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CONTINUOUS SELECTIONS OF SOLUTION SETS TO VOLTERRA INTEGRAL INCLUSIONS IN BANACH SPACES

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Abstract. We consider a nonlinear Volterra integral equation governed by an m-accretive operator and a multivalued perturbation in a separable Banach. The existence of a continuous selection for the corresponding solution map is proved. The case when the m-accretive operator in the integral inclusion depends on time is also discussed.

1. Introduction

In this paper we establish the existence of a continuous selection of the solution set to the nonlinear Volterra integral inclusion

$$u(t) + \int_0^t a(t-s)[Au(s) + F(s,u(s))]ds \ni \xi + g(t), \quad t \in I := [0,T] \tag{1.1}$$

in a Banach space X. Here A denotes an m-accretive operator in X, $F : I \times X \to 2^X \backslash \{\emptyset\}$ is a multivalued perturbation, $a : I \to \mathbb{R}$, $\xi \in X$, $g : I \to X$, and the integral is taken in the sense of Bochner. The case when A depends on time is also considered.

Existence and continuous dependence results for Volterra equations of type (1.1) in infinite dimensional spaces were earlier proved in [2, 3, 1]. Continuous selection theorems for semilinear abstract integrodifferential inclusions have recently been obtained in [4]. As compared to [2], [3], we do not impose any compactness restriction on the semigroup generated by $-A$ (respectively, on the corresponding evolution operator, when A is time-dependent), and allow X to be a general (non-reflexive) Banach space.

We note that in the special case when $a = 1$ and $g = 0$, equation (1.1) reduces to

$$u'(t) + Au(t) + F(t,u(t)) \ni 0, \quad t \in I; u(0) = \xi. \tag{1.2}$$

The existence of continuous selections for the multivalued solution map associated with (1.2) was proved by Staicu [15] in a Hilbert space setting. The present work may be viewed as a direct attempt to extend the theory of [15] to a broader class of nonlinear inclusions in a general Banach space.

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The plan of the paper is as follows. Section 2 contains background material on multifunctions, m-accretive operators and evolution equations. The main results for equation (1.1) and its time dependent counterpart are stated in Section 3. The proofs are carried out in Section 4. Finally, in Section 5 we present an example to which our abstract theory applies.

2. Preliminaries

For further background and details pertaining to this section we refer the reader to [5, 6, 10, 11, 13, 14, 16].

Throughout this paper, X stands for a real separable Banach space with norm $\|\cdot\|$ and dual $(X^*, \|\cdot\|_*)$. If Ω is a subset of X, then the closure of Ω will be denoted by $\overline{\Omega}$, or alternatively by $cl(\Omega)$.

Let $I = [0, T]$ and let $\mathcal{L}$ be the $\sigma-$algebra of all Lebesgue measurable subsets of I. By $C(I, X)$ (resp. $L^1(I, X)$) we denote the Banach space of all continuous (resp. Bochner integrable) functions $u : I \to X$ equipped with the standard norm $\|u\|_\infty = \sup_{t\in I} \|u(t)\|$ (resp. $\|u\|_1 = \int_0^T \|u(t)\|dt$). $W^{1,1}(I, X)$ designates the space of all absolutely continuous functions $u : I \to X$ which can be written as

$$u(t) = u(0) + \int_0^t v(s)ds, \quad t \in I,$$

for some $v \in L^1(I, X)$. We will also use $L^1(I)$, $AC(I)$ and $BV(I)$ to indicate the space of all Lebesgue integrable functions, absolutely continuous functions, and respectively functions with bounded variation from I to $\mathbb{R}$. A subset $K \subset L^1(I, X)$ is called decomposable if for all $u, v \in K$ and $A \in \mathcal{L}$, one has that $u\chi_A + v\chi_{I\setminus A} \in K$, where χ_A stands for the characteristic function of A. The family of all nonempty, closed and decomposable subsets of $L^1(I, X)$ is denoted by $\mathcal{D}$.

The notation 2^X (resp. $\mathcal{P}(X)$) will designate the collection of all (resp. all nonempty closed) subsets of X. The Hausdorff distance on $\mathcal{P}(X)$ is defined by

$$h(A, B) = \max\big\{ \sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\big\}, \quad \forall A, B \in \mathcal{P}(X),$$

where $d(a, B) = \inf\{\|a - b\| : b \in B\}$. $\mathcal{B}(X)$ will denote the $\sigma-$algebra of Borel subsets of X and $\mathcal{L} \otimes \mathcal{B}(X)$ will stand for the $\sigma-$algebra on $I \times X$ generated by all sets of the form $A \times B$ with $A \in \mathcal{L}$ and $B \in \mathcal{B}(X)$.

