq 0} \in G_{1/\alpha}$, so that $(H(n)/a(n))_{n\geq 0}$ is bounded on Ω_1. By Birkhoff's ergodic theorem, $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}h(j) =: \mu_a H^*$ exists on an event of probability one, say Ω_2. Applying Proposition 5.2.2 on the a.s. event $\Omega := \Omega_1 \cap \Omega_2$ yields

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\alpha^{-j}x(j) = \frac{\mu_a H^*}{1-\sum_{j=0}^{\infty}k(j)\alpha^{-(j+1)}} \quad a.s.$$

5.3 Fluctuations and Time Averages

We now explore fluctuations of the solutions to equation (5.1.1). Suppose $(a(n))_{n\geq 0}$ is an increasing sequence with $a(n)\to\infty$ as $n\to\infty$. For any sequence $(y(n))_{n\geq 0}$ define

$$\Lambda_a|y| = \limsup_{n\to\infty} \frac{|y(n)|}{a(n)}. \tag{5.3.1}$$

Our first result illustrates the close coupling of the quantities $\Lambda_a|x|$ and $\Lambda_a|H|$ when k and r are summable.

Theorem 5.3.1. *Suppose a is an increasing sequence with $a(n)\to\infty$ as $n\to\infty$. If $k\in\ell^1(\mathbb{Z}^+)$ obeys (5.1.5) and x is the solution to (5.1.1), then*

(a.) $\Lambda_a|x| = 0$ if and only if $\Lambda_a|H| = 0$;

(b.) $\Lambda_a|x| \in (0,\infty)$ if and only if $\Lambda_a|H| \in (0,\infty)$;

(c.) $\Lambda_a|x| = +\infty$ if and only if $\Lambda_a|H| = +\infty$.

In case (a.) fluctuations in x can be no larger than the size of the fluctuations present in H, and vice versa. Similarly, if we observe fluctuations of a certain order of magnitude in either x or H, fluctuations of the same order must have been present in the other sequence (case (b.)). Finally, fluctuations in the forcing sequence H must lead to fluctuations at least as large in the solution sequence x, and vice versa. While still applicable in a deterministic setting, the result above is more natural and useful when the forcing sequence is random and we employ Theorem 5.3.1 in this context in Section 5.4.2.

The next set of results provide bounds on time–averaged functionals of the solution to (5.1.1). The need for results of this type arises frequently in a variety of applications, particularly when H is a stochastic process (see Appleby and Patterson [14, Section 3.8] for sample applications).

Theorem 5.3.2. *Suppose $k\in\ell^1(\mathbb{Z}^+)$ obeys (5.1.5) and x is the solution to (5.1.1). If $\phi:[0,\infty)\mapsto[0,\infty)$ is an increasing convex function, then*

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}\phi(|x(j)|) \leq \limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}\phi\left(|r|_1\,|H(j)|\right).$$

Similarly,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}\phi(|H(j)|) \leq \limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}\phi\left((1+|k|_1)|x(j)|\right).$$

We remark that the non–unit multipliers inside the argument of ϕ in the result above are not entirely an artefact of the method of proof, although neither is our estimate sharp in general. We illustrate this point with a short example in which the sequence H is random. As usual, we defer the proof to the end.

Example 5.3.3. *Let $\sigma>0$ and suppose H is a sequence of independent and identically distributed normal random variables with mean zero and variance σ^2. Let $x(0)$ be deterministic and suppose $r\in\ell^1(\mathbb{Z}^+)$. By the strong law of large numbers,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}H^2(j) = \sigma^2 \quad a.s.$$

Furthermore, if $\lim_{n\to\infty}\log(n)\sum_{j=n}^{\infty}r^2(j) = 0$, then it can be shown that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}x^2(j) = \sigma^2\sum_{j=0}^{\infty}r^2(j) = \sigma^2\left(|r|_2\right)^2 \quad a.s.,$$

where $|r|_2$ denotes the ℓ^2–norm of r. In the context of this example, applying Theorem 5.3.2 with $\phi(x) = x^2$ would enable us to conclude that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n} x^2(j) \leq (|r|_1)^2 \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n} H^2(j) = \sigma^2 (|r|_1)^2 \quad a.s.$$

Since $(|r|_2)^2 \leq (|r|_1)^2$ in general, usually with strict inequality, the aforementioned lack of sharpness in the conclusion of Theorem 5.3.2 is immediately apparent.

With a slight strengthening of hypotheses on ϕ we can immediately prove a very useful corollary regarding the finiteness of time averaged functionals of the solution to (5.1.1). We first require the following standard definition.

Definition 5.3.1. *A nonnegative measurable function ϕ is called O–regularly varying if*

$$0 < \liminf_{x\to\infty} \frac{\phi(\lambda x)}{\phi(x)} \leq \limsup_{x\to\infty} \frac{\phi(\lambda x)}{\phi(x)} < \infty, \quad \text{for each } \lambda > 1.$$

While the definition above may seem somewhat restrictive, it turns out that if ϕ is increasing and $\limsup_{x\to\infty} \phi(\lambda x)/\phi(x)$ is finite for some $\lambda > 1$, then ϕ is O–regularly varying (see Bingham et al. [27, Corollary 2.0.6, p.65]). We now state the aforementioned corollary to Theorem 5.3.2 without proof.

Corollary 5.3.1. *Suppose $k \in \ell^1(\mathbb{Z}^+)$ obeys (5.1.5) and x is the solution to (5.1.1). If $\phi : [0,\infty) \mapsto [0,\infty)$ is an increasing, convex, and O–regularly varying function, then the following are equivalent:*

(a.)

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n} \phi(|x(j)|) < \infty,$$

(b.)

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n} \phi(|H(j)|) < \infty.$$

5.4 Examples & Stochastic Perturbations

5.4.1 Growth

The space G_λ contains many well–behaved sequences covering a wide range of growth, from very slow to very rapid. To see this, it is useful to introduce notation for iterated logarithms and exponentials. For any positive integer k, we define inductively the iterated logarithm $\log_k(x) = \log(\log_{k-1}(x))$ for $k \geq 2$ and $\log_1(x) = \log(x)$ (for appropriate positive x), and the iterated exponential $\exp_k(x) = \exp(\exp_{k-1}(x))$ for $k \geq 2$ and $\exp_1(x) = \exp(x)$ for all $x > 0$.

For examples in G_1 consider a sequence asymptotic to $H_1(n)$ where

$$H_1(n) = \prod_{i=1}^{j} (\log_i(n))^{\beta_i}$$

where j is a positive integer, and $(\beta_i)_{i=1}^{j}$ are any real numbers such that the first non–zero entry in the sequence β is positive. We also can take a sequence asymptotic to $H_1(n)$, where $\theta > 0$

$$H_2(n) = n^{\theta_1} H_1(n)$$

and there is no restriction on the sequence β in H_1. $H_3(n) = n^{\theta_1}$ for $\theta_1 > 0$ is another example in G_1. Sequences which grow faster than positive powers of n, but slower than exponentially are also admissible. For instance, sequences asymptotic to

$$H_4(n) = \exp(\alpha n^{\theta_2})$$

for $\alpha > 0$ and $\theta_2 \in (0,1)$, or even asymptotic to $H_5(n) = H_4(n)H_2(n)$ (without restriction on θ_1 or β in H_2), are also in G_1. For examples of sequences in G_λ for $\lambda \in (0,1)$, we have sequences which grow geometrically or are dominated by a geometric growth component. Such sequences include those asymptotic to $H_6(n) = \lambda^{-n}$ or

$$H_7(n) = H_6(n)H_4(n)H_2(n)$$

where there is no restriction on θ_1 or β in H_2, nor on θ_1 in H_4. Finally it can be seen that G_0 contains many sequences which grow faster than any geometric sequence. Examples include

$$H_8(n) = H_4(n)$$

where $\alpha > 0$ and $\theta_2 > 1$, $H_9(n) = n!$, and $H_{10}(n) = \exp_j(n)$ for any integer $j \geq 2$.

Example 5.4.1. *The sequences considered so far are deterministic and growing; within the class of growing stochastic processes some sequences reside in G_λ while others do not. Consider for instance the random walk with drift*

$$H(n) = \mu n + \sum_{j=1}^{n} Y(j), \quad n \geq 1$$

where $\mu \neq 0$ and the Y's are independent and identically distributed random variables with $\mathbb{E}[Y(j)] = 0$ and $\mathbb{E}[|Y(j)|] =: \mu_1 < +\infty$. We further assume that

$$\mathbb{P}[Y(n) = c] < 1 \text{ for all } c \in \mathbb{R}, \tag{5.4.1}$$

so that the Y's are meaningfully random, and are not almost surely constant. By the strong law of large numbers,

$$\lim_{n\to\infty} \frac{H(n)}{n} = \mu \neq 0, \quad a.s.$$

so $|H(n)|$ is asymptotic to the sequence $|\mu|n$ which is increasing, and H is clearly in G_1, despite the fact that the sequence $|H|$ is not monotone increasing or even ultimately monotone. To see this, take without loss $\mu > 0$ and suppose that $\mathbb{P}[Y(j) < -\mu] > 0$. This will certainly be true if each Y has a distribution supported on all $\mathbb{R}$. Therefore

$$\mathbb{P}[H(n+1) - H(n) < 0] = \mathbb{P}[\mu + Y(n+1) < 0] > 0,$$

so at each step there is a constant and positive probability that H decreases. Moreover, as the events $\{Y(j) < -\mu\}$ are independent, it is true that

$$\mathbb{P}[H(n+1) < H(n) \text{ for some } n > m] = 1, \text{ for each } m \in \mathbb{N},$$

so not only is H non–monotone almost surely, it is also ultimately non–monotone almost surely.

Example 5.4.2. *An example of a sequence that is not in G_λ (in general) is the geometric random walk. Suppose $\mu > 0$ above and consider the sequence*

$$H(n) = \exp\left(\mu n + \sum_{j=1}^{n} Y(j)\right), \quad n \geq 1 \tag{5.4.2}$$

where the process Y is as in Example 5.4.1. Clearly,

$$\frac{H(n-1)}{H(n)} = e^{-\mu} e^{-Y(n)},$$

so $\lim_{n\to\infty} H(n-1)/H(n)$ *exists if and only if* $Y(n)$ *tends to a finite limit. But as the sequence* Y *consists of independent and identically distributed random variables, which obey the non–degeneracy condition* (5.4.1)*, this is impossible. Hence, the growing geometric random walk is not in* G_λ*, for any* $\lambda \in [0,1]$.

To determine the asymptotic behaviour of x for less regular forcing sequences requires further assumptions on the data, and weaker conclusions. Here is an example of the type of result that can be established: we work with the geometric random walk from (5.4.2) for definiteness.

Theorem 5.4.3. *Suppose that* $\mu > 0$ *and that* H *is the geometric random walk given by* (5.4.2)*. Let* x *denote the solution of* (5.1.1) *and suppose* $x(0) > 0$*. If* k *is non–negative, with* $\sum_{j=0}^{\infty} k(j) < 1$*, then* $x(n) > 0$ *for all* $n \geq 0$ *a.s.,* $x(n) \to \infty$ *as* $n \to \infty$ *a.s. and*

$$\lim_{n\to\infty} \frac{1}{n} \log x(n) = \mu, \quad a.s.$$

Proof. Consider the sequence $(\log H(n))_{n\geq 0}$ and apply the strong law of large numbers to obtain

$$\lim_{n\to\infty} \frac{1}{n} \log H(n) = \mu, \quad \text{a.s.}$$

Suppose that Ω^* is the almost sure event on which this limit prevails. Since k is non–negative, we have that r is summable because $\sum_{j=0}^{\infty} k(j) < 1$. Clearly we have that $x(n) > 0$ for all $n \geq 0$ a.s. and furthermore that $x(n) > H(n)$ for all $n \geq 1$ a.s. Hence

$$\liminf_{n\to\infty} \frac{1}{n} \log x(n) \geq \mu, \quad \text{a.s.}$$

On the other hand, for every $\omega \in \Omega^*$ and $\epsilon > 0$, $H(n) < e^{(\mu+\epsilon)n} =: h_\epsilon(n)$ for all $n \geq N(\epsilon,\omega)$. Then we have

$$x(n+1) < h_\epsilon(n+1) + \sum_{j=0}^{n} k(n-j)x(j), \quad n \geq N(\epsilon,\omega)+1.$$

Define $x^*(\epsilon) = \max_{0\leq j\leq N(\epsilon,\omega)+1} x(j)$, so that

$$x(n+1) < x^*(\epsilon), \quad n = 0,\dots,N(\epsilon,\omega).$$

Hence, with $H_\epsilon(n) := e^{(\mu+\epsilon)n} + x^*(\epsilon)$ for all $n \geq 0$, we have the inequality

$$x(n+1) < H_\epsilon(n+1) + \sum_{j=0}^{n} k(n-j)x(j), \quad n \geq 0; \quad x(0) = \xi > 0.$$

Now, consider the solution of the summation equation

$$x_\epsilon(n+1) = H_\epsilon(n+1) + \sum_{j=0}^{n} k(n-j)x_\epsilon(j), \quad n \geq 0; \quad x_\epsilon(0) = x(0)+1.$$

By construction, $x(n) < x_\epsilon(n)$ for all $n \geq 0$. Moreover, x_ϵ omits the representation

$$x_\epsilon(n) = r(n)[\xi+1] + \sum_{j=1}^{n} r(n-j)H_\epsilon(j), \quad n \geq 0,$$

from (5.1.10). Notice now that H_ϵ is in G_λ for $\lambda = e^{-(\mu+\epsilon)}$. By Theorem 5.2.1,

$$\lim_{n\to\infty} \frac{x_\epsilon(n)}{e^{(\mu+\epsilon)n}} = \frac{1}{1-\sum_{j=0}^{\infty} k(j)e^{-(\mu+\epsilon)(j+1)}} =: L(\epsilon).$$

Hence, for each $\omega \in \Omega^*$ and $\epsilon > 0$, we have

$$\limsup_{n\to\infty} \frac{x(n,\omega)}{e^{(\mu+\epsilon)n}} \le L(\epsilon).$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{n}\log x(n,\omega) \le \mu+\epsilon, \quad \text{for each } \omega \in \Omega^*.$$

Finally, letting $\epsilon \to 0^+$ in the equation above and combining with the limit inferior yields

$$\lim_{n\to\infty} \frac{1}{n}\log x(n) = \mu, \quad \text{a.s.}$$

□

5.4.2 Fluctuation

We first sketch a general framework for dealing with forcing sequences comprised of independent and identically distributed (i.i.d.) random variables and then demonstrate how Theorem 5.3.1 can be applied in the presence of stochastic perturbations.

Suppose H is a sequence of i.i.d. random variables with common distribution function F. For ease of exposition assume the distribution is continuous and supported on all of $\mathbb{R}$.

Since each random variable $H(n)$ has distribution function F we have

$$\mathbb{P}[|H(n)| > Ka(n)] = 1 - F(Ka(n)) + F(-Ka(n)).$$

For each $K \in (0,\infty)$ and sequence $(a(n))_{n\ge 0}$, define

$$S(a,K) = \sum_{n=0}^{\infty}\{1 - F(Ka(n)) + F(-Ka(n))\}. \tag{5.4.3}$$

Since the events $\{|H(n)| > Ka(n)\}$ are independent, the Borel–Cantelli Lemma implies that

$$\mathbb{P}[|H(n)| > Ka(n) \text{ i.o.}] = \begin{cases} 0, & \text{if } S(a,K) < +\infty, \\ 1, & \text{if } S(a,K) = +\infty. \end{cases}$$

Therefore, for each $K > 0$ such that $S(a,K) < +\infty$, there is an a.s. event Ω_K^+ such that

$$\Lambda_a|H| = \limsup_{n\to\infty} \frac{|H(n)|}{a(n)} \le K, \quad \text{on } \Omega_K^+.$$

Similarly, for each $K > 0$ such that $S(a,K) = +\infty$ we have that there is an a.s. event Ω_K^- such that

$$\Lambda_a|H| = \limsup_{n\to\infty} \frac{|H(n)|}{a(n)} \ge K, \quad \text{on } \Omega_K^-.$$

It is sometimes possible, for a carefully–chosen sequence a and number K, to produce a sequence $Ka(n)$ for which $S(a,K)$ is either finite or infinite. This will generate upper and lower bounds on the growth of the sequence $(|H(n)|)_{n\ge 0}$, and thereby (via Theorem 5.3.1) allow conclusions about the growth of the fluctuations of x to be deduced.

We now present a ubiquitous example in which one can find a sequence a for which $\Lambda_a|H| \in (0,\infty)$.

Example 5.4.4. *Suppose that $H(n)$ is a sequence of independent normal random variables with mean zero and variance $\sigma^2 > 0$. If $a(n) = \sqrt{2\log n}$, then it is well–known that, for each $\epsilon \in (0,\sigma)$, we have*

$$S(a, \sigma+\epsilon) < +\infty, \quad S(a,\sigma-\epsilon) = +\infty.$$

Therefore, there are a.s. events $\Omega_\epsilon^{\pm}$ such that

$$\limsup_{n\to\infty} \frac{|H(n)|}{\sqrt{2\log n}} \geq \sigma - \epsilon, \quad \text{a.s. on } \Omega_\epsilon^- \quad \text{and,} \quad \limsup_{n\to\infty} \frac{|H(n)|}{\sqrt{2\log n}} \leq \sigma+\epsilon, \ \text{a.s. on } \Omega_\epsilon^+.$$

Now consider $\Omega^* = \{\cap_{\epsilon\in\mathbb{Q}\cap(0,\sigma)}\Omega_\epsilon^+\} \cap \{\cap_{\epsilon\in\mathbb{Q}\cap(0,\sigma)}\Omega_\epsilon^-\}$. By construction, Ω^* is an almost sure event and

$$\limsup_{n\to\infty} \frac{|H(n)|}{\sqrt{2\log n}} = \sigma, \quad \text{on } \Omega^*. \tag{5.4.4}$$

Hence we can apply Theorem 5.3.1 to (5.1.1) with $a(n) = \sqrt{2\log n}$ to obtain

$$0 < \limsup_{n\to\infty} \frac{|x(n)|}{\sqrt{2\log n}} < +\infty, \quad \text{a.s.}$$

In fact, scrutiny of the proof of Theorem 5.3.1 shows that there are deterministic constants K_1 and K_2 which depend on K (but not on σ) such that

$$K_1\sigma \leq \limsup_{n\to\infty} \frac{|x(n)|}{\sqrt{2\log n}} \leq K_2\sigma, \quad \text{a.s.}$$

In fact, we can generalise the conclusion of the previous example if $(H(n))_{n\geq 0}$ is a sequence of i.i.d. random variables with appropriately "thin tails". We first define the class of super–slowly varying functions as follows:

Definition 5.4.1. *A measurable function ℓ is called ξ-super–slowly varying (at infinity) if the limit*

$$\lim_{x\to\infty} \frac{\ell(x\xi^\delta(x))}{\ell(x)} = 1$$

holds uniformly for some $\Delta > 0$ and each $\delta \in [0,\Delta]$. We sometimes write $\ell \in \xi$–SSV for short.

The class of super–slowly varying functions arises naturally in extreme value theory and, in particular, in the context of inverting asymptotic relations involving slowly varying functions (see Anderson [3], and Bojanic and Seneta [28]). The cited works, as well as the classic volume of Bingham et al. [27, Ch. 3], give various convenient sufficient conditions for the definition above to hold; in many situations these conditions are preferable to verifying the definition directly. For example, if ℓ is slowly varying and ξ is non–decreasing with

$$\frac{\ell(\lambda x)}{\ell(x)} = 1 + o(1/\log\xi(x)), \ \text{as } x\to\infty, \ \text{for some } \lambda > 1, \tag{5.4.5}$$

then ℓ is ξ-super–slowly varying. Similarly, if ℓ is continuously differentiable, the condition (5.4.5) can be replaced by

$$\frac{x\ell'(x)}{\ell(x)} \text{ is } o(1/\log\xi(x)) \text{ as } x\to\infty.$$

We employ super–slow variation here as a convenient way to capture rapid decay in the tails of the distribution function which is sufficiently general that it includes most commonly used thin tailed distributions.

Theorem 5.4.5. *Let $(H(n))_{n\geq 0}$ be a sequence of i.i.d. random variables supported on $\mathbb{R}$ with continuous symmetric distribution function F. Define $G(x) = 1 - F(x)$ for each $x \in \mathbb{R}$. If $G^{-1}(1/x)$ is μ-super–slowly varying as $x \to \infty$ with $\mu : (0,\infty) \mapsto (1,\infty)$ non–decreasing and obeying*

$$\sum_{n=1}^{\infty} \frac{1}{n\mu^{\delta^*}(n)} < \infty, \text{ for some } \delta^* \in (0,\Delta], \tag{5.4.6}$$

then

$$\limsup_{n\to\infty} \frac{|H(n)|}{G^{-1}(1/n)} = 1 \quad a.s.$$

Remark 5.4.1. *Under the hypotheses of Theorem 5.4.5, $G^{-1}(1/x)$ is slowly varying as $x \to \infty$, since $\mu(x) \to \infty$ as $x \to \infty$ is a consequence of assuming that μ is non–decreasing and obeys (5.4.6). Hence $G \in RV_\infty(-\infty)$ and $F \in RV_{-\infty}(-\infty)$. We also note that symmetry in the tails of the distribution function is not an essential feature of the result stated above and this could be relaxed in the spirit of Theorem 5.4.8 below.*

It follows that if H is a sequence of random variables satisfying the hypotheses of Theorem 5.4.5, then applying Theorem 5.3.1 (which requires that k and r are summable) will show that the solution to (5.1.1) obeys

$$0 < \limsup_{n\to\infty} \frac{|x(n)|}{G^{-1}(1/n)} < \infty \quad \text{a.s.},$$

generalising the conclusion of Example 5.4.4.

Example 5.4.6. *We now provide some straightforward examples of common distributions and slowly varying functions which satisfy the hypotheses of Theorem 5.4.5. For the moment let $\mu : [0,\infty) \mapsto (1,\infty)$ be an arbitrary increasing and divergent function.*

First, we take an example of a function which grows particularly rapidly within the class of slowly varying functions but is still easily seen to be super–slowly varying (one could of course construct a distribution function with corresponding tails). Let $G^{-1}(1/x)/\exp(\log^\alpha x) \to 1$ as $x \to \infty$, with $\alpha \in (0,1)$. Now check the definition of super–slow variation (with $\delta = 1$) directly as follows:

$$\begin{aligned}\lim_{x\to\infty} \frac{G^{-1}(1/x\mu(x))}{G^{-1}(1/x)} &= \lim_{x\to\infty} \exp\left(\log^\alpha(x\mu(x)) - \log^\alpha(x)\right) \\ &= \exp\left(\lim_{x\to\infty} \log^\alpha(x\mu(x)) - \log^\alpha(x)\right).\end{aligned}$$

Define $\xi(x) = \log^\alpha(x\mu(x)) - \log^\alpha(x) = (\log x + \log\mu(x))^\alpha - \log^\alpha(x)$ for $x > 1$. If we can choose μ such that $\lim_{x\to\infty} \xi(x) = 0$, then $G^{-1}(1/x) \in \mu$–SSV. Apply the mean value theorem to ξ to yield

$$\xi(x) = \alpha(\log x + \theta_x \log\mu(x))^{\alpha-1}\log\mu(x),$$

for some $\theta_x \in [0,1]$. From this simple estimate we can see that

$$\lim_{x\to\infty} \frac{\xi(x)\log^{1-\alpha} x}{\alpha\log\mu(x)} = 1.$$

Hence we could choose μ such that $\mu(x)/\log^\beta x \to 1$ as $x \to \infty$ with $\beta > 1$ and obtain $\lim_{x\to\infty} \xi(x) = 0$, as required. This choice would also provide a (μ,δ^*) pair satisfying the summability condition (5.4.6), since

$$\sum_{n=1}^{\infty} \frac{1}{n\log^\beta(n+1)} < \infty, \text{ for each } \beta > 1,$$

by the Cauchy condensation test.

In the case of the normal distribution with variance σ^2, $G^{-1}(1/x)/\sqrt{2\sigma^2\log x} \to 1$ as $x \to \infty$. We once more choose $\delta = 1$ and in this instance we require

$$\lim_{x\to\infty} \frac{G^{-1}(1/x\mu(x))}{G^{-1}(1/x)} = \lim_{x\to\infty} \frac{\sqrt{2\sigma^2\log(x\mu(x))}}{\sqrt{2\sigma^2\log(x)}} = \sqrt{1 + \lim_{x\to\infty} \frac{\log(\mu(x))}{\log(x)}} = 1.$$

A sufficient condition for the limit above to hold is $\lim_{x\to\infty}\log(\mu(x))/\log(x) = 0$. Hence we could choose μ such that $\mu(x)/\log^\beta x \to 1$ as $x\to\infty$ with $\beta > 1$, as in the previous example, and satisfy both condition (5.4.6) and the requirement that $G^{-1}(1/x) \in \mu$–SSV.

In many applications, particularly in economics and finance, it is important to understand the behaviour of systems driven or corrupted by random noise which is characterised by slow decay in the tails of the related distribution function — so–called "heavy tailed" distributions. Our next example provides a simple application of our results in such a situation.

Example 5.4.7. *We consider the case of a symmetric heavy tailed distribution with power law decay in the tails. Suppose that $H(n)$ are i.i.d. random variables such that there is $\alpha > 0$ and finite $c_1, c_2 > 0$ for which*

$$\lim_{x\to-\infty} \frac{F(x)}{|x|^{-\alpha}} = c_1, \quad \lim_{x\to+\infty} \frac{1-F(x)}{x^{-\alpha}} = c_2.$$

If a_+ and a_- are sequences such that

$$\sum_{n=0}^{\infty} a_+(n)^{-\alpha} < +\infty, \quad \sum_{n=0}^{\infty} a_-(n)^{-\alpha} = +\infty,$$

then $S(K, a_+) < +\infty$ for all $K > 0$ and $S(K, a_-) = +\infty$ for all $K > 0$. Therefore, for all $K > 0$, $\limsup_{n\to\infty}|H(n)|/a_+(n) \le K$, on Ω_K^+. The event $\Omega^+ = \cap_{K\in\mathbb{Q}^+}\Omega_K^+$ has probability one and

$$\limsup_{n\to\infty} \frac{|H(n)|}{a_+(n)} = 0, \quad \text{on } \Omega^+.$$

On the other hand, for all $K > 0$ there is an a.s. event Ω_K^- such that $\limsup_{n\to\infty}|H(n)|/a_-(n) \ge K$ on Ω_K^-. Consider the event $\Omega^- = \cap_{K\in\mathbb{Z}^+}\Omega_K^-$. Then Ω^- is an almost sure event and we have

$$\limsup_{n\to\infty} \frac{|H(n)|}{a_-(n)} = +\infty, \quad \text{on } \Omega^-.$$

Finally, construct the a.s. event $\Omega^* = \Omega^+ \cap \Omega^-$ and notice that

$$\limsup_{n\to\infty} \frac{|H(n)|}{a_+(n)} = 0, \quad \limsup_{n\to\infty} \frac{|H(n)|}{a_-(n)} = \infty, \text{ on } \Omega^*.$$

Applying part (a.) of Theorem 5.3.1 with $a = a_+$ and part (c.) with $a = a_-$, we obtain

$$\limsup_{n\to\infty} \frac{|x(n)|}{a_+(n)} = 0, \quad \limsup_{n\to\infty} \frac{|x(n)|}{a_-(n)} = +\infty, \quad \text{on } \Omega^*.$$

We now try to choose a_+ and a_- "close" to one another, in an appropriate sense. For every $\epsilon > 0$ sufficiently small take $a_\pm(n)$ to be $a_{\pm\epsilon}(n) = n^{1/\alpha\pm\epsilon}$. First, from the existence of the sequences $a_{\pm\epsilon}$ we conclude that there are a.s. events Ω_ϵ^- and Ω_ϵ^+ such that

$$\limsup_{n\to\infty} \frac{|x(n)|}{n^{1/\alpha-\epsilon}} = +\infty, \text{ on } \Omega_\epsilon^- \quad \text{and} \quad \limsup_{n\to\infty} \frac{|x(n)|}{n^{1/\alpha+\epsilon}} = 0, \text{ on } \Omega_\epsilon^+.$$

Now we seek ϵ–independent limits. We conclude from the limits above that

$$\limsup_{n\to\infty} \frac{\log|x(n)|}{\log n} \geq \frac{1}{\alpha} - \epsilon, \text{ on } \Omega_\epsilon^- \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log|x(n)|}{\log n} \leq \frac{1}{\alpha} + \epsilon, \text{ on } \Omega_\epsilon^+.$$

Finally, by constructing the a.s. event $\Omega^* = \{\cap_{\epsilon\in\mathbb{Q}^+}\Omega_\epsilon^+\} \cap \{\cap_{\epsilon\in\mathbb{Q}^+}\Omega_\epsilon^-\}$, it follows that

$$\limsup_{n\to\infty} \frac{\log|x(n)|}{\log n} = \frac{1}{\alpha}, \quad a.s.$$

While the preceding example is in some ways quite special, it is worthwhile pointing out one aspect which is generic. If the sequence of i.i.d. random variables $(H(n))_{n\geq 0}$ has a generalized power law decay in the tails, it is not possible to find an increasing sequence $(a(n))_{n\geq 0}$ tending to infinity which characterizes the partial maxima of $(H(n))_{n\geq 0}$ in the sense of (5.4.4). Hence there was no possible "smarter choice" of the sequences $(a_\pm(n))_{n\geq 0}$ in Example 5.4.7 that could have given us information of the same quality as in Example 5.4.4; this also demonstrates the practical merit of parts ($a.$) and ($c.$) of Theorem 5.3.1. The following result makes the claim above precise.

Theorem 5.4.8. *Let $(H(n))_{n\geq 0}$ be a sequence of i.i.d. random variables supported on $\mathbb{R}$ with continuous distribution function F. Suppose that one of the following holds:*

(i.) $1 - F \in RV_\infty(-\alpha)$ *for some* $\alpha > 0$ *and* $\lim_{x\to\infty}(1 - F(x))/F(-x) = \infty$*;*

(ii.) $F \in RV_{-\infty}(-\alpha)$ *for some* $\alpha > 0$ *and* $\lim_{x\to\infty}(1 - F(x))/F(-x) = 0$*;*

(iii.) $1 - F \in RV_\infty(-\alpha)$ *for some* $\alpha > 0$ *and* $\lim_{x\to\infty}(1 - F(x))/F(-x) = L \in (0,\infty)$*.*

For each positive, increasing (deterministic) sequence $(a(n))_{n\geq 0}$ which tends to infinity, either

$$\limsup_{n\to\infty} \frac{|H(n)|}{a(n)} = 0 \quad a.s., \quad or \quad \limsup_{n\to\infty} \frac{|H(n)|}{a(n)} = \infty \quad a.s. \tag{5.4.7}$$

We emphasise the fact that a is a deterministic sequence in the result above because one could choose the a.s. increasing random sequence $a(n) = \max_{1\leq j\leq n}|H(j)|$ for each $n \geq 0$ and obtain nontrivial limits in (5.4.7). However, understanding the stochastic process H in terms of the closely related process $(\max_{1\leq j\leq n}|H(j)|)_{n\geq 0}$ clearly does not provide the same insight as a result such as Theorem 5.4.5. The case when the distribution function has symmetric tails is trivially included in case ($iii.$) of the result above and hence it applies to our earlier example of power law decay.

5.5 Linearisation at Infinity

Consider the nonlinear Volterra summation equation given by

$$x(n+1) = H(n+1) + \sum_{j=0}^{n} k(n-j)f(x(j)), \quad n \geq 1; \quad x(0) = \xi, \tag{5.5.1}$$

where f obeys

$$f \in C(\mathbb{R};\mathbb{R}). \tag{5.5.2}$$

Most models of macroeconomic growth have nonlinear equations of motion for one or more state variables; however, in many cases, these equations contain a term which is linear in the state. Furthermore, based on standard economic considerations, the nonlinear terms in these models are generally sublinear at infinity (in the sense that $\lim_{|x|\to\infty} g(x)/x = 0$). For example, consider the seminal discrete time

Solow model with no technological progress or population growth (see Solow [113] or, for a more modern account, Acemoglu [1]). The evolution of output per capita k is given by the nonlinear difference equation

$$k(n+1) = f(k(n)) := (1-\delta)k(n) + s\,g(k(n)), \quad n \geq 0, \tag{5.5.3}$$

where g obeys $\lim_{k\to\infty} g'(k) = 0$ (due to the Inada conditions on the aggregate production function), $\delta \in (0,1)$ is the rate of depreciation per time period and $s \in (0,1)$ is the exogenous savings rate. The structure of the nonlinearity highlighted above is, in some sense, generic and can be found in many of the macroeconomic growth models subsequently developed from the basic Solow model. The aforementioned linear leading order behaviour is also found in analogous macroeconomic models incorporating delays (see d'Halbis et al. [43, Example 4.2] for a neoclassical growth model with delay). Hence it is natural, in this context, to assume that the nonlinear function f in (5.5.1) obeys

$$\lim_{x\to\infty} \frac{f(x)}{x} = 1, \quad \lim_{x\to-\infty} \frac{f(x)}{x} = 1. \tag{5.5.4}$$

Under the hypothesis (5.5.4), it is reasonable to conjecture that x may have similar asymptotic behaviour to its "linearisation at infinity", which solves the equation

$$y(n+1) = H(n+1) + \sum_{j=0}^{n} k(n-j)y(j), \quad n \geq 1; \quad y(0) = \xi. \tag{5.5.5}$$

The next result goes some distance to supporting this claim. It can then be used as a lemma to establish results on the fluctuation and growth of the solution to (5.5.1), once the growth or fluctuation of the sequence H is sufficiently well–understood.

