On the global reachability of structured 2D state space systems

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Dedicated to Professor Fátima Leite on the occasion of her 60th birthday

Abstract

This paper is a contribution to extend the study of structured dynamical systems to the case of systems evolving over two-dimensional domains, i.e., 2D systems. In particular, we consider 2D Fornasini-Marchesini state space models and obtain necessary and sufficient conditions for structured global reachability. Such conditions are stated in terms of the generic rank and the irreducibility of suitably defined matrices and generalize well known results on the reachability of 1D structured systems.

1 Introduction

When modeling phenomena by means of dynamical systems situations often occur where the only available knowledge is on the existence or the absence of relations between the relevant variables. In order to deal with this problem, structured systems were first introduced in [6], and largely studied in

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a number of subsequent contributions, mainly concerned with structured linear systems in state space form [4, 8, 7, 2].

In a structured linear state space system, the entries of the system matrices are either zero or then assume arbitrary values, in which case they are identified with a free parameter. Therefore the natural way to define system theoretic properties in this setting is by considering them in a generic sense, i.e., as holding for almost all the evaluations of the parameters.

In this paper we focus on the study of the reachability of 2D structured systems. In a two-dimensional, or, shortly, 2D system, the system variables evolve over a two-dimensional domain, such as 1D space-time or 2D space. Therefore its dynamics is described on the basis of two independent operators (namely, partial differentiators or partial shift-operators, according to the continuous or discrete nature of the domain). Whereas the recursive computation of one-dimensional signals can be performed point-wise, and the future system evolution only depends on its state (initial conditions) at the current time instant, in general, the computation of 2D signals cannot be performed without the knowledge of initial conditions along suitable (1D) "propagation fronts". This has lead to the definition of 2D state at two different levels: the local state (point-wise defined) and the global (consisting of all the values of the local state on a propagation front). Consequently, the system theoretic properties are also defined both locally (when they refer to the local state) and globally (if they concern the global state). This applies in particular to the property of reachability: global reachability is defined as the possibility of attaining an arbitrary global state starting from zero initial conditions, by using a suitable control sequence, while local reachability refers to the possibility of attaining arbitrary local states.

The results of this paper concern the characterization of global reachability for structured 2D state space systems described by a Fornasini-Marchesini model [3]. Since, to our knowledge, up to now no research has been done on 2D structured systems, our contribution is a first step towards the construction of a full theory of 2D structured systems.

2 Structured dynamical systems

A real *structured matrix* is defined as a matrix whose entries are either fixed zeros or independent parameters, in which case they are referred to as the nonzero entries. The actual value of each of the nonzero entries is unknown, but free, in the sense that each nonzero entry can take any real value (including zero). Therefore a structured matrix M having r nonzero entries can be parameterized by means of a parameter vector $\lambda \in \mathbb{R}^r$ and is denoted by M_λ.

Example 2.1. Let λ_i, $i = 1, 2, 3$, be free parameters. The matrix

$$M_\lambda = \begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & \lambda_3 \end{bmatrix}$$

is a structured matrix where $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$.

However, neither

$$\begin{bmatrix} \lambda_1 & \lambda_1 \\ \lambda_2 & \lambda_3 \end{bmatrix} \text{ nor } \begin{bmatrix} \lambda_1 & 1 \\ 0 & 0 \end{bmatrix}$$

are structured matrices. □

Let us consider a discrete time-invariant system of the form

$$x(t+1) = Ax(t) + Bu(t), \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x(\cdot) \in \mathbb{R}^n$ denotes the state of the system and $u(\cdot) \in \mathbb{R}^m$ the input.

If in the system (2.1) we assume that the matrices A and B are structured matrices having together r nonzero entries, the system can be parameterized by means of a parameter vector $\lambda \in \mathbb{R}^r$. The set of parameterized systems thus obtained is called a *structured system* and is denoted by

$$x(t+1) = A_\lambda x(t) + B_\lambda u(t) \tag{2.2}$$

with $\lambda \in \mathbb{R}^r$, or simply by (A_λ, B_λ).

