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ON THE EXISTENCE OF SOLUTIONS TO A CLASS OF DIFFERENTIAL INCLUSIONS

Abstract. We prove the local existence of solutions to the Cauchy problem for the differential inclusion $\dot{x} \in -\partial V(x)+F(x)+f(t,x)$ where ∂V is the subdifferential of a lower semicontinuous proper convex function V. F is a cyclically upper semicontinuous multifunction and f satisfies Carathèodory conditions.

1. Introduction

Bressan, Cellina and Colombo have proved in [4] the existence of solutions of the Cauchy problem

$$\dot{x}(t) \in F(x(t)) \subset \partial V(x(t)),\ x(0) = x_0 \tag{1.1}$$

where F is a monotonic upper semicontinuous (non necessarily convex-valued, hence not maximal) map contained in the subdifferential of a lower semicontinuous proper convex function, and x is in a finite dimensional space. This result has been generalized by Ancona and Colombo [1] to cover perturbations of the kind

$$\dot{x}(t) \in F(x(t)) + f(t, x(t))$$

with f satisfying Carathéodory conditions. In the joint paper with Cellina [8] we have proved the local existence of solutions for a Cauchy Problem of the form

$$(1.2) \qquad \dot{x}(t) \in -\partial V(x(t)) + F(x(t)),\ F(x) \subset \partial W(x), x(0) = x_0\,,$$

where V is a lower semicontinuous proper convex function (hence ∂V is a maximal monotone map), W is a lower semicontinuous proper convex function and F is an upper semicontinuous compact valued map defined over some neighborhood of x_0.

Purpose of the present paper is to prove the (local) existence of solutions of a Cauchy problem of the form

$$\dot{x}(t) \in -\partial V(x(t)) + F(x(t)) + f(t, x(t)), x(0) = x_0\,,$$

where V and F are as in the problem (1.2) and f satisfies Carathéodory conditions. This result unifies the results in [1], [4] and [8] and its proof follows the one in [8].

2. Assumptions and the Statement of the Main Result

Consider on R^n the Euclidean norm $||\cdot||$ and the scalar product $<.,.>$. For $x \in R^n$ and $r > 0$ set: $B(x,r) = \{y \in R^n : ||y - x|| < r\}$, $B[x,r] = \{y \in R^n : ||y - x|| \leq r\}$, and for a closed subset A of R^n, $B(A,r) = \{y \in R^n : d(y,A) < r\}$, where $d(x,A) = \inf\{||y - x|| : y \in A\}$. Denote by cl A the closure of A, and if A is a closed and convex subset of R^n, then by $m(A)$ we denote the element of minimal norm of A, i.e., such that

$$||m(A)|| = \inf\{||y|| : y \in A\}\,.$$

We consider the Cauchy problem

$$(2.1) \qquad \dot{x}(t) \in -\partial V(x(t)) + F(x(t)) + f(t, x(t)), x(0) = x_0\,,$$

under the following assumptions:

(H_1) $V : R^n \to (-\infty, +\infty]$ is a proper convex lower semicontinuous function, $\partial V : R^n \to 2^{R^n}$ is the subdifferential of V defined by

$$\partial V(x) = \{\xi \in R^n : V(y) - V(x) \geq < \xi, y - x >, \forall y \in R^n\}$$

$x_0 \in D(\partial V)$ and $D(\partial V) := \{x \in R^n : \partial V(x) \neq \emptyset\}$.

(H_2) $F : R^n \to 2^{R^n}$ is an upper semicontinuous multifunction (i.e. for every x and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $y \in B(x, \delta)$ implies $F(y) \subset B(F(x), \varepsilon)$), with closed nonempty values and there exists a proper convex lower semicontinuous function $W : R^n \to (-\infty, +\infty]$ such that $F(x) \subset \partial W(x)$ for every $x \in R^n$, where $\partial W(.)$ is the subdifferential of W.

(H_3) $f : R \times R^n \to R^n$ is a *Carathéodory function*, i.e.: for every $x \in R^n$ $t \to f(t, x)$ is measurable, for a.e. $t \in R$ $x \to f(t, x)$ is continuous and there exists $k \in L^2(R, R)$ such that $\|f(t, x)\| \leq k(t)$ for a.e. $t \in R$ and for all $x \in R^n$.

Remark that since for any compact set K containing x_0 there exists $x^* \in K$ such that $\inf\{V(x) : x \in K\} = V(x^*)$ and since the subdifferential of the function $x \to V(x) - V(x^*)$ coincides with the subdifferential of $V(.)$, we can assume in what follows that $V \geq 0$.

