Automatic structures for semigroup constructions

I. Andrade, L. Descalço and Manuel A. Martins

Universidade de Aveiro,

Departamento de Matemática,

3810-193 Aveiro, Portugal

Tel. +351234370359, Fax. +351234382014

e-mails: iandrade@oninet.pt, luisd@mat.ua.pt, martins@mat.ua.pt

February 2, 2008

Abstract

We survey results concerning automatic structures for semigroup constructions, providing references and describing the corresponding automatic structures. The constructions we consider are: free products, direct products, Rees matrix semigroups, Bruck-Reilly extensions and wreath products.

1 Introduction

The notion of "automaticity" has been widely studied in groups (see [2] and [8] for example), and some progress has been made in understanding the notion in the wider context of semigroups. Many results about automatic semigroups concern automaticity of standard semigroups constructions. We survey these results for free products, direct products, Rees matrix semigroups, Bruck-Reilly extensions and wreath products.

Some references on automatic semigroups are [6] (introduction), [26] (geometric aspects and p-automaticity), [18], [19], [20] (computational and decidability aspects), [5], [10], [12] (semigroup constructions), [13] (other notions of "automaticity" for semigroups) and [11], [15] (examples).

We start by introducing the definitions we require. Given a non empty finite set A, which we call an *alphabet*, we denote by A^+ the free semigroup generated by A consisting of finite sequences of elements of A, which we call *words*, under the concatenation; and by A^* the free monoid generated by A consisting of A^+ together with the empty word ϵ, the identity in A^*. Let S be a semigroup and $\psi : A \to S$ a mapping. We say that A is a *finite generating set for S with respect to ψ* if the unique extension of ψ to a semigroup homomorphism $\psi : A^+ \to S$ is surjective. For $u, v \in A^+$ we write $u \equiv v$ to mean that u and v are equal as words and $u = v$ to mean that u and v represent the same element in the semigroup i.e. that $u\psi = v\psi$. We say that a subset L of A^*, usually called a *language*, is *regular* if there is a finite state automaton accepting L. To be able to deal with automata that accept pairs of words and to define automatic semigroups we need to define the set $A(2,\$) = ((A \cup \{\$\}) \times (A \cup \{\$\})) \backslash \{(\$,\$)\}$ where \$ is a symbol not in A (called the *padding symbol*) and the function $\delta_A : A^* \times A^* \to A(2,\$)^*$ defined by

$$(a_1 \dots a_m, b_1 \dots b_n)\delta_A = \begin{cases} \epsilon & \text{if } 0 = m = n \\ (a_1, b_1) \dots (a_m, b_m) & \text{if } 0 < m = n \\ (a_1, b_1) \dots (a_m, b_m)(\$, b_{m+1}) \dots (\$, b_n) & \text{if } 0 \leq m < n \\ (a_1, b_1) \dots (a_n, b_n)(a_{n+1}, \$) \dots (a_m, \$) & \text{if } m > n \geq 0. \end{cases}$$

Let S be a semigroup and A a finite generating set for S with respect to $\psi : A^+ \to S$. The pair (A, L) is an *automatic structure for S (with respect to ψ)* if

- L is a regular subset of A^+ and $L\psi = S$,
- $L_= = \{(\alpha, \beta) : \alpha, \beta \in L, \alpha = \beta\}\delta_A$ is regular in $A(2,\$)^+$, and
- $L_a = \{(\alpha, \beta) : \alpha, \beta \in L, \alpha a = \beta\}\delta_A$ is regular in $A(2,\$)^+$ for each $a \in A$.

We say that a semigroup is *automatic* if it has an automatic structure.

We say that the pair (A, L) is an *automatic structure with uniqueness* (with respect to ψ) for a semigroup S, if it is an automatic structure and each element in S is represented by an unique word in L (the restriction of ψ to L is a bijection). It is known (see [6]) that, any automatic semigroup admits an automatic structure with uniqueness.

We say that a semigroup is *prefix-automatic* or *p-automatic* if it has an automatic structure (A, L) such that the set

$$L'_= = \{(w_1, w_2)\delta_A : w_1 \in L, w_2 \in \mathrm{Pref}(L), w_1 = w_2\}$$

is also regular, where

$$\mathrm{Pref}(L) = \{w \in A^+ : ww' \in L \text{ for some } w' \in A^*\}.$$

We will now present a result from [14] useful to obtain automatic structures for the constructions considered in the sections following.

We say that T is a *subsemigroup of S of finite Rees index* if the set $S - T$ is finite.

Proposition 1.1 *Let S be a semigroup with a subsemigroup T of finite Rees index. Then S is automatic if and only if T is automatic.*

From an automatic structure (A, L) for T, an automatic structure for S can be easily obtained. Take C to be a finite set of new symbols in bijection with the elements of $S - T$, $A' = A \cup C$ and $L' = L \cup C$. The pair (A', L') is an automatic structure for S.

The converse is not so trivial. We start from an automatic structure with uniqueness (A, L) for S. It was shown in [14] that there exists a constant k such that every element of the set $\{\alpha \in \mathrm{sub}(L) : \ k \leq |\alpha| < 2k\}$ maps to an element of T. The generating set B for S is

$$\{b_\alpha : \alpha \in \mathrm{sub}(L), k \leq |\alpha| < 2k\} \cup \{c_\alpha : \alpha \in L, |\alpha| < k, \alpha \in T\}$$

where the b_α and c_α are new symbols such that each b_α and each c_α maps to the same element of T as the corresponding α. The regular language K is obtained as follows. We take $U = \{\alpha \in L : \alpha \text{ represents an element of } T\}$ and let $\phi : U \to B^*$ be defined by

$$\alpha\phi = \begin{cases} c_\alpha & \text{for } |\alpha| < k \\ b_{a_1 \ldots a_k} b_{a_{k+1} \ldots a_{2k}} \ldots b_{a_{(l-1)k+1} \ldots a_{lk}} b_{a_{lk+1} \ldots a_r} & \text{for } |\alpha| \geq k \end{cases}$$

with $k \leq r - lk < 2k$ and $\alpha \equiv a_1 a_2 \ldots a_r$. Note that $\alpha\phi = \alpha$ in T. Taking $K = U\phi$, the pair (B, K) is an automatic structure for T.

In sections two, three and four we present results about automaticy for free products, direct products and Rees matrix semigroups, respectively. We omit the proofs and just describe briefly how to obtain the corresponding automatic structures. In sections five and six we present results from [9] about automaticity of Bruck-Reilly extensions and wreath products, respectively.

2 Free Products

If S_1 and S_2 are semigroups with presentations $\langle A_1 | R_1 \rangle$ and $\langle A_2 | R_2 \rangle$ respectively (where $A_1 \cap A_2 = \emptyset$), then their free product $S_1 * S_2$ is the semigroup defined by the presentation $\langle A_1 \cup A_2 | R_1 \cup R_2 \rangle$. The elements of the product can bee seen as sequences $s_1 \dots s_m$ of elements of $S_1 \cup S_2$ such that two consecutive elements do not belong to the same factor. The product of sequences $s_1 \dots s_m$, $s'_1 \dots s'_n$ is the concatenation $s_1 \dots s_m s'_1 \dots s'_n$ if s_m and s'_1 do not belong to the same factor; otherwise it is $s_1 \dots s_{m-1} s s'_2 \dots s'_n$, where s is the product of s_m by s'_1 in their common factor.

