Universidade de Aveiro
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Departamento de Matemática

João José Neves Silva Xarez

A factorização monótona-leve das categorias via pré-ordens e ordens parciais

The monotone-light factorization for categories via preordered and ordered sets

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The monotone-light factorization for categories via preordered and ordered sets

dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Dr. George Janelidze, Professor Catedrático, A. Razmadze Mathematical Institute, Georgian Academy of Sciences

o júri

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resumo

Neste trabalho provamos que as adjunções **Cat** ? **Preord** e **Cat** ? **Ord** da categoria de todas as categorias na categoria das pré-ordens e na das ordens, respectivamente, determinam ambas distintos sistemas de factorização "monotone-light" em **Cat**.
Caracterizamos para as duas adjunções acima os morfismos de cobertura trivial, os de cobertura, os verticais, os verticais estáveis, os separáveis, os puramente inseparáveis, os normais e os dissonantes. Daqui se segue que os sistemas de factorização "monotone-light", concordante-dissonante e inseparável-separável em **Cat** coincidem para a adjunção **Cat** ? **Preord**.

abstract

It is shown that the reflections **Cat** ? **Preord** and **Cat** ? **Ord** of the category of all categories into the category of preorders and orders, respectively, determine both distinct monotone-light factorization systems on **Cat**.

We give explicit descriptions of trivial coverings, coverings, vertical, stably-vertical, separable, purely inseparable, normal and dissonant morphisms with respect to those two reflections. It follows that the monotone-light, concordant-dissonant and inseparable-separable factorizations on **Cat** do coincide in the reflection **Cat** ? **Preord**.

Contents

Introduction

Monotone-light factorization of morphisms in an abstract category $\mathbb{C}$, with respect to a full reflective subcategory $\mathbb{X}$, was studied by A. Carboni, G. Janelidze, G. M. Kelly, and R. Paré in [**3**].

According to [**3**], the existence of such factorization requires strong additional conditions on the reflection $\mathbb{C} \to \mathbb{X}$, which hold in the (Galois theory of the) adjunction between compact Hausdorff and Stone spaces, needed to make the classical monotone-light factorization of S. Eilenberg (cf. [**5**]) and G. T. Whyburn (cf. [**16**]) a special case of the categorical one.

In fact, A. Carboni and R. Paré studied in categorical terms the classical monotone-light factorization, relating this factorization system with the reflection **CompHaus** $\to$ **Stone** of compact Hausdorff spaces into Stone spaces. But the connection between adjunctions and factorization systems was already known by M. Kelly (from [**4**]), and surprisingly the (reflective) factorization system associated to **CompHaus** $\to$ **Stone** did not coincide with the one of Carboni and Paré.

The connection between the two distinct factorization systems was to be made by the categorical Galois theory of G. Janelidze: the right-hand class of the former factorization system $(\mathcal{E}', \mathcal{M}^*)$ is the class of all *coverings*, while the right-hand class of the latter $(\mathcal{E}, \mathcal{M})$ only includes the *trivial coverings*.

There are few known similar situations where there is a (categorical) monotone-light factorization. The "purely inseparable-separable" factorization for field extensions is an example, where the associated reflection is the one from the opposite category of finite dimensional algebras over a fixed field to the category of finite sets. There is also another example involving torsion theory (cf. [**3**]).

We showed that monotone-light factorization also does exist for $\mathbb{C} = $ **Cat**, the category of all categories, and $\mathbb{X}$ being either the category **Ord** of (partially) ordered sets, or the category **Preord** of preorders (cf. [**17**]). A crucial observation here is that the reflections **Cat** $\to$ **Ord** and **Cat** $\to$ **Preord** have stable units in the sense of [**3**]. This gives two different factorization systems on **Cat**, and it turns out that the light morphisms (=coverings) with respect to **Preord** are precisely the faithful functors. Therefore Galois theories of categories via orders and preorders are much richer, i.e., have more covering morphisms, than the "standard" one (via sets, regarded as discrete categories).

We will also give explicit descriptions of normal, separable, and other types of morphisms that occur in Galois theory of categories via orders and preorders.

In this way, we were able to conclude that the monotone-light factorization for **Cat** $\to$ **Preord** coincides with the concordant-dissonant and the inseparable-separable factorization, in the sense of G. Janelidze and W. Tholen in [**12**].

This also occurs for the reflection **Preord** $\to$ **Ord** of preorders into orders, which is of course just a simplified version of the reflection of categories into orders, where it does not happen.

So, we have that the composite of two well-behaved adjunctions **Cat** $\to$ **Preord** $\to$ **Ord** is not so well behaved.

Finally, we end our work by studying the case of the reflection of categories into sets, which can be thought as an extension of the previous reflections:
Cat $\to$ **Preord** $\to$ **Ord** $\to$ **Set**.

For this last adjunction we will prove that it also has stable units but it does not have a monotone-light factorization system.

The reason for this failure is that the effective descent morphisms in **Cat** cannot be chosen anymore so that their domain belongs to the "smaller" category, as it was the case for the reflections into preorders and orders, where we could choose effective descent morphisms over any category with domain an ordered set.

Indeed, a functor from a set to a category which is not discrete cannot be an effective descent morphism.

CHAPTER 1

Basic concepts of categorical Galois theory

In this first chapter we put together some well-known notions and results of categorical Galois theory and factorization systems to which we will refer in the next. We have tried to choose these in the most economic way for our purposes.

For the sake of completeness, we also give some proofs in this chapter, whenever these are not too long or too complicated.

1.1. Factorization systems

DEFINITION 1.1. A weak factorization on the category $\mathbb{C}$ is a functor $F : \mathbb{C}^{\mathbf{2}} \to \mathbb{C}$ such that $F \cdot E = 1_{\mathbb{C}}$, with $E : \mathbb{C} \to \mathbb{C}^{\mathbf{2}}$ the canonical embedding of $\mathbb{C}$ into its category of arrows.

For a morphism $\alpha : A \to B$ in $\mathbb{C}$, if we apply F to the general factorization $(\alpha, 1_B) \cdot (1_A, \alpha)$ of $E(\alpha) = (\alpha, \alpha) : 1_A \to 1_B$, we obtain the decomposition $\alpha = m_\alpha \cdot e_\alpha$.

With respect to this decomposition we define two classes of morphisms in $\mathbb{C}$:

- $\mathcal{E} := \{\alpha \mid m_\alpha \textit{ iso}\}$ is the class of all morphisms $\alpha : A \to B$ such that in the above decomposition m_α is an isomorphism;
- $\mathcal{M} := \{\alpha \mid e_\alpha \textit{ iso}\}$ is the class of all morphisms $\alpha : A \to B$ for which e_α is an isomorphism.

A factorization system is a weak factorization such that in every decomposition $\alpha = m_\alpha \cdot e_\alpha$ the morphisms m_α and e_α belong always to $\mathcal{M}$ and $\mathcal{E}$, respectively.

The above definition of a factorization system was presented in [**13**]. There it was also proved that this is equivalent to the more usual definition as a *prefactorization system* $(\mathcal{E}, \mathcal{M})$ on a category $\mathbb{C}$, such that every morphism in $\mathbb{C}$ can be written as $m \cdot e$ with m in $\mathcal{M}$ and e in $\mathcal{E}$ (cf. [**3**] and [**12**]).

Notice that a factorization system is completely determined by any of the two classes in the pair $(\mathcal{E}, \mathcal{M})$ (cf. [**3**, §2.8]).

1.2. Vertical morphisms and trivial coverings

Consider the adjunction

$$(I, H, \eta, \epsilon) : \mathbb{C} \to \mathbb{X}, \tag{1.1}$$

in which $\mathbb{C}$ is a finitely-complete category.

For each object B in $\mathbb{C}$, consider also the induced adjunction

$$(I^B, H^B, \eta^B, \epsilon^B) : \mathbb{C}/B \to \mathbb{X}/I(B), \tag{1.2}$$

from the category of objects over B in $\mathbb{C}$ to the category of objects over $I(B)$ in $\mathbb{X}$, whose right adjoint H^B sends (X, φ) to the pullback of $(H(X), H(\varphi))$ along the unit morphism η_B, and whose left adjoint simply sends (A, α) to $(I(A), I(\alpha))$.

At last, consider the commutative diagram

$$\begin{tikzcd}
A \arrow[dr, "e_\alpha"] \arrow[drr, "\eta_A"] \arrow[ddr, "\alpha"'] & & \\
& C \arrow[r, "d_\alpha"] \arrow[d, "m_\alpha"] & HI(A) \arrow[d, "HI(\alpha)"] \\
& B \arrow[r, "\eta_B"] & HI(B)
\end{tikzcd} \tag{1.3}$$

whose square part is a pullback and which defines a weak factorization on $\mathbb{C}$.

Definition 1.2. With respect to the weak factorization of diagram 1.3 above, we call vertical to the morphisms in $\mathcal{E}$, i.e., to the morphisms $\alpha : A \to B$ in $\mathbb{C}$ characterized by the existence of a morphism d such that

$$\begin{tikzcd}
B \arrow[r, "d"] \arrow[d, "1_B"] & HI(A) \arrow[d, "HI(\alpha)"] \\
B \arrow[r, "\eta_B"] & HI(B)
\end{tikzcd} \tag{1.4}$$

is a pullback diagram.

The following lemma will be useful in characterizing vertical morphisms for our specific adjunctions in the next chapters. It plays also an important role in proving some statements that are crucial for our purposes, later on in this chapter.

Lemma 1.3. *If in the adjunction 1.1 the right adjoint H is fully faithful, i.e., the counit $\varepsilon : IH \to 1$ is an iso*[1]*, a morphism $\alpha : A \to B$ in $\mathbb{C}$ is vertical if and only if $I(\alpha)$ is an isomorphism.*

In particular, every unit morphism η_C is vertical.

[1]Cf. [**14**, §IV.3].

PROOF. If $I(\alpha)$ is an isomorphism then $HI(\alpha)$ is an isomorphism and therefore α is vertical.

On the other hand, if α is vertical then, applying the functor I to the pullback diagram 1.4, we obtain $IHI(\alpha) \cdot I(d) = I(\eta_B)$ and $I(d) \cdot I(\alpha) = I(\eta_A)$.

As for any other adjunction, the triangular identity $\varepsilon_{I(C)} \cdot I(\eta_C) = 1_{I(C)}$ holds for every object C in $\mathbb{C}$. So, being the counit $\varepsilon : IH \to 1$ an iso by hypothesis, $I(\eta_A)$ and $I(\eta_B)$ are isomorphisms. Which implies that $I(\alpha) \cong IHI(\alpha)$ is also an isomorphism.

□

DEFINITION 1.4. With respect to the weak factorization given in diagram 1.3, we call trivial coverings to the morphisms in $\mathcal{M}$, i.e., to the morphisms $\alpha : A \to B$ in $\mathbb{C}$ for which

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "\alpha"] & HI(A) \arrow[d, "HI(\alpha)"] \\ B \arrow[r, "\eta_B"] & HI(B) \end{tikzcd} \qquad (1.5)$$

is a pullback square.

In other words, a morphism $\alpha : A \to B$ is a trivial covering if and only if the unit morphism $\eta^B_{(A,\alpha)}$ of the induced adjunction 1.2 is an isomorphism. Indeed, one easily checks that $\eta^B_{(A,\alpha)} = e_\alpha : (A, \alpha) \to (C, m_\alpha)$ (cf. diagram 1.3).

LEMMA 1.5. *For the adjunction 1.1, when the right adjoint H is fully faithful, a morphism with codomain of the form $H(X)$ is a trivial covering if and only if it is up to an iso of the form $H(\varphi)$, for some morphism φ in $\mathbb{X}$.*

PROOF. As for any other adjunction, the triangular identity $H(\varepsilon_X) \cdot \eta_{H(X)} = 1_{H(X)}$ holds for every object X in $\mathbb{X}$. So, as in this case the counit is an iso, $\eta_{H(X)}$ is an isomorphim for every object X in $\mathbb{X}$. The proof of the lemma follows trivially from this fact.

□

1.3. Admissibility

DEFINITION 1.6. The adjunction 1.1 is said to be admissible if all the induced adjunctions 1.2 have fully faithful right adjoints, that is, for every object B in $\mathbb{C}$ the counit $\varepsilon^B : I^B H^B \to 1$ is an iso.

In the specific case when also the right adjoint H is fully faithful, that is,

$\varepsilon : IH \to 1$ is an iso, the adjunction 1.1 is admissible[2] if and only if in every pullback diagram of the form

$$\begin{tikzcd} B \times_{HI(B)} H(X) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & H(X) \arrow[d, "H(\varphi)"] \\ B \arrow[r, "\eta_B"] & HI(B) \end{tikzcd} \tag{1.6}$$

the projection π_2 is in $\mathcal{E}$. That this is so follows immediately from Lemma 1.3 and $\varepsilon^B_{(X,\varphi)} = \varepsilon_X \cdot I(\pi_2)$, where π_2 is the projection at the pullback diagram 1.6 above.

Lemma 1.7. *If the adjunction 1.1 is admissible then, for every object B in $\mathbb{C}$, the functor H^B in the adjunction 1.2 induces an equivalence $\mathbb{X}/I(B) \sim \mathcal{M}/B$, between the category of objects over $I(B)$ in $\mathbb{X}$ and the full subcategory of $\mathbb{C}/B$ whose objects are the trivial coverings over B.*

Proof. As for any adjunction such that its counit is an iso, $\eta^B_{(A,\alpha)}$ is an isomorphism if and only if there exists an object (X, φ) in $X/I(B)$ with $(A, \alpha) \cong H^B(X, \varphi)$. That is, being fully faithful, H^B induces an equivalence between the comma categories in the statement.

□

Given a morphism $p : E \to B$ in $\mathbb{C}$, consider another adjunction

$$p! \dashv p^* : \mathbb{C}/B \to \mathbb{C}/E, \tag{1.7}$$

in which the right adjoint p^* sends an object (A, α) over B to the pullback of α along the morphism p, and $p!(D, \delta) = p \cdot \delta$.

