HIGHER-ORDER HAHN'S QUANTUM VARIATIONAL CALCULUS

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Abstract. We prove a necessary optimality condition of Euler–Lagrange type for quantum variational problems involving Hahn's derivatives of higher-order.

1. Introduction

Many physical phenomena are described by equations involving nondifferentiable functions, e.g., generic trajectories of quantum mechanics [15]. Several different approaches to deal with nondifferentiable functions are followed in the literature of variational calculus, including the time scale approach, which typically deal with delta or nabla differentiable functions [14, 20, 23], the fractional approach, allowing to consider functions that have no first order derivative but have fractional derivatives of all orders less than one [3, 12, 16], and the quantum approach, which is particularly useful to model physical and economical systems [8, 10, 22].

Roughly speaking, a quantum calculus substitute the classical derivative by a difference operator, which allows to deal with sets of nondifferentiable functions. Several dialects of quantum calculus are available [13, 18]. For motivation to study a nondifferentiable quantum variational calculus we refer the reader to [4, 8, 10].

In 1949 Hahn introduced the difference operator $D_{q,\omega}$ defined by

$$D_{q,\omega}\left[f\right](t) := \frac{f\left(qt+\omega\right) - f\left(t\right)}{\left(q-1\right)t+\omega},$$

where f is a real function, and $q \in (0,1)$ and $\omega > 0$ are real fixed numbers [17]. The Hahn difference operator has been applied successfully in the construction of families of ortogonal polynomials as well as in approximation problems [6, 11, 25]. However, during 60 years, the construction of the proper inverse of Hahn's difference operator remained an open question. Eventually, the problem was solved in 2009 by Aldwoah [1] (see also [2, 7]). Here we introduce the higher-order Hahn's quantum variational calculus, proving the Hahn quantum analog of the higher-order Euler–Lagrange equation. As particular cases we obtain the q-calculus Euler–Lagrange equation [8] and the h-calculus Euler–Lagrange equation [9, 19].

Variational functionals that depend on higher derivatives arise in a natural way in applications of engineering, physics, and economics. Let us consider, for example, the equilibrium of an elastic bending beam. Let us denote by $y(x)$ the deflection

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of the point x of the beam, $E(x)$ the elastic stiffness of the material, that can vary with x, and $\xi(x)$ the load that bends the beam. One may assume that, due to some constraints of physical nature, the dynamics does not depend on the usual derivative $y'(x)$ but on some quantum derivative $D_{q,\omega}[y](x)$. In this condition, the equilibrium of the beam correspond to the solution of the following higher-order Hahn's quantum variational problem:

$$\int_0^L \left[\frac{1}{2}\left(E(x)D^2_{q,\omega}[y](x)\right)^2 - \xi(x)y\left(q^2x+q\omega+\omega\right)\right]dx \longrightarrow \min . \tag{1.1}$$

Note that we recover the classical problem of the equilibrium of the elastic bending beam when $(\omega, q) \to (0,1)$. Problem (1.1) is a particular case of the problem (P) investigated in Section 3. Our higher-order Hahn's quantum Euler–Lagrange equation (Theorem 3.10) gives the main tool to solve such problems.

The paper is organized as follows. In Section 2 we summarize all the necessary definitions and properties of the Hahn difference operator and the associated q,ω-integral. In Section 3 we formulate and prove our main results: in §3.1 we prove a higher-order fundamental Lemma of the calculus of variations with the Hahn operator (Lemma 3.8); in §3.2 we deduce a higher-order Euler–Lagrange equation for Hahn's variational calculus (Theorem 3.10); finally we provide in §3.3 a simple example of a quantum optimization problem where our Theorem 3.10 leads to the global minimizer, which is not a continuous function.

2. Preliminaries

Let $q \in (0,1)$ and $\omega > 0$. We introduce the real number

$$\omega_0 := \frac{\omega}{1-q}.$$

Let I be a real interval containing ω_0. For a function f defined on I, the *Hahn difference operator* of f is given by

$$D_{q,\omega}[f](t) := \frac{f(qt+\omega) - f(t)}{(q-1)t+\omega}, \text{ if } t \neq \omega_0 ,$$

and $D_{q,\omega}[f](\omega_0) := f'(\omega_0)$, provided f is differentiable at ω_0. We sometimes call $D_{q,\omega}[f]$ the *q,ω-derivative of f*, and f is said to be *q,ω-differentiable on I* if $D_{q,\omega}[f](\omega_0)$ exists.

Remark 2.1. The $D_{q,\omega}$ operator generalizes (in the limit) the forward h-difference and the Jackson q-difference operators [13, 18]. Indeed, when $q \to 1$ we obtain the forward h-difference

$$\Delta_h[f](t) := \frac{f(t+h) - f(t)}{h},$$

when $\omega \to 0$ we obtain the Jackson q-difference operator

$$D_q[f](t) := \frac{f(qt) - f(t)}{t(q-1)}, \text{ if } t \neq 0 ,$$

and $D_q[f](0) = f'(0)$ provided $f'(0)$ exists. Notice also that, under appropriate conditions,

$$\lim_{\omega\to 0, q\to 1} D_{q,\omega}[f](t) = f'(t) .$$

The Hahn difference operator has the following properties:

Theorem 2.2 ([1, 2, 7])**.** *Let f and g be q,ω-differentiable on I and $t \in I$. One has:*

