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ADMISSIBLE EQUIVALENCE SYSTEMS*

Abstract

Whenever a logic is the set of theorems of some deductive system, where the latter has an equivalence system, the behavioral theorems of the logic can be determined by means of that equivalence system. In general, this original equivalence system may be too restrictive, because it suffices to check behavioral theorems by means of any admissible equivalence system (that is an equivalence system of the smallest deductive system associated with the given logic). In this paper, we present a range of examples, which show that: 1) there is an admissible equivalence system which is not an equivalence system for the initial deductive system, 2) there is a non-finitely equivalential deductive system with a finite admissible equivalence system, and 3) there is a deductive system with an admissible equivalence systems, such that this deductive system is not even protoalgebraic itself. We use methods and results from algebraic and modal logic.

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1. Introduction

Suppose we have a *propositional language*, i.e. a non-empty set of functional symbols Λ and a countable infinite set of variables. A *theory* is simply some set of well-defined formulas of this language, and a *logic* is a theory which is closed under simultaneous substitutions. A *behavioral theorem* for a logic $\mathcal{L}$ is a pair of formulas, that cannot be distinguished *semantically*, by that we mean that they cannot be distinguished modulo logic $\mathcal{L}$, when substituted into the same arbitrary context. More formally, $\langle \alpha, \beta \rangle$ is a behavioral theorem for $\mathcal{L}$ iff for every Λ-term (context) $t(x, \bar{z})$,

$$t(\alpha, \bar{z}) \in \mathcal{L} \iff t(\beta, \bar{z}) \in \mathcal{L}$$

—the set of all Λ-terms will be denoted henceforth by $\mathcal{T}_\Lambda$. Following the tradition of abstract algebraic logic we denote the set of behavioral theorems for $\mathcal{L}$ by $\Omega\mathcal{L}$. In algebraic specification of software systems, namely in equational setting, behavioral theorems play an important role. Very often, computer scientists are interested in developing automatic tools to check if a given equation is a behavioral theorem (cf. [5] and [9]).

Generally, to determine that a pair $\langle \alpha, \beta \rangle$ is a behavioral theorem for $\mathcal{L}$ would require testing it in infinitely many contexts. In a particular case, when $\mathcal{L}$ has a biimplication-like connective, say $\leftrightarrow$, we have

$$\langle \alpha, \beta \rangle \in \Omega\mathcal{L} \quad \text{iff} \quad (\forall t \in \mathcal{T}_\Lambda : t(\alpha, \bar{z}) \in \mathcal{L} \iff t(\beta, \bar{z}) \in \mathcal{L})$$
$$\text{iff} \quad (\forall t \in \mathcal{T}_\Lambda : t(\alpha, \bar{z}) \leftrightarrow t(\beta, \bar{z}) \in \mathcal{L})$$

In this case we call the set $\Delta(x, y, \bar{p}) := \{t(x, \bar{z}) \leftrightarrow t(y, \bar{z}) \mid t \in \mathcal{T}_\Lambda\} \subseteq \mathrm{Fm}_\Lambda$ a *parameterized defining set* for behavioral theorems of logic $\mathcal{L}$, where all z_i from each vector $\bar{z}$ are among the elements of the potentially infinite list $\bar{p}$ ($\bar{p}$ in $\Delta(x, y, \bar{p})$ is a notational convention meant to show that contexts depend on parameters).

If it is possible to eliminate parameters, then we have simply a *defining set*, i.e., there is $\Delta(x, y) \subseteq \mathrm{Fm}_\Lambda$ such that

$$\langle \alpha, \beta \rangle \in \Omega\mathcal{L} \quad \text{iff} \quad \Delta(\alpha, \beta) \subseteq \mathcal{L}.$$

Sometimes it is possible to further simplify a defining set, by reducing it to a finite set of chosen contexts, such set is called a *finite defining set* for behavioral theorems. Obviously, for practical purposes this is the

most desirable condition. This is the case, for instance, for the classical propositional logic CPL, where

$$\langle \alpha, \beta \rangle \in \Omega(\text{CPL}) \iff \alpha \leftrightarrow \beta \in \text{CPL},$$

therefore $\Delta(x, y) = \{x \leftrightarrow y\}$.

The notion of the defining set for behavioral theorems is related to (but not coincides with) the notion of the *equivalence system* for a deductive system. Deductive systems are often considered in the same contexts as logics. For instance, the classical propositional logic CPL is assumed to be equipped with the *modus ponens* rule: $\alpha, \alpha \to \beta / \beta$, and is called in this context the *classical propositional calculus* (CPC). Similarly, a particular normal modal logic, say $\mathcal{L}$, is considered to be equipped with a deductive apparatus, which usually includes the rules of modus ponens and *necessitation* $\alpha / \Box \alpha$. Therefore it is sometimes beneficial to consider instead of just the logic $\mathcal{L}$ the set of all theories, that extend $\mathcal{L}$ and are closed under postulated inference rules. That leads to the notion of the *abstract deductive system*, the qualification *abstract* relating to the fact, that so presented deductive system does not require any explicit representation of its deductive apparatus. An *equivalence system* for a deductive system $\mathcal{S}$ plays essentially the same role for every theory T of $\mathcal{S}$ as a defining set for a logic. For instance, for the deductive system $K^{\to}$ (all sets of modal formulas extending K and closed under modus ponens) associated with the modal logic K, we have for every theory $T \in K^{\to}$

$$\langle \alpha, \beta \rangle \in \Omega T \iff \{\Box^n(\alpha \leftrightarrow \beta)\}_{n \in \omega} \subseteq T,$$

where $\Box^0 \alpha := \alpha$, $\Box^{n+1} \alpha := \Box(\Box^n \alpha)$. Thus $\Delta(x, y) := \{\Box^n(\alpha \leftrightarrow \beta)\}_{n \in \omega}$ is an (infinite) equivalence system for the deductive system $K^{\to}$ [8, Theorem II.4] and therefore can serve as a defining set for the behavioral theorems of the logic K.