Let S be a separable metric space and let $\mathcal{A}$ denote a $\sigma-$algebra of subsets of S. A multivalued map $G : S \to 2^X\setminus\{\emptyset\}$ is said to be *$\mathcal{A}$-measurable* if for each closed subset C of X, the set $\{s \in S : G(s) \cap C \neq \emptyset\}$ belongs to $\mathcal{A}$.

A function $g : S \to X$ satisfying $g(s) \in G(s)$, for all $s \in S$, is called a selection of G. The multivalued map G is said to be *lower semicontinuous* (l. s. c.) if for every closed set C of X, the set $\{s \in S : G(s) \subset C\}$ is closed in S.

The following two results of [8] will play a key role in the sequel.

Proposition 2.1. *Assume that $F^* : I \times S \to \mathcal{P}(X)$ is $\mathcal{L} \otimes \mathcal{B}(S)$ measurable and that $F^*(t, .)$ is l.s.c. for each $t \in I$. Then the map $\xi \to G_F(\xi)$ given by*

$$G_F(\xi) = \{v \in L^1(I, X) : v(t) \in F^*(t, \xi), \ a.e. \ on \ I\} \tag{2.1}$$

is l.s.c. from S into $\mathcal{D}$ if and only if there exists a continuous map $\beta : S \to L^1(I)$, such that for every $\xi \in S$

$$d(0, F^*(t, \xi)) \le \beta(\xi)(t), \quad a.e. \ on \ I. \tag{2.2}$$

Proposition 2.2. *Let $G : S \to \mathcal{D}$ be a l.s.c. multifunction, and let $\varphi : S \to L^1(I, X)$ and $\psi : S \to L^1(I)$ be continuous maps. Assume that for each $\xi \in S$, the set*

$$H(\xi) = cl\{v \in G(\xi) : \|v(t) - \varphi(\xi)(t)\| < \psi(\xi)(t), \text{ a.e. on } I\} \tag{2.3}$$

is nonempty. Then the map $\xi \to H(\xi)$ (with $H(\xi)$ given by (2.3)*, from S into $\mathcal{D}$, admits a continuous selection.*

The remaining of this section is devoted to a brief discussion of accretive operators and related evolution equations.

Let $A : X \to 2^X$ be a set-valued operator in X and let

$$D(A) := \{x \in X : Ax \neq \emptyset\}, \quad \mathcal{R}(A) := \bigcup_{x \in D(A)} Ax$$

be the *domain* and *range* of A, respectively. We say that A is an *accretive operator* if

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|, \quad \forall \lambda > 0, \, \forall x_i \in D(A), \, \forall y_i \in Ax_i \ (i = 1, 2).$$

If in addition $R(Id + \lambda A) = X$ for all (equivalently, some) $\lambda > 0$, where Id is the identity in X, then A is said to be *m-accretive.*

The accretivity of A is equivalent to the condition

$$\langle y_1 - y_2, x_1 - x_2 \rangle_s \geq 0, \quad \forall x_i \in D(A), \forall y_i \in Ax_i (i = 1, 2),$$

with $\langle ., . \rangle_s$ given by $\langle y, x \rangle_s = \sup\{x^*(y) : x^* \in J(x)\}$, where $J : X \to 2^{X^*}$ is the duality map defined by

$$J(x) = \{x^* \in X^* : x^*(x) = \|x\|^2 = \|x^*\|_*^2\}.$$

Consider now the initial value problem

$$u'(t) + Au(t) \ni f(t), \quad t \in I; u(0) = \xi, \tag{2.4}$$

where A is m-accretive in X, $\xi \in \overline{D(A)}$ and $f \in L^1(I, X)$, whose solutions are meant in the sense of the following definition due to Bénilan [7].

Definition 2.3. A function $u \in C(I, \overline{D(A)})$ is called an *integral solution* of the problem (2.4) if $u(0) = \xi$ and the inequality

$$\|u(t) - x\|^2 \leq \|u(s) - x\|^2 + 2 \int_s^t \langle f(\tau) - y, u(\tau) - x \rangle_s d\tau$$

holds for all x, $y \in X$, with $y \in Ax$, and all $0 \leq s \leq t \leq T$.

It is well-known that the problem (2.4) has a unique integral solution for each $f \in L^1(I, X)$ and each $\xi \in \overline{D(A)}$. The following property of integral solutions will be used in Section 4.

Proposition 2.4. *Let u and v be integral solutions of* (2.4) *that correspond to (ξ, f) and (η, g), respectively (where ξ, $\eta \in \overline{D(A)}$ and f, $g \in L^1(I, X)$). Then*

$$\|u(t) - v(t)\| \leq \|\xi - \eta\| + \int_0^t \|f(\tau) - g(\tau)\| d\tau \tag{2.5}$$

for all $t \in I$.