Free products of semigroups and monoids were considered in [6]. For semigroups the following was shown:

Theorem 2.1 *Let S_1 and S_2 be semigroups. Then $S_1 * S_2$ is automatic if and only if both S_1 and S_2 are automatic.*

The proof of this theorem give us the automatic structures. Suppose that S_1 and S_2 are automatic semigroups, with automatic structures with uniqueness, say (A_1, L_1) and (A_2, L_2) respectively, with A_1 and A_2 disjoint sets. Taking $A = A_1 \cup A_2$ and

$$L \;=\; (L_1 \cup \{\epsilon\})(L_2 L_1)^*(L_2 \cup \{\epsilon\}) \;-\; \{\epsilon\},$$

we obtain a pair (A, L) which is an automatic structure for the semigroup free product $S_1 * S_2$. Conversely, suppose that $S_1 * S_2$ is automatic with an automatic structure (A, L). Letting

$$B = \{a \in A : a \text{ represents an element of } S_1\},$$

the pair $(B, L \cap B^+)$ is an automatic structure for S_1.

The monoid free product is not the same as the semigroup free product. It is the same as the group free product and can be seen as the semigroup free product with the identity subgroups amalgamated. For monoids we have the following:

Theorem 2.2 *The monoid free product $M = M_1 * M_2$ is automatic if and only if both monoids M_1 and M_2 are automatic.*

One implication was proved in [6]. Suppose that (A_1, L_1) and (A_2, L_2) are automatic structures with uniqueness for M_1 and M_2 respectively, with $A_1 \cap A_2 = \{e\}$, e representing the identity element of each M_i, $e \in L_i$ $(i = 1, 2)$, and $\overline{L}_i = L_i - \{e\} \subseteq (A_i - \{e\})^+$ $(i = 1, 2)$. Taking $A = A_1 \cup A_2$ and

$$L \;=\; \{e\}(\overline{L}_1 \cup \{\epsilon\})(\overline{L}_2 \overline{L}_1)^*(\overline{L}_2 \cup \{\epsilon\}),$$

the pair (A, L) is an automatic structure for M.

The converse was shown in [12], answering a question formalized in [6]. Finite generating sets A_i for M_i, $i = 1, 2$, give us a finite generating set $A = A_1 \cup A_2$ for M

with respect to an homomorphism ψ. It is possible to obtain an automatic structure (A, L) for M such that every element of S is represented by a unique element of L. The pair $(A, L(M_i))$, where $L(M_i) = \{w \in L : w\psi \in M_i\}$, is an automatic structure for M_i, $i = 1, 2$.

3 Direct products

The first result about direct products was obtained for monoids in [6] where the authors have shown:

Theorem 3.1 *If M_1 and M_2 are automatic monoids, then their direct product $M_1 \times M_2$ is automatic.*

Since we have the identities, an automatic structure for the product can be obtained from automatic structures for the factors in a natural way. We can start from automatic structures with uniqueness (A_1, L_1) and (A_2, L_2) for M_1 and M_2 respectively, with $A_1 \cap A_2 = \emptyset$, $e_i \in A_i$, e_i representing the identity element of M_i, $e_i \in L_i$, and $L_i - \{e_i\} \subseteq (A_i - \{e_i\})^+$ $(i = 1, 2)$. Let $\overline{L}_i$ denote $L_i - \{e_i\}$ and let $A = A_1 \cup A_2$. For words $\alpha \equiv a_1 \dots a_n \in L_1$ and $\beta \equiv b_1 \dots b_m \in L_2$, we define the word $\alpha\sharp\beta \in A^+$ by

$$\alpha\sharp\beta = \begin{cases} a_1b_1 \dots a_nb_n & \text{if } n = m, \\ a_1b_1 \dots a_nb_ne_1b_{n+1} \dots e_1b_m & \text{if } n < m, \\ a_1b_1 \dots a_mb_ma_{m+1}e_2 \dots a_ne_2 & \text{if } n > m. \end{cases}$$

If $\sigma : A(2, \$)^* \rightarrow A^*$ is the homomorphism defined by

$$(a, b) \mapsto ab, \quad (a, \$) \mapsto ae_2, \quad (\$, b) \mapsto e_1b,$$

then $\alpha\sharp\beta = (\alpha, \beta)\delta_A\sigma$. Let

$$L = \{\alpha\sharp\beta : \alpha \in L_1, \beta \in L_2\} = (L_1 \times L_2)\delta_A\sigma.$$

The pair (A, L) is an automatic structure for $M = M_1 \times M_2$.

Semigroups were then considered in [5] where the authors have proved the following:

Theorem 3.2 *Let S and T be automatic semigroups.*

(i) *If S and T are infinite, then $S \times T$ is automatic if and only if $S^2 = S$ and $T^2 = T$.*

(ii) *If S is finite and T is infinite, then $S \times T$ is automatic if and only if $S^2 = S$.*

In [25], there were established necessary and sufficient conditions for the direct product of semigroups to be finitely generated:

Proposition 3.3 *Let S and T be two semigroups. If both S and T are infinite then $S \times T$ is finitely generated if and only if both S and T are finitely generated, $S^2 = S$ and $T^2 = T$. If S is finite and T is infinite then $S \times T$ is finitely generated if and only if $S^2 = S$ and T is finitely generated.*

Using this result, Theorem 3.2 has the following equivalent formulation:

Theorem 3.4 *The direct product of automatic semigroups is automatic if and only if it is finitely generated.*

The answer to the following converse question is not known even for groups: If the direct product $G_1 \times G_2$ is automatic are both factors G_1 and G_2 necessarily automatic?

Without the identities it is still possible to obtain automatic structures for the product, starting from automatic structures for the factors, although the method is not so natural.

For case (i) in Theorem 3.2, the general ideia is to start from two automatic structures, say (A, L) and (B, K), for the factors S and T and then modify them, using the fact that $S^2 = S$ (and $T^2 = T$) to control the length of the words in the languages, in order to obtain new automatic structures. From the modified automatic structures, say (A', L') and (B', K'), an automatic structure (X, J) for the product $S \times T$ can be obtained by just taking $X = A' \times B'$ and $J = \{(u_1, v_1), \ldots, (u_p, v_p) : (u_i, v_i) \in X, u_1 \ldots u_p \in L', v_1 \ldots v_p \in K'\}$.

For case (ii) we can assume that $S = \{s_1, \ldots, s_m\}$ and take an alphabet $A = \{a_1, \ldots, a_m\}$ to represent the elements in S. Given an automatic structure (B, K) for T, the set $X = A \times B$ is a generating set for $S \times T$. Now, taking $J = \{(u_1, v_1) \ldots (u_p, v_v) : (u_i, v_i) \in X, v_1 \ldots v_p \in K\}$, the pair (X, J) is an automatic structure for the product $S \times T$ (the details can be found in [5]).

4 Rees matrix semigroups

The *Rees matrix semigroup* $S = \mathcal{M}[U; I, J; P]$ over the semigroup U, with $P = (p_{ji})_{j \in J, i \in I}$ a matrix with entries in U, is the semigroup with the support set $I \times U \times J$ and multiplication defined by $(l_1, s_1, r_1)(l_2, s_2, r_2) = (l_1, s_1 p_{r_1 l_2} s_2, r_2)$ where $(l_1, s_1, r_1), (l_2, s_2, r_2) \in I \times U \times J$. We say that U is the *base semigroup* of the Rees matrix semigroup S.

We can obtain an automatic structure for a Rees matrix semigroup S by using the automatic structure for its base semigroup U, as shown in [10]. We observe that the case where U is a group, was firstly considered in [7].

Theorem 4.1 *Let $S = \mathcal{M}[U; I, J; P]$ be a Rees matrix semigroup. If U is an automatic semigroup and if S is finitely generated then S is automatic.*

This theorem has the following equivalent formulation:

Theorem 4.2 *Let $S = \mathcal{M}[U; I, J; P]$ be a Rees matrix semigroup, where I, J are finite sets and $U \backslash V$ is finite, where V is the ideal of U generated by the entries of the matrix P. If U is an automatic semigroup then S is automatic.*

In fact, it is described in [10] how to obtain an automatic structure for the semigroup $S_1 = \mathcal{M}[U^1; I, J; P]$ from an automatic structure with uniqueness for V (U^1 stands for the monoid obtained from U by adding an identity). But note that, since S is finitely generated, I, J and $U - V$ are finite, and so, using Proposition 1.1, an automatic structure for S can then be obtained from an automatic structure for U.