Also, in general, if a logic is a theory of an equivalential deductive system, then its behavioral theorems can be determined by the use of the inherent equivalence system. But the latter system is usually excessive, and in fact the behavioral theorems can be checked by means of any *admissible* equivalence system, which is an equivalence system of the deductive system of admissible rules associated with the given logic (note that the inherent equivalence system is also admissible).

For a quick illustration, let us consider logics Int, CPL and $Triv$ over the language $\Lambda = \{\vee, \wedge, \to, \neg\}$, where Int is the intuitionistic logic and

$Triv$ is the set of all formulas in the language Λ. It is well known, that $Int^{\rightarrow}$ has $\{x \leftrightarrow y\}$ as its equivalence system. The logic CPL is a theory of $Int^{\rightarrow}$, therefore $\{x \leftrightarrow y\}$ is a defining set for behavioral theorems of CPL (so $\{x \leftrightarrow y\}$ is also an admissible equivalence system for CPC := $CPL^{\rightarrow}$). There are no proper admissible equivalence systems for $CPL^{\rightarrow}$, because $CPL^{\rightarrow}$ is *structurally complete* (so $CPL^{\rightarrow}$ is the smallest deductive system containing CPL). The set $Triv := Fm_\Lambda$ is a logic in our definition. It is also a theory of $Int^{\rightarrow}$, therefore has $\{x \leftrightarrow y\}$ as a defining system for its behavioral theorems, but it also has the smaller set $\Delta = \emptyset$ as a defining system (since every pair $\langle \alpha, \beta \rangle$ is a behavioral theorem for $Triv$).

The question arises:

Given a deductive system, are there any proper admissible equivalence systems for it?

(We will call such systems *weak equivalence systems*.) If so, in many cases there will be possible, for the sake of determining of behavioral theorems, to replace the original equivalence system with a finer-tuned weak one.

In general the answer for this question is "yes", as shown for instance in [8].

In this paper we extend on the results of [8] and study the problem of existence of weak equivalence systems for deductive systems over modal signatures. For that goal we review and present a list of some known results about equivalential logics and add to that list several results of our own. We show that even in the simplest case of the modal language, i.e., with one unary modal operator and classical connectives, a variety of possible situations arises, including the cases:

- there is an admissible equivalence system which is not an equivalence system for the initial deductive system (Example 12),
- there is a non-finitely equivalential deductive system with a finite admissible equivalence system (Example 13),
- there is a deductive system with an admissible equivalence system, such that this deductive system is not even protoalgebraic itself (Example 14).

These results indicate that in the case of yet more expressive languages, there can be numerous ways for refining of implicit equivalence systems.

2. Preliminaries and Notation

We use the terminology employed in the theory of non-classical logics.

A *propositional language type* is any non-empty set Λ. The elements of Λ are called *functional symbols* in the algebraic context or *logical connectives* in the logical context. With Λ we associate an *arity* function $\rho : \Lambda \to \omega$ such that ρf is the *arity* of the connective $f \in \Lambda$. For each $n \in \omega$: $\Lambda_n := \{f \in \Lambda \mid \rho f = n\}$. An *algebra* $\mathbf{A}$ of type Λ is a pair $\langle A, \Lambda^{\mathbf{A}} \rangle$, where A is a non-empty set called the *universe* of $\mathbf{A}$ and $\Lambda^{\mathbf{A}} = \langle f^{\mathbf{A}} \mid f \in \Lambda \rangle$ is a list of operations over the set A such that for every $f \in \Lambda_n$, $f^{\mathbf{A}} : A^n \to A$. Members of $\Lambda^{\mathbf{A}}$ are called *basic operations* of $\mathbf{A}$. If $\mathbf{A}$, $\mathbf{B}$ are algebras of the same type, then a mapping $h : A \to B$ is called a *homomorphism of* $\mathbf{A}$ *into* $\mathbf{B}$ (written $h : \mathbf{A} \to \mathbf{B}$), if for every $f \in \Lambda_n$ and every $\langle \bar{a} \rangle \in A^n$, $hf^{\mathbf{A}}\langle \bar{a} \rangle = f^{\mathbf{B}}h\langle \bar{a} \rangle$. A homomorphism $h : \mathbf{A} \to \mathbf{A}$ is called an *endomorphism of* $\mathbf{A}$; if h is also surjective and injective, then h is an *automorphism of* $\mathbf{A}$.

Let X be a non-empty set. The set $\mathrm{Fm}_\Lambda X$ of *formulas (or terms) of type Λ over the set of variables* (or *generators*) X is defined recursively as follows

1. $X \subseteq \mathrm{Fm}_\Lambda X$,
2. if $f \in \Lambda_n$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Fm}_\Lambda X$, then $\langle f, \alpha_1, \ldots, \alpha_n \rangle \in \mathrm{Fm}_\Lambda X$.

Traditionally the formula $\langle f, \alpha_1, \ldots, \alpha_n \rangle$ is written as $f(\alpha_1, \ldots, \alpha_n)$. Formulas will be denoted usually by small Greek letters. We write $\alpha(x_1, \ldots, x_n)$ or $\mathrm{Var}(\alpha) \subseteq \{x_1, \ldots, x_n\}$, if $\alpha \in \mathrm{Fm}_\Lambda \{x_1, \ldots, x_n\}$. The vector $\langle \alpha_1, \ldots, \alpha_k \rangle$ of $\mathrm{Fm}_{\mathcal{L}}^+$ is called a *sequent* and will be written usually in the form $\alpha_1, \ldots, \alpha_{k-1} \triangleright \alpha_k$.