We note that Benilan's original definition of an integral solution [7] required the operator A to be merely accretive. The accretivity alone doesn't generally guarantee the well-posedness of the problem (2.4) and the validity of the inequality (2.5). If, however, A is m-accretive, then problem (2.4) has a unique integral solution, and (2.5) holds.

Now let $\{A(t) : t \in I\}$ be a family of (possibly multivalued) operators on X, of domains $D(A(t))$, with $\overline{D(A(t))} = \overline{D}$ (independent of t) which satisfy the assumption

(H1) (i) $R(Id + \lambda A(t)) = X$, for all $\lambda > 0$ and $t \in I$,
(ii) There exists a continuous function $m_1 : I \to X$ and a continuous nondecreasing function $m_2 : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ := [0, \infty)$) such that

$$\begin{aligned}&\langle y_1 - y_2, x_1 - x_2\rangle_s \\ &\geq -\|m_1(t) - m_1(s)\|\|x_1 - x_2\|m_2(\max\{\|x_1\|, \|x\|_2\}),\end{aligned}$$

for all $x_1 \in D(A(t))$, $y_1 \in A(t)x_1$, $x_2 \in D(A(s))$, $y_2 \in A(s)x_2$, $0 \leq s \leq t \leq T$.

We remark that (H1) implies that $A(t)$ is m-accretive for each $t \in I$. We consider the nonautonomous Cauchy problem

$$u'(t) + A(t)u(t) \ni f(t), \quad t \in I; u(0) = \xi, \tag{2.6}$$

where $A(t)$ satisfy (H1), $\xi \in \overline{D}$ and $f \in L^1(I, X)$.

Definition 2.5. An integral solution of (2.6) is a function $u \in C(I, \overline{D})$ such that $u(0) = \xi$ and the inequality

$$\begin{aligned}\|u(t) - x\|^2 \leq \|u(s) - x\|^2 + 2\int_s^t &[\langle f(\tau) - y, u(\tau) - x\rangle_s \\ &+ C\|u(\tau) - x\|\|m_1(\tau) - m_1(\theta)\|]d\tau\end{aligned}$$

holds for all $0 \leq s \leq t \leq T$, $\theta \in I$, $x \in D(A(\theta))$, $y \in A(\theta)x$, and $C = m_2(\max\{\|x\|, \|u\|_\infty\})$, with m_1 and m_2 as in (H1)(ii).

Recall (cf., e.g., [14]) that (2.6) has a unique integral solution for each $\xi \in \overline{D}$ and $f \in L^1(I, X)$, provided that (H1) is satisfied. Moreover, the following analog of Proposition 2.4 is true.

Proposition 2.6. *Let (H1) be satisfied and let u and v be integral solutions of* (2.6) *corresponding to (ξ, f) and (η, g), respectively (with ξ, $\eta \in \overline{D}$ and f, $g \in L^1(I, X)$). Then the inequality* (2.5) *holds for all $t \in I$.*

3. Main results

We consider the Volterra integral inclusion (1.1) under the following conditions:

(H2) A is an m-accretive operator in X, with domain $D(A)$, and there exists an open subset U of X such that $U_A := U \cap \overline{D(A)}$ is nonempty;
(H3) $a \in AC(I)$ with $a' \in BV(I)$ and $a(0) = 1$;
(H4) (i) $F : I \times X \to \mathcal{P}(X)$ is $\mathcal{L} \otimes \mathcal{B}(X)$ measurable,
(ii) There exists $k \in L^1(I, (0, \infty))$ such that

$$h(F(t, x), F(t, y)) \leq k(t)\|x - y\|, \quad \forall x,\ y \in X, \quad \text{a.e. on } I,$$

(iii) There exists $\beta \in L^1(I, \mathbb{R}_+)$ such that
$$d(0, F(t,0)) \le \beta(t), \quad \text{a.e. on } I\,;$$

(H5) $g \in W^{1,1}(I, X)$ and $g(0) = 0$.

Remark 3.1. The restriction $a(0) = 1$ in (H3) is only made for convenience. The essential condition is $a(0) > 0$; see [9, p. 317].

For each $\xi \in U_A$, we reduce the study of (1.1) to that of the equivalent functional differential inclusion (cf. [2, 9])
$$u'(t) + Au(t) + F(t, u(t)) \ni \Gamma(u)(t), \quad t \in I; \quad u(0) = \xi, \tag{3.1}$$
where $\Gamma : C\big(I, \overline{D(A)}\big) \to L^1(I, X)$ is defined by
$$\Gamma(u)(t) = g'(t) + \int_0^t r(t-s)g'(s)ds - r(0)u(t) + r(t)\xi - \int_0^t u(t-s)dr(s), \tag{3.2}$$
$$r(t) + \int_0^t a'(t-s)r(s)ds = -a'(t). \tag{3.3}$$

Note that by (H3), the function r (as defined in (3.3)) is a function with bounded variation.

