Bulletin of the **S**ection of **L**ogic
Volume 34/4 (2005), pp. 203–211

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ON THE INJECTIVITY OF THE LEIBNIZ OPERATOR

Abstract

The class of *weakly algebrizable logics* is defined as the class of logics having monotonic and injective Leibniz operator. We show that "monotonicity" cannot be discarded on this definition, by presenting an example of a system with injective and non monotonic Leibniz operator.

We also show that the non injectivity of the non protoalgebraic inf-sup fragment of the Classic Propositional Calculus, $CPC_{\wedge\vee}$, holds only from the fact that the empty set is a $CPC_{\wedge\vee}$-filter.

1. Introduction

An important paradigm in algebraic logic is the *Lindenbaum-Tarski process* for building Boolean algebras from the classical propositional logic. A main aim on abstract algebraic logic is the study of the generalization of this process for other deductive systems. This study has lead to the establishment of an algebraic hierarchy defined by properties of the Leibniz operator. The relevant classes of this hirarchy for the present note are the class of protoalgebraic logics and the class of weakly algebraizable logics.

Relevant references about this subject are the papers by Blok and Pigozzi [2] and [3]; Czelakowski's book, *Protoalgebraic Logics* [6] and the paper [5]; Hermann's papers [14] and [15], the book *A General Semantics*

*The authors wish to thank Don Pigozzi and Isabel Ferreirim for some fruitful discussions concerning this work.

[1]Research partially supported by project PRODEP III.

for Sentential Logic by Font and Jansana [10] and the survey [11] by Font, Jansana and Pigozzi.

We follow the notation by Blok and Pigozzi in [1]. Some basic definitions and proofs of well known results will be omitted. However, references are provided.

We begin by stating some definitions and results about algebraic logic we require. Then we study the $\{\vee, \wedge, \neg, \top, \bot\}$-fragment of the Intuitionistic Propositional Calculus, IPC^*, and we show that its Leibniz operator is injective but not monotonic. Therefore, the class of weakly algebraizable logics may not be simply defined as the class of logics with injective Leibniz operator. Finally, we show that the non injectivity of the inf-sup fragment $\mathrm{CPC}_{\wedge\vee}$ of the Classic Propositional Calculus follows from the fact that the empty set is a $\mathrm{CPC}_{\wedge\vee}$-filter.

2. Preliminaries

Let $\Lambda = \{\omega_i : i \in I\}$ be a (countable algebraic similarity) language, a set of finitary connectives with associated natural numbers called their arity, and let $V = \{x_1, x_2, x_3, ...\}$ be a countable infinite set of (propositional) variables. An *algebra* $\mathbf{A}$ *of type* Λ consists of a set A and, for each element $w \in \Lambda$, a function from A^n to A where n is the arity of w.

We denote by $\mathrm{Fm}(\Lambda)$ the set of formulas over Λ with variables in V and, defining the operations on $\mathrm{Fm}(\Lambda)$ in the usual way, we obtain an algebra over the language Λ, called the *formula algebra*, that we denote by $\mathbf{Fm}(\mathbf{\Lambda})$. We write $\delta(x_0, \dots, x_n)$ to denote a formula whose variables belong to the set $\{x_0, \dots, x_n\}$. An *equation over* Λ is a pair $\langle \delta, \epsilon \rangle$, with $\delta, \epsilon \in \mathbf{Fm}(\mathbf{\Lambda})$, which we denote by $\delta \approx \epsilon$.

The algebra $\mathbf{Fm}(\mathbf{\Lambda})$ has the universal mapping property over V, i.e. for any algebra $\mathbf{A}$, of type Λ, with domain A, and any mapping $h : V \to A$, there is a unique homomorphism from $\mathbf{Fm}(\mathbf{\Lambda})$ into $\mathbf{A}$, denoted by h. For each $\delta(x_0, \dots, x_n) \in \mathbf{Fm}(\mathbf{\Lambda})$ we denote by $\delta^{\mathbf{A}}(a_0, \dots, a_n)$ its image by h, for any h such that $h(x_i) = a_i$ $(i = 0, \dots, n)$.