By choosing λ, system (2.2) becomes completely known and can be written as a system of the form (2.1). Thus, for each value of λ, its system theoretic properties can be studied in the usual way. It is clear that these properties may depend on the parameter values and hold for some of them while for others not. In this context, for structured systems, the relevant issue is not whether a property holds for some particular parameter values, but rather whether it is a *generic property*, in the sense that it holds "for almost all parameter values", i.e., it holds for all parameter values except for those in some proper algebraic variety in the parameter space (which is a set with Lebesgue measure zero) [1]. Hence we shall say that the structured system (2.2) has a certain property P if P is a generic property of the system.

In this paper we shall focus on the study of reachability. As is well-known, the system (2.1) is said to be *reachable* if for every $x^* \in \mathbb{R}^n$ there exist $t^* > 0$ and an input sequence $u(t)$, $t = 0, 1, \dots, t^* - 1$, that steers the state from $x(0) = 0$ to $x(t^*) = x^*$.

Characterizations of reachability for completely specified systems of type (2.1) are given by the following results [5].

Theorem 2.2. *The system* (2.1) *is reachable if and only if* $\operatorname{rank} \mathcal{R}^n = n$*, where* $\mathcal{R}^n$ *is the reachability matrix of the system, i.e.,*

$$\mathcal{R}^n := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

Theorem 2.3 (PBH test)**.** *The system* (2.1) *is reachable if and only if*

$$\operatorname{rank} \begin{bmatrix} zI - A & | & B \end{bmatrix} = n, \forall z \in \mathbb{C}.$$

By Theorem 2.2, system (2.2) is reachable if and only if the reachability matrix

$$\mathcal{R}^n = \begin{bmatrix} B_\lambda & A_\lambda B_\lambda & \cdots & A_\lambda^{n-1} B_\lambda \end{bmatrix}$$

has rank n for almost all $\lambda \in \mathbb{R}^r$. But, noting that $\mathcal{R}^n$ is a polynomial matrix in r indeterminates, we can show that this is equivalent to say that there exists $\lambda^* \in \mathbb{R}$ such that $\operatorname{rank} \mathcal{R}^n = n$. This means that the structured system (2.2) is reachable if and only if it is reachable for one choice of λ. However, neither this characterization nor, equivalently, the study of the rank of the polynomial matrix $\mathcal{R}^n$ yield useful tests. This suggests that a different approach should be adopted, for instance based only on the structure of the relevant matrices.

Unfortunately, given two structured matrices A_λ and B_λ the reachability matrix $\mathcal{R}^n$ is not necessarily structured, as is illustrated by the next example.

Example 2.4. Let

$$A_\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} \text{ and } B_\lambda = \begin{bmatrix} \lambda_5 \\ 0 \end{bmatrix}$$

be two structured matrices. Then

$$\mathcal{R}^2 = \begin{bmatrix} B_\lambda & A_\lambda B_\lambda \end{bmatrix} = \begin{bmatrix} \lambda_5 & \lambda_1\lambda_5 \\ 0 & \lambda_3\lambda_5 \end{bmatrix}$$

is not a structured matrix since its nonzero entries are not independent. □

Moreover, define a *structured polynomial matrix* $M_\lambda(z)$ as

$$M_\lambda(z) = M_k^\lambda z^k + \cdots + M_1^\lambda z + M_0^\lambda$$

for some nonnegative integer k, where the matrix $\begin{bmatrix} M_0^\lambda & | & \ldots & | & M_k^\lambda \end{bmatrix}$ is a structured matrix. Then the matrix

$$\begin{bmatrix} zI - A_\lambda & | & B_\lambda \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} z + \begin{bmatrix} -A_\lambda & B_\lambda \end{bmatrix}$$

associated to the pair (A_λ, B_λ) of structured matrices is also not structured.