Theorem 5.5.1. *Let x and y denote the solutions to (5.5.1) and (5.5.5) respectively. Suppose that $k \in \ell^1(\mathbb{Z}^+)$ obeys (5.1.5) and that f obeys (5.5.2) and (5.5.4). If $(a(n))_{n\geq 0}$ is an increasing sequence such that*

$$\limsup_{n\to\infty} \frac{|H(n)|}{a(n)} < +\infty,$$

then

$$\limsup_{n\to\infty} \frac{|y(n)|}{a(n)} < +\infty, \quad \text{and} \quad \lim_{n\to\infty} \frac{x(n)-y(n)}{a(n)} = 0.$$

The result above shows that when $\Lambda_a|H| \in (0,\infty)$, not only is $\Lambda_a|y| \in (0,\infty)$, but also $\Lambda_a|x| = \Lambda_a|y|$; in other words, the solutions to (5.5.1) and (5.5.5) are coupled together so that the difference between them is always $o(a(n))$ as $n \to \infty$. Moreover, we have shown that $x(n)/a(n) \approx y(n)/a(n)$ as $n \to \infty$ and hence we can prove a corollary of Theorem 5.5.1 which constitutes a nonlinear version of Theorem 5.2.2.

Theorem 5.5.2. *Let x denote the solution to (5.5.1), and suppose that $k \in \ell^1(\mathbb{Z}^+)$ obeys (5.1.5). If f obeys (5.5.2) and (5.5.4), and G_λ is defined by (5.1.7), then the following are equivalent:*

(a.) $H \in BG_{a,\lambda}$;

(b.) $x \in BG_{a,\lambda}$.

Moreover, when $H \in BG_{a,\lambda}$,

$$\frac{x(n)}{a(n)} \approx (\lambda_a H)(n) + \sum_{j=1}^{n} r(j)\lambda^j(\lambda_a H)(n-j), \text{ as } n\to\infty, \tag{5.5.6}$$

and similarly, when $x \in BG_{a,\lambda}$,

$$\frac{H(n)}{a(n)} \approx (\lambda_a x)(n) - \sum_{j=0}^{n-1} k(j)\lambda^{j+1}(\lambda_a x)(n-j-1), \ \text{as } n \to \infty. \tag{5.5.7}$$

Similarly, it is straightforward to generate nonlinear versions of Theorem 5.2.1, Proposition 5.2.1, and Proposition 5.2.2 using the same line of arugment used to establish Theorem 5.5.2.

Theorem 5.5.1 has another nice corollary when H obeys the hypotheses of Theorem 5.3.1, and in fact the solution of the nonlinear equation inherits the fluctuation bounds seen in (5.1.1).

Theorem 5.5.3. *Let x denote the solution to (5.5.1) and suppose that $k \in \ell^1(\mathbb{Z}^+)$ obeys (5.1.5). Suppose further that f obeys (5.5.2) and (5.5.4). If $(a(n))_{n\geq 0}$ is an increasing sequence such that $\Lambda_a|H| \in [0,\infty]$, then*

(a.) $\Lambda_a|x| = 0$ *if and only if* $\Lambda_a|H| = 0$.

(b.) $\Lambda_a|x| \in (0,\infty)$ *if and only if* $\Lambda_a|H| \in (0,\infty)$.

(c.) $\Lambda_a|x| = \infty$ *if and only if* $\Lambda_a|H| = \infty$.

The proof of the forward implications of (a.) and (b.) can be read off from Theorem 5.5.1. The reverse implications can be established by rewriting H in terms of x according to

$$H(n+1) = x(n+1) - \sum_{j=0}^{n} k(n-j)f(x(j)), \quad n \geq 1,$$

and using (5.5.2) and (5.5.4), as well as results concerning the growth of convolutions (see Lemma 5.6.1), to bound the right hand side. As to the proof of the last part, take as hypothesis that $\Lambda_a|H| = \infty$. Suppose now that $\Lambda_a|x| < \infty$; then applying part (b.) gives a contradiction, so $\Lambda_a|H| = \infty$ implies $\Lambda_a|x| = \infty$. If, on the other hand we take as hypothesis that $\Lambda_a|x| = \infty$, and suppose that $\Lambda_a|H| < \infty$, applying part (b.) again gives a contradiction. Hence $\Lambda_a|x| = \infty$ implies $\Lambda_a|H| = \infty$.

5.6 Proofs

5.6.1 Proof of Theorem 5.2.1

First note that the proof of Theorem 5.2.2 does not depend on the conclusion of Theorem 5.2.1 whatsoever. If $H \in G_\lambda$, then $H \in BG_{a,\lambda}$ with $a = H$ and $\lambda_a H = 1$. Hence Theorem 5.2.2 implies that

$$\lim_{n\to\infty} \frac{x(n)}{H(n)} = 1 + \sum_{j=1}^{\infty} r(j)\lambda^j. \tag{5.6.1}$$

When $\lambda = 0$, $\lim_{n\to\infty} x(n)/H(n) = 1$. If $\lambda \in (0,1]$, we claim that

$$\sum_{j=0}^{\infty} r(j)\lambda^j = \frac{1}{1-\sum_{j=0}^{\infty} k(j)\lambda^{j+1}},$$

since $r \in \ell^1(\mathbb{Z}^+)$. To see this first note that both $\left(\sum_{j=0}^{N} r(j)\lambda^j\right)_{N\geq 0}$ and $\left(\sum_{j=0}^{N} k(j)\lambda^{j+1}\right)_{N\geq 0}$ are Cauchy sequences because $r \in \ell^1(\mathbb{Z}^+)$ and $k \in \ell^1(\mathbb{Z}^+)$. Furthermore, by the same considerations, the limits of these sequences are finite. Recall that

$$r(n) = \sum_{j=0}^{n-1} k(n-1-j)r(j), \quad n \geq 1, \quad r(0) = 1.$$

Suppose $N > 1$ and calculate as follows:

$$\begin{aligned}\sum_{n=0}^{N} r(n)\lambda^n &= 1 + \sum_{n=1}^{N}\sum_{j=0}^{n-1} k(n-1-j)r(j)\lambda^n = 1 + \sum_{n=1}^{N}\sum_{l=0}^{n-1} k(l)r(n-l-1)\lambda^{l+1}\lambda^{n-l-1}\\
&= 1 + \sum_{l=0}^{N} k(l)\lambda^{l+1}\left(\sum_{n=l+1}^{N} r(n-l-1)\lambda^{n-l-1}\right)\\
&= 1 + \sum_{l=0}^{N} k(l)\lambda^{l+1}\left(\sum_{m=0}^{N-l+1} r(m)\lambda^{m}\right).\end{aligned}$$

Hence

$$\sum_{j=0}^{\infty} r(j)\lambda^j = 1 + \left(\sum_{l=0}^{\infty} k(l)\lambda^{l+1}\right)\left(\sum_{m=0}^{\infty} r(m)\lambda^m\right), \quad \text{when } \lambda \in (0,1].$$

Rearrange to obtain

$$\sum_{j=0}^{\infty} r(j)\lambda^j = \frac{1}{1-\sum_{l=0}^{\infty} k(l)\lambda^{l+1}}, \quad \text{when } \lambda \in (0,1].$$

Note that the right–hand side of the equality above is finite due to (5.1.5). Combine the calculation above with (5.6.1) to obtain

$$\lim_{n\to\infty} \frac{x(n)}{H(n)} = \frac{1}{1-\sum_{l=0}^{\infty} k(l)\lambda^{l+1}}, \quad \text{when } \lambda \in [0,1].$$

The above limit clearly implies that $x \in G_\lambda$ also.

Similarly, if $x \in G_\lambda$, then $x \in BG_{a,\lambda}$ with $a = x$ and $\lambda_a x = 1$. By Theorem 5.2.2 and equation (5.2.4), we have

$$\lim_{n\to\infty} \frac{H(n)}{x(n)} = 1 - \sum_{j=0}^{\infty} k(j)\lambda^{j+1}.$$

By analogous considerations, the limit above suffices to prove the converse claim.

5.6.2 Proof of Theorem 5.2.2

First assume that $H \in BG_{a,\lambda}$ and show that (5.2.3) holds; it is clear that $x \in BG_{a,\lambda}$ follows from (5.2.3). Since $H \in BG_{a,\lambda}$, there is a bounded sequence $((\lambda_a H)(n))_{n\geq 0}$ such that

$$\lim_{n\to\infty} \left|(\lambda_a H)(n) - \frac{H(n)}{a(n)}\right| = 0,$$

for some $(a(n))_{n\geq 0} \in G_\lambda$. From (5.1.10), we have that

$$\frac{x(n)}{a(n)} = \frac{H(n)}{a(n)} + \frac{r(n)x(0)}{a(n)} + \frac{1}{a(n)}\sum_{j=1}^{n-1} r(n-j)H(j), \quad n \geq 2. \tag{5.6.2}$$

Hence

$$\begin{aligned}\left|\frac{x(n)}{a(n)} - (\lambda_a H)(n) - \sum_{l=1}^{n} r(l)\lambda^l(\lambda_a H)(n-l)\right| &\leq \left|\frac{H(n)}{a(n)} - (\lambda_a H)(n)\right| + \left|\frac{r(n)x(0)}{a(n)}\right|\\
&\quad + \left|\sum_{j=1}^{n-1} r(n-j)\frac{H(j)}{a(n)} - \sum_{l=1}^{n} r(l)\lambda^l(\lambda_a H)(n-l)\right|, \quad n \geq 2. \end{aligned}\tag{5.6.3}$$

The first two terms on the right–hand side above clearly tend to zero as $n \to \infty$ and thus it remains to show that the final term also has limit zero. Let $\epsilon > 0$ be arbitrary in what follows. Since $H \in BG_{a,\lambda}$, there exists $N_1(\epsilon) > 2$ such that

$$\left|\frac{H(n)}{a(n)} - (\lambda_a H)(n)\right| < \epsilon, \text{ for all } n \geq N_1(\epsilon). \tag{5.6.4}$$

Given $N_1(\epsilon)$, since $a \in G_\lambda$ and $r \in \ell^1(\mathbb{Z}^+)$, there exists $N_2(\epsilon) > 1$ such that

$$\left|\frac{1}{a(n)}\sum_{j=1}^{N_1-1} r(n-j)H(j)\right| < \epsilon, \text{ for all } n \geq N_2(\epsilon) \text{ and } j \in \{1,\dots,N_3+1\}. \tag{5.6.5}$$

Similarly, because $r \in \ell^1(\mathbb{Z}^+)$, there exists $N_3(\epsilon) > 1$ such that $\sum_{j=N_3}^{\infty} |r(j)| < \epsilon$. Finally, since $a \in G_\lambda$, there is an $N_4(\epsilon) > 1$ such that

$$\left|\frac{a(n-j)}{a(n)} - \lambda^j\right| < \epsilon, \text{ for all } n \geq N_4(\epsilon).$$

First concentrate on the final term on the right–hand side of (5.6.2) and decompose as follows:

$$\begin{aligned}\sum_{j=1}^{n-1} r(n-j)\frac{H(j)}{a(n)} &= \sum_{j=1}^{N_1-1} r(n-j)\frac{H(j)}{a(n)} + \frac{1}{a(n)}\sum_{j=N_1}^{n-1} r(n-j)a(j)\left\{\frac{H(j)}{a(j)} - (\lambda_a H)(j)\right\}\\ &\quad + \frac{1}{a(n)}\sum_{j=N_1}^{n-1} r(n-j)a(j)(\lambda_a H)(j), \quad n \geq N_1+1.\end{aligned}$$

Splitting the final term in the expression above, with $n \geq N_1 + N_3 + 3$, then yields

$$\begin{aligned}\sum_{j=1}^{n-1} r(n-j)\frac{H(j)}{a(n)} &= \sum_{j=1}^{N_1-1} r(n-j)\frac{H(j)}{a(n)} + \frac{1}{a(n)}\sum_{j=N_1}^{n-1} r(n-j)a(j)\left\{\frac{H(j)}{a(j)} - (\lambda_a H)(j)\right\}\\ &\quad + \frac{1}{a(n)}\sum_{j=N_1}^{n-N_3-2} r(n-j)a(j)(\lambda_a H)(j) + \frac{1}{a(n)}\sum_{l=1}^{N_3+1} r(l)a(n-l)(\lambda_a H)(n-l),\end{aligned} \tag{5.6.6}$$

where the order of summation was reversed in the final term. Subtract $\sum_{l=1}^{n} r(l)\lambda^l(\lambda_a H)(n-l)$ from the expression above and take absolute values to obtain

$$\left|\sum_{j=1}^{n-1} r(n-j)\frac{H(j)}{a(n)} - \sum_{l=1}^{n} r(l)\lambda^l(\lambda_a H)(n-l)\right| \leq \left|\sum_{j=1}^{N_1-1} r(n-j)\frac{H(j)}{a(n)}\right| \tag{5.6.7a}$$

$$+ \sum_{j=N_1}^{n-1} |r(n-j)|\left|\frac{a(j)}{a(n)}\right|\left|\frac{H(j)}{a(j)} - (\lambda_a H)(j)\right| \tag{5.6.7b}$$

$$+ \sum_{j=N_1}^{n-N_3-2} |r(n-j)|\left|\frac{a(j)}{a(n)}\right||(\lambda_a H)(j)| \tag{5.6.7c}$$

$$+ \sum_{l=1}^{N_3+1} |r(l)|\left|\frac{a(n-l)}{a(n)} - \lambda^l\right||(\lambda_a H)(n-l)| \tag{5.6.7d}$$

$$+ \sum_{l=N_3+2}^{n} |r(l)|\lambda^l|(\lambda_a H)(n-l)|, \tag{5.6.7e}$$

which is valid for each $n \geq N_1 + N_3 + 3$. Now let $n \geq N_1 + N_2 + N_3 + N_4 + 1$; the term on the right–hand side of (5.6.7a) is less than ϵ for all $n \geq N_2$ by construction (see (5.6.5)). Estimate the right–hand side of (5.6.7b) as follows:

$$\sum_{j=N_1}^{n-1} |r(n-j)| \left|\frac{a(j)}{a(n)}\right| \left|\frac{H(j)}{a(j)} - (\lambda_a H)(j)\right| \le \epsilon \sum_{j=N_1}^{n-1} |r(n-j)| \left|\frac{a(j)}{a(n)}\right| \le \epsilon\, \bar{A}\, |r|_1,$$

where we have used that $a(j)/a(n)$ can be uniformly bounded by $\bar{A} > 0$ and that the summation index started at N_1 (see (5.6.4)). Next estimate the term in (5.6.7c):

$$\sum_{j=N_1}^{n-N_3-2} |r(n-j)| \left|\frac{a(j)}{a(n)}\right| |(\lambda_a H)(j)| \le \bar{\Lambda}\,\bar{A} \sum_{j=N_1}^{n-N_3-2} |r(n-j)| \le \bar{\Lambda}\,\bar{A} \sum_{l=N_3+2}^{\infty} |r(l)| \le \epsilon\, \bar{\Lambda}\,\bar{A},$$

where $\lambda_a H$ was uniformly bounded by $\bar{\Lambda} > 0$ and we also used the definition of $N_3(\epsilon)$. Now note that $n - N_3 - 1 \ge N_4$ and hence that we may estimate from (5.6.7d) as below:

$$\sum_{l=1}^{N_3+1} |r(l)| \left|\frac{a(n-l)}{a(n)} - \lambda^l\right| |(\lambda_a H)(n-l)| \le \epsilon \sum_{l=1}^{N_3+1} |r(l)||(\lambda_a H)(n-l)| \le \epsilon\, \Lambda\, |r|_1.$$

The estimation of (5.6.7e) is handled by simply noting that $\lambda \in [0,1]$, $r \in \ell^1(\mathbb{Z}^+)$, and that $\lambda_a H$ is bounded; thus

$$\sum_{l=N_3+2}^{n} |r(l)|\lambda^l |(\lambda_a H)(n-l)| \le \bar{\Lambda} \sum_{l=N_3+2}^{\infty} |r(l)| \le \bar{\Lambda}\epsilon.$$

Returning to (5.6.7), we have shown that

$$\left|\sum_{j=1}^{n-1} r(n-j)\frac{H(j)}{a(n)} - \sum_{l=1}^{n} r(l)\lambda^l (\lambda_a H)(n-l)\right| \le \epsilon\left(1 + \bar{A}\,|r|_1 + \bar{\Lambda}\,\bar{A} + \Lambda\,|r|_1 + \bar{\Lambda}\right),$$

for each $n \ge N_1 + N_2 + N_3 + N_4 + 1$. Letting $n \to \infty$ and then $\epsilon \to 0$ in the estimate above, we have proven that

$$\lim_{n\to\infty} \left|\sum_{j=1}^{n-1} r(n-j)\frac{H(j)}{a(n)} - \sum_{l=1}^{n} r(l)\lambda^l (\lambda_a H)(n-l)\right| = 0, \text{ for each } \lambda \in [0,1],$$

and hence demonstrated that (5.2.3) holds.

For the converse result assume $x \in BG_{a,\lambda}$ and note that

$$\begin{aligned}\frac{H(n)}{a(n)} &= \frac{x(n)}{a(n)} - \frac{1}{a(n)} \sum_{j=0}^{n-1} k(n-1-j)x(j) \\ &= \frac{x(n)}{a(n)} - k(n-1)x(0) - \frac{1}{a(n)} \sum_{j=1}^{n-1} k(n-1-j)x(j), \quad n \ge 1.\end{aligned}$$

Hence, for $n \ge 1$, we have

$$\begin{aligned}\left|\frac{H(n)}{a(n)} - (\lambda_a x)(n) + \sum_{j=0}^{n-1} k(j)\lambda^{j+1}(\lambda_a x)(n-j-1)\right| &\le \left|\frac{x(n)}{a(n)} - (\lambda_a x)(n)\right| \\ &+ \left|\frac{k(n-1)x(0)}{a(n)}\right| + \left|\sum_{j=1}^{n-1} k(n-1-j)\frac{x(j)}{a(n)} - \sum_{j=0}^{n-1} k(j)\lambda^{j+1}(\lambda_a x)(n-j-1)\right|. \quad (5.6.8)\end{aligned}$$

Now rewrite the final sum on the right–hand side above as follows:

$$\sum_{j=0}^{n-1} k(j)\lambda^{j+1}(\lambda_a x)(n-j-1) = \sum_{l=1}^{n} k(l-1)\lambda^{l}(\lambda_a x)(n-l).$$

Substitute the expression above into (5.6.8) to obtain

$$\left|\frac{H(n)}{a(n)} - (\lambda_a x)(n) + \sum_{j=0}^{n-1} k(j)\lambda^{j+1}(\lambda_a x)(n-j-1)\right| \leq \left|\frac{x(n)}{a(n)} - (\lambda_a x)(n)\right| + \left|\frac{k(n-1)x(0)}{a(n)}\right| + \left|\sum_{j=1}^{n-1} k(n-1-j)\frac{x(j)}{a(n)} - \sum_{l=1}^{n} k(l-1)\lambda^{l}(\lambda_a x)(n-l)\right|.$$

Compare the above estimate with (5.6.3); these estimates are exactly analogous with $k(j-1)$ replaced by $r(j)$, $\lambda_a x$ replaced by $\lambda_a H$, and x replaced by H. Repeat the argument above to complete the proof.

5.6.3 Proof of Proposition 5.2.1

We use elementary but nontrivial properties of almost periodic sequences throughout this argument; the reader is invited to consult Corduneanu [38, Chapter 1] for the requisite proofs.

Assume $H \in PG_{a,\lambda}$. Since $H/a \in \text{AAP}(\mathbb{Z}^+)$,

$$\frac{H(n)}{a(n)} = \pi_H(n) + \phi(n), \quad \text{for each } n \geq 0,$$

for some $\pi_H \in AP(\mathbb{Z})$ and $(\phi(n))_{n\geq 0}$ such that $\phi(n) \to 0$ as $n \to \infty$. Moreover, π_H is bounded (because it is almost periodic). Hence we may take $\lambda_a H = \pi_H$ in (5.1.8), or in other words $PG_{a,\lambda} \subset BG_{a,\lambda}$. Therefore the asymptotic representation (5.2.6) is valid by appealing to Theorem 5.2.2. Thus

$$\frac{x(n)}{a(n)} = \pi_H(n) + \sum_{j=1}^{n} r(j)\lambda^j \pi_H(n-j) + \tilde{\phi}(n), \quad \text{for each } n \geq 0, \tag{5.6.9}$$

where $\left(\tilde{\phi}(n)\right)_{n\geq 0}$ obeys $\tilde{\phi}(n) \to 0$ as $n \to \infty$. It is clear from (5.6.9) that if $\lambda = 0$, then $x/a \in AAP(\mathbb{Z}^+)$ and hence that $x \in PG_{a,\lambda}$. Henceforth assume that $\lambda \in (0,1]$. Consider the sequence $(\pi(n))_{n\geq 0}$ given by

$$\pi(n) = \sum_{j=n+1}^{\infty} r(j)\lambda^{j+1}\pi_H(n-j), \quad \text{for each } n \geq 0,$$

and note that it is well defined because $\pi_H(n)$ is bounded and defined for all $n \in \mathbb{Z}$. Furthermore, since $r \in \ell^1(\mathbb{Z}^+)$ and π_H is bounded, $\lim_{n\to\infty} \pi(n) = 0$. Thus adding $\pi(n)$ to both sides of (5.6.9) yields

$$\frac{x(n)}{a(n)} = \sum_{j=0}^{\infty} r(j)\lambda^j \pi_H(n-j) + \Phi(n), \quad \text{for each } n \geq 0, \tag{5.6.10}$$

where $\Phi(n) = \tilde{\phi}(n) - \pi(n)$ for each $n \geq 0$ and hence $\Phi(n) \to 0$ as $n \to \infty$. We claim that the sequence $(\pi_x(n))_{n\in\mathbb{Z}}$ given by

$$\pi_x(n) = \sum_{j=0}^{\infty} r(j)\lambda^j \pi_H(n-j), \quad n \in \mathbb{Z},$$

is almost periodic. To see this we require the definition of a normal sequence, which we now state:

Definition 5.6.1. *$(\pi_H(n))_{n\in\mathbb{Z}}$ is normal if for any sequence $(\alpha(l))_{l\in\mathbb{Z}}\subset\mathbb{Z}$ there exists a subsequence $(\alpha'(l))_{l\in\mathbb{Z}}$, a sequence $(\bar{\pi}_H(n))_{n\in\mathbb{Z}}$, and an integer $L(\epsilon)$ such that*

$$|\pi_H\left(n+\alpha'(l)\right)-\bar{\pi}_H(n)|<\epsilon \quad \textit{for } l\geq L(\epsilon) \textit{ and each } n\in\mathbb{Z}. \tag{5.6.11}$$

In other words, $\pi_H(n+\alpha'(l))$ converges uniformly with respect to $n\in\mathbb{Z}$ as $l\to\infty$.

A sequence is almost periodic if and only if it is normal. In order to show that π_x is normal, let $(\alpha(l))_{l\in\mathbb{Z}}\subset\mathbb{Z}$ be an arbitrary sequence. Since π_H is normal, there exists a subsequence $(\alpha'(l))_{l\in\mathbb{Z}}$, a sequence $(\bar{\pi}_H(n))_{n\in\mathbb{Z}}$, and an integer $L(\epsilon)$ such that (5.6.11) holds for π_H. Define the sequence $(\bar{\pi}_x(n))_{n\in\mathbb{Z}}$ by

$$\bar{\pi}_x(n)=\sum_{j=0}^{\infty}r(j)\lambda^j\bar{\pi}_H(n-j), \quad n\in\mathbb{Z}.$$

Suppose $l\geq L(\epsilon)$ and estimate as follows

$$|\pi_x(n+\alpha'(l))-\bar{\pi}_x(n)|\leq\sum_{j=0}^{\infty}|r(j)|\,|\pi_H(n-j+\alpha'(l))-\bar{\pi}_H(n-j)|<\epsilon\,|r|_1,$$

for each $n\in\mathbb{Z}$. Hence $\pi_x\in AP(\mathbb{Z})$ and, by (5.6.10), $x/a\in AAP(\mathbb{Z}^+)$. Therefore $x\in PG_{a,\lambda}$, as claimed.

For the converse result, assume that $x\in PG_{a,\lambda}$, so that there exists $a\in G_\lambda$ such that $x/a\in AAP(\mathbb{Z}^+)$. As before, we can apply Theorem 5.2.2 to show that

$$\frac{H(n)}{a(n)}=\pi_x(n)-\sum_{j=0}^{n-1}k(j)\lambda^{j+1}\pi_x(n-j-1)+\phi(n), \quad \text{for each } n\geq 0, \tag{5.6.12}$$

where $\pi_x\in AP(\mathbb{Z})$ and $(\phi(n))_{n\geq 0}$ obeys $\phi(n)\to 0$ as $n\to\infty$. Once more note that if $\lambda=0$, then $H/a\in AAP(\mathbb{Z}^+)$ trivially. Assume henceforth that $\lambda\in(0,1]$ and rewrite (5.6.12) as follows

$$\frac{H(n)}{a(n)}=\pi_x(n)-\sum_{l=1}^{n}k(l-1)\lambda^l\pi_x(n-l)+\phi(n)=\sum_{l=0}^{n}\tilde{r}(l)\lambda^l\pi_x(n-l)+\phi(n),$$

for each $n\geq 0$, where $\tilde{r}(l)=-k(l-1)$ for each $l\geq 0$ and $k(-1)=-1$. Thus $\tilde{r}\in\ell^1(\mathbb{Z}^+)$ with $|\tilde{r}|_1=1+|k|_1$. The representation above is exactly analogous to that of (5.6.9) and the proof proceeds as in the previous case (with $\tilde{r}$ in the role of r and π_x in place of π_H).

5.6.4 Proof of Proposition 5.2.2

Suppose $H\in AG_{a,\lambda}$ with $\lim_{n\to\infty}(\mu_a H)(n)=\mu_a H^*$ and note that $AG_{a,\lambda}\subset BG_{a,\lambda}$. By Theorem 5.2.2, we may write

$$\frac{x(n)}{a(n)}=(\lambda_a H)(n)+\sum_{j=1}^{n}r(j)\lambda^j(\lambda_a H)(n-j)+R(n), \quad n\geq 1, \tag{5.6.13}$$

where $(R(n))_{n\geq 0}$ obeys $\lim_{n\to\infty}R(n)=0$. Note that if $\lambda=0$ we may trivially conclude that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{x(j)}{a(j)}=\mu_a H^*,$$

as claimed. Assume henceforth that $\lambda \in (0,1]$. Thus

$$\frac{1}{n}\sum_{j=1}^{n}\frac{x(j)}{a(j)} = \frac{1}{n}\sum_{j=1}^{n}\sum_{l=0}^{j} r(l)\lambda^{l}(\lambda_a H)(j-l) + \frac{1}{n}\sum_{j=1}^{n} R(j), \quad n \geq 1. \tag{5.6.14}$$

Begin by rewriting the first term on the right–hand side of (5.6.14) as follows:

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n}\sum_{l=0}^{j} r(l)\lambda^{l}(\lambda_a H)(j-l) &= \frac{1}{n}\sum_{k=0}^{n}\sum_{j=1\vee k}^{n} r(j-k)\lambda^{j-k}(\lambda_a H)(k)\\
&= \frac{1}{n}\sum_{j=1}^{n} r(j)\lambda^{j}(\lambda_a H)(0) + \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n} r(j-k)\lambda^{j-k}(\lambda_a H)(k)\\
&= \frac{1}{n}\sum_{j=1}^{n} r(j)\lambda^{j}(\lambda_a H)(0) + \frac{1}{n}\sum_{k=1}^{n}\sum_{i=0}^{n-k} r(i)\lambda^{i}(\lambda_a H)(k)\\
&= \frac{1}{n}\sum_{j=1}^{n} r(j)\lambda^{j}(\lambda_a H)(0) + \frac{1}{n}\sum_{k=1}^{n} \rho(n-k)(\lambda_a H)(k), \quad n \geq 1,
\end{aligned}$$

where $\rho(n) = \sum_{i=0}^{n} r(i)\lambda^{i}$ for each $n \geq 0$. Substitute the above expression into (5.6.14) to obtain

$$\frac{1}{n}\sum_{j=1}^{n}\frac{x(j)}{a(j)} = \frac{1}{n}\sum_{j=1}^{n} R(j) + \frac{1}{n}\sum_{j=1}^{n} r(j)\lambda^{j}(\lambda_a H)(0) + \frac{1}{n}\sum_{k=1}^{n} \rho(n-k)(\lambda_a H)(k), \quad n \geq 1. \tag{5.6.15}$$

Since $r \in \ell^1(\mathbb{Z}^+)$, recall from the proof of Theorem 5.2.1 that

$$\lim_{n\to\infty} \rho(n) = \sum_{i=0}^{\infty} r(i)\lambda^{i} = \frac{1}{1-\sum_{i=0}^{\infty} k(i)\lambda^{i+1}} =: \rho^*.$$

By similar considerations, the first two terms on the right–hand side of (5.6.15) tend to zero as $n \to \infty$. We claim that the third term also tends to a limit, namely

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \rho(n-k)(\lambda_a H)(k) = \frac{\mu_a H^*}{1-\sum_{i=0}^{\infty} k(i)\lambda^{i+1}}.$$

We prove the limit above for $\mu_a H^*$ finite but our conclusions are also true when $\mu_a H^* = \pm\infty$, interpreting the relevant formulae correctly. Suppose $\mu_a H^*$ is finite, let $n \geq 1$, and consider

$$\left|\frac{1}{n}\sum_{k=1}^{n} \rho(n-k)(\lambda_a H)(k) - \frac{1}{n}\sum_{k=1}^{n} \rho^*(\lambda_a H)(k)\right| = \left|\frac{1}{n}\sum_{k=1}^{n} \{\rho(n-k)-\rho^*\}(\lambda_a H)(k)\right|.$$

Now, because $\lim_{n\to\infty} \rho(n) = \rho^*$, there exists $N(\epsilon) > 1$ such that for all $n \geq N(\epsilon)$ we have $|\rho(n)-\rho^*| < \epsilon$, for an arbitrary $\epsilon > 0$. Define $\bar{\rho} = \sup_{n\in\mathbb{Z}^+} |\rho(n)-\rho^*|$ and $\bar{S}_H = \sup_{n\in\mathbb{Z}^+} |(\lambda_a H)(n)|$, recalling that

$\lambda_a H$ is a bounded sequence. Thus

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1}^{n}\{\rho(n-k)-\rho^*\}(\lambda_a H)(k)\right| &\leq \left|\frac{1}{n}\sum_{k=1}^{n-N}\{\rho(n-k)-\rho^*\}(\lambda_a H)(k)\right| \\ &\quad + \left|\frac{1}{n}\sum_{k=n-N+1}^{n}\{\rho(n-k)-\rho^*\}(\lambda_a H)(k)\right| \\ &\leq \frac{\epsilon}{n}\sum_{k=1}^{n-N}|(\lambda_a H)(k)| + \left|\frac{1}{n}\sum_{k=n-N+1}^{n}\{\rho(n-k)-\rho^*\}(\lambda_a H)(k)\right| \\ &\leq \frac{\epsilon(n-N)\bar{S}_H}{n} + \frac{\bar{\rho}\,\bar{S}_H\,N}{n},\end{aligned}$$

for each $n \geq N(\epsilon)+1$. Since it is clear that the right–hand side of the inequality above can be made arbitrarily small for n sufficiently large we have proven that

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\rho(n-k)(\lambda_a H)(k) - \frac{1}{n}\sum_{k=1}^{n}\rho^*(\lambda_a H)(k)\right| = 0,$$

and therefore

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\rho(n-k)(\lambda_a H)(k) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\rho^*(\lambda_a H)(k) = \frac{\mu_H^*}{1-\sum_{i=0}^{\infty}\lambda^{i+1}k(i)},$$

as claimed. Using the limit above and sending $n\to\infty$ in (5.6.15) yields the desired conclusion.

Conversely, assume $x \in AG_{a,\lambda}$. By Theorem 5.2.2, we can write

$$\frac{H(n)}{a(n)} = (\lambda_a x)(n) - \sum_{j=0}^{n-1}k(j)\lambda^{j+1}(\lambda_a x)(n-j-1) + R(n), \quad n \geq 1, \tag{5.6.16}$$

where $(R(n))_{n\geq 0}$ obeys $\lim_{n\to\infty} R(n) = 0$. Once more note that the case $\lambda = 0$ is trivial and assume that $\lambda \in (0,1]$. Define $k(-1) = -1$, so that

$$\begin{aligned}\frac{H(n)}{a(n)} &= -\sum_{j=-1}^{n-1}k(j)\lambda^{j+1}(\lambda_a x)(n-j-1) + R(n) \\ &= -\sum_{l=0}^{n}k(l-1)\lambda^{l}(\lambda_a x)(n-l) + R(n) \\ &= \sum_{l=0}^{n}\tilde{r}(l)\lambda^{l}(\lambda_a x)(n-l) + R(n), \quad n \geq 1,\end{aligned}$$

where $\tilde{r}(l) = -k(l-1)$ for each $l \geq 0$. At this point we have a formula exactly analogous to equation (5.6.13). Repeating the argument from the previous case we arrive at the analogue of equation (5.6.15), namely

$$\frac{1}{n}\sum_{j=1}^{n}\frac{H(j)}{a(j)} = \frac{1}{n}\sum_{j=1}^{n}R(j) + \frac{1}{n}\sum_{j=1}^{n}\tilde{r}(j)\lambda^j(\lambda_a x)(0) + \frac{1}{n}\sum_{j=1}^{n}\tilde{\rho}(n-j)(\lambda_a x)(j), \tag{5.6.17}$$

where $\tilde{\rho}(n) = \sum_{i=0}^{n}\tilde{r}(i)\lambda^i$ for $n \geq 0$. Now notice that

$$\tilde{\rho}(n) = \sum_{i=0}^{n}\tilde{r}(i)\lambda^i = -\sum_{i=0}^{n}k(i-1)\lambda^i = 1 - \sum_{i=1}^{n}k(i-1)\lambda^i = 1 - \sum_{j=0}^{n-1}k(j)\lambda^{j+1},$$

for $n \geq 1$. Hence $\lim_{n\to\infty} \tilde{\rho}(n) = 1 - \sum_{j=0}^{\infty} k(j)\lambda^{j+1}$. By the same argument as before, we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \tilde{\rho}(n-j)(\lambda_a x)(j) = \mu_a x^* \left(1 - \sum_{j=0}^{\infty} k(j)\lambda^{j+1}\right).$$

Therefore, by sending $n \to \infty$ in (5.6.17), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{H(j)}{a(j)} = \mu_a x^* \left(1 - \sum_{j=0}^{\infty} k(j)\lambda^{j+1}\right),$$

as required.