(1) $D_{q,\omega}[f](t) \equiv 0$ *on I if and only if f is constant;*
(2) $D_{q,\omega}[f+g](t) = D_{q,\omega}[f](t) + D_{q,\omega}[g](t)$*;*
(3) $D_{q,\omega}[fg](t) = D_{q,\omega}[f](t)\, g(t) + f(qt+\omega)\, D_{q,\omega}[g](t)$*;*
(4) $D_{q,\omega}\left[\frac{f}{g}\right](t) = \frac{D_{q,\omega}[f](t)\, g(t) - f(t)\, D_{q,\omega}[g](t)}{g(t)\, g(qt+\omega)}$ *if $g(t)\, g(qt+\omega) \neq 0$;*
(5) $f(qt+\omega) = f(t) + (t(q-1)+\omega)\, D_{q,\omega}[f](t)$*.*

For $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ define $[k]_q := \frac{1-q^k}{1-q}$ and let $\sigma(t) = qt + \omega$, $t \in I$. Note that σ is a contraction, $\sigma(I) \subseteq I$, $\sigma(t) < t$ for $t > \omega_0$, $\sigma(t) > t$ for $t < \omega_0$, and $\sigma(\omega_0) = \omega_0$. The following technical result is used several times in our paper:

Lemma 2.3 ([1, 7])**.** *Let $k \in \mathbb{N}$ and $t \in I$. Then,*

(1) $\sigma^k(t) = \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{k\text{-times}}(t) = q^k t + \omega [k]_q$*;*
(2) $\left(\sigma^k(t)\right)^{-1} = \sigma^{-k}(t) = \frac{t - \omega [k]_q}{q^k}$*.*

¿From now on I denotes an interval of $\mathbb{R}$ containing ω_0. Following [1, 2, 7] we define the notion of *q,ω-integral* (also known as the *Jackson–Nörlund integral*) as follows:

Definition 2.4. Let $a, b \in I$ and $a < b$. For $f : I \to \mathbb{R}$ the *q,ω-integral of f* from a to b is given by

$$\int_a^b f(t)\, d_{q,\omega}t := \int_{\omega_0}^b f(t)\, d_{q,\omega}t - \int_{\omega_0}^a f(t)\, d_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t)\, d_{q,\omega}t := (x(1-q) - \omega) \sum_{k=0}^{+\infty} q^k f\left(xq^k + \omega [k]_q\right),\ x \in I,$$

provided that the series converges at $x = a$ and $x = b$. In that case, f is called *q,ω-integrable on* $[a, b]$. We say that f is *q,ω-integrable over I* if it is *q,ω-integrable* over $[a, b]$ for all $a, b \in I$.

Remark 2.5. The *q,ω-integral* generalizes (in the limit) the Jackson q-integral and the Nörlund's sum [18]. When $\omega \to 0$, we obtain the Jackson q-integral

$$\int_a^b f(t)\, d_q t := \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t,$$

where

$$\int_0^x f(t)\, d_q t := x(1-q) \sum_{k=0}^{+\infty} q^k f\left(xq^k\right).$$

When $q \to 1$, we obtain the Nörlund's sum

$$\int_a^b f(t)\, \Delta_\omega t := \int_{+\infty}^b f(t)\, \Delta_\omega t - \int_{+\infty}^a f(t)\, \Delta_\omega t,$$

where

$$\int_{+\infty}^{x} f(t)\,\Delta_\omega t := -\omega \sum_{k=0}^{+\infty} f(x+k\omega).$$

It can be shown that if $f : I \to \mathbb{R}$ is continuous at ω_0, then f is *q,ω-integrable over I* [1, 2, 7].

Theorem 2.6 (Fundamental Theorem of Hahn's Calculus [1, 7])**.** *Assume that $f : I \to \mathbb{R}$ is continuous at ω_0 and, for each $x \in I$, define*

$$F(x) := \int_{\omega_0}^{x} f(t)\,d_{q,\omega}t.$$

Then F is continuous at ω_0. Furthermore, $D_{q,\omega}[F](x)$ exists for every $x \in I$ with $D_{q,\omega}[F](x) = f(x)$. Conversely, $\int_a^b D_{q,\omega}[f](t)\,d_{q,\omega}t = f(b) - f(a)$ for all $a, b \in I$.

The *q,ω-integral* has the following properties:

Theorem 2.7 ([1, 2, 7])**.** *Let $f, g : I \to \mathbb{R}$ be q,ω-*integrable on *I, $a, b, c \in I$ and $k \in \mathbb{R}$. Then,*

(1) $\int_a^a f(t)\,d_{q,\omega}t = 0$;
(2) $\int_a^b kf(t)\,d_{q,\omega}t = k\int_a^b f(t)\,d_{q,\omega}t$;
(3) $\int_a^b f(t)\,d_{q,\omega}t = -\int_b^a f(t)\,d_{q,\omega}t$;
(4) $\int_a^b f(t)\,d_{q,\omega}t = \int_a^c f(t)\,d_{q,\omega}t + \int_c^b f(t)\,d_{q,\omega}t$;
(5) $\int_a^b (f(t)+g(t))\,d_{q,\omega}t = \int_a^b f(t)\,d_{q,\omega}t + \int_a^b g(t)\,d_{q,\omega}t$;
(6) *Every Riemann integrable function f on I is q,ω-*integrable on *I;*
(7) *If $f, g : I \to \mathbb{R}$ are q,ω-differentiable and $a, b \in I$, then*

$$\int_a^b f(t)\,D_{q,\omega}[g](t)\,d_{q,\omega}t = f(t)\,g(t)\Big|_a^b - \int_a^b D_{q,\omega}[f](t)\,g(qt+\omega)\,d_{q,\omega}t.$$

Property 7 of Theorem 2.7 is known as *q,ω-integration by parts.* Note that

$$\int_{\sigma(t)}^{t} f(\tau)\,d_{q,\omega}\tau = (t(1-q)-\omega)\,f(t).$$

Lemma 2.8 (cf. [1, 7])**.** *Let $b \in I$ and f be q,ω-*integrable *over I. Suppose that*

$$f(t) \geq 0 \quad \forall t \in \left\{q^n b + \omega\,[n]_q : n \in \mathbb{N}_0\right\}.$$

(1) *If $\omega_0 \leq b$, then*
$$\int_{\omega_0}^{b} f(t) d_{q,\omega}t \geq 0.$$
(2) *If $\omega_0 > b$, then*
$$\int_{b}^{\omega_0} f(t) d_{q,\omega}t \geq 0.$$

Remark 2.9. There is an inconsistency in [1, 7]. Indeed, Lemma 6.2.7 of [1] is only valid if $b \geq \omega_0$ and $a \leq b$. Similarly with respect to Lemma 3.7 of [7].