Thus, if we wish to make a study of reachability of a structured system by analyzing structured matrices we must use different tools.

The next two concepts are fundamental for this study [2]

Let $A_\lambda \in \mathbb{R}^{n\times n}$ and $B_\lambda \in \mathbb{R}^{n\times m}$ be structured matrices. The pair (A_λ, B_λ) is said to be:

- *reducible*, or to ***be in form I***, if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P^{-1}A_\lambda P = \begin{bmatrix} A_{11}^\lambda & 0 \\ A_{21}^\lambda & A_{22}^\lambda \end{bmatrix} \quad \text{and} \quad PB_\lambda = \begin{bmatrix} 0 \\ B_2^\lambda \end{bmatrix}$$

  where A_{ij}^λ is an $n_i \times n_j$ structured matrix for $i, j = 1, 2$, with $0 < n_1 \leq n$ and $n_1 + n_2 = n$, and where B_2^λ is an $n_2 \times m$ structured matrix.

- *not of full generic row rank*, or to ***be in form II***, if the generic rank of $\begin{bmatrix} A_\lambda & B_\lambda \end{bmatrix}$ is less than n.

Recall that the *generic rank* of a structured matrix M_λ is ρ if it is equal to ρ for almost all $\lambda \in \mathbb{R}^r$. This coincides with the maximal rank that M_λ achieves as a function of the parameter λ.

A necessary and sufficient condition for a pair (A_λ, B_λ) to be in form II is that $\begin{bmatrix} A_\lambda & B_\lambda \end{bmatrix}$ has a zero submatrix of order $k \times l$ where $k + l \geq n + m + 1$ [8].

For structured systems of type (2.2), the following result has been proved (see [4, 6, 8]).

Theorem 2.5. *The structured system* (2.2) *is (generically) reachable if and only if the pair* (A_λ, B_λ) *is neither in form I nor in form II.*

The main goal of this paper is to generalize this theorem for 2D structured systems.

3 2D state space systems

One of the most well-known representations for 2D systems is the Fornasini-Marchesini state space model [3], which is described by the following 2D first order state updating equation

$$\begin{aligned} x(i+1, j+1) &= A_1 x(i, j+1) + A_2 x(i+1, j) \\ &\quad + B_1 u(i, j+1) + B_2 u(i+1, j), \end{aligned} \tag{3.1}$$

with *local states* $x(\cdot,\cdot) \in \mathbb{R}^n$, inputs $u(\cdot,\cdot) \in \mathbb{R}^m$, state matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ and input matrices $B_1, B_2 \in \mathbb{R}^{n\times m}$. In the sequel this 2D system will be denoted by (A_1, A_2, B_1, B_2). The corresponding updating scheme is illustrated in Figure 1.

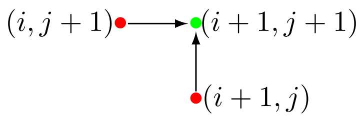

Figure 1

Introducing the *shift operators*

$$\begin{aligned} \sigma_1 x(i,j) &:= x(i+1,j), \\ \sigma_2 x(i,j) &:= x(i,j+1), \end{aligned}$$

the equation (3.1) can be written as

$$\sigma_1 \sigma_2 x = A_1 \sigma_2 x + A_2 \sigma_1 x + B_1 \sigma_2 u + B_2 \sigma_1 u,$$

or, equivalently,

$$\sigma_1 x = \left(A_1 + A_2 \sigma_1 \sigma_2^{-1}\right) x + \left(B_1 + B_2 \sigma_1 \sigma_2^{-1}\right) u.$$

Defining a new operator $\sigma := \sigma_1 \sigma_2^{-1}$ equation (3.1) can be written as

$$\sigma_1 x = (A_1 + A_2 \sigma) \, x + (B_1 + B_2 \sigma) \, u. \tag{3.2}$$

The initial conditions for this equation may be assigned by specifying the values of the state on the separation set $\mathcal{C}_0$, where

$$\mathcal{C}_k = \{(i,j) \in \mathbb{Z}^2 : i + j = k\},$$

see Figure 2.