DEFINITION 2.1. *Let $h \in L^2([0, T], R^n)$ and $x_0 \in D(\partial V)$. By solution of the problem*

$$(P_h) \qquad \dot{x}(t) \in -\partial V(x(t)) + h(t),\ x(0) = x_0$$

we mean any absolutely continuous function $x : [0, T] \to R^n$ such that $x(0) = x_0$, and for a.e. $t \in [0, T]$:

$$x(t) \in D(\partial V) \text{ and } \dot{x}(t) \in -\partial V(x(t)) + h(t).$$

DEFINITION 2.2. *A function $x : [0, T] \to R^n$ is called a solution of the Cauchy problem (2.1) if there exists $g \in L^2([0, T], R^n)$, a selection of $F(x(\cdot))$ (i.e. $g(t) \in F(x(t))$ for a.e. $t \in [0, T]$), such that x is a solution of the problem (P_h), with $h(t) = g(t) + f(t, x(t))$.*

Our main result is the following:

THEOREM. *Let V, F and f satisfy assumptions $(H_1)-(H_3)$. Then for every $x_0 \in D(\partial V)$ there exist $T > 0$ and $x : [0,T] \to R^n$ a solution of the Cauchy problem*

$$\dot{x}(t) \in -\partial V(x(t)) + F(x(t)) + f(t, x(t)), \quad x(0) = x_0 .$$

3. Proof of the Main Result

We shall use the following known results:

LEMMA 3.1. ([2], Theorems 1.2. and 1.3.) Let $V(.)$ satisfy assumptions (H_1). Then for every $x_0 \in D(\partial V)$ and $h \in L^2([0,T], R^n)$ there exists a unique solution $x^h : [0,T] \to R^n$ of the problem

$$(P_h) \qquad \dot{x}(t) \in -\partial V(x(t)) + h(t), x(0) = x_0 .$$

Moreover: $\frac{dx^h}{dt} \in L^2([0,T], R^n), t \to V(x^h(t))$ is absolutely continuous (hence almost everywhere differentiable) on $[0,T]$,

$$(3.1) \qquad \left\| \frac{dx^h(t)}{dt} \right\|^2 = -\frac{d}{dt} V(x^h(t)) + < h(t), \frac{dx^h(t)}{dt} >$$

and

$$(3.2) \qquad \left[\int_0^T \left\| \frac{dx^h(t)}{dt} \right\|^2 dt \right]^{1/2} \leq \left(\int_0^T \|h(t)\|^2 dt \right)^{1/2} + \sqrt{V(x_0)} .$$

If $g, h \in L^2([0,T], R^n)$ and $x^g(.), x^h(.)$ are the corresponding solutions then for any $0 \leq s \leq t \leq T$:

$$(3.3) \qquad \|x^g(t) - x^h(t)\| \leq \|x^g(s) - x^h(s)\| + \int_s^t \|g(u) - h(u)\| ds \qquad \blacksquare$$

Let $x^0 : [0, \infty) \to R^n$ be the unique solution of the problem (P_h) with $h = 0$. Then, by Theorem 3.2.1. in [3], for any $T > 0$ and a.e. $t \in$

$(0,T)$: $\frac{d}{dt}x^0(t) = -m(\partial V(x(t)))$ and $t \to ||m(\partial V(x^0(t)))||$ is nonincreasing. Therefore, for any $t \in [0,T]$,

$$||x^0(t) - x_0|| = ||\int_0^t \dot{x}^0(s)ds|| \le \int_0^t ||m(\partial V(x^0(s)))||ds$$
$$\le \int_0^t ||m(\partial V(x^0(0)))||ds\,,$$

hence

(3.4) $$||x^0(t) - x_0|| \le t||m(\partial V(x_0))||, \text{ for any } t \in [0,T]\,. \quad \blacksquare$$

Let $\mathcal{L}$ be the σ-algebra of Lebesgue measurable subsets of the interval $[0,T]$. A multivalued map $G : [0,T] \to 2^{R^n}$ is called *measurable* if for any closed subset C of R^n the set $\{t \in [0,T] : G(t) \cap C \neq \emptyset\}$ belongs to $\mathcal{L}$.