Lemma 1.8. *If the adjunction 1.1 is admissible then the class $\mathcal{M}$ of trivial coverings is pullback-stable.*

Proof. Consider the two diagrams:

$$\begin{tikzcd} \mathbb{C}/E \arrow[r, "I^E"] \arrow[d, "p!"] & \mathbb{X}/I(E) \arrow[d, "I(p)!"] \\ \mathbb{C}/B \arrow[r, "I^B"] & \mathbb{X}/I(B) \end{tikzcd} \tag{1.8}$$

[2] Also called *semi-left-exact* in [**4**].

$$
\begin{tikzcd}
\mathbb{C}/E & \mathbb{X}/I(E) \arrow[l, "H^E"'] \\
\mathbb{C}/B \arrow[u, "p^*"'] & \mathbb{X}/I(B) \arrow[l, "H^B"'] \arrow[u, "I(p)^*"']
\end{tikzcd}
\qquad (1.9)
$$

where $p : E \to B$ is a general morphism in $\mathbb{C}$.

The diagram 1.8 obviously commutes, and since the diagram 1.9 is obtained from 1.8 by replacing all arrows with their right adjoints, it commutes too, up to a canonical isomorphism.

Then, as H^B and H^E were seen above in Lemma 1.7 to induce equivalences respectively into $\mathcal{M}/B$ and $\mathcal{M}/E$, diagram 1.9 tells us that p^* carries trivial coverings to trivial coverings.

□

PROPOSITION 1.9. *If the adjunction 1.1 is admissible and its counit* $\varepsilon : IH \to 1$ *is an iso then* $(\mathcal{E}, \mathcal{M})$ *is a factorization system.*

PROOF. We have to show that in diagram 1.3 the morphism m_α is a trivial covering and e_α is vertical. The former assertion follows from Lemma 1.8 and Lemma 1.5. The latter from $I(d_\alpha) \cdot I(e_\alpha) = I(\eta_A)$, Lemma 1.3 and admissibility.

□

1.4. Stable units

DEFINITION 1.10. The reflection 1.1 is said to have stable units[3] when the functor $I : \mathbb{C} \to \mathbb{X}$ preserves every pullback of the form

$$
\begin{tikzcd}
A \times_{H(X)} B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "\beta"] \\
A \arrow[r, "\alpha"] & H(X)
\end{tikzcd}
\quad . \qquad (1.10)
$$

We shall show that the specific full reflections we are about to study have all stable units. Hence, next proposition tells that they will always have an $(\mathcal{E}, \mathcal{M})$-factorization system associated, and that we can apply categorical Galois theory to them, since there is admissibility (cf. [**3**] and [**7**]).

PROPOSITION 1.11. *If the adjunction 1.1 has stable units and its counit* $\varepsilon :$ $IH \to 1$ *is an iso then it is admissible.*

[3] In the sense of [**4**] and [**3**].

PROOF. If we apply the functor I to the pullback diagram 1.6 we still obtain a pullback diagram, since the adjunction has stable units. Moreover, by Lemma 1.3 the bottom row $I(\eta_B)$ is then an isomorphism, since the counit is an iso. Hence, $I(\pi_2)$ must also be an isomorphism, i.e., the projection π_2 is in $\mathcal{E}$.

□

1.5. Monotone-light factorization

DEFINITION 1.12. The morphism $p : E \to B$ in $\mathbb{C}$ is said to be an effective descent morphism or (E, p) is called a monadic extension of B if the pullback functor $p^* : \mathbb{C}/B \to \mathbb{C}/E$ is monadic, i.e., the comparison functor associated to the adjunction 1.7 is an equivalence of categories (cf. [**14**, §VI.3]).

We give in the example 1.13 just below the only two characterizations of monadic extensions that will be needed in our work.

EXAMPLE 1.13. The effective descent morphisms in the category of preordered sets **Preord** are known to be the morphisms $p : E \to B$ such that for every $b_2 \to b_1 \to b_0$ in B there exists $e_2 \to e_1 \to e_0$ in E with $p(e_i) = b_i$, for $i = 0, 1, 2$ (cf. [**10**]).

Consider, for instance, the following functor $p : 3 \cdot \mathbf{3} \to \mathbf{5}$, which is the obvious projection from the coproduct of three copies of the ordinal number $\mathbf{3}$ to the ordinal number $\mathbf{5}$:[4]

$$\begin{tikzcd}
 & & 4 & & 4 \\
 & 3 & 3 \arrow[u] & & 3 \arrow[u] \\
2 & 2 \arrow[u] & 2 \arrow[u] \arrow[rr, "p"] & & 2 \arrow[u] \\
1 \arrow[u] & 1 \arrow[u] & & & 1 \arrow[u] \\
0 \arrow[u] & & & & 0 \arrow[u]
\end{tikzcd}$$

And the effective descent morphisms in **Cat** are known to be the functors surjective on composable triples of morphisms (cf. [**11**]).

Consider, for instance, the following functor $p : 2 \cdot \mathbf{4} \to \mathbf{5}$, which is the obvious projection from the coproduct of two copies of the ordinal number $\mathbf{4}$ to the ordinal number $\mathbf{5}$:

[4] Where the obvious composite and identity morphisms are not represented for the sake of simplicity.

$$\begin{tikzcd}
 & 4 & & 4 \\
3 & 3 \arrow[u] & & 3 \arrow[u] \\
2 \arrow[u] & 2 \arrow[u] \arrow[rr, "p"] & & 2 \arrow[u] \\
1 \arrow[u] & 1 \arrow[u] & & 1 \arrow[u] \\
0 \arrow[u] & & & 0 \arrow[u]
\end{tikzcd}$$

DEFINITION 1.14. We define two new classes of morphisms with respect to the adjunction 1.1:

- $\mathcal{E}'$ is the class of stably-vertical morphisms, i.e., of all morphisms $\alpha : A \to B$ in $\mathbb{C}$ such that every pullback of α is in $\mathcal{E}$ ($\mathcal{E}'$ is therefore the largest pullback-stable class contained in $\mathcal{E}$);
- $\mathcal{M}^*$ is the class of all coverings, i.e., of all morphisms $\alpha : A \to B$ in $\mathbb{C}$ such that some pullback $p^*(A, \alpha)$ of α along a monadic extension (E, p) of B is in $\mathcal{M}$.

It is immediate from the following proposition that the defined coverings are stable under pullbacks whenever trivial coverings are, for instance, when there is admissibility (cf. Lemma 1.8).

This proposition is very simple to prove if one knows that effective descent morphisms are pullback-stable in any finitely-complete category, a result presented in [**15**] (cf. [**3**, §6.1]).

PROPOSITION 1.15. *For any pullback-stable class $\mathcal{N}$ of morphisms in a finitely-complete category $\mathbb{C}$, the class $\mathcal{N}^*$ of all morphisms $\alpha : A \to B$ in $\mathbb{C}$ such that some pullback $p^*(A, \alpha)$ of α along a monadic extension (E, p) of B is in $\mathcal{N}$, is again pullback-stable.*

Next, we have another proposition giving information about coverings in the stable-units case. Its proof is given in [**3**, §5.4].

PROPOSITION 1.16. *If the adjunction 1.1 has stable units then every covering morphism whith codomain of the form $H(X)$ is a trivial covering.*

When the pair $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system, it is said to arise by simultaneously *stabilizing* $\mathcal{E}$ and *localizing* $\mathcal{M}$. We will call *monotone-light* to the *stable* factorization systems obtained through this general process.

The following proposition was proved in [**3**, §6]. And the next theorem is the main result of that same paper. Notice that both the proposition and the theorem apply more generally to *any* factorization system $(\mathcal{E}, \mathcal{M})$ on a finitely-complete category $\mathbb{C}$.

PROPOSITION 1.17. *When the adjunction 1.1 is under the conditions of Proposition 1.9, the pair $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system if and only if every morphism $\alpha : A \to B$ in $\mathbb{C}$ has a factorization $\alpha = m^* \cdot e'$ with $m^* \in \mathcal{M}^*$ and $e' \in \mathcal{E}'$.*

We call the $(\mathcal{E}, \mathcal{M})$-factorization of $\alpha = me$ stable if $e \in \mathcal{E}'$. And locally stable if there is some monadic extension (E, p) of B for which the $(\mathcal{E}, \mathcal{M})$-factorization of the pullback $p^*(A, \alpha)$ of α along p is stable.

THEOREM 1.18. *When the adjunction 1.1 is under the conditions of Proposition 1.9, the pair $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system if and only if every $(\mathcal{E}, \mathcal{M})$-factorization is locally stable.*

1.6. The categorical form of fundamental theorem of Galois theory

We will assume at this section that the adjunction 1.1 is admissible and that the category $\mathbb{X}$ in it is finitely-complete.

The following two definitions collect all that is needed to state the categorical form of fundamental theorem of Galois theory.

DEFINITION 1.19.

(a) An *internal precategory* P in the category $\mathbb{X}$ is a diagram

$$\begin{tikzcd} P_2 \arrow[r, shift left=2, "p"] \arrow[r, "m"] \arrow[r, shift right=2, "q"'] & P_1 \arrow[r, shift left=2, "d"] & P_0 \arrow[l, "e"'] \arrow[l, phantom, shift right=2, "c"] \end{tikzcd} \tag{1.11}$$

in $\mathbb{X}$ with $de = 1 = ce$, $dp = cq$, $dm = dq$, and $cm = cp$.

The internal precategory P is called a *category* precisely when the square represented by the equation $dp = cq$ is a pullback and the "composition" morphism m satisfies the associativity and unit laws.

An internal category P in $\mathbb{X}$ is called a *groupoid*, or a *preorder*, or an *equivalence relation*, when the elementary formulations of these properties in terms of internal diagrams in $\mathbb{X}$ hold. Of course that an internal category is an equivalence relation precisely when it is both a groupoid and a preorder.

(b) A morphism $f : P \to Q$ of internal precategories in $\mathbb{X}$, given by components $f_i : P_i \to Q_i$, is said to be a *discrete opfibration* if each of the squares

$$\begin{tikzcd}
P_2 \arrow[r, "p"] \arrow[d, "f_2"] & P_1 \arrow[r, "d"] \arrow[d, "f_1"] & P_0 \arrow[d, "f_0"] \\
Q_2 \arrow[r, "p"'] & Q_1 \arrow[r, "d"'] & Q_0
\end{tikzcd} \tag{1.12}$$

is a pullback.

(c) Given an internal precategory Q in $\mathbb{X}$, we write $\mathbb{X}^Q$ for the full subcategory of the category $PreCat(\mathbb{X})/Q$, of internal precategories in $\mathbb{X}$ over Q, determined by the discrete opfibrations $P \to Q$.

For reasons clear to those familiar with internal category theory, $\mathbb{X}^Q$ is called the *category of internal actions of the precategory Q in $\mathbb{X}$*.

(d) The kernel-pair $(d, c) : E \times_B E \to E$ of any map $p : E \to B$ extends in an obvious way to an equivalence relation

$$\begin{tikzcd}
E \times_B E \times_B E \arrow[r, shift left=4, "p"] \arrow[r, "m"] \arrow[r, shift right=4, "q"'] & E \times_B E \arrow[r, shift left=4, "d"] \arrow[r, shift right=4, "c"'] & P_0 \arrow[l, "e"']
\end{tikzcd} \tag{1.13}$$

in $\mathbb{C}$.

Consider in particular that $p : E \to B$ is an effective descent morphism.

Then, the internal precategory $I(Eq(p))$ in $\mathbb{X}$, that is, the image under the left adjoint $I : \mathbb{C} \to \mathbb{X}$ of the internal equivalence relation associated to the kernel-pair of p, is to be called the *Galois precategory of the extension (E,p)* and denoted by $Gal(E, p)$.

Definition 1.20. Consider the monadic extension (E, p) in $\mathbb{C}$.

A morphism $\alpha : A \to B$ in $\mathbb{C}$ is said to be split over (E, p), if the pullback of α along p is a trivial covering of E: $p^*(A, \alpha) \in \mathcal{M}/E$.

The full subcategory of $\mathcal{M}^*/B$ determined by the morphisms in $\mathbb{C}$ split over (E, p) will be denoted by $Spl(E, p)$. In other words $Spl(E, p)$ is defined as the pullback

$$\begin{array}{ccc} Spl(E,p) & \longrightarrow & \mathcal{M}/E \\ \downarrow & & \downarrow \\ \mathbb{C}/B & \xrightarrow{p^*} & \mathbb{C}/E \end{array} \tag{1.14}$$

where the vertical arrows are inclusion functors.

The next simpler version (we restrict ourselves to the *absolute* case where all morphisms are considered) of the fundamental theorem of categorical Galois theory developed by Janelidze, will give us more information on the category $\mathbb{C}/B$ using the category $\mathbb{X}$ than Lemma 1.7. In effect, notice that the category $Spl(E,p)$ is larger than $\mathcal{M}/B$, since the trivial coverings are pullback-stable when there is admissibility.

THEOREM 1.21. *If the adjunction 1.1 is admissible, and not only the category $\mathbb{C}$ but also $\mathbb{X}$ has finite limits, and $p : E \to B$ is an effective descent morphism in $\mathbb{C}$, then there is an equivalence of categories*

$$Spl(E,p) \simeq \mathbb{X}^{Gal(E,p)}$$

from the category $Spl(E,p)$ of the morphisms over B in $\mathbb{C}$ split by (E,p), to the category of internal actions of the precategory $Gal(E,p) := I(Eq(p))$ in $\mathbb{X}$.

Confer [**3**, §5] for the proof of the fundamental theorem.

1.7. Locally semisimple coverings

At this section we reproduce the two crucial definitions and the central result of the paper [**9**], restricting ourselves to the *absolute* case where all morphisms in the category $\mathbb{C}$ are considered, and switching from regular epis to effective descent morphisms.

This last change is needed because we are going to work on the category of all small categories **Cat**, which is not a regular category.