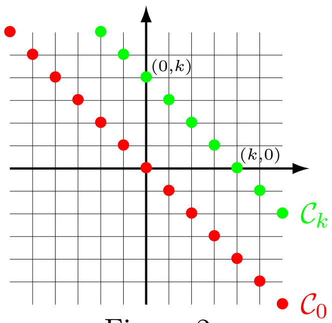

Figure 2

Defining the *global state* on the separation set $\mathcal{C}_k$ as

$$\mathcal{X}_k(t) := \big(x(k+t, -t)\big)_{t \in \mathbb{Z}}$$

and the global input as

$$\mathcal{U}_k(t) := \big(u(k+t, -t)\big)_{t\in\mathbb{Z}},$$

by (3.2), the global state evolution is given by

$$\mathcal{X}_{k+1} = A(\sigma)\mathcal{X}_k + B(\sigma)\mathcal{U}_k, \tag{3.3}$$

where $A(\sigma) = A_1 + A_2\sigma$, $B(\sigma) = B_1 + B_2\sigma$, and the action of σ on $\mathcal{X}_k$ is given by

$$\begin{aligned}\sigma\mathcal{X}_k(t) &= \big(x(k+(t+1), -(t+1))\big)_{t\in\mathbb{Z}}\\ &= \mathcal{X}_k(t+1).\end{aligned}$$

The action of σ on $\mathcal{U}_k$ is analogous.

Denote by $\mathbb{R}\big[\big[z, z^{-1}\big]\big]$ the set of bilateral Laurent formal power series in the indeterminate z with coefficients in $\mathbb{R}$ and define the $z-$transform $\mathcal{Z}$: $(\mathbb{R})^{\mathbb{Z}} \to \mathbb{R}\big[\big[z, z^{-1}\big]\big]$ by

$$\mathcal{Z}[\mathcal{W}_k] := \sum_{t=-\infty}^{+\infty} \mathcal{W}_k(t)z^{-t}$$

which will be denoted by $W_k(z)$, with $k \in \mathbb{Z}$. For vector signals in $\big(\mathbb{R}^l\big)^{\mathbb{Z}}$ the $z-$transform is defined componentwise.

Then

$$\begin{aligned}\mathcal{Z}[\sigma\mathcal{X}_k] &= \textstyle\sum_{t=-\infty}^{+\infty} \sigma\mathcal{X}_k(t)z^{-t}\\ &= \textstyle\sum_{t=-\infty}^{+\infty} \mathcal{X}_k(t+1)z^{-t}\\ &= z\textstyle\sum_{t=-\infty}^{+\infty} \mathcal{X}_k(t+1)z^{-(t+1)}\\ &= z\textstyle\sum_{t=-\infty}^{+\infty} \mathcal{X}_k(t)z^{-t}\\ &= zX_k(z)\end{aligned}$$

and hence, by (3.3), we obtain

$$X_{k+1}(z) = A(z)X_k(z) + B(z)U_k(z), \tag{3.4}$$

where $A(z) = A_1 + A_2 z$, $B(z) = B_1 + B_2 z$ and $U_k(z) := \mathcal{Z}[\mathcal{U}_k]$.

4 Local and global reachability

When dealing with 2D systems, the concept of reachability is naturally introduced in two different forms: a weak (local) and a strong (global) form which refer, respectively, to single local states and to global states. These notions are defined next as in [3].

