Sharp embeddings of Besov spaces involving only slowly varying smoothness

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Abstract

We prove sharp embeddings of Besov spaces $B^{0,b}_{p,r}$ involving only a slowly varying smoothness b into Lorentz-Karamata spaces. As consequences of our results, we obtain the growth envelope of the Besov space $B^{0,b}_{p,r}$.

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1 Introduction

The aim of this paper is to find sharp embeddings of Besov spaces $B^{0,b}_{p,r} = B^{0,b}_{p,r}(\mathbb{R}^n)$ (involving the zero classical smoothness and a slowly varying smoothness b) into Lorentz-Karamata spaces $L^{loc}_{p,q;\tilde{b}}$ (here $\tilde{b}$ might be another slowly varying function) provided that $1 \le p < \infty$, $1 \le r \le \infty$ and $0 < q \le \infty$. As consequences of our results, we obtain the growth envelope of the Besov space $B^{0,b}_{p,r}$. In distinction to the case when the classical smoothness is positive, we show that we cannot describe all embeddings in question in terms of growth envelopes.

The paper is a direct continuation of [3], where such a problem was solved for the particular case when $b(t) = \ell^{\beta}(t)$ and $\tilde{b} = \ell^{\gamma}(t)$ with $\ell(t) := 1 + |\ln t|$, $t > 0$. In that paper we have made use of the fact that the smoothness is of logarithmic form and a corresponding discretization of quasi-norms involved in the problem to prove the sharp embeddings. However, in this paper, where a more general setting is treated, we do not use a discretization, our method to prove the embeddings is a more straightforward.

To solve the problem, we use Kolyada's inequality (see [17]) and its converse form (see [3, Proposition 3.5] or Proposition 8.1 mentioned below) to characterize the given embedding as a weighted inequality involving a certain integral operator (see Theorem 3.5 below). We convert this inequality to a reverse Hardy inequality and we solve it to prove the embedding in question.

To prove its sharpness, we test the mentioned weighted inequality with convenient test functions and we also use some known characterizations of weighted inequalities involving quasi-concave operators.

Embeddings of Besov spaces into rearrangement invariant spaces were considered by Goldman [11], Goldman and Kerman [12], and Netrusov [19]. These authors used different methods and considered a more general setting. However, as mentioned in [11], a characterization of embeddings in question can be obtained from [19] only when $q = r$. Furthermore, the methods used in [11] also do not allow to consider the full range of parameters. Indeed, after a careful checking, one can see that the restriction $1 < p \le r$ appears in the relevant theorem (cf. Theorem 3 of [11]).

The paper is organized as follows. Section 2 contains notation, basic definitions and preliminary assertions. In Section 3 we present main results (Theorems 3.1, 3.2, 3.3 and 3.5). Section 4 is devoted to the proof of Theorem 3.5. Sufficiency part of Theorem 3.1 is proved in Section 5, while the proof of the necessity part of Theorem 3.1 is given in Section 6. Theorem 3.2 is proved in Section 7. Finally, the proof of Theorem 3.3 is given in Section 8.

2 Notation, basic definitions and preliminaries

For two non-negative expressions (i.e. functions or functionals) $\mathcal{A}$ and $\mathcal{B}$, the symbol $\mathcal{A} \precsim \mathcal{B}$ (or $\mathcal{A} \succsim \mathcal{B}$) means that $\mathcal{A} \le c\,\mathcal{B}$ (or $c\,\mathcal{A} \ge \mathcal{B}$), where c is a positive constant independent of appropriate quantities involved in $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \precsim \mathcal{B}$ and $\mathcal{A} \succsim \mathcal{B}$, we write $\mathcal{A} \approx \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*. Throughout the paper we use the abbreviation LHS$(*)$ (RHS$(*)$) for the left- (right-) hand side of the relation $(*)$. Furthermore, we adopt the convention that $\frac{0}{0} = 0$ and $0.\infty = 0$.

Given a set A, its characteristic function is denoted by χ_A. Given two sets A and B, we write $A \Delta B$ for their symmetric difference. For $a \in \mathbb{R}^n$ and $r \ge 0$, the notation $B(a, r)$ stands for the closed ball in $\mathbb{R}^n$ centred at a with the radius r. The volume of $B(0, 1)$ in $\mathbb{R}^n$ is denoted by V_n though, in general, we use the notation $|\cdot|_n$ for Lebesgue measure in $\mathbb{R}^n$.

Let Ω be a Borel subset of $\mathbb{R}^n$. The symbol $\mathcal{M}_0(\Omega)$ is used to denote the family of all complex-valued or extended real-valued (Lebesgue-)measurable functions defined and finite a.e. on Ω. By $\mathcal{M}_0^+(\Omega)$ we mean the subset of

$\mathcal{M}_0(\Omega)$ consisting of those functions which are non-negative a.e. on Ω. If $\Omega = (a,b) \subset \mathbb{R}$, we write simply $\mathcal{M}_0(a,b)$ and $\mathcal{M}_0^+(a,b)$ instead of $\mathcal{M}_0((a,b))$ and $\mathcal{M}_0^+((a,b))$, respectively. By $\mathcal{M}_0^+(a,b;\downarrow)$ or $\mathcal{M}_0^+(a,b;\uparrow)$ we mean the collection of all $f \in \mathcal{M}_0^+(a,b)$ which are non-increasing or non-decreasing on (a,b), respectively. Finally, by $AC(a,b)$ we denote the family of all functions which are locally absolutely continuous on (a,b) (that is, absolutely continuous on any closed subinterval of (a,b)).

For $f \in \mathcal{M}_0(\mathbb{R}^n)$, we define the *non-increasing rearrangement* f^* by

$$f^*(t) := \inf\{\lambda \geq 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|_n \leq t\}, \quad t \geq 0.$$

The corresponding *maximal function* f^{**} is given by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,ds \tag{2.1}$$

and is also non-increasing on the interval $(0,\infty)$.

Given a Borel subset Ω of $\mathbb{R}^n$ and $0 < r \leq \infty$, $L_r(\Omega)$ is the usual *Lebesgue space* of measurable functions for which the quasi-norm

$$\|f\|_{r,\Omega} := \begin{cases} (\int_\Omega |f(t)|^r\,dt)^{1/r} & \text{if } 0 < r < \infty \\ \operatorname{ess\,sup}_{t\in\Omega}|f(t)| & \text{if } r = \infty \end{cases}$$

is finite. When $\Omega = \mathbb{R}^n$, we simplify $L_r(\Omega)$ to L_r and $\|\cdot\|_{r,\Omega}$ to $\|\cdot\|_r$.

Definition 2.1 *Let (α,β) be one of the intervals $(0,\infty)$, $(0,1)$ or $(1,\infty)$. A function $b \in \mathcal{M}_0^+(\alpha,\beta)$, $0 \not\equiv b \not\equiv \infty$, is said to be slowly varying on (α,β), notation $b \in SV(\alpha,\beta)$, if, for each $\varepsilon > 0$, there are functions $g_\varepsilon \in \mathcal{M}_0^+(\alpha,\beta;\uparrow)$ and $g_{-\varepsilon} \in \mathcal{M}_0^+(\alpha,\beta;\downarrow)$ such that*

$$t^\varepsilon b(t) \approx g_\varepsilon(t) \quad \text{and} \quad t^{-\varepsilon} b(t) \approx g_{-\varepsilon}(t) \quad \text{for all } \ t \in (\alpha,\beta).$$

Here we follow the definition of $SV(0,+\infty)$ given in [8]; for other definitions see, for example, [1, 4, 5, 20]. The family of all slowly varying functions includes not only powers of iterated logarithms and the broken logarithmic functions of [7] but also such functions as $t \to \exp\left(|\log t|^a\right)$, $a \in (0,1)$. (The last mentioned function has the interesting property that it tends to infinity more quickly than any positive power of the logarithmic function.)

We shall need some properties of slowly varying functions.

Lemma 2.2 *Let $b \in SV(0,1)$.*

1. *Given $\alpha > 0$ and $\beta \in \mathbb{R}$, then the functions $t \mapsto b(t^\alpha)$ and $t \mapsto (b(t))^\beta$ are also in $SV(0,1)$; given $a \in SV(0,1)$, then $ab \in SV(0,1)$.*

2. *If* $\varepsilon > 0$*, then* $t^{\varepsilon} b(t) \to 0$ *as* $t \to 0+$*.*

3. *The extension of* b *by* 1 *outside of* $(0,1)$ *gives a function in* $SV(0,\infty)$*. (Such an extension will be assumed throughout this lemma, whenever* b *is considered in points outside of* $(0,1)$*.)*

4. *The functions* b *and* b^{-1} *are bounded in the interval* $(\delta, 1]$ *for any* $\delta \in (0,1)$*.*

5. *Given* $c > 0$*, then* $b(ct) \approx b(t)$ *for all* $t \in (0,1)$*.*

6. *If* $\varepsilon > 0$ *and* $0 < r \le \infty$*, then*

$$\|t^{\varepsilon - 1/r} b(t)\|_{r,(0,T)} \approx T^{\varepsilon} b(T) \quad \textit{and} \quad \|t^{-\varepsilon - 1/r} b(t)\|_{r,(T,2)} \approx T^{-\varepsilon} b(T)$$

for all $T \in (0,1]$*.*

7. *If* $0 < r \le \infty$*, then the function* $B(t) := \|\tau^{-1/r} b(\tau)\|_{r,(t,2)}$*,* $t \in (0,1)$*, belongs to* $SV(0,1)$ *and the estimate* $b(t) \lesssim B(t)$ *holds for all* $t \in (0,1)$*.*

8. $\displaystyle \limsup_{t \to 0+} \frac{\int_t^1 s^{-1} b(s)\, ds}{b(t)} = \infty.$

Proof. We only prove assertion 8 here, as some of the others are easy consequences of Definition 2.1, and the proofs of the rest of them can be found, e.g., in [8, Proposition 2.2] and [13, Lemma 2.1].

Assume that assertion 8 does not hold. Then there exist $b \in SV(0,1)$, $c_1 > 0$ and $t_0 \in (0,1)$ such that $\int_t^1 s^{-1} b(s)\, ds \le c_1\, b(t)$ for all $t \in (0, t_0)$. Since $\int_t^1 s^{-1} b(s)\, ds \approx \int_t^2 s^{-1} b(s)\, ds$ for all $t \in (0, t_0)$,

$$\exists c_2 > 0 : \qquad f(t) := \int_t^2 s^{-1} b(s)\, ds \;\le\; c_2 b(t) \quad \forall t \in (0, t_0). \tag{2.2}$$

Consequently, given $\varepsilon \in (0, c_2^{-1})$, the function $t \mapsto t^{\varepsilon} f(t)$ (which belongs to $AC(0, t_0)$) is decreasing on $(0, t_0)$. Indeed, by (2.2), $(t^{\varepsilon} f(t))' = t^{\varepsilon - 1}(\varepsilon f(t) - b(t)) < 0$ for all $t \in (0, t_0)$. However, by assertion 7, $f \in SV(0,1)$ and, by assertion 2, $\lim_{t \to 0+} t^{\varepsilon} f(t) = 0$. Thus, $f \equiv 0$ on $(0, t_0)$, which is a contradiction. Hence, assertion 8 holds. □

More properties and examples of slowly varying functions can be found in [23, Chapt. V, p. 186], [1], [4], [5], [18], [20] and [8].

Throughout the paper, we adopt the following.

Convention 2.3 *If* $b \in SV(0,1)$*, then we assume that* b *is extended by* 1 *in the interval* $[1, \infty)$*.*

Let $p, q \in (0, +\infty]$ and let $b \in SV(0,1)$. The *Lorentz-Karamata* space $L^{loc}_{p,q;b}$ is defined to be the set of all measurable functions $f \in \mathbb{R}^n$ such that

$$\|t^{1/p-1/q}\, b(t)\, f^*(t)\|_{q;(0,1)} < +\infty. \tag{2.3}$$

Note that Lorentz-Karamata spaces involve as particular cases the generalized Lorentz-Zygmund spaces, the Lorentz spaces, the Zygmund classes and Lebesgue spaces (cf., e.g., [4]).

Given $f \in L_p$, $1 \le p < \infty$, the *first difference operator* Δ_h of step $h \in \mathbb{R}^n$ transforms f in $\Delta_h f$ defined by

$$(\Delta_h f)(x) := f(x+h) - f(x), \quad x \in \mathbb{R}^n,$$

whereas the *modulus of continuity* of f is given by

$$\omega_1(f,t)_p := \sup_{\substack{h\in\mathbb{R}^n\\ |h|\le t}} \|\Delta_h f\|_p, \quad t > 0.$$

Definition 2.4 *Let $1 \le p < \infty$, $1 \le r \le \infty$ and let $b \in SV(0,1)$ be such that*

$$\|t^{-1/r} b(t)\|_{r,(0,1)} = \infty. \tag{2.4}$$

The Besov space $B^{0,b}_{p,r} = B^{0,b}_{p,r}(\mathbb{R}^n)$ consists of those functions $f \in L_p$ for which the norm

$$\|f\|_{B^{0,b}_{p,r}} := \|f\|_p + \|t^{-1/r} b(t)\, \omega_1(f,t)_p\|_{r,(0,1)} \tag{2.5}$$

is finite.