DEFINITION 1.22. A class $\mathbb{X}_0$ of objects in a category $\mathbb{C}$ is said to be a *generalized semisimple class* if, for any pullback diagram

$$\begin{array}{ccc} E \times_B A & \xrightarrow{v} & A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow{p} & B \end{array} \tag{1.15}$$

in $\mathbb{C}$ in which p is an effective descent morphism, the following conditions hold:

(a) if E and A are in $\mathbb{X}_0$ so is $E \times_B A$,

(b) if B, E and $E \times_B A$ are in $\mathbb{X}_0$ then so is A.

Example 1.23. The preordered sets, the ordered sets and the sets constitute generalized semisimple classes of objects in the category of all categories **Cat**. And the ordered sets are in addition a generalized semisimple class of the category of preordered sets **Preord**.

It is easy to check that they obey the first condition of the immediately above Definition 1.22 .

The second condition holds since they correspond to reflections having stable units, as we shall show. Hence, being $u : E \times_B A$ a trivial covering by Lemma 1.5, α is a covering; which implies by Proposition 1.16 that α is trivial, and so again by Lemma 1.5 that it is of the form $H(\varphi)$ for some φ in $\mathbb{X}$.

Definition 1.24.

(a) A morphism $\alpha : A \to B$ in $\mathbb{C}$ is said to be a *semisimple covering* if $A, B \in \mathbb{X}_0$.

(b) A morphism $\alpha : A \to B$ is said to be a *locally semisimple covering* if there exists a pullback diagram like 1.15 at previous definition in which p is an effective descent morphism and $u : E \times_B A \to E$ is a semisimple covering; we will also say that (A, α) is *split over* (E, p).

Theorem 1.25. *Let $\mathbb{X}_0$ be a generalized semisimple class of objects in a category $\mathbb{C}$, and $p : E \to B$ be an effective descent morphism in $\mathbb{C}$ with $E \in \mathbb{X}_0$.*
And let $\mathbb{X}$ denote the full subcategory of $\mathbb{C}$ determined by the objects in $\mathbb{X}_0$.

Then, there exists a category equivalence

$$Cov(B) \simeq \mathbb{X}^{Eq(p)}$$

from the full subcategory of $\mathbb{C}/B$, determined by the locally semisimple coverings over B, to the category of internal actions of the internal equivalence relation $Eq(p)$, determined by the kernel pair of p, in $\mathbb{X}$.

1.8. Inseparable-separable factorization

Definition 1.26. Consider the following commutative diagram

$$
\begin{tikzcd}
A \arrow[dr, "\delta_\alpha"] \arrow[drr, "1_A"] \arrow[ddr, "1_A"'] & & \\
& A \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\
& A \arrow[r, "\alpha"] & B
\end{tikzcd}
\tag{1.16}
$$

where (u, v) is the kernel-pair of the morphism $\alpha : A \to B$.

Then, with respect to the adjunction 1.1,

- α is called a *separable* morphism if δ_α is a trivial covering;
- α is called a *purely inseparable* morphism if δ_α is vertical;
- α is called a *normal* morphism if $u : A \times_B A \to A$ is a trivial covering.

We will denote the classes of separable, purely inseparable and normal morphisms by Sep, Pin and $Normal$, respectively.

In the following two remarks the reader should refer to the diagram 1.16 at the previous definition.

Remark 1.27. Notice that every normal morphism is separable, since $u \cdot \delta_\alpha = 1_A$ and trivial coverings are weakly left cancellable (i.e., $g \cdot f, g \in \mathcal{M} \Rightarrow f \in \mathcal{M}$). And if trivial coverings are pullback stable then they are of course normal:

$$\mathcal{M} \subseteq Normal \subseteq Sep.$$

Remark 1.28. If in the adjunction 1.1 the right adjoint is fully faithful then every stably-vertical morphism is a purely inseparable one:

$$\mathcal{E}' \subseteq Pin.$$

Indeed, as $I(u) \cdot I(\delta_\alpha) = 1_{I(A)}$ and being the morphism $u : A \times_B A \to A$ vertical then by Lemma 1.3 the fibred product δ_α is vertical.

Throughout the rest of this section, let our finitely-complete $\mathbb{C}$ have coequalizers and suppose in addition that the adjunction 1.1 is under the conditions of Proposition 1.9.

For the factorization system $(\mathcal{E}, \mathcal{M})$ we can form a derived *weak* factorization system as follows:

$$
\begin{tikzcd}
 & {} \arrow[dr, "m_{\delta_\alpha}"] & & & {} \arrow[dr, "m^*_\alpha"] & \\
A \arrow[ur, "e_{\delta_\alpha}"] \arrow[rr, "\delta_\alpha"'] & & A \times_B A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & A \arrow[ur, "e'_\alpha"] \arrow[rr, "\alpha"'] & & B
\end{tikzcd}
\tag{1.17}
$$

Here, e'_α is just the coequalizer of $(u \cdot m_{\delta_\alpha}, v \cdot m_{\delta_\alpha})$, where (u, v) is the kernel-pair of α and $m_{\delta_\alpha} \cdot e_{\delta_\alpha}$ is of course the $(\mathcal{E}, \mathcal{M})$-factorization of the morphism δ_α of diagram 1.16.

Definition 1.29. A morphism $\alpha : A \to B$ is called inseparable with respect to the adjunction 1.1 if, in its decomposition given at diagram 1.17, m^*_α is an isomorphism.

So, we define the class of inseparable morphisms

$$Ins := \{\alpha \mid m^*_\alpha \ iso\}.$$

The proofs of next lemma and proposition can be found in [**12**, §3,§4]. Notice nevertheless that the first inclusion in the lemma is trivial.

Lemma 1.30. *For the factorization given at diagram 1.17 the following two inclusions and one equality hold, where RegEpi is the class of regular epimorphisms in $\mathbb{C}$:*

- $Pin \cap RegEpi \subseteq Ins \subseteq \mathcal{E} \cap RegEpi$;
- $Sep = \{\alpha \mid e'_\alpha \ iso\}$.

Proposition 1.31. *The pair (Ins, Sep) is a factorization system if and only if the class of morphisms Ins is closed under composition.*

The statement of the next proposition and its proof are given in [**11**]. Its following corollary will be used in proving that the two reflections of categories into preordered sets and of these into ordered sets have both inseparable-separable factorization systems.

Proposition 1.32. *A composite $\beta \cdot \alpha$ of regular epimorphisms in $\mathbb{C}$ is also a regular epi whenever α is a pullback stable epimorphism.*

In particular, the class of stably-regular epimorphisms in $\mathbb{C}$ is closed under composition. And if the stably-vertical morphisms are all epis, $\mathcal{E}' \subseteq Epi$, then the class of stably-vertical regular epimorphisms $\mathcal{E}' \cap RegEpi$ is closed under composition.

Example 1.33. It is known that the class of stably-regular epis in the category of all small categories **Cat** is the class of all functors surjective on composable pairs of morphisms.

Corollary 1.34. *If the counit $\varepsilon : IH \to 1$ is an iso, every stably-vertical morphism is an epi and every vertical regular epi is stably-vertical, $\mathcal{E} \cap RegEpi \subseteq$*

$\mathcal{E}' \subseteq Epi$, then there is an inseparable-separable factorization system (Ins, Sep) with $Ins = \mathcal{E}' \cap RegEpi = \mathcal{E} \cap RegEpi$.

PROOF. We know from Remark 1.28 that $\mathcal{E}' \subseteq Pin$. Therefore, it follows from Lemma 1.30 that

$$\mathcal{E}' \cap RegEpi \subseteq Ins \subseteq \mathcal{E} \cap RegEpi.$$

Then, the given hypothesis $\mathcal{E} \cap RegEpi \subseteq \mathcal{E}'$ implies that

$$Ins = \mathcal{E}' \cap RegEpi = \mathcal{E} \cap RegEpi.$$

As the stably-vertical morphisms are by hypothesis all epis, we conclude from the previous Proposition 1.32 that $Ins = \mathcal{E}' \cap RegEpi$ is closed under composition.

Which in turn implies by Proposition 1.31 that (Ins, Sep) is a factorization system.

□

1.9. Concordant-dissonant factorization

The next proposition is stated and proved in [**3**, §3.9].

PROPOSITION 1.35. *If $(\mathcal{F}, \mathcal{N})$ is a factorization system on $\mathbb{C}$, for which $\mathcal{F}$ is contained in the class of epimorphisms, then there is a factorization system $(\bar{\mathcal{E}}, \bar{\mathcal{M}})$ such that*

- *$\bar{\mathcal{E}} = \mathcal{E} \cap \mathcal{F}$ is the class of vertical morphisms in $\mathcal{F}$, and*
- *a morphism $\alpha : A \to B$ is in $\bar{\mathcal{M}}$ if and only if the following arrow $\langle \alpha, \eta_A \rangle : A \to B \times HI(A)$ with components $\alpha : A \to B$ and $\eta_A : A \to HI(A)$ is in $\mathcal{N}$:*

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[d, "\langle \alpha, \eta_A \rangle"] \arrow[dr, "\eta_A"] & \\ B & B \times I(A) \arrow[l] \arrow[r] & I(A) \end{tikzcd}$$

The $(\bar{\mathcal{E}}, \bar{\mathcal{M}})$-factorization of a morphism $\alpha : A \to B$ in $\mathbb{C}$ is then given by $(m_\alpha \cdot n) \cdot f$, with $n \cdot f$ the $(\mathcal{F}, \mathcal{N})$-factorization of e_α and $m_\alpha \cdot e_\alpha$ the $(\mathcal{E}, \mathcal{M})$-factorization of α:

$$\begin{tikzcd}
 & {} \arrow[ddr, "m_\alpha"] & \\
{} \arrow[ur, "n"] & & \\
A \arrow[u, "f"] \arrow[uur, "e_\alpha"'] \arrow[rr, "\alpha"'] & & B
\end{tikzcd} \tag{1.18}$$

The following definition generalizes slightly the sense of *concordant* and *dissonant* as presented in [**12**]. In this way, the class of *concordant* morphisms can include not only regular epis but also extremal epis.

DEFINITION 1.36. If the class of morphisms $\mathcal{N}$ at previous Proposition 1.35 equals the class $Mono$ of all monomorphisms, as it is the case when $\mathbb{C}$ is a regular category and $(\mathcal{F}, \mathcal{N}) = (RegEpi, Mono)$, then we call *concordant* to a morphism in $\mathbb{C}$ which belongs to

$$Conc := \bar{\mathcal{E}} = \mathcal{E} \cap \mathcal{F} \subseteq Epi,$$

and *dissonant* if it belongs to

$$Diss := \bar{\mathcal{M}} \supseteq \mathcal{M} \cup Mono.$$

CHAPTER 2

The reflection of Cat into Preord

In this chapter, all notions and classes of morphisms defined in previous chapter 1, are to be considered, unless stated otherwise, with respect to the following full reflection:

$$(I, H, \eta, \epsilon) : \mathbf{Cat} \to \mathbf{Preord}, \tag{2.1}$$

where:

- $H(X)$ is the preordered set X regarded as a category;
- $I(A) = A_0$ is the preordered set of objects a in A,

  in which $a \leq a'$ if and only if there exists a morphism from a to a';
- $\eta_A : A \to HI(A)$ is the unique functor with $\eta_A(a) = a$

  for each object a in A;
- $\epsilon : IH \to 1$ is the identity natural transformation.

2.1. Preamble

2.1.1. Monotone-light factorization.

Every map $\alpha : A \to B$ of compact Hausdorff spaces has a factorization $\alpha = me$ such that $m : C \to B$ has totally disconnected fibres and $e : A \to C$ has only connected ones. This is known as the classical monotone-light factorization of S. Eilenberg [**5**] and G. T. Whyburn [**16**].

Consider now, for an arbitrary functor $\alpha : A \to B$, the factorization $\alpha = me$ such that m is a faithful functor and e is a full functor bijective on objects. We shall show that this familiar factorization for categories is as well monotone-light, meaning that both factorizations are special and very similar cases of the categorical monotone-light factorization in an abstract category $\mathbb{C}$, with respect to a full reflective subcategory $\mathbb{X}$, as was studied in [**3**].

It is well known that any full reflective subcategory $\mathbb{X}$ of a category $\mathbb{C}$ gives rise, under mild conditions, to a factorization system $(\mathcal{E}, \mathcal{M})$. Hence, each of the two reflections **CompHaus** $\to$ **Stone**, of compact Hausdorff spaces into Stone spaces, and **Cat** $\to$ **Preord**, of categories into preorders, yields its own reflective factorization system. For the former reflection, the maps in $\mathcal{E}$ are known to be those which induce a bijection between the connected components; and a map $\alpha : A \to B$ is in $\mathcal{M}$ if, for each connected component H of B, every connected component of $\alpha^{-1}(H)$ is mapped by α homeomorphically onto H. Explicit descriptions of the same two classes for the latter reflection are given ahead in this chapter.

Moreover, the process of simultaneously stabilizing $\mathcal{E}$ and localizing $\mathcal{M}$, in the sense of [**3**], was already known to produce a new non-reflective and stable factorization system $(\mathcal{E}', \mathcal{M}^*)$ for the adjunction **CompHaus** $\to$ **Stone**, which is just the (**Monotone**, **Light**)-factorization mentioned above. But this process does not work in general, the monotone-light factorization for the reflection **CompHaus** $\to$ **Stone** being just one of a few known examples. Nevertheless, we shall prove that the (**Full and Bijective on Objects**, **Faithful**)-factorization for categories is another instance of a successful simultaneous stabilization and localization.

What guarantees the success is the following pair of conditions, which hold in both cases:

(1) the reflection $I : \mathbb{C} \to \mathbb{X}$ has stable units (in the sense of [**4**]);

(2) for each object B in $\mathbb{C}$, there is a monadic extension[1] (E, p) of B such that E is in the full subcategory $\mathbb{X}$.

Indeed, the two conditions (1) and (2) trivially imply that the $(\mathcal{E}, \mathcal{M})$-factorization is locally stable, which is a necessary and sufficient condition for $(\mathcal{E}', \mathcal{M}^*)$ to be a factorization system (cf. Theorem 1.18, which is the central result of [**3**]).