Definition 3.2. A function $u \in C(I, \overline{D(A)})$ is said to be an integral solution of the equation (1.1) (equivalently, (3.1)) if there exists $\widehat{f} \in L^1(I, X)$ with $\widehat{f}(t) \in F(t, u(t))$, *a. e.* on I, such that u is an integral solution, in the sense of Definition 2.3, of the problem (2.4) with $\Gamma(u)(t) - \widehat{f}(t)$ in place of $f(t)$.

In the following, $\mathcal{S}(\xi)$ denotes the set of all integral solutions of the equation (1.1), which is viewed as a subset of $C(I, \overline{D(A)})$, for each $\xi \in U_A$.

Theorem 3.3. *Let assumptions (H2), (H3), (H4), (H5) be satisfied. Then there exists* $u : I \times U_A \to \overline{D(A)}$ *such that:*
$$u(., \xi) \in \mathcal{S}(\xi), \quad \forall \xi \in U_A, \tag{3.4}$$
$$\xi \to u(., \xi) \text{ is continuous from } U_A \text{ into } C(I, \overline{D(A)}). \tag{3.5}$$

Remark 3.4. (i) In the case when $a = 1$, $g = 0$ and X is a Hilbert space we recover [15, Theorem 2.4].
(ii) A similar result can be derived for the Volterra integral equation
$$u(t) + \int_0^t a(t-s)[Au(s) + F(s, u(s))]ds \ni g(\xi) + \int_0^t p(s)ds, \quad t \in I,$$
where $g : U_A \to X$ is continuous, $p \in L^1(I, X)$, and conditions (H2), (H3) and (H4) are satisfied. For simplicity, we have restricted our attention to equations of the form (1.1).

Next, we are concerned with the time-dependent analog of (1.1), namely
$$u(t) + \int_0^t a(t-s)[A(s)u(s) + F(s, u(s))]ds \ni \xi + g(t), \quad t \in I, \tag{3.6}$$
where $\{A(t) : t \in I\}$ is a family of m-accretive operators in X that satisfy assumption (H1), while a, F and g are subject to conditions (H3), (H4) and (H5),

respectively, and $\xi \in \overline{D}$. As in [1, 9] we can replace (3.6) by an equivalent functional differential equation of the form (3.1) (with $A(t)$ in place of A), where $\Gamma : C(I, \overline{D}) \to L^1(I, X)$ is given by (3.2).

Definition 3.5. A function $u \in C(I, \overline{D})$ is called an integral solution of equation (3.6) if there exists $\widehat{f} \in L^1(I, X)$ with $\widehat{f}(t) \in F(t, u(t))$, *a. e.* on I, such that u is an integral solution, in the sense of Definition 2.5, of the problem (2.6) where $f(t)$ is replaced by $\Gamma(u)(t) - \widehat{f}(t)$.

For each $\xi \in \overline{D}$, let $\mathcal{T}(\xi)$ denote the set of all integral solutions of the equation (3.6)), which is regarded as a subset of $C(I, \overline{D})$. The following counterpart of Theorem 3.3 is valid.

Theorem 3.6. *Let conditions (H1), (H3), (H4), (H5) be satisfied. In addition assume that there exists an open subset V of X such that $V_A := V \cap \overline{D}$ is nonempty. Then there exists $u : I \times V_A \to \overline{D}$ such that*

$$u(., \xi) \in \mathcal{T}(\xi), \quad \forall \xi \in V_A, \tag{3.7}$$

$$\xi \to u(., \xi) \text{ is continuous from } V_A \text{ into } C(I, \overline{D}). \tag{3.8}$$

4. Proofs

Proof of Theorem 3.3. We adapt the technique used in [8, 15] to handle (3.1), which is the functional differential inclusion equivalent of the integral equation (1.1).

Fix $\varepsilon > 0$ and set $\varepsilon_n := \varepsilon / 2^{n+1}$, $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all nonnegative integers. For $\xi \in U_A$, let $u_0(., \xi) : I \to \overline{D(A)}$ be the unique integral solution of

$$u'(t) + Au(t) \ni \Gamma(u)(t), \quad t \in I; \quad u(0) = \xi.$$

The existence and uniqueness of $u_0(., \xi)$ follows from [9, Prop. 1 and Theorem 1], on account of (H2), (H4) and (H5). Set

$$\alpha(\xi)(t) := \beta(t) + k(t)\|u_0(t, \xi)\|, \quad m(t) := \int_0^t k(s)ds, t \in I, \tag{4.1}$$

where $k(.)$ and $\beta(.)$ are as in (H4) (ii) and (iii), respectively. Also define

$$\gamma(t) := |r(0)| + var\{r : [0, t]\}, \quad t \in I; \quad M := e^{\int_0^t \gamma(s)ds}, \tag{4.2}$$

where the function $r(.)$ satisfies (3.3) and $var\{r : [0, t]\}$ indicates the total variation of r over $[0, t]$. Let $f_{-1}(\xi)(t) \equiv 0$.