Remark 2.5 (i) *Note that only the case when* (2.4) *holds is of interest. Indeed, otherwise $B^{0,b}_{p,r} \equiv L_p$ since*

$$\omega_1(f,t)_p \le 2\|f\|_p \quad \text{for all } t > 0 \text{ and } f \in L_p. \tag{2.6}$$

(ii) *An equivalent norm results on $B^{0,b}_{p,r}(\mathbb{R}^n)$ if the modulus of continuity $\omega_1(f,\cdot)_p$ in* (2.5) *is replaced by the k-th order modulus of continuity $\omega_k(f,\cdot)_p$, where $k \in \{2,3,4,\ldots\}$. Indeed, this is a corollary of the Marchaud theorem* (cf. [2, Thm. 4.4, Chapt. 5]) *and the Hardy-type inequality from* Lemma 4.1 (*with $P = Q$, $b_1 = b_2$*) *below.*

(iii) *Let the function $b \in SV(0,\infty)$ satisfy*

$$\|t^{-1/r} b(t)\|_{r,(1,\infty)} < \infty. \tag{2.7}$$

Then the functional

$$\|f\|_p + \|t^{-1/r} b(t)\, \omega_1(f,t)_p\|_{r,(0,\infty)} \tag{2.8}$$

is an equivalent norm on $B^{0,b}_{p,r}(\mathbb{R}^n)$. *Indeed, this follows from* (2.7) *and* (2.6).

Note also that assumption (2.7) *is natural. Otherwise the space of all functions on* $\mathbb{R}^n$ *for which norm* (2.8) *is finite is trivial* (*that is, it consists only of the zero element*). *This is a consequence of the estimate*

$$\omega_1(f,1)_p \|t^{-1/r}b(t)\|_{r,(1,\infty)} \le \|t^{-1/r}b(t)\omega_1(f,t)_p\|_{r,(1,\infty)}.$$

In the next definition (we refer to [14] for details — see also [22, Chapt. II]) we need the notion of a Borel measure μ associated with a non-decreasing function $g:(a,b)\to\mathbb{R}$, where $-\infty \le a < b \le \infty$. We mean by this the unique (non-negative) measure μ on the Borel subsets of (a,b) such that $\mu([c,d]) = g(d+) - g(c-)$ for all $[c,d]\subset(a,b)$.

Definition 2.6 *Let* $(A, \|\cdot\|_A) \subset \mathcal{M}_0(\mathbb{R}^n)$ *be a quasi-normed space such that* $A \not\hookrightarrow L_\infty$. *A positive, non-increasing, continuous function* h *defined on some interval* $(0,\varepsilon]$, $\varepsilon\in(0,1)$, *is called the* (local) growth envelope function *of the space* A *provided that*

$$h(t) \approx \sup_{\|f\|_A\le 1} f^*(t) \quad \textit{for all} \quad t\in(0,\varepsilon].$$

Given a growth envelope function h *of the space* A (*determined up to equivalence near zero*) *and a number* $u\in(0,\infty]$, *we call the pair* (h,u) *the* (local) growth envelope *of the space* A *when the inequality*

$$\Big(\int_{(0,\varepsilon)} \Big(\frac{f^*(t)}{h(t)}\Big)^q \, d\mu_H(t)\Big)^{1/q} \lesssim \|f\|_A$$

(*with the usual modification when* $q=\infty$) *holds for all* $f\in A$ *if and only if the positive exponent* q *satisfies* $q\ge u$. *Here* μ_H *is the Borel measure associated with the non-decreasing function* $H(t) := -\ln h(t)$, $t\in(0,\varepsilon)$. *The component* u *in the growth envelope pair is called the* fine index.

3 Main Results

Theorem 3.1 *Let* $1\le p<\infty$, $1\le r\le\infty$, $0<q\le\infty$ *and let* $b\in SV(0,1)$ *satisfy* (2.4). *Put* $b(t)=1$ *if* $t\in[1,2)$. *Define, for all* $t\in(0,1)$,

$$b_r(t) := \|s^{-1/r}b(s^{1/n})\|_{r,(t,2)} \tag{3.1}$$

and

$$\tilde{b}(t) := \begin{cases} b_r(t)^{1-r/q+r/\max\{p,q\}} b(t^{1/n})^{r/q-r/\max\{p,q\}} & \textit{if } r\ne\infty \\ b_\infty(t) & \textit{if } r=\infty \end{cases}. \tag{3.2}$$

Then the inequality

$$\|t^{1/p-1/q}\tilde{b}(t)f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{3.3}$$

holds for all $f \in B^{0,b}_{p,r}$ *if and only if* $q \geq r$.

The next result shows that the embedding given by (3.3) is sharp.

Theorem 3.2 *Let* $1 \leq p < \infty$, $1 \leq r \leq q \leq \infty$ *and let* $b \in SV(0,1)$ *satisfy* (2.4). *Put* $b(t) = 1$ *if* $t \in [1,2)$, *define* b_r *and* $\tilde{b}$ *by* (3.1) *and* (3.2). *Let* $\kappa \in \mathcal{M}_0^+(0,1;\downarrow)$. *Then the inequality*

$$\|t^{1/p-1/q}\tilde{b}(t)\kappa(t)f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{3.4}$$

holds for all $f \in B^{0,b}_{p,r}$ *if and only if* κ *is bounded.*

As consequences of our results, we are able to determine the growth envelope of the Besov space $B^{0,b}_{p,r}$.

Theorem 3.3 *Let* $1 \leq p < \infty$, $1 \leq r \leq \infty$ *and let* $b \in SV(0,1)$ *satisfy* (2.4). *Put* $b(t) = 1$ *if* $t \in [1,2)$ *and define* b_r *by* (3.1). *Then the growth envelope of* $B^{0,b}_{p,r}$ *is the pair*

$$(t^{-1/p}\, b_r(t)^{-1}, \max\{p,r\}).$$

Remark 3.4 (i) *Strictly speaking,* $t^{-\frac{1}{p}} b_r(t)^{-1}$ *might not have all the properties associated to a growth envelope function mentioned in* Definition 2.6 *but, with the help of part 6 of* Lemma 2.2, *it is possible to show that there is always an equivalent function defined on* $(0,1)$, *namely,*

$$h(t) := \int_t^2 s^{-1/p-1}\, b_r(s)^{-1}\, ds,$$

which does.

(ii) *Put* $H(t) := -\ln h(t)$ *for* $t \in (0,\varepsilon)$, *where* $\varepsilon \in (0,1)$ *is small enough. Since* $H'(t) \approx \frac{1}{t}$ *for a.e.* $t \in (0,\varepsilon)$ (*cf.* (8.5) *below*), *the measure* μ_H *associated with the function* H *satisfies* $d\mu_H(t) \approx \frac{dt}{t}$. *Thus, by* Definition 2.6, Theorem 3.3 *and part* (i) *of this remark,*

$$\|t^{1/p-1/q}b_r(t)f^*(t)\|_{q,(0,\varepsilon)} \lesssim \|f\|_{B^{0,b}_{p,r}} \quad \text{for all } f \in B^{0,b}_{p,r} \tag{3.5}$$

if and only if

$$q \geq \max\{p,r\}. \tag{3.6}$$

Hence, if (3.6) *holds, then inequality* (3.5) *gives the same result as inequality* (3.3) *of* Theorem 3.1 (*since* (3.6) *implies that* $\tilde{b} = b_r$). *However, if* $r \leq q < p$, *then inequality* (3.5) *does not hold, while inequality* (3.3) *does. This means that the embeddings of Besov spaces* $B^{0,b}_{p,r}$ *given by* Theorem 3.1 *cannot be described in terms of growth envelopes when* $1 \leq r \leq q < p < \infty$.

Our approach to embeddings of Besov spaces $B^{0,b}_{p,r}$ is based on the following theorem.

Theorem 3.5 *Let $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$ and let $b \in SV(0,1)$ satisfy (2.4). Assume that ω is a measurable function on $(0,1)$.*

(i) *Then*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{3.7}$$

for all $f \in B^{0,b}_{p,r}$ if and only if

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_p + \left\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(u) - f^*(t))^p \, du \Big)^{1/p} \right\|_{r,(0,1)} \tag{3.8}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ such that $|\mathrm{supp} f|_n \le 1$.

(ii) *Moreover, when $p = 1$, then inequality (3.7) holds for all $f \in B^{0,b}_{1,r}$ if and only if*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \left\| t^{-1/r} b(t^{1/n}) \int_0^t f^*(u) \, du \right\|_{r,(0,1)} \tag{3.9}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ such that $|\mathrm{supp} f|_n \le 1$.

4 Proof of Theorem 3.5

We shall need the following Hardy-type inequality, which is a consequence of [21, Thm. 6.2].

Lemma 4.1 *Let $1 \le P \le Q \le \infty$, $\nu \in \mathbb{R} \setminus \{0\}$ and let $b_1, b_2 \in SV(0,1)$. Then the inequality*

$$\left\| t^{\nu - 1/Q} \, b_2(t) \int_t^1 g(s) \, ds \right\|_{Q,(0,1)} \lesssim \|t^{\nu + 1 - 1/P} b_1(t) g(t)\|_{P,(0,1)}$$

holds for all $g \in \mathcal{M}_0^+(0,1)$ if and only if $\nu > 0$ and $b_2 \lesssim b_1$ on $(0,1)$.

We refer to [15, Thm. 2.4] for the next auxiliary result.

Lemma 4.2 *Let $0 < Q \le P \le 1$, $\Phi \in \mathcal{M}_0^+(\mathbb{R}_+ \times \mathbb{R}_+)$ and $v, w \in \mathcal{M}_0^+(0,\infty)$. Then the inequality*

$$\Big[\int_0^\infty \Big(\int_0^\infty \Phi(x,y) h(y) \, dy \Big)^P w(x) \, dx \Big]^{1/P} \lesssim \Big[\int_0^\infty h(x)^Q v(x) \, dx \Big]^{1/Q} \tag{4.1}$$

holds for every $h \in \mathcal{M}_0^+(0,\infty;\uparrow)$ if and only, for all $R > 0$,

$$\Big[\int_0^\infty \Big(\int_R^\infty \Phi(x,y) \, dy \Big)^P w(x) \, dx \Big]^{1/P} \lesssim \Big[\int_R^\infty v(x) \, dx \Big]^{1/Q}. \tag{4.2}$$

We shall also need the next assertion.

Lemma 4.3 (see [3, Proposition 4.2]) *Given $p > 0$ and a non-increasing function $g : (0, \infty) \to \mathbb{R}$, the function*

$$t \mapsto \int_0^t (g(s) - g(t))^p \, ds \tag{4.3}$$

is non-decreasing on $(0, \infty)$. In particular, if $f \in \mathcal{M}_0(\mathbb{R}^n)$, then the functions

$$t \to \int_0^t (f^*(s) - f^*(t))^p \, ds \tag{4.4}$$

and

$$t \to t(f^{**}(t) - f^*(t)) \tag{4.5}$$

are non-decreasing on $(0, \infty)$.

To prove Theorem 3.5 we shall also make use of the following lemma concerning RHS(3.8).

Lemma 4.4 *Let $1 \le p < \infty$, $1 \le r \le \infty$, and let $b \in SV(0,1)$. Then*

$$\begin{aligned} &\Big\| t^{1-1/r} b(t) \Big(\int_{t^n}^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\ &\approx \|f\|_p + \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(u) - f^*(t))^p \, du \Big)^{1/p} \Big\|_{r,(0,1)} \end{aligned} \tag{4.6}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ with $|\operatorname{supp} f|_n \le 1$.