We start from an automatic structure with uniqueness (B, K) for V, where $B = \{b_1, \ldots, b_n\}$ is a set of semigroup generators for V. Then we write each b_h ($h \in N = \{1, \ldots, n\}$) as $b_h = s_h p_{\rho_h \lambda_h} s'_h$ where $s_h, s'_h \in U^1, \rho_h \in J, \lambda_h \in I$. Let $S_1 = \mathcal{M}[U^1; I, J; P]$. Given $(l, s, r) \in I \times V \times J$ we can write $s = b_{\alpha_1} \ldots b_{\alpha_h}$ where $b_{\alpha_1} \ldots b_{\alpha_h}$ is a word in K. So we can write

$$(l, s, r) = (l, s_{\alpha_1}, \rho_{\alpha_1})(\lambda_{\alpha_1}, s'_{\alpha_1} s_{\alpha_2}, \rho_{\alpha_2}) \ldots (\lambda_{\alpha_h}, s'_{\alpha_h}, r).$$

Since $U^1 \backslash V$ is finite and non empty we can write $U^1 \backslash V = \{x_1, \ldots, x_m\}$ with $m \geq 1$. We define a set $A = C \cup D$ of semigroup generators for S_1 by

$$C = \{c_{li} : l \in I, i \in N\} \cup \{d_{ij} : i, j \in N\} \cup \{e_{jr} : j \in N, r \in J\},$$
$$D = \{f_{lhr} : l \in I, h \in \{1, \ldots, m\}, r \in J\}$$

with

$$\psi : A^+ \to S_1,\ c_{li} \mapsto (l, s_i, \rho_i),\quad d_{ij} \mapsto (\lambda_i, s'_i s_j, \rho_j),$$
$$e_{jr} \mapsto (\lambda_j, s'_j, r), f_{lhr} \mapsto (l, x_h, r).$$

Defining the language $L = L_1 \cup D$ to represent the elements of S_1 with

$$L_1 = \{c_{l\alpha_1} d_{\alpha_1\alpha_2} \ldots d_{\alpha_{h-1}\alpha_h} e_{\alpha_h r} : b_{\alpha_1} \ldots b_{\alpha_h} \in K, h \geq 1, l \in I, r \in J\},$$

the pair (A, L) is an automatic structure for S_1.

It was also shown in [10] that, in some particular situations, it is possible to obtain an automatic structure for the base semigroup, from the automatic structure for the construction.

Theorem 4.3 *Let $S = \mathcal{M}[U; I, J; P]$ be a semigroup, and suppose that there is an entry p in the matrix P such that $pU^1 = U$. If S is automatic then U is automatic.*

We start from an automatic structure with uniqueness (A, L) for the semigroup $S_1 = \mathcal{M}[U^1; I, J; P]$, where $A = \{a_1, \ldots, a_n\}$ is a generating set for S_1 with respect to

$$\psi : A^+ \to S_1,\ a_h \mapsto (i_h, s_h, j_h)\ (h = 1, \ldots, n).$$

The set

$$B = \{b_1, \ldots, b_n\} \cup \{c_{ji} : j \in J, i \in I\}$$

is a generating set for U^1 with respect to

$$\phi : B^+ \to U^1; b_h \mapsto s_h, c_{ji} \mapsto p_{ji}\ (h = 1, \ldots, n,\ j \in J, i \in I).$$

Without loss of generality we can assume that $p_{11} = p$. Let

$$L_{11} = L \cap (\{1\} \times U^1 \times \{1\})\psi^{-1}.$$

Let

$$f : A^+ \to B^+; a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_h} \mapsto b_{\alpha_1} c_{j_{\alpha_1} i_{\alpha_2}} b_{\alpha_2} \ldots c_{j_{\alpha_{h-1}} i_{\alpha_h}} b_{\alpha_h}.$$

Taking $K = L_{11} f$, the pair (B, K) is an automatic structure with uniqueness for U^1 with respect to ϕ.

Theorem 4.4 *Let $S = \mathcal{M}[U; I, J; P]$ be a Rees matrix semigroup. If S is prefix-automatic then U is automatic.*

We start from a prefix-automatic structure with uniqueness (A, L) for S (see [26]). We define $A, \psi, B, \phi, L_{11}, f$ and K as above just replacing U^1 by U and S_1 by S in the definitions, and assume that $\psi\restriction_A$ is injective. The pair (B, K) is a (prefix-)automatic structure with uniqueness for U with respect to ϕ.

5 Bruck-Reilly extensions

Let T be a monoid and $\theta : T \mapsto T$ be a monoid homomorphism. The set

$$\mathbb{N}_0 \times T \times \mathbb{N}_0$$

with the operation defined by

$$(m, t_1, n)(p, t_2, q) = (m - n + k, (t_1\theta^{k-n})(t_2\theta^{k-p}), q - p + k)\ (k = max\{n, p\}),$$

where θ^0 denotes the identity map on M, is called the Bruck–Reilly extension of T determined by θ and is denoted by $\mathrm{BR}(T,\theta)$. The semigroup $\mathrm{BR}(T,\theta)$ is a monoid with identity $(0,1_T,0)$, denoting by 1_T the identity of T. This is a generalization of the constructions from [3, 17, 21], also considered in [1].

Theorem 5.1 *If T is a finite monoid, then any Bruck–Reilly extension of T is automatic.*

Proof. Let $T=\{t_1,\ldots,t_l\}$ and let $\overline{T}=\{\overline{t_1},\ldots,\overline{t_l}\}$ be an alphabet in bijection with T. We define the alphabet $A=\{b,c\}\cup\overline{T}$ and the regular language

$$L=\{c^m\overline{t}b^n : m,n\geq 0, \overline{t}\in\overline{T}\}$$

on A. Defining the homomorphism

$$\psi : A^+\to \mathrm{BR}(T,\theta);\ \overline{t}\mapsto(0,t,0),\ c\mapsto(1,1_T,0),\ b\mapsto(0,1_T,1)$$

it is clear that A is a generating set for $\mathrm{BR}(T,\theta)$ with respect to ψ and, in fact, given an element $(m,t,n)\in\mathbb{N}_0\times T\times\mathbb{N}_0$, the unique word in L representing it is $c^m\overline{t}b^n$.

In order to prove that (A,L) is an automatic structure with uniqueness for $BR(T,\theta)$ we only have to prove that, for each generator $a\in A$, the language L_a is regular. To prove that L_b is regular we observe that

$$(c^m\overline{t_i}b^n)b=(m,t_i,n)(0,1_T,1)=(m,t_i,n+1)=c^m\overline{t_i}b^{n+1}$$

and so we can write

$$\begin{aligned}L_b&=\bigcup_{i=1}^{l}\{(c^m\overline{t_i}b^n,c^m\overline{t_i}b^{n+1})\delta_A : n,m\in\mathbb{N}_0\}\\&=\bigcup_{i=1}^{l}(\{(c,c)\}^*\cdot\{(\overline{t_i},\overline{t_i})\}\cdot\{(b,b)\}^*\cdot\{(\$,b)\})\end{aligned}$$

which is a finite union of regular languages and so is regular. With respect to L_c we have

$$\begin{aligned}&(c^m\overline{t})c=(m,t,0)(1,1_T,0)=(m+1,t\theta,0)=c^{m+1}\overline{t\theta},\\&(c^m\overline{t}b^{n+1})c=(m,t,n+1)(1,1_T,0)=(m,t,n)=c^m\overline{t}b^n\ (n,m\in\mathbb{N}_0;\overline{t}\in\overline{T})\end{aligned}$$

and so we can write

$$\begin{aligned}L_c =& \bigcup_{i=1}^{l} \{(c^m \overline{t_i}, c^{m+1}\overline{t_i\theta})\delta_A : m \in \mathbb{N}_0\} \cup \\ & \bigcup_{i=1}^{l} \{(c^m \overline{t_i} b^{n+1}, c^m \overline{t_i} b^n)\delta_A : m, n \in \mathbb{N}_0\} \\ =& \bigcup_{i=1}^{l} (\{(c,c)\}^* \cdot \{(\overline{t_i}, c)(\$, \overline{t_i\theta})\}) \cup \\ & \bigcup_{i=1}^{l} (\{c,c)\}^* \cdot \{(\overline{t_i}, \overline{t_i})\} \cdot \{(b,b)\}^* \cdot \{(b,\$)\})\end{aligned}$$

and we conclude that L_c is a regular language as well.