We can induce the structure of an algebra on $\mathrm{Fm}_\Lambda X$ by associating with each $f \in \Lambda_n$ an n-ary operation $f^{\mathbf{Fm}_\Lambda X}$ on the set $\mathrm{Fm}_\Lambda X$ defined by $f^{\mathbf{Fm}_\Lambda X}\langle \bar{\alpha} \rangle = f(\bar{\alpha})$. The superscript in this case is usually omitted. This algebra $\mathbf{Fm}_\Lambda X$ is called the *algebra of formulas (terms) of type Λ over the set of variables X*. We fix a countable infinite set $\mathrm{Var} = \{x_0, x_1, x_2, \ldots\}$ of *propositional variables*. Then $\mathbf{Fm}_\Lambda \mathrm{Var}$ is called *the formula algebra over the language of type Λ* and will be denoted $\mathbf{Fm}_\Lambda$. The universe of $\mathbf{Fm}_\Lambda$ is denoted as Fm_Λ.

The algebra $\mathbf{Fm}_\Lambda X$ is in fact an *absolutely free algebra over the set X* in the class of all algebras of type Λ. This means that, for every algebra $\mathbf{A}$ of type Λ, an arbitrary mapping $h : X \to A$ can be uniquely extended to

a homomorphism $h : \mathbf{Fm}_\Lambda X \to \mathbf{A}$. In particular any homomorphism $h : \mathbf{Fm}_\Lambda X \to \mathbf{A}$ is determined by the mapping $h : X \to A$. A homomorphism $h : \mathbf{Fm}_\Lambda \to \mathbf{A}$ is called an *evaluation* (or *valuation*); a homomorphism $h : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$ is called a *substitution*.

A non-empty family $\mathcal{C} \subseteq \mathcal{P}(A)$ (where $\mathcal{P}(A)$ is the powerset of A) is *upward-directed* if for every pair $X, Y \in \mathcal{C}$ there is $Z \in \mathcal{C}$ such that $X, Y \subseteq Z$. A subset $\mathcal{C} \subseteq \mathcal{P}(A)$ is *algebraic* if $\bigcup \mathcal{D} \in \mathcal{C}$ for every non-empty upward-directed subfamily $\mathcal{D} \subseteq \mathcal{C}$. A family $\mathcal{C} \subseteq \mathcal{P}(A)$ is called a *closure system over A* if $\bigcap \mathcal{D} \in \mathcal{C}$ for every subfamily $\mathcal{D} \subseteq \mathcal{C}$. In particular, $\bigcap \emptyset := A \in \mathcal{C}$. A closure system $\mathcal{C}$ over Fm_Λ is *(surjectively) invariant* if for any (surjective) substitution σ and any $T \in \mathcal{C}$, $\sigma^{-1} T := \{\alpha \mid \sigma\alpha \in T\} \in \mathcal{C}$, or, in other words, if $\sigma^{-1}\mathcal{C} \subseteq \mathcal{C}$ for all (surjective) substitutions $\sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$.

Every closure system $\mathcal{C}$, as a family of subsets ordered under set-inclusion, forms a complete lattice. The infimum of a family $\{X_i\}_{i \in I} \subseteq \mathcal{C}$ is its intersection $\bigcap_{i \in I} X_i$, and its supremum is $\bigvee_{i \in I}^{\mathcal{C}} X_i := \bigcap \{T \in \mathcal{C} \mid \bigcup_{i \in I} X_i \subseteq T\}$; its largest element is A, and its smallest element is $\bigcap \mathcal{C}$.

A *deductive system* is a pair $\mathcal{S} = \langle \mathbf{Fm}_\Lambda, \mathrm{Th}\,\mathcal{S} \rangle$ such that $\mathrm{Th}\,\mathcal{S} \subseteq \mathcal{P}(\mathrm{Fm}_\Lambda)$ is an algebraic invariant closure system over Fm_Λ. (Note that some authors define the deductive system without the condition for $\mathrm{Th}\,\mathcal{S}$ being algebraic. They would call our deductive systems *finitary*). The elements of $\mathrm{Th}\,\mathcal{S}$ are called *$\mathcal{S}$-theories*. For any deductive system $\mathcal{S}$ and all $T \in \mathrm{Th}\,\mathcal{S}$, $[T)_{\mathrm{Th}\,\mathcal{S}} := \{U \in \mathrm{Th}\,\mathcal{S} \mid T \subseteq U\}$ denote a principal filter of the lattice $\mathrm{Th}\,\mathcal{S}$ generated by T. If $\mathcal{S}$ is a deductive system, we denote $\mathrm{Thm}\,\mathcal{S} := \bigcap \mathrm{Th}\,\mathcal{S}$—the set of *theorems* of $\mathcal{S}$. If the language Λ is fixed we identify $\mathcal{S}$ with $\mathrm{Th}\,\mathcal{S}$.

Given a deductive system $\mathcal{S}$ we define a relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_\Lambda) \times \mathrm{Fm}_\Lambda$, between sets of formulas and individual formulas, in the following way

$$\Gamma \vdash_{\mathcal{S}} \alpha \text{ iff } \alpha \in \bigcap \{T \in \mathrm{Th}\,\mathcal{S} \mid \Gamma \subseteq T\}.$$

The relation $\vdash_{\mathcal{S}}$ is called the *consequence relation of $\mathcal{S}$*. When $\mathcal{S}$ is clear from context we simply write $\vdash$. This relation can be extended to the relation between sets of formulas as follows:

$$\Gamma \vdash_{\mathcal{S}} \Delta \text{ iff } \Gamma \vdash_{\mathcal{S}} \alpha \text{ for every } \alpha \in \Delta.$$

A *rule r* is a pair $\langle \Gamma, \alpha \rangle$ (we write it as $\Gamma \vdash \alpha$), where $\Gamma \subseteq \mathrm{Fm}_\Lambda$ and $\alpha \in \mathrm{Fm}_\Lambda$.