A *matrix over* Λ is a pair $\mathcal{A} = \langle \mathbf{A}, \mathbf{F} \rangle$, where $\mathbf{A}$ is an algebra of type Λ and F is a subset of A. The algebra $\mathbf{A}$ is called the *algebraic reduct of* $\mathcal{A}$ and the set F is called the *designated filter of* $\mathcal{A}$. A congruence relation θ on $\mathbf{A}$ is *compatible with* F if for any two elements $a, a' \in A$ such that $a \, \theta \, a'$, either $a, a' \in F$ or $a, a' \notin F$.

For a matrix $\mathcal{A} = \langle \mathbf{A}, \mathbf{F} \rangle$ the *Leibniz congruence on* $\mathbf{A}$ *over* F is the largest congruence relation on $\mathbf{A}$ compatible with F; we denote it by $\Omega_{\mathbf{A}}(F)$ or simply by $\Omega(F)$. For an algebra $\mathbf{A}$ the *Leibniz operator* $\Omega_{\mathbf{A}}$ is defined by

$$\begin{array}{rccc} \Omega_{\mathbf{A}}: & \mathcal{P}(\mathcal{A}) & \rightarrow & \mathrm{Cong}(\mathbf{A}) \\ & F & \mapsto & \Omega_{\mathbf{A}}(F). \end{array}$$

The Leibniz operator $\Omega_{\mathbf{A}}$ is *injective* if it is an injective map and it is called *monotonic* if

$$\forall\, F, G \in \mathcal{P}(A), F \subseteq G \Rightarrow \Omega_{\mathbf{A}}(F) \subseteq \Omega_{\mathbf{A}}(G).$$

A (*finitary*) *deductive system* is a pair $\mathcal{S} = \langle \Lambda, \vdash_{\mathcal{S}} \rangle$, where Λ is a language and $\vdash_{\mathcal{S}}$ is a structural (or substitution-invariant) finitary consequence relation (a finitary deductive system can also be defined by axioms and inference rules, see [2]). We write $\Delta \vdash_{\mathcal{S}} \delta$ (or simply $\Delta \vdash \delta$) to mean that the pair $\langle \Delta, \delta \rangle$ belongs to $\vdash_{\mathcal{S}}$.

By a *theorem of* $\mathcal{S}$ we mean a formula φ such that $\emptyset \vdash_{\mathcal{S}} \varphi$; we denote by $\mathrm{Thm}(\mathcal{S})$ the set of all theorems. A set of formulas Γ is said to be closed under the consequence relation $\vdash_{\mathcal{S}}$ if $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\varphi \in \Gamma$ and it is called a *theory of* $\mathcal{S}$ or an $\mathcal{S}$-theory. The set of all theories of S is denoted by $\mathrm{Th}(\mathcal{S})$. Given any set of formulas Γ, the set of all consequences of Γ, $\mathrm{Con}_{\mathcal{S}}(\Gamma)$, is the smallest theory that contains Γ. Clearly, $\mathrm{Con}_{\mathcal{S}}(\Gamma) = \{\varphi : \Gamma \vdash_{\mathcal{S}} \varphi\}$.