We now fix an arbitrary $\overline{t} \in \overline{T}$ and prove that $L_{\overline{t}}$ is regular. For any words $c^m\overline{t_\alpha}b^n, c^p\overline{t_\beta}b^q \in L$ we have

$$c^m\overline{t_\alpha}b^n\overline{t} = c^p\overline{t_\beta}b^q$$

if and only if $m = p, n = q$, and $t_\alpha(t\theta^n) = t_\beta$, because

$$c^m\overline{t_\alpha}b^n\overline{t} = (m, t_\alpha, n)(0, t, 0) = (m, t_\alpha(t\theta^n), n).$$

Since T is finite the set $\{t\theta^n : n \in \mathbb{N}_0\}$ is finite as well. Taking j to be minimum such that the set $C_j = \{k \geq j : t\theta^j = t\theta^{k+1}\}$ is non empty and k to be the minimum element of C_j, we will now show that

$$\{t\theta^n : n \in \mathbb{N}_0\} = \{t, t\theta, \ldots, t\theta^j, \ldots, t\theta^k\}.$$

Given $n \geq j$ we have $n = j + h$ with $h \geq 0$ and, dividing h by $k + 1 - j$, we obtain $n = j + q(k + 1 - j) + r$ with $q \geq 0$ and $0 \leq r < k + 1 - j$. We now prove, by induction on q, that $t\theta^{j+r+q(k+1-j)} = t\theta^{j+r}$ for $q \geq 0$. For $q = 0$ it holds trivially and for $q > 0$ we have

$$\begin{aligned}t\theta^{j+r+q(k+1-j)} &= t\theta^{j+r+k+1-j+(q-1)(k+1-j)} = (t\theta^r)(t\theta^{k+1})(t\theta^{(q-1)(k+1-j)}) \\ &= (t\theta^r)(t\theta^j)(t\theta^{(q-1)(k+1-j)}) = t\theta^{j+r+(q-1)(k+1-j)}.\end{aligned}$$

We can then write

$$\begin{aligned}L_{\overline{t}} =& \bigcup_{n=0}^{j-1} \{(c^m\overline{t_\alpha}b^n, c^m\overline{t_\alpha(t\theta^n)}b^n)\delta_A : m \in \mathbb{N}_0, t_\alpha \in T\} \cup \\ & \bigcup_{n=j}^{k} \{(c^m\overline{t_\alpha}b^{n+q(k+1-j)}, c^m\overline{t_\alpha(t\theta^n)}b^{n+q(k+1-j)})\delta_A : m, q \in \mathbb{N}_0, t_\alpha \in T\} \\ =& \bigcup_{n=0}^{j-1} (\{(c,c)\}^* \cdot \{(\overline{t_\alpha}, \overline{t_\alpha(t\theta^n)}) : t_\alpha \in T\} \cdot \{(b,b)\}^*) \cup \\ & \bigcup_{n=j}^{k} (\{(c,c)\}^* \cdot \{(\overline{t_\alpha}, \overline{t_\alpha(t\theta^n)}) : t_\alpha \in T\} \cdot \{(b,b)^n\} \cdot \{(b,b)^{k+1-j}\}^*)\end{aligned}$$

and since all sets in this union are regular we conclude that $L_{\overline{t}}$ is regular as well. ■

From now on we assume that T is an automatic monoid and we fix an automatic structure (X, K) with uniqueness for T, where $X = \{x_1, \ldots, x_n\}$ is a set of semigroup generators for T with respect to the homomorphism

$$\phi : X^+ \to T.$$

We define the alphabet

$$A = \{b, c\} \cup X \tag{1}$$

to be a set of semigroup generators for $\mathrm{BR}(T, \theta)$ with respect to the homomorphism

$$\psi : A^+ \to \mathrm{BR}(T, \theta), x_i \mapsto (0, x_i\phi, 0), c \mapsto (1, 1_T, 0), b \mapsto (0, 1_T, 1),$$

and the regular language

$$L = \{c^i w b^j : w \in K; i, j \in \mathbb{N}_0\} \tag{2}$$

on A^+, which is a set of unique normal forms for $\mathrm{BR}(T, \theta)$, since we have $(c^i w b^j)\psi = (i, w\phi, j)$ for $w \in K$, $i, j \in \mathbb{N}_0$. As usual, to simplify notation, we will avoid explicit use of the homomorphisms ψ and ϕ, associated with the generating sets, and it will be clear from the context whenever a word $w \in X^+$ is being identified with an element of T, with an element of $\mathrm{BR}(T, \theta)$ or considered as a word. In particular, for a word $w \in X^+$ we write $w\theta$ instead of $(w\phi)\theta$, seeing θ also as a homomorphism $\theta : X^+ \to T$, and we will often write (i, w, j) instead of $(i, w\phi, j)$ for $i, j \in \mathbb{N}_0$.

For (A, L) to be an automatic structure for $\mathrm{BR}(T, \theta)$ the languages

$$\begin{aligned}
L_b = \ & \{(c^i w b^j, c^i w b^{j+1})\delta_A : w \in K; i, j \in \mathbb{N}_0\},\\
L_c = \ & \{(c^i w b^{j+1}, c^i w b^j)\delta_A : w \in K; i, j \in \mathbb{N}_0\} \cup \\
& \{(c^i w_1, c^{i+1} w_2)\delta_A : w_1, w_2 \in K; i \in \mathbb{N}_0; w_2 = w_1\theta\},\\
L_{x_r} = \ & \{(c^i w_1 b^j, c^i w_2 b^j)\delta_A : (w_1, w_2)\delta_X \in K_{x_r\theta^j}; i, j \in \mathbb{N}_0\} \ (x_r \in X),
\end{aligned}$$

must be regular. The language L_b is regular, since we have

$$L_b = \{(c, c)\}^* \cdot \{(w, w)\delta_X : w \in K\} \cdot \{(b, b)\}^* \cdot \{(\$, b)\},$$

but there is no obvious reason why the languages L_c and L_{x_r} should also be regular. We will consider particular situations where (A, L) is an automatic structure for $\mathrm{BR}(T, \theta)$.

Theorem 5.2 *If T is an automatic monoid and $\theta : T \to T;\ t \mapsto 1_T$ then* $\mathrm{BR}(T, \theta)$ *is automatic.*

To show this we use the notion of *padded product of languages* and an auxiliary result whose proof can be found in [9]. Fixing an alphabet A, and given two regular languages M, N in $(A^* \times A^*)\delta$, the *padded product of languages* M and N is

$$M \odot N = \{(w_1 w_1', w_2 w_2')\delta : (w_1, w_2)\delta \in M, (w_1', w_2')\delta \in N\}$$

The result is the following:

Lemma 5.3 *Let A be an alphabet and let M, N be regular languages on $(A^* \times A^*)\delta$. If there exists a constant C such that, for any two words $w_1, w_2 \in A^*$ we have*

$$(w_1, w_2)\delta \in M \;\Rightarrow\; ||w_1| - |w_2|| \leq C,$$

then the language $M \odot N$ is regular.