5.6.5 Proof of Theorem 5.3.1

A bound on convolution growth

We note that results very similar to Lemma 5.6.1 and Theorem 5.3.1 are part of the well–established theory in the area of Volterra equations and hence the following results are in some sense "known". Nonetheless, providing our own proofs and formulations is most convenient from a presentational viewpoint. Furthermore, our interest in stochastic equations (cf. Section 5.4) and linearisation (cf. Section 5.5) strongly motivates both the results of this section and our presentational emphasis on unifying the cases when $\Lambda_a|H|$ (resp. $\Lambda_a|x|$) is zero, finite, or infinite.

We first prove a preliminary lemma.

Lemma 5.6.1. *Suppose that a is an increasing sequence with $a(n) \to \infty$ as $n \to \infty$ and that k is summable. If $\Lambda_a|H| \in [0,\infty)$, then*

$$\limsup_{n\to\infty} \left| \frac{1}{a(n)} \sum_{j=1}^{n} k(n-j)H(j) \right| \leq \sum_{j=0}^{\infty} |k(j)| \cdot \Lambda_a|H|. \tag{5.6.18}$$

Proof. Since $\Lambda_a|H| \in [0,\infty)$, it follows for every $\epsilon > 0$ that there is $N(\epsilon) > 0$ such that

$$|H(n)| \leq (\epsilon + \Lambda_a|H|)a(n), \quad n \geq N(\epsilon).$$

Suppose $n \geq N(\epsilon)$ and estimate as follows:

$$\begin{aligned}
\left| \frac{1}{a(n)} \sum_{j=1}^{n} k(n-j)H(j) \right| &\leq \frac{1}{a(n)} \sum_{j=1}^{N-1} |k(n-j)||H(j)| + \frac{1}{a(n)} \sum_{j=N}^{n} |k(n-j)||H(j)| \\
&\leq \frac{1}{a(n)} \sum_{j=1}^{N-1} |k(n-j)| \cdot \sup_{1\leq j\leq N-1} |H(j)| \\
&\quad + (\epsilon + \Lambda_a|H|) \sum_{j=N}^{n} |k(n-j)| \frac{a(j)}{a(n)} \\
&\leq \frac{1}{a(n)} \sum_{l=n-(N-1)}^{n-1} |k(l)| \sup_{1\leq j\leq N-1} |H(j)| \\
&\quad + (\epsilon + \Lambda_a|H|) \sum_{l=0}^{n-N} |k(l)|.
\end{aligned}$$

Since k is summable and $a(n) \to \infty$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \left| \frac{1}{a(n)} \sum_{j=1}^{n} k(n-j)H(j) \right| \leq (\epsilon + \Lambda_a|H|) \sum_{l=0}^{\infty} |k(l)|.$$

Since $\epsilon > 0$ is arbitrary, letting $\epsilon \to 0^+$ yields the result. $\square$

5.6.6 Proof of Theorem 5.3.1

Suppose that $\Lambda_a|H| \in [0,\infty)$. Since r is summable and $\Lambda_a|H| \in [0,\infty)$, we have from Lemma 5.6.1 that

$$\limsup_{n\to\infty} \left| \frac{1}{a(n)} \sum_{j=1}^{n} r(n-j)H(j) \right| \leq \sum_{j=0}^{\infty} |r(j)| \cdot \Lambda_a|H|.$$

Since r is summable and $a(n) \to \infty$ as $n \to \infty$, from the representation (5.1.10) we get

$$\limsup_{n\to\infty} \frac{|x(n)|}{a(n)} \leq \sum_{j=0}^{\infty} |r(j)| \cdot \Lambda_a|H|.$$

Hence

$$\Lambda_a|x| \leq \sum_{j=0}^{\infty} |r(j)| \cdot \Lambda_a|H|. \tag{5.6.19}$$

Suppose on the other hand that $\Lambda_a|x| \in [0,\infty)$. Rearranging (5.1.1) yields

$$H(n+1) = x(n+1) - \sum_{j=0}^{n} k(n-j)x(j) = x(n+1) - k(n)x(0) - \sum_{j=1}^{n} k(n-j)x(j).$$

By Lemma 5.6.1 we have

$$\limsup_{n\to\infty} \left| \frac{1}{a(n)} \sum_{j=1}^{n} k(n-j)x(j) \right| \leq \sum_{j=0}^{\infty} |k(j)| \cdot \Lambda_a|x|.$$

Since a is increasing and $k(n) \to 0$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \frac{|H(n+1)|}{a(n+1)} \leq \limsup_{n\to\infty} \left\{ \frac{|x(n+1)|}{a(n+1)} + \frac{|k(n)|}{a(n+1)}|x(0)| + \frac{1}{a(n)} \left| \sum_{j=1}^{n} k(n-j)x(j) \right| \cdot \frac{a(n)}{a(n+1)} \right\},$$

and hence

$$\limsup_{n\to\infty} \frac{|H(n+1)|}{a(n+1)} \leq \Lambda_a|x| + \sum_{j=0}^{\infty} |k(j)| \cdot \Lambda_a|x|$$

Therefore

$$\Lambda_a|H| \leq \Lambda_a|x| \left(1 + \sum_{j=0}^{\infty} |k(j)| \right). \tag{5.6.20}$$

To prove part (a.) of the result, suppose that $\Lambda_a|H| = 0$. By (5.6.19), we have that $\Lambda_a|x| = 0$. On the other hand, if $\Lambda_a|x| = 0$, by (5.6.20), we have $\Lambda_a|H| = 0$.

To prove part (b.), we start by showing that $\Lambda_a|H| \in (0,\infty)$ implies $\Lambda_a|x| \in (0,\infty)$. Suppose therefore that $\Lambda_a|H| \in (0,\infty)$. Then by (5.6.19), we have that $\Lambda_a|x| \in [0,\infty)$. Suppose that $\Lambda_a|x| = 0$.

Then by part (a.), we have $\Lambda_a|H| = 0$, a contradiction. Thus we must have $\Lambda_a|x| \in (0,\infty)$.

To prove the converse statement, we suppose now that $\Lambda_a|x| \in (0,\infty)$. By (5.6.20) it follows that $\Lambda_a|H| \in [0,\infty)$. If we assume that $\Lambda_a|H| = 0$, then by part (a.), we have that $\Lambda_a|x| = 0$, which gives a contradiction. Therefore we must have $\Lambda_a|H| \in (0,\infty)$.

To prove part (c.), we start by showing that $\Lambda_a|H| = +\infty$ implies $\Lambda_a|x| = +\infty$. Suppose not, so that $\Lambda_a|x| \in [0,\infty)$. Then the argument used to deduce (5.6.20) is valid and we have that $\Lambda_a|H| < +\infty$, which is a contradiction. To prove the reverse implication, we have by hypothesis that $\Lambda_a|x| = +\infty$. Suppose now that $\Lambda_a|H| < +\infty$. Then the argument used to prove (5.6.19) is valid, and we have that $\Lambda_a|x| < +\infty$, which is a contradiction.

5.6.7 Proof of Theorem 5.3.2

Estimating from (5.1.10) we have

$$|x(n)| \leq \sum_{j=0}^{n} |r(n-j)||H(j)| \leq |r|_1 \sum_{j=0}^{n} \frac{|r(n-j)|}{\sum_{l=0}^{n}|r(l)|}|H(j)|, \quad n \geq 1. \tag{5.6.21}$$

Now apply ϕ to the expression above and use Jensen's inequality to obtain

$$\phi(|x(n)|) \leq \phi\left(|r|_1 \sum_{j=0}^{n} \frac{|r(n-j)|}{\sum_{l=0}^{n}|r(l)|}|H(j)|\right) \leq \frac{1}{\sum_{l=0}^{n}|r(l)|}\sum_{j=0}^{n}|r(n-j)|\phi\left(|r|_1|H(j)|\right),$$

for $n \geq 1$. Since $r \in \ell^1(\mathbb{Z}^+)$, there exists an $N_1(\epsilon) > 1$ such that $\sum_{l=0}^{n}|r(l)| > (1-\epsilon)|r|_1$ for all $n \geq N_1(\epsilon)$ and hence $1/\sum_{l=0}^{N_1}|r(l)| < 1/(1-\epsilon)|r|_1$ for all $n \geq N_1(\epsilon)$, with $\epsilon \in (0,1)$ arbitrary. Returning to (5.6.21), we have that

$$\phi(|x(n)|) \leq \frac{1}{(1-\epsilon)|r|_1}\sum_{j=0}^{n}|r(n-j)|\phi\left(|r|_1|H(j)|\right), \quad n \geq N_1(\epsilon).$$

With N sufficiently large, summing over the previous inequality yields

$$\begin{aligned}
\sum_{n=N_1}^{N}\phi(|x(n)|) &\leq \frac{1}{(1-\epsilon)|r|_1}\sum_{n=N_1}^{N}\sum_{j=0}^{n}|r(n-j)|\phi\left(|r|_1|H(j)|\right) \\
&= \frac{1}{(1-\epsilon)|r|_1}\sum_{j=0}^{N}\left\{\sum_{n=N_1\wedge j}^{N}|r(n-j)|\right\}\phi\left(|r|_1|H(j)|\right) \\
&\leq \frac{1}{1-\epsilon}\sum_{j=0}^{N}\phi\left(|r|_1|H(j)|\right).
\end{aligned}$$

Adding the remaining terms to the sums on the left–hand side of the above inequality, we have

$$\sum_{n=0}^{N}\phi(|x(n)|) \leq \frac{1}{1-\epsilon}\sum_{j=0}^{N}\phi\left(|r|_1|H(j)|\right) + \sum_{n=0}^{N_1-1}\phi(|x(n)|).$$

Therefore

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}\phi(|x(n)|) \leq \frac{1}{1-\epsilon}\limsup_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N}\phi\left(|r|_1|H(j)|\right),$$

and letting $\epsilon \to 0^+$ gives the desired conclusion.

The second claim is proven analogously; first note that we can write

$$H(n) = \sum_{j=0}^{n} \rho(n-j)x(j), \quad n \geq 1,$$

where $\rho(j) = -k(j+1)$, $\rho(0) = 1$ and $|\rho|_1 = 1 + |k|_1$. Now apply the argument above with ρ in place of r to complete the proof.

5.6.8 Justification of Example 5.3.3

In this section we use the standard probabilistic notation $x \overset{d}{\sim} N(\mu, \sigma^2)$ to denote the random variable x having a normal distribution with mean μ and variance σ^2.

We cannot proceed by direct calculation due to the nonstationarity of $(x(n))_{n\geq 0}$. Instead, by extending the filtration in a suitable manner, write

$$x(n) = x^*(n) + R(n), \quad \text{for each } n \geq 1, \tag{5.6.22}$$

where $x^*(n) = \sum_{j=-\infty}^{n} r(n-j)H(j)$ for $n \geq 0$ and $R(n) = r(n)x(0) - \sum_{j=-\infty}^{0} r(n-j)H(j)$ for $n \geq 0$. Since $r \in \ell^1(\mathbb{Z}^+) \subset \ell^2(\mathbb{Z}^+)$, we can use the dominated convergence theorem to show that

$$\mathbb{E}[x^*(n)] = 0, \quad \mathrm{Var}[x^*(n)] = \mathbb{E}[(x^*)^2(n)] = \sigma^2 \sum_{l=0}^{\infty} r^2(l), \quad \text{for each } n \geq 0.$$

In fact, $(x^*(n))_{n\geq 0}$ is a strongly stationary sequence, since

$$x^*(n) \overset{d}{\sim} N\left(0, \sigma^2 \sum_{l=0}^{\infty} r^2(l)\right)$$

for each $n \geq 0$. Hence, by Birkhoff's ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} x^*(j) = 0 \quad \text{a.s.,} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} (x^*)^2(j) = \sigma^2 \sum_{l=0}^{\infty} r^2(l) \quad \text{a.s.}$$

From (5.6.22), we have

$$\frac{1}{n} \sum_{j=0}^{n} x^2(j) = \frac{1}{n} \sum_{j=0}^{n} (x^*)^2(j) + \frac{2}{n} \sum_{j=0}^{n} x^*(j)R(j) + \frac{1}{n} \sum_{j=0}^{n} R^2(j), \quad n \geq 1.$$

Thus if we can show that $\lim_{n\to\infty} R(n) = 0$ a.s. we will have proven that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} x^2(j) = \sigma^2 \sum_{l=0}^{\infty} r^2(l) = \sigma^2 (|r|_2)^2 \quad a.s.$$

Define $R_1(n) = \sum_{j=-\infty}^{0} r(n-j)H(j)$ for each $n \geq 0$. Hence

$$R^2(n) = r^2(n)x^2(0) - 2r(n)x(0)R_1(n) + R_1^2(n), \quad n \geq 0.$$

From the equality above, we see that $\lim_{n\to\infty} R_1(n) = 0$ a.s. would imply that $\lim_{n\to\infty} R(n) = 0$ a.s., since $r(n) \to 0$ as $n \to \infty$. Hence it remains to prove that $\lim_{n\to\infty} R_1(n) = 0$ a.s. The dominated convergence theorem can be used to show that

$$\mathbb{E}[R_1(n)] = 0, \quad \mathbb{E}[R_1^2(n)] = \sigma^2 \sum_{l=n}^{\infty} r^2(l), \quad n \geq 0.$$

In fact, $R_1(n) \overset{d}{\sim} N\left(0, \sigma^2 \sum_{l=n}^{\infty} r^2(l)\right)$ for each $n \geq 0$. To see this, fix $n \geq 0$ and define the sequence $R_1^N(n) = \sum_{j=-N}^{0} r(n-j)$ for $N \geq 0$. Clearly, $R_1^N(n) \to R_1(n)$ a.s. as $N \to \infty$ and $R_1^N(n) \overset{d}{\sim} N\left(0, \sigma^2 \sum_{l=n}^{n+N} r^2(l)\right)$ for each $N \geq 0$. By explicitly writing down the characteristic function we can see that $\lim_{N\to\infty} R_1^N(n) = R_1(n)$ is normal with mean zero and variance $\sigma^2 \sum_{l=n}^{\infty} r^2(l)$ for each $n \geq 0$.

By a standard Borel–Cantelli argument the side condition $\log(n) \sum_{l=n}^{\infty} r^2(l) \to 0$ as $n \to \infty$ can then be used to show that $R_1(n) \to 0$ a.s. as $n \to \infty$, completing the proof (alternatively this can be deduced from Example 5.4.4 with an appropriate choice of $(a(n))_{n\geq 0}$ and constant K).

5.6.9 Proof of Theorem 5.4.5

From (5.4.3), we have

$$S(a, K) = \sum_{n=1}^{\infty} F(-Ka(n)) + G(Ka(n)),$$

where $K > 0$ is a constant, a is a positive increasing sequence which tends to infinity, and $G(x) = 1 - F(x)$ for each $x \in \mathbb{R}$. Choose $K = 1$ and note that the symmetry of the distribution function means that the finiteness of $S(a, 1)$ is equivalent to the finiteness of $\sum_{n=1}^{\infty} G(a(n))$.

For each $n > 1$, take $a_1(n) = G^{-1}(1/n)$ and note that this yields an increasing, positive sequence which tends to infinity as $n \to \infty$. Furthermore,

$$\sum_{n=1}^{\infty} G(a_1(n)) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

and hence the Borel–Cantelli Lemma implies that

$$\limsup_{n\to\infty} \frac{|H(n)|}{G^{-1}(1/n)} \geq 1 \text{ a.s.} \tag{5.6.23}$$

Now, because $G^{-1}(1/x) \in \mu$–SSV, there exists $\Delta > 0$ such that

$$\lim_{n\to\infty} \frac{G^{-1}(1/n\mu^{\delta}(n))}{G^{-1}(1/n)} = 1, \text{ for each } \delta \in [0, \Delta]. \tag{5.6.24}$$

Choosing $a_2(n) = G^{-1}(1/n\mu^{\delta^*}(n))$ for each $n > 1$ once more yields a positive, increasing and divergent sequence (where δ^* is chosen to be the number in $(0, \Delta]$ whose existence was assumed in (5.4.6)). Thus

$$\sum_{n=1}^{\infty} G(a_2(n)) < \infty,$$

by hypothesis. It follows from (5.6.24) that

$$\limsup_{n\to\infty} \frac{|H(n)|}{G^{-1}(1/n)} = \limsup_{n\to\infty} \frac{|H(n)|}{G^{-1}(1/n\mu^{\delta^*}(n))} \leq 1 \text{ a.s.}$$

Therefore, combining the above inequality with (5.6.23), we have

$$\limsup_{n\to\infty} \frac{|H(n)|}{G^{-1}(1/n)} = 1 \text{ a.s.},$$

as claimed.

5.6.10 Proof of Theorem 5.4.8

Suppose that $(i.)$ holds and let $(a(n))_{n\geq 0}$ be an arbitrary positive increasing sequence which tends to infinity. In the notation of (5.4.3), we have

$$S(a,K)=\sum_{n=1}^{\infty}\left\{1-F(Ka(n))+F(a(n))\right\}, \text{ for each } K>0.$$

Define $G(x)=1-F(x)$ for $x\in(-\infty,\infty)$ and

$$S_N(a,K)=\sum_{n=1}^{N}\left\{G(Ka(n))+F(a(n))\right\}, \text{ for each } K>0 \text{ and } N\geq 1.$$

Since the summands are non–negative, $\lim_{N\to\infty}S_N(a,K)=S(a,K)$ either converges to a finite limit or to $+\infty$. Now consider the following dichotomy: either

$$S(a,K)=\infty \text{ for each } K\in(0,\infty), \tag{5.6.25}$$

or

$$\text{there exists a } K^*\in(0,\infty) \text{ such that } S(a,K^*)<\infty. \tag{5.6.26}$$

If (5.6.25) holds, then

$$\limsup_{n\to\infty}\frac{|H(n)|}{a(n)}=\infty \text{ a.s.},$$

by a simple application of the Borel–Cantelli Lemma. We claim that if (5.6.26) holds, then

$$S(a,K)<\infty \text{ for each } K\in(0,\infty),$$

and hence that

$$\limsup_{n\to\infty}\frac{|H(n)|}{a(n)}=0 \text{ a.s.}$$

Let $K\in(0,\infty)$ be arbitrary. By hypothesis, $\lim_{n\to\infty}G(Ka(n))/F(-Ka(n))=\infty$ and there exists $N_1>2$ such that $F(-Ka(n))<G(Ka(n))$ for all $n\geq N_1$. Thus

$$\begin{aligned} S_N(a,K)&=\sum_{n=1}^{N_1-1}\left\{G(Ka(n))+F(-Ka(n))\right\}+\sum_{n=N_1}^{N}\left\{G(Ka(n))+F(-Ka(n))\right\}\\ &<\sum_{n=1}^{N_1-1}\left\{G(Ka(n))+F(-Ka(n))\right\}+2\sum_{n=N_1}^{N}G(Ka(n)), \text{ for each } N>N_1. \end{aligned} \tag{5.6.27}$$

Using the regular variation of G, we have

$$\lim_{n\to\infty}\frac{G(K^*a(n))}{G(Ka(n))}=\lim_{n\to\infty}\frac{G(K^*a(n))}{G(a(n))}\frac{G(a(n))}{G(Ka(n))}=(K^*)^{-\alpha}K^{\alpha}=:\kappa>0.$$

Hence for each $\epsilon\in(0,\kappa)$ there exists an $N_2(\epsilon)>N_1$ such that for all $n\geq N_2(\epsilon)$,

$$G(Ka(n))<\frac{1}{\kappa-\epsilon}G(K^*a(n)).$$

Plugging the estimate above into (5.6.27) yields

$$\begin{aligned}
S_N(a,K) &< \sum_{n=1}^{N_1-1}\{G(Ka(n))+F(-Ka(n))\}+2\sum_{n=N_1}^{N_2-1}G(Ka(n)) \\
&\quad+\frac{2}{\kappa-\epsilon}\sum_{n=N_2}^{N}G(K^*a(n)) \\
&\le \sum_{n=1}^{N_1-1}\{G(Ka(n))+F(-Ka(n))\}+2\sum_{n=N_1}^{N_2-1}G(Ka(n)) \\
&\quad+\frac{2}{\kappa-\epsilon}S_N(a,K^*),\ \text{ for each } N>N_2.
\end{aligned}$$

Finally, let $N\to\infty$ in the estimate above to see that $S(a,K)<\infty$ for each $K\in(0,\infty)$.

The arguments needed to tackle cases (*ii.*) and (*iii.*) are exactly analogous to those given above and hence the details are omitted.

5.6.11 Proof of Theorem 5.5.1

If y is the solution of (5.5.5), then y is given by

$$y(n)=r(n)\xi+\sum_{j=1}^{n}r(n-j)H(j),\quad n\ge 1$$

where r is the solution of (5.1.3). From Theorem 5.3.1 it follows from the fact that $\Lambda_a|H|<+\infty$ that

$$\limsup_{n\to\infty}\frac{|y(n)|}{a(n)}=:\Lambda_a|y|<+\infty.$$

Define $z(n)=x(n)-y(n)$ for $n\ge 0$. Then $z(0)=0$ and for $n\ge 0$ we have from (5.5.1) and (5.5.5) that

$$z(n+1)=\sum_{j=0}^{n}k(n-j)[f(x(j))-y(j)]=\sum_{j=0}^{n}k(n-j)[\phi(x(j))+z(j)],$$

where $\phi(x):=f(x)-x$. By (5.5.4) we have that $\phi(x)/x\to 0$ as $|x|\to\infty$ and that ϕ is continuous. Hence we have

$$z(n+1)=G(n+1)+\sum_{j=0}^{n}k(n-j)z(j),\quad n\ge 0;\quad z(0)=0 \tag{5.6.28}$$

and

$$G(n+1):=\sum_{j=0}^{n}k(n-j)\phi(x(j)),\quad n\ge 0. \tag{5.6.29}$$

Therefore, we have from (5.1.10) and (5.6.28) that z has the representation

$$z(n)=\sum_{j=1}^{n}r(n-j)G(j)=\sum_{l=0}^{n-1}r(n-l-1)G(l+1),\quad n\ge 1. \tag{5.6.30}$$

We next seek to estimate G, and thereby deduce a linear summation inequality for z. Define $|k|_1:=\sum_{j=0}^{\infty}|k(j)|$, $|r|_1:=\sum_{j=0}^{\infty}|r(j)|$, and choose $\epsilon>0$ so that $\epsilon|k|_1|r_1|<1/2$. For every $\epsilon>0$, by the properties of ϕ, there exists a $\Phi(\epsilon) > 0$ such that $|\phi(x)| \le \Phi(\epsilon) + \epsilon|x|$ for all $x \in \mathbb{R}$. Hence

$$\begin{aligned}|G(n+1)| &\le \sum_{j=0}^{n} |k(n-j)||\phi(x(j))| \le \sum_{j=0}^{n} |k(n-j)| \{\Phi(\epsilon) + \epsilon|x(j)|\} \\ &\le |k|_1\Phi(\epsilon) + \epsilon\sum_{j=0}^{n} |k(n-j)||z(j)| + \epsilon\sum_{j=0}^{n} |k(n-j)||y(j)|.\end{aligned}$$

Now by defining $c(n) = \sum_{j=0}^{n} |r(n-j)||k(j)|$, we have

$$\begin{aligned}|z(n+1)| &\le \sum_{l=0}^{n} |r(n-l)||G(l+1)| \\ &\le \sum_{l=0}^{n} |r(n-l)| \left\{ |k|_1\Phi(\epsilon) + \epsilon\sum_{j=0}^{l} |k(l-j)||z(j)| + \epsilon\sum_{j=0}^{l} |k(l-j)||y(j)| \right\} \\ &\le |r|_1|k|_1\Phi(\epsilon) + \epsilon\sum_{l=0}^{n}\sum_{j=0}^{l} |r(n-l)||k(l-j)||y(j)| \\ &\quad + \epsilon\sum_{l=0}^{n}\sum_{j=0}^{l} |r(n-l)||k(l-j)||z(j)| \\ &= |r|_1|k|_1\Phi(\epsilon) + \epsilon\sum_{j=0}^{n} c(n-j)|y(j)| + \epsilon\sum_{j=0}^{n} c(n-j)|z(j)|.\end{aligned}$$

Therefore we have for $n \ge 0$ the estimate

$$|z(n+1)| \le |r|_1|k|_1\Phi(\epsilon) + \epsilon\sum_{j=0}^{n} c(n-j)|y(j)| + \epsilon\sum_{j=0}^{n} c(n-j)|z(j)|.$$

Define

$$H_2(n+1) = |r|_1|k|_1\Phi(\epsilon) + \epsilon\sum_{j=0}^{n} c(n-j)|y(j)|, \quad n \ge 0$$

and

$$r_2(n+1) = \epsilon\sum_{j=0}^{n} c(n-j)r(j), \quad n \ge 0; \quad r_2(0) = 1. \tag{5.6.31}$$

Let z_2 be the solution of the summation equation

$$z_2(n+1) = H_2(n+1) + \epsilon\sum_{j=0}^{n} c(n-j)z_2(j), \quad n \ge 0; \quad z_2(0) = 1.$$

Then $|z(n)| \le z_2(n)$ for all $n \ge 0$. Moreover, z_2 has the representation

$$z_2(n) = r_2(n) + \sum_{j=1}^{n} r_2(n-j)H_2(j), \quad n \ge 1. \tag{5.6.32}$$

To determine the asymptotic behaviour of z_2, we use the representation (5.6.32). This requires knowledge of the asymptotic behaviour of H_2. We also need to check that r_2 is summable. Since $c(n) \ge 0$ for each $n \ge 0$ and

$$\epsilon\sum_{n=0}^{\infty} c(n) = \epsilon|k|_1|r|_1 < \frac{1}{2}$$

it follows that r_2 is summable, and of course $r(n) \to 0$ as $n \to \infty$. Since c is summable, and $\Lambda_a|y| < +\infty$, we have from Lemma 5.6.1 that

$$\limsup_{n\to\infty} \frac{1}{a(n)} \left| \sum_{j=1}^{n} c(n-j)|y(j)| \right| \le \sum_{j=0}^{\infty} c(j) \cdot \Lambda_a|y|.$$

Therefore as

$$H_2(n+1) = |r|_1|k|_1\Phi(\epsilon) + \epsilon c(n)|y(0)| + \epsilon \sum_{j=1}^{n} c(n-j)|y(j)|,$$

and $c(n) \to 0$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \frac{|H_2(n+1)|}{a(n)} \le \epsilon \sum_{j=0}^{\infty} c(j) \cdot \Lambda_a|y|.$$

Also, because a is increasing,

$$\Lambda_a|H_2| := \limsup_{n\to\infty} \frac{|H_2(n)|}{a(n)} \le \epsilon \sum_{j=0}^{\infty} c(j) \cdot \Lambda_a|y|.$$

Since r_2 is summable and $\Lambda_a|H_2| < +\infty$, applying Lemma 5.6.1 once more yields

$$\limsup_{n\to\infty} \frac{1}{a(n)} \left| \sum_{j=1}^{n} r_2(n-j)H_2(j) \right| \le \sum_{j=0}^{\infty} r_2(j)\Lambda_a|H_2| \le \sum_{j=0}^{\infty} r_2(j) \cdot \epsilon \sum_{j=0}^{\infty} c(j) \cdot \Lambda_a|y|. \tag{5.6.33}$$

We wish to identify a bound on the right hand side in terms of ϵ and quantities which are explicitly ϵ–independent. We start by estimating the sum of r_2. Since $r_2(n) \ge 0$ for all $n \ge 0$, by (5.6.31) and the fact that $r_2(0) = 1$ we have

$$\sum_{n=0}^{\infty} r_2(n) - 1 = \sum_{n=0}^{\infty} r_2(n+1) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \epsilon c(n-j) r_2(j) = \epsilon \sum_{n=0}^{\infty} c(n) \cdot \sum_{n=0}^{\infty} r_2(n).$$

Hence as $\sum_{n=0}^{\infty} c(n) = |k|_1|r|_1$,

$$\sum_{n=0}^{\infty} r_2(n) = \frac{1}{1-\epsilon|k|_1|r|_1},$$

and combining this with (5.6.33) yields

$$\limsup_{n\to\infty} \frac{1}{a(n)} \left| \sum_{j=1}^{n} r_2(n-j)H_2(j) \right| \le \frac{1}{1-\epsilon|k|_1|r|_1} \cdot \epsilon|k|_1|r|_1\Lambda_a|y|.$$

Thus by (5.6.32) and the fact that $|z(n)| \le z_2(n)$, we get

$$\limsup_{n\to\infty} \frac{|z(n)|}{a(n)} \le \frac{1}{1-\epsilon|k|_1|r|_1} \cdot \epsilon|k|_1|r|_1\Lambda_a|y|.$$

Since y, k, r, z, and a are ϵ–independent, letting $\epsilon \to 0^+$ gives

$$\limsup_{n\to\infty} \frac{|z(n)|}{a(n)} = 0,$$

which proves the result.

5.6.12 Proof of Theorem 5.5.2

Suppose that $H \in BG_{a,\lambda}$. By Theorem 5.2.2, the solution y of (5.5.5) obeys

$$\frac{y(n)}{a(n)} \approx (\lambda_a y)(n) + \sum_{j=1}^{n} r(j)\lambda^j (\lambda_a y)(n-j), \quad \text{as } n \to \infty.$$

Furthermore, by Theorem 5.5.1, $x(n)/a(n) \approx y(n)/a(n)$ as $n \to \infty$. Thus we immediately have that (5.5.6) holds and hence that $x \in BG_{a,\lambda}$.

Conversely, suppose that $x \in BG_{a,\lambda}$. For each $n \geq 0$,

$$\begin{aligned} \frac{H(n+1)}{a(n+1)} &= \frac{x(n+1)}{a(n+1)} - \sum_{j=0}^{n} k(n-j)\frac{f(x(j))}{a(n+1)} \\ &= \frac{x(n+1)}{a(n+1)} - \sum_{j=0}^{n} \frac{k(n-j)}{a(n+1)}\left\{f(x(j)) - x(j)\right\} + \sum_{j=0}^{n} k(n-j)\frac{x(j)}{a(n+1)}. \end{aligned}$$

Hence

$$\begin{aligned} &\left| \frac{H(n+1)}{a(n+1)} - (\lambda_a x)(n+1) + \sum_{j=0}^{n} k(j)\lambda^{j+1}(\lambda_a x)(n-j-1) \right| \leq \\ &\qquad \left| \frac{x(n+1)}{a(n+1)} - (\lambda_a x)(n+1) \right| + \left| \sum_{j=0}^{n} \frac{k(n-j)}{a(n+1)}\left\{x(j) - f(x(j))\right\} \right| \\ &\qquad + \left| \sum_{j=0}^{n} k(j)\lambda^{j+1}(\lambda_a x)(n-j-1) - \sum_{j=0}^{n} k(n-j)\frac{x(j)}{a(n+1)} \right|. \end{aligned} \tag{5.6.34}$$

The first term on the right–hand side of (5.6.34) tends to zero as $n \to \infty$ by hypothesis and the final term tends to zero as $n \to \infty$ by the same argument used in Theorem 5.2.2. Thus it remains to show that

$$\lim_{n \to \infty} \left| \sum_{j=0}^{n} \frac{k(n-j)}{a(n+1)}\left\{x(j) - f(x(j))\right\} \right| = 0.$$

Let $\phi(x) = x - f(x)$ for each $x \in \mathbb{R}$ and note that ϕ is continuous by hypothesis. By dint of (5.5.4), for each $\epsilon > 0$ there exists a $\Phi(\epsilon) > 0$ such that

$$|\phi(x)| \leq \epsilon x + \Phi(\epsilon), \quad \text{for each } x \in \mathbb{R}.$$

Now estimate as follows

$$\begin{aligned} \left| \sum_{j=0}^{n} \frac{k(n-j)}{a(n+1)}\left\{x(j) - f(x(j))\right\} \right| &\leq \sum_{j=0}^{n} \left| \frac{k(n-j)}{a(n+1)} \right| \left\{\epsilon |x(j)| + \Phi(\epsilon)\right\} \\ &\leq \epsilon |k|_1 \bar{x} + \frac{\Phi(\epsilon)|k|_1}{|a(n+1)|}, \end{aligned}$$

where $\bar{x} > 0$ uniformly bounds $x(j)/a(n+1)$ (which is possible since $x \in BG_{a,\lambda}$). Now letting $n \to \infty$ and then $\epsilon \downarrow 0$ in the estimate above yields the desired conclusion.

Chapter 6

Blow–up and Superexponential Growth in Volterra Equations

6.1 Introduction

This chapter concerns the blow–up and asymptotic behaviour of positive solutions to initial value problems of the form

$$x'(t) = \int_0^t w(t-s)f(x(s))\,ds, \quad t \geq 0; \quad x(0) = \psi > 0. \tag{6.1.1}$$

We assume that the nonlinearity, f, obeys

$$f \in C((0,\infty);(0,\infty)), \quad f \text{ is asymptotically increasing}, \quad \lim_{x\to\infty} \frac{f(x)}{x} = \infty. \tag{6.1.2}$$

The positivity and monotonicity hypotheses in (6.1.2) are natural when studying growing solutions to (6.1.1). Moreover, $f(x)/x \to \infty$ as $x \to \infty$ is necessary for the existence of a solution to (6.1.1) which blows up in finite–time (see Appendix A.1). It is well–known that the behaviour of the kernel near zero is crucial in the analysis of blow–up problems of the type studied in this chapter [32]. In the present work, we assume

$$w(0) > 0, \quad w \in C(\mathbb{R}^+;\mathbb{R}^+). \tag{6.1.3}$$

Since w is assumed to be continuous in order to determine asymptotic behaviour, solutions to (6.1.1) will be continuously differentiable, in contrast to the absolutely continuous solutions considered in Chapters 2 and 3 of this thesis. In the notation of Chapters 2 and 3, $\mu(ds) = w(s)ds$ with ds denoting the Lebesgue measure. *This hypothesis is intrinsic to the results of this chapter.* For example, if (6.1.1) contained an $f(x(t))$ term on the right–hand side (which would correspond to a point mass at zero for a measure–valued kernel), the asymptotic behaviour of solutions to (6.1.1) would follow the ODE $y'(t) = f(y(t))$. By contrast, we presently show that the asymptotic behaviour of solutions to (6.1.1) is in fact analogous to that of a particular second order ODE.