Definition 4.1. The 2D state space model (3.1) is

- ***locally reachable*** if, upon assuming $\mathcal{X}_0 \equiv 0$, for every $x^* \in \mathbb{R}^n$ there exists $(i,j) \in \mathbb{Z}^2$, with $i+j>0$, and an input sequence $u(\cdot,\cdot)$ such that $x(i,j)=x^*$. In this case, we say that x^* is reachable in $i+j$ steps.

- ***globally reachable*** if, upon assuming $\mathcal{X}_0 \equiv 0$, for every global state sequence $\mathcal{X}^*$ with values in $\mathbb{R}^n$ there exists $k \in \mathbb{Z}_+$ and an input sequence $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_{k-1}$ such that the global state $\mathcal{X}_k$ coincides with $\mathcal{X}^*$. In this case, we say that $\mathcal{X}^*$ is reachable in k steps.

Clearly, global reachability implies local reachability. In this paper we shall focus on this global property.

Bearing in mind that, if $\mathcal{X}_0 \equiv 0$, then

$$\begin{aligned} X_k(z) &= \sum_{l=0}^{k-1} A^{k-1-l}(z)B(z)U_l(z) \\ &= \begin{bmatrix} B(z) & A(z)B(z) & \cdots & A^{k-1}(z)B(z) \end{bmatrix} \begin{bmatrix} U_{k-1}(z) \\ U_{k-2}(z) \\ \vdots \\ U_0(z) \end{bmatrix}, \end{aligned}$$

it is easy to see that the global state $\mathcal{X}^*$ is reachable in k steps if and only if

$$X^*(z) = \mathcal{Z}[\mathcal{X}^*] \in \operatorname{Im} \mathcal{R}^k(z)$$

with $\mathcal{R}^k(z) := \begin{bmatrix} B(z) & A(z)B(z) & \cdots & A^{k-1}(z)B(z) \end{bmatrix}$.

The matrix

$$\mathcal{R}^n(z) = \begin{bmatrix} B(z) & A(z)B(z) & \cdots & A^{n-1}(z)B(z) \end{bmatrix},$$

where n is the dimension of the local state and the polynomial matrices $A(z)$ and $B(z)$ are defined as in (3.4), is called ***global reachability matrix*** of the 2D system (A_1, A_2, B_1, B_2).

In the following theorem [3], global reachability is characterized in terms of the global reachability matrix.

Theorem 4.2. *The 2D system (A_1, A_2, B_1, B_2) is global reachable if and only if the polynomial matrix $\mathcal{R}^n(z)$ has rank[1] n, i.e.,* $\operatorname{rank} \mathcal{R}^n(z) = n$.

[1]We recall that the rank of a polynomial matrix $\mathcal{R}(z)$ is defined as $\max_{\lambda \in \mathbb{C}} \operatorname{rank} \mathcal{R}(\lambda)$

5 2D structured systems

In the sequel we consider 2D systems of the form (3.1), where the matrices A_1, A_2, B_1 and B_2 are structured, i.e., their entries are either fixed zeros or independent free parameters. In this case, the polynomial matrices $A_\lambda(z) = A_1^\lambda + A_2^\lambda z$ and $B_\lambda(z) = B_1^\lambda + B_2^\lambda z$ are structured matrices too. Moreover, their evaluations for any $\nu^* \in \mathbb{C}$, yield matrices $A_\lambda(\nu^*)$ and $B_\lambda(\nu^*)$ that are also structured.

Similar to the 1D case, we say that a 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is (*globally / locally*) *reachable* if it is generically (globally / locally) reachable, i.e., if it is reachable for almost all $\lambda \in \mathbb{R}^r$. Again this is equivalent to say that $(A_1^{\lambda^*}, A_2^{\lambda^*}, B_1^{\lambda^*}, B_2^{\lambda^*})$ is reachable for at least one value $\lambda^* \in \mathbb{R}^r$.