Proof. of Theorem 5.2 To show that the pair (A, L) defined by (1) and (2) is an automatic structure for $\mathrm{BR}(T, \theta)$ we just have to prove that the languages L_c and L_{x_r} $(x_r \in X)$ are regular. But now, denoting by w_{1_T} the unique word in K representing 1_T, we have

$$\begin{aligned} L_c &= \{(c^i w b^{j+1}, c^i w b^j)\delta_A : w \in K; i, j \in \mathbb{N}_0\} \cup \{(c^i w, c^{i+1} w_{1_T})\delta_A : w \in K; i \in \mathbb{N}_0\} \\ &= (\{(c,c)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(b,\$)\}) \cup \\ &\quad ((\{(c,c)\}^* \cdot \{(\$,c)\}) \odot (K \times \{w_{1_t}\})\delta_X), \end{aligned}$$

which is a regular language by Lemma 5.3. We have

$$\begin{aligned} L_{x_r} &= \{(c^i w b^j, c^i w b^j)\delta_A : w \in K, i \in \mathbb{N}_0, j \in \mathbb{N}\} \cup \\ &\quad \{(c^i w_1, c^i w_2)\delta_A : (w_1, w_2)\delta_X \in K_{x_r}; i \in \mathbb{N}_0\} \\ &= (\{(c,c)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^+) \cup \\ &\quad (\{(c,c)\}^* \cdot K_{x_r}) \end{aligned}$$

because, for any $c^i w b^j \in L$ with $j \geq 1$, we have

$$(c^i w b^j) x_r = (i, w, j)(0, x_r, 0) = (i, w(x_r \theta^j), j) = (i, w, j) = c^i w b^j$$

and for $c^i w \in L$ we have

$$(c^i w) x_r = (i, w, 0)(0, x_r, 0) = (i, w x_r, 0).$$

Therefore L_{x_r} is also a regular language and so $\mathrm{BR}(T, \theta)$ is automatic. ■

Theorem 5.4 *If T is an automatic monoid and θ is the identity in T then $\mathrm{BR}(T, \theta)$ is automatic.*

Proof. We use the generating set A defined by equation (1) but we now define $L = \{c^i b^j w : w \in K\}$ observing that, since θ is the identity, for any $x_r \in X$, we have

$$x_r c = (0, x_r, 0)(1, 1_T, 0) = (1, x_r\theta, 0) = (1, x_r, 0) = (1, 1_T, 0)(0, x_r, 0) = cx_r,$$
$$x_r b = (0, x_r, 0)(0, 1_T, 1) = (0, x_r, 1) = (0, x_r\theta, 1) = (0, 1_T, 1)(0, x_r, 0) = bx_r.$$

The language L is regular and it is a set of unique normal forms for $\mathrm{BR}(T, \theta)$. Also the languages

$$\begin{aligned}
L_b &= \{(c^i b^j w, c^i b^{j+1} w)\delta_A : w \in K; i, j \in \mathbb{N}_0\} \\
&= (\{(c, c)\}^* \cdot \{(b, b)\}^* \cdot \{(\$, b)\}) \odot \{(w, w)\delta_X : w \in K\}, \\
L_c &= \{(c^i b^{j+1} w, c^i b^j w)\delta_A : w \in K; i, j \in \mathbb{N}_0\} \cup \\
&\quad \{(c^i w, c^{i+1} w)\delta_A : i \in \mathbb{N}_0, w \in K\} \\
&= ((\{(c, c)\}^* \cdot \{(b, b)\}^* \cdot \{(b, \$)\}) \odot \{(w, w)\delta_X : w \in K\}) \cup \\
&\quad ((\{(c, c)\}^* \cdot \{(\$, c)\}) \odot \{(w, w)\delta_X : w \in K\}), \\
L_{x_r} &= \{(c^i b^j w_1, c^i b^j w_2)\delta_A : (w_1, w_2)\delta_X \in K_{x_r}\} \\
&= (\{(c, c)\}^* \cdot \{(b, b)\}^*) \cdot K_{x_r}
\end{aligned}$$

are regular, by Lemma 5.3, and so (A, L) is an automatic structure for $\mathrm{BR}(T, \theta)$. ∎

We say that a semigroup T is of *finite geometrical type* (fgt) (see [26]) if for every $t_1 \in T$, there exists $k \in \mathbb{N}$ such that the equation

$$xt_1 = t_2$$

has at most k solutions for every $t_2 \in M$.

To prove next theorem we will use the following two auxiliary results from [9]:

Lemma 5.5 *Let T be a* fgt *monoid with an automatic structure with uniqueness (X, K). Then for every $w \in X^+$ there is a constant C such that $(w_1, w_2)\delta_X \in K_w$ implies $||w_1| - |w_2|| < C$.*

Lemma 5.6 *Let S be a finite semigroup, X be a finite set and $\psi : X^+ \to S$ be a surjective homomorphism. For any $s \in S$ the set $s\psi^{-1}$ is a regular language.*

Theorem 5.7 *Let T be a fgt automatic monoid and let $\theta : T \to T$ be a monoid homomorphism. If $T\theta$ is finite then* $\mathrm{BR}(T, \theta)$ *is automatic.*

Proof. We will prove that the pair (A, L) defined by (1) and (2) is an automatic structure for $\mathrm{BR}(T, \theta)$. We have

$$\begin{aligned}
L_c = &\{(c^i w b^{j+1}, c^i w b^j)\delta_A : w \in K; i, j \in \mathbb{N}_0\} \cup \\
&\{(c^i w_1, c^{i+1} w_2)\delta_A : w_1, w_2 \in K; i \in \mathbb{N}_0; w_2 = w_1\theta\}
\end{aligned}$$

and, since the language

$$\begin{aligned}&\{(c^i w b^{j+1}, c^i w b^j)\delta_A : w \in K; i, j \in \mathbb{N}_0\} = \\ &\{(c,c)^i\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(b,\$)\}\end{aligned}$$

is regular, we just have to prove that the language

$$M = \{(c^i w_1, c^{i+1} w_2)\delta_A : w_1, w_2 \in K; i \in \mathbb{N}_0; w_2 = w_1\theta\}$$

is also regular. For any $t \in T\theta$ let w_t be the unique word in K representing t. Let

$$\begin{aligned}N =&\{(w_1, w_2)\delta_X : w_1, w_2 \in K; w_2 = w_1\theta\} = \\ &\bigcup_{t\in T\theta} \{(w_1, w_2)\delta_X : w_1, w_2 \in K; w_2 = w_1\theta = t\} = \\ &\bigcup_{t\in T\theta} \{(w_1, w_t)\delta_X : w_1 \in K; w_1 \in (t\theta^{-1})\phi^{-1}\} = \\ &\bigcup_{t\in T\theta} (((t\theta^{-1})\phi^{-1} \cap K) \times \{w_t\})\delta_X.\end{aligned}$$

We can define $\psi : X^+ \to T\theta;\ w \mapsto w\phi\theta$ and, since $T\theta$ is finite, for any $t \in T\theta$, we can apply Lemma 5.6 and conclude that $(t\theta^{-1})\phi^{-1} = t\psi^{-1}$ is regular. Therefore, N is a regular language and, since we have

$$\begin{aligned}M =&\{(c^i w_1, c^{i+1} w_2)\delta_A : (w_1, w_2)\delta_X \in N; i \in \mathbb{N}_0\} = \\ &(\{(c,c)\}^* \cdot \{(\$, c)\}) \odot N,\end{aligned}$$

by Lemma 5.3, M is a regular language as well. We will now prove that the language

$$L_{x_r} = \{(c^i w_1 b^j, c^i w_2 b^j)\delta_A : (w_1, w_2)\delta_X \in K_{x_r\theta^j}; i, j \in \mathbb{N}_0\}$$

is regular. Since $T\theta$ is finite we can, as in the proof of Lemma 5.1, take j, k to be minimum with $x_r\theta^j = x_r\theta^{k+1}$ and $j \leq k$, and we have $x_r\theta^{j+r+q(k+1-j)} = x_r\theta^{j+r}$ for $j \leq j + r < k + 1$ and $q \geq 0$. Therefore, we can write

$$\begin{aligned}L_{x_r} =&\bigcup_{n=0}^{j-1} \{(c^i w_1 b^n, c^i w_2 b^n)\delta_A : (w_1, w_2)\delta_X \in K_{x_r\theta^n}; i \in \mathbb{N}_0\} \cup \\ &\bigcup_{n=j}^{k} \{(c^i w_1 b^{n+q(k+1-j)}, c^i w_2 b^{n+q(k+1-j)})\delta_A : (w_1, w_2)\delta_X \in K_{x_r\theta^n}; i, q \in \mathbb{N}_0\} \\ =&\bigcup_{n=0}^{j-1} (\{(c,c)\}^* \cdot (K_{x_r\theta^n} \odot \{(b,b)\}^*)) \cup \\ &\bigcup_{n=j}^{k} (\{(c,c)\}^* \cdot (K_{x_r\theta^n} \odot (\{(b,b)^n\} \cdot \{(b,b)^{k+1-j}\}^*))).\end{aligned}$$