Proof. Put $S = \{f \in \mathcal{M}_0(\mathbb{R}^n)\colon |\operatorname{supp} f|_n \le 1\}$. If $f \in S$, then function (4.4) is non-decreasing on $(0, \infty)$. Therefore, for all $t \in (0,1)$ and every $f \in S$,

$$\begin{aligned} &\Big(\int_{t^n}^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, \frac{ds}{s} \Big)^{1/p} \\ &\ge \Big(\int_0^{t^n} (f^*(u) - f^*(t^n))^p \, du \Big)^{1/p} \Big(\int_{t^n}^2 s^{-p/n} \, \frac{ds}{s} \Big)^{1/p} \\ &\approx t^{-1} \Big(\int_0^{t^n} (f^*(u) - f^*(t^n))^p \, du \Big)^{1/p}. \end{aligned} \tag{4.7}$$

Together with the change of variables $t^n = \tau$, this implies that, for all $f \in S$,

$$\text{LHS(4.6)} \gtrsim \Big\| \tau^{-1/r} b(\tau^{1/n}) \Big(\int_0^\tau (f^*(u) - f^*(\tau))^p \, d\tau \Big)^{1/p} \Big\|_{r,(0,1)}. \tag{4.8}$$

If $f \in S$, then $f^*(s) = 0$ for all $s \in [1, \infty)$. Thus, for all $t \in (0,1)$ and every $f \in S$,

$$
\begin{aligned}
&\Big(\int_{t^n}^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, \frac{ds}{s} \Big)^{1/p} \\
&\geq \Big(\int_1^2 s^{-p/n} \int_0^s f^*(u)^p \, du \, \frac{ds}{s} \Big)^{1/p} \\
&\geq \Big(\int_0^1 f^*(u)^p \, du \Big)^{1/p} \Big(\int_1^2 s^{-p/n} \, \frac{ds}{s} \Big)^{1/p} \\
&\approx \|f\|_p.
\end{aligned}
$$

Consequently,

$$
\mathrm{LHS}(4.6) \gtrsim \|f\|_p \, \|t^{1-1/r} b(t)\|_{r,(0,1)} \approx \|f\|_p \quad \text{for all } f \in S.
$$

This estimate and (4.8) show that

$$
\mathrm{LHS}(4.6) \gtrsim \mathrm{RHS}(4.6) \quad \text{for all } f \in S. \tag{4.9}
$$

Now, we are going to prove the reverse estimate. Given $f \in S$, we put

$$
h(s) = h_f(s) := \int_0^s (f^*(u) - f^*(s))^p \, du, \; s \in (0,2). \tag{4.10}
$$

Then

$$
\begin{aligned}
\mathrm{LHS}(4.6) &\approx \Big\| \tau^{1/n-1/r} b(\tau^{1/n}) \Big(\int_\tau^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\
&\lesssim \Big\| \tau^{1/n-1/r} b(\tau^{1/n}) \Big(\int_\tau^1 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\
&+ \| \tau^{1/n-1/r} b(\tau^{1/n}) \Big(\int_1^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\
&\leq \Big\| \tau^{1/n-1/r} b(\tau^{1/n}) \Big(\int_\tau^1 s^{-p/n} h(s) \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\
&+ \Big\| \tau^{1/n-1/r} b(\tau^{1/n}) \Big(\int_1^2 s^{-p/n} \int_0^s f^*(u)^p \, du \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\
&=: N_1 + N_2.
\end{aligned} \tag{4.11}
$$

Moreover,

$$
\begin{aligned}
N_2 &\leq \Big(\int_0^2 f^*(u)^p \, du \Big)^{1/p} \Big(\int_1^2 s^{-p/n} \frac{ds}{s} \Big)^{1/p} \| \tau^{1/n-1/r} b(\tau^{1/n}) \|_{r,(0,1)} \\
&\approx \|f\|_p \quad \text{for all } f \in S.
\end{aligned} \tag{4.12}
$$

To estimate N_1, we distinguish two cases.

(i) Assume that $r/p \in [1, +\infty]$. Then, using Lemma 4.1 (with $P = Q = r/p$, $\nu = p/n$, $b_2(t) = b_1(t) = b(t^{1/n})$, $g(s) = s^{-p/n-1}h(s)$), we obtain, for all $f \in S$,

$$\begin{aligned} N_1^p &= \Big\| \tau^{p/n-p/r} b(\tau^{1/n})^p \int_\tau^1 g(s)\, ds \Big\|_{r/p,(0,1)} \\ &\lesssim \| \tau^{p/n+1-p/r} b(\tau^{1/n})^p g(\tau) \|_{r/p,(0,1)} \\ &= \| \tau^{-p/r} b(\tau^{1/n})^p h(\tau) \|_{r/p,(0,1)} \\ &\approx \| \tau^{-1/r} b(\tau^{1/n}) h(\tau)^{1/p} \|_{r,(0,1)}^p \\ &= \Big\| \tau^{-1/r} b(\tau^{1/n}) \Big(\int_0^\tau (f^*(u) - f^*(\tau))^p\, du \Big)^{1/p} \Big\|_{r,(0,1)}^p. \end{aligned} \tag{4.13}$$

Combining estimates (4.11)–(4.13), we see that

$$\mathrm{LHS}(4.6) \lesssim \mathrm{RHS}(4.6) \quad \text{for all } f \in S. \tag{4.14}$$

(ii) Assume that $r/p \in (0, 1)$. First we prove that, for all $f \in S$,

$$\begin{aligned} N_1^p &= \Big\| \tau^{p/n-p/r} b(\tau^{1/n})^p \int_\tau^1 s^{-p/n-1} h(s)\, ds \Big\|_{r/p,(0,1)} \\ &\lesssim \| \tau^{-p/r} b(\tau^{1/n})^p h(\tau) \|_{r/p,(0,2)} =: N_3 \end{aligned} \tag{4.15}$$

The function h given by (4.10) is non-decreasing on $(0, \infty)$. Thus, to verify (4.15), we apply Lemma 4.2. On putting $Q = P = r/p$ and

$$\begin{aligned} w(x) &= \chi_{(0,1)}(x) x^{r/n-1} b(x^{1/n})^r, \\ v(x) &= \chi_{(0,2)}(x) x^{-1} b(x^{1/n})^r, \\ \Phi(x, y) &= \chi_{(x,1)}(y) y^{-p/n-1} \end{aligned}$$

for all $x, y \in (0, \infty)$, we see that inequality (4.15) can be rewritten as (4.1). Consequently, by Lemma 4.2, inequality (4.15) holds for every $h \in \mathcal{M}_0^+(0, \infty; \uparrow)$ provided that condition (4.2) is satisfied.

Making use of Lemma 2.2, we obtain that, for all $R > 0$,

$$\mathrm{LHS}(4.2) \lesssim \Big[b(R^{1/n})^p + \Big(\int_R^1 x^{-1} b(x^{1/n})^r\, dx \Big)^{p/r} \Big] \chi_{(0,1)}(R)$$

and

$$\mathrm{RHS}(4.2) \approx \Big[\int_R^2 x^{-1} b(x^{1/n})^r\, dx \Big]^{p/r} \chi_{(0,2)}(R).$$

Therefore, condition (4.2) is satisfied, which means that inequality (4.15) holds.

To finish the proof, it is sufficient to show that

$$N_3^{1/p} \lesssim \mathrm{RHS}(4.6) \quad \text{for all } f \in S. \tag{4.16}$$

The definition of N_3 and (4.10) imply that, for all $f \in S$,

$$\begin{aligned} N_3^{1/p} &= \|\tau^{-1/r} b(\tau^{1/n}) h(\tau)^{1/p}\|_{r,(0,2)} \\ &\approx \|\tau^{-1/r} b(\tau^{1/n}) h(\tau)^{1/p}\|_{r,(0,1)} + \|\tau^{-1/r} b(\tau^{1/n}) h(\tau)^{1/p}\|_{r,(1,2)} \\ &\approx \Big\|\tau^{-1/r} b(\tau^{1/n}) \Big(\int_0^\tau (f^*(u) - f^*(\tau))^p \, du\Big)^{1/p} \Big\|_{r,(0,1)} \\ &\quad + \Big\|\tau^{-1/r} b(\tau^{1/n}) \Big(\int_0^\tau (f^*(u) - f^*(\tau))^p \, du\Big)^{1/p} \Big\|_{r,(1,2)}. \end{aligned} \tag{4.17}$$

Comparing this estimate with RHS(4.6), we see that it is enough to verify that

$$\Big\|\tau^{-1/r} b(\tau^{1/n}) \Big(\int_0^\tau (f^*(u) - f^*(\tau))^p \, du\Big)^{1/p} \Big\|_{r,(1,2)} \lesssim \|f\|_p \tag{4.18}$$

for all $f \in S$. However, such an estimate is an easy consequence of the facts that function (4.4) is non-decreasing on $(0, \infty)$, that $|\operatorname{supp} f|_n \le 1$, and that $\|\tau^{-1/r} b(\tau^{1/n})\|_{r,(1,2)} < \infty$. □

The next lemma provides another expression equivalent to RHS (3.8). (We shall need this assertion in Section 5.)

Lemma 4.5 *Let $1 \le p < \infty$, $1 \le r \le \infty$, and let $b \in SV(0,1)$. Then*

$$\begin{aligned} \|f\|_p &+ \Big\|t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(u) - f^*(t))^p \, du\Big)^{1/p} \Big\|_{r,(0,1)} \\ &\approx \Big\|t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(u) - f^*(t))^p \, du\Big)^{1/p} \Big\|_{r,(0,2)} \end{aligned} \tag{4.19}$$

for all $f \in S := \{f \in \mathcal{M}_0(\mathbb{R}^n)\colon |\operatorname{supp} f|_n \le 1\}$.

Proof. Since RHS(4.19) $= N_3^{1/p}$ (cf. (4.15) and (4.10)), we see from (4.17) and (4.18) that

$$\text{RHS(4.19)} \lesssim \text{LHS(4.19)} \quad \text{for all } f \in S.$$

On the other hand, since

$$\begin{aligned} \text{RHS(4.19)} &\ge \Big\|t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(u) - f^*(t))^p \, du\Big)^{1/p} \Big\|_{r,(1,2)} \\ &\ge \Big(\int_0^1 f^*(u)^p \, du\Big)^{1/p} \|t^{-1/r} b(t^{1/n})\|_{r,(1,2)} \\ &\approx \|f\|_p, \end{aligned}$$

it is clear that

$$\text{RHS(4.19)} \gtrsim \text{LHS(4.19)} \quad \text{for all } f \in S.$$

$\square$

We shall need the following variant of Lemmas 4.4 and 4.5.

Lemma 4.6 *Let $1 \le p < \infty$, $1 \le r \le \infty$ and let $b \in SV(0,\infty)$. Then*

$$
\begin{aligned}
&\|f\|_p + \Big\| t^{-1/r} b(t) \Big(\int_0^t f^*(s)^p \, ds \Big)^{1/p} \Big\|_{r,(0,1)} \\
&\approx \Big\| t^{-1/r} b(t) \Big(\int_0^t f^*(s)^p \, ds \Big)^{1/p} \Big\|_{r,(0,1)} \\
&\approx \Big\| t^{-1/r} b(t) \Big(\int_0^t f^*(s)^p \, ds \Big)^{1/p} \Big\|_{r,(0,2)} \qquad (4.20)
\end{aligned}
$$

for all $f \in S := \{f \in \mathcal{M}_0(\mathbb{R}^n) \colon |\operatorname{supp} f|_n \le 1\}$.

Proof. Since, for all $f \in S$,

$$
\begin{aligned}
&\Big\| t^{-1/r} b(t) \Big(\int_0^t f^*(s)^p \, ds \Big)^{1/p} \Big\|_{r,(1/2,1)} \\
&\ge \Big(\int_0^{1/2} f^*(s)^p \, ds \Big)^{1/p} \| t^{-1/r} b(t) \|_{r,(1/2,1)} \\
&\approx \Big(\int_0^{1/2} f^*(s)^p \, ds \Big)^{1/p} \ge \frac{1}{2^{1/p}} \Big(\int_0^1 f^*(s)^p \, ds \Big)^{1/p} = \frac{1}{2^{1/p}} \|f\|_p, \qquad (4.21)
\end{aligned}
$$

the first estimate in (4.20) is clear. Furthermore, for all $f \in S$,

$$
\begin{aligned}
\Big\| t^{-1/r} b(t) \Big(\int_0^t f^*(s)^p \, ds \Big)^{1/p} \Big\|_{r,(1,2)} &\le \Big(\int_0^2 f^*(s)^p \, ds \Big)^{1/p} \| t^{-1/r} b(t) \|_{r,(1,2)} \\
&\approx \Big(\int_0^2 f^*(s)^p \, ds \Big)^{1/p} = \|f\|_p. \qquad (4.22)
\end{aligned}
$$

The second estimate in (4.20) is a consequence of (4.22) and (4.21). $\square$

The last result which we need to prove Theorem 3.5 reads as follows.

Proposition 4.7 *Let $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$ and let $b \in SV(0,1)$ satisfy (2.4). Assume that ω is a measurable function on $(0,1)$. Then*

$$
\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \qquad (4.23)
$$

for all $f \in B^{0,b}_{p,r}$ if and only if

$$
\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \Big\| t^{1-1/r} b(t) \Big(\int_{t^n}^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \qquad (4.24)
$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ such that $|\operatorname{supp} f|_n \le 1$.

Proof is analogous to that of Proposition 3.6 in [3] (where the slowly varying function b was of logarithmic type). $\square$

Proof of Theorem 3.5. Part (i) of Theorem 3.5 follows from Proposition 4.7 and Lemma 4.4.