Actually, we shall prove that the reflection **Cat** $\to$ **Preord** also has stable units, as the reflection **CompHaus** $\to$ **Stone** was known to have. And, for the reflection **Cat** $\to$ **Preord**, the monadic extension (E, p) of B may be chosen to be the obvious projection from the coproduct $E = \mathbf{Cat}(\mathbf{4}, B) \cdot \mathbf{4}$ of sufficiently many copies of the ordinal number **4**, one copy for each triple of composable morphisms in B. As for **CompHaus** $\to$ **Stone**, it was chosen to be the canonical surjection from the Stone-Čech compactification $E = \beta|B|$ of the underlying set of B.

In both cases these monadic extensions are precisely the counit morphisms of the following adjunctions from **Set**: the unique (up to an isomorphism) adjunction $\mathbf{Cat}(\mathbf{4}, -) \dashv (-) \cdot \mathbf{4} : \mathbf{Set} \to \mathbf{Cat}$ which takes the terminal object 1 to the ordinal number **4**, and the adjunction $|\cdot| \dashv \beta : \mathbf{Set} \to \mathbf{CompHaus}$, where the standard forgetful functor $|\cdot|$ is monadic, respectively.

Notice that this perfect matching exists in spite of the fact that **CompHaus** is an exact category and **Cat** is not even regular.[2]

The reader may even extend the analogy, to the explicit descriptions of the classes in $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}^*)$, by simply making the following naive correspondence between some concepts of spaces and categories: "point"/"arrow"; "connected component"/"hom-set"; "fibre"/"inverse image of an arrow"; "connected"/"in the same hom-set"; "totally disconnected"/"every two arrows are in distinct hom-sets".

The two reflections may be considered as admissible Galois structures, in the

[1]It is said that (E, p) is a monadic extension of B, or that p is an effective descent morphism, if the pullback functor $p^* : \mathbb{C}/B \to \mathbb{C}/E$ is monadic.

[2]A monadic extension in **CompHaus** is just an epimorphism, i.e., a surjective mapping, while on **Cat** epimorphisms, regular epimorphisms and monadic extensions are distinct classes.

sense of categorical Galois theory as presented in [**7**], since having stable units implies admissibility.

Therefore, in both cases, for every object B in $\mathbb{C}$, one knows that the full subcategory $TrivCov(B)$ of $\mathbb{C}/B$, determined by the trivial coverings of B (i.e., the morphisms over B in $\mathcal{M}$), is equivalent to $\mathbb{X}/I(B)$ (cf. Lemma 1.7).

Moreover, the categorical form of the fundamental theorem of Galois theory gives us even more information on each $\mathbb{C}/B$ using the subcategory $\mathbb{X}$. It states that the full subcategory $Spl(E,p)$ of $\mathbb{C}/B$, determined by the morphisms split by the monadic extension (E,p) of B, is equivalent to the category $\mathbb{X}^{Gal(E,p)}$ of internal actions of the Galois precategory of (E,p) (cf. section 1.6 and Theorem 1.21).

In fact, conditions (1) and (2) above imply that $Gal(E,p)$ is really an internal groupoid in $\mathbb{X}$ (see section 5.3 of [**3**]).

And, as all the monadic extensions (E,p) of B described above are projective[3], one has in both cases that $Spl(E,p) = Cov(B)$, the full subcategory of $\mathbb{C}/B$ determined by the coverings of B (i.e., the morphisms over B in $\mathcal{M}^*$).

Condition (1) implies as well that any covering over an object which belongs to the subcategory is a trivial covering (cf. Proposition 1.16).

An easy consequence of this previous statement, condition (2), and of the fact that coverings are pullback stable, is that a morphism $\alpha : A \to B$ is a covering over B if and only if, for every morphism $\phi : X \to B$ with X in the subcategory $\mathbb{X}$, the pullback $X \times_B A$ of α along ϕ is also in $\mathbb{X}$ (cf. Lemma 2.9).

In particular, when the reflection has stable units, a monadic extension (E,p) as in condition (2) is a covering if and only if the kernel-pair of p is in the full subcategory $\mathbb{X}$ of $\mathbb{C}$.

Thus, since the monadic extensions considered for the two cases are in fact coverings, one concludes that $Gal(\mathbf{Cat}(\mathbf{4},B)\cdot\mathbf{4},p)$ and $Gal(\beta|B|,p)$ are not just internal groupoids, but internal equivalence relations in **Preord** and **Stone**, respectively.

In symbols, specifically for the reflection $\mathbf{Cat} \to \mathbf{Preord}$:

- $Faithful(B) \simeq \mathbf{Preord}^{Gal(\mathbf{Cat}(\mathbf{4},B)\cdot\mathbf{4},p)}$, for a general category B, and
- $Faithful(X) \simeq \mathbf{Preord}/X$, when X is a preorder.

As for $\mathbf{CompHaus} \to \mathbf{Stone}$:

- $Light(B) \simeq \mathbf{Stone}^{Gal(\beta|B|,p)}$, for a general compact Hausdorff space B, and
- $Light(X) \simeq \mathbf{Stone}/X$, when X is a Stone space.

The fact that $Gal(\beta|B|,p)$ is an internal equivalence relation in **Stone** was

[3] I.e., for each monadic extension (A,f) of B there exists a morphism $g : E \to A$ with $fg = p$.

already stated in [**9**].

Actually, the Stone spaces constitute what was defined there to be a generalized semisimple class of objects in **CompHaus**, and such that every compact Hausdorff space B is a quotient of a Stone space (in fact, the above effective descent morphisms $\beta|B| \to B$ are of course regular epis).

In this way, the equivalence $Light(B) \simeq \mathbf{Stone}^{Gal(\beta|B|,p)}$ is just a special case of its main Theorem 3.1. Which can be easily extended to non-exact categories, by using monadic extensions instead of regular epis, and in such a manner that the equivalence $Faithful(B) \simeq \mathbf{Preord}^{Gal(\mathbf{Cat}(\mathbf{4},B)\cdot\mathbf{4},p)}$ is also a special case of it.

Hence, the preordered sets do constitute a generalized semisimple class of objects in **Cat**, the faithful functors coincide with the locally semisimple coverings[4], $Gal(\mathbf{Cat}(\mathbf{4},B)\cdot\mathbf{4},p)$ is an internal equivalence relation in **Preord**, and the reflection $\mathbf{Cat} \to \mathbf{Preord}$ stands now as an interesting non-exact example of the case studied in [**9**].

In fact, remark that we have proved implicitly above in this preamble the following proposition:

Proposition 2.1. *Theorem 1.25 follows from Theorem 1.21 if:*

- *the first condition in Definition 1.22 holds for the category* $\mathbb{C}$*,*
- *the adjunction 1.1 has stable units and its counit is an iso, and*
- *in the monadic extension* (E,p) *of* B*,* $E = H(X)$ *for some* $X \in \mathbb{X}$*.*

We shall show that faithful functors are the covering morphisms with respect to the reflection $\mathbf{Cat} \to \mathbf{Preord}$ of categories into preorders. And that they also constitute the right-hand side of the monotone-light factorization system $(\mathcal{E}', \mathcal{M}^*)$ on **Cat**, which arises by simultaneously stabilizing $\mathcal{E}$ and localizing $\mathcal{M}$ in the reflective factorization system $(\mathcal{E}, \mathcal{M})$ associated to $\mathbf{Cat} \to \mathbf{Preord}$.

In fact, $\mathbf{Cat} \to \mathbf{Preord}$ stands as a non-exact counterpart of the reflection $\mathbf{CompHaus} \to \mathbf{Stone}$, of compact Hausdorff spaces into Stone spaces, in what concerns categorical Galois theory.

2.1.2. Inseparable-separable and concordant-dissonant factorizations.

Moreover, both reflections have concordant-dissonant factorization systems $(Conc, Diss)$, in the sense of [**12**, §2.11], where $Conc$ is the class $\mathcal{E} \cap RegEpi$ of regular epimorphisms in the left-hand side $\mathcal{E}$ of the reflective factorization system $(\mathcal{E}, \mathcal{M})$.

One concludes from Corollary 2.11 in [**12**] that $(Conc, Diss)$ is a factorization

[4]Under the condition that one replaces regular epis by monadic extensions in the Definition 2.1 of [**9**].

system on **CompHaus**, since **CompHaus** is an exact category and so it has a regular epi-mono factorization system $(RegEpi, Mono)$.

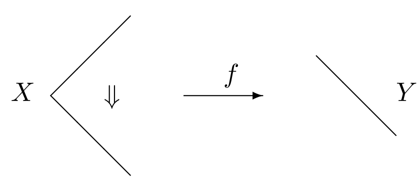
On the other hand, the existence of an extremal epi-mono factorization system $(ExtEpi, Mono)$ on **Cat** implies that $(\mathcal{E} \cap ExtEpi, Diss)$ is also a factorization system on **Cat** (see [**3**, §3.9] which generalizes Corollary 2.11 in [**12**]), where $\mathcal{E} \cap ExtEpi$ is the class of extremal epis in $\mathcal{E}$.

Finally, remark that the two classes $\mathcal{E} \cap ExtEpi$ and $Conc$ coincide on **Cat**. This follows easily from the known characterizations of extremal epis and regular epis on **Cat**, and from Proposition 2.7.

So far, the analogy between the two reflections continues.

Now, notice that for **Cat** $\to$ **Preord** the concordant morphisms are exactly the monotone morphisms, i.e., the full functors bijective on objects, but for **CompHaus** $\to$ **Stone** it is not so.

Indeed, consider the map



which bends a closed segment in the Euclidean plane through its middle point, identifying in this way its two halves.

It is a concordant map, i.e., a surjection whose fibres are contained in connected components[5], since X has only one component. But it is not monotone, i.e., a map whose fibres are all connected, since every point of Y, excepted one of the vertices, has disconnected two-point fibres.

Hence, we have:

- $(\mathcal{E}', \mathcal{M}^*) = (Conc, Diss)$, for **Cat** $\to$ **Preord**;
- $\mathcal{M}^*$ contains strictly the maps in $Diss$, i.e., the maps whose fibres meet the connected components in at most one point, for **CompHaus** $\to$ **Stone**.

One also knows from [**12**, §4.1] that

$$Pin \cap RegEpi = Pin^* \subseteq Ins \subseteq Conc \ ,$$

where Ins and Pin are respectively the classes of inseparable and purely inseparable morphisms on **Cat**.

By Proposition 2.16 a functor is in Pin if and only if it is injective on objects. So, one easily concludes that

$$Pin^* = Ins = Conc = \mathcal{E}' \ .$$

[5]Every given description of a class of maps of compact Hausdorff spaces, was either taken from [**3**, §7] or obtained from the Example 5.1 in [**12**]. At the latter, those descriptions were stated for the reflection of topological spaces into hereditarily disconnected ones, which extends **CompHaus** $\to$ **Stone**.

And, by Proposition 1.31, the monotone-light factorization on **Cat**, besides being also a concordant-dissonant factorization, is in addition an inseparable-separable factorization:

$$(\mathcal{E}', \mathcal{M}^*) = (Conc, Diss) = (Pin^*, Sep) = (Ins, Sep) \ .$$

Proposition 2.15 gives a direct proof of this fact by stating that the separable morphisms on **Cat** are just the faithful functors, i.e., the light morphisms on **Cat**.

Notice that, for an inseparable-separable factorization $m^*_\alpha \cdot e'_\alpha$ of any functor $\alpha : A \to B$ (see diagram 1.17), e'_α is the coequalizer of $(u \cdot m_{\delta_\alpha}, v \cdot m_{\delta_\alpha})$, where (u, v) is the kernel-pair of α and $m_{\delta_\alpha} \cdot e_{\delta_\alpha}$ is the reflective $(\mathcal{E}, \mathcal{M})$-factorization of the fibred product $\delta_\alpha : A \to A \times_B A$ (see [**12**, §3.2]).

Hence, one has two procedures for obtaining the monotone-light factorization of a functor via preorders:

- the one given in diagram 1.17, corresponding to the inseparable-separable factorization;
- another one in diagram 1.18, associated with the concordant-dissonant factorization considered above: $\alpha = (m_\alpha \cdot n) \cdot f$, such that $e_\alpha = n \cdot f$ is the extremal epi-mono factorization of e_α, and $\alpha = m_\alpha \cdot e_\alpha$ is the reflective $(\mathcal{E}, \mathcal{M})$-factorization of α.

As for the reflection **CompHaus** $\to$ **Stone**, the classes Pin and Pin^* are not closed under composition, and so they cannot be part of a factorization system. This was shown in [**12**, §5.1] with a counterexample. One can also infer from the same counterexample that $Ins \neq Pin^*$.[6]

Remark that, as far as separable and dissonant morphisms are concerned, and unlike the reflection **CompHaus** $\to$ **Stone**, the reflection **Top** $\to$ $\mathbf{T_0}$ of topological spaces into T_0-spaces is analogous to the reflection **Cat** $\to$ **Preord**. Indeed, one has as well for the reflection **Top** $\to$ $\mathbf{T_0}$ that (see [**12**, §5.4]):

$$Conc = Ins = Pin^* \quad \text{and} \quad Diss = Sep \ .$$

2.1.3. Normal morphisms.

We shall also give in this chapter an explicit description of normal morphisms of categories via preorders. That is, of functors α which are split over themselves, meaning that the pullback $u : A \times_B A \to A$ of $\alpha : A \to B$ along itself is a trivial covering, in the sense of categorical Galois theory as presented in [**7**], with respect to the adjunction **Cat** $\to$ **Preord**.

Notice that a trivial covering is a normal morphism, since the trivial coverings are pullback stable[7]. And every normal morphism is a separable morphism, since

[6] Cf. the last sentence in the previous footnote.

[7] It is so because the adjunction **Cat** $\to$ **Preord** is semi-left-exact, in the sense of [**4**] and [**3**], or admissible in the sense of categorical Galois theory.

$u \cdot \delta_\alpha = 1_A$ is a trivial covering[8] and trivial coverings are weakly left-cancellable[9] (cf. Definition 1.26 and Remark 1.27).

One easily checks with Proposition 2.20 that the effective descent morphisms above are normal morphisms, with respect to the reflection of categories into preorders.