If Γ is finite, say $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$, the rule is called *finitary*; in this case a rule can be written as $\alpha_1, \ldots, \alpha_n \vdash \alpha$ or $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$.

Let Λ be a propositional language. *A theory* is any set of formulas $T \subseteq \mathrm{Fm}_\Lambda$. *A logic* $\mathcal{L}$ is any theory $\mathcal{L} \subseteq \mathrm{Fm}_\Lambda$ such that it is closed under substitutions, i.e., for all $\sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$: $\sigma\mathcal{L} \subseteq \mathcal{L}$. Note that for every deductive system $\mathcal{S}$, the set of theorems $\mathrm{Thm}\,\mathcal{S}$ is always a logic.

A rule $\Gamma \vdash \alpha$ is *compatible* with the theory $T \subseteq \mathrm{Fm}_\Lambda$ if, for all substitutions $\sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$,

$$\sigma\Gamma \subseteq T \implies \sigma\alpha \in T.$$

If a rule r is compatible with the theory T, then we say that T is *closed under r* or that T is *a theory of r*. Every rule r defines a deductive system $\langle \mathbf{Fm}_\Lambda, \mathrm{Th(r)} \rangle$ of all theories of r. If a rule r is compatible with the logic $\mathcal{L}$, then we say that r is *admissible* for the logic $\mathcal{L}$. We denote $\mathrm{Adm}\,\mathcal{L}$ the set of all finitary admissible rules for $\mathcal{L}$. Let $\mathcal{L}^{\mathrm{ad}}$ be the finitary deductive system of all theories that are closed under all finitary rules admissible for $\mathcal{L}$. For a deductive system $\mathcal{S}$, denote

$$\mathcal{S}^{\mathrm{ad}} := (\mathrm{Thm}\,\mathcal{S})^{\mathrm{ad}}.$$

It is easy to see that

LEMMA 1. $\mathcal{L}^{\mathrm{ad}}$ *is the smallest deductive system (i.e., it has the smallest set of theories), such that*

$$\{\sigma^{-1}\mathcal{L} \mid \sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda\} \subseteq \mathcal{L}^{\mathrm{ad}}.$$

3. Weak Equivalence Systems

DEFINITION 2. Let $\mathcal{S}$ be a deductive system. A pair of formulas $\langle \alpha, \beta \rangle$ is a *behavioral theorem* for a theory $T \in \mathrm{Th}\,\mathcal{S}$ iff for every term $t(x, \bar{z})$

$$t(\alpha, \bar{z}) \in T \iff t(\beta, \bar{z}) \in T.$$

The set of all behavioral theorems for a theory T forms a congruence, known in algebraic logic as the *Leibniz congruence for T*, notationally ΩT.

DEFINITION 3. A set of formulas $\Delta(x, y) \subseteq \mathrm{Fm}_\Lambda\{x, y\}$ is an *equivalence system for a deductive system* $\mathcal{S}$ if the following hold

1. $\vdash_\mathcal{S} \Delta(x, x);$ (EQ1)

2. $x, \Delta(x, y) \vdash_\mathcal{S} y;$ (EQ2)

3. $\Delta(x, y) \vdash_\mathcal{S} \Delta(y, x);$ (EQ3)

4. $\Delta(x, y), \Delta(y, z) \vdash_\mathcal{S} \Delta(x, z);$ (EQ4)

5. $\langle \Delta(x_i, y_i) \rangle_{i \in n} \vdash_\mathcal{S} \Delta(o\langle x_i \rangle_{i \in n}, o\langle x_i \rangle_{i \in n})$ for each $o \in \Lambda_n$.   (EQ5)

In such case we also say that $\mathcal{S}$ is Δ-*equivalential*. A deductive system $\mathcal{S}$ is *equivalential* if it is Δ-equivalential for some set of formulas Δ. If, in addition, Δ can be chosen finite, then $\mathcal{S}$ is called *finitely equivalential*.

If for a deductive system $\mathcal{S}$ there is a finite set of formulas $\Delta(x, y)$ such that only EQ1 and EQ2 hold, then $\mathcal{S}$ is called *protoalgebraic* and Δ is called a *protoequivalence* system for $\mathcal{S}$. Protoalgebraic deductive systems is the most basic type of deductive system for which some weak algebraizability phenomenon occurs (cf. [1]).

Note that every EQ1–5 represents the condition that every theory of $\mathcal{S}$ is closed under every rule in the respective collection of rules. For example, EQ3 means that every $\mathcal{S}$-theory is closed under all rules of the kind

$$\Delta(x, y) \vdash \alpha(y, x), \text{ where } \alpha(x, y) \in \Delta(x, y).$$

It is well known that an equivalence system defines the Leibniz congruence for every theory T. This is stated in the following lemma.

LEMMA 4. ([3]) *Let* $\mathcal{S}$ *be a deductive system and* $\Delta(x, y)$ *be an equivalence system for* $\mathcal{S}$. *Then for every* $T \in \mathrm{Th}\,\mathcal{S}$

$$\Omega T = \{\langle \alpha, \beta \rangle \mid \Delta(\alpha, \beta) \subseteq T\}.$$

DEFINITION 5. A set of formulas is called an *admissible equivalence system for the deductive system* $\mathcal{S}$ if it is an equivalence system for $\mathcal{S}^{\mathrm{ad}}$.

Note that every equivalence system for a deductive system $\mathcal{S}$ is also an admissible equivalence system for $\mathcal{S}$, but not every admissible equivalence system for $\mathcal{S}$ is an equivalence system for $\mathcal{S}$. This distinction is captured in the following definition

DEFINITION 6. We say that an admissible equivalence system Δ for a deductive system $\mathcal{S}$ is *weak*, if

1. $\mathcal{S}^{\mathrm{ad}}$ is Δ-equivalential,
2. $\mathcal{S}$ is not Δ-equivalential.