Now, we put $p = 1$ and we prove part (ii) of Theorem 3.5.

By Proposition 4.7, it is enough to verify that RHS(4.24) $\approx$ RHS(3.9). Lemmas 4.4 and 4.6 imply that, for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ with $|\mathrm{supp} f|_n \le 1$,

$$\begin{aligned} \mathrm{RHS}(4.24) &\approx \|f\|_1 + \left\| t^{-1/r} b(t^{1/n}) \int_0^t (f^*(u) - f^*(t))\, du \right\|_{r,(0,1)} \\ &\le \|f\|_1 + \left\| t^{-1/r} b(t^{1/n}) \int_0^t f^*(u)\, du \right\|_{r,(0,1)} \\ &\approx \mathrm{RHS}(3.9). \end{aligned}$$

To prove the reverse estimate, we apply Fubini's theorem to obtain, for all $t > 0$ and $f \in \mathcal{M}_0(\mathbb{R}^n)$,

$$\begin{aligned} & t \int_{t^n}^\infty s^{-1/n} \int_0^s (f^*(u) - f^*(s))\, du \, \frac{ds}{s} \\ & \qquad \int_0^{t^n} f^*(u)\, du + (n-1) \int_{t^n}^\infty f^*(u) u^{-1/n}\, du\,. \end{aligned}$$

Hence, for all $t > 0$ and $f \in \mathcal{M}_0(\mathbb{R}^n)$,

$$\int_0^{t^n} f^*(u)\, du \lesssim t \int_{t^n}^\infty s^{-1/n} \int_0^s (f^*(u) - f^*(s))\, du \, \frac{ds}{s}\,. \tag{4.25}$$

Using a change of variables, (4.25) and Lemma 4.4, we arrive at

$$\begin{aligned} \mathrm{RHS}(3.9) &\lesssim \left\| t^{1-1/r} b(t) \int_{t^n}^\infty s^{-1/n} \int_0^s (f^*(u) - f^*(s))\, du \, \frac{ds}{s} \right\|_{r,(0,1)} \\ &\le \left\| t^{1-1/r} b(t) \int_{t^n}^2 s^{-1/n} \int_0^s (f^*(u) - f^*(s))\, du \, \frac{ds}{s} \right\|_{r,(0,1)} \\ &\quad + \|f\|_1 \|t^{1-1/r} b(t)\|_{r,(0,1)} \int_2^\infty s^{-1/n} \frac{ds}{s} \\ &\approx \mathrm{RHS}(4.24) + \|f\|_1 \\ &\approx \mathrm{RHS}(4.24) \quad \text{for all } f \in \mathcal{M}_0(\mathbb{R}^n) \text{ with } |\mathrm{supp} f|_n \le 1. \end{aligned}$$

$\square$

5 Proof of the sufficiency part of Theorem 3.1

We shall need the following reverse Hardy inequality, which is a particular case of [6, Thm. 5.1].

Lemma 5.1 *Let* $0 < Q \le P \le 1$, $w, u \in \mathcal{M}_0^+(0,2)$ *and let* $\|u\|_{Q,(t,2)} < +\infty$ *for all* $t \in (0,2)$. *Then the inequality*

$$\|gw\|_{P,(0,2)} \lesssim \Big\| u(x) \int_0^x g(y)\,dy \Big\|_{Q,(0,2)} \tag{5.1}$$

holds for all $g \in \mathcal{M}_0^+(0,2)$ *if and only if*

$$\mathcal{B} := \sup_{x \in (0,2)} \|w\|_{P',(x,2)} \|u\|_{Q,(x,2)}^{-1} < \infty, \tag{5.2}$$

where $P' = P/(1-P)$ *if* $P \in (0,1)$ *and* $P' = \infty$ *if* $P = 1$.

We shall also use the next result on the boundedness of the identity operator between the cones of non-negative and non-decreasing functions in weighted Lebesgue spaces.

Lemma 5.2 (see [16, Proposition 2.1(i)]) *Let* $0 < P \le Q < \infty$ *and let* $w, v \in \mathcal{M}_0^+(0,\infty)$. *Then there exists a constant* C *such that the inequality*

$$\Big[\int_0^\infty g(x)^Q w(x)\,dx\Big]^{1/Q} \le C\Big[\int_0^\infty g(x)^P v(x) dx\Big]^{1/P} \tag{5.3}$$

holds for all $g \in \mathcal{M}_0^+(0,\infty;\uparrow)$ *if and only if*

$$A := \sup_{R>0} \Big(\int_R^\infty w(t)\,dt\Big)^{1/Q} \Big(\int_R^\infty v(t)\,dt\Big)^{-1/P} < \infty. \tag{5.4}$$

Moreover, if C *is the least constant for which* (5.3) *holds, then* $C = A$.

We shall also need the following assertions.

Lemma 5.3 (see [3, Proposition 4.5]) *If* $1 < p < \infty$, *then*

$$\int_0^t (f^{**}(s) - f^*(s))^p\,ds \lesssim \int_0^t (f^*(s) - f^*(t))^p\,ds \lesssim \int_0^{2t} (f^{**}(s) - f^*(s))^p\,ds$$

for all $t > 0$ *and* $f \in L_p$.

Lemma 5.4 (see [15, Theorems 2.1 and 2.3]) *Let* $1 \le Q \le P < \infty$, $\Phi \in \mathcal{M}_0^+(\mathbb{R}_+ \times \mathbb{R}_+)$ *and let* $v, w \in \mathcal{M}_0^+(0,\infty)$.

(i) *The inequality*

$$\Big[\int_0^\infty g^P(x) w(x)\,dx\Big]^{1/P} \lesssim \Big[\int_0^\infty \Big(\int_0^\infty \Phi(x,y) g(y)\,dy\Big)^Q v(x)\,dx\Big]^{1/Q} \tag{5.5}$$

holds for all $g \in \mathcal{M}_0^+(0,\infty;\downarrow)$ *if and only if, for all* $R > 0$,

$$\Big[\int_0^R w(x)\,dx\Big]^{1/P} \lesssim \Big[\int_0^\infty \Big(\int_0^R \Phi(x,y)\,dy\Big)^Q v(x)\,dx\Big]^{1/Q}. \tag{5.6}$$

(ii) *Inequality* (5.5) *holds for all* $g \in \mathcal{M}_0^+(0,\infty;\uparrow)$ *if and only if, for all* $R > 0$,

$$\Big[\int_R^\infty w(x)\,dx\Big]^{1/P} \lesssim \Big[\int_0^\infty \Big(\int_R^\infty \Phi(x,y)\,dy\Big)^Q v(x)\,dx\Big]^{1/Q}. \tag{5.7}$$

Proof of the sufficiency part of Theorem 3.1. Assume that $q \geq r$. Put

$$\omega(t) := t^{1/p-1/q}\tilde{b}(t), \qquad t \in (0,1). \tag{5.8}$$

By Theorem 3.5 it is enough to verify that inequality (3.8) holds for all $f \in S_p := \{f \in L_p : |\mathrm{supp}\, f|_n \leq 1\}$. Moreover, the inequality $f^* \leq f^{**}$ and the identity (see [3, (16)])

$$f^{**}(t) - f^{**}(1) = \int_t^1 \frac{f^{**}(s) - f^*(s)}{s}\,ds$$

for all $f \in L_p$ and $t \in (0,1)$ imply that

$$\begin{aligned} \mathrm{LHS}(3.8) &\leq \|\omega(t) f^{**}(t)\|_{q,(0,1)} \\ &\leq f^{**}(1)\ \|t^{1/p-1/q}\tilde{b}(t)\|_{q,(0,1)} \\ &\quad + \Big\|t^{1/p-1/q}\,\tilde{b}(t)\int_t^1 \frac{f^{**}(s) - f^*(s)}{s}\,ds\Big\|_{q,(0,1)}. \end{aligned} \tag{5.9}$$

Since $|\mathrm{supp}\, f|_n \leq 1$ if $f \in S_p$, $1 \leq p < \infty$ and $\tilde{b} \in SV(0,1)$, we get

$$f^{**}(1)\ \|t^{1/p-1/q}\,\tilde{b}(t)\|_{q,(0,1)} \lesssim \|f\|_p \quad \text{for all } f \in S_p. \tag{5.10}$$

Our assumptions $q \geq r$ and $1 \leq r \leq \infty$ show that $q \in [1,\infty]$. Therefore, using Lemma 4.1 (with $P = Q = q$, $\nu = 1/p$, $b_1(t) = b_2(t) = \tilde{b}(t)$ and $g(s) = [f^{**}(s) - f^*(s)]/s$), we arrive at

$$\begin{aligned} &\Big\|t^{1/p-1/q}\tilde{b}(t)\int_t^1 \frac{f^{**}(s) - f^*(s)}{s}\,ds\Big\|_{q,(0,1)} \\ &\lesssim \|t^{1/p-1/q}\tilde{b}(t)[f^{**}(t) - f^*(t)]\|_{q,(0,1)} \quad \text{for all } f \in S_p. \end{aligned} \tag{5.11}$$

Combining estimates (5.9)–(5.11), we obtain that

$$\mathrm{LHS}(3.8) \lesssim \|f\|_p + \|t^{1/p-1/q}\tilde{b}(t)[f^{**}(t) - f^*(t)]\|_{q,(0,1)} \quad \text{for all } f \in S_p. \tag{5.12}$$

Together with Lemma 4.5, this implies that inequality (3.8) will be satisfied if we prove that, for all $f \in S_p$,

$$\|t^{1/p-1/q}\tilde{b}(t)[f^{**}(t) - f^*(t)]\|_{q,(0,1)} \lesssim \Big\|t^{-1/r}b(t^{1/n})\Big(\int_0^t (f^*(u) - f^*(t))^p\,du\Big)^{1/p}\Big\|_{r,(0,2)}. \tag{5.13}$$

Moreover, if $p \in (1,\infty)$, then the first estimate in Lemma 5.3 shows that (5.13) is a consequence of the inequality

$$\|t^{1/p-1/q}\tilde{b}(t)[f^{**}(t) - f^*(t)]\|_{q,(0,1)} \lesssim \Big\|t^{-1/r}\,b(t^{1/n})\Big(\int_0^t (f^{**}(u) - f^*(u))^p\,du\Big)^{1/p}\Big\|_{r,(0,2)} \quad \text{for all } f \in S_p. \tag{5.14}$$

(i) Assume that $p \in (1,\infty)$ and $r \le q \le p$. Then $q/p \in (0,1]$ and (5.14) can be rewritten as

$$\|t^{1-p/q}\tilde{b}(t)^p g(t)\|_{q/p,(0,1)} \lesssim \Big\|t^{-p/r}b(t^{1/n})^p \int_0^t g(u)\,du\Big\|_{r/p,(0,2)}, \tag{5.15}$$

where the function $g \in \mathcal{M}_0^+(0,2)$ is given by

$$g(t) = g_f(t) := [f^{**}(t) - f^*(t)]^p, \quad t \in (0,2).$$

To verify (5.15), we apply Lemma 5.1. On putting $P = q/p$, $Q = r/p$, $w(x) = x^{1-p/q}\tilde{b}(x)^p\chi_{(0,1)}(x)$, $u(x) = x^{-p/r}b(x^{1/n})^p$ for all $x \in (0,2)$, we see that inequality (5.15) coincides with (5.1). Consequently, by Lemma 5.1, inequality (5.15) holds on $\mathcal{M}_0^+(0,2)$ provided that condition (5.2) is satisfied, that is, when

$$\|t^{1-p/q}\tilde{b}(t)^p\chi_{(0,1)}(t)\|_{\frac{q}{p-q},(x,2)} \lesssim \|t^{-p/r}b(t^{1/n})^p\|_{\frac{r}{p},(x,2)} \tag{5.16}$$

for all $x \in (0,2)$. Since, for all $x \in (0,2)$,

$$\mathrm{LHS}(5.16) \lesssim b_r(x)^p\chi_{(0,1)}(x) \quad \text{and} \quad \mathrm{RHS}(5.16) = b_r(x)^p,$$

condition (5.16) holds. Consequently, inequality (5.15) (and also (5.14) and (5.13)) is satisfied for all $f \in S_p$. Therefore, inequality (3.8) holds on S_p.

(ii) Assume that $r \le q$ and $1 < p < q < \infty$. Then $\tilde{b} = b_r$.