LEMMA 3.2. ([8], Lemma 3.2.) Let $\{\delta_n(\cdot) : n \in N\}$ be a sequence of measurable functions, $\delta_n : [0,T] \to R^n$ and assume that there exists a function $\alpha \in L^1([0,T], R^n)$ such that for a.e. $t \in [0,T]$

$$||\delta_n(t)|| \le \alpha(t)\,.$$

Let

$$\psi(t) = \bigcap_{i \in N} [cl(\bigcup_{n \ge i} \{\delta_n(t)\})]\,.$$

Then:

(i) for a.e. $t \in [0,T]$, $\psi(t)$ is a nonempty compact subset of R^n and $t \to \psi(t)$ is measurable.

(ii) If $G : [0,T] \to 2^{R^n}$ is a multifunction with closed nonempty values such that $d(\delta_n(t), G(t)) \to 0$ for $n \to \infty$ then $\psi(t) \subset G(t)$.

Proof of the theorem. Let $x_0 \in D(\partial V)$ and let $W : R^n \to (-\infty, +\infty]$ satisfy (H_2). Then, as in [4], there exist $r > 0$ and $M < \infty$ such that W is Lipschitzean with Lipschitz constant M on $B(x_0, r)$ and, since $F(x) \subset \partial W(x)$ it follows that F is bounded by M on $B(x_0, r)$. Let $m(\partial V(x_0))$ be the element of minimal norn of $\partial V(x_0)$ and let $T > 0$ be such that

$$\int_0^T (k(s) + M + \|m(\partial V(x_0))\|)ds \le r \, , \tag{3.5}$$

where $k(.)$ is given by (H_3). Our purpose is to prove that there exists $x : [0, T] \to B[x_0, r]$, a solution to the Cauchy problem (2.1).

Let $n \in N, t_0^n = 0$ and, for $k = 1, \ldots, n$, let $t_k^n = k\frac{T}{n}$ and $I_k^n = (t_{k-1}^n, t_k^n]$. Take $y_0^n \in F(x_0)$ and define $h_1^n : I_1^n \to R^n$ by $h_1^n(t) = y_0^n + f(t, x_0)$. Since $h_1^n \in L^2(I_1^n, R^n)$, by Lemma 3.1, there exists $x_1^n : I_1^n \to R^n$, the unique solution of the problem

$$\dot{x}(t) \in -\partial V(x(t)) + h_1^n(t), \; x(0) = x_0 \, . \tag{P_1^n}$$

Then by (3.3) we obtain that for any $t \in [t_0^n, t_1^n]$,

$$\|x_1^n(t) - x^0(t)\| \le \int_0^t \|h_1^n(s)\| ds \le \int_0^t (M + \|f(s, x_0)\|)ds \le \int_0^t (M + k(s))ds$$

and by using (3.4) and (3.5) we obtain

$$\|x_1^n(t) - x_0\| \le \int_0^t (M + \|m(\partial V(x_0))\| + k(s))ds \le r \, .$$

Analogously for $k = 2, \ldots, n$ take $y_{k-1}^n \in F(x_{k-1}^n(t_{k-1}^n))$; define $h_k^n : I_k^n \to R^n$ by $h_k^n(t) = y_{k-1}^n + f(t, x_{k-1}^n(t_{k-1}^n))$ and set $x_k^n : I_k^n \to R^n$ to be the unique solution of the problem

$$\dot{x}(t) \in -\partial V(x(t)) + h_k^n(t), \; x(t_{k-1}^n) = x_{k-1}^n(t_{k-1}^n) \, . \tag{P_k^n}$$

Then by (3.3), (3.4) and (3.5) we obtain that, for $t \in [t_{k-1}^n, t_k^n]$,

$$\|x_k^n(t) - x_0\| \leq \|x_k^n(t) - x^0(t)\| + \|x^0(t) - x_0\| \leq \|x_k^n(t_{k-1}^n) - x^0(t_{k-1}^n)\|$$

$$+ \int_{t_{k-1}^n}^t \|h_k^n(s)\| ds + t\|m(\partial V(x_0))\|$$

$$\leq \int_0^{t_{k-1}^n} [M + k(s)] ds + \int_{t_{k-1}^n}^t [M + k(s)] ds + t\|m(\partial V(x_0))\|$$

$$= \int_0^t (M + \|m(\partial V(x_0))\| + k(s)) ds \leq r$$

Define for $t \in [0, T]$:

$$x_n(t) = \sum_{k=1}^n x_k^n(t)\chi_{I_k^n}(t),\ h_n(t) = \sum_{k=1}^n h_k^n(t)\chi_{I_k^n}(t),\ a_n(t) = \sum_{k=1}^n t_{k-1}^n \chi_{I_k^n}(t)\,,$$

$$g_n(t) = h_n(t) - f(t, x_n(a_n(t)))\,.$$

By the construction we have

(3.6) $\quad \dot{x}(t) \in -\partial V(x_n(t)) + h_n(t)$ a.e. on $[0, T], x_n(0) = x_0$