There is a rich and active literature on blow–up problems in Volterra integral equations (VIEs) (see, for example, the survey articles [107, 108], and the recent papers [69, 81]). Much of this interest originally stems from the connection between VIEs and partial differential equations (PDEs) of parabolic–type in which the source term has a highly localised spacial dependence. The following example gives a simple and explicit illustration of the coupling between PDEs and VIEs.

Example 6.1.1. *Consider the scalar boundary value problem*

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \delta(x-x_0) f(u(x,t)), \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ \lim_{x\to\pm\infty} u(x,t) &= 0, \quad t \in (0,\infty), \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{6.1.4}$$

where δ denotes the Dirac delta function. If G denotes the Green's function for the homogeneous PDE (i.e. setting $f \equiv 0$ in (6.1.4)), then

$$\begin{aligned} u(x,t) = &\int_0^t \int_{\mathbb{R}} G(x,t|\xi,s)\delta(\xi - x_0) f(u(\xi,s))\, d\xi\, ds \\ &+ \int_{\mathbb{R}} G(x,t|\xi,0) u_0(\xi)\, d\xi, \quad t > 0, \quad x \in \mathbb{R}. \end{aligned}$$

Let $x = x_0$ and use the sifting property of the Dirac delta to obtain

$$u(x_0,t) = \int_0^t G(x_0,t|x_0,s) f(u(x_0,s))\, ds + \int_{\mathbb{R}} G(x_0,t|\xi,0) u_0(\xi)\, d\xi, \quad t > 0. \tag{6.1.5}$$

Now set $v(t) = u(x_0,t)$ and note that the Green's function for the homogeneous PDE is given by $G(x_0,t|x_0,s) = G(0,t-s) = \Theta(t-s)/\sqrt{4\pi(t-s)}$, where Θ denotes the Heavyside step–function. Thus, at x_0, (6.1.4) can be reduced to the study of the Volterra integral equation

$$v(t) = \int_0^t \frac{f(v(s))}{\sqrt{4\pi(t-s)}}\, ds + H(t), \quad t \geq 0,$$

where $H(t) = \int_{\mathbb{R}} e^{-x^2/4t} v_0(x)\, dx$ for $t \geq 0$.

PDEs such as (6.1.4) arise in the modelling of combustion in a reactive–diffusive medium (see [107, 108] and the references therein). In Example 6.1.1, the presence of the Dirac delta in the source term is intended to model very intense heating narrowly concentrated at a particular point – the heating of a combustible material via a thin wire or laser are the archetypal physical examples. In this context, a blow–up solution represents the scenario in which the energy entering the system via the source term, $f(v)$ in (6.1.4), outweighs the ability of the medium to dissipate this energy and a literal explosion occurs in the physical system. More complicated PDEs of parabolic type can be similarly linked with associated VIEs. Indeed, the example above generalises to combustion models incorporating moving heat sources, higher dimensions, and weaker localisation of the source [68, 94, 96]; in almost all cases, the leading order behaviour in such models is governed by a nonlinear VIE of the general form

$$x(t) = x(0) + H(t) + \int_0^t W(t-s) f(x(s))\, ds, \quad t \geq 0. \tag{6.1.6}$$

Equation (6.1.1) is a special case of (6.1.6). In particular, if $W \in C^1([0,\infty);[0,\infty))$ with $W(0) = 0$ and $H \equiv 0$, differentiation of (6.1.6) yields (6.1.1) with $w = W'$. Similarly, integration of (6.1.1) yields (6.1.6) with $H \equiv 0$. After analysing the unforced equation (6.1.1), we later extend our results to the case of nontrivial H (see Section 6.4).

According to the survey of Roberts [107], research on blow–up problems of the type discussed above has mainly sought to answer the following questions:

(1.) Under what conditions do solutions blow–up?

(2.) At what time do solutions blow–up?

(3.) What is the asymptotic behaviour of solutions at blow–up?

Being the most fundamental, (1.) has naturally attracted the most attention and thus blow–up criteria for both general and specialised classes of VIEs are very well understood (see, for example, [84]). We revisit (1.) for the Volterra integro–differential equation (VIDE) (6.1.1) and prove necessary and sufficient conditions for finite–time explosion of solutions. However, as we explain in more detail in Section 6.2, our conditions can be recovered from existing general conditions for (6.1.6). As noted earlier, our problem obeys (6.1.6) with $W(t) = \int_0^t w(s)\,ds$ for $t \geq 0$ and $H \equiv 0$. Hence, $W(t) \sim w(0)t$ as $t \to 0^+$ and a sharp result of Brunner and Yang [32] can be applied. We still find it useful to independently prove our own blow–up criteria for (6.1.1) in order to gain preliminary insight into the behaviour of solutions. Moreover, our method of proof is different to that which Brunner and Yang used to tackle the related VIE problem. We do not address (2.) – estimation of the blow-up time – in the present work, but this is also a very active area of investigation (see [83, 85] and the references therein) and represents an interesting open problem for general nonlinear VIEs.

Our main contribution is to provide a comprehensive answer to (3.) for equations of the form (6.1.1), and furthermore to understand the behaviour of nonexplosive solutions. The asymptotic behaviour of blow–up solutions has attracted considerable attention, both for VIEs and PDEs. Chadam et al. provide *one–sided* growth estimates for the general nonlinear parabolic equation $u_t - \Delta u = f(u)$ but such results only put an upper bound on the rate of growth of solutions [33]. Roberts [106], and Olmstead and Roberts [109] study VIEs with parametric families of nonlinearities and kernels. They employ integral transform methods to estimate the growth rates of solutions but this work relies on conjecturing the leading order behaviour of solutions and finding a consistent "asymptotic balance" from the original equation, so the full proof of these conclusions remains open. Mydlarczyk provides very good estimates on the size of solutions to (6.1.6) in the presence of a blow-up with a power–type kernel [92, 93]. However, these estimates do not give a sharp characterisation of the asymptotic growth rate of solutions. In particular, Mydlarczyk's results lead to conclusions of the form

$$0 < \liminf_{t\to T^-} A(x(t)) < \limsup_{t\to T^-} A(x(t)) < \infty,$$

where A is an appropriately chosen monotone function and T is the blow–up time. Evtukhov and Samoilenko also study the power kernel case but specialise to regularly varying nonlinearities, in fact their particular interest is n–th order equations [47]. In this special case, they improve upon Mydlarczyk's results by proving that

$$\lim_{t\to\omega} B_\omega(x(t)) = 1, \quad \omega \in \{T, \infty\},$$

for an appropriately chosen function B_ω. To the best of our knowledge, this is the most complete result available in the extant literature.

We first outline our asymptotic growth estimates for $H \equiv 0$. Under (6.1.2) and (6.1.3), we identify a *decreasing* function F_B such that

$$\lim_{t\to T^-} \frac{F_B(x(t))}{T-t} = \sqrt{2w(0)}, \tag{6.1.7}$$

where T is the blow–up time. Similarly, in the nonexplosive case, we identify an *increasing* function F_U such that

$$\lim_{t\to\infty} \frac{F_U(x(t))}{t} = \sqrt{2w(0)}, \tag{6.1.8}$$

under the additional assumption that $w \in L^1(\mathbb{R}^+;\mathbb{R}^+)$. The functions F_B and F_U depend only on f and hence can be estimated from the problem data. Furthermore, our assumptions on the nonlinearity are nonparametric and allow a good deal of generality, while still yielding strong conclusions. In the nonexplosive case, the rate of unbounded growth depends positively on the value of $w(0)$. Interestingly, in spite of the dependence of these growth rates on w, the presence of a blow–up is completely *independent* of the value of $w(0)$ and the structure of the kernel under (6.1.3).

If $H \in C^1([0,\infty);[0,\infty))$ is nontrivial, then (6.1.7) is unchanged (since H is bounded on compact intervals). However, in the nonexplosive case, H can impact the growth rate of solutions. When H is sufficiently small the growth rate from (6.1.8) is preserved and we characterise these rate preserving perturbations. For larger forcing terms the growth rate of solutions can be enhanced and if the forcing term is sufficiently large the solution even tracks it asymptotically (in the sense that $x \sim H$). In fact, results precisely identifying the influence of additive perturbations on asymptotic growth rates of solutions are new, even for nonlinear ordinary differential equations (i.e. $w \equiv 1$) [15].

6.2 Blow–up Conditions

Definition 6.2.1. *A solution to (6.1.1) blows up in finite–time if there exists $T \in (0,\infty)$ such that $x \in C([0,\infty);[0,\infty))$ but $\lim_{t\to T^-}|x(t)| = \infty$. The minimal such T is called the blow–up time.*

The following result characterises the finite–time blow–up of solutions to (6.1.1).

Theorem 6.2.1. *Suppose (6.1.2) and (6.1.3) hold. Solutions to (6.1.1) blow up in finite–time if and only if*

$$\int_\eta^\infty \frac{du}{\sqrt{\int_0^u f(s)\,ds}} < \infty, \quad \textit{for some } \eta > 0. \tag{6.2.1}$$

Under (6.1.2), the negation of (6.2.1) is of course

$$\int_\eta^\infty \frac{du}{\sqrt{\int_0^u f(s)\,ds}} = \infty, \quad \text{for all } \eta > 0, \tag{6.2.2}$$

and, by Theorem 6.2.1, condition (6.2.2) guarantees that solutions to (6.1.1) are global; we record condition (6.2.2) for future reference.

Theorem 6.2.1 is a special case of the following result for Volterra integro–differential equations of Hammerstein type due to Brunner and Yang.

Theorem 6.2.2 (Brunner and Yang [32, Theorem 3.9])**.** *Suppose $\psi > 0$, $h(t) \geq 0$ for $t \geq 0$, $w(t) = t^{\beta-1}w_1(t) \geq 0$ for $t \geq 0$, $\beta > 0$, and w_1 is bounded on every compact interval with $\inf_{s\in[0,\delta]} w_1(s) > 0$ for some $\delta > 0$. Suppose that $G : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ is continuous (uniformly in its second argument), increasing in its second argument, and satisfies $\lim_{u\to\infty} G(0,u)/u = \infty$. Solutions to*

$$u'(t) = h(t) + \int_0^t w(t-s)G(s,u(s))\,ds, \quad t \geq 0; \quad u(0) = \psi, \tag{6.2.3}$$

blow–up in finite–time if and only if there exists a $t^* > 0$ such that

$$\int_0^{t^*} h(s)\,ds + \min_{u\in[0,\infty)}\left(\int_0^{t^*} W(t^*-s)G(s,u)\,ds - u\right) > 0, \quad W(t) = \int_0^t w(s)\,ds, \tag{6.2.4}$$

and

$$\int_\eta^\infty \left(\frac{u}{G(t^*,u)}\right)^{1/(1+\beta)} \frac{du}{u} < \infty, \quad \textit{for some } \eta > 0. \tag{6.2.5}$$

To recover Theorem 6.2.1 from Theorem 6.2.2, set $h \equiv 0$, $\beta = 1$, and $G(s,u) = f(u)$. Thus (6.2.4) holds if $\min_{u\in[0,\infty)} \left(f(u) \int_0^{t^*} W(s)\,ds - u \right) > 0$ and we can always choose a $t^* > 0$ sufficiently large to satisfy this condition since $\int_0^\infty W(s)\,ds = \infty$. Under our specialisations, condition (6.2.5) reduces to the finiteness of the integral

$$\int_1^\infty \frac{dx}{\sqrt{xf(x)}} \tag{6.2.6}$$

The following proposition, whose proof is deferred, shows that the finiteness of the integrals in (6.2.1) and (6.2.6) are equivalent for the relevant class of nonlinear functions.

Proposition 6.2.1. *If $f \in C((0,\infty);(0,\infty))$ is increasing, then*

$$\int_1^\infty \frac{dx}{\sqrt{xf(x)}} < \infty \quad \textit{if and only if} \quad \int_1^\infty \frac{dx}{\sqrt{\int_0^x f(s)\,ds}} < \infty. \tag{6.2.7}$$

Proof of Proposition 6.2.1. Since f is positive and increasing, we have the following estimate

$$\int_0^x f(s)\,ds > \int_{\epsilon x}^x f(s)\,ds \ge x(1-\epsilon)f(\epsilon x), \quad x \ge 1,$$

for each $\epsilon \in (0,1)$. Thus

$$\int_1^N \frac{dx}{\sqrt{\int_0^x f(s)\,ds}} \le \int_1^N \frac{dx}{\sqrt{x(1-\epsilon)f(\epsilon x)}}, \quad N > 1.$$

Now make the substitution $u = \epsilon x$ to obtain

$$\int_1^N \frac{dx}{\sqrt{x(1-\epsilon)f(\epsilon x)}} = \frac{1}{\epsilon}\int_\epsilon^{\epsilon N} \frac{du}{\sqrt{\frac{(1-\epsilon)}{\epsilon} uf(u)}} < \sqrt{\frac{\epsilon}{(1-\epsilon)}} \int_\epsilon^N \frac{du}{\sqrt{uf(u)}}$$

Therefore

$$\int_1^\infty \frac{dx}{\sqrt{xf(x)}} < \infty \text{ implies } \int_1^\infty \frac{dx}{\sqrt{\int_0^x f(s)\,ds}} < \infty.$$

For the converse result, use the elementary estimate

$$\int_0^x f(s)\,ds \le xf(x), \quad x \ge 1.$$

Hence

$$\int_1^N \frac{dx}{\sqrt{\int_0^x f(s)\,ds}} \ge \int_1^N \frac{dx}{\sqrt{xf(x)}}, \quad N > 1,$$

and letting $N \to \infty$ yields

$$\int_1^\infty \frac{dx}{\sqrt{xf(x)}} = \infty \text{ implies } \int_1^\infty \frac{dx}{\sqrt{\int_0^x f(s)\,ds}} = \infty,$$

as required. □

While the conclusion of Theorem 6.2.1 is known, unlike Theorem 6.2.2, its proof yields considerable insight into the rate at which solutions to (6.1.1) grow, which is in fact our primary interest. The proof of Theorem 6.2.2 proceeds by integrating (6.2.3) to obtain an integral equation of the form

$$u(t) = u(0) + H(t) + \int_0^t W(t-s)G(s,u(s))\,ds, \quad t \geq 0.$$

The integral equation above is then discretised along a sequence $(t_n)_{n\geq 1}$ upon which the solution to (6.2.3) grows geometrically, i.e. $u(t_n) = R^n$ for each $n \geq 1$ and some $R > 1$. In all cases, $\lim_{n\to\infty} t_{n+1} - t_n = 0$ and moreover, if there is a global solution, $h_n = t_{n+1} - t_n$ tends to zero so fast that $\sum_{n=1}^{\infty} h_n < \infty$, contradicting the existence of a global solution. Conversely, in the presence of a blow–up solution, $(h_n)_{n\geq 1}$ is proven not to be summable using similar difference inequalities. Hence $\lim_{n\to\infty} t_n = \infty$, contradicting the assumption that the solution explodes in finite–time. In both cases, the summability of the sequence $(h_n)_{n\geq 1}$ hinges on (6.2.5). Naturally, some rough rates of growth are implicit in the constructions described above, but it is difficult to see how to obtain sharp estimates on rates of asymptotic growth of solutions from this approach, even for the simpler equation (6.1.1).

In contrast, we exploit the enhanced differential structure of (6.1.1) and employ comparison equations of the form

$$\begin{aligned} z'(t) &= C\int_{t-\delta}^{t} f(z(s))\,ds, \quad t \geq T^* \geq 0, \quad \text{with } \delta > 0 \text{ and } C > 0; \\ z(t) &= \psi(t), \quad t \leq T^*, \end{aligned} \tag{6.2.8}$$

to establish sharp blow–up conditions. The fact that comparison equations such as (6.2.8) yield sharp blow–up criteria suggests that these bounded delay equations are promising candidates for investigating the more subtle issue of asymptotic behaviour at blow–up for equation (6.1.1). Under very mild continuity assumptions,

$$z''(t) = Cf(z(t)) - Cf(z(t-\delta)), \quad t > T^* > \delta. \tag{6.2.9}$$

Solutions of (6.1.1) and (6.2.8) will grow extremely rapidly when $f(x)/x \to \infty$ as $x \to \infty$ so we conjecture that the delayed term in (6.2.9) is negligible asymptotically. Following this line of reasoning, we expect the second order ODE

$$z''(t) = f(z(t)), \quad t \geq T^*; \quad z(T^*) = \psi > 0,$$

to give a good asymptotic approximation to solutions of (6.1.1); this approximation is at the heart of our analysis and the definitions which follow are the product of our efforts to systematically exploit this idea.

Definition 6.2.2. *We say $g \in C((0,\infty);(0,\infty))$ exhibits* superexponential growth *if $g(x) \to \infty$ as $x \to \infty$ and*

$$\lim_{x\to\infty} \frac{g(x-\epsilon)}{g(x)} = 0, \quad \textit{for each } \epsilon > 0.$$

Continuous, positive functions which obey $g'(x)/g(x) \to \infty$ as $x \to \infty$ exhibit superexponential growth. However, this convenient sufficient condition for superexponential growth is not necessary.

Definition 6.2.3. $\phi \in C((0,\infty);(0,\infty))$ preserves superexponential growth *if for each function g which exhibits superexponential growth and each $\epsilon > 0$, we have*

$$\lim_{x\to\infty} \frac{\phi(g(x-\epsilon))}{\phi(g(x))} = 0.$$

If $\phi, f \in C((0,\infty);(0,\infty))$ obey $\phi \sim f$ and ϕ preserves superexponential growth, then so does f. The following simple lemma records several important classes of nonlinear functions which preserve superexponential growth and frequently arise in applications.

Proposition 6.2.2. *If $\phi \in C([0,\infty);[0,\infty))$ obeys one of the following conditions:*

(i.) $x \mapsto \phi(x)/x$ *is eventually increasing,*

(ii.) ϕ *is increasing and convex,*

(iii.) $\phi \in RV_\infty(\alpha)$ *for some* $\alpha > 0$,

then ϕ *preserves superexponential growth.*

Remark 6.2.1. *By $x \mapsto \phi(x)/x$ eventually increasing we mean that there exists a number X such that $x \mapsto \phi(x)/x$ is increasing on $[X,\infty)$. For example, if $\phi(x) = \exp(x)$ for $x \geq 0$, then $\lim_{x\to\infty} \phi(x)/x = \infty$, but $\lim_{x\to 0^+} \phi(x)/x = \infty$! However, $x \mapsto \exp(x)/x$ is increasing on $(1,\infty)$ and thus $\phi(x)/x$ is eventually increasing.*

In the next section, we demonstrate the utility of the preceding definitions in providing precise estimates on the rate of asymptotic growth or explosion of solutions to (6.1.1).

6.3 Growth Rates of Solutions

In order to compute rates of growth of solutions, define the functions

$$F_B(x) = \int_x^\infty \frac{du}{\sqrt{\int_0^u f(s)\,ds}}, \qquad \text{for each } x > 0, \tag{6.3.1}$$

and

$$F_U(x) = \int_1^x \frac{du}{\sqrt{\int_0^u f(s)\,ds}}, \qquad \text{for each } x > 0. \tag{6.3.2}$$

F_B characterises the rate of growth to infinity of solutions which blow–up in finite time, while F_U captures rates of growth of unbounded but nonexplosive solutions. In order to compute growth rates, we ask that the nonlinearity preserves superexponential growth, in the sense of Definition 6.2.3. As discussed in Section 6.3, preservation of superexponential growth is a relatively mild hypothesis satisfied by broad classes of nonlinearities commonly found in applications (see Proposition 6.2.2).

Theorem 6.3.1. *Suppose (6.1.2) and (6.1.3) hold. If (6.2.1) holds and f preserves superexponential growth, then solutions to (6.1.1) blow up in finite–time and obey*

$$\lim_{t\to T^-} \frac{F_B(x(t))}{T-t} = \sqrt{2w(0)},$$

where T denotes the blow–up time.

When studying the growth rate to infinity of non–explosive solutions, we further suppose that

$$w \in L^1(\mathbb{R}^+;\mathbb{R}^+), \quad ||w||_{L^1} = \mathcal{W}. \tag{6.3.3}$$

If w does not have finite L^1–norm, then it can contribute to faster growth in the convolution term when the solution is global. Assuming (6.3.3) rules this out and allows us to prove the following analogue of Theorem 6.3.1 for non–explosive solutions.

Theorem 6.3.2. *Suppose (6.1.2), (6.1.3), and (6.3.3) hold. If (6.2.2) holds and f preserves superexponential growth, then solutions to (6.1.1) obey $x \in C([0,\infty);(0,\infty))$ and*

$$\lim_{t\to\infty} \frac{F_U(x(t))}{t} = \sqrt{2w(0)}. \tag{6.3.4}$$

The final result of this section shows that when $w(0) = 0$ and (6.2.2) holds, solutions to (6.1.1) do not blow–up in finite–time. Furthermore, the rate of growth of solutions to (6.1.1) must be strictly slower than the case when $w(0) > 0$. More precisely, our new assumption on the kernel is as follows:

$$w \in C([0,\infty);[0,\infty)), \quad w(0) = 0, \quad w(t) > 0 \text{ for } t \in (0,\delta] \text{ for some } \delta > 0. \tag{6.3.5}$$

Theorem 6.3.3. *Suppose* (6.1.2), (6.3.3), *and* (6.3.5) *hold. If* (6.2.2) *holds, solutions to* (6.1.1) *obey* $x \in C([0,\infty);(0,\infty))$. *If f also preserves superexponential growth, then*

$$\lim_{t\to\infty} \frac{F_U(x(t))}{t} = 0. \tag{6.3.6}$$

6.4 Extensions to Perturbed Equations

We now consider the case when a nonautonomous forcing term is added to (6.1.1), i.e.

$$x'(t) = h(t) + \int_0^t w(t-s)f(x(s))\,ds, \quad t \geq 0; \quad x(0) = \psi > 0, \tag{6.4.1}$$

and demonstrate that the results of Section 6.2 are preserved under "small" perturbations. In the results which follow, some hypotheses in Theorem 6.2.2 which arise from the integral equation approach are unnecessary, although we are only treating the case $\beta = 1$. In particular we do not require h to be nonnegative and hence solutions to (6.4.1) are no longer necessarily monotone; due to the nature of our comparison arguments this relaxation does not present any additional difficulties. Furthermore, since our nonlinearity is not of Hammerstein type, condition (6.2.4) always holds. We assume that the forcing term, h, obeys

$$h \in C(\mathbb{R};\mathbb{R}), \quad H(t) := \int_0^t h(s)\,ds \geq 0 \quad \text{for each } t \geq 0, \tag{6.4.2}$$

Results regarding the finite–time blow–up of solutions require no additional hypotheses. However, for results regarding rates of growth we ask that the nonlinearity obeys

$$f \in C((0,\infty);(0,\infty)), \quad f \text{ is increasing}, \quad \lim_{x\to\infty} \frac{f(x)}{x} = \infty, \tag{6.4.3}$$

in order to simplify and shorten the proofs.

Our first result regarding solutions to the forced Volterra equation (6.4.1) shows that the blow–up condition and rate of explosion are unchanged by forcing terms obeying (6.4.2).

Theorem 6.4.1. *Suppose* (6.1.2), (6.1.3), *and* (6.4.2) *hold. If* (6.2.1) *holds, then solutions to* (6.4.1) *blow up in finite–time. If we further suppose that f preserves superexponential growth and* (6.4.3) *holds, then solutions to* (6.4.1) *obey*

$$\lim_{t\to T^-} \frac{F_B(x(t))}{T-t} = \sqrt{2w(0)},$$

where T denotes the blow–up time.

Previously we assumed that f preserves superexponential growth when proving results regarding the rate of growth of solutions; henceforth we replace this hypothesis with the assumption that

$$x \mapsto f(x)/x \quad \text{is eventually increasing}. \tag{6.4.4}$$

By Proposition 6.2.2, f preserves superexponential growth when (6.4.4) holds. As we show presently, the stronger hypothesis (6.4.4) allows us to characterise the perturbation terms which preserve the rate of growth when $h \equiv 0$, i.e. the asymptotic relation (6.3.4) still holds, in the non–explosive case. Our next result also shows that our blow–up conditions remain necessary in the presence of a nonautonomous forcing term.

Theorem 6.4.2. *Suppose* (6.1.2), (6.1.3), *and* (6.4.2) *hold. If* (6.2.2) *holds, then solutions to* (6.4.1) *obey* $x \in C([0,\infty);(0,\infty))$. *If we further suppose* $x \mapsto f(x)/x$ *is eventually increasing,* (6.4.3) *holds, and* w *obeys* (6.3.3), *then the following are equivalent:*

(i.)

$$\limsup_{t\to\infty} \frac{F_U(H(t))}{t} \le \sqrt{2w(0)},$$

(ii.)

$$\lim_{t\to\infty} \frac{F_U(x(t))}{t} = \sqrt{2w(0)}.$$

Our next result treats the case when the solution to (6.4.1) is global and the non–autonomous forcing term is "large", in the sense that

$$\lim_{t\to\infty} \frac{F_U(H(t))}{t} = K > \sqrt{2w(0)}. \tag{6.4.5}$$

When (6.4.5) holds, the growth rate of the solution is essentially the same as that of the forcing term.

Theorem 6.4.3. *Suppose* (6.1.3), (6.2.2), (6.4.2) *and* (6.4.3) *hold. If we further suppose* $x \mapsto f(x)/x$ *is eventually increasing,* w *obeys* (6.3.3), *and* (6.4.5) *holds, then*

$$\lim_{t\to\infty} \frac{F_U(x(t))}{t} = \lim_{t\to\infty} \frac{F_U(H(t))}{t} > \sqrt{2w(0)}. \tag{6.4.6}$$

When the forcing term is so large that $\lim_{t\to\infty} F_U(H(t))/t = \infty$ it is more difficult to prove precise results. Of course, a simple extension of the argument of Theorem 6.4.3 shows $\lim_{t\to\infty} F_U(x(t))/t = \infty$, but better estimates on the growth rate appear to require a stronger hypothesis. If H is asymptotic to an increasing function $\tilde{H}$ and

$$\lim_{t\to\infty} \frac{\int_0^t \bar{F}(K\tilde{H}(s))^{1/2}\,ds}{\tilde{H}(t)} = 0, \qquad \text{for some } K > 1, \tag{6.4.7}$$

where $\bar{F}$ is given by (6.6.4), then $\lim_{t\to\infty} F_U(H(t))/t = \infty$, and we can prove the following result.

Theorem 6.4.4. *Suppose* (6.1.3), (6.2.2), (6.4.2) *and* (6.4.3) *hold. If we further suppose* $x \mapsto f(x)/x$ *is eventually increasing,* w *obeys* (6.3.3), *and* H *is asymptotic to an increasing, superexponential function* $\tilde{H} \in C^2((0,\infty);(0,\infty))$ *such that* (6.4.7) *holds, then*

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = 1. \tag{6.4.8}$$

Roughly speaking, Theorem 6.4.4 gives an easy to check sufficient condition for the solution to (6.4.1) to inherit the leading order asymptotic behaviour of the forcing term, as opposed to inheriting the asymptotics of the unforced equation (6.1.1). However, it is not immediately clear how the crucial hypothesis for Theorem 6.4.4, i.e. (6.4.7), relates to conditions on the size of the perturbation term involving F_U, such as (6.4.5); the following proposition explains the connection between these different hypotheses.

Proposition 6.4.1. *Let (6.2.2), (6.4.2) and (6.4.3) hold. If $x \mapsto f(x)/x$ is eventually increasing and*

$$\lim_{t\to\infty} \frac{\int_0^t \bar{F}(KH(s))^{1/2}\,ds}{KH(t)} = \alpha_K \in (0,\infty), \quad \text{for some } K>1, \tag{6.4.9}$$

then

$$\lim_{t\to\infty} \frac{F_U(H(t))}{t} = \frac{1}{\alpha_K}.$$

Furthermore, if $\alpha_K = 0$, then $\lim_{t\to\infty} F_U(H(t))/t = \infty$. Similarly, $\lim_{t\to\infty} F_U(H(t))/t = 0$ when $\alpha_K = \infty$.

A priori, the limit function α_K in (6.4.9) depends on K. However, due to the conclusion of Proposition 6.4.1 and the uniqueness of limits, α_K is actually independent of K; henceforth we modify the hypothesis (6.4.9) to reflect this fact.

Our final result uses Proposition 6.4.1 to show that our hypotheses regarding the size of the forcing term can be thought of as lying on a continuous spectrum and track the transition between the system retaining the dynamics of (6.1.1) and the perturbation dominating the long–term behaviour. The proof of Theorem 6.4.5 merely requires combining the conclusion of Proposition 6.4.1 with Theorems 6.4.2, 6.4.3 and 6.4.4.

Theorem 6.4.5. *Let (6.1.3), (6.2.2), (6.4.2) and (6.4.3) hold. Suppose further that $x \mapsto f(x)/x$ is eventually increasing, w obeys (6.3.3), and H is asymptotic to an increasing, superexponential function $\tilde{H} \in C^2((0,\infty);(0,\infty))$. Finally, suppose*

$$\lim_{t\to\infty} \frac{\int_0^t \bar{F}(K\tilde{H}(s))^{1/2}\,ds}{K\tilde{H}(t)} = \alpha \in [0,\infty], \quad \text{for some } K>1. \tag{6.4.10}$$

(i.) If $\alpha = 0$, then $\lim_{t\to\infty} x(t)/H(t) = 1$;

(ii.) If $\alpha \in (0,\infty)$, then

$$\lim_{t\to\infty} \frac{F_U(x(t))}{t} = \max\left\{\sqrt{2w(0)}, 1/\alpha\right\};$$

(iii.) If $\alpha = \infty$, then

$$\lim_{t\to\infty} \frac{F_U(x(t))}{t} = \sqrt{2w(0)}.$$

6.5 Examples

Before addressing the proofs of our main results we pause to provide some simple examples of their application.

Since our results are insensitive to the structure of the memory, once $w(0)$ is fixed, the examples which follow do not require a functional form for w (as long as $w \in C((0,\infty);(0,\infty))$ and $w \in L^1(\mathbb{R}^+;\mathbb{R}^+)$ in the nonexplosive case). For example, with $\omega > 0$ arbitrary, the following kernels would be admissible in what follows:

$$\begin{aligned}
w_1(t) &= \omega(1+t)^{-\alpha}, \quad \alpha \geq 0, \quad t \geq 0,\\
w_2(t) &= \omega\exp(-t^{\gamma}), \quad \gamma > 0, \quad t \geq 0,\\
w_3(t) &= \omega/\Gamma(t+1), \quad \gamma > 0, \quad t \geq 0.
\end{aligned}$$

where Γ denotes the Gamma function.

Example 6.5.1. *Suppose $f(x) = (1+x)^{\beta}$ for $x > 0$ and for some $\beta > 1$. Choose any w obeying (6.1.3). Note that this choice of f obeys (6.1.2) and also preserves superexponential growth; to see this* *check any of (i. − iii.) in Proposition 6.2.2. We first check condition (6.2.1) to determine whether or not solutions to (6.1.1) blow–up in finite–time. First note that*

$$\sqrt{\int_0^u f(s)\,ds} = \left(\int_0^u (1+s)^\beta\,ds\right)^{1/2} = \left(\frac{(u+1)^{\beta+1}-1}{\beta+1}\right)^{1/2}, \quad u \geq 0.$$

For $\eta > 0$ arbitrary and $N > 0$ sufficiently large, we have

$$\int_\eta^N \frac{du}{\sqrt{\int_0^u f(s)\,ds}} = \sqrt{\beta+1}\int_\eta^N \left((u+1)^{\beta+1}-1\right)^{-1/2} du.$$

As $u \to \infty$, $\left((u+1)^{\beta+1}-1\right)^{-1/2} \sim u^{-(\beta+1)/2}$ and (6.2.1) holds since

$$\int_\eta^\infty u^{-(\beta+1)/2}\,du = \frac{2\eta^{(1-\beta)/2}}{\beta-1} < \infty, \qquad \textit{for each } \eta > 0 \textit{ and } \beta > 1.$$

Therefore, by Theorem 6.2.1, solutions to (6.1.1) blow–up for every w obeying (6.1.3). It can be shown that

$$F_B(x) \sim \frac{2(\beta-1)}{\sqrt{\beta+1}}\,x^{(1-\beta)/2}, \qquad \textit{as } x \to \infty.$$

Thus, by Theorem 6.3.1, solutions to (6.1.1) obey

$$\lim_{t\to T^-} \frac{x(t)^{(1-\beta)/2}}{T-t} = \frac{1}{\beta-1}\sqrt{\frac{(\beta+1)w(0)}{2}}, \quad \beta > 1, \quad w(0) > 0, \tag{6.5.1}$$

for some $T \in (0,\infty)$. Furthermore, solutions to (6.4.1) will still obey (6.5.1) for any perturbation term h obeying (6.4.2).

In this example it is possible to "invert" the asymptotic relation (6.5.1) to obtain the leading order behaviour of the solution at blow–up. In other words, (6.5.1) can be improved to

$$x(t) \sim \left(\frac{1}{\beta-1}\sqrt{\frac{(\beta+1)w(0)}{2}}\right)^{2/(1-\beta)} (T-t)^{2/(1-\beta)}, \qquad \textit{as } t \to T^-.$$

Example 6.5.2. *Now suppose $f(x) = (x+e)\log(x+e)$ for $x > 0$ and let w obey (6.1.3). Once again, it is straightforward to verify that f satisfies (6.1.2) and preserves superexponential growth. Moreover, $x \mapsto f(x)/x = (x+e)\log(x+e)/x$ is eventually increasing.*

We first check condition (6.2.1) to see if solutions to (6.1.1) blow–up in finite–time. Direct computation shows that

$$\int_\eta^N \frac{du}{\sqrt{\int_0^u f(s)\,ds}} = 2\int_\eta^N \frac{du}{\sqrt{(u+e)^2\left(2\log(u+e)-1\right)-e^2}}, \qquad N > \eta > 0.$$

As $u \to \infty$,

$$\sqrt{(u+e)^2\left(2\log(u+e)-1\right)-e^2} \sim u\sqrt{2\log(u)}.$$

Thus (6.2.1) does not hold because

$$\int_\eta^N \frac{du}{u\sqrt{2\log(u)}} = \sqrt{2}\left(\sqrt{\log(N)} - \sqrt{\log(\eta)}\right) \to \infty, \qquad \textit{as } N \to \infty.$$

Therefore, by Theorem 6.2.1, solutions to (6.1.1) are global if w obeys (6.1.3). It can also be shown *that*

$$F_U(x) \sim 2\sqrt{2\log(x)}, \qquad \text{as } x \to \infty.$$

Hence, by Theorem 6.3.2, solutions to (6.1.1) obey

$$\lim_{t\to\infty} \frac{\log(x(t))^{1/2}}{t} = \frac{\sqrt{w(0)}}{2}. \tag{6.5.2}$$

Equation (6.5.2) is of course equivalent to saying that $\log(x(t)) \sim w(0)t^2/4$ *as* $t \to \infty$.