Let Λ and $\Lambda' \subseteq \Lambda$ be two languages and let $\mathcal{S}$ be a deductive system over Λ. The Λ'*-fragment* $\mathcal{S}'$ *of* $\mathcal{S}$ is the deductive system with language Λ' and consequence relation $\vdash_{\mathcal{S}'}$ defined by:

$$\Gamma \vdash_{\mathcal{S}'} \varphi \text{ iff } \Gamma \vdash_{\mathcal{S}} \varphi \ \ (\Gamma \subseteq \mathrm{Fm}(\Lambda'),\ \varphi \in \mathrm{Fm}(\Lambda')).$$

Given a matrix $\mathcal{A} = \langle \mathbf{A}, \mathbf{F} \rangle$, a formula φ is a (semantic) *consequence of* a set of formulas Γ, and we write $\Gamma \models_{\mathcal{A}} \varphi$, if for every mapping $h : V \rightarrow A$, $h(\varphi) \in F$ whenever $h(\delta) \in F$ for every $\delta \in \Gamma$. We call $\mathcal{A}$ a *model of* a formula φ if $\emptyset \models_{\mathcal{A}} \varphi$ and we write $\mathcal{A} \models \varphi$.

Let K be a class of matrices over a language Λ. A formula φ is a *consequence of* a set of formulas Γ in K, and we write $\Gamma \models_K \varphi$, if for every matrix $\mathcal{A} \in \mathcal{K}$, $\Gamma \models_{\mathcal{A}} \varphi$.

Let $\mathcal{S}$ be a deductive system over a language Λ. A matrix $\mathcal{A}$ is an $\mathcal{S}$*-matrix* (or a model of $\mathcal{S}$) if the consequence in $\mathcal{S}$ implies the semantic consequence in $\mathcal{A}$, i.e., $\Gamma \vdash \varphi \Rightarrow \Gamma \models_{\mathcal{A}} \varphi$. The class of all $\mathcal{S}$-matrices is denoted by $\mathrm{Mod}(\mathcal{S})$. An $\mathcal{S}$-matrix $\mathcal{A} = \langle \mathbf{A}, \mathbf{F} \rangle$ is *reduced* if $\Omega(F) = id_A$ and we denote by $\mathrm{Mod}^*(\mathcal{S})$ the class of all reduced $\mathcal{S}$-matrices. A set $F \subseteq A$ such that $\langle \mathbf{A}, \mathbf{F} \rangle \in \mathrm{Mod}(\mathcal{S})$ is called an *$\mathcal{S}$-filter of* $\mathbf{A}$. The set of all $\mathcal{S}$-filters of $\mathbf{A}$ is denoted by $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$.

3. Injectivity for non protoalgebraic logics

A deductive system $\mathcal{S}$ is said to be *protoalgebraic* if the restriction of $\Omega_{\mathbf{Fm}}$ to the set of theories of $\mathcal{S}$ is monotonic. The subclass of the protoalgebraic logics for which this restriction of $\Omega_{\mathbf{Fm}}$ is also injective is called the class of *weakly algebraizable logics*. We say that a deductive system $\mathcal{S}$ has *injective Leibniz operator* (for simplicity, we say that $\mathcal{S}$ is *injective*) if for any algebra $\mathbf{A}$, the restriction of $\Omega_{\mathbf{A}}$ to the set $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ is injective. In the context of protoalgebraic logics it was shown that the injectivity of the restriction of $\Omega_{\mathbf{Fm}}$ to the set of theories of $\mathcal{S}$ implies the deductive system to be injective (see [7], Theorem 4.7).

Lemma 1. ([6]). *Let $\mathcal{S}$ be a deductive system. Then $\mathcal{S}$ is protoalgebraic if and only if for all $\mathcal{F} \subseteq \mathrm{Th(S)}$, $\Omega(\bigcap \mathcal{F}) = \bigcap \{\Omega(T) : T \in \mathcal{F}\}$.*

As a consequence we have:

Proposition 2. *If $\mathcal{S}$ is protoalgebraic then $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under (arbitrary) intersections.*

The converse of this proposition holds in the case where the S-theories are definable by a set of equations in the sense of the following definition:

Definition 3. Let K be a set of matrices over a language Λ. We say that the filters of the matrices in K are *equationally definable by a set of equations* $E = \{\delta_i(x) \approx \varepsilon_i(x) : i \in I\}$ if for each matrix $\mathcal{A} = \langle \mathbf{A}, \mathbf{F} \rangle \in \mathbf{K}$ we have:

$$F = \Big\{a \in A \,:\, \text{for all } \delta \approx \varepsilon \in E(x),\ \delta^{\mathbf{A}}(a) \equiv \varepsilon^{\mathbf{A}}(a)\ \big(\,\Omega(F)\,\big)\Big\}.$$

This notion of equational definability generalizes the concept of explicit definability of the truth predicate introduced by Czelakowski and

Jansana in [7] (see also [6]). Moreover, if all the matrices in K are reduced the two notions coincide.