This fact combined with the existing admissibility implies that, for every object B in **Cat**, $Faithful(B) = \mathcal{M}^*/B$ is reflective in $\mathbf{Cat}/B$, by Theorem 7.1 in [**8**]. And, as $\mathcal{M}^*$ is a pullback-stable class closed under composition, we could conclude *a priori* that $\mathcal{M}^*$ is the right-hand class of a certain factorization system, before knowing that it was part of a *stable* factorization system: see [**3**, §2.12].

2.2. Monotone-light factorization for categories via preorders

2.2.1. The reflection of Cat into Preord has stable units.

The following obvious lemma will be used many times below:

LEMMA 2.2. *A commutative diagram*

$$\begin{tikzcd} D \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\ C \arrow[r, "\gamma"] & B \end{tikzcd} \qquad (2.2)$$

in **Cat** *is a pullback square if and only if its object version*

$$\begin{tikzcd} D_0 \arrow[r, "v_0"] \arrow[d, "u_0"] & A_0 \arrow[d, "\alpha_0"] \\ C_0 \arrow[r, "\gamma_0"] & B_0 \end{tikzcd} \qquad (2.3)$$

is a pullback square in **Set**, *and also its hom-set version*

$$\begin{tikzcd} Hom_D(d,d') \arrow[r] \arrow[d] & Hom_A(v(d),v(d')) \arrow[d] \\ Hom_C(u(d),u(d')) \arrow[r] & Hom_B(\alpha v(d),\alpha v(d')) \end{tikzcd} \qquad (2.4)$$

for arbitrary objects d and d' in D, where the maps are induced by the arrow functions of the functors in diagram 2.2.

[8]The morphism δ_α is of course the fibred product of Definition 1.26.

[9]I.e., $(gf, g \in TrivCov \Rightarrow f \in TrivCov)$ for all f, g.

Proposition 2.3. *The adjunction 2.1 has stable units, that is, the functor* $I : \mathbf{Cat} \to \mathbf{Preord}$ *preserves every pullback of the form*

$$\begin{tikzcd} A \times_{H(X)} B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & H(X) \end{tikzcd} \quad .$$

Proof. Since the reflector I does not change the sets of objects (i.e., the underlying set of $I(A)$ is the same as the set of objects in A), the underlying sets of the two preorders $I(A \times_{H(X)} B)$ and $I(A) \times_{IH(X)} I(B)$ are both equal to $A_0 \times_{H(X)_0} B_0$.

Moreover, for any pair of objects (a, b) and (a', b') in $A_0 \times_{H(X)_0} B_0$, we observe that:

$(a, b) \leq (a', b')$ in $I(A \times_{H(X)} B) \Leftrightarrow$

there exist two morphisms $f : a \to a'$ in A and $g : b \to b'$ in B

such that $\alpha(f) = \beta(g) \Leftrightarrow$

(since $H(X)$ has no parallel arrows!)

there exist two morphisms $f : a \to a'$ in A and $g : b \to b'$ in $B \Leftrightarrow$

$a \leq a'$ in $I(A)$ and $b \leq b'$ in $I(B) \Leftrightarrow$

$(a, b) \leq (a', b')$ in $I(A) \times_{IH(X)} I(B)$.

□

2.2.2. Trivial coverings and vertical morphisms.

In this section we give explicit descriptions of the two classes $\mathcal{M}$ and $\mathcal{E}$, which do constitute a factorization system, as we now know after proving Proposition 2.3 and recalling Propositions 1.11 and 1.9.

Proposition 2.4. *A functor* $\alpha : A \to B$ *belongs to* $\mathcal{M}$ *if and only if for every two objects* a *and* a' *in* A *with* $Hom_A(a, a')$ *nonempty, the map* $Hom_A(a, a') \to Hom_B(\alpha(a), \alpha(a'))$ *induced by* α *is a bijection.*

We will also express this by saying that α *is a trivial covering if and only if* α *is a faithful and "almost full" functor.*

Proof. According to Lemma 2.2, the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "\alpha"] & HI(A) \arrow[d, "HI(\alpha)"] \\ B \arrow[r, "\eta_B"] & HI(B) \end{tikzcd} \tag{2.5}$$

is a pullback square in **Cat** if and only if the diagram

$$\begin{tikzcd} A_0 \arrow[r, "1"] \arrow[d, "\alpha_0"] & A_0 \arrow[d, "\alpha_0"] \\ B_0 \arrow[r, "1"] & B_0 \end{tikzcd} \tag{2.6}$$

and the diagrams

$$\begin{tikzcd} Hom_A(a,a') \arrow[r] \arrow[d] & Hom_{HI(A)}(a,a') \arrow[d] \\ Hom_B(\alpha(a),\alpha(a')) \arrow[r] & Hom_{HI(B)}(\alpha(a),\alpha(a')) \end{tikzcd} \tag{2.7}$$

for arbitrary objects a and a' in A, whose maps are induced by the arrow functions of the functors at diagram 2.5, are all pullback squares in **Set**.

We then observe:

the functor $\alpha : A \to B$ belongs to $\mathcal{M} \Leftrightarrow$

(since the diagram 2.6 is obviously a pullback square)

for every two objects a and a' in A,

the diagram 2.7 is a pullback square $\Leftrightarrow$

(if $Hom_A(a,a')$ is empty then $Hom_{HI(A)}(a,a')$ is also empty!)

for every two objects a and a' in A, provided $Hom_A(a,a')$ is nonempty,

the diagram 2.7 is a pullback square $\Leftrightarrow$

(if $Hom_A(a,a')$ is nonempty then $Hom_{HI(A)}(a,a') \cong 1 \cong Hom_{HI(B)}(\alpha(a),\alpha(a'))$)

for every two objects a and a' in A, provided $Hom_A(a,a')$ is nonempty,

the induced map $Hom_A(a,a') \to Hom_B(\alpha(a),\alpha(a'))$ is a bijection.

□

REMARK 2.5. In the internal diagrams, we are going to present in the examples below, the identity morphisms will always be omitted. Sometimes some other morphisms will also be omitted, but in that case we will always mention the fact.

EXAMPLE 2.6. The following functor $\alpha : A \to B$, such that $\alpha(a_0) = b_1$ and $\alpha(a_i) = b_i$, for $i = 1, 2, 3$, is a trivial covering:

$$
\begin{tikzcd}
a_0 \arrow[d] & & b_0 \arrow[d, shift left] \\
a_1 & & b_1 \arrow[u, shift left] \arrow[dd] \\
{} \arrow[rr, "\alpha"] & & {} \\
a_2 & & b_2 \\
a_3 \arrow[u] & & b_3 \arrow[u]
\end{tikzcd}
$$

(Notice that we have omitted in the diagram the unique arrow from b_0 to b_2)

In fact, α is faithful and all the maps $Hom_A(a_i, a_j) \to Hom_A(\alpha(a_i), \alpha(a_j))$ it induces are surjections except for the three cases where $Hom_A(a_1, a_2)$, $Hom_A(a_1, a_0)$ and $Hom_A(a_0, a_2)$ are empty.

If one adds an idempotent morphism to any of the three objects b_1, b_2 and b_3 in B, the functor α will be no longer almost full. And if one adds an idempotent to an object in A, one loses the faithfulness.

PROPOSITION 2.7. *A functor $\alpha : A \to B$ belongs to $\mathcal{E}$ if and only if the following two conditions hold:*

(1) *the functor α is bijective on objects;*
(2) *for every two objects a and a' in A, if $Hom_B(\alpha(a), \alpha(a'))$ is nonempty then so is $Hom_A(a, a')$.*

PROOF. The condition 2 reformulated in terms of the functor I becomes:

- *for every two objects a and a' in $I(A)$, $a \leq a'$ in $I(A)$ if and only if*

  $\alpha(a) \leq \alpha(a')$ in $I(B)$.

Therefore, the two conditions together are satisfied if and only if $I(\alpha)$ is an isomorphism, i.e., α is in $\mathcal{E}$ (cf. Lemma 1.3).

□

EXAMPLE 2.8. The functor bijective on objects $\alpha : \mathbf{2} \to \downarrow\downarrow$, from the ordinal number $\mathbf{2}$ to the category consisting of two parallel arrows, is vertical but not full:

$$
\begin{tikzcd}
0 \arrow[d] & & {} \arrow[d, shift left] \arrow[d, shift right] \\
1 \arrow[r, phantom, "\xrightarrow{\alpha}"] & & {}
\end{tikzcd}
$$

2.2.3. Coverings and stably-vertical morphisms.

The following simple Lemma 2.9 refers to the general case of adjunction 1.1 in previous chapter 1. It is needed to prove next Lemma 2.10, from which the characterization of coverings at Proposition 2.11 becomes an easy task.

LEMMA 2.9. *If the adjunction 1.1 has stable units then the following pullback* $u : H(X) \times_B A \to H(X)$

$$
\begin{tikzcd}
H(X) \times_B A \arrow[r] \arrow[d, "u"'] & A \arrow[d, "\alpha"] \\
H(X) \arrow[r, "\varphi"] & B
\end{tikzcd}
$$

of any covering morphism $\alpha : A \to B$ *along a morphism of the form* $\varphi : H(X) \to B$ *is a trivial covering, with respect to the adjunction 1.1.*

PROOF. The lemma follows immediately from Propositions 1.15 and 1.16. $\square$

LEMMA 2.10. *A functor* $\alpha : A \to B$ *in* **Cat** *is a covering if and only if, for every functor* $\varphi : X \to B$ *over* B *from any preorder* X*, the following pullback* $X \times_B A$

$$
\begin{tikzcd}
X \times_B A \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\
X \arrow[r, "\varphi"] & B
\end{tikzcd}
$$

of α *along* φ *is also a preorder.*

PROOF. By Proposition 2.3 and Lemmas 1.5 and 2.9, if one shows that for every category B in **Cat** there is a monadic extension (X, p) of B with X a preorder, then the proof will be done.

Hence, we complete the proof by presenting, for each category B in **Cat**, a monadic extension (X, p) of B with X a preorder (cf. Example 1.13):

make X the coproduct of all composable triples of morphisms in B,

and then let p be the obvious projection of X into B. $\square$

PROPOSITION 2.11. *A functor* $\alpha : A \to B$ *in* **Cat** *is a covering if and only if it is faithful.*

PROOF. We have:

the functor $\alpha : A \to B$ in **Cat** is a covering $\Leftrightarrow$ (by Lemma 2.10)

for every functor $\varphi : X \to B$ from a preorder X,

the following pullback $X \times_B A$ is a preorder:

$$\begin{tikzcd} X \times_B A \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\ X \arrow[r, "\varphi"] & B \end{tikzcd}$$

$\Leftrightarrow$

for every functor $\varphi : X \to B$ from a preorder X,

for any (x, a) and (x', a') in $X \times_B A$,

$Hom_{X \times_B A}((x, a), (x', a'))$ has at most one element $\Leftrightarrow$

for every functor $\varphi : X \to B$ from a preorder X,

if f is the unique morphism from x to x' in X,

and if any two morphisms $g : a \to a'$ and $h : a \to a'$ in A

are such that $\alpha(g) = \varphi(f) = \alpha(h)$, then $g = h \Leftrightarrow$

the functor $\alpha : A \to B$ is faithful.

$\square$

PROPOSITION 2.12. *A functor $\alpha : A \to B$ belongs to $\mathcal{E}'$ if and only if it is a full functor bijective on objects.*

PROOF. We have:

a functor $\alpha : A \to B$ belongs to $\mathcal{E}' \Leftrightarrow$

(according to the definition of $\mathcal{E}'$ and Lemma 1.3)

for every pullback u of α, $I(u)$ is an isomorphism $\Leftrightarrow$

$I(\alpha)$ is an isomorphism and I preserves every pullback of $\alpha \Leftrightarrow$

(according to Proposition 2.7)

α is bijective on objects

and

$Hom_A(a, a')$ is empty if and only if $Hom_B(\alpha(a), \alpha(a'))$ is so,

for arbitrary a and a' in A

and

(by Lemma 2.2, and since the reflector I does not change the sets of objects) for every pullback

$$\begin{tikzcd} C \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\ C \arrow[r, "\gamma"] & B \end{tikzcd} \tag{2.8}$$

of α, the hom-set version

$$\begin{tikzcd} Hom_{I(C\times_B A)}((c,a),(c',a')) \arrow[r] \arrow[d] & Hom_{I(A)}(a,a') \arrow[d] \\ Hom_{I(C)}(c,c') \arrow[r] & Hom_{I(B)}(\alpha(a),\alpha(a')) \end{tikzcd} \tag{2.9}$$

of its image by I is also a pullback square in **Set**,

for arbitrary objects (c,a) and (c',a') in $C \times_B A \Leftrightarrow$

α is bijective on objects

and

$Hom_{I(A)}(a,a') \cong Hom_{I(B)}(\alpha(a),\alpha(a'))$, for arbitrary a and a' in A

and

for every pullback $C \times_B A$ of α, $Hom_{I(C\times_B A)}((c,a),(c',a')) \cong Hom_{I(C)}(c,c')$

for arbitrary objects (c,a) and (c',a') in $C \times_B A \Leftrightarrow$

α is bijective on objects

and

α is full.

$\square$

Notice that the following lemma, as it was the case in the precedent lemma 2.9, refers to the general adjunction 1.1 in chapter 1.

LEMMA 2.13. *If the adjunction 1.1 has stable units, its counit $\varepsilon : IH \to 1$ is an iso, and for each object B in $\mathbb{C}$ there is a monadic extension (E,p) of B such that E is of the form $H(X)$, then $(\mathcal{E}',\mathcal{M}^*)$ is a factorization system, with respect to adjunction 1.1.*

PROOF. Consider the pullback $u : H(X)\times_B A \to H(X)$ of an arbitrary morphism $\alpha : A \to B$ along a monadic extension as in the statement.