We will construct two sequences of functions $(u_n(., \xi))_{n \in \mathbb{N}} \subset C(I, \overline{D(A)})$ and $(f_n(\xi))_{n \in \mathbb{N}} \subset L^1(I, X)$ satisfying the following conditions:

(C1) $u_n(., \xi) : I \to \overline{D(A)}$ is the unique integral solution of the problem

$$u'(t) + Au(t) \ni \Gamma(u)(t) - f_{n-1}(\xi)(t), \quad u(0) = \xi; \tag{4.3}$$

(C2) $\xi \to f_n(\xi)$ is continuous from U_A into $L^1(I, X)$;

(C3) $f_n(\xi)(t) \in F(t, u_n(t, \xi))$, for all $\xi \in U_A$, a.e. on I;

(C4) $\|f_n(\xi)(t) - f_{n-1}(\xi)(t)\| \le k(t)\beta_n(\xi)(t)$, for all $\xi \in U_A$, a.e. on I,

where

$$\beta_0(\xi)(t) := (\alpha(\xi)(t) + \varepsilon_0)(k(t))^{-1},$$

and, for $n \geq 1$,

$$\beta_n(\xi)(t) = M^n \int_0^t \alpha(\xi)(s) \frac{[m(t)-m(s)]^{n-1}}{(n-1)!} ds + M^n T \frac{[m(t)]^{n-1}}{(n-1)!} \sum_{i=0}^{n} \varepsilon_i, \tag{4.4}$$

with $\alpha(\xi)(.)$, $m(.)$ and M defined in (4.1) and (4.2).

We remark that $u_0(.,\xi)$ is the integral solution of (4.3) with $n=0$. We claim that the map $\xi \to u_0(.,\xi)$ is continuous from U_A into $C(I,\overline{D(A)})$. Indeed, for $\xi_1, \xi_2 \in U_A$, we can invoke (2.5) to deduce, for $t \in I$,

$$\|u_0(t,\xi_1) - u_0(t,\xi_2)\| \leq \|\xi_1 - \xi_2\| + \int_0^t \|\Gamma(u_0(.,\xi_1))(s) - \Gamma(u_0(.,\xi_2))(s)\| ds. \tag{4.5}$$

It is easily seen that the definition of Γ (cf. (3.2)) implies

$$\begin{aligned} &\int_0^t \|\Gamma(u_0(.,\xi_1))(s) - \Gamma(u_0(.,\xi_2))(s)\| ds \\ &\leq r(t)\|\xi_1 - \xi_2\| + \int_0^t \gamma(s) \|u_0(.,\xi_1) - u_0(.,\xi_2)\|_\infty(s) ds, \end{aligned} \tag{4.6}$$

where $\|u\|_\infty(s) := \sup_{\tau \in [0,s]} \|u(\tau)\|$ is the norm in $C([0,s],X)$ and $\gamma(.)$ is given by (4.2). Since $r(.) \in BV(I)$, one has that $\|r\|_\infty := \sup_{t \in I} |r(t)| < \infty$. Using (4.6) in (4.5) and applying Gronwall's lemma, we conclude that

$$\|u_0(.,\xi_1) - u_0(.,\xi_2)\|_\infty \leq M(1 + \|r\|_\infty)\|\xi_1 - \xi_2\|.$$

This yields the continuity of $\xi \to u_0(.,\xi)$ from U_A into $C(I,\overline{D(A)})$, as claimed.

Next, by (H4) (ii), (iii) and (4.1), we have

$$d(0, F(t, u_0(t,\xi))) \leq \alpha(\xi)(t), \quad \text{a.e. on } I, \tag{4.7}$$

where it is to be noted that $\alpha(.)$ is continuous as a function from U_A into $L^1(I)$. Define the multifunctions G_0, $H_0 : U_A \to 2^{L^1(I,X)}$ by

$$G_0(\xi) := \{v \in L^1(I,X) : v(t) \in F(t, u_0(t,\xi)), \text{ a.e. on } I\}, \tag{4.8}$$

$$H_0(\xi) := cl\{v \in G_0(\xi) : \|v(t)\| < \alpha(\xi)(t) + \varepsilon_0, \text{ a.e. on } I\}. \tag{4.9}$$