First we prove that, for all $f \in S_p$,

$$\|t^{1/p-1/q}\tilde{b}(t)[f^{**}(t) - f^*(t)]\|_{q,(0,1)} \lesssim \Big\|t^{-1/q}b(t^{1/n})^{r/q}b_r(t)^{(q-r)/q}\Big(\int_0^t (f^{**}(s) - f^*(s))^p\,ds\Big)^{1/p}\Big\|_{q,(0,2)}. \tag{5.17}$$

Denoting $g(t) := (t\,[f^{**}(t) - f^{*}(t)])^p$, $t > 0$, we see that it is enough to show that the inequality

$$\begin{aligned} &\|t^{1-p-p/q} b_r(t)^p g(t)\|_{q/p,(0,1)} \\ &\lesssim \Big\| t^{-p/q} b(t^{1/n})^{rp/q} b_r(t)^{(q-r)p/q} \int_0^t s^{-p} g(s)\, ds \Big\|_{q/p,(0,2)} \end{aligned} \tag{5.18}$$

holds for all $g \in \mathcal{M}_0^+(0,\infty;\uparrow)$.

To verify (5.18), we apply Lemma 5.4(ii). On putting $Q = P = q/p$,

$$\begin{aligned} w(x) &= \chi_{(0,1)}(x)\, x^{q/p-q-1} b_r(x)^q, \\ v(x) &= \chi_{(0,2)}(x)\, x^{-1} b(x^{1/n})^r b_r(x)^{q-r}, \\ \Phi(x,y) &= \chi_{(0,x)}(y)\, y^{-p} \end{aligned}$$

for all $x, y \in (0,\infty)$, inequality (5.18) coincides with (5.5). Consequently, by Lemma 5.4(ii), inequality (5.18) holds on $\mathcal{M}_0^+(0,\infty;\uparrow)$ provided that condition (5.7) is satisfied. This is the case since, for all $R > 0$,

$$\mathrm{LHS}(5.7) \lesssim R^{1-p}\, b_r(R)^p \chi_{(0,1)}(R)$$

and, for all $R \in (0,1)$,

$$\begin{aligned} \mathrm{RHS}(5.7) &= \Big(\int_R^2 x^{-1} b(x^{1/n})^r b_r(x)^{q-r} \Big(\int_R^x y^{-p}\, dy \Big)^{q/p} dx \Big)^{p/q} \\ &\geq \Big(\int_{\frac{3}{2}R}^2 x^{-1} b(x^{1/n})^r b_r(x)^{q-r}\, dx \Big(\int_R^{\frac{3}{2}R} y^{-p}\, dy \Big)^{q/p} \Big)^{p/q} \\ &\approx R^{1-p}\, b_r(\tfrac{3}{2}R)^p \\ &\approx R^{1-p}\, b_r(R)^p. \end{aligned}$$

Consequently, inequality (5.18) (and (5.17) as well) is proved.

Recall that our aim is to show that inequality (5.14) holds. As LHS(5.17) = LHS(5.14), inequality (5.14) will be satisfied provided that we prove that

$$\mathrm{RHS}(5.17) \lesssim \mathrm{RHS}(5.14). \tag{5.19}$$

On putting $h(t) := (\int_0^t (f^{**}(s) - f^{*}(s))^p\, ds)^{1/p}$, $t > 0$, we see that (5.19) is a consequence of the inequality

$$\begin{aligned} &\Big(\int_0^2 t^{-1} b(t^{1/n})^r\, b_r(t)^{q-r} h(t)^q\, dt \Big)^{1/q} \\ &\lesssim \Big(\int_0^2 t^{-1} b(t^{1/n})^r h(t)^r\, dt \Big)^{1/r} \quad \text{for all } h \in \mathcal{M}_0^+(0,\infty;\uparrow). \end{aligned} \tag{5.20}$$

To prove (5.20), we apply Lemma 5.2 with $P = r$, $Q = q$ and

$$w(t) = \chi_{(0,2)}(t)\, t^{-1} b(t^{1/n})^r\, b_r(t)^{q-r}, \qquad v(t) = \chi_{(0,2)}(t)\, t^{-1} b(t^{1/n})^r$$

for all $t \in (0,\infty)$. In our case, for all $R > 0$,

$$\Big(\int_R^\infty w(t)\,dt\Big)^{1/Q} \lesssim b_r(R)\,\chi_{(0,2)}(R),$$
$$\Big(\int_R^\infty v(t)\,dt\Big)^{1/P} = b_r(R)\,\chi_{(0,2)}(R),$$

which implies that condition (5.4) is satisfied. Consequently, inequality (5.20) holds for all $h \in \mathcal{M}_0^+(0,\infty;\uparrow)$.

(iii) The case $p \in (1,\infty)$, $1 \le r \le \infty$, $q = \infty$. Again, it is enough to show that (5.13) holds for all $f \in S_p$. In our case,

$$\tilde b(t) = b_r(t) = \|s^{-1/r} b(s^{1/n})\|_{r,(t,2)}, \quad t \in (0,1).$$

By Hölder's inequality,

$$f^{**}(t) - f^*(t) \le t^{-1/p}\Big(\int_0^t (f^*(u) - f^*(t))^p\,du\Big)^{1/p} \quad \text{for all} \;\; t > 0. \tag{5.21}$$

Thus, applying (5.21) and the monotonicity of function (4.4), we obtain

$$\begin{aligned} &t^{1/p}\,\tilde b(t)\,[f^{**}(t) - f^*(t)] \\ &\le \|s^{-1/r} b(s^{1/n})\|_{r,(t,2)}\Big(\int_0^t (f^*(u) - f^*(t))^p\,du\Big)^{1/p} \\ &\le \Big\| s^{-1/r} b(s^{1/n})\Big(\int_0^s (f^*(u) - f^*(s))^p\,du\Big)^{1/p}\Big\|_{r,(t,2)} \end{aligned}$$

for all $t \in (0,1)$. This implies that, for all $f \in S_p$,

$$\begin{aligned} &\|t^{1/p}\,\tilde b(t)\,[f^{**}(t) - f^*(t)]\|_{\infty,(0,1)} \\ &\le \Big\| s^{-1/r} b(s^{1/n})\Big(\int_0^s (f^*(u) - f^*(s))^p\,du\Big)^{1/p}\Big\|_{r,(0,2)}, \end{aligned} \tag{5.22}$$

which is desired inequality (5.13).

(iv) The case $1 = p \le r \le q < \infty$. In this case $\tilde b = b_r$. By Theorem 3.5 (ii), it is sufficient to show that inequality (3.9) (with ω given by (5.8)), that is,

$$\|t^{1-1/q} b_r(t) f^*(t)\|_{q,(0,1)} \lesssim \Big\| t^{-1/r} b(t^{1/n}) \int_0^t f^*(s)\,ds \Big\|_{r,(0,1)}, \tag{5.23}$$

holds for all $f \in S_1 := \{f \in L_1 : |\operatorname{supp} f|_n \le 1\}$. Putting $Q = r$, $P = q$, $g = f^*$,

$$\begin{aligned} w(x) &= \chi_{(0,1)}(x)\, x^{q-1} b_r(x)^q, \\ v(x) &= \chi_{(0,1)}(x)\, x^{-1} b(x^{1/n})^r, \\ \Phi(x,y) &= \chi_{(0,x)}(y) \end{aligned}$$

for all $x, y \in (0,\infty)$, we see that inequality (5.23) coincides with (5.5). Thus, by Lemma 5.4(i), inequality (5.23) holds on S_1 provided that condition (5.6) is satisfied. This is the case since, for all $R > 0$,

$$\mathrm{LHS}(5.6) \approx R\, b_r(R)\, \chi_{(0,1)}(R) + \chi_{[1,\infty)}(R)$$

and

$$\mathrm{RHS}(5.6) \approx [R\, b(R^{1/n}) + R\, b_r(R)]\, \chi_{(0,1)}(R) + \chi_{[1,\infty)}(R).$$

(v) The case $p = 1$, $1 \le r \le \infty$, $q = \infty$. As in part (iv), we see that it is enough to verify that inequality (5.23) with $q = \infty$ holds on S_1. By Lemma 4.6 (with p=1), this will be the case if

$$\|t\, b_r(t)\, f^*(t)\|_{\infty,(0,1)} \lesssim \Big\|t^{-1/r} b(t^{1/n}) \int_0^t f^*(s)\, ds\Big\|_{r,(0,2)} \quad \text{for all } f \in S_1. \tag{5.24}$$

Using the formula for b_r, the trivial estimate $t\, f^*(t) \le \int_0^t f^*(\tau)\, d\tau$, $t > 0$, and the monotonicity of the function $t \mapsto \int_0^t f^*(\tau)\, d\tau$, $t > 0$, we obtain

$$\begin{aligned} t\, b_r(t) f^*(t) &\le \|s^{-1/r} b(s^{1/n})\|_{r,(t,2)} \int_0^t f^*(\tau)\, d\tau \\ &\le \Big\|s^{-1/r} b(s^{1/n}) \Big(\int_0^s f^*(\tau)\, d\tau \Big)\Big\|_{r,(t,2)} \end{aligned}$$

for all $t \in (0,1)$. This implies that inequality (5.24) holds on S_1. $\square$

6 Proof of the necessity part of Theorem 3.1

We shall need the following function:

$$\ell(t) := 1 + |\ln t|, \quad t \in (0,\infty).$$

Note that $\ell \in SV(0,\infty)$.

First we prove two technical lemmas.

Lemma 6.1 *Let $1 \le p < \infty$ and let $b \in SV(0,1)$. Define $b(t) = 1$ for $t \in [1,2)$ and put*

$$b_\infty(t) := \operatorname*{ess\,sup}_{s \in (t,2)} b(s^{1/n}), \quad t \in (0,1),$$

$$v(t) := t\, b(t^{1/n})^p \chi_{(0,1)}(t) + \ell(t) \chi_{[1,\infty)}(t), \quad t \in (0,\infty),$$

$$\phi(t) := \operatorname*{ess\,sup}_{s \in (0,t)} \big(s \operatorname*{ess\,sup}_{\tau \in (s,\infty)} \frac{v(\tau)}{\tau}\big), \quad t \in (0,\infty). \tag{6.1}$$

Then

$$\phi(t) \approx \begin{cases} t\, b_\infty(t)^p & \textit{for all } t \in (0,1] \\ \ell(t) & \textit{for all } t \in (1,\infty) \end{cases}. \tag{6.2}$$

Proof. Assume first that $t \in (0,1]$. Then, using assertions 4, 1, 7 and 6 of Lemma 2.2,

$$\begin{aligned}
\phi(t) &= \operatorname*{ess\,sup}_{s\in(0,t)} \Big(s \max\{ \operatorname*{ess\,sup}_{\tau\in(s,1)} b(\tau^{1/n})^p,\ \operatorname*{ess\,sup}_{\tau\in[1,\infty)} \frac{\ell(\tau)}{\tau}\}\Big)\\
&\approx \operatorname*{ess\,sup}_{s\in(0,t)} (s\, b_\infty(s)^p)\\
&= \|s^{1-1/\infty} b_\infty(s)^p\|_{\infty,(0,t)}\\
&\approx t\, b_\infty(t)^p \quad \text{for all } t \in (0,1].
\end{aligned}$$

Assume now that $t \in (1,\infty)$. Then

$$\begin{aligned}
\phi(t) &= \max\{ \operatorname*{ess\,sup}_{s\in(0,1)} \big(s \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{v(\tau)}{\tau}\big),\ \operatorname*{ess\,sup}_{s\in[1,t)} \big(s \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{v(\tau)}{\tau}\big)\}\\
&\approx \max\{\phi(1), \ell(t)\}\\
&\approx \ell(t) \quad \text{for all } t \in (1,\infty).
\end{aligned}$$

□

In the next lemma we consider the maximal function b_∞^{**} given by (cf. (2.1)) $b_\infty^{**}(t) := t^{-1}\int_0^t b_\infty(\tau)\,d\tau$, $t \in (0,1]$, where b_∞ is the function from Lemma 6.1. By part 6 of Lemma 2.2, $b_\infty^{**} \approx b_\infty$ on $(0,1]$. Moreover, $b_\infty^{**} \in AC(0,1)$.

Lemma 6.2 *Let p, b, b_∞, v and ϕ be the same as in Lemma 6.1. Assume that (2.4) with $r = \infty$ holds. Let $0 < q < \infty$ and ν be the measure on $[0,\infty)$ which satisfies*

$$d\nu(t) = \begin{cases} -b_\infty^{**}(t)^{-q-1}(b_\infty^{**})'(t)\,dt & \text{if } 0 < t \le 1\\ t^{q/p-1}\ell^{-q/p-1}(t)\,dt & \text{if } t > 1 \end{cases}. \tag{6.3}$$

Then

$$\int_{[0,\infty)} \frac{d\nu(s)}{s^{q/p}+t^{q/p}} \approx \frac{1}{\phi(t)^{q/p}} \quad \textit{for all } t \in (0,\infty).$$

Proof. Since $(b_\infty^{**})'(t) = t^{-1}(b_\infty(t) - b_\infty^{**}(t)) \le 0$ *a.e.* on $(0,1)$, the measure ν is non-negative.