Then, in the following factorization $u = u' \cdot \eta_{H(X)\times_B A}$

$$\begin{tikzcd}
 & H(X)\times_B A \arrow[r] \arrow[dl, "\eta_{H(X)\times_B A}"'] \arrow[dd] & A \arrow[dd, "\alpha"] \\
HI(H(X)\times_B A) \arrow[dr, "u'"'] & & \\
 & H(X) \arrow[r, "p"] & B
\end{tikzcd}$$

through the universal arrow $\eta_{H(X)\times_B A}$ from $H(X)\times_B A$ to H, the unit $\eta_{H(X)\times_B A}$ is stably-vertical and u' is a trivial covering. Indeed, the fact that every unit morphism is stably-vertical, under the conditions of the statement, follows trivially from Lemma 1.3 and Definition 1.10. Since u' is a morphism in $H(\mathbb{X})$ it is a trivial covering by Lemma 1.5.

Hence, being every $(\mathcal{E},\mathcal{M})$-factorization locally stable, $(\mathcal{E}',\mathcal{M}^*)$ is a factorization system by Theorem 1.18.

□

Conclusion 2.14. *As follows from the previous Lemma 2.13, $(\mathcal{E}',\mathcal{M}^*)$ is a factorization system.*

Moreover, Propositions 2.11 and 2.12 also tell us that it is a well-known one.

Notice therefore that we could draw the same conclusion by using the characterizations of coverings and stably-vertical morphisms given in those two propositions and Proposition 1.17.

2.3. Inseparable-separable and concordant-dissonant factorizations

2.3.1. Separable morphisms.

Proposition 2.15. *A functor $\alpha : A\to B$ in* **Cat** *is a separable morphism if and only if it is faithful.*

Proof. According to Definition 1.26 and Proposition 2.4 one has to show that the following functor $\delta_\alpha : A\to A\times_B A$

$$\begin{tikzcd}
A \arrow[dr, "\delta_\alpha"] \arrow[drr, "1_A"] \arrow[ddr, "1_A"'] & & \\
 & A\times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\
 & A \arrow[r, "\alpha"] & B
\end{tikzcd}$$

is faithful and "almost full" if and only if α is faithful.

We then observe:

for every two objects a and a' in A with $Hom_A(a, a')$ nonempty,
the map $Hom_A(a, a') \to Hom_{A\times_B A}((a, a), (a', a'))$ induced by δ_α
is a bijection $\Leftrightarrow$
(since the maps $Hom_A(a, a') \to Hom_{A\times_B A}((a, a), (a', a'))$
induced by δ_α are obviously injective)
for every two objects a and a' in A with $Hom_A(a, a')$ nonempty,
the map $Hom_A(a, a') \to Hom_{A\times_B A}((a, a), (a', a'))$ induced by δ_α
is a surjection $\Leftrightarrow$
(since if $Hom_A(a, a')$ is empty then so is $Hom_{A\times_B A}((a, a), (a', a')))$
the maps $Hom_A(a, a') \to Hom_{A\times_B A}((a, a), (a', a'))$ induced by δ_α
are all surjections $\Leftrightarrow$
for every two morphisms f and g in A with the same domain and codomain,
if $\alpha(f) = \alpha(g)$ then $f = g \Leftrightarrow$
α is a faithful functor.

□

2.3.2. Purely inseparable morphisms.

Proposition 2.16. *A functor $\alpha : A \to B$ in* **Cat** *is a purely inseparable morphism if and only if its object function is injective.*

Proof. According to Definition 1.26 and Proposition 2.7, $\alpha : A \to B$ is purely inseparable if and only if the following two conditions hold:

(1) the following functor $\delta_\alpha : A \to A \times_B A$

$$\begin{tikzcd}
A \arrow[dr, "\delta_\alpha"] \arrow[drr, "1_A"] \arrow[ddr, "1_A"'] & & \\
& A \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\
& A \arrow[r, "\alpha"] & B
\end{tikzcd}$$

is bijective on objects;

(2) for every two objects a and a' in A, if $Hom_{A\times_B A}(\delta_\alpha(a), \delta_\alpha(a'))$ is nonempty then so is $Hom_A(a, a')$.

We then observe:

the functor α is purely inseparable $\Leftrightarrow$

(the condition (2) holds trivially, since $\delta_\alpha(a) = (a, a)$ for every object a in A,

and the morphisms in $A \times_B A$ are the ordered pairs (f, g)

of morphisms in A such that $\alpha(f) = \alpha(g)$)

the functor δ_α is bijective on objects $\Leftrightarrow$

(since the object function of δ_α is injective)

the functor δ_α is surjective on objects $\Leftrightarrow$

the functor α is injective on objects.

□

2.3.3. The light, separable and dissonant morphisms coincide.

According to the given explicit descriptions above, it is easy to verify that every stably-vertical functor is an epi, in fact it is even a stably-regular epi (cf. Example 1.33), $\mathcal{E}' \subseteq RegEpi$. And that every vertical regular epi is stably-vertical, $\mathcal{E} \cap RegEpi \subseteq \mathcal{E}'$, since every epimorphism in **Cat** which is a bijection on objects must also be a full functor.

Hence, the conditions of Corollary 1.34 hold and we can conclude that (Ins, Sep) is a factorization system with $Ins = \mathcal{E}'$.

On the other hand, the known existence of an extremal epi-mono factorization system $(ExtEpi, Mono)$ on **Cat** implies by Proposition 1.35 that $Conc := \mathcal{E} \cap ExtEpi$ is the left-hand class of a factorization system on **Cat**.

But it is very easy to check that $Ins = \mathcal{E} \cap RegEpi = \mathcal{E} \cap ExtEpi$. Therefore, we conclude that our monotone-light factorization, besides being inseparable-separable as just seen, is in addition concordant-dissonant:

Conclusion 2.17. *(Ins, Sep) is an inseparable-separable factorization system.*

It is in fact the monotone-light factorization of previous section 2.2. And it is also a concordant-dissonant factorization:

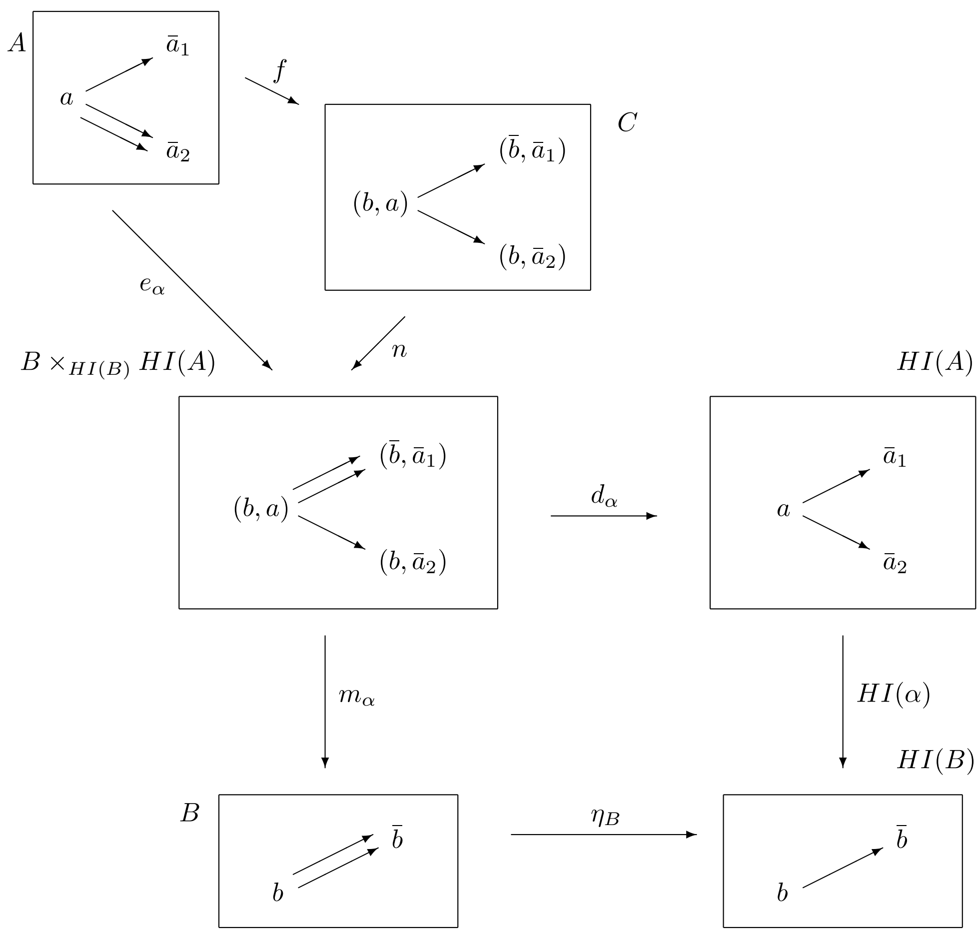
$$(\mathcal{E}', \mathcal{M}^*) = (Ins, Sep) = (Conc, Diss).$$

EXAMPLE 2.18. Every functor from and to one-object categories, i.e., a monoid homomorphism or in particular a group homomorpism, is vertical.

Therefore, its $(\mathcal{E}, \mathcal{M})$-factorization is the trivial one.

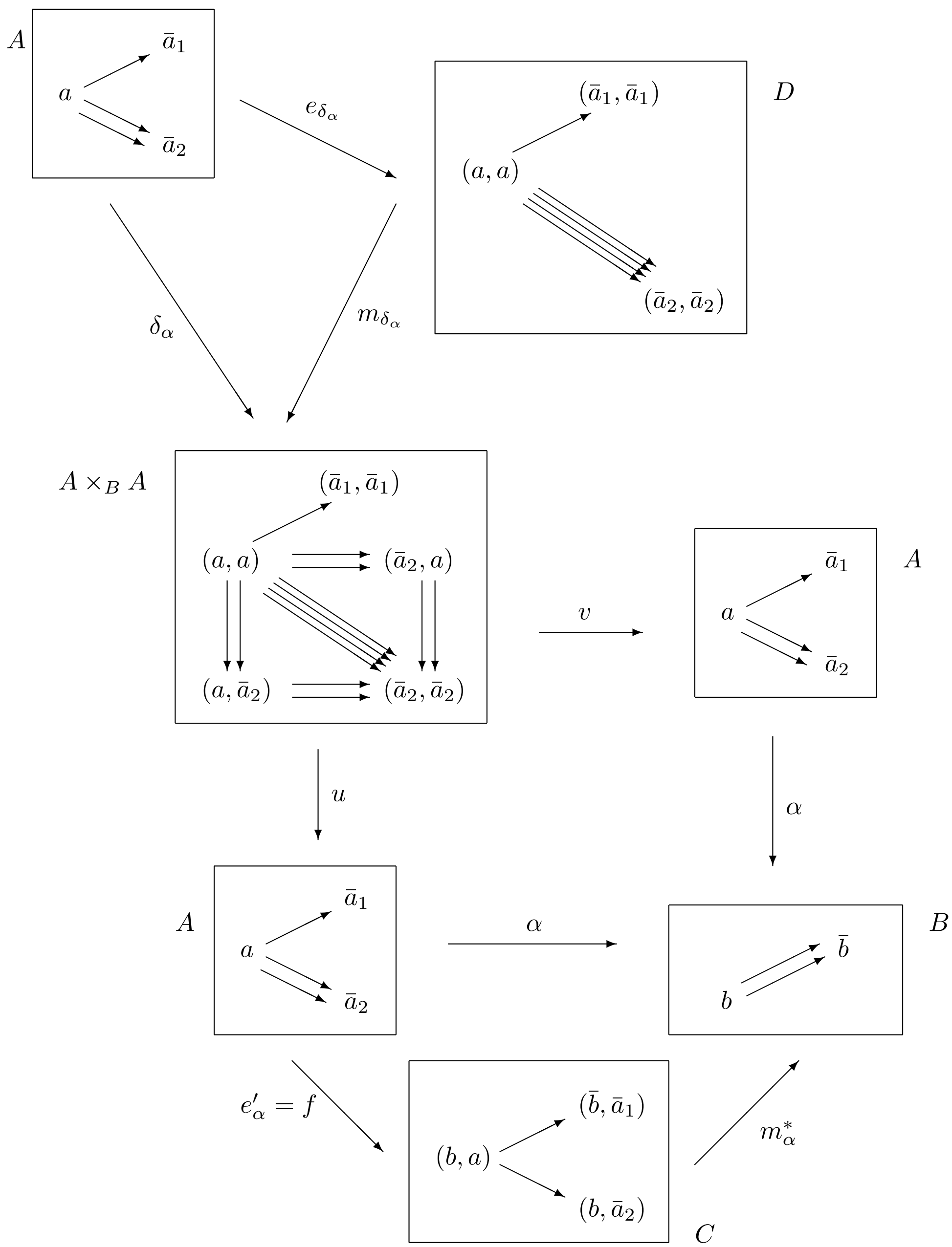
On the other hand, its $(\mathcal{E}', \mathcal{M}^*)$-factorization is the usual epi-mono factorization of an homomorphism.

EXAMPLE 2.19. The pullback diagram



displays, for a specific functor $\alpha : A \to B$ such that $\alpha(\bar{a}_1) = \bar{b}$ and $\alpha(a) = \alpha(\bar{a}_2) = b$, both the $(\mathcal{E}, \mathcal{M})$-factorization $m_\alpha \cdot e_\alpha$ and the $(Conc, Diss)$-factorization $(m_\alpha \cdot n) \cdot f$ (cf. diagrams 1.3 and 1.18).

As we now know, the $(Conc, Diss)$-factorization $\alpha = (m_\alpha \cdot n) \cdot f$ is also its (Ins, Sep)-factorization, and so it can also be obtained using the diagram 1.17:


A
a
$\bar{a}_1$
$\bar{a}_2$
e_{δ_α}
D
(a, a)
$(\bar{a}_1, \bar{a}_1)$
$(\bar{a}_2, \bar{a}_2)$
δ_α
m_{δ_α}
$A \times_B A$
(a, a)
$(\bar{a}_1, \bar{a}_1)$
$(\bar{a}_2, a)$
$(a, \bar{a}_2)$
$(\bar{a}_2, \bar{a}_2)$
v
A
a
$\bar{a}_1$
$\bar{a}_2$
u
α
A
a
$\bar{a}_1$
$\bar{a}_2$
α
B
b
$\bar{b}$
$e'_\alpha = f$
(b, a)
$(\bar{b}, \bar{a}_1)$
$(b, \bar{a}_2)$
C
m^*_α

2.4. Normal morphisms

PROPOSITION 2.20. *A functor $\alpha : A \to B$ in* **Cat** *is a normal morphism if and only if the following two conditions hold:*

(1) *α is a faithful functor;*

(2) *for every two morphisms $f : a \to a'$ and $\bar{f} : \bar{a} \to \bar{a}'$ in A, if $\alpha(f) = \alpha(\bar{f})$ then $\alpha(Hom_A(a, a')) = \alpha(Hom_A(\bar{a}, \bar{a}'))$.*

Of course that, for instance, $\alpha(Hom_A(a, a'))$ stands for the subset of $Hom_B(\alpha(a), \alpha(a'))$ whose elements are in the image of α.