Remark 2.10. In general it is not true that

$$\left|\int_a^b f(t)\,d_{q,\omega}t\right| \leq \int_a^b |f(t)|\,d_{q,\omega}t, \quad a, b \in I.$$

For a counterexample see [1, 7]. This illustrates well the difference with other non-quantum integrals, e.g., the time scale integrals [21, 24].

For $s \in I$ we define

$$[s]_{q,\omega} := \left\{ q^n s + \omega \, [n]_q : n \in \mathbb{N}_0 \right\} \cup \{\omega_0\} . \tag{2.1}$$

The following definition and lemma are important for our purposes.

Definition 2.11. Let $s \in I$ and $g : I \times (-\bar{\theta}, \bar{\theta}) \to \mathbb{R}$. We say that $g(t, \cdot)$ is differentiable at θ_0 uniformly in $[s]_{q,\omega}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |\theta - \theta_0| < \delta \Rightarrow \left| \frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t, \theta_0) \right| < \varepsilon$$

for all $t \in [s]_{q,\omega}$, where $\partial_2 g = \dfrac{\partial g}{\partial \theta}$.

Lemma 2.12 (cf. [22])**.** *Let $s \in I$. Assume that $g : I \times (-\bar{\theta}, \bar{\theta}) \to \mathbb{R}$ is differentiable at θ_0 uniformly in $[s]_{q,\omega}$, and $\int_{\omega_0}^{s} \partial_2 g(t, \theta_0) \, d_{q,\omega} t$ exist. Then,*

$$G(\theta) := \int_{\omega_0}^{s} g(t, \theta) \, d_{q,\omega} t,$$

for θ near θ_0, is differentiable at θ_0 with $G'(\theta_0) = \int_{\omega_0}^{s} \partial_2 g(t, \theta_0) \, d_{q,\omega} t$.

3. Main Results

We define the q,ω-derivatives of higher-order in the usual way: the rth q,ω-derivative ($r \in \mathbb{N}$) of $f : I \to \mathbb{R}$ is the function $D^r_{q,\omega}[f] : I \to \mathbb{R}$ given by $D^r_{q,\omega}[f] := D_{q,\omega}[D^{r-1}_{q,\omega}[f]]$, provided $D^{r-1}_{q,\omega}[f]$ is q,ω-differentiable on I and where $D^0_{q,\omega}[f] := f$.

Let $a, b \in I$ and $a < b$. We introduce the linear space $\mathcal{Y}^r = \mathcal{Y}^r([a,b], \mathbb{R})$ by

$$\mathcal{Y}^r := \left\{ y : I \to \mathbb{R} \mid D^i_{q,\omega}[y], i = 0, \ldots, r, \text{ are bounded on } [a,b] \text{ and continuous at } \omega_0 \right\}$$

endowed with the norm $\|y\|_{r,\infty} := \sum_{i=0}^{r} \left\| D^i_{q,\omega}[y] \right\|_\infty$, where $\|y\|_\infty := \sup_{t \in [a,b]} |y(t)|$. The following notations are in order: $\sigma(t) = qt + \omega$, $y^\sigma(t) = y^{\sigma^1}(t) = (y \circ \sigma)(t) = y(qt + \omega)$, and $y^{\sigma^k} = y \circ y^{\sigma^{k-1}}$, $k = 2, 3, \ldots$ Our main goal is to establish necessary optimality conditions for the higher-order q,ω-variational problem[1]

$$\begin{gathered} \mathcal{L}[y] = \int_a^b L\left(t, y^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t), \ldots, D^r_{q,\omega}[y](t)\right) d_{q,\omega} t \longrightarrow \text{extr} \\ y \in \mathcal{Y}^r([a,b], \mathbb{R}) \\ y(a) = \alpha_0, \quad y(b) = \beta_0, \\ \vdots \\ D^{r-1}_{q,\omega}[y](a) = \alpha_{r-1}, \quad D^{r-1}_{q,\omega}[y](b) = \beta_{r-1}, \end{gathered} \tag{P}$$

where $r \in \mathbb{N}$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 0, \ldots, r-1$, are given.

Definition 3.1. We say that y is an admissible function for (P) if $y \in \mathcal{Y}^r([a,b], \mathbb{R})$ and y satisfies the boundary conditions $D^i_{q,\omega}[y](a) = \alpha_i$ and $D^i_{q,\omega}[y](b) = \beta_i$ of problem (P), $i = 0, \ldots, r-1$.

[1] In problem (P) "extr" denotes "extremize" (i.e., minimize or maximize).

The Lagrangian L is assumed to satisfy the following hypotheses:

(H1) $(u_0,\ldots,u_r)\to L(t,u_0,\ldots,u_r)$ is a $C^1(\mathbb{R}^{r+1},\mathbb{R})$ function for any $t\in[a,b]$;
(H2) $t\to L(t,y(t),D_{q,\omega}[y](t),\ldots,D^r_{q,\omega}[y](t))$ is continuous at ω_0 for any admissible y;
(H3) functions $t\to\partial_{i+2}L(t,y(t),D_{q,\omega}[y](t),\cdots,D^r_{q,\omega}[y](t))$, $i=0,1,\cdots,r$, belong to $\mathcal{Y}^1([a,b],\mathbb{R})$ for all admissible y.