Setting $F^*(t,\xi) := F(t, u_0(t,\xi))$ and invoking assumptions (H4) (i), (ii), [11, Proposition 2.66, p.61], the continuity of $\alpha(.)$ on U_A and (4.7), we conclude by applying Proposition 2.1 that $G_0(.)$ is lower semicontinuous from U_A into $\mathcal{D}$ and the set $H_0(\xi)$ is nonempty. Therefore, by Proposition 2.2, there exists $h_0 \in C(U_A, L^1(I,X))$ such that $h_0(\xi) \in H_0(\xi)$, $\forall \xi \in U_A$. Set

$$f_0(\xi)(t) := h_0(\xi)(t), \quad \forall \xi \in U_A, t \in I, \tag{4.10}$$

and remark that, by virtue of (4.8), (4.9), (4.10) and the fact that F is closed valued, $f_0(.)$ is continuous from U_A into $L^1(I,X)$, $f_0(\xi)(t) \in F(t, u_0(t,\xi))$ and $\|f_0(\xi)(t)\| \leq k(t)\beta_0(\xi)(t)$, *a.e. on* I. Recalling that $u_0(.,\xi)$ is the integral solution of (4.3) with $n=0$, we conclude that conditions $(C_1)-(C_4)$ hold for $n=0$.

We now proceed inductively. Assume that the functions $\{u_0, \quad u_1, \dots, u_n\}$ and $\{f_0, \quad f_1, \dots, f_n\}$ have been constructed such that conditions $(C_1)-(C_4)$ are satisfied. For $\xi \in U_A$, let $u_{n+1}(.,\xi) : I \to \overline{D(A)}$ be the unique integral solution of (4.3) with $n+1$ in place of n. (Taking into account that $f_n(\xi) \in L^1(I,X)$, we can again invoke [9, Prop. 1 and Theorem 1] to establish the existence and uniqueness of $u_{n+1}(.,\xi)$). Inasmuch as $u_n(.,\xi)$ and $u_{n+1}(.,\xi)$ satisfy (4.3), and (4.3) with $n+1$ instead of n, respectively, we can apply Proposition 2.4 to obtain, for $t \in I$,

$$\begin{aligned} \|u_{n+1}(t,\xi) - u_n(t,\xi)\| \le \int_0^t \|\Gamma(u_{n+1}(.,\xi))(s) - \Gamma(u_n(.,\xi))(s)\| ds \\ + \int_0^t \|f_n(\xi)(s) - f_{n-1}(\xi)(s)\| ds. \end{aligned} \tag{4.11}$$

From (3.2) it follows that

$$\begin{aligned} &\int_0^t \|\Gamma(u_{n+1}(.,\xi))(s) - \Gamma(u_n(.,\xi))(s)\| ds \\ &\le \int_0^t \gamma(s)\|u_{n+1}(.,\xi) - u_n(.,\xi)(s)\|_\infty(s) ds. \end{aligned} \tag{4.12}$$

Combining (4.11) with (4.12) and using Gronwall's lemma, we arrive at

$$\|u_{n+1}(t,\xi) - u_n(t,\xi)\| \le M \int_0^t \|f_n(\xi)(s) - f_{n-1}(\xi)(s)\| ds, t \in I. \tag{4.13}$$

Employing property (C4) in (4.13), we have

$$\|u_{n+1}(t,\xi) - u_n(t,\xi)\| \le M \int_0^t k(s)\beta_n(\xi)(s) ds, t \in I. \tag{4.14}$$

If $n = 0$, this implies, by virtue of (4.4),

$$\|u_1(t,\xi) - u_0(t,\xi)\| \le M \int_0^t \alpha(\xi)(s) ds < \beta_1(\xi)(t), \quad \text{a.e. on } I. \tag{4.15}$$

If $n > 0$, then (4.14) and (4.4) lead to

$$\begin{aligned} \|u_{n+1}(t,\xi) - u_n(t,\xi)\| \le M^{n+1} \int_0^t k(s) \int_0^s \alpha(\xi)(\tau) \frac{[m(s) - m(\tau)]^{n-1}}{(n-1)!} d\tau ds \\ + M^{n+1} T[\sum_{i=0}^n \varepsilon_i] \int_0^t k(s) \frac{[m(s)]^{n-1}}{(n-1)!} ds, \end{aligned} \tag{4.16}$$

for $t \in I$. Recalling the definition of m (cf. (4.1)), and interchanging the order of integration in the first term on the right-hand side of (4.16), we get

$$\begin{aligned} &\|u_{n+1}(t,\xi) - u_n(t,\xi)\| \\ &\le M^{n+1} \int_0^t \alpha(\xi)(\tau) \frac{[m(t) - m(\tau)]^n}{n!} d\tau + M^{n+1} T[\sum_{i=0}^n \varepsilon_i] \frac{[m(t)]^n}{n!} \\ &< \beta_{n+1}(\xi)(t), \quad \text{a.e. on } I. \end{aligned} \tag{4.17}$$

By (4.15), (4.17), (C3) and (H4) (ii), it follows that

$$d(f_n(\xi)(t), F(t, u_{n+1}(t,\xi))) < k(t)\beta_{n+1}(\xi)(t), \quad \text{a.e. on } I, \tag{4.18}$$

and subsequently

$$d(0, F(t, u_{n+1}(t,\xi))) \le \|f_n(\xi)(t)\| + k(t)\beta_{n+1}(\xi)(t), \quad \text{a.e. on } I, \tag{4.19}$$

where the expression on the right-hand side of (4.19) is continuous from U_A into $L^1(I)$ (cf. (C2), (4.1) and (4.4)).