4. Theories of admissible rules

Consider the following closure operators on families of sets. Let $\mathcal{C} \subseteq \mathcal{P}(A)$, then

- $\mathbf{X}\,\mathcal{C} := \{$intersections of arbitrary subfamilies of $\mathcal{C}\}$,
- $\hat{\mathbf{U}}\,\mathcal{C} := \{$unions of upward-directed subfamilies of $\mathcal{C}\}$.

If, in addition, $\mathcal{C}$ is a family of theories, i.e., $\mathcal{C} \subseteq \mathcal{P}(\mathrm{Fm}_\Lambda)$, then let

- $\Sigma^{-1}\mathcal{C} := \{\sigma^{-1}T \mid \sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda,\ T \in \mathcal{C}\}$.

Note that if $\mathcal{C} \subseteq \mathcal{P}(A)$, then $A \in \mathbf{X}\mathcal{C}$, $\mathcal{C} \subseteq \mathbf{X}\mathcal{C}$, $\mathcal{C} \subseteq \hat{\mathbf{U}}\mathcal{C}$.

We can provide now a simple characterization for $\mathcal{L}^{\mathrm{ad}}$.

THEOREM 7. *Let $\mathcal{L}$ be a logic. Then*

$$\mathcal{L}^{\mathrm{ad}} = \hat{\mathbf{U}}\mathbf{X}\Sigma^{-1}\mathcal{L}.$$

PROOF. Since the operators $\mathbf{X}$, $\hat{\mathbf{U}}$ and Σ^{-1} preserve compatibility with the finitary rules, the family of theories $\hat{\mathbf{U}}\mathbf{X}\Sigma^{-1}\mathcal{L} \subseteq \mathcal{L}^{\mathrm{ad}}$. Therefore it suffices to show that $\hat{\mathbf{U}}\mathcal{C}$ is closed under intersections and inverse substitutions, whenever $\mathcal{C}$ is an invariant closure system over Fm_Λ. Indeed, the latter would mean that $\hat{\mathbf{U}}\mathbf{X}\Sigma^{-1}\mathcal{L}$ is a deductive system. Since $\Sigma^{-1}\mathcal{L} \subseteq \hat{\mathbf{U}}\mathbf{X}\Sigma^{-1}\mathcal{L}$, then $\mathcal{L}^{\mathrm{ad}} \subseteq \hat{\mathbf{U}}\mathbf{X}\Sigma^{-1}\mathcal{L}$, by Lemma 1.

Suppose we have a set $\{\mathcal{C}_i \mid i \in I\}$ of upward-directed families $\mathcal{C}_i = \{T_j\}_{j \in J_i}$, $i \in I$, where all $J_i, i \in I$ can be chosen so that $J_i \cap J_j = \emptyset$, if $i \neq j$. Suppose $S_i = \bigcup_{j \in J_i}\{T_j\}$. We want to show that $\bigcap_{i \in I} S_i \in \hat{\mathbf{U}}\mathcal{C}$. We have

$$\bigcap_{i \in I} S_i = \bigcap_{i \in I} \bigcup_{j \in J_i} T_j = \bigcup_{a \in \prod_{i \in I} J_i} \bigcap_{i \in I} T_{a(i)}.$$

Let $T_a := \bigcap_{i \in I} T_{a(i)}$, for every $a \in \prod_{i \in I} J_i$. Let us show that the family $\{T_a \mid a \in \prod_{i \in I} J_i\}$ is upward-directed. Take T_a and T_b. For every $i \in I$,

$T_{a(i)}, T_{b(i)}$ belong to the upward-directed set $\mathcal{C}_i$. Therefore, there exist $c \in \prod_{i \in I} J_i$, such that $T_{a(i)} \cup T_{b(i)} \subseteq T_{c(i)} \in \mathcal{C}_i$. Then

$$T_a \cup T_b = \left(\bigcap_{i \in I} T_{a(i)}\right) \bigcup \left(\bigcap_{i \in I} T_{a(i)}\right) \subseteq \bigcap_{i \in I} \left(T_{a(i)} \bigcup T_{b(i)}\right) \subseteq \bigcap_{i \in I} T_{c(i)} = T_c.$$

For proving invariance, let $S \in \hat{\mathbf{U}}\mathcal{C}$, i.e., $S = \bigcup_{i \in I} \{T_i\}$, for an upward-directed family $\{T_i\}_{i \in I} \subseteq \mathcal{C}$. Then

$$\sigma^{-1}S = \sigma^{-1}(\bigcup_{i \in I} T_i) = \bigcup_{i \in I} \sigma^{-1}T_i.$$

But $\{\sigma^{-1}T_i\}_{i \in I}$ is upward-directed, whenever $\{T_i\}_{i \in I}$ is:

$$T_i \cup T_j \subseteq T_k \implies \sigma^{-1}T_i \cup \sigma^{-1}T_j = \sigma^{-1}(T_i \cup T_j) \subseteq \sigma^{-1}T_k,$$

hence $\sigma^{-1}S \in \hat{\mathbf{U}}\mathcal{C}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

DEFINITION 8. Let $\mathcal{L}$ be a logic. We say that the set $\Omega\mathcal{L}$ of behavioral theorems of $\mathcal{L}$ is *explicitly defined* in $\mathcal{L}$ iff there is a set of formulas $\Delta(x, y)$ in two variables such that

$$\langle \alpha, \beta \rangle \in \Omega\mathcal{L} \iff \Delta(\alpha, \beta) \subseteq \mathcal{L}.$$

The set Δ is called *a defining set for behavioral theorems of $\mathcal{L}$*.