Now we consider the effect of forcing terms on the asymptotic growth rate captured by (6.5.2). Firstly suppose h obeys (6.4.2) and $H(t) \sim t^\alpha$ as $t \to \infty$, for some $\alpha > 0$. Then

$$\limsup_{t\to\infty} \frac{F_U(H(t))}{t} = \limsup_{t\to\infty} \frac{2\sqrt{2\log(t^\alpha)}}{t} = 0, \quad \alpha > 0.$$

Hence, by Theorem 6.4.2, solutions to (6.4.1) still obey (6.5.2) for any perturbation tending to infinity no faster than a power.

Next choose h obeying (6.4.2) such that $H(t) \sim c_1 \exp(t^\alpha)$ as $t \to \infty$, for some $c_1 > 0$ and some $\alpha > 0$. In this case

$$\begin{aligned}\limsup_{t\to\infty} \frac{F_U(H(t))}{t} &= \limsup_{t\to\infty} \frac{2\sqrt{2\log(c_1\exp(t^\alpha))}}{t} = \lim_{t\to\infty} 2\sqrt{2}\, t^{(\alpha/2)-1} \\ &= \begin{cases} 0, & \alpha \in (0,2), \\ 2\sqrt{2}, & \alpha = 2, \\ \infty, & \alpha > 2. \end{cases}\end{aligned}$$

By Theorem 6.4.2, solutions to (6.4.2) continue to obey (6.5.2) for $\alpha \in (0,2)$. When $\alpha = 2$, (6.5.2) still holds if $w(0) \geq 4$, but if $w(0) < 4$, then

$$\lim_{t\to\infty} \frac{\log(x(t))^{1/2}}{t} = 1.$$

Finally, when $\alpha > 2$, we must resort to checking condition (6.4.7) in the hopes of applying Theorem 6.4.4. Applying L'Hôpital's rule (with $K > 1$ arbitrary) and performing the necessary integration yields

$$\lim_{t\to\infty} \frac{\int_0^t \bar{F}(K\tilde{H}(s))^{1/2} ds}{\tilde{H}(t)} = \lim_{t\to\infty} \frac{\sqrt{\int_0^{KH(t)} f(u)\,du}}{H'(t)} = \lim_{t\to\infty} \frac{K\exp(t^\alpha)t^{\alpha/2}}{t^{\alpha-1}\exp(t^\alpha)} = 0,$$

since $\alpha - 1 > \alpha/2$ for $\alpha > 2$. Hence condition (6.4.7) holds for each $K > 1$, H is superexponential for $\alpha > 1$, and therefore Theorem 6.4.4 implies that the solution to (6.4.1) obeys

$$\lim_{t\to\infty} \frac{x(t)}{\exp(t^\alpha)} = c_1, \quad \alpha > 2.$$

6.6 Preliminary Results and Lemmas

We first characterise the behaviour of solutions of two auxiliary equations, namely

$$y'(t) = \int_{t-\delta}^t w(t-s)f(y(s))\,ds, \quad t \geq 0; \quad y(t) = \psi(t), \quad t \in [-\delta, 0], \tag{6.6.1}$$

and

$$z'(t) = C \int_{t-\delta}^{t} f(z(s))\, ds, \quad t \geq 0; \quad z(t) = \psi(t), \quad t \in [-\delta, 0], \tag{6.6.2}$$

for some $C > 0$ and $\delta > 0$. We often use solutions to equations of the form (6.6.1) and (6.6.2) as comparison solutions for the more complex Volterra equations (6.1.1) and (6.4.1). The hypotheses on the nonlinearity are as before and the initial function, denoted by ψ, is assumed positive throughout, i.e.

$$\psi \in C([-\delta, 0]; (0, \infty)). \tag{6.6.3}$$

The function $\bar{F}$ given by

$$\bar{F}(x) = \int_0^x f(s)\, ds, \quad \text{for each } x > 0, \tag{6.6.4}$$

appears frequently in our proofs and inherits useful asymptotic properties from f, as noted in the following corollary.

Corollary 6.6.1. *If (6.1.2) holds, then $\bar{F}$ preserves superexponential growth.*

Corollary 6.6.1 follows directly from Proposition 6.2.2 by noting that $\bar{F}$ is the integral of a positive and increasing function, and thus is both increasing and convex itself.

Lemma 6.6.1. *Let $C > 0$ and $\delta > 0$, and suppose that (6.1.2) and (6.6.3) hold. If a solution to (6.6.2) obeys $z \in C([-\delta, \infty); (0, \infty))$, then z exhibits superexponential growth.*

Proof of Lemma 6.6.1. Assuming $z \in C([-\delta, \infty); (0, \infty))$ and (6.6.3) implies that $t \mapsto z(t)$ is increasing for $t \in [0, \infty)$ and hence that $\lim_{t\to\infty} z(t) = \infty$. Suppose $\sigma \in (0, \delta]$; let $t > 2\delta$ and integrate (6.6.2) from $t - \sigma$ to t to obtain

$$\begin{aligned} z(t) - z(t-\sigma) &= C \int_{t-\sigma}^{t} \int_{s-\delta}^{s} f(z(u))\, du\, ds \\ &= C \int_{t-\sigma-\delta}^{t} \int_{(t-\sigma)\vee u}^{t\wedge(u+\delta)} f(z(u))\, ds\, du, \qquad \text{for each } t > 2\delta. \end{aligned}$$

Using the positivity of z and (6.1.2) yields the lower bound

$$z(t) - z(t-\sigma) \geq C \int_{t-\sigma}^{t} \int_{(t-\sigma)\vee u}^{t\wedge(u+\delta)} f(z(u))\, ds\, du \geq C \int_{t-\sigma}^{t} (t-u) f(z(u))\, du,$$

for each $t > 2\delta$. The estimate above can (equivalently) be written as

$$\frac{z(t)}{z(t-\sigma)} \geq 1 + \frac{C \int_{t-\sigma}^{t} (t-u) f(z(u))\, du}{z(t-\sigma)}, \qquad \text{for each } t > 2\delta.$$

By (6.1.2), there exists a continuous, increasing function ϕ such that $\phi(x) \sim f(x)$ as $x \to \infty$. Since $\lim_{t\to\infty} z(t) = \infty$, for each $\epsilon \in (0, 1)$, there exists $T(\epsilon) > 0$ such that $f(z(t)) > (1-\epsilon)\phi(z(t))$ for all $t \geq T(\epsilon)$. Thus, by making the substitution $\alpha = t - u$ and using the monotonicity of ϕ, it can be shown that

$$\int_{t-\sigma}^{t} (t-u) f(z(u))\, du = \int_0^{\sigma} \alpha f(z(t-\alpha))\, d\alpha > \frac{(1-\epsilon)\sigma^2}{2} \phi(z(t-\sigma)),$$

for each $t > 2\delta + T(\epsilon) + \sigma$. Hence

$$\frac{z(t)}{z(t-\sigma)} > 1 + \frac{C\,(1-\epsilon)\sigma^2}{2} \frac{\phi(z(t-\sigma))}{z(t-\sigma)}, \quad t > 2\delta + T(\epsilon) + \sigma.$$

Since $f(x)/x \to \infty$ as $x \to \infty$ and $\lim_{t\to\infty} z(t-\sigma) = \infty$ for each $\sigma \in (0,\delta]$, taking the liminf in the inequality above shows that $\lim_{t\to\infty} z(t)/z(t-\sigma) = \infty$. Therefore, by the positivity of z,

$$\lim_{t\to\infty} \frac{z(t-\sigma)}{z(t)} = 0, \text{ for each } \sigma \in (0,\delta]. \tag{6.6.5}$$

Finally, since z is monotonically increasing, $z(t-\delta) \geq z(t-\sigma)$ for each $\sigma > \delta$, for t sufficiently large. Hence, from (6.6.5),

$$0 = \limsup_{t\to\infty} \frac{z(t-\delta)}{z(t)} \geq \limsup_{t\to\infty} \frac{z(t-\sigma)}{z(t)} \geq 0, \text{ for each } \sigma > \delta.$$

Thus (6.6.5) holds for all $\sigma > 0$ and z obeys Definition 6.2.2, as required. □

From Lemma 6.6.1 and Corollary 6.6.1, we immediately have the following useful lemma which we record now for future use.

Lemma 6.6.2. *Let $C > 0$ and $\delta > 0$, and suppose that (6.1.2) and (6.6.3) hold. If the solution to (6.6.2) obeys $z \in C([-\delta,\infty);(0,\infty))$, then*

$$\lim_{t\to\infty} \frac{\bar{F}(z(t-\delta))}{\bar{F}(z(t))} = 0,$$

where $\bar{F}$ is defined by (6.6.4).

Lemma 6.6.3. *Let $\delta > 0$, and suppose that (6.1.2), (6.1.3), and (6.6.3) hold. If (6.2.1) holds, then solutions to (6.6.1) blows up in finite–time.*

Proof of Lemma 6.6.3. Under the stated hypotheses there is a continuous solution to (6.6.1) on a maximal interval $[-\delta, T)$ for some $T > 0$. Suppose $T = \infty$.

Let $t \geq \delta$ and estimate as follows:

$$\begin{aligned} y'(t) &= \int_{t-\delta}^{t} w(t-s) f(y(s))\, ds = \int_{0}^{\delta} w(u) f(y(t-u))\, du \\ &\geq \inf_{s\in[0,\delta]} w(s) \int_{0}^{\delta} f(y(t-u))\, du. \end{aligned}$$

Define $\underline{w}(\delta) = \inf_{s\in[0,\delta]} w(s)$ and note that (6.1.3) guarantees $\underline{w}(\delta) > 0$. Hence

$$y'(t) > \underline{w}(\delta) \int_{t-\delta}^{t} f(y(s))\, ds, \quad t \geq \delta.$$

By (6.1.2), there exists a continuous, increasing function ϕ such that for each $\epsilon > 0$, $f(y(u)) > (1-\epsilon)\phi(y(u))$ for each $u \geq T_1(\epsilon) + \delta$. Let $\epsilon \in (0, 1/2)$ and define the lower comparison solution z by

$$z'(t) = \underline{w}(\delta)(1-2\epsilon) \int_{t-\delta}^{t} \phi(z(s))\, ds, \quad t \geq T_1(\epsilon) + \delta; \quad z(t) = \frac{y(t)}{2}, \tag{6.6.6}$$

for $t \in [0, T_1(\epsilon) + \delta]$. By construction, $z(t) < y(t)$ for each $t \in [0, T_1 + \delta]$. Hence, by a simple time of the first breakdown argument, $z(t) < y(t)$ for all $t \geq 0$. Due to the continuity of ϕ, $z \in C^2((T_1 + \delta, \infty); (0, \infty))$ and because $\phi \circ z$ is increasing

$$z''(t) = \underline{w}(\delta)(1-2\epsilon)\{\phi(z(t)) - \phi(z(t-\delta))\} > 0, \quad \text{for each } t > T_1 + \delta,$$

so z is convex on $(T_1+\delta,\infty)$. Using the convexity of z, we have

$$\begin{aligned}
(z'(t))^2 &= \underline{\mathrm{w}}(\delta)(1-2\epsilon)\int_{t-\delta}^{t}\phi(z(s))z'(t)\,ds \\
&\geq \underline{\mathrm{w}}(\delta)(1-2\epsilon)\int_{t-\delta}^{t}\phi(z(s))z'(s)\,ds \\
&= \underline{\mathrm{w}}(\delta)(1-2\epsilon)\int_{z(t-\delta)}^{z(t)}\phi(u)\,du \\
&= \underline{\mathrm{w}}(\delta)(1-2\epsilon)\left\{\bar{\Phi}(z(t))-\bar{\Phi}(z(t-\delta))\right\}, \qquad \text{for each } t>T_1+2\delta, \qquad (6.6.7)
\end{aligned}$$

where $\bar{\Phi}(x)=\int_0^x\phi(s)\,ds$. The function q given by $q(t)=z(t+T_1+\delta)$ for $t\geq -T_1-\delta$ solves (6.6.2) with $C=\underline{\mathrm{w}}(\delta)(1-2\epsilon)$ and $\psi=y/2$; thus Lemmas 6.6.1 and 6.6.2 apply to q, so that

$$\lim_{t\to\infty}\frac{q(t-\delta)}{q(t)}=0, \quad \lim_{t\to\infty}\frac{\bar{\Phi}(q(t-\delta))}{\bar{\Phi}(q(t))}=0.$$

It follows readily that

$$\lim_{t\to\infty}\frac{\bar{\Phi}(z(t-\delta))}{\bar{\Phi}(z(t))}=0.$$

Combining the limit above with (6.6.7) yields

$$\liminf_{t\to\infty}\frac{(z'(t))^2}{\bar{\Phi}(z(t))}\geq \underline{\mathrm{w}}(\delta)(1-2\epsilon)>0.$$

It follows that there exists a $T^*(\epsilon)>0$ such that for each $\epsilon\in(0,1/2)$

$$\frac{(z'(t))^2}{\bar{\Phi}(z(t))}>(1-\epsilon)\underline{\mathrm{w}}(\delta)(1-2\epsilon), \quad t\geq T^*(\epsilon).$$

Taking the square root across the inequality above and integrating from $T^*(\epsilon)$ to some fixed $t>T^*(\epsilon)$ we obtain

$$\int_{T^*}^{t}\frac{z'(s)ds}{\bar{\Phi}(z(s))^{1/2}}=\int_{z(T^*)}^{z(t)}\bar{\Phi}(u)^{-1/2}\geq (t-T^*)\sqrt{(1-\epsilon)\underline{\mathrm{w}}(\delta)(1-2\epsilon)}, \quad t>T^*.$$

Since $z(t)\to\infty$ as $t\to\infty$ and $\bar{F}(x)\sim\bar{\Phi}(x)$ as $x\to\infty$, taking the liminf in the inequality above gives

$$\int_{z(T^*)}^{\infty}\bar{F}(u)^{-1/2}\,du=\infty,$$

in contradiction to (6.2.1). Therefore $T<\infty$, as claimed. □

Lemma 6.6.4. *Let $C>0$ and $\delta>0$, and suppose that (6.1.2) and (6.6.3) hold. If (6.2.2) holds, then solutions to (6.6.2) obey $z\in C([-\delta,\infty);(0,\infty))$. Solutions to (6.6.1) obey $y\in C([-\delta,\infty);(0,\infty))$.*

Proof of Lemma 6.6.4. First consider equation (6.6.2). By (6.6.3), there exists a maximal $T\in(0,\infty]$ such that $z\in C([-\delta,T);(0,\infty))$ and $\lim_{t\to T^-}z(t)=\infty$. Suppose $T\in(0,\infty)$. By (6.1.2), there exists an increasing, continuous function ϕ such that $f(x)<\kappa\,\phi(x)$ for some $\kappa>0$, for each $x>0$. Define $\phi_\kappa(x)=\kappa\,\phi(x)$ for each $x>0$ and note that

$$\int_1^\infty\frac{du}{\sqrt{\int_0^u\phi_\kappa(s)\,ds}}=\infty$$

is equivalent to (6.2.2), since $f \sim \phi$. Let $\psi = 1 + \sup_{s\in[0,T/2]} z(s)$ and define the function α by

$$\alpha'(t) = \sqrt{2K_1 \int_1^{\alpha(t)} \phi_\kappa(u)\,du}, \quad t \geq 0; \quad \alpha(t) = \psi, \quad t \leq 0,$$

with

$$K_1 = \max\left(2,\ \frac{(\delta\,\phi_\kappa(\psi))^2}{2\int_1^\psi \phi_\kappa(u)\,du}\right).$$

Both ψ and K_1 are larger than 1, and (6.2.2) implies $\alpha \in C((-\infty,\infty);(0,\infty))$. In fact, due to the continuity of ϕ_κ, $\alpha \in C^2((0,\infty);(0,\infty))$. Furthermore, $\alpha'(t) > 0$ for $t \geq 0$ and due to our choice of ψ, $\alpha(t) > z(t)$ for each $t \in [-\delta, T/2]$. Now consider the function

$$A_\alpha(t) = \int_{t-\delta}^t \phi_\kappa(\alpha(u))\,du, \quad t \geq 0.$$

Differentiating A_α, estimating, and using the fact that $\alpha''(t) = K_1\phi_\kappa(\alpha(t))$ for $t > 0$ yields

$$A'_\alpha(t) = \phi_\kappa(\alpha(t)) - \phi_\kappa(\alpha(t-\delta)) < K_1\phi_\kappa(\alpha(t)) = \alpha''(t), \quad t > 0.$$

Integrating from 0 to t we obtain

$$\begin{aligned} A_\alpha(t) - A_\alpha(0) &= \int_{t-\delta}^t \phi_\kappa(\alpha(u))\,du - \int_{-\delta}^0 \phi_\kappa(\alpha(u))\,du \\ &\leq \alpha'(t) - \sqrt{2K_1 \int_1^{\alpha(0)} \phi_\kappa(u)\,du} = \alpha'(t) - \alpha'(0), \quad t \geq 0. \end{aligned}$$

Rearrangement shows that this is equivalent to

$$\alpha'(t) \geq \int_{t-\delta}^t \phi_\kappa(\alpha(u))\,du + \sqrt{2K_1 \int_1^\psi \phi_\kappa(u)\,du} - \int_{-\delta}^0 \phi_\kappa(\psi)\,du, \quad t \geq 0. \tag{6.6.8}$$

Note that

$$\sqrt{2K_1 \int_1^\psi \phi_\kappa(u)\,du} - \int_{-\delta}^0 \phi_\kappa(\psi)\,du > 0$$

if and only if $K_1 > (\delta\,\phi_\kappa(\psi))^2/2\int_1^\psi \phi_\kappa(u)\,du$, which is guaranteed by our earlier choice of K_1. Hence inequality (6.6.8) implies that

$$\alpha'(t) > \int_{t-\delta}^t \phi_\kappa(\alpha(u))\,du, \quad t \geq 0. \tag{6.6.9}$$

Now suppose there is a minimal $T_B \in (T/2, T)$ such that $\alpha(T_B) = z(T_B)$. Since $\alpha(t) > z(t)$ for each $t \in [-\delta, T/2]$, it must be the case that $z'(T_B) \geq \alpha'(T_B)$. Thus

$$z'(T_B) = \int_{T_B-\delta}^{T_B} f(z(u))\,du \leq \int_{T_B-\delta}^{T_B} \phi_\kappa(\alpha(u))\,du < \alpha'(T_B),$$

where the final *strict* inequality follows from (6.6.9). But this implies that $z'(T_B) < \alpha'(T_B) \leq z'(T_B)$, a contradiction. Thus $z(t) < \alpha(t)$ for each $t \in [-\delta, \infty)$ and there cannot exist a $T \in (0,\infty)$ such that $\lim_{t\to T^-} z(t) = \infty$, since (6.2.2) ensures that α is bounded on compact intervals. Therefore we must have $T = \infty$, as claimed.

Now consider (6.6.1). By hypothesis, $y \in C([-\delta, T); (0, \infty))$ for some $T > 0$. For each $t \in (0, T)$,

$$y'(t) = \int_{t-\delta}^{t} w(t-s) f(y(s))\, ds = \int_{0}^{\delta} w(u) f(y(t-u))\, du$$
$$< 2 \sup_{s \in [0,\delta]} w(s) \int_{0}^{\delta} f(y(t-u))\, du.$$

Define $\bar{w}(\delta) = \sup_{s \in [0,\delta]} w(s) > 0$ and hence define the upper comparison solution z by

$$z'(t) = 2\bar{w}(\delta) \int_{t-\delta}^{t} \phi_\kappa(z(s))\, ds, \quad t \geq 0; \quad z(t) = \sup_{u \in [-\delta, 0]} \psi(u) + 1, \quad t \in [-\delta, 0].$$

By the arguments above, $z \in C([-\delta, \infty); (0, \infty))$ and, by construction, $z(t) > y(t)$ for each $t \in [-\delta, T)$. Hence y cannot explode in finite–time and the claim is proven. □

Our final lemma identifies the growth rate of solutions to (6.6.2). The corresponding results for (6.1.1) and (6.4.1) consist of carefully constructing comparison solutions using equations of the form of (6.6.2) and then invoking this lemma.

Lemma 6.6.5. *Suppose that the hypotheses of Lemma 6.6.1 hold. If f preserves superexponential growth, then the solution $z \in C([-\delta, \infty); (0, \infty))$ to (6.6.2) obeys*

$$\lim_{t \to \infty} \frac{F_U(z(t))}{t} = \sqrt{2C}.$$

Proof of Lemma 6.6.5. Due to the continuity of f, $z \in C^2((\delta, \infty); (0, \infty))$ and

$$z''(t) = Cf(z(t)) - Cf(z(t-\delta)), \quad t > \delta.$$

By Lemma 6.6.1,

$$\lim_{t \to \infty} \frac{z(t-\delta)}{z(t)} = 0,$$

and hence, because f preserves superexponential growth, we have

$$\lim_{t \to \infty} \frac{f(z(t-\delta))}{f(z(t))} = 0.$$

Thus

$$\lim_{t \to \infty} \frac{z''(t)}{Cf(z(t))} = 1. \tag{6.6.10}$$

It follows from (6.1.2) that $\int_0^{z(t)} f(u)\, du \to \infty$ as $t \to \infty$ and hence $z'(t) \to \infty$ as $t \to \infty$ by integration of (6.6.10). Now use L'Hôpital's rule to obtain

$$\lim_{t \to \infty} \frac{(z'(t))^2}{2C \int_0^{z(t)} f(u)\, du} = \lim_{t \to \infty} \frac{z''(t)}{Cf(z(t))} = 1.$$

Therefore

$$\lim_{t \to \infty} \frac{z'(t)}{\sqrt{\int_0^{z(t)} f(u)\, du}} = \sqrt{2C}.$$

It follows that for each $\epsilon > 0$ there exists $T^*(\epsilon) > 0$ such that

$$\sqrt{2C} - \epsilon < \frac{z'(t)}{\sqrt{\int_0^{z(t)} f(u)\,du}} < \epsilon + \sqrt{2C}, \quad t \geq T^*(\epsilon).$$

Suppose $t > T^*(\epsilon)$ and integrate the inequality above to yield

$$(\sqrt{2C} - \epsilon)(t - T^*) < \int_{T^*}^{t} \frac{z'(u)\,du}{\sqrt{\int_0^{z(u)} f(s)\,ds}} < (\epsilon + \sqrt{2C})(t - T^*), \quad t > T^*(\epsilon).$$

By making the substitution $y = z(u)$ it is straightforward to show that

$$(\sqrt{2C} - \epsilon)\frac{t - T^*}{t} + \frac{F_U(z(T^*))}{t} < \frac{F_U(z(t))}{t} < (\epsilon + \sqrt{2C})\frac{t - T^*}{t} + \frac{F_U(z(T^*))}{t},$$

for $t > T^*(\epsilon)$. The claim follows by letting $t \to \infty$ and then $\epsilon \to 0^+$ in the inequalities above. □

6.7 Proofs of Main Results

Proof of Theorem 6.2.1. **Sufficiency of Conditions:** Suppose (6.2.1) holds. By the usual considerations, $x \in C([0,T);(0,\infty))$ for some (maximal) $T \in (0,\infty]$. Suppose $T = \infty$ and let $\tau > 0$ be arbitrary. By (6.1.3) and positivity, we have

$$\begin{aligned} x'(t) &= \int_0^t w(t-s)f(x(s))\,ds = \int_{t-\tau}^{t} w(t-s)f(x(s))\,ds + \int_0^{t-\tau} w(t-s)f(x(s))\,ds \\ &> \int_{t-\tau}^{t} w(t-s)f(x(s))\,ds, \quad t \geq \tau. \end{aligned}$$

Let ϕ denote any monotone increasing, continuous function obeying $f(x) \sim \phi(x)$ as $x \to \infty$. Since $x(t) \to \infty$ as $t \to \infty$, for each $\epsilon \in (0,1)$ there exists $T_1(\epsilon) > 0$ such that $f(x(t)) > (1-\epsilon)\phi(x(t))$ for each $t \geq T_1(\epsilon)$. Hence

$$x'(t) > (1-\epsilon)\int_{t-\tau}^{t} w(t-s)\phi(x(s))\,ds, \quad t \geq T_1(\epsilon) + \tau.$$

Define the lower comparison solution y by

$$y'(t) = (1-\epsilon)\int_{t-\tau}^{t} w(t-s)\phi(y(s))\,ds, \quad t \geq T_1(\epsilon) + \tau; \quad y(t) = x_L(t), \quad t \in [0, T_1 + \tau],$$

where x_L obeys $x_L'(t) = \int_0^t w(t-s)f(x_L(s))\,ds$ for $t \in [0, T_1 + \tau]$ and $x_L(0) = x(0)/2$. By construction, $y(t) < x(t)$ for $t \geq 0$. Let $y_\tau(t) = y(t + \tau + T_1)$ for each $t \geq -T_1 - \tau$ and note that y_τ solves (6.6.1) with $\delta = \tau + T_1$ and $\psi = x_L$. Hence Lemma 6.6.3 applies to y_τ and there exists a $T_\tau < \infty$ such that $\lim_{t \to T_\tau^-} y_\tau(t) = \infty$, contradicting the assumption that $T = \infty$ and completing the proof.

Necessity of Conditions: Suppose (6.2.2) holds. As usual, our hypotheses guarantee a well defined solution to (6.1.1) on some maximal interval $[0,T)$ with $T \in (0,\infty]$. Assume, contrary to our claim, that $T < \infty$. Let $\delta \in (0,T)$ and estimate the derivative of x for $t \in (\delta, T)$ as follows:

$$\begin{aligned} x'(t) &= \int_{t-\delta}^{t} w(t-s)f(x(s))\,ds + \int_0^{t-\delta} w(t-s)f(x(s))\,ds \\ &\leq \bar{w}(\delta)\int_{t-\delta}^{t} f(x(s))\,ds + \bar{M}(\delta), \end{aligned} \tag{6.7.1}$$

where $\bar{w}(\delta) = \sup_{s\in[0,\delta]} w(s)$ and $\bar{M}(\delta) = \sup_{t\in[0,T]} \int_0^{t-\delta} w(t-s)f(x(s))\,ds$. Note that

$$\limsup_{t\to T^-} \frac{\bar{M}(\delta)}{\bar{w}(\delta)\int_{t-\delta}^t f(x(s))\,ds} =: C(\delta) \in [0,\infty),$$

which can be observed by simply taking suprema of continuous functions over the compact interval $[0,T]$. Combining the limit superior above with (6.7.1) yields

$$\limsup_{t\to T^-} \frac{x'(t)}{\bar{w}(\delta)\int_{t-\delta}^t f(x(s))\,ds} \le 1 + C(\delta) < \infty, \qquad \text{for each } \delta \in (0,T).$$

Thus, for each $\epsilon > 0$, there exists $T^*(\epsilon) \in (\delta, T)$ such that

$$x'(t) < (1+\epsilon)(1+C(\delta))\bar{w}(\delta)\int_{t-\delta}^t f(x(s))\,ds, \quad t \in [T^*(\epsilon), T).$$

Taking $\epsilon = \delta$ in the estimate above gives

$$x'(t) < (1+\delta)(1+C(\delta))\bar{w}(\delta)\int_{t-\delta}^t f(x(s))\,ds, \quad t \in [T^*(\delta), T).$$

By hypothesis, there is an increasing, continuous function ϕ such that $f(x) < \kappa\,\phi(x)$ for some $\kappa > 0$, for each $x > 0$. As before let $\phi_\kappa(x) = \phi_\kappa(x)$ for each $x > 0$. Hence

$$x'(t) < (1+\delta)(1+C(\delta))\bar{w}(\delta)\int_{t-\delta}^t \phi_\kappa(x(s))\,ds, \quad t \in [T^*(\delta), T).$$

Now define the upper comparison solution z according to

$$z'(t) = (1+2\delta)(1+C(\delta))\bar{w}(\delta)\int_{t-\delta}^t \phi_\kappa(z(s))\,ds, \quad t \ge 0;$$
$$z(t) = Z^* := 1 + \sup_{u\in[0,T^*(\delta)]} x(u), \quad t \in [-\delta, 0].$$

By construction, $x(t) < z(t)$ for all $t \in [0,T)$. However, since z solves (6.6.2) with $C = (1+2\delta)(1+C(\delta))\bar{w}(\delta)$ and $\psi \equiv Z^*$, Lemma 6.6.4 implies that $z \in C([-\delta,\infty);(0,\infty))$. Therefore the assumption that $T < \infty$ leads to a contradiction and the proof is complete. □

Proof of Theorem 6.3.1. By hypothesis there exists $T \in (0,\infty)$ such that $x \in C([0,T);(0,\infty))$ and $\lim_{t\to T^-} x(t) = \infty$. We first show that $\lim_{t\to T^-} x'(t) = \infty$. For an arbitrary $\delta \in (0,T)$, construct a lower bound on x' of the form

$$x'(t) > \kappa\,\underline{w}(\delta)\int_{t-\delta}^t \phi(x(u))\,du + \underline{\mathrm{M}}(\delta), \quad t \in (\delta, T), \tag{6.7.2}$$

where $\underline{\mathrm{M}}(\delta) = \inf_{t\in[0,T]} \int_0^{t-\delta} w(t-s)f(x(s))\,ds$, $\kappa > 0$, and ϕ is monotone increasing and continuous. We have $\lim_{t\to T^-} \phi(x(t)) = \infty$ and hence

$$\lim_{t\to T^-} \frac{d}{dt}\int_{t-\delta}^t \phi(x(u))\,du = \infty.$$

Thus the function $t \mapsto \int_{t-\delta}^t \phi(x(u))\,du$ is increasing on some interval (T^*, T) and must have a limit as $t \to T^-$. However, if $\lim_{t\to T^-} \int_{t-\delta}^t \phi(x(u))\,du$ is finite, integration of (6.7.1) yields

$$x(t) \le A + Bt, \text{ for } t \in (T_1^*, T),$$

for some positive constants A and B. But the solution blows-up in finite–time, a contradiction. Therefore, $\lim_{t\to T^-}\int_{t-\delta}^t \phi(x(u))\,du = \infty$ and hence $\lim_{t\to T^-} x'(t) = \infty$, as claimed.

Now let $\delta \in (0,T)$ be arbitrary and estimate as follows:

$$\begin{aligned} x'(t) &= \int_{t-\delta}^t w(t-s)f(x(s))\,ds + \int_0^{t-\delta} w(t-s)f(x(s))\,ds \\ &\le \bar{w}(\delta)\int_{t-\delta}^t f(x(u))\,du + \bar{M}(\delta) \quad t\in(\delta,T), \end{aligned} \tag{6.7.3}$$

where $\bar{w}(\delta) = \sup_{u\in[0,\delta]} w(u)$ and $\bar{M}(\delta) = \sup_{t\in[0,T]}\int_0^{t-\delta} w(t-s)f(x(s))\,ds$. Note that $\bar{M}(\delta)$ is finite for each $\delta\in(0,T)$. By (6.7.3) and the fact that $x'(t)\to\infty$ as $t\to T^-$, we have $\int_{t-\delta}^t f(x(u))\,du\to\infty$ as $t\to T^-$, for each $\delta\in(0,T)$. Thus $\int_0^t f(x(u))\,du\to\infty$ as $t\to T^-$ and, by applying L'Hôpital's rule,

$$\lim_{t\to T^-}\frac{\int_{t-\delta}^t f(x(u))\,du}{\int_0^t f(x(u))\,du} = \lim_{t\to T^-}\frac{f(x(t)) - f(x(t-\delta))}{f(x(t))} = 1, \qquad \text{for each } \delta\in(0,T).$$

Dividing across by $\bar{w}(\delta)\int_0^t f(x(u))\,du$ in (6.7.3) and taking the limsup thus yields

$$\limsup_{t\to T^-}\frac{x'(t)}{\bar{w}(\delta)\int_0^t f(x(u))\,du} \le 1, \qquad \text{for each } \delta\in(0,T).$$

Letting $\delta\to 0^+$ in the limit above shows that

$$\limsup_{t\to T^-}\frac{x'(t)}{w(0)\int_0^t f(x(u))\,du} \le 1.$$

Similarly, we can obtain the following lower estimate on the derivative

$$x'(t) > \int_{t-\delta}^t w(t-s)f(x(s))\,ds \ge \underline{w}(\delta)\int_{t-\delta}^t f(x(u))\,du, \quad t\in(\delta,T),$$

where $\underline{w}(\delta) = \inf_{u\in[0,\delta]} w(u) > 0$. Following the same steps as above quickly reveals that

$$\liminf_{t\to T^-}\frac{x'(t)}{w(0)\int_0^t f(x(u))\,du} \ge 1.$$

Therefore

$$\lim_{t\to T^-}\frac{x'(t)}{w(0)\int_0^t f(x(s))\,ds} = 1. \tag{6.7.4}$$

We claim that (6.7.4) implies

$$\lim_{t\to T^-}\frac{x'(t)}{\sqrt{2w(0)\int_0^{x(t)} f(s)\,ds}} = 1. \tag{6.7.5}$$

Using (6.7.4), (6.7.5) is equivalent to

$$\lim_{t\to T^-}\frac{w(0)\int_0^t f(x(s))\,ds}{\sqrt{2w(0)\int_0^{x(t)} f(s)\,ds}} = 1.$$

Letting $I(t) = \int_0^t f(x(s))\,ds$, the limit above is in turn equivalent to

$$\lim_{t\to T^-} \frac{[w(0)I(t)]^2}{2w(0)\int_0^{x(t)} f(s)\,ds} = 1.$$

However, since $I(t) \to \infty$ as $t \to T^-$ and $\int_0^{x(t)} f(s)\,ds \to \infty$ as $t \to T^-$, applying L'Hôpital's rule yields

$$\lim_{t\to T^-} \frac{[(w(0)I(t)]^2}{2w(0)\int_0^{x(t)} f(s)\,ds} = \lim_{t\to T^-} \frac{2[w(0)]^2 I(t)I'(t)}{2w(0)x'(t)f(x(t))} = \lim_{t\to T^-} \frac{w(0)\int_0^t f(x(s))\,ds}{x'(t)} = 1,$$

where the final equality follows from (6.7.4). Thus (6.7.4) implies (6.7.5), as claimed.