(i) Let $t \in (1,\infty)$. In view of (6.2), we need to show that

$$I = I(t) := \int_{[0,\infty)} \frac{d\nu(s)}{s^{q/p}+t^{q/p}} \approx \ell(t)^{-q/p} \quad \text{for all } t \in (1,\infty).$$

Split the integral in the following three terms:

$$\begin{aligned} I_1 &:= \int_{(0,1)} \frac{-b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)}{s^{q/p}+t^{q/p}}\,ds\,, \\ I_2 &:= \int_{(1,t)} \frac{s^{q/p-1}\ell^{-q/p-1}(s)}{s^{q/p}+t^{q/p}}\,ds\,, \\ I_3 &:= \int_{(t,\infty)} \frac{s^{q/p-1}\ell^{-q/p-1}(s)}{s^{q/p}+t^{q/p}}\,ds\,. \end{aligned}$$

Since $(b_\infty^{**}(s)^{-q})' = -q\,b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)$ for a.e. $s \in (0,1)$ and $b_\infty^{**\,-q}$ is non-decreasing on $[0,1]$,

$$\begin{aligned} I_1 &\le t^{-q/p}\int_0^1 -b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)\,ds \\ &\le \frac{1}{q}\,t^{-q/p}b_\infty^{**}(1)^{-q} \\ &\approx t^{-q/p} \le \ell(t)^{-q/p} \quad \text{for all } t \in (1,\infty). \end{aligned}$$

Furthermore, for all $t \in (1,\infty)$,

$$\begin{aligned} I_2 &\le t^{-q/p}\int_1^t s^{q/p-1}\ell^{-q/p-1}(s)\,ds \\ &\le t^{-q/p}\int_0^t s^{q/p-1}\ell^{-q/p-1}(s)\,ds \\ &\approx \ell(t)^{-q/p-1} \le \ell(t)^{-q/p} \end{aligned}$$

and

$$I_3 \le \int_t^\infty s^{-1}\ell^{-q/p-1}(s)\,ds \approx \ell(t)^{-q/p}.$$

So, we have got the estimate of I by $\ell(t)^{-q/p}$ from above. To prove the reverse estimate, note that

$$I_3 \ge \frac{1}{2}\int_t^\infty s^{-1}\ell^{-q/p-1}(s)\,ds \approx \ell(t)^{-q/p} \quad \text{for all } t \in (1,\infty).$$

(ii) Consider now $t \in (0,1]$. By (6.2), we need to show that

$$J = J(t) := \int_{[0,\infty)} \frac{d\nu(s)}{s^{q/p}+t^{q/p}} \approx t^{-q/p}b_\infty(t)^{-q} \quad \text{for all } t \in (0,1].$$

Again, we split the integral in three terms:

$$\begin{aligned} J_1 &:= \int_{(0,t)} \frac{-b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)}{s^{q/p}+t^{q/p}}\,ds\,, \\ J_2 &:= \int_{[t,1]} \frac{-b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)}{s^{q/p}+t^{q/p}}\,ds\,, \\ J_3 &:= \int_{(1,\infty)} \frac{s^{q/p-1}\ell^{-q/p-1}(s)}{s^{q/p}+t^{q/p}}\,ds\,. \end{aligned}$$

As before,

$$\begin{aligned} J_1 &\le t^{-q/p}\int_0^t -b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)\,ds \\ &\lesssim t^{-q/p}b_\infty^{**}(t)^{-q} \approx t^{-q/p}b_\infty(t)^{-q} \quad \text{for all } t\in(0,1]. \end{aligned}$$

Using the fact that $b_\infty^{**}\in AC(0,1)$, the integration by parts, assertions 6 and 1 of Lemma 2.2 together with the definition of slowly varying functions, we obtain , for all $t\in(0,1]$,

$$\begin{aligned} J_2 &\le \int_t^1 -s^{-q/p}b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)\,ds \\ &\lesssim b_\infty^{**}(1)^{-q}+\int_t^2 s^{-q/p-1}b_\infty^{**}(s)^{-q}\,ds \\ &\approx 1+t^{-q/p}b_\infty^{**}(t)^{-q} \approx t^{-q/p}b_\infty(t)^{-q}, \end{aligned}$$

$$J_3 \le \int_1^\infty s^{-1}\ell^{-q/p-1}(s)\,ds \approx 1 \lesssim t^{-q/p}b_\infty(t)^{-q}.$$

So, we have got the estimate of J by $t^{-q/p}b_\infty(t)^{-q}$ from above. To prove the converse estimate, we apply the fact $b_\infty^{**}\in AC(0,1)$ and hypothesis (2.4), to arrive at

$$\begin{aligned} J_1 &\ge \frac{1}{2}t^{-q/p}\int_0^t -b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)\,ds \\ &\approx t^{-q/p}b_\infty^{**}(t)^{-q} \approx t^{-q/p}b_\infty(t)^{-q} \quad \text{for all } t\in(0,1]. \end{aligned}$$

□

Remark 6.3 *It will be useful to note that, for* $1\le p<\infty$, $1\le r\le\infty$, $b\in SV(0,1)$ *and* $f\in\mathcal{M}_0(\mathbb{R}^n)$ *such that* $|\mathrm{supp} f|_n\le 1$, RHS(3.8) *may be estimated from above by*

$$\Big\| t^{-1/r}b(t^{1/n})\Big(\int_0^t f^*(u)^p\,du\Big)^{1/p}\Big\|_{r,(0,1)}.$$

Indeed, this is an immediate consequence of Lemma 4.6.

We shall also make use of the next two assertions which are consequences of more general results of Gogatishvili and Pick [9, Thm. 4.2 (ii),(iii)], [10, Thm. 1.8 (i)]:

Proposition 6.4 *Let* $P,Q\in(0,\infty)$, *let* v,w *be non-negative measurable functions on* $[0,\infty)$ *such that* $V(t):=\int_0^t v(s)\,ds$ *and* $W(t):=\int_0^t w(s)\,ds$ *are finite for all* $t>0$. *Assume that, for all* $t\in(0,\infty)$,

$$\int_{[0,1]}\frac{v(s)}{s^P}\,ds = \int_{[1,\infty)} v(s)\,ds = \infty$$

and

$$\int_{[0,\infty)} \frac{v(s)}{s^P + t^P}\, ds < \infty.$$

(i) *Let* $1 \le Q < P < \infty$ *and* $R = PQ/(P-Q)$. *Then the inequality*

$$\Big(\int_0^\infty w(t) f^*(t)^Q\, dt\Big)^{1/Q} \lesssim \Big(\int_0^\infty v(t) f^{**}(t)^P\, dt\Big)^{1/P} \tag{6.4}$$

holds for all measurable f *on* $\mathbb{R}^n$ *if and only if*

$$\int_0^\infty \frac{t^R \sup_{y\in[t,\infty)} y^{-R} W(y)^{R/Q}}{(V(t) + t^P \int_t^\infty s^{-P} v(s)\, ds)^{R/P+2}}\, V(t) \int_t^\infty s^{-P} v(s)\, ds\, t^{P-1}\, dt < \infty. \tag{6.5}$$

(ii) *Let* $0 < P \le Q < 1$. *Then the inequality* (6.4) *holds for all measurable* f *on* $\mathbb{R}^n$ *if and only if*

$$\sup_{t\in(0,\infty)} \frac{W(t)^{1/Q} + t\,(\int_t^\infty W(s)^{Q/(1-Q)} w(s) s^{-Q/(1-Q)}\, ds)^{(1-Q)/Q}}{(V(t) + t^P \int_t^\infty s^{-P} v(s)\, ds)^{1/P}} < \infty. \tag{6.6}$$

Proposition 6.5 *Let* $1 \le Q < \infty$, *let* v, w *be non-negative, locally integrable functions on* $(0,\infty)$ *and* $W(t) := \int_0^t w(s)\, ds$, $t > 0$. *Define the quasi-concave function*

$$\phi(t) := \operatorname*{ess\,sup}_{s\in(0,t)}\, s \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{v(\tau)}{\tau}, \quad t \in (0,\infty).^1 \tag{6.7}$$

Assume that ϕ *is non-degenerate, that is,*

$$\lim_{t\to 0+} \phi(t) = \lim_{t\to\infty} \frac{1}{\phi(t)} = \lim_{t\to\infty} \frac{\phi(t)}{t} = \lim_{t\to 0+} \frac{t}{\phi(t)} = 0. \tag{6.8}$$

Let ν *be a non-negative Borel measure on* $[0,\infty)$ *such that*

$$\frac{1}{\phi(t)^Q} \approx \int_{[0,\infty)} \frac{d\nu(s)}{s^Q + t^Q} \quad \text{for all } t \in (0,\infty). \tag{6.9}$$

Then the inequality

$$\Big(\int_0^\infty w(t) f^*(t)^Q\, dt\Big)^{1/Q} \lesssim \operatorname*{ess\,sup}_{t\in(0,\infty)} v(t) f^{**}(t) \tag{6.10}$$

holds for all measurable f *on* $\mathbb{R}^n$ *if and only if*

$$\int_{[0,\infty)} \sup_{s\in(t,\infty)} \frac{W(s)}{s^Q}\, d\nu(t) < \infty. \tag{6.11}$$

[1]Recall that ϕ is quasi-concave if ϕ is equivalent to a function in $\mathcal{M}_0^+(0,\infty;\uparrow)$ while $\phi(t)/t$ is equivalent to a function in $\mathcal{M}_0^+(0,\infty;\downarrow)$.

Proof of the necessity part of Theorem 3.1. Assume that (3.3) holds for all $f \in B^{0,b}_{p,r}$. Then, by Theorem 3.5 with $\omega(t) = t^{1/p-1/q}\tilde{b}(t)$,

$$\|t^{1/p-1/q}\tilde{b}(t)f^*(t)\|_{q,(0,1)} \lesssim \|f\|_p + \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(u) - f^*(t))^p \, du \Big)^{1/p} \Big\|_{r,(0,1)} \quad (6.12)$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ such that $|\mathrm{supp} f|_n \le 1$. Our aim is to prove that $r \le q$.

(i) Assume that $0 < q < p < \infty$ and $1 \le r < \infty$. Consider the function $g(t) := t^{-1/p} b_r(t)^{-1-r/p} b(t^{1/n})^{r/p}$, $t \in (0,1)$, with $b_r(t)$ from (3.1). By assertions 1 and 7 of Lemma 2.2, the function $t \mapsto b_r(t)^{-1-r/p} b(t^{1/n})^{r/p}$, $t \in (0,1)$, belongs to $SV(0,1)$. Thus, by Definition 2.1, there is $\varphi \in \mathcal{M}^+_0(0,1;\downarrow)$ such that $g \approx \varphi$ on $(0,1)$.

By (2.4) and (3.1), there is $y_0 \in (0,1)$ such that $\ln b_r(y)^r > 0$ for all $y \in (0, y_0)$. Put, for any $y \in (0, y_0)$,

$$f_y(x) := \varphi(y)\chi_{[0,y]}(V_n|x|^n) + \varphi(V_n|x|^n)\chi_{(y,1)}(V_n|x|^n), \quad x \in \mathbb{R}^n. \quad (6.13)$$

Hence,

$$f^*_y(t) = \varphi(y)\chi_{[0,y]}(t) + \varphi(t)\chi_{(y,1)}(t), \quad t > 0. \quad (6.14)$$

Since $f_y \in \mathcal{M}_0(\mathbb{R}^n)$ and $|\mathrm{supp} f_y|_n = 1$, inequality (6.12) holds for all functions f_y, $y \in (0, y_0)$. Inserting f_y into (6.12), we obtain, for all $y \in (0, y_0)$,

$$\begin{aligned} \mathrm{LHS}(6.12) &= \|t^{1/p-1/q}\tilde{b}(t) f^*_y(t)\|_{q,(0,1)} \\ &\gtrsim \Big(\int_y^1 t^{-1} b_r(t)^{-r} b(t^{1/n})^r \, dt \Big)^{1/q} \\ &\approx (\ln b_r(y)^r)^{1/q}. \end{aligned} \quad (6.15)$$

On the other hand, since φ is non-increasing on $(0,1)$,

$$\begin{aligned} \|f_y\|_p &= \|f^*_y\|_p \\ &\le \Big(\int_0^1 \varphi(t)^p \, dt \Big)^{1/p} \\ &\approx \Big(\int_0^1 t^{-1} b_r(t)^{-p-r} b(t^{1/n})^r \, dt \Big)^{1/p} \\ &\approx b_r(1)^{-1} \approx 1 \quad \text{for all } y \in (0, y_0). \end{aligned} \quad (6.16)$$

Moreover, since also $(f_y^*(u) - f_y^*(t))^p = 0$ when $0 < u < t \le y$,

$$\begin{aligned}
&\Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f_y^*(u) - f_y^*(t))^p \, du \Big)^{1/p} \Big\|_{r,(0,1)} \\
&\le \Big(\int_y^1 t^{-1} b(t^{1/n})^r \Big(\int_0^t f_y^*(u)^p \, du \Big)^{r/p} dt \Big)^{1/r} \\
&\lesssim \Big(\int_y^1 t^{-1} b(t^{1/n})^r \Big(\int_0^t u^{-1} b_r(u)^{-p-r} b(u^{1/n})^r \, du \Big)^{r/p} dt \Big)^{1/r} \\
&\approx \Big(\int_y^1 t^{-1} b(t^{1/n})^r b_r(t)^{-r} \, dt \Big)^{1/r} \\
&\approx (\ln b_r(y)^r)^{1/r}. \qquad (6.17)
\end{aligned}$$

Thus, inserting f_y into (6.12), we obtain, for all $y \in (0, y_0)$,

$$\mathrm{RHS}(6.12) \lesssim (\ln b_r(y)^r)^{1/r}.$$

Together with (6.15), this yields

$$(\ln b_r(y)^r)^{1/q} \lesssim (\ln b_r(y)^r)^{1/r} \quad \text{for all } y \in (0, y_0).$$

Since $\lim_{y \to 0+} \ln b_r(y)^r = \infty$ (cf. (2.4) and (3.1)), the last estimate can hold only if $q \ge r$.