PROOF. According to Definition 1.26 and Proposition 2.4, we have to show that conditions (1) and (2) hold if and only if the following pullback $u : A \times_B A \to A$

$$\begin{tikzcd} A \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\ A \arrow[r, "\alpha"] & B \end{tikzcd}$$

of $\alpha : A \to B$ along itself, is faithful and "almost full".

Claim 1. The maps $Hom_{A\times_B A}((a, \bar{a}), (a', \bar{a}')) \to Hom_A(a, a')$, induced by u for every two objects $(a, \bar{a})$ and $(a', \bar{a}')$ in $A \times_B A$, are all *injections* if and only if (1) holds.

Indeed, we have

the maps $Hom_{A\times_B A}((a, \bar{a}), (a', \bar{a}')) \to Hom_A(a, a')$ induced by u

are all injections $\Leftrightarrow$

(since u is displayed as $u(h, \bar{h}) = h$, for every morphism $(h, \bar{h})$ in $A \times_B A$,

i.e., for any pair $(h, \bar{h})$ of morphisms in A such that $\alpha(h) = \alpha(\bar{h})$)

for any three morphisms $h : a \to a'$ and $\bar{h}_1, \bar{h}_2 : \bar{a} \to \bar{a}'$ in A,

if $\alpha(h) = \alpha(\bar{h}_1)$ and $\alpha(h) = \alpha(\bar{h}_2)$ then $\bar{h}_1 = \bar{h}_2 \Leftrightarrow$

α is a faithful functor.

Claim 2. The maps $Hom_{A\times_B A}((a, \bar{a}), (a', \bar{a}')) \to Hom_A(a, a')$, induced by u for every two objects $(a, \bar{a})$ and $(a', \bar{a}')$ in $A \times_B A$, provided $Hom_{A\times_B A}((a, \bar{a}), (a', \bar{a}'))$

is *nonempty*, are all *surjections* if and only if (2) holds.

We have

> the maps $Hom_{A\times_B A}((a,\bar{a}),(a',\bar{a}')) \to Hom_A(a,a')$, induced by u for every two objects $(a,\bar{a})$ and $(a',\bar{a}')$ in $A\times_B A$, provided $Hom_{A\times_B A}((a,\bar{a}),(a',\bar{a}'))$ is nonempty, are all surjections $\Leftrightarrow$
>
> for every two morphisms $f: a\to a'$ and $\bar{f}:\bar{a}\to\bar{a}'$ in A such that $\alpha(f)=\alpha(\bar{f})$, the map $Hom_{A\times_B A}((a,\bar{a}),(a',\bar{a}')) \to Hom_A(a,a')$ induced by u is always a surjection $\Leftrightarrow$
>
> for every two morphisms $f: a\to a'$ and $\bar{f}:\bar{a}\to\bar{a}'$ in A, if $\alpha(f)=\alpha(\bar{f})$ then for every morphism h in $Hom_A(a,a')$ there is some morphism $\bar{h}$ in $Hom_A(\bar{a},\bar{a}')$ such that $\alpha(\bar{h})=\alpha(h)$ $\Leftrightarrow$
>
> for every two morphisms $f: a\to a'$ and $\bar{f}:\bar{a}\to\bar{a}'$ in A,
> if $\alpha(f)=\alpha(\bar{f})$ then $\alpha(Hom_A(a,a'))\subseteq\alpha(Hom_A(\bar{a},\bar{a}'))$ $\Leftrightarrow$
> (by the symmetry in f and $\bar{f}$!)
> for every two morphisms $f: a\to a'$ and $\bar{f}:\bar{a}\to\bar{a}'$ in A,
> if $\alpha(f)=\alpha(\bar{f})$ then $\alpha(Hom_A(a,a'))=\alpha(Hom_A(\bar{a},\bar{a}'))$.

□

EXAMPLE 2.21. The following functor $\alpha: A\to B$, such that and $\alpha(f)=\alpha(g)$ is normal but it is not a trivial covering:

$$\begin{tikzcd}
a_0 \arrow[d, "f"'] & & & \\
a_1 & & & b_1 \arrow[dd, shift right] \arrow[dd, shift left] \\
 & {} \arrow[r, "\alpha"] & {} & \\
a_2 \arrow[d, "g"'] & & & b_2 \\
a_3 & & &
\end{tikzcd}$$

Indeed, α is faithful and any two maps $Hom_A(a_i,a_j)\to Hom_A(\alpha(a_i),\alpha(a_j))$ induced by it which have the same codomain also have the same image, but α is not almost full.

Notice that the Example 3.23 in next chapter 3 also holds in the present context of the reflection of categories into preorders, showing that normal morphisms are not closed under composition.

CHAPTER 3

The reflection of Cat into Ord

In this chapter, every notion and class of morphisms defined in chapter 1 is to be considered, unless stated otherwise, with respect to the following full reflection:

$$(I, H, \eta, \epsilon) : \mathbf{Cat} \to \mathbf{Ord}, \tag{3.1}$$

where:

- $H(X)$ is the ordered set X regarded as a category;
- $I(A) = A_0/\sim$ is the ordered set of classes $[a]$ of objects a in A, in which $[a] \leq [a']$ if and only if there exists a morphism from a to a';
- $\eta_A : A \to HI(A)$ associates with each object $a \in A_0$ its equivalence class under $[a]$;
- ϵ is the canonical isomorphism $IH \to 1$.

3.1. Preamble

3.1.1. Monotone-light factorization.

Considerations completely analogous to the ones given at the beginning of previous chapter 2 remain valid for the reflection **Cat** $\to$ **Ord**, of categories into orders.

In fact this reflection is the composite of the following two:

$$\mathbf{Cat} \to \mathbf{Ord} = \mathbf{Cat} \to \mathbf{Preord} \to \mathbf{Ord},$$

the reflection of categories into preordered sets already studied and the reflection of preordered sets into ordered sets. Observe that the latter one is a simplified version of the "larger" one in what concerns categorical Galois theory.

For each one of the three reflections just mentioned, as for the known case of the reflection **CompHaus** $\to$ **Stone**, the process of simultaneously stabilizing the class of vertical morphisms and localizing the class of trivial coverings produces a new non-reflective and stable factorization system.

And what guarantees the success, in each one of the four, are the same two conditions already mentioned in the preamble to the previous chapter:

(1) the reflection $I : \mathbb{C} \to \mathbb{X}$ has stable units (in the sense of [**4**]);

(2) for each object B in $\mathbb{C}$, there is a monadic extension[1] (E,p) of B such that E is in the full subcategory $\mathbb{X}$.

Indeed, we shall prove that the reflection $\mathbf{Cat} \to \mathbf{Ord}$ also has stable units; and remark that the counit morphisms of the adjunction $\mathbf{Cat}(\mathbf{4},-) \dashv (-)\cdot\mathbf{4} : \mathbf{Set} \to \mathbf{Cat}$ continue to make the second condition hold, for the reflection into orders. As for the reflection of preorders into orders one can of course choose the counits of the adjunction $\mathbf{Preord}(\mathbf{2},-) \dashv (-)\cdot\mathbf{2} : \mathbf{Set} \to \mathbf{Preord}$ which takes the terminal object 1 to the ordinal number $\mathbf{2}$.

Exactly in the same way as before, one concludes that $Gal(\mathbf{Cat}(\mathbf{4},B)\cdot\mathbf{4},p)$ and $Gal(\mathbf{Preord}(\mathbf{2},B)\cdot\mathbf{2},p)$ are internal equivalence relations in $\mathbf{Ord}$.

In symbols, for the reflection $\mathbf{Cat} \to \mathbf{Ord}$:

- $\mathcal{M}^*/B \simeq \mathbf{Ord}^{Gal(\mathbf{Cat}(\mathbf{4},B)\cdot\mathbf{4},p)}$, for a general category B, and
- $\mathcal{M}^*/X \simeq \mathbf{Ord}/X$, when X is an ordered set.

As for $\mathbf{Preord} \to \mathbf{Ord}$:

- $\mathcal{M}^*/B \simeq \mathbf{Ord}^{Gal(\mathbf{Preord}(\mathbf{2},B)\cdot\mathbf{2},p)}$, for a general preordered set B, and
- $\mathcal{M}^*/X \simeq \mathbf{Ord}/X$, when X is an ordered set.

We are also going to present, for the reflections $\mathbf{Cat} \to \mathbf{Ord}$ and $\mathbf{CompHaus} \to \mathbf{Stone}$, another naïve correspondence between some concepts of spaces and categories which, if the reader is patient enough to compare the explicit descriptions of $\mathcal{E}$, $\mathcal{M}$, $\mathcal{E}'$ and $\mathcal{M}^*$, will reveal a surprising analogy: "point"/"any identity arrow 1_a"; "connected component"/"full subcategory determined by any [a]"; "fibre"/"inverse image of an identity arrow"; "connected"/"in the same full subcategory determined by some [a]"; "totally disconnected"/"every two arrows are in distinct hom-sets and there are no two arrows with reversed domain and codomain".

Notice finally that these two reflections of categories into orders and of preorders into orders are also non-exact examples of the case studied in [**9**]. In fact, the conditions of Proposition 2.1 are also verified for the two reflections to be studied in this chapter.

See the ending remarks in section 2.1.1.

3.1.2. Inseparable-separable and concordant-dissonant factorizations.

For the reflection $\mathbf{Preord} \to \mathbf{Ord}$, which is a simplified version of the reflection of categories into orders, we will show that there exist a monotone-light and an inseparable-separable factorization system, and that in fact they coincide with the concordant-dissonant factorization system.

Hence, the reflection of preorders into orders behaves exactly as the reflection

[1]It is said that (E,p) is a monadic extension of B, or that p is an effective descent morphism, if the pullback functor $p^* : \mathbb{C}/B \to \mathbb{C}/E$ is monadic.

of categories into preorders already studied:

$$(\mathcal{E}', \mathcal{M}^*) = (Conc, Diss) = (Pin^*, Sep) = (Ins, Sep) \ .$$

And the reason for the equality $(\mathcal{E}', \mathcal{M}^*) = (Ins, Sep)$ is the same as before (cf. Corollary 1.34):

$$\mathcal{E}' \subseteq \mathcal{E}' \cap RegEpi \subseteq Ins \subseteq \mathcal{E} \cap RegEpi \subseteq \mathcal{E}'.$$

In fact, **Preord** $\rightarrow$ **Ord** is a simplified version of the reflection of topological spaces into T_0-spaces $\mathbf{Top} \rightarrow \mathbf{T_0}$, since preorders are up to an iso the Alexandroff spaces, **Preord** $\cong$ **Alexandroff**, and orders are up to an iso the T_0-Alexandroff spaces, $\mathbf{Ord} \cong \mathbf{T_0} - \mathbf{Alexandroff}$.

We know from [**12**] that for the reflection $\mathbf{Top} \rightarrow \mathbf{T_0}$ one has $(Conc, Diss) = (Ins, Sep) = (Quotient\ maps\ with\ indiscrete\ fibres,\ Fibres\ are\ T_0 - spaces)$ and $Ins = Conc = \mathcal{E} \cap RegEpi$, as for the reflections **Cat** $\rightarrow$ **Preord** and **Preord** $\rightarrow$ **Ord**.

But, for the composite

$$\mathbf{Cat} \rightarrow \mathbf{Ord} = \mathbf{Cat} \rightarrow \mathbf{Preord} \rightarrow \mathbf{Ord},$$

of the two well-behaved reflections, the facts are surprisingly quite different:

- $(\mathcal{E}', \mathcal{M}^*)$ and $(Conc, Diss)$ are factorization systems;
- the dissonant functors are strictly included in the separable, which are strictly included in the coverings, $Diss \subset Sep \subset \mathcal{M}^*$.

Remark that these strict inclusions imply that the existing factorization systems are all distinct.

The question about the existence of an inseparable-separable factorization system (Ins, Sep) remains open; but if the answer to that question is yes then such factorization system is a distinct one!

Remark that in this case we mean by a concordant morphism not a vertical functor which is regular epi, but a vertical functor which is an extremal epimorphism, in the sense of our generalization of the meaning of $Conc$ in Definition 1.36.

So, for the case of the reflection **Cat** $\rightarrow$ **Ord** the (partial) analogy seems to be with the reflection **CompHaus** $\rightarrow$ **Stone** (see section 2.1.2 in the preamble of the previous chapter, where we give the facts about this reflection from the compact Hausdorff spaces to the Stone ones).

3.1.3. Normal morphisms.

We refer the reader to the remarks given in the homonymous section 2.1.3 of last chapter 2, which apply exactly to the adjunction into orders of this chapter.

In particular, effective descent morphisms remain normal.