By (6.7.5), for each $\epsilon \in (0,1)$, there exists $\bar{T}(\epsilon) \in (0,T)$ such that

$$1-\epsilon < \frac{x'(t)}{\sqrt{2w(0)\int_0^{x(t)} f(s)\,ds}} < 1+\epsilon, \quad t \in (\bar{T}, T).$$

Let t and T_L be such that $\bar{T} < t < T_L < T$ and integrate the inequalities above from t to T_L; this yields

$$(1-\epsilon)(T_L - t)\sqrt{2w(0)} < \int_t^{T_L} \frac{x'(u)\,du}{\sqrt{\int_0^{x(u)} f(s)\,ds}} < (1+\epsilon)(T_L - t)\sqrt{2w(0)},$$

for $\bar{T} < t < T_L < T$. Make the substitution $y = x(u)$ in the integral to obtain

$$(1-\epsilon)(T_L - t)\sqrt{2w(0)} < \int_{x(t)}^{x(T_L)} \frac{dy}{\sqrt{\int_0^y f(s)\,ds}} < (1+\epsilon)(T_L - t)\sqrt{2w(0)},$$

for $\bar{T} < t < T_L < T$. Now let $T_L \to T^-$ and divide across by $T - t$ to show that

$$(1-\epsilon)\sqrt{2w(0)} < \frac{1}{T-t}\int_{x(t)}^{\infty} \frac{dy}{\sqrt{\int_0^y f(s)\,ds}} = \frac{F_B(x(t))}{T-t} < (1+\epsilon)\sqrt{2w(0)}, \quad \bar{T} < t < T.$$

Letting $\epsilon \to 0^+$ in the inequalities above completes the proof. □

Proof of Theorem 6.3.2. By hypothesis, $x \in C([0,\infty);(0,\infty))$. Let $\delta > 0$ be arbitrary and estimate as follows:

$$\begin{aligned} x'(t) &= \int_0^{t-\delta} w(t-s)f(x(s))\,ds + \int_{t-\delta}^t w(t-s)f(x(s))\,ds > \int_{t-\delta}^t w(t-s)f(x(s))\,ds \\ &= \int_0^\delta w(u)f(x(t-u))\,du \geq \underline{w}(\delta)\int_{t-\delta}^t f(x(s))\,ds, \quad t \geq \delta, \end{aligned}$$

where $\underline{w}(\delta) = \inf_{u\in[0,\delta]} w(u) > 0$. By (6.1.2), there exists a continuous, increasing function ϕ such that, for each $\epsilon \in (0,1)$, $f(x(s)) > (1-\epsilon)\phi(x(s))$ for all $s \geq T(\epsilon)$. Hence

$$x'(t) > (1-\epsilon)\underline{w}(\delta)\int_{t-\delta}^t \phi(x(s))\,ds, \quad t \geq T(\epsilon) + \delta.$$

Now define the lower comparison solution z_- by

$$z_-'(t) = (1-\epsilon)\underline{w}(\delta)\int_{t-\delta}^t \phi(z_-(s))\,ds, \quad t \geq T+\delta; \quad z_-(t) = x(t)/2, \quad t \in [0, T+\delta].$$

It can be shown that $z_-(t) < x(t)$ for all $t \geq 0$ and, by applying Lemma 6.6.5 to z_-, we have

$$\sqrt{2(1-\epsilon)\underline{w}(\delta)} = \liminf_{t\to\infty} \frac{F_U(z_-(t))}{t} \leq \liminf_{t\to\infty} \frac{F_U(x(t))}{t}, \tag{6.7.6}$$

for each $\delta > 0$. Letting $\delta \to 0^+$ and $\epsilon \to 0^+$ in the inequality above, and using the continuity of w, we obtain

$$\liminf_{t\to\infty} \frac{F_U(x(t))}{t} \geq \sqrt{2w(0)}.$$

Remark 6.7.1. *Technically, Lemma 6.6.5 applies to the function $t \mapsto z_-(t+T+\delta)$ and gives*

$$\sqrt{2\underline{w}(\delta)} = \lim_{t\to\infty} \frac{F_U(z_-(t+T+\delta))}{t} = \lim_{t\to\infty} \frac{F_U(z_-(t+T+\delta))}{t+T+\delta} = \lim_{s\to\infty} \frac{F_U(z_-(s))}{s},$$

and this is what we are actually using in equation (6.7.6).

We now tackle the corresponding limsup. By (6.1.2), there exists a continuous, increasing function ϕ such that, for each $\epsilon > 0$, $f(x(t)) < (1+\epsilon)\phi(x(t))$ for each $t \geq T_1(\epsilon)$. Furthermore, we can find a $\kappa > 0$ such that $f(x) < \kappa\,\phi(x)$ for each $x > 0$. Let $\delta > 0$, and begin by estimating as follows:

$$\begin{aligned}
x'(t) &= \int_0^{t-\delta} w(t-s) f(x(s))\,ds + \int_{t-\delta}^{t} w(t-s) f(x(s))\,ds \\
&\leq \kappa\,\phi(x(t-\delta)) \int_0^{t-\delta} w(t-s)\,ds + \int_{t-\delta}^{t} w(t-s) f(x(s))\,ds \\
&< \kappa\,\mathcal{W}\,\phi(x(t-\delta)) + \bar{w}(\delta) \int_{t-\delta}^{t} f(x(s))\,ds \\
&\leq \kappa\,\mathcal{W}\,\phi(x(t-\delta)) + (1+\epsilon)\bar{w}(\delta) \int_{t-\delta}^{t} \phi(x(s))\,ds, \quad t \geq \delta + T_1(\epsilon),
\end{aligned}$$

where $\bar{w}(\delta) = \sup_{u\in[0,\delta]} w(u) > 0$. Thus

$$x'(t) < \kappa\,\mathcal{W}\phi(x(t-\delta)) + (1+\epsilon)\bar{w}(\delta) \int_{t-\delta}^{t} \phi(x(s))\,ds, \quad t \geq T_1(\epsilon) + \delta. \tag{6.7.7}$$

Now make the following lower estimate on the second term in (6.7.7):

$$\bar{w}(\delta) \int_{t-\delta}^{t} \phi(x(s))\,ds > \bar{w}(\delta) \int_{t-\delta/2}^{t} \phi(x(s))\,ds \geq \frac{\delta}{2}\phi(x(t-\delta/2)), \quad t \geq T_1(\epsilon) + \delta.$$

Hence

$$0 \leq \limsup_{t\to\infty} \frac{\kappa\,\mathcal{W}\,\phi(x(t-\delta))}{\bar{w}(\delta)\int_{t-\delta}^{t}\phi(x(s))\,ds} \leq \lim_{t\to\infty} \frac{2\kappa\,\mathcal{W}}{\delta} \frac{\phi(x(t-\delta))}{\phi(x(t-\delta/2))} = 0,$$

where the final limit can be established by repeating verbatim the argument from Lemma 6.6.1. Combining the limit above with (6.7.7) then yields

$$\limsup_{t\to\infty} \frac{x'(t)}{\bar{w}(\delta)\int_{t-\delta}^{t}\phi(x(s))\,ds} \leq 1 + \epsilon.$$

Hence, for each $\epsilon > 0$, there exists $T^*(\epsilon,\delta) > 0$ such that

$$x'(t) < (1+\epsilon)^2 \bar{w}(\delta) \int_{t-\delta}^{t} \phi(x(s))\,ds, \quad t \geq T^*(\epsilon,\delta).$$

Now fix $\epsilon = \delta > 0$, so that

$$x'(t) < (1+\delta)^2 \bar{w}(\delta) \int_{t-\delta}^{t} \phi(x(s))\, ds, \quad t \geq T^*(\delta).$$

Define the upper comparison solution z_+ by

$$z_+'(t) = (1+\delta)^2 \bar{w}(\delta) \int_{t-\delta}^{t} \phi(z_+(s))\, ds, \quad t \geq 0;$$
$$z_+(t) = \sup_{u \in [0, T^*(\delta)]} x(u) + 1, \quad t \in [-\delta, 0].$$

By construction, $z_+(t) > x(t)$ for each $t \geq 0$ and, by applying Lemma 6.6.5, we obtain

$$\limsup_{t\to\infty} \frac{F_U(x(t))}{t} \leq \limsup_{t\to\infty} \frac{F_U(z_+(t))}{t} \leq \sqrt{2(1+\delta)^2\bar{w}(\delta)}, \text{ for each } \delta > 0,$$

once more using that $\phi \sim f$. Therefore, by letting $\delta \to 0^+$,

$$\limsup_{t\to\infty} \frac{F_U(x(t))}{t} \leq \sqrt{2w(0)},$$

and combining this with the corresponding liminf yields the result. □

Proof of Theorem 6.3.3. By (6.1.2), we can find a continuous, increasing function ϕ obeying $\phi \sim f$ such that $f(x) < \kappa\,\phi(x)$ for some $\kappa > 1$, for each $x > 0$. Let $\epsilon > 0$ be arbitrary and define $w_\epsilon(t) = \kappa\, w(t) + \kappa\,\epsilon e^{-t}$ for $t \geq 0$. Now construct the upper comparison solution x_ϵ by letting

$$x_\epsilon'(t) = \int_0^t w_\epsilon(t-s)\phi(x_\epsilon(s))\, ds, \quad t \geq 0; \quad x_\epsilon(0) = x(0) + \epsilon.$$

By Theorem 6.2.1, $x_\epsilon \in C([0,\infty);(0,\infty))$. Furthermore, because $x_\epsilon(t) > x(t)$ for each $t \geq 0$, $x \in C([0,\infty);(0,\infty))$ also. To see this suppose that $x(T_B) = x_\epsilon(T_B)$ but that $x(t) < x_\epsilon(t)$ for each $t \in [0, T_B)$ (for some $T_B > 0$). Hence $x'(T_B) \geq x_\epsilon'(T_B)$ but

$$\begin{aligned} x'(T_B) \geq x_\epsilon'(T_B) &= \int_0^{T_B} w_\epsilon(T_B - s)\phi(x_\epsilon(s))\, ds \\ &= \int_0^{T_B} \kappa\, w(T_B - s)\phi(x_\epsilon(s))\, ds + \kappa\,\epsilon \int_0^{T_B} e^{-(T_B - s)}\phi(x_\epsilon(s))\, ds \\ &> \int_0^{T_B} w(T_B - s) f(x(s))\, ds = x'(T_B), \end{aligned}$$

a contradiction. Note that since $\phi \sim f$, we have

$$F_U(x) \sim \int_1^x \frac{du}{\sqrt{\int_0^u \phi(s)\, ds}}, \text{ as } x \to \infty.$$

By applying Theorem 6.3.2 to x_ϵ, we have

$$\lim_{t\to\infty} \frac{F_U(x_\epsilon(t))}{t} = \sqrt{2w_\epsilon(0)} = \sqrt{2\kappa\,\epsilon},$$

and by monotonicity of F_U,

$$0 \leq \limsup_{t\to\infty} \frac{F_U(x(t))}{t} \leq \lim_{t\to\infty} \frac{F_U(x_\epsilon(t))}{t} = \sqrt{2\kappa\,\epsilon}.$$

Letting $\epsilon \to 0^+$ in the inequality above yields

$$\lim_{t\to\infty} \frac{F_U(x(t))}{t} = 0,$$

as required. $\square$

Proof of Theorem 6.4.1. We first show that (6.2.1) is a sufficient condition for the finite–time blow–up of solutions to (6.4.1). Begin, as in the proof of necessity in Theorem 6.2.1, by assuming $T = \infty$. Now let $\tau > 0$ be arbitrary and make the lower estimate

$$x'(t) > \int_{t-\tau}^{t} w(t-s)f(x(s))\,ds + h(t), \quad t \geq \tau.$$

By (6.1.2), there exists an increasing, positive function ϕ asymptotic to f and a finite, positive constant C such that

$$C = \inf_{x\in[x(0)/2,\infty)} \frac{f(x)}{\phi(x)}.$$

Hence $C\,\phi(x) \leq f(x)$ for each $x \geq x(0)/2$. Now let $\varphi(x) = C\,\phi(x)$ for each $x \geq x(0)/2$. Thus

$$x'(t) > \int_{t-\tau}^{t} w(t-s)\varphi(x(s))\,ds + h(t), \quad t \geq \tau,$$

because $x(t) \geq x(0) > x(0)/2$ for all $t \geq 0$. Integration then yields

$$x(t) \geq x(\tau) + \int_{\tau}^{t}\int_{u-\tau}^{u} w(u-s)\varphi(x(s))\,ds\,du + H_\tau(t), \quad t \geq \tau, \tag{6.7.8}$$

where $H_\tau(t) = \int_\tau^t h(s)\,ds$. Now define the lower comparison solution y according to

$$y(t) = y(\tau) + \int_{\tau}^{t}\int_{u-\tau}^{u} w(u-s)\varphi(y(s))\,ds\,du, \quad t \geq \tau; \quad y(t) = x(0)/2, \quad t \in [0,\tau].$$

Of course, y also obeys the delayed integro–differential equation

$$y'(t) = \int_{t-\tau}^{t} w(t-s)\varphi(y(s))\,ds, \quad t \geq \tau; \quad y(t) = x(0)/2, \quad t \in [0,\tau].$$

By integrating (6.4.1), note that $x(t) \geq x(0)$ for each $t > 0$ due to (6.4.2) and $y(t) < x(t)$ for each $t \in [0,\tau]$, by construction. Using the continuity of h, choose $\tau > 0$ sufficiently small that $\int_0^\tau h(s)\,ds \leq x(0)/4$ and suppose $T_B > \tau$ is the minimal time such that $y(T_B) = x(T_B)$. Thus

$$\begin{aligned}
y(T_B) &= y(\tau) + \int_{\tau}^{T_B}\int_{u-\tau}^{u} w(u-s)\varphi(y(s))\,ds\,du = x(T_B)\\
&\geq x(\tau) + H_\tau(T_B) + \int_{\tau}^{T_B}\int_{u-\tau}^{u} w(u-s)\varphi(x(s))\,ds\,du\\
&\geq x(0) + H(T_B) - \int_0^\tau h(s)\,ds + \int_{\tau}^{T_B}\int_{u-\tau}^{u} w(u-s)\varphi(x(s))\,ds\,du\\
&\geq x(0) - \int_0^\tau h(s)\,ds + \int_{\tau}^{T_B}\int_{u-\tau}^{u} w(u-s)\varphi(y(s))\,ds\,du\\
&> x(0)/2 + \int_{\tau}^{T_B}\int_{u-\tau}^{u} w(u-s)\varphi(y(s))\,ds\,du = y(T_B),
\end{aligned}$$

a contradiction. Thus $y(t) < x(t)$ for each $t \geq 0$. At this point, following the proof of necessity in Theorem 6.2.1 shows that $T = \infty$ produces a contradiction and hence $T \in (0,\infty)$, as required.

We have shown $x \in C([0,T);(0,\infty))$ for some $T \in (0,\infty)$ with $\lim_{t\to T^-} x(t) = \infty$, so we now proceed to show that the rate of growth of the solution as it approaches the blow–up time is the same as that of the unperturbed equation. Since h is continuous, there exists $a_1 > 0$ such that

$$|h(t)| \le a_1, \quad \text{for all } t \in [0,T].$$

Hence, following the line of argument from the proof of Theorem 6.3.1, we obtain the upper estimate

$$x'(t) \le a_1 + \bar{w}(\delta) \int_{t-\delta}^{t} f(x(s))\,ds + \int_0^{t-\delta} w(t-s) f(x(s))\,ds, \quad t \in (\delta, T),$$

for each $\delta \in (0,T)$ and with $\bar{w}(\delta) = \sup_{u\in[0,\delta]} w(u)$. Now let

$$a_2(\delta) = \sup_{t\in[\delta,T]} \int_0^{t-\delta} w(t-s) f(x(s))\,ds$$

and note that $a_2(\delta)$ is bounded for each $\delta \in (0,T)$. Thus

$$x'(t) < a_3(\delta) + \bar{w}(\delta) \int_{t-\delta}^{t} f(x(s))\,ds, \quad t \in (\delta, T), \tag{6.7.9}$$

where $a_3(\delta) = 1 + a_1 + a_2(\delta)$. Similarly, once again reusing the arguments from the proof of Theorem 6.3.1, we have

$$x'(t) > -a_1 + \underline{w}(\delta) \int_{t-\delta}^{t} f(x(s))\,ds, \quad t \in (\delta, T), \tag{6.7.10}$$

where $\underline{w}(\delta) = \inf_{u\in[0,\delta]} w(u)$. Define $I(t) = \int_0^t f(x(s))\,ds$ for each $t \in [0,T)$ and note that I is an increasing function due to the positivity of $f \circ x$. Hence $\lim_{t\to T^-} I(t)$ exists; suppose $\lim_{t\to T^-} I(t) = I^* \in (0,\infty)$. By positivity, $\int_{t-\delta}^t f(x(s))\,ds < I(t) \le I^*$ for each $t \in (\delta,T)$ and thus we obtain the inequalities

$$-a_1 < x'(t) < a_3(\delta) + \bar{w}(\delta) I^*, \quad t \in (\delta, T).$$

But by simply integrating the above inequalities we rule out the finite–time explosion of x, a contradiction. Therefore $\lim_{t\to T^-} I(t) = \infty$ and, as in the proof of Theorem 6.3.1, it follows from L'Hôpital's rule that

$$\lim_{t\to T^-} \frac{\int_{t-\delta}^t f(x(s))\,ds}{\int_0^t f(x(s))\,ds} = 1.$$

Combining the fact above with (6.7.9) and (6.7.10) quickly yields

$$1 \le \liminf_{t\to\infty} \frac{x'(t)}{\underline{w}(\delta) \int_0^t f(x(s))\,ds}, \quad \limsup_{t\to T^-} \frac{x'(t)}{\bar{w}(\delta) \int_0^t f(x(s))\,ds} \le 1.$$

Letting $\delta \to 0^+$ in the inequalities above shows that

$$\lim_{t\to\infty} \frac{x'(t)}{w(0) \int_0^t f(x(s))\,ds} = 1.$$

We are now in the same position as at equation (6.7.4) in the proof of Theorem 6.3.1. Repeating verbatim the arguments which follow equation (6.7.4) completes the proof. □

We now establish several useful lemmas which are needed for the proof of Theorem 6.4.2.

Lemma 6.7.1. *Suppose* (6.4.3) *and* (6.2.2) *hold. Each solution, y, to the initial value problem*

$$y'(t) = \sqrt{\bar{F}(y(t))}, \quad t \geq 0; \quad y(0) = 1, \tag{6.7.11}$$

obeys

$$\lim_{t\to\infty} \frac{y((1-\eta)t)}{y(t)} = 0, \quad \text{for each } \eta \in (0,1).$$

Proof of Lemma 6.7.1. Under the given hypotheses, the initial value problem (6.7.11) has a unique solution $y \in C([0,\infty);(0,\infty))$. By L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{x^2} = \lim_{x\to\infty} \frac{f(x)}{2x} = \infty,$$

where the final equality is due to (6.4.3). Thus $\sqrt{\bar{F}(x)}/x \to \infty$ as $x \to \infty$ and

$$\lim_{t\to\infty} \frac{y'(t)}{y(t)} = \frac{\sqrt{\bar{F}(y(t))}}{y(t)} = \infty.$$

Therefore y exhibits superexponential growth and $\lim_{t\to\infty} y(t-1)/y(t) = 0$ ($\epsilon = 1$ in Definition 6.2.2). By monotonicity,

$$\frac{y((1-\eta)t)}{y(t)} < \frac{y(t-1)}{y(t)}, \quad t > \frac{1}{\eta},$$

for each $\eta \in (0,1)$. Therefore $\lim_{t\to\infty} y((1-\eta)t)/y(t) = 0$, as claimed. □

Lemma 6.7.2. *Suppose a and b are continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^+/\{0\}$ which obey $a(t)/b(t) \to 0$ as $t \to \infty$. If $f : \mathbb{R}^+ \mapsto (0,\infty)$ is a continuous function such that $x \mapsto f(x)/x$ is increasing for $x \geq 0$, then*

$$\lim_{t\to\infty} \frac{f(a(t))}{f(b(t))} = 0.$$

Proof of Lemma 6.7.2. By hypothesis, for each $\epsilon \in (0,1)$ there exists a $\bar{T}(\epsilon) > 0$ such that

$$a(t) < \epsilon\, b(t) < b(t), \quad t \geq \bar{T}(\epsilon).$$

Thus, by the monotonicity of $f(x)/x$,

$$\frac{f(a(t))}{a(t)} < \frac{f(b(t))}{b(t)}, \quad t \geq \bar{T}(\epsilon).$$

Therefore, for $t \geq \bar{T}(\epsilon)$,

$$\frac{f(a(t))}{f(b(t))} = \frac{f(a(t))}{a(t)}\frac{a(t)}{f(b(t))} < \frac{f(b(t))}{b(t)}\frac{a(t)}{f(b(t))} = \frac{a(t)}{b(t)} < \epsilon,$$

as required. □

Proof of Theorem 6.4.2. Firstly, we claim that (6.2.2) implies $x \in C([0,\infty);(0,\infty))$ in the presence of perturbations obeying (6.4.2). Under our standing hypotheses, there exists a maximal $T \in (0,\infty]$ such that $x \in C([0,T);(0,\infty))$. Now follow the line of argument from the proof of necessity in Theorem 6.2.1. Suppose $T < \infty$ and estimate x' for $t \in (\delta, T)$, for some $\delta \in (0,T)$, as follows:

$$x'(t) < \bar{w}(\delta) \int_{t-\delta}^{t} f(x(s))\, ds + \bar{M}(\delta) + h(t),$$

where $\bar{w}(\delta) = \sup_{s\in[0,\delta]} w(s)$ and $\bar{M}(\delta) = \sup_{t\in[0,T]} \int_0^{t-\delta} w(t-s)f(x(s))\,ds$. Since h is continuous there exists $a_1 > 0$ such that $|h(t)| \le a_1$ for all $t \in [0,T]$ and hence

$$x'(t) < \bar{w}(\delta)\int_{t-\delta}^{t} f(x(s))\,ds + \bar{M}(\delta) + a_1, \quad t \in (\delta, T). \tag{6.7.12}$$

Beginning at equation (6.7.1), repeat the argument from the proof of necessity in Theorem 6.2.1 verbatim to conclude that $x \in C([0,\infty);(0,\infty))$. Furthermore, by (6.4.2) and a straightforward comparison argument with an equation of the form (6.6.2), $\lim_{t\to\infty} x(t) = \infty$.

We now show that $(i.)$ implies $(ii.)$. Suppose that

$$\limsup_{t\to\infty} \frac{F_U(H(t))}{t} = K \in \left(0, \sqrt{2w(0)}\right).$$

For each $\delta > 0$, there exists $T_1(\delta) > 0$ such that

$$H(t) < F_U^{-1}\left((1+\delta)Kt\right) = \bar{H}(t), \quad t \ge T_1(\delta).$$

Remark 6.7.2. *In the case that $K = 0$, we would have*

$$H(t) < F_U^{-1}\left(\delta t\right) =: \bar{H}(t), \quad t \ge T_1(\delta),$$

for each $\delta > 0$ and the proof would proceed as in the case $K \in \left(0, \sqrt{2w(0)}\right)$.

Integrating (6.4.1) yields

$$x(t) = x(0) + H(t) + \int_0^t W(t-s)f(x(s))\,ds, \quad t \ge 0,$$

where $W(t) = \int_0^t w(u)\,du$. Hence

$$\begin{aligned} x(t) &< x(0) + \bar{H}(t) + \int_{T_1}^t W(t-s)f(x(s))\,ds + \int_0^{T_1} W(t-s)f(x(s))\,ds \\ &\le x(0) + \bar{H}(t) + \int_{T_1}^t W(t-s)f(x(s))\,ds + \mathcal{W}\int_0^{T_1} f(x(s))\,ds, \quad t > T_1(\delta). \end{aligned} \tag{6.7.13}$$

Let $x^* = x(0) + \mathcal{W}\int_0^{T_1} f(x(s))\,ds$ and define the upper comparison solution x_+ by

$$x_+(t) = 1 + x^* + \bar{H}(t) + \int_{T_1}^t W(t-s)f(x_+(s))\,ds, \quad t \ge T_1(\delta).$$

By construction, $x(t) < x_+(t)$ for $t \ge T_1(\delta)$. Under (6.4.3), $\bar{H} \in C^1((T_1,\infty);(0,\infty))$ and thus $x_+ \in C^1((T_1,\infty);(0,\infty))$. Hence differentiation yields

$$x_+'(t) = \bar{H}'(t) + \int_{T_1}^t w(t-s)f(x_+(s))\,ds, \quad t > T_1(\delta).$$

Since x_+ is increasing, it follows that

$$x_+'(t) > \underline{w}(\delta)\int_{t-\delta}^t f(x_+(s))\,ds, \quad t \ge T_1(\delta) + \delta,$$

where $\underline{w}(\delta) = \inf_{u\in[0,\delta]} w(u)$. Following the proof of Lemma 6.6.1, we then obtain

$$\lim_{t\to\infty} \frac{x_+(t-\delta)}{x_+(t)} = 0, \text{ for each } \delta > 0.$$

Proposition 6.2.2 applies to f since $x \mapsto f(x)/x$ is increasing and thus

$$\lim_{t\to\infty} \frac{f(x_+(t-\delta))}{f(x_+(t))} = 0, \text{ for each } \delta > 0.$$

Now, for $t \geq T_1(\delta) + \delta$, estimate as follows:

$$\begin{aligned} x_+'(t) &= \bar{H}'(t) + \int_{T_1}^{t-\delta} w(t-s) f(x_+(s))\,ds + \int_{t-\delta}^{t} w(t-s) f(x_+(s))\,ds \\ &\leq \bar{H}'(t) + \mathcal{W} f(x_+(t-\delta)) + \bar{w}(\delta) \int_{t-\delta}^{t} f(x_+(s))\,ds, \end{aligned} \tag{6.7.14}$$

where $\bar{w}(\delta) = \sup_{u\in[0,\delta]} w(u)$. As noted in earlier arguments

$$\frac{f(x_+(t-\delta))}{\int_{t-\delta}^t f(x_+(s))\,ds} < \frac{f(x_+(t-\delta))}{\int_{t-\delta/2}^t f(x_+(s))\,ds} \leq \frac{2f(x_+(t-\delta))}{\delta f(x_+(t-\delta/2))},$$

and hence $\lim_{t\to\infty} f(x_+(t-\delta))/\int_{t-\delta}^t f(x_+(s))\,ds = 0$ for each $\delta > 0$. Thus, from (6.7.14), we derive the estimate

$$x_+'(t) < \bar{H}'(t) + \bar{w}(\delta)(1+\delta) \int_{t-\delta}^{t} f(x_+(s))\,ds, \quad t \geq T_2(\delta) > T_1(\delta) + \delta,$$

where $T_2(\delta)$ is sufficiently large to guarantee that

$$\frac{\mathcal{W} f(x_+(t-\delta))}{\int_{t-\delta}^t f(x_+(s))\,ds} < \delta \bar{w}(\delta), \quad t \geq T_2(\delta).$$

The solution to the initial value problem

$$y'(t) = \sqrt{\bar{F}(y(t))}, \quad t \geq 0; \quad y(0) = 1, \tag{6.7.15}$$

is given by $y(t) = F_U^{-1}(t)$ for each $t \geq 0$. Furthermore,

$$y''(t) = \frac{1}{2} f(y(t)), \quad t > 0.$$

Now express the derivatives of $\bar{H}$ in terms of y as follows:

$$\bar{H}'(t) = K(1+\delta) y'(K(1+\delta)t), \quad \bar{H}''(t) = \frac{K^2(1+\delta)^2}{2} f(y(K(1+\delta)t)), \quad t \geq T_1(\delta).$$

For short we write $\bar{y}(t) = y(K(1+\delta)t)$ henceforth. Let x_u be the solution to

$$\begin{aligned} x_u'(t) &= \bar{H}'(t) + \bar{w}(\delta)(1+\delta) \int_{t-\delta}^{t} f(x_u(s))\,ds, \quad t \geq T_2(\delta); \\ x_u(t) &= 1 + x_+(T_2), \quad t \in [0, T_2]. \end{aligned}$$

By construction, $x(t) < x_+(t) < x_u(t)$ for $t \geq T_2(\delta)$. Differentiation yields

$$x_u''(t) = \bar{H}''(t) + \bar{w}(\delta)(1+\delta)\left\{f(x_u(t)) - f(x_u(t-\delta))\right\}, \quad t > T_2(\delta).$$

With the notation introduced above, we have

$$\begin{aligned} x_u''(t) &= \frac{K^2(1+\delta)^2}{2} f(\bar{y}(t)) + \bar{w}(\delta)(1+\delta) f(x_u(t)) \left\{ 1 - \frac{f(x_u(t-\delta))}{f(x_u(t))} \right\} \\ &> \bar{w}(\delta)(1+\delta) f(x_u(t)) \left\{ 1 - \frac{f(x_u(t-\delta))}{f(x_u(t))} \right\}, \quad t > T_2(\delta). \end{aligned} \tag{6.7.16}$$

It is easily demonstrated that $x_u(t-\delta)/x_u(t) \to 0$ as $t \to \infty$ and hence $f(x_u(t-\delta))/f(x_u(t)) \to 0$ as $t \to \infty$, since f obeys the hypotheses of Proposition 6.2.2. Thus

$$\liminf_{t\to\infty} \frac{x_u''(t)}{f(x_u(t))} \geq \bar{w}(\delta)(1+\delta), \quad \text{for each } \delta > 0, \tag{6.7.17}$$

and there exists $T_3(\delta) > T_2(\delta)$ such that

$$\frac{x_u''(t)}{f(x_u(t))} > \bar{w}(\delta)(1+\delta)(1-\delta/4)^{1/2}, \quad t \geq T_3(\delta). \tag{6.7.18}$$

Let $C(\delta) = \bar{w}(\delta)(1+\delta)(1-\delta/4)^{1/2}$ and calculate as follows:

$$\frac{d}{dt}\left\{ \frac{(x_u'(t))^2}{2} - C(\delta)\bar{F}(x_u(t)) \right\} = \{ x_u''(t) - C(\delta) f(x_u(t)) \} \, x_u'(t) > 0, \quad t > T_3(\delta),$$

using equation (6.7.18). Hence,

$$\frac{(x_u'(t))^2}{2\bar{F}(x_u(t))} > C(\delta), \quad t > T_3(\delta),$$

or equivalently

$$\frac{x_u'(t)}{\sqrt{\bar{F}(x_u(t))}} > \sqrt{2C(\delta)}, \quad t > T_3(\delta).$$

Asymptotic integration now yields

$$\liminf_{t\to\infty} \frac{F_U(x_u(t))}{t} \geq \sqrt{2C(\delta)}.$$

Thus there exists $T_4(\delta) > T_3(\delta)$ such that

$$\frac{F_U(x_u(t))}{t} > \sqrt{2C(\delta)}\sqrt{(1-\delta/4)} = \sqrt{2\bar{w}(\delta)(1+\delta)(1-\delta/4)}, \quad t > T_4(\delta). \tag{6.7.19}$$

Therefore

$$\frac{\bar{y}(t)}{x_u(t)} = \frac{F_U^{-1}(K(1+\delta)t)}{x_u(t)} < \frac{F_U^{-1}(K(1+\delta)t)}{F_U^{-1}\left(\sqrt{2\bar{w}(\delta)(1+\delta)(1-\delta/4)}\, t\right)}, \quad t \geq T_4(\delta).$$

We want to choose $\delta \in (0,1)$ small enough that

$$K(1+\delta) < \sqrt{2\bar{w}(\delta)(1+\delta)(1-\delta/4)}.$$

Since $K < \sqrt{2w(0)}$, it is sufficient to choose δ small enough that

$$\delta < \min\left(\frac{\alpha - 1}{1+\alpha/4}, 1 \right), \text{ where } \alpha = \frac{2w(0)}{K^2} > 1.$$

Hence, by applying Lemma 6.7.1, we have $\lim_{t\to\infty} \bar{y}(t)/x_u(t) = 0$ for each $\delta > 0$ sufficiently small.

It then follows from Lemma 6.7.2 that $f(\bar{y}(t))/f(x_u(t)) \to 0$ as $t \to \infty$. Combining this limit with (6.7.16) gives

$$\lim_{t\to\infty} \frac{x_u''(t)}{f(x_u(t))} = \bar{w}(\delta)(1+\delta),$$

and there exists $T_5(\delta) > T_4(\delta)$ such that

$$\bar{w}(\delta) < \frac{x_u''(t)}{f(x_u(t))} < \bar{w}(\delta)(1+\delta)^2, \quad t \geq T_5(\delta). \tag{6.7.20}$$

Now, for $t \geq T_5(\delta)$,

$$\frac{d}{dt}\left\{\frac{(x_u'(t))^2}{2} - \bar{w}(\delta)(1+\delta)^2\bar{F}(x_u(t))\right\} = \left\{x_u''(t) - \bar{w}(\delta)(1+\delta)^2 f(x_u(t))\right\} x_u'(t) < 0,$$

by (6.7.20). Hence

$$x_u'(t) < \sqrt{2\bar{w}(\delta)(1+\delta)^2\bar{F}_U(x_u(t))}, \quad t \geq T_5(\delta).$$

Asymptotic integration readily yields

$$\limsup_{t\to\infty} \frac{F_U(x_u(t))}{t} \leq \sqrt{2\bar{w}(\delta)(1+\delta)^2}.$$

Now note that $x(t) < x_u(t)$ for each $t \geq T_2(\delta)$. Therefore, letting $\delta \to 0^+$, we have

$$\limsup_{t\to\infty} \frac{F_U(x(t))}{t} \leq \sqrt{2w(0)}.$$

When $K = \sqrt{2w(0)}$, define $\bar{H}(t) = F_U^{-1}(K(1+\delta/2)t)$ for $t \geq T_1(\delta)$. The argument proceeds analogously until equation (6.7.17). Now asymptotic integration can be used to show that

$$\liminf_{t\to\infty} \frac{F_U(x_u(t))}{t} \geq \sqrt{2w(0)(1+2\delta)}, \quad \text{for each } \delta > 0.$$

At this point we want to show that for $\delta > 0$ sufficiently small

$$\limsup_{t\to\infty} \frac{\bar{y}(t)}{x_u(t)} \leq \limsup_{t\to\infty} \frac{F_U^{-1}\left(\sqrt{2w(0)}(1+\delta/2)t\right)}{F_U^{-1}\left(\sqrt{2w(0)(1+2\delta)(1-\delta/4)}\right)} = 0,$$

by applying Lemma 6.7.1. Thus it is sufficient to choose $\delta > 0$ such that

$$\sqrt{2w(0)}(1+\delta/2) < \sqrt{2w(0)(1+2\delta)(1-\delta/4)}.$$

In fact, this is equivalent to choosing $\delta \in (0,1)$. The proof concludes as in the case $K \in \left(0, \sqrt{2w(0)}\right)$.