Since T is fgt, by Lemma 5.5 there is a constant C such that

$$(w_1, w_2)\delta_X \in K_{x_r\theta^n} \;\Rightarrow\; ||w_1| - |w_2|| < C$$

for any $n = 0, \ldots, k$, and therefore we can apply Lemma 5.3 and we conclude that L_{x_r} is a regular language. ■

Since automatic groups are characterized by the fellow traveller property and Bruck–Reilly extensions of groups are somehow "almost groups" the following is a natural question: Is a Bruck–Reilly extension of a group automatic if and only if it has the fellow traveller property?

6 Wreath products

We consider the automaticity of the wreath product of semigroups, $S \operatorname{wr} T$, in the case where T is a finite semigroup. We start by giving the necessary and sufficient conditions, obtained in [23], for the wreath product, to be finitely generated, when T is finite. Finite generation of the wreath product is related to finite generation of the diagonal S-act. We use the conditions obtained for the case where the diagonal S-act is not finitely generated to prove that, in this case, the wreath product $S \operatorname{wr} T$ is automatic whenever it is finitely generated and S is an automatic semigroup.

We start by giving the definitions we require. If S is a semigroup and X is a set, then the set S^X of all mappings $X \to S$ forms a semigroup under *component-wise* multiplication of mappings: for $f, g \in S^X$, $fg : X \to S$; $x \mapsto (xf)(xg)$; this semigroup is called the *Cartesian power* of S by X. If S has a distinguished idempotent e, then the *support* of $f \in S^X$ relative to e is defined by

$$\operatorname{supp}_e(f) = \{x \in X : xf \neq e\}.$$

The set

$$S^{(X)_e} = \{f \in S^X : |\operatorname{supp}_e(f)| < \infty\}$$

is a subsemigroup of S^X; it is called the *direct power* of S relative to e (the subscript e is usually omitted). If X is finite of size n then S^X and $S^{(X)_e}$ coincide, and they are isomorphic to the semigroup $S^{(n)}$ consisting of n-tuples of elements of S under the component-wise multiplication. In this context, we write $S^{(X)_e}$ even if S has no idempotents; we can think of this as computing supports with respect to an identity adjoined to S.

The *unrestricted wreath product* $S \operatorname{Wr} T$ of two semigroups is the set $S^T \times T$ under multiplication

$$(f, t)(g, u) = (f\,{}^t g, tu),$$

where ${}^{t}g \in S^T$ is defined by

$$(x)\, {}^{t}g = (xt)g.$$

Let $e \in S$ be a distinguished idempotent. The *(restricted) wreath product* $S_e \operatorname{wr} T$ (with respect to e) is the subsemigroup of $S \operatorname{Wr} T$ generated by the set $\{(f,t) \in S \operatorname{Wr} T : |\operatorname{supp}_e(f)| < \infty\}$ (again the subscript e is often omitted).

The wreath product $S \operatorname{wr} T$ coincides with the unrestricted wreath product $S \operatorname{Wr} T$ in the case where T is finite, as observed in [27, Chapter 3].

An *action* of a semigroup S on a set X is a mapping $X \times S \to X$, $(x,s) \mapsto xs$, satisfying $(xs_1)s_2 = x(s_1s_2)$. The set X, together with an action, is called an *S-act*. It is said to be generated by a set $U \subseteq X$ if $US^1 = X$, and *finitely generated* if there exists a finite such U.

The *diagonal act* of a semigroup S is the set $S \times S$ with the action $(s_1,s_2)s = (s_1s, s_2s)$. The diagonal acts of infinite groups, free semigroups, free commutative semigroups and completely simple semigroups are not finitely generated. On the other hand, the diagonal act of the full transformation monoid $T_{\mathbb{N}}$ on positive integers can be generated by a single element; see [4]. In [22] the authors give an example of an infinite, finitely presented monoid with a finitely generated diagonal act.

We will only state the conditions obtained in [23] for the case where T is finite and S is infinite.

Proposition 6.1 *Let S be an infinite semigroup and let T be a finite non-trivial semigroup. If the diagonal S-act is finitely generated then $S \operatorname{wr} T$ is finitely generated if and only if the following conditions are satisfied:*

(i) *$S^2 = S$ and $T^2 = T$;*

(ii) *S is finitely generated.*

If the diagonal S-act is not finitely generated then $S \operatorname{wr} T$ is finitely generated if and only if the following conditions are satisfied:

(i) *$S^2 = S$;*

(ii) *S is finitely generated;*

(iii) *every element of T is contained in the principal right ideal generated by a right identity.*

We will now consider the automaticity of the wreath product $S \operatorname{wr} T$ in the case where T is finite. In the case where S is also finite, $S \operatorname{wr} T$ is finite as well, and, in particular, it is automatic. We will consider the case where S is infinite and the diagonal S-act is not finitely generated.

Theorem 6.2 *If S and T are semigroups satisfying the following conditions:*

(i) *T is finite;*

(ii) *S is automatic;*

(iii) *the diagonal S-act is not finitely generated;*

(iv) *the wreath product $S \operatorname{wr} T$ is finitely generated;*

then $S \operatorname{wr} T$ is automatic.

To prove this theorem we will need some notation and a result from [10]. A *generalized sequential machine* (gsm for short) is a six-tuple $\mathcal{A} = (Q, A, B, \mu, q_0, T)$ where Q, A and B are finite sets, (called the *states*, the *input alphabet* and the *output alphabet* respectively), μ is a (partial) function from $Q \times A$ to finite subsets of $Q \times B^+$, $q_0 \in Q$ is the *initial state* and $T \subseteq Q$ is the *set of terminal states.* We can read $(q', u) \in (q, a)\mu$ in the following way: if $\mathcal{A}$ is in state q and receives input a, then it can move into state q' and output u.

We can interpret $\mathcal{A}$ as a directed labelled graph with vertices Q, and an edge $q \xrightarrow{(a,u)} q'$ for every pair $(q', u) \in (q, a)\mu$. For a path

$$\pi \ : \ q_1 \xrightarrow{(a_1,u_1)} q_2 \xrightarrow{(a_2,u_2)} q_3 \ldots \xrightarrow{(a_n,u_n)} q_{n+1}$$

we define

$$\Phi(\pi) = a_1 a_2 \ldots a_n, \ \Sigma(\pi) = u_1 u_2 \ldots u_n.$$

For $q, q' \in Q$, $u \in A^+$ and $v \in B^+$ we write $q \xrightarrow{(u,v)}_+ q'$ to mean that there exists a path π from q to q' such that $\Phi(\pi) \equiv u$ and $\Sigma(\pi) \equiv v$, and we say that (u, v) is the *label* of the path. We say that a path is *successful* if it has the form $q \xrightarrow{(u,v)}_+ t$ with $t \in T$.