Definition 3.2. We say that y_* is a local minimizer (resp. local maximizer) for problem (P) if y_* is an admissible function and there exists $\delta>0$ such that

$$\mathcal{L}[y_*]\le\mathcal{L}[y]\quad(\text{resp. }\mathcal{L}[y_*]\ge\mathcal{L}[y]\,)$$

for all admissible y with $\|y_*-y\|_{r,\infty}<\delta$.

Definition 3.3. We say that $\eta\in\mathcal{Y}^r([a,b],\mathbb{R})$ is a *variation* if $\eta(a)=\eta(b)=0$, $\ldots$, $D^{r-1}_{q,\omega}[\eta](a)=D^{r-1}_{q,\omega}[\eta](b)=0$.

We define the q,ω-interval from a to b by

$$[a,b]_{q,\omega}:=\left\{q^na+\omega[n]_q:n\in\mathbb{N}_0\right\}\cup\left\{q^nb+\omega[n]_q:n\in\mathbb{N}_0\right\}\cup\{\omega_0\},$$

i.e., $[a,b]_{q,\omega}=[a]_{q,\omega}\cup[b]_{q,\omega}$, where $[a]_{q,\omega}$ and $[b]_{q,\omega}$ are given by (2.1).

3.1. **Higher-order fundamental lemma of Hahn's variational calculus.** The chain rule, as known from classical calculus, does not hold in Hahn's quantum context (see a counterexample in [1, 7]). However, we can prove the following.

Lemma 3.4. *If f is q,ω-differentiable on I, then the following equality holds:*

$$D_{q,\omega}[f^\sigma](t)=q\left(D_{q,\omega}[f]\right)^\sigma(t),\quad t\in I.$$

Proof. For $t\ne\omega_0$ we have

$$\left(D_{q,\omega}[f]\right)^\sigma(t)=\frac{f(q(qt+\omega)+\omega)-f(qt+\omega)}{(q-1)(qt+\omega)+\omega}=\frac{f(q(qt+\omega)+\omega)-f(qt+\omega)}{q((q-1)t+\omega)}$$

and

$$D_{q,\omega}[f^\sigma](t)=\frac{f^\sigma(qt+\omega)-f^\sigma(t)}{(q-1)t+\omega}=\frac{f(q(qt+\omega)+\omega)-f(qt+\omega)}{(q-1)t+\omega}.$$

Therefore, $D_{q,\omega}[f^\sigma](t)=q\left(D_{q,\omega}[f]\right)^\sigma(t)$. If $t=\omega_0$, then $\sigma(\omega_0)=\omega_0$. Thus,

$$\left(D_{q,\omega}[f]\right)^\sigma(\omega_0)=\left(D_{q,\omega}[f]\right)(\sigma(\omega_0))=\left(D_{q,\omega}[f]\right)(\omega_0)=f'(\omega_0)$$

and $D_{q,\omega}[f^\sigma](\omega_0)=[f^\sigma]'(\omega_0)=f'(\sigma(\omega_0))\,\sigma'(\omega_0)=qf'(\omega_0)$. □

Lemma 3.5. *If $\eta\in\mathcal{Y}^r([a,b],\mathbb{R})$ is such that $D^i_{q,\omega}[\eta](a)=0$ (resp. $D^i_{q,\omega}[\eta](b)=0$) for all $i\in\{0,1,\ldots,r\}$, then $D^{i-1}_{q,\omega}[\eta^\sigma](a)=0$ (resp. $D^{i-1}_{q,\omega}[\eta^\sigma](b)=0$) for all $i\in\{1,\ldots,r\}$.*

Proof. If $a=\omega_0$ the result is trivial (because $\sigma(\omega_0)=\omega_0$). Suppose now that $a\ne\omega_0$ and fix $i\in\{1,\ldots,r\}$. Note that

$$D^i_{q,\omega}[\eta](a)=\frac{\left(D^{i-1}_{q,\omega}[\eta]\right)^\sigma(a)-D^{i-1}_{q,\omega}[\eta](a)}{(q-1)a+\omega}.$$

Since, by hypothesis, $D^{i}_{q,\omega}[\eta](a)=0$ and $D^{i-1}_{q,\omega}[\eta](a)=0$, then $\left(D^{i-1}_{q,\omega}[\eta]\right)^{\sigma}(a)=0$. Lemma 3.4 shows that

$$\left(D^{i-1}_{q,\omega}[\eta]\right)^{\sigma}(a)=\left(\frac{1}{q}\right)^{i-1}D^{i-1}_{q,\omega}[\eta^{\sigma}](a).$$

We conclude that $D^{i-1}_{q,\omega}[\eta^{\sigma}](a)=0$. The case $t=b$ is proved in the same way. □

Lemma 3.6. *Suppose that $f\in\mathcal{Y}^{1}([a,b],\mathbb{R})$. One has*

$$\int_a^b f(t)\,D_{q,\omega}[\eta](t)\,d_{q,\omega}t=0$$

for all functions $\eta\in\mathcal{Y}^{1}([a,b],\mathbb{R})$ such that $\eta(a)=\eta(b)=0$ if and only if $f(t)=c$, $c\in\mathbb{R}$, for all $t\in[a,b]_{q,\omega}$.