(3.7) $\quad x_n(t) \in D(\partial V) \cap B(x_0, r)$ a.e. on $[0, T]$

(3.8) $\quad g_n(t) \in F(x_n(a_n(t)))$ a.e. on $[0, T]$

and by (3.6) and (3.2) we obtain

$$\left(\int_0^T \left\|\frac{dx_n(t)}{dt}\right\| dt\right)^{1/2} \leq \left(\int_0^T (M + k(s))^2 dt\right)^{1/2} + \sqrt{V(x_0)} =: N\,.$$

Therefore $\left\| \frac{dx_n}{dt} \right\|_{L^2} \leq N$ and since $||x_n||_\infty \leq r + ||x_0||$, we can assume that $(x_n, \dot{x}_n)$ is precompact in $C([0,T], R^n) \times L^2([0,T], R^n)$, the first space with the sup norm and the second with the weak topology. Then there exists a subsequence (again denote by) x_n and an absolutely continuous function $x : [0,T] \to B[x_0, r]$ such that

(3.9) $\qquad x_n$ converges to x uniformly on compact subsets on $[0,T]$

(3.10) $\qquad \dot{x}_n$ converges weakly in L^2 to $\dot{x}$.

Since $||g_n(t)|| \leq M$ on $[0,T]$, we can assume that

(3.11) $\qquad g_n$ converges weakly in L^2 to some g .

Since $a_n(t) \to t$ uniformly and $x \to f(t,x)$ is continuous, we obtain that $f(t, x_n(a_n(t)))$ converges to $f(t, x(t))$ uniformly with respect to t on compact subsets on $[0,T]$. Moreover since

$$d((x_n(t), g_n(t)), \text{graph } F) \leq ||x_n(a_n(t)) - x_n(t)||$$

we have that

$$d((x_n(t), g_n(t)), \text{ graph } F)) \text{ converges to } 0 \text{ for } n \to \infty .$$

and by the Convergence theorem ([3], p. 60) it follows that

(3.12) $$g(t) \in coF(x(t)) \subset \partial W(x(t))$$

where *co* stands for the convex hull, and,

(3.13) $$\dot{x}(t) \in -\partial V(x(t)) + g(t) + f(t, x(t)) .$$

Since $g(t) \in \partial W(x(t))$ by Lemma 3.3 in ([5], p. 73) we obtain that $\frac{d}{dt}W(x(t)) = < \dot{x}(t), g(t) >$, hence,

$$\int_0^T < \dot{x}(s), g(s) > ds = W(x(T)) - W(x_0)\,. \tag{3.14}$$

By the definition of ∂W,

$$\begin{aligned} W(x_n(t_k^n)) - W(x_n(t_{k-1}^n)) &\geq < y_{k-1}^n, \int_{t_{k-1}^n}^{t_k^n} \dot{x}_n(s) ds > \\ &= \int_{t_{k-1}^n}^{t_k^n} < g_n(s), \dot{x}_n(s) > ds\,. \end{aligned}$$

and by adding for $k = 1, \ldots, n$ we obtain

$$W(x_n(T)) - W(x_0) \geq \int_0^T < g_n(s), \dot{x}_n(s) > ds\,. \tag{3.15}$$

Using the lower semicontinuity of W in $x(T)$ and the convergence of x_n to x, by (3.14) and (3.15) it follows

$$\limsup_{n\to\infty} \int_0^T < \dot{x}_n(s), g_n(s) ds > \leq \int_0^T < \dot{x}(s), g(s) > ds\,. \tag{3.16}$$

On the other hand by (3.6) and (3.1) we obtain

$$\begin{aligned} \left\| \frac{dx_n(t)}{dt} \right\|^2 &= -\frac{d}{dt}V(x_n(t)) + < h_n(t), \frac{dx_n(t)}{dt} > \\ &= -\frac{d}{dt}V(x_n(t)) + < g_n(t) + f(t, x_n(a_n(t)), \frac{dx_n(t)}{dt} > \end{aligned}$$

hence

$$\begin{aligned} \int_0^T \|\dot{x}_n(s)\|^2 ds = &\int_0^T < g_n(s), \dot{x}_n(s) > ds \\ &+ \int_0^T < f(s, x_n(a_n(s))), \dot{x}_n(s) > ds \\ &- V(x_n(T)) + V(x_0)\,. \end{aligned} \tag{3.17}$$