To prove the corresponding limit inferior, positivity of H yields the trivial lower bound

$$x(t) \geq x(0) + \int_0^t W(t-s)f(x(s))\,ds, \quad t \geq 0,$$

where $W(t) = \int_0^t w(s)\,ds$. Now define the lower comparison solution x_- by

$$x_-(t) = x(0)/2 + \int_0^t W(t-s)f(x_-(s))\,ds, \quad t \geq 0.$$

It is straightforward to show that $x_-(t) \le x(t)$ for all $t \ge 0$. Furthermore,

$$x_-'(t) = \int_0^t w(t-s) f(x_-(s))\, ds, \quad t \ge 0.$$

Hence Theorem 6.3.2 applies to x_- and yields

$$\lim_{t\to\infty} \frac{F_U(x_-(t))}{t} = \sqrt{2w(0)}.$$

Therefore

$$\liminf_{t\to\infty} \frac{F_U(x(t))}{t} \ge \lim_{t\to\infty} \frac{F_U(x_-(t))}{t} = \sqrt{2w(0)},$$

and combining this with the corresponding limit superior establishes that

$$\lim_{t\to\infty} \frac{F_U(x(t))}{t} = \sqrt{2w(0)},$$

as required.

Finally, we show that $(ii.)$ implies $(i.)$, under our standing assumptions. By positivity, $x(t) > H(t)$ for each $t \ge 0$. Thus, owing to the monotonicity of F_U, $F_U(x(t)) > F_U(H(t))$ for each $t \ge 0$. Therefore

$$\limsup_{t\to\infty} \frac{F_U(H(t))}{t} \le \limsup_{t\to\infty} \frac{F_U(x(t))}{t} = \sqrt{2w(0)},$$

as required. □

Proof of Theorem 6.4.3. The existence of a global solution is guaranteed by Theorem 6.4.2. By positivity, $x(t) > H(t)$ for each $t \ge 0$. Hence, due to the monotonicity of F_U,

$$\liminf_{t\to\infty} \frac{F_U(x(t))}{t} \ge \lim_{t\to\infty} \frac{F_U(H(t))}{t} = K.$$

It remains to prove the corresponding limit superior. By (6.4.5), for each $\epsilon \in (0,1)$ there is a $T_1(\epsilon) > 0$ such that

$$H(t) < F_U^{-1}\left(K(1+\epsilon)t\right), \quad t \ge T_1(\epsilon).$$

Thus, similarly to (6.7.13), we have the initial upper estimate

$$x(t) \le x(0) + F_U^{-1}\left(K(1+\epsilon)t\right) + \mathcal{W}\int_0^{T_1} f(x(s))\, ds + \int_{T_1}^t W(t-s) f(x(s))\, ds, \ \ t \ge T_1(\epsilon).$$

Now let $x^* = x(0) + \mathcal{W}\int_0^{T_1} f(x(s))\, ds$ and define the upper comparison solution

$$x_+(t) = 1 + x^* + F_U^{-1}\left(K(1+\epsilon)t\right) + \int_{T_1}^t W(t-s) f(x_+(s))\, ds, \quad t \ge T_1(\epsilon).$$

In order to streamline notation, define $\phi(x) = \bar{F}(x)^{1/2}$ for each $x \ge 0$. Hence

$$x_+'(t) = K(1+\epsilon)\phi\left(F_U^{-1}(K(1+\epsilon)t)\right) + \int_{T_1}^t w(t-s) f(x_+(s))\, ds, \quad t > T_1(\epsilon).$$

Clearly, x_+ is increasing on $[T_1, \infty)$. Thus, by positivity and (6.1.3),

$$x_+'(t) > \underline{w}(\epsilon) \int_{t-\epsilon}^t f(x_+(s))\, ds, \quad t > T_1(\epsilon) + \epsilon, \tag{6.7.21}$$

where $\underline{w}(\epsilon) = \inf_{s\in[0,\epsilon]} w(s)$. Repeating the arguments of Lemma 6.6.1 and recalling that $f(x)/x$

increasing is sufficient to preserve superexponential growth, we have

$$\lim_{t\to\infty} \frac{x_+(t-\delta)}{x_+(t)} = 0, \quad \lim_{t\to\infty} \frac{f(x_+(t-\delta))}{f(x_+(t))} = 0, \quad \text{for each } \delta > 0.$$

Now make a simple upper estimate on x'_+ as follows:

$$x'_+(t) < K(1+\epsilon)\phi\left(F_U^{-1}(K(1+\epsilon)t)\right) + \mathcal{W} f(x_+(t-\epsilon)) + \bar{w}(\epsilon)\int_{t-\epsilon}^t f(x_+(s))\,ds, \tag{6.7.22}$$

for $t > T_1(\epsilon)+\epsilon$, where $\bar{w}(\epsilon) = \sup_{s\in[0,\epsilon]} w(s)$. As in previous arguments, it is straightforward to show that

$$\lim_{t\to\infty} \frac{f(x_+(t-\epsilon))}{\int_{t-\epsilon}^t f(x_+(s))\,ds} = 0.$$

Thus, for each $\eta > 0$, there exists $T_2(\eta,\epsilon) > 0$ such that

$$\mathcal{W} f(x_+(t-\epsilon)) < \eta\bar{w}(\epsilon)\int_{t-\epsilon}^t f(x_+(s))\,ds, \quad t \geq T_2(\eta,\epsilon).$$

Let $\eta = \epsilon$ and set $T_3(\epsilon) = T_1(\epsilon) + T_2(\epsilon)$. Hence

$$x'_+(t) < K(1+\epsilon)\phi\left(F_U^{-1}(K(1+\epsilon)t)\right) + (1+\epsilon)\bar{w}(\epsilon)\int_{t-\epsilon}^t f(x_+(s))\,ds, \quad t \geq T_3(\epsilon). \tag{6.7.23}$$

Now define $K_\epsilon = K(1+2\epsilon)$ and $w_\epsilon = (1+\epsilon)\bar{w}(\epsilon)$. By Lemma 6.7.1,

$$\lim_{t\to\infty} \frac{F_U^{-1}\left(K(1+\epsilon)t\right)}{F_U^{-1}\left(K(1+2\epsilon)t\right)} = 0, \quad \text{for each } \epsilon > 0.$$

It follows that there exists $T_4(\epsilon) > 0$ such that

$$\frac{F_U^{-1}\left(K(1+\epsilon)t\right)}{F_U^{-1}\left(K(1+2\epsilon)t\right)} < \frac{1}{2}\frac{K_\epsilon - \frac{2w_\epsilon}{K_\epsilon}}{K(1+\epsilon)} =: \kappa(\epsilon), \quad t \geq T_4(\epsilon). \tag{6.7.24}$$

Note that $\kappa(\epsilon) > 0$ if and only if

$$\frac{(1+2\epsilon)K^2}{1+\epsilon} > 2\bar{w}(\epsilon) > 2w(0).$$

Thus, since $K > \sqrt{2w(0)}$, $0 < \kappa(\epsilon) < 1$ for each $\epsilon > 0$ sufficiently small. Let $T(\epsilon) = T_3(\epsilon) + T_4(\epsilon)$ and define the comparison solution

$$x_\epsilon(t) = F_U^{-1}\left(K_\epsilon(t-T) + F_U(x_*)\right), \quad t \geq T(\epsilon),$$

where $x_* = 1 + F_U^{-1}\left(K_\epsilon T(\epsilon)\right) + x_+(T(\epsilon)+\epsilon)$. By definition, $x_\epsilon(t) > F_U^{-1}(K_\epsilon t)$ for $t \geq T(\epsilon)$. Hence

$$\frac{F_U^{-1}\left(K(1+\epsilon)t\right)}{x_\epsilon(t)} < \frac{F_U^{-1}\left(K(1+\epsilon)t\right)}{F_U^{-1}\left(K(1+2\epsilon)t\right)} < \kappa(\epsilon), \quad t \geq T(\epsilon), \tag{6.7.25}$$

by equation (6.7.24).

Remark 6.7.3. *We claim $x \mapsto \phi(x)/x = \left(\bar{F}(x)\right)^{1/2}/x$ is increasing for $x > 0$. Define*

$$\Psi(x) = \frac{\phi(x)^2}{x^2} = \frac{\int_0^x f(u)\,du}{x^2}, \quad x > 0$$

Continuity of f means that $\Psi \in C^1((0,\infty);(0,\infty))$ and

$$\Psi'(x) = \frac{xf(x) - 2\int_0^x f(u)\,du}{x^3}, \quad x > 0.$$

Since $x \mapsto f(x)/x$ is increasing, $f(u)/u < f(x)/x$, for each $u \in [0,x)$. It follows that

$$\int_0^x f(u)\,du \le \int_0^x \frac{u}{x} f(x)\,du = \frac{f(x)}{x}\frac{x^2}{2} = \frac{xf(x)}{2},$$

and therefore $\Psi'(x) > 0$ for each $x > 0$.

Since ϕ and $x \mapsto \phi(x)/x$ are increasing, (6.7.25) implies

$$\frac{\phi\left(F_U^{-1}(K(1+\epsilon)t)\right)}{\phi(x_\epsilon(t))} < \frac{\phi(\kappa(\epsilon)x_\epsilon(t)}{\phi(x_\epsilon(t))} = \kappa(\epsilon)\frac{\phi(\kappa(\epsilon)x_\epsilon(t))/\kappa(\epsilon)x_\epsilon(t)}{\phi(x_\epsilon(t))/x_\epsilon(t)} < \kappa(\epsilon), \quad t \ge T(\epsilon).$$

Therefore

$$\phi\left(F_U^{-1}(K(1+\epsilon)t)\right) < \kappa(\epsilon)\phi(x_\epsilon(t)), \quad t \ge T(\epsilon). \tag{6.7.26}$$

By virtue of monotonicity,

$$x_\epsilon(t) \ge x_\epsilon(T(\epsilon)) = x_*(\epsilon) > x_+(T+\epsilon) \ge x_+(t), \quad t \in [T, T+\epsilon].$$

We now claim that

$$x_\epsilon'(t) > K(1+\epsilon)\phi\left(F_U^{-1}(K(1+\epsilon)t)\right) + w_\epsilon \int_{t-\epsilon}^t f(x_\epsilon(s))\,ds, \quad t \ge T(\epsilon) + \epsilon.$$

By construction, $x_\epsilon'(t) = K(1+2\epsilon)\phi(x_\epsilon(t))$ for each $t > T(\epsilon)$. Thus, for $t \ge T(\epsilon)+\epsilon$,

$$\begin{aligned} w_\epsilon \int_{t-\epsilon}^t f(x_\epsilon(s))\,ds &= w_\epsilon \int_{x_\epsilon(t-\epsilon)}^{x_\epsilon(t)} \frac{f(u)}{K(1+2\epsilon)\phi(u)}\,du = \frac{2w_\epsilon}{K_\epsilon}\int_{x_\epsilon(t-\epsilon)}^{x_\epsilon(t)} \phi'(u)\,du \\ &= \frac{2w_\epsilon}{K_\epsilon}\left(\phi(x_\epsilon(t)) - \phi(x_\epsilon(t-\epsilon))\right), \end{aligned}$$

where we have used the fact that $f(x) = 2\phi(x)\phi'(x)$. Positivity then yields

$$w_\epsilon \int_{t-\epsilon}^t f(x_\epsilon(s))\,ds < \frac{2w_\epsilon}{K_\epsilon}\phi(x_\epsilon(t)), \quad t \ge T(\epsilon)+\epsilon.$$

Therefore

$$\begin{aligned} &x_\epsilon'(t) - K(1+\epsilon)\phi\left(F_U^{-1}(K(1+\epsilon)t)\right) - w_\epsilon \int_{t-\epsilon}^t f(x_\epsilon(s))\,ds \\ &\qquad > K_\epsilon\phi(x_\epsilon(t)) - (1+\epsilon)K\phi\left(F_U^{-1}(K(1+\epsilon)t)\right) - \frac{2w_\epsilon}{K_\epsilon}\phi(x_\epsilon(t)) > \frac{1}{2}\left(K_\epsilon - \frac{2w_\epsilon}{K_\epsilon}\right) > 0, \end{aligned}$$

for $t \ge T(\epsilon)+\epsilon$, as required. Finally, we have established that $x(t) < x_\epsilon(t)$ for each $t \ge T(\epsilon)$ and hence $F_U(x(t)) < F_U(x_\epsilon(t)) = K(1+2\epsilon)(t-T) + F_U(x_*(\epsilon))$ for each $t \ge T(\epsilon)$. Taking the limsup and letting $\epsilon \to 0^+$ yields

$$\limsup_{t\to\infty} \frac{F_U(x(t))}{t} \le K,$$

as claimed. □

Proof of Theorem 6.4.4. By hypothesis, there exists a $T_1(\epsilon) > 0$ such that $H(t) < (1+\epsilon)\tilde{H}(t)$ for each $t \ge T_1(\epsilon)$. Now follow the proof of Theorem 6.4.3 with $\bar{H}(t) = (1+\epsilon)\tilde{H}(t)$ down to equation (6.7.14).

Doing so shows $x(t) < x_+(t)$ for each $t \geq T_1(\epsilon)$, where

$$x_+(t) = 1 + x^* + (1+\epsilon)\tilde{H}(t) + \int_{T_1}^t W(t-s)f(x_+(s))\,ds, \quad t \geq T_1(\epsilon).$$

Furthermore, with $\bar{w}(\epsilon) := \sup_{s\in[0,\epsilon]} w(s)$ for each $\epsilon > 0$, x_+ obeys

$$x_+'(t) < (1+\epsilon)\tilde{H}'(t) + (1+\epsilon)\bar{w}(\epsilon)\int_{t-\epsilon}^t f(x_+(s))\,ds, \quad t \geq T_2(\epsilon) > T_1(\epsilon)+\epsilon,$$

where $T_2(\epsilon)$ is chosen so that $\mathcal{W}f(x_+(t-\epsilon)) < \epsilon\bar{w}(\epsilon)\int_{t-\epsilon}^t f(x_+(s))\,ds$ for $t \geq T_2(\epsilon)$. It is possible to find such a $T_2(\epsilon)$ since

$$\lim_{t\to\infty} \frac{\mathcal{W}f(x_+(t-\epsilon))}{\bar{w}(\epsilon)\int_{t-\epsilon}^t f(x_+(s))\,ds} = 0, \quad \text{for each } \epsilon > 0,$$

by the usual considerations. Now let x_ϵ be the solution to

$$x_\epsilon'(t) = (1+\epsilon)\tilde{H}'(t) + (1+\epsilon)\bar{w}(\epsilon)\int_{t-\epsilon}^t f(x_\epsilon(s))\,ds, \quad t \geq T_2(\epsilon), \tag{6.7.27}$$

where $x_\epsilon(t) = 1 + \max_{s\in[T_2-\epsilon,T_2]} x_+(s)$ for $t \in [T_2-\epsilon, T_2]$. By construction, $x_+(t) < x_\epsilon(t)$ for each $t \geq T_2 - \epsilon$ and hence $x(t) < x_+(t) < x_\epsilon(t)$ for each $t \geq T_2$. Let $H_\epsilon(t) = (1+\epsilon)\tilde{H}(t)$ and $w_\epsilon = (1+\epsilon)\bar{w}(\epsilon)$, and integrate (6.7.27) to obtain

$$x_\epsilon(t) = x_\epsilon(T_2) + H_\epsilon(t) - H_\epsilon(T_2) + w_\epsilon\int_{T_2}^t\int_{s-\epsilon}^s f(x_\epsilon(u))\,du\,ds, \quad t \geq T_2(\epsilon).$$

If

$$\liminf_{t\to\infty} \frac{x_\epsilon(t)}{H_\epsilon(t)} = \infty, \tag{6.7.28}$$

then $H_\epsilon(t)/x_\epsilon(t) \to 0$ as $t \to \infty$ and hence

$$\lim_{t\to\infty} \frac{w_\epsilon\int_{T_2}^t\int_{s-\epsilon}^s f(x_\epsilon(u))\,du\,ds}{x_\epsilon(t)} = 1. \tag{6.7.29}$$

It is easily shown that x_ϵ and $f\circ x_\epsilon$ are superexponential. Define $J(t) = \int_{t-\epsilon}^t f(x_\epsilon(u))\,du$ for $t \geq T_2(\epsilon)$. Since $f\circ x_\epsilon$ is superexponential,

$$\lim_{t\to\infty} \frac{J'(t)}{f(x_\epsilon(t))} = 1.$$

Hence for every $\eta \in (0,1)$ there exists a $T_6(\eta)$ such that

$$(1-\eta)f(x_\epsilon(t)) < J'(t) < (1+\eta)f(x_\epsilon(t)),$$

and, owing to (6.7.29), we can also ensure that $T_6(\eta)$ is sufficiently large that

$$(1-\eta)w_\epsilon\int_{T_2}^t J(s)\,ds < x_\epsilon(t) < (1+\eta)w_\epsilon\int_{T_2}^t J(s)\,ds,$$

for each $t \geq T_6(\eta)$. Define $K(t) = w_\epsilon\int_{T_2}^t J(s)\,ds$ for $t \geq T_2(\epsilon)$, so $K'(t) = w_\epsilon J(t) > 0$ and $K''(t) = w_\epsilon J'(t)$ for $t > T_2(\epsilon)$. Thus, for $t \geq T_6(\eta)$,

$$(1-\eta)w_\epsilon f(x_\epsilon(t)) < K''(t) < (1+\eta)w_\epsilon f(x_\epsilon(t)),$$

and
$$(1-\eta)K(t) < x_\epsilon(t) < (1+\eta)K(t).$$

Hence, for $t \geq T_6(\eta)$, $(1-\eta)w_\epsilon f((1-\eta)K(t)) < K''(t) < w_\epsilon f((1+\eta)K(t))$. Let $K_\eta(t) = (1+\eta)K(t)$ and calculate as follows:

$$\frac{d}{dt}\left\{\frac{1}{2}K_\eta'(t)^2 - (1+\eta)w_\epsilon \bar{F}(K_\eta(t))\right\} = K_\eta'(t)\left\{K_\eta''(t) - (1+\eta)w_\epsilon f(K_\eta(t))\right\} < 0,$$

for $t \geq T_6(\eta)$. Therefore

$$\frac{1}{2}K_\eta'(t)^2 - (1+\eta)w_\epsilon \bar{F}(K_\eta(t)) \leq \frac{1}{2}K_\eta'(T_6)^2 - (1+\eta)w_\epsilon \bar{F}(K_\eta(T_6)), \quad t \geq T_6(\eta),$$

and it follows that
$$\limsup_{t\to\infty} \frac{K_\eta'(t)}{\left(\bar{F}(K_\eta(t))\right)^{1/2}} \leq \sqrt{2(1+\eta)w_\epsilon}.$$

Standard asymptotic integration immediately shows that

$$\limsup_{t\to\infty} \frac{F_U(K_\eta(t))}{t} \leq \sqrt{2(1+\eta)w_\epsilon}.$$

Since $x_\epsilon(t) < (1+\eta)K(t) = K_\eta(t)$ for t sufficiently large, $\limsup_{t\to\infty} F_U(x_\epsilon(t))/t \leq \sqrt{2(1+\eta)w_\epsilon}$ owing to the monotonicity of F_U. Now let $\eta \to 0^+$ to obtain

$$\limsup_{t\to\infty} \frac{F_U(x_\epsilon(t))}{t} \leq \sqrt{2w_\epsilon},$$

We have assumed (6.7.28) holds and hence there exists $T(M,\epsilon)$ such that $x_\epsilon(t) > M\tilde{H}(t)$ for any $M > 1$ when $t \geq T(M,\epsilon)$. Therefore

$$\limsup_{t\to\infty} \frac{F_U(M\tilde{H}(t))}{t} \leq \sqrt{2w_\epsilon},$$

for any $M > 1$. Since $\tilde{H}$ is ϵ–independent and $F_U \in \mathrm{RV}_\infty(0)$, this implies

$$\limsup_{t\to\infty} \frac{F_U(\tilde{H}(t))}{t} \leq \sqrt{2w(0)},$$

by letting $\epsilon \to 0^+$. However, by (6.4.7), $\lim_{t\to\infty} F_U(\tilde{H}(t))/t = \infty$, a contradiction. Thus (6.7.28) cannot hold and we must have
$$1 \leq \liminf_{t\to\infty} \frac{x_\epsilon(t)}{H_\epsilon(t)} =: \lambda < \infty.$$

Suppose $\lambda > 1$. Hence $\limsup_{t\to\infty} H_\epsilon(t)/x_\epsilon(t) = 1/\lambda \in (0,1)$ and moreover

$$\liminf_{t\to\infty} \frac{w_\epsilon \int_{T_2}^t \int_{s-\epsilon}^s f(x_\epsilon(u))\,du\,ds}{x_\epsilon(t)} = 1 - \limsup_{t\to\infty} \frac{H_\epsilon(t)}{x_\epsilon(t)} = 1 - \frac{1}{\lambda} > 0.$$

In our previous notation this reads

$$\liminf_{t\to\infty} \frac{K(t)}{x_\epsilon(t)} = 1 - \frac{1}{\lambda} > 0.$$

Recalling that $\lim_{t\to\infty} J'(t)/f(x_\epsilon(t)) = 1$ independently of (6.7.28), there exists $T_8(\eta)$ such that

$$(1-\eta)f(x_\epsilon(t)) < J'(t) < f(x_\epsilon(t)), \quad t \geq T_8(\eta),$$

and $T_9(\eta)$ such that
$$\frac{K(t)}{x_\epsilon(t)} > \left(1-\frac{1}{\lambda}\right)(1-\eta) =: \frac{1}{\Lambda_\eta}, \quad t \geq T_9(\eta).$$
Let $T_{10}(\eta) = T_8(\eta)+T_9(\eta)$ so that $x_\epsilon(t) < \Lambda_\eta K(t)$ for each $t \geq T_{10}(\eta)$. Therefore $K''(t) < w_\epsilon f(x_\epsilon(t)) < w_\epsilon f(\Lambda_\eta K(t))$ for $t \geq T_{10}(\eta)$. If $K_\eta(t) := \Lambda_\eta K(t)$, then $K_\eta''(t) = w_\epsilon \Lambda_\eta K''(t) < w_\epsilon \Lambda_\eta f(K_\eta(t))$ for $t \geq T_{10}(\eta)$. Thus, for $t \geq T_{10}(\eta)$, $K_\eta'(t) = \Lambda_\eta K'(t) = w_\epsilon \Lambda_\eta J(t) > 0$. Furthermore,
$$\frac{d}{dt}\left\{\frac{1}{2}K_\eta'(t)^2 - w_\epsilon \Lambda_\eta \bar{F}(K_\eta(t))\right\} = K_\eta'(t)\left\{K_\eta''(t) - w_\epsilon\Lambda_\eta f(K_\eta(t))\right\} < 0, \quad t \geq T_{10}(\eta).$$
As before, it is now straightforward to establish that
$$\limsup_{t\to\infty} \frac{K_\eta'(t)}{\left(\bar{F}(K_\eta(t))\right)^{1/2}} \leq \sqrt{2w_\epsilon \Lambda_\eta},$$
and by asymptotic integration
$$\limsup_{t\to\infty} \frac{F_U(K_\eta(t))}{t} = \limsup_{t\to\infty} \frac{F_U(\Lambda_\eta K(t))}{t} \leq \sqrt{2w_\epsilon \Lambda_\eta}.$$
Since $x_\epsilon(t) < \Lambda_\eta K(t)$, $\limsup_{t\to\infty} F_U(x_\epsilon(t))/t \leq \sqrt{2w_\epsilon\Lambda_\eta}$ for $\eta < (\lambda-1)\wedge 1$. Now $x_\epsilon(t) > (\lambda-\eta)H_\epsilon(t) > H_\epsilon(t)$ for $t \geq T_{10}(\eta)$. Hence
$$\limsup_{t\to\infty} \frac{F_U(H_\epsilon(t))}{t} \leq \sqrt{2w_\epsilon \Lambda_\eta},$$
and letting $\eta \to 0^+$ yields
$$\limsup_{t\to\infty} \frac{F_U(\tilde{H}(t))}{t} \leq \limsup_{t\to\infty} \frac{F_U((1+\epsilon)\tilde{H}(t))}{t} \leq \sqrt{2w_\epsilon\Lambda_0} = \sqrt{2w_\epsilon}\sqrt{\frac{\lambda}{\lambda-1}},$$
by monotonicity of F_U. Thus letting $\epsilon \to 0^+$ gives
$$\limsup_{t\to\infty} \frac{F_U(\tilde{H}(t))}{t} \leq \sqrt{2w(0)}\sqrt{\frac{\lambda}{\lambda-1}}.$$
However, by (6.4.7), $\lim_{t\to\infty} F_U(\tilde{H}(t))/t = \infty$, a contradiction. Therefore $\lambda = 1$, or in other words, $\liminf_{t\to\infty} x_\epsilon(t)/H_\epsilon(t) = 1$. Hence $\liminf_{t\to\infty} x_\epsilon(t)/\tilde{H}(t) = 1+\epsilon$. Next define u_ϵ by
$$u_\epsilon(t) = x_\epsilon(t) - (1+\epsilon)\tilde{H}(t), \quad t \geq T_2(\epsilon).$$
It follows that $u_\epsilon'(t) = x_\epsilon'(t) - (1+\epsilon)\tilde{H}(t) = w_\epsilon \int_{t-\epsilon}^t f(x_\epsilon(s))\,ds > 0$ for $t \geq T_2(\epsilon)$ and
$$u_\epsilon''(t) = w_\epsilon\left\{f(x_\epsilon(t)) - f(x_\epsilon(t-\epsilon))\right\} > 0, \quad t \geq T_2(\epsilon). \tag{6.7.30}$$
Thus there exists $T_3(\epsilon) > T_2(\epsilon)$ such that $u_\epsilon(t) > 0$ for $t \geq T_3(\epsilon)$. Moreover, $\liminf_{t\to\infty} u_\epsilon(t)/\tilde{H}(t) = 0$. Next, let $t \geq T_3$ be arbitrary, and for any $\theta > T$ and $t \in [T,\theta]$, define
$$V(t,\theta) = \frac{1}{2}u_\epsilon'(t)^2 - w_\epsilon \bar{F}\left(u_\epsilon(t) + (1+\epsilon)\tilde{H}(\theta)\right).$$
Assume for the moment that $T \geq T_3$ is arbitrary. Differentiate V to obtain
$$\frac{d}{dt}V(t,\theta) = u_\epsilon'(t)\left\{u_\epsilon''(t) - w_\epsilon f\left(u_\epsilon(t) + (1+\epsilon)\tilde{H}(\theta)\right)\right\}.$$

Now, from (6.7.30), $u_\epsilon''(t) < w_\epsilon f(x_\epsilon(t)) = w_\epsilon f\left(u_\epsilon(t) + (1+\epsilon)\tilde{H}(t)\right)$. Since $u_\epsilon'(t) > 0$, monotonicity of $\tilde{H}$ and f imply that

$$\frac{d}{dt}V(t,\theta) < u_\epsilon'(t) w_\epsilon \left\{f\left(u_\epsilon(t) + (1+\epsilon)\tilde{H}(t)\right) - f\left(u_\epsilon(t) + (1+\epsilon)\tilde{H}(\theta)\right)\right\} \le 0,$$

for $t \in [T,\theta]$. Therefore

$$\frac{d}{dt}V(t,\theta) \le 0, \quad t \in [T,\theta].$$

Hence $V(t,\theta) \le V(T,\theta)$ for $t \in [T,\theta]$. Setting $\theta = t$ gives $V(t,t) \le V(T,t)$ for $t \ge T$. Thus

$$\frac{1}{2}u_\epsilon'(t)^2 - w_\epsilon \bar{F}\left(u_\epsilon(t) + (1+\epsilon)\tilde{H}(t)\right) \le \frac{1}{2}u_\epsilon'(T)^2 - w_\epsilon \bar{F}\left(u_\epsilon(T) + (1+\epsilon)\tilde{H}(t)\right).$$

Since $T \ge T_3(\epsilon)$, $u_\epsilon(T) > 0$ and by monotonicity

$$\bar{F}\left(u_\epsilon(T) + (1+\epsilon)\tilde{H}(t)\right) > \bar{F}\left((1+\epsilon)\tilde{H}(t)\right), \quad t \ge T.$$

It follows that, for all $t \ge T$,

$$\frac{1}{2}u_\epsilon'(t)^2 - w_\epsilon \bar{F}\left(u_\epsilon(t) + (1+\epsilon)\tilde{H}(t)\right) \le \frac{1}{2}u_\epsilon'(T)^2 - w_\epsilon \bar{F}\left((1+\epsilon)\tilde{H}(t)\right).$$

Hence, for $t \ge T > T_3(\epsilon)$,

$$\frac{1}{2}u_\epsilon'(t)^2 \le \frac{1}{2}u_\epsilon'(T)^2 + w_\epsilon \left\{\bar{F}\left(u_\epsilon(t) + (1+\epsilon)\tilde{H}(t)\right) - \bar{F}\left((1+\epsilon)\tilde{H}(t)\right)\right\}.$$

Fixing $T = T_3(\epsilon)$ yields

$$\frac{1}{2}u_\epsilon'(t)^2 \le \frac{1}{2}u_\epsilon'(T_3)^2 + w_\epsilon \left\{\bar{F}\left(u_\epsilon(t) + (1+\epsilon)\tilde{H}(t)\right) - \bar{F}\left((1+\epsilon)\tilde{H}(t)\right)\right\}, \quad t \ge T_3(\epsilon). \tag{6.7.31}$$

Since $u_\epsilon(t) > 0$, $u_\epsilon'(t) > 0$ and $u_\epsilon''(t) > 0$, for $t \ge T_2(\epsilon)$, we have $\lim_{t\to\infty} u_\epsilon(t) = \infty$. By the mean value theorem there exists $\theta_{\epsilon,t} \in [0, u_\epsilon(t)]$ such that

$$\begin{aligned} a_\epsilon(t) &:= w_\epsilon \left\{\bar{F}\left(u_\epsilon(t) + (1+\epsilon)\tilde{H}(t)\right) - \bar{F}\left((1+\epsilon)\tilde{H}(t)\right)\right\} \\ &= w_\epsilon \bar{F}'\left((1+\epsilon)\tilde{H}(t) + \theta_{\epsilon,t}\right) u_\epsilon(t) > w_\epsilon f\left((1+\epsilon)\tilde{H}(t)\right) u_\epsilon(t), \end{aligned}$$

and therefore $\lim_{t\to\infty} a_\epsilon(t) = \infty$. Now, for every $\eta > 0$, there is a $\tilde{T}_4(\eta,\epsilon) > T_3(\epsilon)$ such that

$$\frac{1}{2}u_\epsilon'(T_3)^2 < \eta a_\epsilon(t), \quad t \ge \tilde{T}_4(\eta,\epsilon).$$

Fix $\eta = \epsilon$ so that $T_4(\epsilon) := \tilde{T}_4(\eta,\epsilon)$ and recall that $H_\epsilon(t) = (1+\epsilon)\tilde{H}(t)$. Thus

$$\frac{1}{2}u_\epsilon'(t)^2 \le \frac{1}{2}u_\epsilon'(T_3)^2 + a_\epsilon(t) = w_\epsilon(1+\epsilon)\left\{\bar{F}\left(u_\epsilon(t) + H_\epsilon(t)\right) - \bar{F}\left(H_\epsilon(t)\right)\right\}, \quad t \ge T_4(\epsilon).$$

Therefore

$$u_\epsilon'(t) < \sqrt{2w_\epsilon(1+\epsilon)}\left\{\bar{F}\left(u_\epsilon(t) + H_\epsilon(t)\right) - \bar{F}\left(H_\epsilon(t)\right)\right\}^{1/2} < \sqrt{2w_\epsilon(1+\epsilon)}\bar{F}(x_\epsilon(t))^{1/2},$$

for $t \ge T_4(\epsilon)$, because $u_\epsilon(t) = x_\epsilon(t) - (1+\epsilon)\tilde{H}(t) = x_\epsilon(t) - H_\epsilon(t)$. Thus

$$x_\epsilon'(t) = u_\epsilon'(t) + H_\epsilon'(t) < \sqrt{2w_\epsilon(1+\epsilon)}\bar{F}(x_\epsilon(t))^{1/2} + H_\epsilon'(t), \quad t \ge T(\epsilon). \tag{6.7.32}$$

Finally, apply Theorem A.2.1 to the ordinary differential inequality (6.7.32) – this can be done rigorously in terms of the related differential equation (i.e. equality in (6.7.32)) with a trivial comparison argument. The key hypothesis for Theorem A.2.1 is (A.2.4) and with regard to (6.7.32) this amounts to checking that (6.4.7) holds. Thus applying Theorem A.2.1 to x_ϵ yields

$$\limsup_{t\to\infty} \frac{x_\epsilon(t)}{(1+\epsilon)\tilde{H}(t)} \le 1.$$

Since $x(t) < x_\epsilon(t)$ for $t \ge T(\epsilon)$, it follows that

$$\limsup_{t\to\infty} \frac{x(t)}{\tilde{H}(t)} \le 1+\epsilon.$$

Send $\epsilon \to 0^+$ in the estimate to show that $\limsup_{t\to\infty} x(t)/H(t) \le 1$. Since $\liminf_{t\to\infty} x(t)/H(t) \ge 1$ by (6.4.2), this completes the proof. $\square$

Proof of Proposition 6.2.2. $(i.)$ Suppose g exhibits superexponential growth and $x \mapsto \phi(x)/x$ is increasing on $[X,\infty)$. By positivity, $\lim_{x\to\infty} g(x)/g(x-\epsilon) = \infty$ for each $\epsilon > 0$. Hence there exists $T^*(\epsilon, X) > 0$ such that $g(x) > g(x-\epsilon) > X^*$ for each $x \ge T^*(\epsilon, X)$. Since $x \mapsto \phi(x)/x$ is increasing,

$$\frac{\phi(g(x-\epsilon))}{g(x-\epsilon)} < \frac{\phi(g(x))}{g(x)}, \quad x \ge T^*(\epsilon, X).$$

But the inequality above holds if and only if

$$\frac{\phi(g(x-\epsilon))}{\phi(g(x))} < \frac{g(x-\epsilon)}{g(x)}, \quad x \ge T^*(\epsilon, X).$$

Letting $x \to \infty$ in the inequality above completes the proof of part $(i.)$.