The importance of the admissible equivalence systems for $\mathcal{L}^{\mathrm{ad}}$ is shown by the following lemma

PROPOSITION 9. *Let $\mathcal{L}$ be a logic and $\Delta(x, y)$ a finite subset of $\mathrm{Fm}_\Lambda\{x, y\}$. Then $\Delta(x, y)$ is a defining set for behavioral theorems of $\mathcal{L}$ iff $\Delta(x, y)$ is an equivalence system for $\mathcal{L}^{\mathrm{ad}}$.*

PROOF. ($\Rightarrow$) Suppose $\Delta(x, y)$ explicitly defines the behavioral theorems of $\mathcal{L}$. It is straightforward to show that $\mathcal{L}$ is closed under the rules EQ1–5. For illustration, we will prove condition EQ3.

Let σ be a substitution. Suppose that $\sigma(\Delta(x, y)) = \Delta(\sigma(x), \sigma(y)) \subseteq \mathcal{L}$. By hypothesis,

$$
\begin{aligned}
\Delta(\sigma(x), \sigma(y)) \subseteq \mathcal{L} \quad &\Leftrightarrow \quad \langle \sigma(x), \sigma(y) \rangle \in \Omega\mathcal{L} \\
&\Leftrightarrow \quad \langle \sigma(y), \sigma(x) \rangle \in \Omega\mathcal{L} \\
&\Leftrightarrow \quad \Delta(\sigma(y), \sigma(x)) \subseteq \mathcal{L} \\
&\Leftrightarrow \quad \sigma(\Delta(y, x)) \subseteq \mathcal{L}
\end{aligned}
$$

Moreover, every $T \in \Sigma^{-1}\mathcal{L}$ is closed under the rules EQ1–5. As above, we only show for EQ3. Let $T \in \Sigma^{-1}\mathcal{L}$. That is, $T = \tau^{-1}\mathcal{L}$ for some

substitution τ. Let σ be another substitution. Suppose that $\sigma(\Delta(x,y)) \subseteq \tau^{-1}\mathcal{L}$. Hence, $\tau(\sigma(\Delta(x,y))) = \Delta(\tau(\sigma(x)), \tau(\sigma(y))) \subseteq \tau(\tau^{-1}\mathcal{L}) \subseteq \mathcal{L}$. By previous result, $\tau(\sigma(\Delta(y,x))) \subseteq \mathcal{L}$. Therefore,

$$\sigma(\Delta(y,x)) \subseteq \tau^{-1}(\tau(\sigma(\Delta(y,x)))) \subseteq \tau^{-1}\mathcal{L}$$

Since operators $\hat{\mathbf{U}}$ and $\mathbf{X}$ preserve compatibility with finitary rules, it follows from Theorem 7, that every theory of $\mathcal{L}^{\mathrm{ad}}$ is closed under the rules EQ1–5.

($\Leftarrow$) Suppose Δ is an equivalence system for $\mathcal{L}^{\mathrm{ad}}$. It follows from Lemma 4, that Δ explicitly defines, in the sense of Lemma 4, ΩT for any theory T of $\mathcal{L}^{\mathrm{ad}}$. In particular, Δ explicitly defines behavioral theorems of $\mathcal{L}$. $\qquad\square$

LEMMA 10. *If Δ is a finite equivalence system for a deductive system $\mathcal{S}$, then Δ is a finite equivalence system for $\mathcal{S}^{\mathrm{ad}}$.*

PROOF. By Definition 3, every theory of $\mathcal{S}$ is a theory of rules EQ1–5. It follows from Lemma 1 that $\mathcal{S}^{\mathrm{ad}} \subseteq \mathcal{S}$. Therefore all theories of $\mathcal{S}^{\mathrm{ad}}$ are also theories for the rules EQ1–5, and Δ is an equivalence system for $\mathcal{S}^{\mathrm{ad}}$. $\qquad\square$

We will construct our examples based on the following result

LEMMA 11. *Δ is a finite weak equivalence system for a deductive system $\mathcal{S}$ iff there is a deductive system $\mathcal{Q} \subseteq \mathcal{S}$ such that*

1. *$\mathcal{S}^{\mathrm{ad}} \subseteq \mathcal{Q} \subseteq \mathcal{S}$,*
2. *Δ is a finite equivalence system for $\mathcal{Q}$,*
3. *Δ is not an equivalence system for $\mathcal{S}$.*

PROOF. ($\Rightarrow$) If Δ is a finite weak equivalence system then it is an equivalence system for $\mathcal{S}^{\mathrm{ad}}$, and $\mathcal{S}^{\mathrm{ad}}$ satisfies all the necessary conditions.

($\Leftarrow$) Since $\mathcal{S}^{\mathrm{ad}} \subseteq \mathcal{Q}$, then $\mathcal{S}^{\mathrm{ad}}$ is Δ-equivalential. Therefore $\mathcal{S}$ has Δ as an admissible equivalence system, but not as an equivalence system. $\qquad\square$

5. Examples

In this section we will provide some examples from the area of modal propositional logics that will support the claims we made in the introduction.

Let $\Lambda = \{\vee^2, \wedge^2, \rightarrow^2, \neg^1, \square^1\}$ be the modal propositional language (where functional symbols are shown with arities) and

(MP)	$x, x \rightarrow y \vdash y$	modus ponens
(RE)	$x \leftrightarrow y \vdash \square x \leftrightarrow \square y$	extensionality rule
(NR)	$x \vdash \square x$	necessitation rule.

By a *modal logic* $\mathcal{L}$ we understand an invariant set $\mathcal{L} \subsetneq \mathrm{Fm}_\Lambda$ of modal formulas containing all classical tautologies and closed under MP. A modal logic $\mathcal{L}$ is called *classical* if $\mathcal{L}$ is closed under RE. We will need the following modal logics

- E is the least classical modal logic,
- K is the least modal logic containing the formula $\square(x \rightarrow y) \rightarrow (\square x \rightarrow \square y)$ and closed under NR.