As in the 1D case, the notions of matrix pairs in *form I* and in *form II* play an important role in the characterization of 2D reachability, now applied to the polynomial matrix pair $(A_\lambda(z), B_\lambda(z))$. In the polynomial case the definitions remain the same as in the constant case, with the difference that the (generic) rank of $\begin{bmatrix} A_\lambda(z) & B_\lambda(z) \end{bmatrix}$ is to be understood as its rank as a polynomial matrix.

Lemma 5.1. *Let $\nu^* \in \mathbb{C} \setminus \{0\}$. Then the pair of structured matrices $(A_\lambda(z), B_\lambda(z))$ is neither in form I nor in form II if and only if the pair of structured matrices $(A_\lambda(\nu^*), B_\lambda(\nu^*))$ is not in form I nor in form II, where $A_\lambda(z) = A_1^\lambda + A_2^\lambda z$ and $B_\lambda(z) = B_1^\lambda + B_2^\lambda z$.*

Proof. If $\nu^* \in \mathbb{C}\backslash\{0\}$ both implications are obvious since the pairs of structured matrices $(A_\lambda(z), B_\lambda(z))$ and $(A_\lambda(\nu^*), B_\lambda(\nu^*))$ have the same zero structure. □

Remark 5.2. The "if" part also holds for $\nu^* = 0$. In fact, if $\nu^* = 0$ then $A_\lambda(0) = A_1^\lambda$ and $B_\lambda(0) = B_1^\lambda$. Since all the zero entries of the matrix $A_\lambda(z)$ are also zero in A_1^λ and the same happens between $B_\lambda(z)$ and B_1^λ, the set of zero entries for the pair $(A_\lambda(z), B_\lambda(z))$ is contained in the set of zero entries of $(A_\lambda(0), B_\lambda(0)) = (A_1^\lambda, B_1^\lambda)$. The result is easily obtained by the definition of form I and the characterization of form II.

The following example shows that the "only if" part does not hold for $\nu^* = 0$.

Example 5.3. Consider the structured matrices

$$A_1^\lambda = \begin{bmatrix} 0 & 0 \\ \lambda_1 & \lambda_2 \end{bmatrix}, A_2^\lambda = \begin{bmatrix} \lambda_3 & \lambda_4 \\ 0 & 0 \end{bmatrix}, B_1^\lambda = \begin{bmatrix} 0 \\ \lambda_5 \end{bmatrix} \text{ and } B_2^\lambda = \begin{bmatrix} \lambda_6 \\ 0 \end{bmatrix}.$$

Since all the entries of $(A_1^\lambda + A_2^\lambda, B_1^\lambda + B_2^\lambda)$ are free, this pair is neither in form I nor in form II and by Lemma 5.1 the same holds for the pair

$(A_\lambda(z), B_\lambda(z))$. However, its clear that the pair $(A_\lambda(0), B_\lambda(0)) = \left(A_1^\lambda, B_1^\lambda\right)$ is in form I and II. □

The next theorem characterizes the global reachability of 2D structured systems

Theorem 5.4. *A 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable if and only if the pair of structured matrices $(A_\lambda(z), B_\lambda(z))$ is neither in form I nor in form II, where $A_\lambda(z) = A_1^\lambda + A_2^\lambda z$ and $B_\lambda(z) = B_1^\lambda + B_2^\lambda z$.*

Proof. By definition, the 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable if there exists $\lambda^* \in \mathbb{R}^r$ such that the 2D system $(A_1^{\lambda^*}, A_2^{\lambda^*}, B_1^{\lambda^*}, B_2^{\lambda^*})$ is globally reachable.