Analogously, by (3.13) and (3.1) it follows that

$$\begin{aligned}\int_0^T \|\dot{x}(s)\|^2 ds = \int_0^T &< g(s), \dot{x}(s) > ds \\ &+ \int_0^T < f(s, x(s)), \dot{x}(s) > ds \\ &- V(x(T)) + V(x_0)\,. \end{aligned} \tag{3.18}$$

and by (3.17), (3.16), the continuity of $x \to f(t, x)$ the lower semicontinuity of V and (3.18) we obtain

$$\limsup_{n\to\infty} \|\dot{x}_n\|_{L^2} \leq \|\dot{x}\|_{L^2}\,.$$

Since, by the weak convergence of $\dot{x}_n$ to $\dot{x}$, $\liminf_{n\to\infty} \|\dot{x}_n\|_{L^2} \leq \|\dot{x}\|_{L^2}$ we have that

$$\lim_{n\to\infty} \|\dot{x}_n\|_{L^2} = \|\dot{x}\|_{L^2}\,.$$

Therefore $\dot{x}_n$ converges to $\dot{x}$ in L^2-norm, hence (Theorem IV.9 in ([6], p. 58)) a subsequence (denoted again by) $\dot{x}_n$ converges pointwise almost everywhere on $[0, T]$ to $\dot{x}$ and there exists $\lambda \in L^2([0, T], R^n)$ such that $\|\dot{x}_n(t)\| \leq \lambda(t)$.

Now, as in [8], we apply Lemma 3.2. for δ_n given by $\delta_n(t) = f(t, x_n(a_n(t))) + g_n(t) - \dot{x}_n(t)$ and $G(t) := F(x(t)) - \dot{x}(t) + f(t, x(t))$. By construction, $\delta_n(t) \in F(x_n(a_n(t))) - \dot{x}_n(t) + f(t, x_n(a_n(t)))$, hence $\|\delta_n(t)\| \leq M + \lambda(t) + k(t) =: \alpha(t)$, and

$$\begin{aligned} d(\delta_n(t), G(t)) = d(\delta_n(t) + \dot{x}(t) - f(t, x(t)), F(x(t)) \leq \|\dot{x}_n(t) - \dot{x}(t)\| \\ + \|f(t, x_n(a_n(t))) - f(t, x(t))\| + d^*(F(x_n(a_n(t))), F(x(t)))\,, \end{aligned}$$

where $d^*(A, B) = \sup\,\{d(a, B) : a \in A\}$.

Since $\dot{x}_n(t) \to \dot{x}(t), x_n(a_n(t)) \to x(t)$, the map $x \to f(t, x)$ is continuous and F is upper semicontinuous we have that

$$d(\delta_n(t), G(t)) \to 0 \text{ for } n \to \infty\,.$$

Then Lemma 3.2. implies that

$$\psi(t) := n \bigcap_{n\in N} cl(\bigcup_{n\geq i} \{\delta_n(t)\})$$

is nonempty, compact, contained in $G(t)$ and $t \to \psi(t)$ is measurable.

Taking $G^*(t) = \partial V(x(t)) \cap B(0,\alpha(t))$ we have that $\delta_n(t) \in \partial V(x_n(t)) \cap B(0,\alpha(t))$ and since $x \to \partial V(x) \cap B(0,\alpha(t))$ is upper semicontinuous, it follows that

$$d(\delta_n(t), G^*(t)) \to 0 \text{ for } n \to \infty$$

hence, by Lemma 3.2., $\psi(t) \subset \partial V(x(t)) \cap B(0,\alpha(t))$.

Let $\sigma(\cdot)$ be a measurable selection of $\psi(\cdot)$ and set $h(t) := \dot{x}(t) + \sigma(t) - f(t,x(t))$. Then $\sigma(\cdot)$, and also $h(.)$, belong to $L^2([0,T],R^n)$, and by definition of G, we have that $h(t) \in F(x(t))$.

Therefore $\dot{x} = -\sigma(t) + h(t) + f(t,x(t)) \in -\partial V(x(t)) + h(t) + f(t,x(t))$ and the proof is complete. ∎

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