A modal logic is called *normal* if it contains K and is closed under NR.

- T is the least normal modal logic containing K and the formula $\square x \rightarrow x$,
- $K4$ is the least normal modal logic containing K and the formula $\square x \rightarrow \square\square x$,
- $S4$ is the least normal modal logic containing both the logic $K4$ and T.

Note that some of the deductive systems we will consider next are well known in the theory of modal logics (under various names). For instance, the *weak consequence relation* $\mathcal{L}^w$ (or $\mathcal{L}^\rightarrow$ as in Introduction and in [8]) corresponds to $[\mathcal{L})_{\mathrm{MP}}$ in our current notation (recall that $[\mathcal{L})_{\mathrm{MP}} = \{T \in \mathrm{Th}(\mathrm{MP}) \mid \mathcal{L} \subseteq \mathrm{T}\}$) and the *strong consequence relation* $\mathcal{L}^s$ (denoted also $\mathcal{L}^{\rightarrow,\square}$) corresponds to $[\mathcal{L})_{\mathrm{MP+NR}}$.

EXAMPLE 12. ([8]) *There exists a non-equivalential deductive system with a finite weak equivalence system.*

PROOF. The deductive system $[E)_{\mathrm{MP}}$ is not equivalential [8, Corollary II.3], but $[E)_{\mathrm{MP+RE}}$ is finitely equivalential with the equivalence system $\Delta = \{x \leftrightarrow y\}$ (see the remark after [8, Corollary II.3]).

Note that instead of $[E)_{\mathrm{MP+RE}}$, we can take $([E)_{\mathrm{MP}})^{\mathrm{ad}}$, since $([E)_{\mathrm{MP}})^{\mathrm{ad}} \subseteq [E)_{\mathrm{MP+RE}}$. $\square$

EXAMPLE 13. *There exists a non-finitely equivalential deductive system with a finite weak equivalence system.*

PROOF. While $[K)_{\mathrm{MP}}$ does not have a finite equivalential system [8, Corollary III.2], it has an infinite equivalence system [8, Theorem II.4]

$$\Delta_\omega(x,y) = \{\Box^n(x \leftrightarrow y)\}_{n \in \omega}.$$

It is easy to see that $[K)_{\mathrm{MP+NR}}$ is finitely equivalential with

$$\Delta(x,y) = \{x \leftrightarrow y\}$$

as it equivalence system. Indeed, for every theory T of $[K)_{\mathrm{MP+NR}}$, by application of the rule NR,

$$\Delta_\omega(\alpha,\beta) \subseteq T \iff \Delta(\alpha,\beta) \subseteq T.$$

Thus $[K)_{\mathrm{MP}}$ is a non-finitely equivalential deductive system that has a finite weak equivalence system.

Analogous to Example 12, we can take $([K)_{\mathrm{MP}})^{\mathrm{ad}}$ instead of $[K)_{\mathrm{MP+RE}}$. $\Box$

Similar to the above, but using [8, Theorem III.1], instead of [8, Corollary III.2], we can show that the deductive system $[T)_{\mathrm{MP}}$ based on Feys-von Wright's logic T is also a non-finitely equivalential deductive system with a finite weak equivalence system.

By a *structurally complete* normal modal logic we understand a normal modal logic $\mathcal{L}$ such that $\mathcal{L}^{\mathrm{ad}} = [\mathcal{L})_{\mathrm{MP+NR}}$. (In other words, all admissible rules of $\mathcal{L}$ are derivable using MP, NR and usual structural rules.)

EXAMPLE 14. *There exists a non-protoalgebraic deductive system with a finite weak equivalence system.*

PROOF. Let $\mathcal{L}$ be any structurally complete normal modal logic extending $K4$ (such logics exist, cf. [4]). By definition of structural completeness $\mathcal{L}^{\mathrm{ad}} = [\mathcal{L})_{\mathrm{MP+NR}}$.

Let $\mathcal{L}^{\emptyset}$ be the deductive system of all sets of modal formulas that extend $\mathcal{L}$:

$$\mathcal{L}^{\emptyset} := [\mathcal{L})_{\mathcal{P}(\mathrm{Fm}_\Lambda)}.$$

Note that the theories of $\mathcal{L}^{\emptyset}$ are not necessarily closed under modus ponens. Then $\mathcal{L}^{\emptyset}$ is strictly bigger than $\mathcal{L}^{\rightarrow} := [\mathcal{L})_{\mathrm{MP}}$. Indeed, consider the theory $T := \mathcal{L} \cup \{(x \rightarrow x) \rightarrow y\} \in \mathcal{L}^{\emptyset}$. Then $x \rightarrow x \in \mathcal{L} \subseteq T$, but $y \notin \mathcal{L}$ (since

$\mathcal{L} \neq \mathrm{Fm}_\Lambda$) and therefore $y \notin T$. Thus T is not closed under modus ponens, therefore $T \in \mathcal{L}^\emptyset$, but $T \notin \mathcal{L}^\rightarrow$.

Under the assumptions that $K4 \subseteq \mathcal{L}$, $\mathcal{L}^{\mathrm{ad}} = [\mathcal{L})_{\mathrm{MP+NR}}$ is equivalential with the parameter-free equivalence system $\Delta(x, y) = \{x \leftrightarrow y\}$.

On the other hand, for any set of formulas $\Delta(x, y) \subseteq \mathcal{L}$, the theory $T := \mathcal{L} \cup \Delta(x, y) \cup \{x\} \in \mathcal{L}^\emptyset$ shows that Δ cannot be a protoequivalence system for $\mathcal{L}^\emptyset$, since otherwise $y \in \mathcal{L} \cup \Delta(x, y)$, by EQ2. Thus if $y \in \mathcal{L}$, then $\mathcal{L} = \mathrm{Fm}_\Lambda$. If $y \in \Delta(x, y)$, then $x \in \Delta(x, x) \subseteq \mathcal{L}$, by EQ1, and again $\mathcal{L} = \mathrm{Fm}_\Lambda$, but a modal logic, by definition, cannot be totally inconsistent. □

An immediate question arises:

Suppose we have an equivalential system Δ for a deductive system $\mathcal{S}$. Is it true that every weak equivalence system Δ' for $\mathcal{S}$ can be obtained as a subset of Δ?