$(ii.)$ If ϕ is increasing and convex, then there exists an increasing and positive function Φ such that

$$\phi(x) = \phi(0) + \int_0^x \Phi(u)\, du, \quad x \ge 0.$$

If g exhibits superexponential growth there exists $T(\epsilon, M) > 0$ such that

$$0 < g(x-\epsilon) < \frac{g(x)}{M}, \quad x \ge T(\epsilon, M),$$

for an arbitrary $M > 1$. For $x \ge T$, we have

$$\frac{\phi(g(x-\epsilon))}{\phi(g(x))} < \frac{\phi(g(x)/M)}{\phi(g(x))} = \frac{\phi(0) + \int_0^{g(x)/M} \Phi(u)\, du}{\phi(0) + \int_0^{g(x)} \Phi(u)\, du}.$$

However, note that

$$\int_0^{g(x)/M} \Phi(u)\, du = \frac{1}{M}\int_0^{g(x)} \Phi(y/M)\, dy \le \frac{1}{M}\int_0^{g(x)} \Phi(y)\, dy,$$

with the monotonicity of Φ (and $M > 1$) clinching the final inequality. Combining this with our earlier estimate yields

$$\frac{\phi(g(x-\epsilon))}{\phi(g(x))} < \frac{\phi(0) + \frac{1}{M}\int_0^{g(x)} \Phi(u)\, du}{\phi(0) + \int_0^{g(x)} \Phi(u)\, du}, \quad x \ge T(\epsilon, M).$$

Therefore

$$\limsup_{x\to\infty} \frac{\phi(g(x-\epsilon))}{\phi(g(x))} \le \frac{1}{M},$$

and since $M > 1$ was arbitrary, this completes the proof of part $(ii.)$.

$(iii.)$ First assume that g exhibits superexponential growth, so that

$$\lim_{x\to\infty} \frac{g(x-\epsilon)}{g(x)} = 0, \quad \text{for each } \epsilon > 0.$$

Fix $\epsilon > 0$. For each $\eta > 0$, there exists $T^*(\eta) > 0$ such that

$$0 < g(x) < \eta g(x-\epsilon), \quad x \geq T^*(\eta).$$

Since $\phi \in \mathrm{RV}_\infty(\alpha)$ for some $\alpha > 0$, there exists a monotone increasing function $\varphi \in \mathrm{RV}_\infty(\alpha)$ such that $\lim_{x\to\infty} \varphi(x)/\phi(x) = 1$. Hence

$$\frac{\varphi(g(x-\epsilon))}{\varphi(g(x))} < \frac{\varphi(\eta g(x))}{\varphi(g(x))}, \quad x \geq T^*(\eta).$$

Thus

$$\limsup_{x\to\infty} \frac{\varphi(g(x-\epsilon))}{\varphi(g(x))} \leq \limsup_{x\to\infty} \frac{\varphi(\eta g(x))}{\varphi(g(x))} = \eta^\alpha,$$

where we have used that $g(x) \to \infty$ as $x \to \infty$ and that $\varphi \in \mathrm{RV}_\infty(\alpha)$. Now let $\eta \to 0^+$ to see that

$$\lim_{x\to\infty} \frac{\varphi(g(x-\epsilon))}{\varphi(g(x))} = 0, \quad \text{for each } \epsilon > 0.$$

Finally, observe that

$$0 = \lim_{x\to\infty} \frac{\varphi(g(x-\epsilon))}{\varphi(g(x))} = \lim_{x\to\infty} \frac{\varphi(g(x-\epsilon))}{\phi(g(x-\epsilon))} \frac{\phi(g(x-\epsilon))}{\phi(g(x))} \frac{\phi(g(x))}{\varphi(g(x))} = \lim_{x\to\infty} \frac{\phi(g(x-\epsilon))}{\phi(g(x))},$$

since $\lim_{x\to\infty} \varphi(x)/\phi(x) = 1$ and $g(x) \to \infty$ as $x \to \infty$. □

Proof of Proposition 6.4.1. Suppose $\alpha_K \in (0,\infty)$ for some (fixed) $K > 1$. By (6.4.10), for each $\epsilon \in (0,1)$ there exists a $T(\epsilon) > 0$ such that

$$(1-\epsilon)K\alpha_K H(t) < \int_0^t \bar{F}(KH(s))^{1/2} ds < (1+\epsilon)K\alpha_K H(t), \quad t \geq T(\epsilon). \tag{6.7.33}$$

Hence

$$H(t) < \frac{1}{(1-\epsilon)K\alpha_K} \int_0^t \bar{F}(KH(s))^{1/2} ds, \quad t \geq (T_\epsilon).$$

Define the upper comparison solution H_ϵ as the solution to the integral equation

$$H_\epsilon(t) = H_\epsilon(T) + \frac{1}{(1-\epsilon)^2 K\alpha_K} \int_T^t \bar{F}(KH_\epsilon(s))^{1/2} ds, \quad t \geq T(\epsilon),$$
$$H_\epsilon(T) = 1 + \sup_{u\in[0,T(\epsilon)]} H(u) + \sup_{u\in[0,T(\epsilon)]} \frac{1}{(1-\epsilon)K\alpha_K} \int_0^u \bar{F}(KH(s))^{1/2} ds.$$

By construction, $H(t) < H_\epsilon(t)$ for each $t \geq 0$. Furthermore,

$$H_\epsilon'(t) = \frac{1}{(1-\epsilon)^2 K\alpha_K} \bar{F}(KH_\epsilon(t))^{1/2}, \quad t > T(\epsilon).$$

Integrating the equation above then yields

$$\int_T^t \frac{KH_\epsilon'(s)\, ds}{\bar{F}(KH_\epsilon(t))^{1/2}} = \frac{t-T}{(1-\epsilon)^2 \alpha_K}, \quad t \geq T(\epsilon).$$

Integration by substitution quickly reveals that

$$F_U(KH_\epsilon(t)) - F_U(KH_\epsilon(T)) = \frac{t-T}{(1-\epsilon)^2\alpha_K}, \quad t \geq T(\epsilon).$$

Since $F_U \in \mathrm{RV}_\infty(0)$, we obtain

$$\lim_{t\to\infty} \frac{F_U(H_\epsilon(t))}{t} = \frac{1}{(1-\epsilon)^2\alpha_K}.$$

Now since $H(t) < H_\epsilon(t)$ for $t \geq 0$ and F_U is increasing,

$$\limsup_{t\to\infty} \frac{F_U(H(t))}{t} \leq \lim_{t\to\infty} \frac{F_U(H_\epsilon(t))}{t} = \frac{1}{(1-\epsilon)^2\alpha_K}.$$

Letting $\epsilon \to 0^+$ then shows that

$$\limsup_{t\to\infty} \frac{F_U(H(t))}{t} \leq \frac{1}{\alpha_K}.$$

For the corresponding limit inferior, begin by observing that (6.7.33) implies

$$H(t) > \frac{1}{(1+\epsilon)K\alpha_K} \int_T^t \bar{F}(KH(s))^{1/2} ds, \quad t \geq T(\epsilon).$$

Now define the lower comparison solution H_ϵ by

$$H_\epsilon(t) = \frac{H(T)}{2} + \frac{1}{(1+\epsilon)^2 K\alpha_K} \int_T^t \bar{F}(KH(s))^{1/2} ds, \quad t \geq T(\epsilon).$$

By construction, $H_\epsilon(t) < H(t)$ for $t \geq T(\epsilon)$ and an asymptotic integration argument analogous to that employed above shows that

$$\lim_{t\to\infty} \frac{F_U(H_\epsilon(t))}{t} = \frac{1}{(1+\epsilon)^2\alpha_K}.$$

It follows readily that

$$\liminf_{t\to\infty} \frac{F_U(H(t))}{t} \geq \frac{1}{\alpha_K},$$

completing the proof when $\alpha_K \in (0,\infty)$. The limiting cases $\alpha_K = 0$ and $\alpha_K = \infty$ can be dealt with using the argument above and formally taking limits at the appropriate moments. $\square$

Appendices

Appendix A

Auxiliary Blow–up Results

A.1 Necessity of Superlinearity for Blow–Up

In this section we assume that the nonlinearity obeys

$$f \in C((0,\infty);(0,\infty)) \text{ and } f \text{ is increasing.} \tag{f_+}$$

We claim that, under (f_+) and (6.1.3), $\lim_{x\to\infty} f(x)/x = \infty$ is a necessary condition for the finite–time blow–up of solutions to (6.1.1); more precisely

Theorem A.1.1. *Suppose (f_+) and (6.1.3) hold, and that there exists a minimal $T \in (0,\infty)$ such that the unique solution to (6.1.1), x, obeys $x \in C([0,T);(0,\infty))$ and $\lim_{t\to T^-} x(t) = \infty$. Then*

$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty.$$

In order to prove Theorem A.1.1, we first show that $f(x)/x \to \infty$ as $x \to \infty$ is a necessary condition for the finite–time blow–up of solutions to delay differential equations of the form (6.6.2).

Theorem A.1.2. *Let $C > 0$ and δ. Suppose (f_+) and (6.6.3) hold, and that there exists a minimal $T \in (0,\infty)$ such that the unique solution to (6.6.2), z, obeys $z \in C([0,T);(0,\infty))$ and $\lim_{t\to T^-} z(t) = \infty$. Then*

$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty.$$

Remark A.1.1. *When f obeys (f_+), by Proposition 6.2.1, we have*

$$\int_1^\infty \frac{dx}{\sqrt{xf(x)}} < \infty \quad \textit{if and only if} \quad \int_1^\infty \frac{dx}{\sqrt{\int_0^x f(s)\,ds}} < \infty.$$

Hence $\int_1^\infty dx/\sqrt{xf(x)} < \infty$ implies that $\lim_{x\to\infty} x/\sqrt{xf(x)} = 0$. It follows that $x/f(x) \to 0$ as $x \to \infty$ and hence that $\lim_{x\to\infty} x/f(x) = 0$. Therefore, if $\int_1^\infty dx/\sqrt{xf(x)} < \infty$, then $\lim_{x\to\infty} f(x)/x = \infty$. We record this fact for use in the proofs of Theorem A.1.1 and Theorem A.1.2.

Proof of Theorem A.1.2. Let C, δ, and ψ be given. Either

$$\lim_{N\to\infty} F_U(N) < \infty \quad \text{or} \quad \lim_{N\to\infty} F_U(N) = \infty.$$

Suppose the latter holds. Since $\lim_{N\to\infty} F_U(N) = \infty$, we may apply the construction of Lemma 6.6.4

(which does not require that $f(x)/x$ as $x \to \infty$) to find a function $\alpha \in C([-\delta,\infty);(0,\infty))$ such that

$$z(t) < \alpha(t), \qquad \text{for each } t \in [-\delta, T).$$

This contradicts the assumption that $T < \infty$ and hence $\lim_{N\to\infty} F_U(N) < \infty$. But from our earlier remark this implies that $\lim_{x\to\infty} f(x)/x = \infty$, as required. □

Proof of Theorem A.1.1. By hypothesis there exists $T \in (0,\infty)$ such that $x \in C([0,T);(0,\infty))$ and $\lim_{t\to T^-} x(t) = \infty$. Hence we can apply the construction from the proof of necessity in Theorem 6.2.1 to show that

$$\limsup_{t\to\infty} \frac{x'(t)}{\bar{w}(\delta)\int_{t-\delta}^{t} f(x(s))\,ds} < \infty, \qquad \text{for each } \delta \in (0,T),$$

where $\bar{w}(\delta) = \sup_{u\in[0,\delta]} w(u)$. Once more following the arguments from the proof of necessity in Theorem 6.2.1, there exists a $C > 0$, $\delta > 0$, and ψ obeying (6.6.3) such that z solves (6.6.2) and obeys

$$x(t) < z(t), \qquad \text{for each } t \in [-\delta, T \wedge T_z),$$

where $T_z \in (0,\infty]$ is the blow–up time of z. But $T < \infty$ forces $T_z < \infty$, or in other words, z must blow–up in finite time. Therefore, we may apply Theorem A.1.2 to z and conclude that $f(x)/x \to \infty$, as claimed. □

A.2 Superlinear ODE Asymptotics

In order to keep the presentation of Chapter 6 relatively self–contained, we state without proof the following result of Appleby and Patterson [15, Theorem 4].

Theorem A.2.1. *Consider the nonlinear ordinary differential equation*

$$x'(t) = f(x(t)) + h(t), \quad t \geq 0; \quad x(0) = \psi > 0. \tag{A.2.1}$$

Suppose that

$$\begin{aligned} &f \in C((0,\infty);(0,\infty)), \quad f \text{ is increasing}, \quad \lim_{x\to\infty} f_1(x) = \infty \\ &\text{and } x \mapsto f_1(x) := f(x)/x \text{ is ultimately increasing,} \end{aligned} \tag{A.2.2}$$

and

$$h \in C((0,\infty);\mathbb{R}), \quad H(t) = \int_0^t h(s)\,ds \geq 0 \text{ for each } t \geq 0. \tag{A.2.3}$$

If the solution to (A.2.1) exists for all $t \in \mathbb{R}^+$, H is asymptotic to an increasing function $\tilde{H}$, and

$$\lim_{t\to\infty} \frac{\int_0^t f(K\tilde{H}(s))\,ds}{\tilde{H}(t)} = 0 \text{ for some } K > 1, \tag{A.2.4}$$

then $\lim_{t\to\infty} x(t)/H(t) = 1$.

Bibliography

[1] D. Acemoglu. *Introduction to Modern Economic Growth.* Princeton University Press, 2008.

[2] R. P. Agarwal. *Difference Equations and Inequalities: Theory, Methods, and Applications.* CRC Press, 2000.

[3] C. W. Anderson. Super-slowly varying functions in extreme value theory. *J. R. Stat. Soc. Ser. B. Stat. Methodol.*, 40(2):197–202, 1978.

[4] J. A. Appleby, M. Riedle, and C. Swords. Bubbles and crashes in a Black–Scholes model with delay. *Finance and Stochastics*, 17(1):1–30, 2013.

[5] J. A. D. Appleby. On regularly varying and history-dependent convergence rates of solutions of a Volterra equation with infinite memory. *Adv. Difference Equ.*, 2010(1):1–31, 2010.

[6] J. A. D. Appleby and E. Buckwar. A constructive comparison technique for determining the asymptotic behaviour of linear functional differential equations with unbounded delay. *Differ. Equ. Dyn. Syst.*, 28(3):271–301, 2010.

[7] J. A. D. Appleby, J. Cheng, and A. Rodkina. Characterisation of the asymptotic behaviour of scalar linear differential equations with respect to a fading stochastic perturbation. *Discrete Contin. Dynam. Syst. Suppl.*, pages 79–90, 2011.

[8] J. A. D. Appleby, I. Győri, and D. W. Reynolds. On exact rates of decay of solutions of linear systems of Volterra equations with delay. *J. Math. Anal. Appl.*, 320(1):56–77, 2006.

[9] J. A. D. Appleby, I. Győri, and D. W. Reynolds. On exact convergence rates for solutions of linear systems of Volterra difference equations. *J. Difference Equ. Appl.*, 12(12):1257–1275, 2006.

[10] J. A. D. Appleby and K. Krol. Long memory in a linear stochastic Volterra differential equation. *J. Math. Anal. Appl.*, 380(2):814–830, 2011.

[11] J. A. D. Appleby, M. J. McCarthy, and A. Rodkina. Growth rates of delay–differential equations and uniform Euler schemes. In *Proceedings of the 14th International Conference on Difference Equations and Applications, Istanbul*, pages 117–124, 2009.

[12] J. A. D. Appleby and D. D. Patterson. Subexponential growth rates in functional differential equations. *Discrete Contin. Dyn. Syst. Suppl.*, pages 56–65, 2015.

[13] J. A. D. Appleby and D. D. Patterson. Growth and fluctuation in perturbed nonlinear Volterra equations. *arXiv preprint arXiv:1612.00515*, pages 1–36, 2016.

[14] J. A. D. Appleby and D. D. Patterson. On the admissibility of unboundedness properties of forced deterministic and stochastic sublinear Volterra summation equations. *Electron. J. Qual. Theory Differ. Equ.*, 2016(63):1–44, 2016.

[15] J. A. D. Appleby and D. D. Patterson. Growth rates of solutions of superlinear ordinary differential equations. *Appl. Math. Lett.*, 71:30–37, 2017.

[16] J. A. D. Appleby and D. D. Patterson. Hartman–Wintner growth results for sublinear functional differential equations. *Electron. J. Differential Equations*, 2017(21):1–45, 2017.

[17] J. A. D. Appleby and D. D. Patterson. Large fluctuations and growth rates of linear Volterra summation equations. *J. Difference Equ. Appl.*, 23(6):1047–1080, 2017.

[18] J. A. D. Appleby and D. D. Patterson. Memory dependent growth in sublinear Volterra differential equations. *J. Integral Equations Appl.*, 29(4):531–584, 2017.

[19] J. A. D. Appleby and D. D. Patterson. Growth rates of sublinear functional and Volterra differential equations. *SIAM J. Math. Anal., forthcoming*, 2018.

[20] J. A. D. Appleby and M. Riedle. Stochastic Volterra differential equations in weighted spaces. *J. Integral Equ. Appl.*, 22(1):1–17, 2010.

[21] R. T. Baillie. Long memory processes and fractional integration in econometrics. *J. Econometrics*, 73(1):5–59, 1996.

[22] R. E. Bellman and K. L. Cooke. *Differential–difference Equations.* Rand Corporation, 1963.

[23] J. Benhabib and A. Rustichini. Vintage capital, investment, and growth. *J. Econ. Theory*, 55(2):323–339, 1991.

[24] V. I. Berdyshev, S. V. Emel'yanov, A. B. Kurzhanski, E. F. Mishchenko, and Y. S. Osipov. Nikolai Nikolaevich Krasovskii. *Proc. Steklov Inst. Math.*, 268(1):1–14, 2010.

[25] J. Bertoin. *Lévy Processes*, volume 121. Cambridge University Press, 1998.

[26] I. Bihari. A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. *Acta Math. Hungar.*, 7(1):81–94, 1956.

[27] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27. Cambridge University Press, 1989.

[28] R. Bojanic and E. Seneta. Slowly varying functions and asymptotic relations. *J. Math. Anal. Appl.*, 34(2):302–315, 1971.

[29] P. J. Brockwell and R. A. Davis. *Time Series: Theory and Methods.* Springer Science & Business Media, 2013.

[30] P. J. Brockwell and A. Lindner. Existence and uniqueness of stationary Lévy-driven CARMA processes. *Stochastic Process. Appl.*, 119(8):2660–2681, 2009.

[31] H. Brunner and S. Maset. Time transformations for delay differential equations. *Discrete Contin. Dyn. Syst.*, 25(3):751–775, 2009.

[32] H. Brunner and Z. Yang. Blow-up behavior of Hammerstein–type Volterra integral equations. *J. Integral Equations Appl.*, 24(4):487, 2012.

[33] J. M. Chadam, A. Peirce, and H.-M. Yin. The blowup property of solutions to some diffusion equations with localized nonlinear reactions. *J. Math. Anal. Appl.*, 169(2):313–328, 1992.

[34] G. E. Chatzarakis, K. Takasi, and I. P. Stavroulakis. Precise asymptotic behavior of strongly decreasing solutions of first-order nonlinear functional differential equations. *Electron. J. Differential Equations*, 2014(206):1–14, 2014.

[35] J. Chover, P. Ney, and S. Wainger. Functions of probability measures. *J. Anal. Math.*, 26(1):255–302, 1973.

[36] F.-C. Cîrstea and V. Rădulescu. Boundary blow-up in nonlinear elliptic equations of Bieberbach–Rademacher type. *Trans. Amer. Math. Soc.*, 359(7):3275–3286, 2007.

[37] C. Corduneanu. *Integral Equations and Stability of Feedback Systems.* Academic Press, Inc., 1973.

[38] C. Corduneanu. *Almost Periodic Functions.* Chelsea Pub. Co., 1989.

[39] C. Corduneanu. *Integral Equations and Applications*, volume 148. Cambridge University Press, 1991.

[40] R. Cross, H. Mc Namara, and A. Pokrovskii. Modelling macroeconomic flows related to large ensembles of elementary exchange operations. *Phys. B*, 403(2):451–455, 2008.

[41] R. Cross, H. McNamara, A. Pokrovskii, and D. Rachinskii. A new paradigm for modelling hysteresis in macroeconomic flows. *Phys. B*, 403(2):231–236, 2008.

[42] J. M. Cushing. *Integrodifferential Equations and Delay Models in Population Dynamics*, volume 20. Springer Science & Business Media, 2013.

[43] H. d'Albis, E. Augeraud-Véron, and H. J. Hupkes. Multiple solutions in systems of functional differential equations. *J. Math. Econom.*, 52:50–56, 2014.

[44] J. Diblík, M. Ružičková, E. Schmeidel, and M. Zbaszyniak. Weighted asymptotically periodic solutions of linear Volterra difference equations. *Abstr. Appl. Anal.*, 2011:1–14, 2011.

[45] Z. Ding and C. W. Granger. Modeling volatility persistence of speculative returns: a new approach. *J. Econometrics*, 73(1):185–215, 1996.

[46] S. Elaydi. *An Introduction to Difference Equations.* Springer Science & Business Media, 2005.

[47] V. Evtukhov and A. Samoilenko. Asymptotic representations of solutions of nonautonomous ordinary differential equations with regularly varying nonlinearities. *Differ. Equ.*, 47(5):627–649, 2011.

[48] É. I. Gol'dengerŝhel'. Discrete analog of an integral equation of Volterra type on the half-axis. *Uspekhi Mat. Nauk*, 21(2):223–225, 1966.

[49] J. R. Graef. Oscillation, nonoscillation, and growth of solutions of nonlinear functional differential equations of arbitrary order. *J. Math. Anal. Appl.*, 60:398–409, 1977.

[50] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra Integral and Functional Equations*, volume 34. Cambridge University Press, 1990.

[51] I. Győri and E. Awwad. On the boundedness of the solutions in nonlinear discrete Volterra difference equations. *Adv. Difference Equ.*, 2012(1):1–20, 2012.

[52] I. Győri and L. Horváth. New limit formulas for the convolution of a function with a measure and their applications. *J. Inequal. Appl.*, 2008(1):1, 2008.

[53] I. Győri and D. W. Reynolds. Sharp conditions for boundedness in linear discrete Volterra equations. *J. Difference Equ. Appl.*, 15(11-12):1151–1164, 2009.

[54] I. Győri and D. W. Reynolds. On admissibility of the resolvent of discrete Volterra equations. *J. Difference Equ. Appl.*, 16(12):1393–1412, 2010.

[55] I. Győri and D. W. Reynolds. On asymptotically periodic solutions of linear discrete Volterra equations. *Fasc. Math.*, 44(1):53–67, 2010.

[56] J. Haldane. A contribution to the theory of price fluctuations. *Rev. Econ. Stud.*, 1(3):186–195, 1934.

[57] J. K. Hale. History of delay equations. In O. Arino, M. Hbid, and E. A. Dads, editors, *Delay Differential Equations and Applications*, pages 1–28. Springer Netherlands, 2006.

[58] K. B. Hannsgen. Indirect Abelian theorems and a linear Volterra equation. *Trans. Amer. Math. Soc.*, 142:539–555, 1969.

[59] P. Hartman. *Ordinary Differential Equations.* SIAM, second edition, 2002.

[60] P. Hartman and A. Wintner. Asymptotic integration of ordinary nonlinear differential equations. *Amer. J. Math.*, 77:692–724, 1955.

[61] H. R. Henríquez. Asymptotically almost-periodic solutions of abstract differential equations. *J. Math. Anal. Appl.*, 160(1):157–175, 1991.

[62] C. I. Jones. R&D-based models of economic growth. *J. Polit. Econ.*, 103(4):759–784, 1995.

[63] C. I. Jones. Time series tests of endogenous growth models. *Q. J. Econ.*, 110(2):495–525, 1995.

[64] C. I. Jones and J. C. Williams. Too much of a good thing? The economics of investment in R&D. *J. Econ. Growth*, 5(1):65–85, 2000.

[65] M. Kalecki. A macroeconomic theory of the business cycle. *Econometrica*, 1(3):327–344, 1935.

[66] M. Kalecki. A theory of the business cycle. *Rev. Econ. Stud.*, 1(4):77–97, 1937.

[67] J. Karamata. Sur un mode de croissance régulière. Théorèmes fondamentaux. *Bull. Soc. Math. France*, 61:55–62, 1933.

[68] C. Kirk and W. Olmstead. Blow-up in a reactive-diffusive medium with a moving heat source. *Z. Angew. Math. Phys.*, 53(1):147–159, 2002.

[69] C. Kirk, W. Olmstead, C. Roberts, et al. A system of nonlinear Volterra equations with blow-up solutions. *J. Integral Equations Appl.*, 25(3):377–393, 2013.

[70] V. Kolmanovskii and A. Myshkis. *Applied Theory of Functional Differential Equations*, volume 85. Springer Science & Business Media, 2012.

[71] Y. Kuang. *Delay Differential Equations: With Applications in Population Dynamics*, volume 191. Academic Press, 1993.

[72] T. Kusano and H. Onose. Oscillatory and asymptotic behavior of sublinear retarded differential equations. *Hiroshima Math. J.*, 3:343—355, 1974.

[73] V. Lakshmikantham and S. Leela. *Differential and Integral Inequalities*, volume 55 I-II. Academic press, 1969.

[74] W. W. Leontief. *The Structure of American economy, 1919-1939.* Oxford University Press, 1949.

[75] W. W. Leontief. *Input–output Economics.* Oxford University Press, 1966.

[76] J. Levin and J. Nohel. On a system of integrodifferential equations occurring in reactor dynamics. *Journal of Mathematics and Mechanics (US) changed to Indiana Univ. Math. J.*, 9(3):347–368, 1960.

[77] H. C. Lin and L. F. Shampine. R&D–based calibrated growth models with finite-length patents: A novel relaxation algorithm for solving an autonomous FDE system of mixed type. *Comput. Econ.*, pages 1–36, 2016.

[78] O. Lipovan. Integral inequalities for retarded Volterra equations. *J. Math. Anal. Appl.*, 322(1):349–358, 2006.

[79] O. Lipovan. Asymptotic properties of solutions to some nonlinear integral equations of convolution type. *Nonlinear Anal.*, 69(7):2179–2183, 2008.

[80] O. Lipovan. On the asymptotic behaviour of solutions to some nonlinear integral equations of convolution type. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal*, 16(2):147–154, 2009.

[81] J. Ma. Blow-up solutions of nonlinear Volterra integro-differential equations. *Math. Comput. Model.*, 54(11):2551–2559, 2011.

[82] M. C. Mackey. Commodity price fluctuations: price dependent delays and nonlinearities as explanatory factors. *J. Econom. Theory*, 48(2):497–509, 1989.

[83] T. Małolepszy. Blow-up solutions in one-dimensional diffusion models. *Nonlinear Anal.*, 95:632–638, 2014.

[84] T. Małolepszy and W. Okrasiński. Blow-up conditions for nonlinear Volterra integral equations with power nonlinearity. *Appl. Math. Lett.*, 21(3):307–312, 2008.

[85] T. Małolepszy and W. Okrasiñski. Blow-up time for solutions to some nonlinear Volterra integral equations. *J. Math. Anal. Appl.*, 366(1):372–384, 2010.

[86] X. Mao. *Stochastic Differential Equations and Applications.* Horwood Pub., 2nd edition, 2007.

[87] V. Marić. *Regular Variation and Differential Equations*, volume 1726. Springer, 2000.

[88] T. Marquardt and R. Stelzer. Multivariate CARMA processes. *Stochastic Process. Appl.*, 117(1):96–120, 2007.

[89] S. Matucci and P. Řehák. Asymptotics of decreasing solutions of coupled p–Laplacian systems in the framework of regular variation. *Ann. Mat. Pura Appl.*, 193(3):837–858, 2014.

[90] S. Matucci and P. Řehák. Extremal solutions to a system of n nonlinear differential equations and regularly varying functions. *Math. Nachr.*, 288(11-12):1413–1430, 2015.

[91] M. Métivier and J. Pellaumail. On a stopped Doob's inequality and general stochastic equations. *Ann. Probab.*, 9(1):96–114, 1980.

[92] W. Mydlarczyk. A condition for finite blow-up time for a Volterra integral equation. *J. Math. Anal. Appl.*, 181(1):248 – 253, 1994.

[93] W. Mydlarczyk. The blow-up solutions of integral equations. *Colloq. Math.*, 79(1):147–156, 1999.

[94] W. Mydlarczyk, W. Okrasiński, and C. Roberts. Blow-up solutions to a system of nonlinear Volterra equations. *J. Math. Anal. Appl.*, 301(1):208–218, 2005.

[95] A. D. Myshkis. General theory of differential equations with retarded arguments. *Uspekhi Mat. Nauk*, 4(5):99–141, 1949.

[96] W. Olmstead and C. A. Roberts. Explosion in a diffusive strip due to a source with local and nonlocal features. *Methods Appl. Anal.*, 3:345–357, 1996.

[97] F. Olver, D. Lozier, R. Boisvert, and C. Clark. *NIST Handbook of Mathematical Functions, US Department of Commerce, National Institute of Standards and Technology, Washington, DC, 2010*. Cambridge University Press, 2010.

[98] B. Pachpatte. *Inequalities for Dfferential and Integral Equations*, volume 197. Academic Press, 1998.

[99] O. Perron. Die stabilitätsfrage bei differentialgleichungen. *Math. Z.*, 32(1):703–728, 1930.

[100] M. Pituk. The Hartman–Wintner theorem for functional differential equations. *J. Differential Equations*, 155(1):1—16, 1999.

[101] P. E. Protter. Stochastic integration and differential equations. *Stoch. Model. Appl. Probab.*, 21, 2004.

[102] V. D. Rădulescu. Singular phenomena in nonlinear elliptic problems: from blow-up boundary solutions to equations with singular nonlinearities. *Handbook of Differential Equations: Stationary Partial Differential Equations*, 4:485–593, 2007.

[103] P. Řehák. Nonlinear differential equations in the framework of regular variation. http://users.math.cas.cz/~rehak/ndefrv/ndefrv.pdf, 2014. Accessed: 05–04–2017.

[104] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*, volume 293. Springer Science & Business Media, 1999.

[105] D. W. Reynolds. On asymptotic constancy for linear discrete summation equations. *Comput. Math. Appl.*, 64(7):2335–2344, 2012.

[106] C. A. Roberts. Characterizing the blow-up solutions for nonlinear Volterra integral equations. *Nonlinear Anal.*, 30(2):923–933, 1997.

[107] C. A. Roberts. Analysis of explosion for nonlinear Volterra equations. *J. Comput. Appl. Math.*, 97(1-2):153–166, 1998.

[108] C. A. Roberts. Recent results on blow-up and quenching for nonlinear Volterra equations. *J. Comput. Appl. Math.*, 205(2):736–743, 2007.

[109] C. A. Roberts and W. Olmstead. Growth rates for blow-up solutions of nonlinear Volterra equations. *Quart. Appl. Math.*, 54(1):153–159, 1996.

[110] P. A. Samuelson. *Les fondements de l'analyse économique, Vol. 2: Stabilité des systemes et théorie dynamique*. Gauthier-Villars, 1971.

[111] W. R. Schneider. The general solution of a non–linear integral equation of convolution type. *Z. Angew. Math. Phys.*, 33(1):140–142, 1982.

[112] D. F. Shea and S. Wainger. Variants of the Wiener-Lévy theorem, with applications to stability problems for some Volterra integral equations. *Amer. J. Math.*, 97(2):312–343, 1975.

[113] R. M. Solow. A contribution to the theory of economic growth. *Q. J. Econ.*, 70(1):65–94, 1956.

[114] Y. Song. Almost periodic solutions of discrete Volterra equations. *J. Math. Anal. Appl.*, 314(1):174–194, 2006.

[115] Y. Song and C. T. Baker. Linearized stability analysis of discrete Volterra equations. *J. Math. Anal. Appl.*, 294(1):310–333, 2004.

[116] Y. Song and C. T. Baker. Admissibility for discrete Volterra equations. *J. Difference Equ. Appl.*, 12(5):433–457, 2006.

[117] K. Takaŝi and J. Manojlović. Precise asymptotic behavior of solutions of the sublinear Emden–Fowler differential equation. *Appl. Math. Comput.*, 217(9):4382–4396, 2011.

[118] H. R. Thieme. Asymptotic estimates of the solutions of nonlinear integral equations and asymptotic speeds for the spread of populations. *J. Reine Angew. Math.*, 306(94):21, 1979.

[119] H. R. Thieme. Density–dependent regulation of spatially distributed populations and their asymptotic speed of spread. *J. Math. Biol.*, 8(2):173–187, 1979.

[120] A. Vecchio. On the resolvent kernel of Volterra discrete equations. *Funct. Differ. Equ.*, 6(1-2):191–201, 1999.

[121] J. S. Wong and R. Wong. Asymptotic solutions of linear Volterra integral equations with singular kernels. *Trans. Amer. Math. Soc.*, 189:185–200, 1974.

[122] Y. Xing, M. Han, and G. Zheng. Initial value problem for first–order integro–differential equation of Volterra type on time scales. *Nonlinear Anal.*, 60(3):429–442, 2005.