For $\xi \in U_A$, we define

$$G_{n+1}(\xi) = \{v \in L^1(I,X) : v(t) \in F(t, u_{n+1}(t,\xi)), \text{ a.e. on } I\},$$

$$H_{n+1}(\xi) = cl\{v \in G_{n+1}(\xi) : \|v(t) - f_n(\xi)(t)\| < k(t)\beta_{n+1}(\xi)(t), \text{ a.e. on } I\}. \quad (4.20)$$

Clearly, $G_{n+1}(.)$ is lower semicontinuous from U_A into $\mathcal{D}$ and $H_{n+1}(\xi)$ is nonempty, because of (4.18) and (4.19). Therefore, one can apply Proposition 2.2 to derive the existence of $h_{n+1} \in C(U_A, L^1(I, X))$ such that $h_{n+1}(\xi) \in H_{n+1}(\xi)$, for all $\xi \in U_A$.

Setting $f_{n+1}(\xi)(t) = h_{n+1}(\xi)(t)$, for all $\xi \in U_A$ and almost all $t \in I$, we conclude that $f_{n+1}(.)$ is continuous from U_A into $L^1(I, X)$ and $f_{n+1}(\xi)(t) \in F(t, u_{n+1}(t, \xi))$, a.e. on I; hence $f_{n+1}(\xi)$ and $u_{n+1}(., \xi)$ satisfy conditions $(C_1) - (C_3)$. Condition (C4) is also satisfied on account of (4.20), and the induction argument has been completed.

By (C4), (4.1) and (4.4) we now successively obtain

$$\begin{aligned} \|f_n(\xi) - f_{n-1}(\xi)\|_1 &\leq \int_0^T k(s)\beta_n(\xi)(s)ds \\ &= M^n \int_0^T \alpha(\xi)(s) \frac{[m(T) - m(s)]^n}{n!} ds + M^n T [\sum_{i=0}^n \varepsilon_i] \frac{[m(T)]^n}{n!} \\ &\leq \frac{M^n (\|k\|_1)^n}{n!} (\|\alpha(\xi)\|_1 + T\varepsilon). \end{aligned} \quad (4.21)$$

From the above inequality, it follows that $(f_n(\xi))_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(I, X)$, hence it converges in $L^1(I, X)$ to some function $f(\xi) \in L^1(I, X)$. Then, for a subsequence (again denoted by $(f_n(\xi))_{n\in\mathbb{N}}$), we have

$$f_n(\xi)(t) \to f(\xi)(t), \quad \text{as } n \to \infty, \text{ a.e. on } I. \quad (4.22)$$

Next, from (4.13) and (4.21) it follows that

$$\|u_{n+1}(., \xi) - u_n(., \xi)\|_\infty \leq \frac{M^{n+1}(\|k\|_1)^n}{n!} (\|\alpha(\xi)\|_1 + T\varepsilon)$$

and, since the map $\xi \to \|\alpha(\xi)\|_1$ is continuous, this implies that $(u_n(., \xi))_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(I, X)$, locally uniformly in ξ. Therefore, if we denote by $u(., \xi)$ its limit, then $\xi \to u(., \xi)$ is continuous from U_A into $C(I, X)$.

Since the multifunction F is closed valued and since, by (C3) and (H4) (ii),

$$d(f_n(\xi)(t), F(t, u(t, \xi))) \leq k(t)\|u_n(t, \xi) - u(t, \xi)\|,$$

passing to the limit as $n \to \infty$, we have by (4.22) that

$$f(\xi)(t) \in F(t, u(t, \xi)), \quad \text{a.e on } I.$$

Finally, let $u^*(., \xi)$ be the unique integral solution of

$$u'(t) + Au(t) \ni \Gamma(u)(t) - f(\xi)(t), \quad t \in I; u(0) = \xi.$$

Since $u_{n+1}(., \xi)$ satisfies (4.3) with $n + 1$ instead of n, we obtain, with the help of Proposition 2.4 (compare to (4.13)),

$$\|u_{n+1}(., \xi) - u^*(., \xi)\|_\infty \leq M\|f_n(\xi) - f(\xi)\|_1, \quad \xi \in U_A. \quad (4.23)$$