(ii) Now we prove the necessity of the condition $q \ge r$ when $1 \le p \le q \le \infty$ and $1 \le r < \infty$. On the contrary, suppose that $q < r$. Hence, $1 \le p \le q < r < \infty$.

¿From (6.12) and Remark 6.3, we arrive at

$$\| t^{1/p-1/q} \tilde{b}(t) f^*(t) \|_{q,(0,1)} \lesssim \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t f^*(u)^p \, du \Big)^{1/p} \Big\|_{r,(0,1)} \qquad (6.18)$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ with $|\mathrm{supp} f|_n \le 1$. One can see that (6.18) remains true if we omit the assumption $|\mathrm{supp} f|_n \le 1$. (Indeed, if $f \in \mathcal{M}_0(\mathbb{R}^n)$, take $f_1 := f^*(V_n |\cdot|^n) \chi_{[0,1)}(V_n |\cdot|^n)$. Consequently, $f_1^*(t) = f(t)$ for all $t \in (0,1)$, and $|\mathrm{supp} f_1|_n \le 1$. Thus, applying (6.18) to f_1, we obtain the result.) Let $g \in \mathcal{M}_0(\mathbb{R}^n)$ and $f := |g|^{1/p}$. Then (6.18) yields

$$\| t^{1-p/q} \tilde{b}(t)^p g^*(t) \|_{q/p,(0,1)} \lesssim \| t^{-p/r+1} b(t^{1/n})^p g^{**}(t) \|_{r/p,(0,1)} \qquad (6.19)$$

for all $g \in \mathcal{M}_0(\mathbb{R}^n)$ (or even for any measurable function g on $\mathbb{R}^n$). Equation (6.19) implies that the inequality

$$\Big(\int_0^\infty w(t) g^*(t)^{q/p} \, dt \Big)^{p/q} \lesssim \Big(\int_0^\infty v(t) g^{**}(t)^{r/p} \, dt \Big)^{p/r} \qquad (6.20)$$

holds for all measurable g on $\mathbb{R}^n$, where, for all $t \in (0,\infty)$,

$$w(t) = t^{q/p-1} b_r(t)^q \chi_{(0,1)}(t) \quad \text{and} \quad v(t) = t^{r/p-1} b(t^{1/n})^r \chi_{(0,1)}(t) + \chi_{[1,\infty)}(t).$$

By Proposition 6.4(i), with $Q = q/p$ and $P = r/p$, inequality (6.20) holds only if

$$\infty > \int_0^1 \frac{t^{\frac{rq}{(r-q)p}} \sup_{y\in[t,1)} y^{-\frac{rq}{(r-q)p}} \Big(\int_0^y s^{\frac{q}{p}-1} b_r(s)^q\, ds\Big)^{\frac{r}{r-q}}}{\Big(\int_0^t s^{\frac{r}{p}-1} b(s^{\frac{1}{n}})^r\, ds + t^{\frac{r}{p}}\Big(\int_t^1 s^{-1} b(s^{\frac{1}{n}})^r\, ds + \int_1^\infty s^{-\frac{r}{p}}\, ds\Big)\Big)^{\frac{q}{r-q}+2}}$$
$$\times \int_0^t s^{\frac{r}{p}-1} b(s^{\frac{1}{n}})^r\, ds \int_t^1 s^{-1} b(s^{\frac{1}{n}})^r\, ds\, t^{\frac{r}{p}-1}\, dt \;=:\; I.$$

However, choosing $t_0 \in (0,1)$ in such a way that

$$\int_1^2 s^{-1} b(s^{1/n})^r\, ds \le \int_{t_0}^1 s^{-1} b(s^{1/n})^r\, ds$$

(which is possible, due to (2.4)), using assertion 7 of Lemma 2.2, (2.4) and (3.1), we obtain

$$\begin{aligned} I \;&\gtrsim\; \int_0^{t_0} \frac{b_r(t)^{\frac{rq}{r-q}}\, b(t^{1/n})^r\, b_r(t)^r\, t^{-1}}{(b(t^{1/n})^r + b_r(t)^r + \frac{p}{r-p})^{\frac{q}{r-q}+2}}\, dt \\ &\gtrsim\; \int_0^{t_0} t^{-1} b(t^{1/n})^r b_r(t)^{-r}\, dt \;=\; \infty, \end{aligned}$$

which is a contradiction. Consequently, $q \ge r$.

(iii) Assume that $1 \le p \le q \le \infty$ and $r = \infty$. Thus, we want to prove that $q = \infty$. On the contrary, suppose that $q < \infty$. Hence, $1 \le p \le q < \infty$. Proceeding as in part (ii), instead of (6.20), the inequality

$$\Big(\int_0^\infty w(t) g^*(t)^{q/p}\, dt\Big)^{p/q} \lesssim \operatorname*{ess\,sup}_{t\in(0,\infty)} v(t) g^{**}(t) \tag{6.21}$$

holds for all measurable g on $\mathbb{R}^n$, where, for all $t \in (0,\infty)$,

$$w(t) = t^{q/p-1} b_\infty(t)^q \chi_{(0,1)}(t) \quad \text{and} \quad v(t) = t\, b(t^{1/n})^p \chi_{(0,1)}(t) + \ell(t)\chi_{[1,\infty)}(t).$$

In order to apply Proposition 6.5, consider the function given by (6.1). By Lemma 6.1, assertion 2 of Lemma 2.2, (2.4) and (3.1), (6.8) holds. Let ν be the measure given by (6.3). By Lemma 6.2, assumption (6.9) (with $Q = q/p$) is satisfied. Consequently, inequality (6.21) holds only if

$$\infty > \int_0^\infty \sup_{s\in(t,\infty)} \frac{\int_0^s \tau^{\frac{q}{p}-1} b_\infty(\tau)^q \chi_{(0,1)}(\tau)\, d\tau}{s^{\frac{q}{p}}}\, d\nu(t) \;=:\; I.$$

However, using that $(-\ln b_\infty^{**}(t))' = -b_\infty^{**}(t)^{-1}(b_\infty^{**})'(t)$ for a.e. $t \in (0,1)$, (2.4) and (3.1), we arrive at

$$\begin{aligned} I \;&\geq\; \int_0^1 \big(\sup_{s\in(t,1)} b_\infty(s)^q\big)\,(-b_\infty^{**}(t)^{-q-1}(b_\infty^{**})'(t))\,dt \\ &\approx\; \int_0^1 -b_\infty^{**}(t)^{-1}(b_\infty^{**})'(t)\,dt \;=\; \infty, \end{aligned}$$

which is a contradiction. Consequently, $q = \infty = r$.

(iv) Finally, we want to show that it is not possible to have $0 < q < p < \infty$ and $r = \infty$.

By the necessity part of Theorem 3.1 proved in part (iii) (and applied with $q = p$), given $c \in (0,\infty)$, there exists $f \in B^{0,b}_{p,\infty}$ such that

$$\|t^{1/p-1/p} b_\infty(t) f^*(t)\|_{p,(0,1)} > c\,\|f\|_{B^{0,b}_{p,\infty}}.$$

As

$$L^{loc}_{p,q;b_\infty} \hookrightarrow L^{loc}_{p,p;b_\infty} \quad \text{when } 0 < q < p < \infty \tag{6.22}$$

(which can be proved analogously as in the case of Lorentz spaces), we see that, for any $c \in (0,\infty)$, there exists $f \in B^{0,b}_{p,\infty}$ satisfying

$$\|t^{1/p-1/q} b_\infty(t) f^*(t)\|_{q,(0,1)} > c\,\|f\|_{B^{0,b}_{p,\infty}}.$$

However, this contradicts our assumption that (3.3) is valid for all $f \in B^{0,b}_{p,r}$. □

7 Proof of Theorem 3.2

In view of Theorem 3.1, the sufficiency of the condition that κ is bounded is obvious. We are going to prove that this condition is also necessary. To this end, suppose that (3.4) holds for all $f \in B^{0,b}_{p,r}$. Together with Theorem 3.5(i) (where we take $\omega(t) = t^{1/p-1/q}\tilde{b}(t)\kappa(t)$) and Remark 6.3, this implies that

$$\|t^{1/p-1/q}\tilde{b}(t)\kappa(t) f_y^*(t)\|_{q,(0,1)} \lesssim \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t f_y^*(u)^p\,du \Big)^{1/p} \Big\|_{r,(0,1)} \tag{7.1}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ such that $|\mathrm{supp} f|_n \leq 1$.

(i) The case $p \leq q$ (that is, $1 \leq p < \infty$, $1 \leq r \leq q \leq \infty$ and $p \leq q$). For any $y \in (0,1)$, define $f_y(x) := \chi_{[0,y)}(V_n|x|^n)$, $x \in \mathbb{R}^n$. Hence, $f_y^*(t) = \chi_{[0,y)}(t)$, $t > 0$. Since $f_y \in \mathcal{M}_0(\mathbb{R}^n)$ and $|\mathrm{supp} f_y|_n \leq 1$ for all $y \in (0,1)$, inequality (7.1) holds for all functions f_y, $y \in (0,1)$. Inserting

f_y into (7.1) and using assertions 1, 7 and 6 of Lemma 2.2, we obtain, for all $y \in (0,1)$,

$$\text{LHS}(7.1) \;\geq\; \kappa(y)\, \|t^{1/p-1/q} b_r(t)\|_{q,(0,y)} \;\approx\; \kappa(y)\, y^{1/p}\, b_r(y)$$

and

$$\begin{aligned} \text{RHS}(7.1) \;&\lesssim\; \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t 1\, du\Big)^{1/p} \Big\|_{r,(0,y)} \\ &\quad + \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^y 1\, du\Big)^{1/p} \Big\|_{r,(y,1)} \\ &\lesssim\; y^{1/p}\, b(y^{1/n}) + y^{1/p}\, b_r(y) \;\approx\; y^{1/p}\, b_r(y), \end{aligned}$$

which implies that κ is bounded.

(ii) The case $q < p$ (that is, $1 \leq r \leq q < p < \infty$). In particular, q and r are finite. Consider the function $g(t) := t^{-1/p} b_r(t)^{-2r-r/p} b(t^{1/n})^{r/p}$, $t \in (0,1)$. By parts 1 and 7 of Lemma 2.2, the function $t \mapsto b_r(t)^{-2r-r/p} b(t^{1/n})^{r/p}$, $t \in (0,1)$, belongs to $SV(0,1)$. Thus, by Definition 2.1, there is $\varphi \in \mathcal{M}_0^+(0,1;\downarrow)$ such that $g \approx \varphi$ on $(0,1)$. Put, for any $y \in (0,1)$,

$$f_y(x) := \varphi(V_n|x|^n)\chi_{[0,y)}(V_n|x|^n), \quad x \in \mathbb{R}^n.$$

Hence,

$$f_y^*(t) = \varphi(t)\chi_{[0,y)}(t), \quad t > 0.$$

Since $f_y \in \mathcal{M}_0(\mathbb{R}^n)$ and $|\text{supp} f_y|_n \leq 1$, inequality (7.1) holds for all functions f_y, $y \in (0,1)$. Inserting f_y into (7.1) and using the facts that $b_r^{q-2qr} \in AC(0,1)$, $(b_r(t)^{q-2rq})' = (2q - \frac{q}{r})\, t^{-1} b(t^{1/n})^r b_r(t)^{q-2rq-r}$ for a.e. $t \in (0,1)$, (2.4) and (3.1), we obtain, for all $y \in (0,1)$,

$$\text{LHS}(7.1) \;\approx\; \Big(\int_0^y t^{-1} b(t^{1/n})^r b_r(t)^{q-r-2rq} \kappa(t)^q\, dt\Big)^{1/q} \;\gtrsim\; \kappa(y) b_r(y)^{1-2r}$$

and, similarly,

$$\begin{aligned} \text{RHS}(7.1) \;&\lesssim\; \|t^{-1/r} b(t^{1/n}) b_r(t)^{-2r}\|_{r,(0,y)} + \|t^{-1/r} b(t^{1/n}) b_r(y)^{-2r}\|_{r,(y,1)} \\ &\lesssim\; b_r(y)^{1-2r}. \end{aligned}$$

These estimates and (7.1) imply that κ is bounded. □

8 Proof of Theorem 3.3

We shall need the following assertion, which we call the "inverse Kolyada inequality".