It is not difficult to see that equationally definability implies injectivity in the following sense:

PROPOSITION 4. *Let $\mathcal{S}$ be a deductive system over Λ. If the class of all models of $\mathcal{S}$ has equationally definable filters by a set of equations, then for any algebra $\mathbf{A}$, the restriction of $\Omega_{\mathbf{A}}$ to the $\mathcal{S}$-filters of $\mathbf{A}$ is injective.*

For the class of all models having the algebraic part being the set of formulas, with filters equationally definable, protoalgebraicity (monotonicity) can be characterized by the condition of $\Omega(\mathrm{Th}(\mathcal{S}))$ being closed under finite intersections.

THEOREM 5. *Let $\mathcal{S}$ be a deductive system over Λ. Assume that the class of all models of $\mathcal{S}$, of the form $\langle \mathbf{Fm}(\mathbf{\Lambda}), \mathbf{T}\rangle$ has equationally definable filters by a set of equations E. Then, $\mathcal{S}$ is protoalgebraic if and only if $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under finite intersections.*

PROOF. By Proposition 2, for any protoalgebraic system $\mathcal{S}$, $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under finite intersections.

Assume now that $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under finite intersections. Let $T, G \in \mathrm{Th}(\mathcal{S})$ such that $T \subseteq G$ and $\mathcal{F} = \{F, G\}$. Since $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under finite intersections, $\bigcap \Omega(\mathcal{F}) = \Omega(T) \cap \Omega(G) = \Omega(H)$, for some $H \in \mathrm{Th}(\mathcal{S})$. Moreover, $a \in H$ if and only if for all $\delta \approx \varepsilon \in E$, $\delta^{\mathbf{A}}(a) \equiv \varepsilon^{\mathbf{A}}(a)\ \big(\Omega(H)\big)$. That is, for all $\delta \approx \varepsilon \in E$, $\delta^{\mathbf{A}}(a) \equiv \varepsilon^{\mathbf{A}}(a)\ \big(\Omega(T) \cap \Omega(G)\big)$. This implies that for all $\delta \approx \varepsilon \in E$, $\delta^{\mathbf{A}}(a) \equiv \varepsilon^{\mathbf{A}}(a)\ \big(\Omega(T)\big)$. Hence $a \in T$ and so $H \subseteq T$. Let now $a \in T$. Since $T \subseteq G$, $a \in G$. Hence, for all $\delta \approx \varepsilon \in E$, $\delta^{\mathbf{A}}(a) \equiv \varepsilon^{\mathbf{A}}(a)\ \big(\Omega(T)\big)$ and for all $\delta \approx \varepsilon \in E$, $\delta^{\mathbf{A}}(a) \equiv \varepsilon^{\mathbf{A}}(a)\ \big(\Omega(G)\big)$. Then for all $\delta \approx \varepsilon \in E$, $\delta^{\mathbf{A}}(a) \equiv \varepsilon^{\mathbf{A}}(a)\ \big(\Omega(T) \cap \Omega(G)\big)$ which means that $\delta^{\mathbf{A}}(a) \equiv \varepsilon^{\mathbf{A}}(a)\ \big(\Omega(H)\big)$ and so $T \subseteq H$.