The gsm $\mathcal{A}$ induces a mapping $\eta_{\mathcal{A}} : \mathcal{P}(A^+) \longrightarrow \mathcal{P}(B^+)$ from subsets of A^+ into subsets of B^+ defined by

$$X\eta_{\mathcal{A}} = \{v \in B^+ \ : \ (\exists u \in X)(\exists t \in T)(q_0 \xrightarrow{(u,v)}_+ t)\}.$$

It is well known that if X is regular then so is $X\eta_{\mathcal{A}}$; see [16]. Similarly, $\mathcal{A}$ induces a mapping $\zeta_{\mathcal{A}} : \mathcal{P}(A^+ \times A^+) \longrightarrow \mathcal{P}(B^+ \times B^+)$ defined by

$$Y\zeta_{\mathcal{A}} = \{(w, z) \in B^+ \times B^+ \ : \ (\exists (u, v) \in Y)(w \in u\eta_{\mathcal{A}} \ \& \ z \in v\eta_{\mathcal{A}})\}.$$

The next lemma asserts that, under certain conditions, this mapping also preserves regularity.

Lemma 6.3 *Let $\mathcal{A} = (Q, A, B, \mu, q_0, T)$ be a* gsm*, and let $\pi_A : (A^* \times A^*)\delta_A \longrightarrow A^* \times A^*$ be the inverse of δ_A. Suppose that there is a constant C such that for any two paths α_1, α_2 in $\mathcal{A}$, we have*

$$|\Phi(\alpha_1)| = |\Phi(\alpha_2)| \Longrightarrow ||\Sigma(\alpha_1)| - |\Sigma(\alpha_2)|| \leq C. \qquad (3)$$

If $M \subseteq (A^+ \times A^+)\delta_A$ is a regular language in $A(2,\$)^+$ then $N = M\pi_A\zeta_{\mathcal{A}}\delta_B$ is a regular language in $B(2,\$)^+$.

Also the following simple fact, from [9], will be used in our proof.

Lemma 6.4 *Let S be an automatic semigroup such that $S^2 = S$. Then S has an automatic structure with uniqueness (A, K) such that $K \cap A = \emptyset$.*

Proof of Theorem 6.2. We assume, without loss of generality, that $T = \{t_1, \ldots, t_m\}$ with $m > 1$. By using Proposition 6.1 we know that S is finitely generated and $S^2 = S$. So, by Theorem 3.2, we conclude that the direct product $S^{|T|}$ is automatic. Let (F, K) be an automatic structure for $S^{|T|}$ with uniqueness with $F = \{f_1, \ldots, f_k\}$. Since $S^2 = S$, we can use Lemma 6.4, and assume that K does not have words of length 1. Given $t \in T$, using again Proposition 6.1, there is a right identity $e \in T$ such that $t = eq$ for some $q \in T$. So we can define a generating set

$$Y = \{e_1, \ldots, e_m\} \cup \{q_1, \ldots, q_m\}$$

for T such that $t_i = e_i q_i$ for $i = 1, \ldots, m$ and $e_1, \ldots, e_m$ represent (not necessarily distinct) right identities in T. We define a new alphabet A by

$$A = \{(f, e_i) : f \in F, i = 1, \ldots, m\} \cup \{(f, q_i) : f \in F, i = 1, \ldots, m\}$$

and a language L on A by

$$L = \bigcup_{i=1,\ldots,m} \{(f_{\alpha_1}, e_i) \ldots (f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i) : f_{\alpha_1} \ldots f_{\alpha_n} \in K\}.$$

We will prove that the pair (A, L) is an automatic structure for $S \operatorname{wr} T$ (with uniqueness). To see that A generates $S \operatorname{wr} T$ and that L is a set of unique representatives for $S \operatorname{wr} T$ we observe that, given $(f, t_i) \in S \operatorname{wr} T$ there is only one word $f_{\alpha_1} \ldots f_{\alpha_n}$ in K such that $f = f_{\alpha_1} \ldots f_{\alpha_n}$. So there is only one word in L representing (f, t_i) which is

$$(f_{\alpha_1}, e_i) \ldots (f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i).$$

To prove that L is a regular language we now define a gsm $\mathcal{A}$ such that $K\eta_{\mathcal{A}} = L$. Let

$$\mathcal{A} = (Q, F, A, \mu, q_0, \{\chi\})$$

with $Q = \{q_0, \dots, q_m\} \cup \{\chi\}$, where q_0 is the initial state, χ is the only final state and μ is a partial function from $Q \times F$ to finite subsets of $Q \times A^+$ defined by:

$$\begin{aligned}(q_0, f)\mu &= \{(q_i, (f, e_i))\} \ (i = 1, \dots, m),\\ (q_i, f)\mu &= \{(q_i, (f, e_i)), (\chi, (f, q_i))\} \ (i = 1, \dots, m).\end{aligned}$$

We will now prove that $L_{(f,e_r)}$ is a regular language, for $(f, e_r) \in A$. If we define

$$L^{(i)}_{(f,e_r)} = L_{(f,e_r)} \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+)\delta_A \ (i = 1, \dots, m)$$

then we can write

$$L_{(f,e_r)} = \bigcup_{i=1,\dots,m} L^{(i)}_{(f,e_r)}$$

and it suffices to prove that, for each $i \in \{1, \dots, m\}$, the language $L^{(i)}_{(f,e_r)}$ is regular. To achieve that, we will use Lemma 6.3. We start by showing that

$$L^{(i)}_{(f,e_r)} = K_{\bar{w}}\pi_F\zeta_{\mathcal{A}}\delta_A \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+ \cdot \{(f, q_i) : f \in F\})\delta_A$$

where $\bar{w}$ is the word in K that represents ${}^{q_i}f \in S^{|T|}$. Let

$$(f_{\alpha_1}, e_i) \dots (f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_j) \dots (f_{\beta_{s-1}}, e_j)(f_{\beta_s}, q_j) \in L.$$

Then

$$\begin{aligned}
&((f_{\alpha_1}, e_i) \dots (f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_j) \dots (f_{\beta_{s-1}}, e_j)(f_{\beta_s}, q_j))\delta_A \in L^{(i)}_{(f,e_r)}\\
\Leftrightarrow\ & f_{\alpha_1} \dots f_{\alpha_n}\, {}^{q_i}f = f_{\beta_1} \dots f_{\beta_s} \ \&\ e_i q_i e_r = e_j q_j\\
\Leftrightarrow\ & f_{\alpha_1} \dots f_{\alpha_n}\, {}^{q_i}f = f_{\beta_1} \dots f_{\beta_s} \ \&\ e_i q_i = e_j q_j\\
\Leftrightarrow\ & f_{\alpha_1} \dots f_{\alpha_n}\, {}^{q_i}f = f_{\beta_1} \dots f_{\beta_s} \ \&\ t_i = t_j\\
\Leftrightarrow\ & (f_{\alpha_1} \dots f_{\alpha_n}, f_{\beta_1} \dots f_{\beta_s})\delta_F \in K_{\bar{w}} \ \&\ i = j\\
\Leftrightarrow\ & ((f_{\alpha_1}, e_i) \dots (f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_j) \dots (f_{\beta_{s-1}}, e_j)(f_{\beta_s}, q_j))\delta_A \in\\
& K_{\bar{w}}\pi_F\zeta_{\mathcal{A}}\delta_A \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+ \cdot \{(f, q_i) : f \in F\})\delta_A
\end{aligned}$$

We conclude, by Lemma 6.3, that $L^{(i)}_{(f,e_r)}$ is a regular language. For a generator $(f, q_r) \in A$ will we prove that $L_{(f,q_r)}$ is regular in a similar way. We can write

$$L_{(f,q_r)} = \bigcup_{i=1,\dots,m} L^{(i)}_{(f,q_r)}$$

where

$$L^{(i)}_{(f,q_r)} = L_{(f,q_r)} \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+)\delta_A \ (i = 1, \dots, m).$$