Although it is true for all our examples, the answer in general is no. Instead of equality, we have that every weak equivalence system Δ' is *equivalent* to some $\Delta'' \subseteq \Delta$ modulo $\mathcal{S}^{\mathrm{ad}}$. For the case of infinitely equivalential deductive systems the answer follows from known properties of equivalential systems (cf. [3, Notes 3.1.7]).

LEMMA 15. *Suppose a deductive system $\mathcal{S}$ is equivalential with an infinite equivalence system Δ. For every finite weak equivalence system Δ' for $\mathcal{S}$, there is a finite $\Delta'' \subseteq \Delta$, which is equivalent to Δ' modulo $\mathcal{S}^{\mathrm{ad}}$.*

PROOF.   By [3, Notes 3.1.7(1)], every two equivalence systems are deductively equivalent. Therefore, since Δ and Δ' are equivalence systems for $\mathcal{S}^{\mathrm{ad}}$ (Δ is by inheritance from $\mathcal{S}$) and $\mathcal{S}^{\mathrm{ad}}$ is finitary, Δ' is deductively equivalent to a finite sunset Δ'' of Δ, therefore Δ'' is an equivalence system for $\mathcal{S}^{\mathrm{ad}}$, by [3, Notes 3.1.7(2)].   □

6.   Discussion

Admissible theories have been thoroughly investigated in the field of non-classical logics by algebraic methods (cf. [12]). The key to the admissibility theory of a logic $\mathcal{L}$ is the family $\Sigma^{-1}\mathcal{L}$ of its inverse images under substi-

tutions. For modal and superintuitionistic logics the problem of describing admissible theories is traditionally reduced to description of pertinent filters of free algebras of finite rank $\mathcal{F}_n(\lambda) := \mathcal{F}_n(\mathrm{var}(\lambda))$, $n \in \omega$ of the corresponding variety of algebras $\mathrm{var}(\lambda)$ (cf. [12]). More direct but also less developed approach would be to start with the free algebra $\mathcal{F}_\omega(\lambda)$ and to consider the quasiequational theory of $\mathcal{F}_\omega(\lambda)$. This theory is determined by a family of corresponding congruences on $\mathcal{F}_\omega(\lambda)$. But $\mathcal{F}_\omega(\lambda)$ is a homomorphic image of the absolutely free algebra of countable rank Fm_Λ (the formulae algebra), under the canonical homomorphism which sends propositional variables to generators. Therefore we can pull the congruences back along this homomorphism to obtain a family of congruences on Fm_Λ. The last step for obtaining a Hilbert-style deductive system is to take for every congruence the congruence class containing the constant term $\top := x \to x$.

For illustration, let us look at the case of the admissible theories of Int, i.e., of intuitionistic propositional logic Int with modus ponens as its only inference rule. Int^{ad} does not have a finite basis for inference rules [11] (but it is recursively axiomatizable by so called de Jongh-Visser rules [7]). It is known that Int is not structurally complete [6], i.e., Int^{ad} is a proper subsystem of $Int^\to$. $Int^\to$, in its turn, is finitely axiomatizable by axioms of Int and modus ponens, it is also $\{x \leftrightarrow y\}$-equivalential (even $\{x \leftrightarrow y\}$-algebraizable). It is not known whether Int^{ad} has an equivalence system not interderivably equivalent to $\{x \leftrightarrow y\}$ modulo Int^{ad}.

In the realm of modal logics, there are three major deductive systems usually associated with $S4$: $S4^{\mathrm{ad}} \subsetneq S4^{\to,\square} \subsetneq S4^\to$. Similarly to its intuitionistic counterpart, $S4^{\mathrm{ad}}$ does not have a finite basis for inference rules [11], but it is recursively axiomatizable [10], while $S4^{\to,\square}$ and $S4^\to$ are finitely axiomatizable. $S4^\to$ is $\{\square(x \leftrightarrow y)\}$-equivalential and $S4^{\to,\square}$ is $\{\square(x \leftrightarrow y)\}$-algebraizable. It is not known whether $S4^{\to,\square}$ has a weak equivalence system.

7. Conclusions and Open Problems

In this paper we studied the notion of weak equivalence system for a deductive system which expands a similar notion used in modal logics [8]. Each weak equivalence system for $\mathcal{S}$ is an equivalence system for the admissible part $\mathcal{S}^{\mathrm{ad}}$ of $\mathcal{S}$. So admissible theories are crucial for characterizing weak

equivalence systems. We outlined the basic properties of admissible theories and showed some connections with the notions of abstract algebraic logic. This is just preliminary work, in the sense that our proposal will require more relevant outcomes, but the approach seems to be powerful. The results and applications obtained are promising. We hope to carry on a consistent algebraic theory of admissible equivalence systems in the near future that we will report in forthcoming papers.

We finish by summing up here the list of open problems that seem to be important for the topic.

- Is it true that every algebraizable deductive system does not have a weak equivalential system?
- For the other direction: Is it true that an equivalential deductive system with no weak equivalence system must be algebraizable?
- The deductive systems of the kind $\mathcal{S}^{\mathrm{ad}}$ are really important for description of admissible rules and solving logical equations. What is the structure of $\mathcal{S}^{\mathrm{ad}}$, at least for classical deductive systems?
- Is there a weak equivalence system for $Int^{\rightarrow}$? $S4^{\rightarrow,\square}$?

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