Therefore $T = H$ and thus $\Omega(T) = \Omega(H) = \Omega(T) \cap \Omega(G) \subseteq \Omega(G)$. Hence, S is protoalgebraic. □

3.1. Injective deductive systems

One important question related to the class of weakly algebraizable logics can be formulated: "Is injectivity enough to guarantee monotonicity?" We will show below that the answer is "no" by giving an example of a deductive system that is injective but not protoalgebraic.

Very few examples of non protoalgebraic deductive systems have been investigated. Among them we have: the inf-sup fragment of the Classic Propositional Calculus ($\mathrm{CPC}_{\wedge\vee}$) (see [9], [10] and [13]); Belnap's Logic (see [8] and [10]), the $\{\vee, \wedge, \neg, \top, \bot\}$-fragment of the Intuitionistic Propositional Calculus IPC, denoted by IPC^* (see [2], [10] and [17]). Recently, Positive Modal Logics and some Subintuitionistic Logics have also been investigated (see [4], [16]).

If a deductive system does not have theorems, then each algebra $\mathbf{A}$ has non injective Leibniz operator, since the Leibniz congruence for both the empty set and the universal set is the universal congruence. An immediate consequence is that an injective deductive system must have theorems. It is well known that the deductive system IPC^* has theorems. We will show that it is injective.

Proposition 6. *Let $\mathcal{S}$ be a deductive system such that for every algebra $\mathbf{A}$ there is at most one reduced $\mathcal{S}$-matrix with algebraic reduct $\mathbf{A}$. Then $\mathcal{S}$ is injective.*

Proof. Suppose that $\mathcal{S}$ is not injective. That is, there exists an algebra $\mathbf{A}$ such that $\Omega_{\mathbf{A}}$ is not injective. Hence, there are $F_1, F_2 \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ such that $F_1 \neq F_2$ and $\Omega_{\mathbf{A}}(F_1) = \Omega_{\mathbf{A}}(F_2)$. Let $\mathcal{A}_\infty = \langle \mathbf{A}, \mathbf{F_1} \rangle$ and $\mathcal{A}_\in = \langle \mathbf{A}, \mathbf{F_2} \rangle$. Clearly, $\mathcal{A}_\infty$ and $\mathcal{A}_\infty$ are S-matrices. Moreover, the matrices $\langle \mathbf{A}/\mathbf{\Omega_A(F_1)}, \mathbf{F_1}/\mathbf{\Omega_A(F_1)} \rangle$ and $\langle \mathbf{A}/\mathbf{\Omega_A(F_2)}, \mathbf{F_2}/\mathbf{\Omega_A(F_2)} \rangle$ are reduced S-matrices with the same algebraic reduct $\mathbf{A}/\mathbf{\Omega_A(F_1)}\,(= \mathbf{A}/\mathbf{\Omega_A(F_2)}\,)$ and such that $F_1/\Omega_{\mathbf{A}}(F_1) \neq F_2/\Omega_{\mathbf{A}}(F_2)$, a contradiction. □

The following result, recalled by Rebagliato and Verdú in [17], characterizes the reduced IPC^*-matrices (PCDL denotes the variety of the *pseudocomplemented distributive lattices*).

Theorem 7. ([17]) *The following are equivalent:*

(i) *$\mathcal{A} = \langle \mathbf{A}, \mathbf{F} \rangle$ is a reduced IPC^*-matrix;*

(ii) (a) $\mathbf{A} \in$ PCDL*;*
(b) *$F = \{1\}$ and*
(c) *$\forall\ a, b \in A$, if $a < b$ then there exists $c \in A$, $c \neq 1$, that satisfies $c \wedge a = a$ and $(c \vee b) \wedge \neg(\neg a \wedge b) = \neg(\neg a \wedge b)$.*

From this theorem we conclude that for every algebra $\mathbf{A}$ there is at most one reduced IPC^*-matrix with algebraic reduct $\mathbf{A}$. Moreover, when such reduced matrix exists, it is equal to $\langle \mathbf{A}, \{\mathbf{1}\}\rangle$. Hence, by Proposition 6, IPC* is injective. Therefore, IPC* is an example of a deductive system which is non protoalgebraic but injective.