Proposition 8.1 (see [3, Prop. 3.5]) (i) *Let $f \in L_1$ and let $F(x) := f^*(V_n|x|^n)$, $x \in \mathbb{R}^n$. Then, for all $t > 0$ and $f \in L_1$,*

$$\begin{aligned}\omega_1(F,t)_1 &\lesssim n\int_0^{t^n} f^*(s)\,ds + (n-1)\,t\int_{t^n}^\infty f^*(s)s^{-1/n}\,ds\\ &= t\Big(\int_{t^n}^\infty s^{-1/n}\int_0^s (f^*(u)-f^*(s))\,du\,\frac{ds}{s}\Big).\end{aligned} \tag{8.1}$$

(ii) *Let $1 < p < \infty$, $f \in L_p$ and let $F(x) = f^{**}(V_n|x|^n)$, $x \in \mathbb{R}^n$. Then, for all $t > 0$ and $f \in L_p$,*

$$\omega_1(F,t)_p \lesssim t\Big(\int_{t^n}^\infty s^{-p/n}\int_0^s (f^*(u)-f^*(s))^p\,du\,\frac{ds}{s}\Big)^{1/p}.$$

Proof of Theorem 3.3 Put $A := B^{0,b}_{p,r}$. By Theorem 3.1 with $q = \infty$, $t^{1/p}b_r(t)f^*(t) \lesssim 1$ for all $t \in (0,1)$ and $f \in A$ with $\|f\|_A \le 1$. Therefore,

$$\sup_{\|f\|_A\le 1} f^*(t) \lesssim t^{-\frac{1}{p}}b_r(t)^{-1} \quad \text{for all } t \in (0,1). \tag{8.2}$$

On the other hand, consider, for $y \in (0,1/2)$, $f_y \in L_p(\mathbb{R}^n)$ with $f_y^* = \chi_{[0,y)}$. It is easy to see that, for all $t > 0$ and $y \in (0,1/2)$,

$$t\Big(\int_{t^n}^\infty s^{-p/n}\int_0^s (f_y^*(u)-f_y^*(s))^p\,du\,\frac{ds}{s}\Big)^{1/p} \approx \min\{y^{1/p}, t\,y^{1/p-1/n}\}. \tag{8.3}$$

Defining

$$F_y(x) := \begin{cases} f_y^{**}(V_n|x|^n) & \text{if } 1<p<\infty \\ f_y^*(V_n|x|^n) & \text{if } p=1\end{cases}, \quad x \in \mathbb{R}^n,$$

using Proposition 8.1, (8.3), assertions 6 and 7 of Lemma 2.2, and hypothesis (2.4), we obtain

$$\begin{aligned}\|F_y\|_A &= \|F_y\|_p + \|t^{-\frac{1}{r}}\,b(t)\,\omega_1(F_y,t)_p\|_{r,(0,1)}\\ &\lesssim y^{\frac{1}{p}} + \|t^{-\frac{1}{r}}\,b(t)\,\min\{y^{\frac{1}{p}}, t\,y^{\frac{1}{p}-\frac{1}{n}}\}\|_{r,(0,1)}\\ &\le y^{\frac{1}{p}} + y^{\frac{1}{p}-\frac{1}{n}}\,\|t^{1-\frac{1}{r}}\,b(t)\|_{r,(0,y^{1/n})} + y^{\frac{1}{p}}\,\|t^{-\frac{1}{r}}\,b(t)\|_{r,(y^{1/n},1)}\\ &\approx y^{\frac{1}{p}}\,(1 + b(y^{\frac{1}{n}}) + \|t^{-\frac{1}{r}}\,b(t)\|_{r,(y^{1/n},1)})\\ &\lesssim y^{\frac{1}{p}}\|t^{-\frac{1}{r}}\,b(t)\|_{r,(y^{1/n},2)}\end{aligned}$$

for all small enough y. Consequently, for all such y,

$$\Big\|\,y^{-\frac{1}{p}}\,\|t^{-\frac{1}{r}}\,b(t)\|^{-1}_{r,(y^{1/n},2)}\,F_y\,\Big\|_A \lesssim 1.$$

Together with the inequality $F_y^{**} \geq f_y^* = \chi_{[0,y)}$, this implies that

$$\sup_{\|f\|_A \leq 1} f^*(t) \gtrsim y^{-\frac{1}{p}} \|s^{-\frac{1}{r}} b(s)\|^{-1}_{r,(y^{1/n},2)} \chi_{[0,y)}(t)$$

for all $t > 0$ and $y \in (0, y_0)$, say (for some y_0 appropriately chosen in $(0, 1/2)$). Thus, taking $y = 2t$ for every $t \in (0, y_0/2)$ and using changes of variables, (2.4), (3.1) and assertion 5 of Lemma 2.2, we obtain

$$\begin{aligned} \sup_{\|f\|_A \leq 1} f^*(t) &\gtrsim t^{-\frac{1}{p}} \|s^{-\frac{1}{r}} b(s)\|^{-1}_{r,((2t)^{1/n},2)} \\ &\approx t^{-\frac{1}{p}} b_r(t)^{-1} \quad \text{for all } t \text{ small enough.} \end{aligned}$$

Together with (8.2), this results in

$$\sup_{\|f\|_A \leq 1} f^*(t) \approx t^{-\frac{1}{p}} b_r(t)^{-1} \quad \text{for all small } t > 0. \tag{8.4}$$

Since $\lim_{t \to 0+} t^{-\frac{1}{p}} b_r(t)^{-1} = \infty$ (cf. assertion 2 of Lemma 2.2), the first consequence of (8.4) is that $A \not\hookrightarrow L_\infty$. Further, as

$$h(t) := \int_t^2 s^{-1/p-1} b_r(s)^{-1} \, ds, \quad t \in (0,1),$$

is a positive, non-increasing and continuous function equivalent to $t^{-\frac{1}{p}} b_r(t)^{-1}$ on $(0,1)$ (cf. Remark 3.4), (8.4) also shows that $h(t)$ is a growth envelope function of the space $A = B^{0,b}_{p,r}$.

To calculate the fine index (cf. Definition 2.6), consider the function $H(t) := -\ln h(t)$, $t \in (0,1)$. Since

$$H'(t) = -\frac{h'(t)}{h(t)} = -\frac{-t^{-\frac{1}{p}-1} b_r(t)^{-1}}{\int_t^2 s^{-1/p-1} b_r(s)^{-1} \, ds} \approx \frac{1}{t} \quad \text{for } a.e.\ t \in (0,1), \tag{8.5}$$

we obtain $d\mu_H(t) = H'(t)\, dt \approx \frac{1}{t}\, dt$ on $(0,1)$. Thus, applying the “if” part of Theorem 3.1 with $q \in [\max\{p,r\}, \infty]$, we get

$$\begin{aligned} &\Big(\int_0^1 \Big(\frac{f^*(t)}{h(t)} \Big)^q d\mu_H(t) \Big)^{1/q} \\ &\approx \Big(\int_0^1 t^{\frac{q}{p}-1} b_r(t)^q f^*(t)^q \, dt \Big)^{1/q} \\ &\lesssim \|f\|_A \quad \text{for all } f \in A \end{aligned} \tag{8.6}$$

(with the usual modification when $q = \infty$).

It remains to show that (8.6) cannot hold for $q \in (0, \max\{p,r\})$.

(i) The case $p \le q < r$. The result follows from Theorem 3.1.

(ii) The case $q < p < r$. By part (i) applied to $p = q < r$, we get that the inequality

$$\Big(\int_0^1 (t^{1/p}\, b_r(t)\, f^*(t))^p\, \frac{dt}{t} \Big)^{1/p} \lesssim \|f\|_A \quad \text{cannot hold for all } f \in A.$$

Since also, by the monotonicity of Lorentz-Karamata spaces (cf. (6.22)),

$$\Big(\int_0^1 (t^{1/p}\, b_r(t)\, f^*(t))^p\, \frac{dt}{t} \Big)^{1/p} \lesssim \Big(\int_0^1 (t^{1/p}\, b_r(t)\, f^*(t))^q\, \frac{dt}{t} \Big)^{1/q}$$

for all $f \in \mathcal{M}_0^+(\mathbb{R}^n)$ if $q < p$, estimate (8.6) cannot hold in the considered case.

(iii) The case $q < r \le p$. By Theorem 3.1, the inequality

$$\Big(\int_0^1 t^{\frac{q}{p}-1}\, b_r(t)^{q-r+\frac{rq}{p}}\, b(t^{\frac{1}{n}})^{r-\frac{rq}{p}}\, f^*(t)^q\, dt \Big)^{1/q} \lesssim \|f\|_A$$

cannot hold for all $f \in A$. Since, by assertion 7 of Lemma 2.2,

$$b_r(t)^{-r+\frac{rq}{p}}\, b(t^{\frac{1}{n}})^{r-\frac{rq}{p}} \lesssim 1 \quad \text{for all } t \in (0,1),$$

estimate (8.6) also cannot hold.

(iv) The case $r \le q < p$. If (8.6) would hold, then, by Theorem 3.5(i) and Remark 6.3,

$$\|t^{1/p-1/q} b_r(t) f^*(t)\|_{q,(0,1)} \lesssim \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t f^*(u)^p\, du \Big)^{1/p} \Big\|_{r,(0,1)}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ with $|\text{supp} f|_n \le 1$. Following the idea to arrive from (6.18) to (6.20), we see that estimate (6.20) would also hold for all measurable functions g on $\mathbb{R}^n$, where now

$$w(t) = t^{q/p-1} b_r(t)^q \chi_{(0,1)}(t)$$

and

$$v(t) = t^{r/p-1} b(t^{1/n})^r \chi_{(0,1)}(t) + t^{-1}\chi_{[1,\infty)}(t).$$

for all $t \in (0,\infty)$. By Proposition 6.4(ii) (with $Q = q/p$ and $P = r/p$), inequality (6.20) would be satisfied only if

$$\begin{aligned} \infty \;&>\; \sup_{t\in(0,1)} \frac{\Big(\int_0^t s^{\frac{q}{p}-1} b_r(s)^q\, ds \Big)^{\frac{p}{q}} + t \Big(\int_t^1 s^{\frac{q}{p}\frac{q}{p-q}} b_r(s)^{q\frac{q}{p-q}} s^{\frac{q}{p}-1} b_r(s)^q s^{-\frac{q}{p-q}}\, ds \Big)^{\frac{p-q}{q}}}{\Big(\int_0^t s^{\frac{r}{p}-1} b(s^{\frac{1}{n}})^r\, ds + t^{\frac{r}{p}} \Big(\int_t^1 s^{-1} b(s^{\frac{1}{n}})^r\, ds + \int_1^\infty s^{-\frac{r}{p}} s^{-1}\, ds \Big) \Big)^{\frac{p}{r}}} \\ &=: \; \sup_{t\in(0,1)} I. \end{aligned}$$

Using assertions 1, 6 and 7 of Lemma 2.2, (2.4) and (3.1), we obtain, for all $t \in (0,1)$, that

$$
\begin{aligned}
I &\gtrsim \frac{t\, b_r(t)^p + t\left(\int_t^1 s^{-1}\, b_r(s)^{\frac{pq}{p-q}}\, ds\right)^{\frac{p-q}{q}}}{t\,(b(t^{\frac{1}{n}})^r + b_r(t)^r + 1)^{\frac{p}{r}}} \\
&\gtrsim \frac{\left(\int_t^1 s^{-1}\, b_r(s)^{\frac{pq}{p-q}}\, ds\right)^{\frac{p-q}{q}}}{b_r(t)^p} \\
&= \left(\frac{\int_t^1 s^{-1}\, b_r(s)^{\frac{pq}{p-q}}\, ds}{b_r(t)^{\frac{pq}{p-q}}}\right)^{\frac{p-q}{q}} .
\end{aligned}
$$

However, by part 8 of Lemma 2.2, the last fraction is unbounded on some interval $(0, t_0)$, $t_0 > 0$. Hence, (6.20), and consequently (8.6), cannot hold. □

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