Then, by Theorem 4.2, $\operatorname{rank} \mathcal{R}_{\lambda^*}^n(z) = n$, where $\mathcal{R}_{\lambda^*}^n(z)$ is the global reachability matrix of the 2D system $(A_1^{\lambda^*}, A_2^{\lambda^*}, B_1^{\lambda^*}, B_2^{\lambda^*})$. Note that, in this case, the set

$$\mathcal{L} := \{\eta \in \mathbb{C} : \operatorname{rank} \mathcal{R}_{\lambda^*}^n(\eta) < \operatorname{rank} \mathcal{R}_{\lambda^*}^n(z)\}$$

corresponds to the common zeros of the $n \times n$ minors of $\mathcal{R}_{\lambda^*}^n(z)$, and is hence a finite set. Thus $\operatorname{rank} \mathcal{R}_{\lambda^*}^n(z) = n$ means that there exist $\nu^* \in \mathbb{C} \setminus \mathcal{L}$ such that

$$\operatorname{rank} \mathcal{R}_{\lambda^*}^n(\nu^*) = n.$$

By Theorem 2.2, the system corresponding to the pair $\left(A_{\lambda^*}(\nu^*), B_{\lambda^*}(\nu^*)\right)$ is reachable, for all $\nu^* \in \mathbb{C} \setminus \mathcal{L}$.

Thus, by definition, $\left(A_\lambda(\nu^*), B_\lambda(\nu^*)\right)$ is a structured system which is reachable.

By Theorem 2.5 we have that the pair of structured matrices $\left(A_\lambda(\nu^*), B_\lambda(\nu^*)\right)$ is neither in form I nor in form II and, by Lemma 5.1, $\left(A_\lambda(z), B_\lambda(z)\right)$ is neither in form I nor in form II. The converse implication is analogous. □

Example 5.5. Let

$$A_1^\lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{bmatrix}, A_2^\lambda = \begin{bmatrix} 0 & \lambda_3 & 0 \\ \lambda_4 & 0 & 0 \\ 0 & \lambda_5 & 0 \end{bmatrix}, B_1^\lambda = \begin{bmatrix} \lambda_6 \\ 0 \\ 0 \end{bmatrix} \text{ and } B_2^\lambda = \begin{bmatrix} 0 \\ 0 \\ \lambda_7 \end{bmatrix},$$

and define

$$A_\lambda(z) = A_1^\lambda + A_2^\lambda z = \begin{bmatrix} \lambda_1 & \lambda_3 z & 0 \\ \lambda_4 z & 0 & 0 \\ 0 & \lambda_2 + \lambda_5 z & 0 \end{bmatrix} \text{ and } B_\lambda(z) = B_1^\lambda + B_2^\lambda z = \begin{bmatrix} \lambda_6 \\ 0 \\ \lambda_7 z \end{bmatrix}.$$

It is easy to check that the pair $(A_\lambda(z), B_\lambda(z))$ is irreducible, and so it is not in form I. Moreover, the generic rank of the structured polynomial matrix $\begin{bmatrix} A_\lambda(z) & B_\lambda(z) \end{bmatrix}$ is 3, i.e., the pair $(A_\lambda(z), B_\lambda(z))$ is not in form II.

By the previous theorem, the 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable. Indeed, the global reachability matrix of that system is

$$\mathcal{R}^2(z) = \begin{bmatrix} B(z) & A(z)B(z) & A^2(z)B(z) \end{bmatrix} = \begin{bmatrix} \lambda_6 & \lambda_1\lambda_6 & \lambda_1^2\lambda_6 + \lambda_3\lambda_4\lambda_6 z^2 \\ 0 & \lambda_4\lambda_6 z & \lambda_1\lambda_4\lambda_6 z \\ \lambda_7 z & 0 & (\lambda_2 + \lambda_5 z)\lambda_4\lambda_6 z \end{bmatrix}.$$

Considering $\lambda^* = (0, 1, 0, 1, 0, 1, 0)$, the polynomial matrix

$$\mathcal{R}^2_{\lambda^*}(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix}$$

has rank 3 and hence, by Theorem 4.2, the 2D system $(A_1^{\lambda^*}, A_2^{\lambda^*}, B_1^{\lambda^*}, B_2^{\lambda^*})$ is globally reachable which, by definition, implies that the 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable. □

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