Proof. The implication "⇐" is obvious. We prove "⇒". We begin noting that

$$\underbrace{\int_a^b f(t)\,D_{q,\omega}[\eta](t)\,d_{q,\omega}t}_{=0}=\underbrace{f(t)\,\eta(t)\Big|_a^b}_{=0}-\int_a^b D_{q,\omega}[f](t)\,\eta^{\sigma}(t)\,d_{q,\omega}t.$$

Hence,

$$\int_a^b D_{q,\omega}[f](t)\,\eta(qt+\omega)\,d_{q,\omega}t=0$$

for any $\eta\in\mathcal{Y}^{1}([a,b],\mathbb{R})$ such that $\eta(a)=\eta(b)=0$. We need to prove that, for some $c\in\mathbb{R}$, $f(t)=c$ for all $t\in[a,b]_{q,\omega}$, that is, $D_{q,\omega}[f](t)=0$ for all $t\in[a,b]_{q,\omega}$. Suppose, by contradiction, that there exists $p\in[a,b]_{q,\omega}$ such that $D_{q,\omega}[f](p)\neq0$.
(1) If $p\neq\omega_0$, then $p=q^{k}a+\omega[k]_q$ or $p=q^{k}b+\omega[k]_q$ for some $k\in\mathbb{N}_0$. Observe that $a(1-q)-\omega$ and $b(1-q)-\omega$ cannot vanish simultaneously.
(a) Suppose that $a(1-q)-\omega\neq0$ and $b(1-q)-\omega\neq0$. In this case we can assume, without loss of generality, that $p=q^{k}a+\omega[k]_q$ and we can define

$$\eta(t)=\begin{cases}D_{q,\omega}[f]\left(q^{k}a+\omega[k]_q\right) & \text{if } t=q^{k+1}a+\omega[k+1]_q\\ 0 & \text{otherwise.}\end{cases}$$

Then,

$$\begin{aligned}&\int_a^b D_{q,\omega}[f](t)\cdot\eta(qt+\omega)\,d_{q,\omega}t\\&\qquad=-\left(a(1-q)-\omega\right)q^{k}D_{q,\omega}[f]\left(q^{k}a+\omega[k]_q\right)\cdot D_{q,\omega}[f]\left(q^{k}a+\omega[k]_q\right)\neq0,\end{aligned}$$

which is a contradiction.
(b) If $a(1-q)-\omega\neq0$ and $b(1-q)-\omega=0$, then $b=\omega_0$. Since $q^{k}\omega_0+\omega[k]_q=\omega_0$ for all $k\in\mathbb{N}_0$, then $p\neq q^{k}b+\omega[k]_q$ $\forall k\in\mathbb{N}_0$ and, therefore,

$$p=q^{k}a+\omega[k]_{q,\omega}\ \text{ for some } k\in\mathbb{N}_0.$$

Repeating the proof of (a) we obtain again a contradiction.
(c) If $a(1-q)-\omega=0$ and $b(1-q)-\omega\neq0$ then the proof is similar to (b).

(2) If $p = \omega_0$ then, without loss of generality, we can assume $D_{q,\omega}[f](\omega_0) > 0$. Since

$$\lim_{n\to+\infty}\left(q^n a + \omega [k]_q\right) = \lim_{n\to+\infty}\left(q^n b + \omega [k]_q\right) = \omega_0$$

(see [1]) and $D_{q,\omega}[f]$ is continuous at ω_0, then

$$\lim_{n\to+\infty} D_{q,\omega}[f]\left(q^n a + \omega [k]_q\right) = \lim_{n\to+\infty} D_{q,\omega}[f]\left(q^n b + \omega [k]_q\right) = D_{q,\omega}[f](\omega_0) > 0.$$

Thus, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ one has $D_{q,\omega}[f]\left(q^n a + \omega [k]_q\right) > 0$ and $D_{q,\omega}[f]\left(q^n b + \omega [k]_q\right) > 0$.

(a) If $\omega_0 \neq a$ and $\omega_0 \neq b$, then we can define

$$\eta(t) = \begin{cases} D_{q,\omega}[f]\left(q^N b + \omega [N]_q\right) & \text{if} \quad t = q^{N+1}a + \omega [N+1]_q \\ D_{q,\omega}[f]\left(q^N a + \omega [N]_q\right) & \text{if} \quad t = q^{N+1}b + \omega [N+1]_q \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} &\int_a^b D_{q,\omega}[f](t)\,\eta(qt+\omega)\,d_{q,\omega}t \\ &\quad = (b-a)(1-q)\,q^N D_{q,\omega}[f]\left(q^N b + \omega [N]_q\right) \cdot D_{q\omega}[f]\left(q^N a + \omega [N]_q\right) \neq 0, \end{aligned}$$

which is a contradiction.

(b) If $\omega_0 = b$, then we define

$$\eta(t) = \begin{cases} D_{q,\omega}[f](\omega_0) & \text{if} \quad t = q^{N+1}a + \omega [N+1]_q \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} &\int_a^b D_{q,\omega}[f](t)\,\eta(qt+\omega)\,d_{q,\omega}t \\ &\quad = -\int_{\omega_0}^a D_{q,\omega}[f](t)\,\eta(qt+\omega)\,d_{q,\omega}t \\ &\quad = -\left(a(1-q)-\omega\right) q^N D_{q,\omega}[f]\left(q^N a + \omega [k]_q\right) \cdot D_{q,\omega}[f](\omega_0) \neq 0, \end{aligned}$$

which is a contradiction.