Hence, letting $n \to \infty$ in (4.23) we obtain that $u(t, \xi) = u^*(t, \xi)$ for each $t \in I$. Then, we conclude that (3.4) holds and since $\xi \to u(., \xi)$ is continuous from U_A into $C(I, X)$, it follows that (3.5) is satisfied, as well, and the proof is complete. $\square$

Proof of Theorem 3.6. As specified in Section 3, we consider the functional differential equivalent of (3.6), namely

$$u'(t) + A(t)u(t) + F(t, u(t)) \ni \Gamma(u)(t), \quad t \in I; u(0) = \xi. \tag{4.24}$$

The theory of [9, p. 323-24], can be adapted to justify the equivalence between (3.6) and (4.24) under assumption (H1); see also [1]. The proof then follows that of Theorem 3.3, with the mention that $\overline{D}$, V_A and Proposition 2.6 are now used in place of $\overline{D(A)}$, U_A and Proposition 2.4, respectively. The details are left to the reader. □

5. An example

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary Γ, and let $\rho : \mathbb{R} \to \mathbb{R}$ satisfy

(H6) $\rho \in C(\mathbb{R})$, $\rho(0) = 0$, ρ is nondecreasing.

Let $X = L^1(\Omega)$, and define the operator $A : D(A) \subset X \to X$ by

$$Au = -\Delta\rho(u), \quad D(A) = \{u \in L^1(\Omega) : \rho(u) \in W_0^{1,1}(\Omega), \Delta\rho(u) \in L^1(\Omega)\}. \tag{5.1}$$

It is well-known (see, e.g., [16, Example 1.5.5]) that A is m-accretive in X, with $\overline{D(A)} = X$.

Next, let $f_i : I \times \Omega \times \mathbb{R} \to \mathbb{R}$ $(I = [0, T],\ i = 1, 2)$ be given functions satisfying $f_1 \leq f_2$ on $I \times \Omega \times \mathbb{R}$ and the following conditions

(H7) (i) $(t, x) \to f_i(t, x, r)$ is measurable for all $r \in \mathbb{R}$,

(ii) There exists $k : I \times \Omega \to (0, \infty)$, $k \in L^1(I, L^\infty(\Omega))$ such that

$$|f_i(t, x, r) - f_i(t, x, \overline{r})| \leq k(t, x)|r - \overline{r}| \quad \text{a.e. on } I \times \Omega, \text{ for all } r, \overline{r} \text{ in } \mathbb{R},$$

(iii) $f_i(., ., 0) \in L^1(I \times \Omega)$.

Introduce the multifunction $\widehat{f} : I \times \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ by

$$\widehat{f}(t, x, r) = [f_1(t, x, r), f_2(t, x, r)] \tag{5.2}$$

and define $F : I \times X \to 2^X$ by

$$F(t, u)(x) = \{v \in X : v(x) \in \widehat{f}(t, x, u(x)), \text{ a.e. on } \Omega\}. \tag{5.3}$$

By (H7) (i)-(iii), (5.2) and (5.3), it is an easy exercise to show that F satisfies (H4). (One uses the definition of the Hausdorff distance, [11, Theorem 7.26, p. 237] and measurability arguments similar to those in [12, p. 97]).

Finally, let $\xi : \Omega \to \mathbb{R}$ and $\overline{g} : I \times \Omega \to \mathbb{R}$ satisfy

(H8) $\xi \in L^1(\Omega)$,

(H9) $\overline{g} \in W^{1,1}(I, L^1(\Omega))$; $\overline{g}(0, x) = 0$, a.e. on Ω

and set

$$g(t)(x) = \overline{g}(t, x) \quad \text{for all } t \in I \text{ and a.a. } x \in \Omega. \tag{5.4}$$

Obviously, by (H9), condition (H5) is verified.

Consider the problem

$$\begin{gathered} u(t, x) + \int_0^t a(t - s)[-\Delta\rho(u(s, x)) + \widehat{f}(s, x, u(s, x))]ds \\ \ni \xi(x) + \overline{g}(t, x) \text{ on } I \times \Omega\,, \\ u(t, x) = 0, \quad \text{on } I \times \Gamma\,, \end{gathered} \tag{5.5}$$

where a satisfies (H3). In view of the above discussion, it is clear that (5.5) can be rewritten in the abstract form (1.1) in the Banach space $X = L^1(\Omega)$, with A, F and g defined by (5.1), (5.3) and (5.4), respectively. Consequently, an application of Theorem 3.3 (with $U_A = X$) yields following result.

Theorem 5.1. *Under assumptions (H3), (H6)–(H9), Problem* (5.5) *has an integral solution* $u(.,\xi) \in C(I, L^1(\Omega))$ *such that* $\xi \to u(.,\xi)$ *is continuous from* $L^1(\Omega)$ *into* $C(I, L^1(\Omega))$.

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