We let $i \in \{1, \dots, m\}$ arbitrary and we will prove that $L^{(i)}_{(f,q_r)}$ is a regular language. Let j the unique element in $\{1, \dots, m\}$ such that $e_i q_i q_r = e_j q_j$ and let $\bar{w}$ be the word in K that represents ${}^{q_i}f \in S^{|T|}$. Let

$$(f_{\alpha_1}, e_i) \dots (f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_k) \dots (f_{\beta_{s-1}}, e_k)(f_{\beta_s}, q_k) \in L.$$

Then

$$
\begin{aligned}
&((f_{\alpha_1}, e_i)\dots(f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_k)\dots(f_{\beta_{s-1}}, e_k)(f_{\beta_s}, q_k))\delta_A \in L^{(i)}_{(f,q_r)}\\
\Leftrightarrow\ & f_{\alpha_1}\dots f_{\alpha_n}\, {}^{q_i}f = f_{\beta_1}\dots f_{\beta_s} \ \&\ e_i q_i q_r = e_k q_k\\
\Leftrightarrow\ & (f_{\alpha_1}\dots f_{\alpha_n}, f_{\beta_1}\dots f_{\beta_s})\delta_F \in K_{\bar{w}} \ \&\ e_i q_i q_r = e_k q_k \ \&\ k = j\\
\Leftrightarrow\ & ((f_{\alpha_1}, e_i)\dots(f_{\alpha_{n-1}}, e_i)(f_{\alpha_n}, q_i), (f_{\beta_1}, e_k)\dots(f_{\beta_{s-1}}, e_k)(f_{\beta_s}, q_k))\delta_A \in\\
& K_{\bar{w}}\pi_F\zeta_{\mathcal{A}}\delta_A \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+ \cdot \{(f, q_j) : f \in F\})\delta_A
\end{aligned}
$$

We can use again Lemma 6.3 to conclude that, for each i, the language

$$
L^{(i)}_{(f,q_r)} = K_{\bar{w}}\pi_F\zeta_{\mathcal{A}}\delta_A \cap (A^+ \cdot \{(f, q_i) : f \in F\} \times A^+ \cdot \{(f, q_j) : f \in F\})\delta_A
$$

is regular. ■

In the case where the semigroups S and T are monoids, necessary and sufficient conditions for the wreath product $S \operatorname{wr} T$ to be finitely generated are given in [24].

Proposition 6.5 *Let S and T be monoids, and let G be the group of units of T. Then the wreath product $S \operatorname{wr} T$ is finitely generated if and only if both S and T are finitely generated, and either S is trivial, or $T = VG$ for some finite subset V of T.*

By using this result, our theorem has the following consequence:

Corollary 6.6 *Let S be an automatic monoid and T be a finite monoid. Then the wreath product $S \operatorname{wr} T$ is automatic.*

Proof. We assume that S is not trivial. We can apply Proposition 6.5, with $V = T$, and so $S \operatorname{wr} T$ is finitely generated. Moreover, the three conditions in Proposition 6.1, for the case where the diagonal S-act is not finitely generated, hold trivially since S and T are monoids. The proof of our theorem is based on these conditions and therefore the wreath product $S \operatorname{wr} T$ is automatic. ■

It is still an open question whether or not the wreath product $S \operatorname{wr} T$ is automatic when it is finitely generated. Of course, from the above result, it only remains to consider the case where the diagonal S-act is finitely generated. In [24] and [27] we can find some examples of wreath products with finitely generated diagonal S-act which, as the authors observe, is in some way the less common case. Another interesting problem is that of the automaticity of the wreath product in the case where the semigroup T is also infinite. A natural starting point here is to use Proposition 6.5 and investigate the case where S and T are monoids.

References

[1] I. M. Araújo and N. Ruškuc, *Finite presentability of Bruck-Reilly extensions of groups*, J. Algebra **242** (2001), 20–30.

[2] G. Baumslag, S. M. Gersten, M. Shapiro, and H. Short, *Automatic groups and amalgams*, J. Pure Appl. Algebra **76** (1991), 229–316.

[3] R. H. Bruck, *A survey of binary systems*, volume 20 of Ergebnisse der Math., Neue Folge, Springer, Berlin, 1958.

[4] S. Bulman-Fleming and K. McDowell, *Solution: Problem E3311*, Amer. Math. Monthly **97** (1990), 617.

[5] C. M. Campbell, E. F. Robertson, N. Ruškuc, and R. M. Thomas, *Direct products of automatic semigroups*, J. Austral. Math. Soc. Ser. A **69** (2000), 19–24.

[6] ———, *Automatic semigroups*, Theoretical Computer Science **250** (2001), 365–391.

[7] C.M. Campbell, E.F. Robertson, N. Ruskuc, and R.M. Thomas, *Automatic completely-simple semigroups.*, Acta Math. Hung. **95** (2002), no. 3, 201–215.

[8] J. W. Cannon, D. B. A. Epstein, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston, *Word processing in groups*, Jones and Bartlett Publishers, 1992.

[9] L. Descalço, *Automatic semigroups: Constructions and subsemigroups*, Ph.D. thesis, University of St Andrews, 2002.

[10] L. Descalço and N. Ruškuc, *On automatic Rees matrix semigroups*, Comm. Algebra **30** (2002), 1207–1226.

[11] ———, *Subsemigroups of the bicyclic monoid*, Internat. J. Algebra Comput. **15** (2005), 37–57.

[12] A. J. Duncan, E. F. Robertson, and N. Ruškuc, *Automatic monoids and change of generators*, Math. Proc. Cambridge Philos. Soc. **127** (1999), 403–409.

[13] M. Hoffmann, *Automatic semigroups*, Ph.D. thesis, University of Leicester, 2001.

[14] M. Hoffmann, N. Ruškuc, and R. M. Thomas, *Automatic semigroups with subsemigroups of finite rees index*, Internat. J. Algebra Comput **12** (2002), 463–476.

[15] M. Hoffmann and R. M. Thomas, *Automaticity and commutative semigroups*, Glasgow J. Math **44** (2002), 167–176.

[16] J. E. Hopcroft and J. D. Ullman, *Introduction to automata theory, languages, and computation*, Addison-Wesley, 1979.

[17] W. D. Munn, *On simple inverse semigroups*, Semigroup Forum **1** (1970), 63–74.

[18] F. Otto, *On s-regular prefix-rewriting systems and automatic structures*, Computing and combinatorics (Tokyo, 1999), Lecture Notes in Comput. Sci., 1627, Springer: Berlin, 1999, pp. 422–431.

[19] ______, *On Dehn functions of finitely presented bi-automatic monoids*, J. Autom. Lang. Comb. **5** (2000), 405–419.

[20] F. Otto, A. Sattler-Klein, and K. Madlener, *Automatic monoids versus monoids with finite convergent presentations*, Rewriting Techniques and Applications (Tsukuba, 1998), Lecture Notes in Comput. Sci., 1379, Springer: Berlin, 1998, pp. 32–46.

[21] N. R. Reilly, *Bisimple ω-semigroups*, Proc. Glasgow Math. Assoc. **7** (1966), 160–167.

[22] E. F. Robertson, N. Ruškuc, and M. R. Thomson, *On diagonal acts of monoids*, Bull. Austral. Math. Soc. **63** (2001), 167–175.

[23] ______, *On finite generation and other finiteness conditions for wreath products of semigroups*, Comm. Algebra **30** (2002), 3851–3873.

[24] ______, *Finite generation and presentability of wreath products of monoids*, J. Algebra **266** (2003), 382–392.

[25] E. F. Robertson, N. Ruškuc, and J. Wiegold, *Generators and relations of direct products of semigroups*, Trans. Amer. Math. Soc. **350** (1998), 2665–2685.

[26] P. V. Silva and B. Steinberg, *A geometric characterization of automatic monoids*, Q. J. Math **55** (2004), 333–356.

[27] R. Thomson, *Finiteness conditions of wreath products of semigroups and related properties of diagonal acts*, Ph.D. thesis, University of St Andrews, 2001.