(c) When $\omega_0 = a$, the proof is similar to (b). □

Lemma 3.7 (Fundamental lemma of Hahn's variational calculus)**.** *Let* $f, g \in \mathcal{Y}^1([a,b],\mathbb{R})$. *If*

$$\int_a^b \left(f(t)\,\eta^\sigma(t) + g(t)\,D_{q,\omega}[\eta](t)\right) d_{q,\omega}t = 0$$

for all $\eta \in \mathcal{Y}^1([a,b],\mathbb{R})$ *such that* $\eta(a) = \eta(b) = 0$*, then*

$$D_{q,\omega}[g](t) = f(t) \quad \forall t \in [a,b]_{q,\omega}.$$

Proof. Define the function A by $A(t) := \int_{\omega_0}^{t} f(\tau)\, d_{q,\omega}\tau$. Then $D_{q,\omega}[A](t) = f(t)$ for all $t \in [a,b]$ and

$$\begin{aligned}\int_a^b A(t)\, D_{q,\omega}[\eta](t)\, d_{q,\omega}t &= A(t)\,\eta(t)\Big|_a^b - \int_a^b D_{q,\omega}[A](t)\,\eta^{\sigma}(t)\, d_{q,\omega}t \\ &= -\int_a^b D_{q,\omega}[A](t)\,\eta^{\sigma}(t)\, d_{q,\omega}t \\ &= -\int_a^b f(t)\,\eta^{\sigma}(t)\, d_{q,\omega}t.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_a^b \left(f(t)\,\eta^{\sigma}(t) + g(t)\, D_{q,\omega}[\eta](t)\right) d_{q,\omega}t = 0 \\ &\qquad\Leftrightarrow \int_a^b \left(-A(t) + g(t)\right) D_{q,\omega}[\eta](t)\, d_{q,\omega}t = 0.\end{aligned}$$

By Lemma 3.6 there is a $c \in \mathbb{R}$ such that $-A(t) + g(t) = c$ for all $t \in [a,b]_{q,\omega}$. Hence $D_{q,\omega}[A](t) = D_{q,\omega}[g](t)$ for $t \in [a,b]_{q,\omega}$, which provides the desired result: $D_{q,\omega}[g](t) = f(t) \quad \forall t \in [a,b]_{q,\omega}$. □

We are now in conditions to deduce the higher-order fundamental Lemma of Hahn's quantum variational calculus.

Lemma 3.8 (Higher-order fundamental lemma of Hahn's variational calculus)**.** *Let* $f_0, f_1, \ldots, f_r \in \mathcal{Y}^1([a,b], \mathbb{R})$*. If*

$$\int_a^b \left(\sum_{i=0}^{r} f_i(t)\, D_{q,\omega}^{i}\left[\eta^{\sigma^{r-i}}\right](t)\right) d_{q,\omega}t = 0$$

for any variation η*, then*

$$\sum_{i=0}^{r} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^{i}[f_i](t) = 0$$

for all $t \in [a,b]_{q,\omega}$*.*

Proof. We proceed by mathematical induction. If $r = 1$ the result is true by Lemma 3.7. Assume that

$$\int_a^b \left(\sum_{i=0}^{r+1} f_i(t)\, D_{q,\omega}^{i}\left[\eta^{\sigma^{r+1-i}}\right](t)\right) d_{q,\omega}t = 0$$

for all functions η such that $\eta(a) = \eta(b) = 0, \ldots, D_{q,\omega}^{r}[\eta](a) = D_{q,\omega}^{r}[\eta](b) = 0$. Note that

$$\begin{aligned}&\int_a^b f_{r+1}(t)\, D_{q,\omega}^{r+1}[\eta](t)\, d_{q,\omega}t \\ &\qquad = f_{r+1}(t)\, D_{q,\omega}^{r}[\eta](t)\Big|_a^b - \int_a^b D_{q,\omega}[f_{r+1}](t)\left(D_{q,\omega}^{r}[\eta]\right)^{\sigma}(t)\, d_{q,\omega}t \\ &\qquad = -\int_a^b D_{q,\omega}[f_{r+1}](t)\left(D_{q,\omega}^{r}[\eta]\right)^{\sigma}(t)\, d_{q,\omega}t\end{aligned}$$

and, by Lemma 3.4,

$$\int_a^b f_{r+1}(t)\, D_{q,\omega}^{r+1}[\eta](t)\, d_{q,\omega}t = -\int_a^b D_{q,\omega}[f_{r+1}](t) \left(\frac{1}{q}\right)^r D_{q,\omega}^r[\eta^\sigma](t)\, d_{q,\omega}t.$$

Therefore,

$$\begin{aligned}
&\int_a^b \left(\sum_{i=0}^{r+1} f_i(t)\, D_{q,\omega}^i\left[\eta^{\sigma^{r+1-i}}\right](t)\right) d_{q,\omega}t \\
&= \int_a^b \left(\sum_{i=0}^{r} f_i(t)\, D_{q,\omega}^i\left[\eta^{\sigma^{r+1-i}}\right](t)\right) d_{q,\omega}t \\
&\quad - \int_a^b D_{q,\omega}[f_{r+1}](t)\left(\frac{1}{q}\right)^r D_{q,\omega}^r[\eta^\sigma](t)\, d_{q,\omega}t \\
&= \int_a^b \left[\sum_{i=0}^{r-1} f_i(t)\, D_{q,\omega}^i\left[(\eta^\sigma)^{\sigma^{r-i}}\right](t)\, d_{q,\omega}t \right.\\
&\quad \left. + \left(f_r - \left(\frac{1}{q}\right)^r D_{q,\omega}[f_{r+1}]\right)(t)\, D_{q,\omega}^r[\eta^\sigma](t)\right] d_{q,\omega}t.
\end{aligned}$$