3.2. Non protoalgebraic deductive systems without theorems

As we pointed out above, there are non protoalgebraic systems without theorems, which obviously are non injective. Since the non injectivity is shown by using that $\Omega(\emptyset) = \Omega(\mathrm{Fm}(\Lambda))$, it is natural to investigate the injectivity of the restriction of the Leibniz operator to the set of non empty $\mathcal{S}$-filters.

We say that a deductive system is *quasi-injective* if for every algebra $\mathbf{A}$, the restriction of $\Omega_{\mathbf{A}}$ to $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A}) \setminus \{\emptyset\}$, $\Omega_{\mathbf{A}} : \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}) \setminus \{\emptyset\} \to \mathrm{Cong}(\mathbf{A})$, is injective.

We will need the following result, proved by Font and Jansana in [10], that characterizes the algebraic reducts of the $\mathrm{CPC}_{\wedge\vee}$-matrices:

THEOREM 8. ([9]) *The class of algebraic reducts of the reduced* $\mathrm{CPC}_{\wedge\vee}$-*matrices is the class of distributive lattices with maximum* 1 *such that for every* $a, b \in A$, *if* $a < b$ *then there is* $c \in A$, *with* $a \vee c \neq 1$ *and* $b \vee c = 1$.

On the other hand, if $\mathbf{A}$ is not a trivial algebra then $\mathcal{A} = \langle \mathbf{A}, \emptyset\rangle$ can not be a reduced $\mathrm{CPC}_{\wedge\vee}$-matrix, since $\Omega_{\mathbf{A}}(\emptyset) = A^2 \neq \Delta_{\mathbf{A}}$.

Next proposition is a weaker version of a result in [9] that characterizes $\mathrm{CPC}_{\wedge\vee}$-matrices:

PROPOSITION 9. ([9]) *Let* $\mathbf{A}$ *be an algebra and* $\mathcal{A} = \langle \mathbf{A}, \mathbf{F}\rangle$ *a reduced* $\mathrm{CPC}_{\wedge\vee}$-*matrix. If* $\mathbf{A}$ *is non trivial then* $F = \{1\}$, *otherwise* $F = \emptyset$.

Now, we are able to state our final result about the injectivity of the Leibniz operator on $\mathrm{CPC}_{\wedge\vee}$.

THEOREM 10. $\mathrm{CPC}_{\wedge\vee}$ *is quasi-injective.*

PROOF. Let $\mathbf{A}$ be an algebra. If $\mathbf{A}$ is not trivial the statement follows from Proposition 9 and Proposition 6. If $A = \{a\}$ then $\mathrm{Fi}_{\mathrm{CPC}_{\wedge\vee}}(\mathbf{A})\setminus\{\emptyset\} = \emptyset$ and, obviously, $\Omega_{\mathbf{A}} : \mathrm{Fi}_{\mathrm{CPC}_{\wedge\vee}}(\mathbf{A}) \setminus \{\emptyset\} \to \mathrm{Cong}(\mathbf{A})$ is injective. □

In the previous subsection we saw that the non injectivity of the Leibniz operator of the deductive system $CPC_{\wedge\vee}$ follows only from the fact that the empty set is a $CPC_{\wedge\vee}$-filter. It is not expected that it happens for any deductive systems without theorems, since $CPC_{\wedge\vee}$ is a very particular case of a non protoalgebraic fragment of an algebraizable logic. In fact, Belnap's logic, [8], is an example of a deductive system, without theorems, which is not quasi-injective. Moreover, not all fragments of an algebraizable logic are quasi-injective. For example, the trivial fragment of CPC, $CPC_{\emptyset}$, is not quasi-injective. It would be interesting to find a characterization of quasi-injective deductive systems.

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