By Lemma 3.5, η^σ is a variation. Hence, using the induction hypothesis,

$$\begin{aligned}
&\sum_{i=0}^{r-1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i[f_i](t) \\
&\qquad + (-1)^r \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} D_{q,\omega}^r\left[\left(f_r - \frac{1}{q^r} D_{q,\omega}[f_{r+1}]\right)\right](t) \\
&= \sum_{i=0}^{r-1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i[f_i](t) + (-1)^r \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} D_{q,\omega}^r[f_r](t) \\
&\qquad + (-1)^{r+1} \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} \frac{1}{q^r} D_{q,\omega}^r[D_{q,\omega}[f_{r+1}]](t) \\
&= 0
\end{aligned}$$

for all $t \in [a,b]_{q,\omega}$, which leads to

$$\sum_{i=0}^{r+1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i[f_i](t) = 0, \;\; t \in [a,b]_{q,\omega}.$$

□

3.2. **Higher-order Hahn's quantum Euler–Lagrange equation.** For a variation η and an admissible function y, we define the function $\phi : (-\bar{\epsilon}, \bar{\epsilon}) \to \mathbb{R}$ by

$$\phi(\epsilon) = \phi(\epsilon, y, \eta) := \mathcal{L}[y + \epsilon\eta].$$

The first variation of the variational problem (P) is defined by

$$\delta\mathcal{L}[y, \eta] := \phi'(0).$$

Observe that

$$\mathcal{L}\left[y+\epsilon\eta\right]=\int_a^b L\Bigg(t,y^{\sigma^r}\left(t\right)+\epsilon\eta^{\sigma^r}\left(t\right),D_{q,\omega}\left[y^{\sigma^{r-1}}\right]\left(t\right)+\epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right]\left(t\right),\ldots,D_{q,\omega}^r\left[y\right]\left(t\right)+\epsilon D_{q,\omega}^r\left[\eta\right]\left(t\right)\Bigg)d_{q,\omega}t$$
$$=\mathcal{L}_b\left[y+\epsilon\eta\right]-\mathcal{L}_a\left[y+\epsilon\eta\right]$$

with

$$\mathcal{L}_\xi\left[y+\epsilon\eta\right]=\int_{\omega_0}^\xi L\Bigg(t,y^{\sigma^r}\left(t\right)+\epsilon\eta^{\sigma^r}\left(t\right),D_{q,\omega}\left[y^{\sigma^{r-1}}\right]\left(t\right)+\epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right]\left(t\right),\ldots,D_{q,\omega}^r\left[y\right]\left(t\right)+\epsilon D_{q,\omega}^r\left[\eta\right]\left(t\right)\Bigg)d_{q,\omega}t,$$

$\xi\in\{a,b\}$. Therefore,

$$\delta\mathcal{L}\left[y,\eta\right]=\delta\mathcal{L}_b\left[y,\eta\right]-\delta\mathcal{L}_a\left[y,\eta\right]. \tag{3.1}$$

Considering (3.1), the following lemma is a direct consequence of Lemma 2.12:

Lemma 3.9. *For a variation η and an admissible function y, let*

$$g\left(t,\epsilon\right):=L\Bigg(t,y^{\sigma^r}\left(t\right)+\epsilon\eta^{\sigma^r}\left(t\right),D_{q,\omega}\left[y^{\sigma^{r-1}}\right]\left(t\right)+\epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right]\left(t\right),\ldots,D_{q,\omega}^r\left[y\right]\left(t\right)+\epsilon D_{q,\omega}^r\left[\eta\right]\left(t\right)\Bigg),$$

$\epsilon\in(-\bar{\epsilon},\bar{\epsilon})$. *Assume that:*
(1) $g\left(t,\cdot\right)$ is differentiable at 0 uniformly in $t\in\left[a,b\right]_{q,\omega}$;
(2) $\mathcal{L}_a\left[y+\epsilon\eta\right]=\int_{\omega_0}^a g\left(t,\epsilon\right)d_{q,\omega}t$ and $\mathcal{L}_b\left[y+\epsilon\eta\right]=\int_{\omega_0}^b g\left(t,\epsilon\right)d_{q,\omega}t$ exist for $\epsilon\approx 0$;
(3) $\int_{\omega_0}^a \partial_2 g\left(t,0\right)d_{q,\omega}t$ and $\int_{\omega_0}^b \partial_2 g\left(t,0\right)d_{q,\omega}t$ exist.
Then

$$\phi'\left(0\right)=\delta\mathcal{L}\left[y,\eta\right]=\int_a^b\Bigg(\sum_{i=0}^r\partial_{i+2}L\left(t,y^{\sigma^r}\left(t\right),D_{q,\omega}\left[y^{\sigma^{r-1}}\right]\left(t\right),\ldots,D_{q,\omega}^r\left[y\right]\left(t\right)\right)\cdot D_{q,\omega}^i\left[\eta^{\sigma^{r-i}}\right]\left(t\right)\Bigg)d_{q,\omega}t,$$

where $\partial_i L$ denotes the partial derivative of L with respect to its ith argument.

The following result gives a necessary condition of Euler–Lagrange type for an admissible function to be a local extremizer for (P).

Theorem 3.10 (Higher-order Hahn's quantum Euler–Lagrange equation)**.** *Under hypotheses (H1)–(H3) and conditions (1)–(3) of Lemma 3.9 on the Lagrangian L,* *if $y_* \in \mathcal{Y}^r$ is a local extremizer for problem* (P)*, then y_* satisfies the q,ω-Euler–Lagrange equation*

(3.2)
$$\sum_{i=0}^{r}(-1)^i\left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}}D_{q,\omega}^i\left[\partial_{i+2}L\right]\left(t,y^{\sigma^r}(t),D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t),\ldots,D_{q,\omega}^r[y](t)\right)=0$$

for all $t \in [a,b]_{q,\omega}$.

Proof. Let y_* be a local extremizer for problem (P) and η a variation. Define $\phi:(-\bar{\epsilon},\bar{\epsilon})\to\mathbb{R}$ by $\phi(\epsilon):=\mathcal{L}[y_*+\epsilon\eta]$. A necessary condition for y_* to be an extremizer is given by $\phi'(0)=0$. By Lemma 3.9 we conclude that

$$\int_a^b\left(\sum_{i=0}^{r}\partial_{i+2}L\left(t,y^{\sigma^r}(t),D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t),\ldots,D_{q,\omega}^r[y](t)\right)\right.$$
$$\left.\cdot D_{q,\omega}^i\left[\eta^{\sigma^{r-i}}\right](t)\right)d_{q,\omega}t=0$$

and (3.2) follows from Lemma 3.8. □

Remark 3.11. In practical terms the hypotheses of Theorem 3.10 are not so easy to verify *a priori*. One can, however, assume that all hypotheses are satisfied and apply the q,ω-Euler–Lagrange equation (3.2) heuristically to obtain a *candidate*. If such a candidate is, or not, a solution to problem (P) is a different question that always requires further analysis (see an example in §3.3).

When $\omega\to0$ one obtains from (3.2) the higher-order q-Euler–Lagrange equation:

$$\sum_{i=0}^{r}(-1)^i\left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}}D_q^i\left[\partial_{i+2}L\right]\left(t,y^{\sigma^r}(t),D_q\left[y^{\sigma^{r-1}}\right](t),\ldots,D_q^r[y](t)\right)=0$$

for all $t\in\{aq^n:n\in\mathbb{N}_0\}\cup\{bq^n:n\in\mathbb{N}_0\}\cup\{0\}$. The higher-order h-Euler–Lagrange equation is obtained from (3.2) taking the limit $q\to1$:

$$\sum_{i=0}^{r}(-1)^i\Delta_h^i\left[\partial_{i+2}L\right]\left(t,y^{\sigma^r}(t),\Delta_h\left[y^{\sigma^{r-1}}\right](t),\ldots,\Delta_h^r[y](t)\right)=0$$

for all $t\in\{a+nh:n\in\mathbb{N}_0\}\cup\{b+nh:n\in\mathbb{N}_0\}$. The classical Euler–Lagrange equation [26] is recovered when $(\omega,q)\to(0,1)$:

$$\sum_{i=0}^{r}(-1)^i\frac{d^i}{dt^i}\partial_{i+2}L\left(t,y(t),y'(t),\ldots,y^{(r)}(t)\right)=0$$

for all $t\in[a,b]$.

We now illustrate the usefulness of our Theorem 3.10 by means of an example that is not covered by previous available results in the literature.

3.3. **An Example.** Let $q=\frac{1}{2}$ and $\omega=\frac{1}{2}$. Consider the following problem:

(3.3)
$$\mathcal{L}[y]=\int_{-1}^{1}\left(y^\sigma(t)+\frac{1}{2}\right)^2\left(\left(D_{q,\omega}[y](t)\right)^2-1\right)^2d_{q,\omega}t\longrightarrow\min$$

over all $y\in\mathcal{Y}^1$ satisfying the boundary conditions

(3.4)
$$y(-1)=0\quad\text{and}\quad y(1)=-1.$$

This is an example of problem (P) with $r = 1$. Our q,ω-Euler–Lagrange equation (3.2) takes the form

$$D_{q,\omega}\left[\partial_3 L\right]\left(t, y^{\sigma}\left(t\right), D_{q,\omega}\left[y\right]\left(t\right)\right) = \partial_2 L\left(t, y^{\sigma}\left(t\right), D_{q,\omega}\left[y\right]\left(t\right)\right).$$

Therefore, we look for an admissible function y_* of (3.3)-(3.4) satisfying

$$D_{q,\omega}\left[4\left(y^{\sigma}+\frac{1}{2}\right)^{2}\left(\left(D_{q,\omega}\left[y\right]\right)^{2}-1\right)D_{q,\omega}\left[y\right]\right](t) = 2\left(y^{\sigma}(t)+\frac{1}{2}\right)\left(\left(D_{q,\omega}\left[y\right](t)\right)^{2}-1\right) \tag{3.5}$$

for all $t \in [-1,1]_{q,\omega}$. It is easy to see that

$$y_*(t) = \begin{cases} -t & \text{if } t \in (-1,0)\cup(0,1] \\ 0 & \text{if } t = -1 \\ 1 & \text{if } t = 0 \end{cases}$$

is an admissible function for (3.3)-(3.4) with

$$D_{q,\omega}\left[y_*\right](t) = \begin{cases} -1 & \text{if } t \in (-1,0)\cup(0,1] \\ 1 & \text{if } t = -1 \\ -3 & \text{if } t = 0, \end{cases}$$

satisfying the q,ω-Euler–Lagrange equation (3.5). We now prove that the *candidate* y_* is indeed a minimizer for (3.3)-(3.4). Note that here $\omega_0 = 1$ and, by Lemma 2.8 and item (3) of Theorem 2.7,

$$\mathcal{L}\left[y\right] = \int_{-1}^{1}\left(y^{\sigma}(t)+\frac{1}{2}\right)^{2}\left(\left(D_{q,\omega}\left[y\right](t)\right)^{2}-1\right)^{2} d_{q,\omega}t \geq 0 \tag{3.6}$$

for all admissible functions $y \in \mathcal{Y}^1\left([-1,1],\mathbb{R}\right)$. Since $\mathcal{L}\left[y_*\right] = 0$, we conclude that y_* is a minimizer for problem (3.3)-(3.4).

It is worth to mention that the minimizer y_* of (3.3)-(3.4) is not continuous while the classical calculus of variations [26], the calculus of variations on time scales [14, 20, 23], or the nondifferentiable scale variational calculus [4, 5, 10], deal with functions which are necessarily continuous. As an open question, we pose the problem of determining conditions on the data of problem (P) assuring, *a priori*, the minimizer to be regular.

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