3.2. Monotone-light factorization for categories via orders

3.2.1. The reflection of Cat into Ord has stable units.

PROPOSITION 3.1. *The adjunction 3.1 has stable units, that is, the functor* $I : \mathbf{Cat} \to \mathbf{Ord}$ *preserves every pullback of the form*

$$\begin{tikzcd} A \times_{H(X)} B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & H(X) \end{tikzcd} .$$

PROOF. We have to show that the map

$$\langle I(\pi_1), I(\pi_2)\rangle : I(A \times_{H(X)} B) \to I(A) \times_{IH(X)} I(B)$$

is an isomorphism of orders. Displaying this map as

$$\langle I(\pi_1), I(\pi_2)\rangle([(a, b)]) = ([a], [b]) ,$$

we then observe:

$([a], [b]) \leq ([a'], [b'])$ in $I(A) \times_{IH(X)} I(B) \Rightarrow$

$[a] \leq [a']$ in $I(A)$ and $[b] \leq [b']$ in $I(B) \Rightarrow$

there exist $a \to a'$ in A and $b \to b'$ in $B \Rightarrow$

there exists $(a, b) \to (a', b')$ in $A \times B \Rightarrow$

(since $H(X)$ has no parallel arrows!)

there exists $(a, b) \to (a', b')$ in $A \times_{H(X)} B \Rightarrow$

$[(a, b)] \leq [(a', b')]$ in $I(A \times_{H(X)} B)$,

and so the map $\langle I(\pi_1), I(\pi_2)\rangle$ induces an isomorphism between $I(A \times_{H(X)} B)$ and its image in $I(A) \times_{IH(X)} I(B)$. After that we only need to show that $\langle I(\pi_1), I(\pi_2)\rangle$ is surjective. For objects a in A and b in B we have:

$([a], [b])$ belongs to $I(A) \times_{IH(X)} I(B) \Rightarrow$

$[\alpha(a)] = [\beta(b)]$ in $IH(X) \Rightarrow$

(since $\epsilon : IH \to 1$ is an isomorphism!)

$\alpha(a) = \beta(b)$ in $X \Rightarrow$

(a, b) belongs to $A \times_{H(X)} B \Rightarrow$

$[(a, b)]$ belongs to $I(A \times_{H(X)} B)$

and hence has $([a], [b])$ as its image under $\langle I(\pi_1), I(\pi_2)\rangle$,

which proves the surjectivity.

∎

3.2.2. Trivial coverings and vertical morphisms.

Before giving explicit descriptions of trivial coverings and vertical morphisms, we notice again, exactly as we did for the reflection into preorders in chapter 2, that by Propositions 3.1, 1.11 and 1.9 the pair $(\mathcal{E}, \mathcal{M})$ is a factorization system.

PROPOSITION 3.2. *A functor $\alpha : A \to B$ belongs to $\mathcal{M}$ if and only if the following maps induced by α are bijections:*

(1) $[a] \to [\alpha(a)]$, *for every object a in A;*

(2) $Hom_A(a, a') \to Hom_B(\alpha(a), \alpha(a'))$, *for every two objects a and a' in A provided $Hom_A(a, a')$ is nonempty.*

Equivalently, α is a trivial covering if and only if the following two conditions hold:

- *for every object a in A, α induces an isomorphism between the full subcategories of A and B determined by $[a]$ and $[\alpha(a)]$ respectively;*
- *α is faithful and "almost full", which means that the maps $Hom_A(a, a') \to Hom_B(\alpha(a), \alpha(a'))$ are required to be bijective whenever $Hom_A(a, a')$ is nonempty.*

PROOF. Consider again the pullback diagram 1.3:

$$\begin{tikzcd}
A \arrow[dr, "e_\alpha"] \arrow[drr, "\eta_A"] \arrow[ddr, "\alpha"'] & & \\
& C \arrow[r, "d_\alpha"] \arrow[d, "m_\alpha"] & HI(A) \arrow[d, "HI(\alpha)"] \\
& B \arrow[r, "\eta_B"] & HI(B)
\end{tikzcd}$$

The functor $\alpha : A \to B$ is a trivial covering if and only if $e_\alpha = \langle \alpha, \eta_A \rangle$ is an isomorphism, i.e., it is a fully faithful functor bijective on objects. Accordingly it suffices to check the following three simple claims:

Claim 1. The functor $\langle \alpha, \eta_A \rangle : A \to C$ is injective on objects if and only if the maps in (1) are injective.

Indeed, we have

$\langle \alpha, \eta_A \rangle$ is an injection on objects $\Leftrightarrow$

$(\alpha(a), [a]) = (\alpha(a'), [a'])$ implies $a = a'$ $\Leftrightarrow$

if $[a] = [a']$, then $\alpha(a) = \alpha(a')$ implies $a = a'$ $\Leftrightarrow$

all the maps $[a] \to [\alpha(a)]$ are injective.

Claim 2. The functor $\langle a, \eta_A \rangle : A \to C$ is surjective on objects if and only if the maps in (1) are surjective.

We have

$\langle a, \eta_A \rangle$ is a surjection on objects $\Leftrightarrow$

for every object a in A and every object b in B with $[b] = [\alpha(a)]$,

there exists a' in A with $(\alpha(a'), [a']) = (b, [a]) \Leftrightarrow$

for every object a in A and every b in $[\alpha(a)]$,

there exists a' in $[a]$ with $\alpha(a') = b \Leftrightarrow$

all the maps $[a] \to [\alpha(a)]$ are surjective.

Claim 3. The functor $\langle a, \eta_A \rangle : A \to C$ is fully faithful if and only if all the maps in (2) with nonempty $Hom_A(a, a')$ are bijective.

For, consider the commutative diagram

$$\begin{tikzcd}
Hom_A(a,a') \arrow[r] \arrow[dr] & Hom_C((\alpha(a),[a]),(\alpha(a'),[a'])) \arrow[d] \\
& Hom_B(\alpha(a),\alpha(a'))
\end{tikzcd}$$

obtained from the hom-set version of the left hand triangle in diagram 1.3, for arbitrary a and a'.

If $Hom_A(a, a')$ is empty, then so is $Hom_C((\alpha(a), [a]), (\alpha(a'), [a']))$; otherwise the vertical arrow is a bijection.

Therefore the horizontal arrow is a bijection if and only if either $Hom_A(a, a')$ is empty, or $Hom_A(a, a') \to Hom_B(\alpha(a), \alpha(a'))$ is a bijection, as desired.

$\square$

EXAMPLE 3.3. The following *faithful* functor $\alpha : A \to B$, such that $\alpha(a_3) = b_2$, $\alpha(\bar{a}_3) = \bar{b}_2$, $\alpha(a_i) = b_i$ and $\alpha(\bar{a}_i) = \bar{b}_i$, for $i = 0, 1, 2$, is a trivial covering:

$$\begin{tikzcd}
a_0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \bar{a}_0 \arrow[d] & & b_0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \bar{b}_0 \arrow[d] \\
a_1 & \bar{a}_1 \arrow[l, shift left] \arrow[l, shift right] & {} \arrow[r, "\alpha"] & {} & b_1 & \bar{b}_1 \arrow[l, shift left] \arrow[l, shift right] \\
a_2 \arrow[u] \arrow[r, shift left] & \bar{a}_2 \arrow[u] \arrow[l, shift left] & & b_2 \arrow[u] \arrow[r, shift left] & \bar{b}_2 \arrow[u] \arrow[l, shift left] \\
a_3 \arrow[r, shift left] & \bar{a}_3 \arrow[l, shift left]
\end{tikzcd}$$

(We have omitted the *unique* left and right diagonal arrows in the squares at the above diagram)

In fact, α is faithful, almost full and it induces an iso between the full subcategories of A and B determined by the equivalence classes on the objects.

PROPOSITION 3.4. *A functor $\alpha : A \to B$ belongs to $\mathcal{E}$ if and only if the following two conditions hold:*

(1) *for every two objects a and a' in A, if $Hom_B(\alpha(a), \alpha(a'))$ is nonempty then so is $Hom_A(a, a')$;*

(2) *for every object b in B, there exists an object a in A such that b is in $[\alpha(a)]$.*

PROOF. The conditions (1) and (2) reformulated in terms of the functor I become:

(1) the map $I(\alpha) : I(A) \to I(B)$ induces an isomorphism of orders between $I(A)$ and its image in $I(B)$;

(2) the map $I(\alpha) : I(A) \to I(B)$ is surjective.

Therefore, the two conditions together are satisfied if and only if $I(\alpha)$ is an isomorphism, that is, according to Lemma 1.3, α is vertical.

□

EXAMPLE 3.5. The following functor $\alpha : A \to B$, such that $\alpha(a_0) = \alpha(a_1) = b_1$ and $\alpha(a_2) = \alpha(a_3) = b_2$ is vertical:

$$\begin{tikzcd}
a_0 \arrow[d, shift left] & & & \\
a_1 \arrow[u, shift left] \arrow[d, shift left] \arrow[d, shift right] & \arrow[r, "\alpha"] & & b_1 \arrow[d] \\
a_2 \arrow[d, shift right] & & & b_2 \arrow[d, shift right] \\
a_3 \arrow[u, shift right] & & & b_3 \arrow[u, shift right]
\end{tikzcd}$$

(Notice that we have omitted in the diagram the *unique* arrows from a_0 to a_2 and to a_3, from a_1 to a_3, and from b_1 to b_3)

Indeed, α induces a surjection on the full subcategories of B determined by the equivalence classes on objects, and it never takes an empty hom-set into a non-empty hom-set.

Remark also that α is not stably-vertical with respect to the reflection from categories into preorders, since it is not bijective on objects.

3.2.3. Coverings and stably-vertical morphisms.

The next Lemma 3.6 makes easier the characterization of coverings in Proposition 3.7. Its proof is *mutatis mutandis* the one given for Lemma 2.10.

LEMMA 3.6. *A functor $\alpha : A \to B$ in* **Cat** *is a covering if and only if, for every functor $\varphi : X \to B$ over B from any order X, the following pullback $X \times_B A$*

$$\begin{tikzcd} X \times_B A \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\ X \arrow[r, "\varphi"] & B \end{tikzcd}$$

of α along φ is also an order.

PROPOSITION 3.7. *A functor $\alpha : A \to B$ in* **Cat** *is a covering if and only if the following two conditions hold:*

(1) *the functor $\alpha : A \to B$ is faithful;*
(2) *for every two morphisms, with reversed domain and codomain, $f : a \to a'$ and $f' : a' \to a$ in A, if $\alpha(f) = 1_{\alpha(a)} = \alpha(f')$ then $a' = a$.*

PROOF. We have:

the functor $\alpha : A \to B$ in **Cat** is a covering $\Leftrightarrow$ (by Lemma 3.6)

for every functor $\varphi : X \to B$ from any order X,

the pullback $X \times_B A$ is an order $\Leftrightarrow$

for every functor $\varphi : X \to B$ from any order X,

for every two objects (x, a) and (x', a') in $X \times_B A$,

the following two conditions hold:

(i) $Hom_{X\times_B A}((x, a), (x', a'))$ has at most one element;
(ii) if both $Hom_{X\times_B A}((x, a), (x', a'))$ and $Hom_{X\times_B A}((x', a'), (x, a))$ are nonempty then $(x, a) = (x', a')$.

$\Leftrightarrow$ (as X is an order, if $x \neq x'$ and there exists $x \to x'$ then $Hom_X(x', x) = \emptyset$,

and so $Hom_{X\times_B A}((x', a'), (x, a)) = \emptyset$)

for every functor $\varphi : X \to B$ from any order X,

the following two conditions hold:

(i) for every two objects (x, a) and (x', a') in $X \times_B A$,

$Hom_{X \times_B A}((x, a), (x', a'))$ has at most one element;

(ii') for every x in X and every two objects a and a' in A

such that $\alpha(a) = \varphi(x) = \alpha(a')$, if both $Hom_{X \times_B A}((x, a), (x, a'))$ and $Hom_{X \times_B A}((x, a'), (x, a))$ are nonempty then $a' = a$.

Accordingly it suffices to check the next two claims:

Claim 1. The condition (1) in the statement holds if and only if for every functor $\varphi : X \to B$ from any order X the condition (i) above holds.

Indeed, we have

for every functor $\varphi : X \to B$ from any order X,

for every two objects (x, a) and (x', a') in $X \times_B A$,

$Hom_{X \times_B A}((x, a), (x', a'))$ has at most one element $\Leftrightarrow$

for every functor $\varphi : X \to B$ from any order X,

if f is the unique morphism from x to x' in X,

and if any two morphisms $g : a \to a'$ and $h : a \to a'$ in A

are such that $\alpha(g) = \varphi(f) = \alpha(h)$ then $g = h \Leftrightarrow$

the functor $\alpha : A \to B$ is faithful.

Claim 2. The condition (2) in the statement holds if and only if for every functor $\varphi : X \to B$ from any order X the condition (ii') above holds.

We have

for every functor $\varphi : X \to B$ from any order X,

for every x in X and every two objects a and a' in A

such that $\alpha(a) = \varphi(x) = \alpha(a')$,

if both $Hom_{X \times_B A}((x, a), (x, a'))$ and $Hom_{X \times_B A}((x, a'), (x, a))$

are nonempty then $a' = a \Leftrightarrow$

(since the only morphism $x \to x$ in an order X is the identity one 1_x)

for every two morphisms, with reversed domain and codomain,

$f : a \to a'$ and $f' : a' \to a$ in A, if $\alpha(f) = 1_{\alpha(a)} = \alpha(f')$ then $a' = a$.

□

EXAMPLE 3.8. The following functor $\alpha : A \to B$, which takes the four non-identity morphisms in A to the non-identity *idempotent* morphism g in B, is a covering:

$$\begin{tikzcd} a_0 \arrow[loop right] \arrow[d, shift right] & & \\ a_1 \arrow[loop right] \arrow[u, shift right] & \arrow[r, "\alpha"] & b \arrow[loop right, "g"] \end{tikzcd}$$

Remark that, if the non-identity morphisms in A did go instead to the identity morphism in B then α would not be a covering morphism. And notice also that α is not a trivial covering, in fact it is vertical.

EXAMPLE 3.9. Suppose that, in the following diagram, the monoid associated to each object is the additive one of non-negative integers. And that $Hom_A(a_0, a_1)$ and $Hom_A(a_1, a_0)$ are both just the set of positive integers. The composite of any two morphisms is then given by the integers sum.

Hence, the functor $\alpha : A \to B$, which takes any integer in A to the same integer in B, is a covering:

$$\begin{tikzcd} a_0 \arrow[loop right, "1\cdots"] \arrow[d, shift right, "\cdots 1"'] & & \\ a_1 \arrow[loop right, "1\cdots"] \arrow[u, shift right, "1\cdots"'] & \arrow[r, "\alpha"] & b \arrow[loop right, "1\cdots"] \end{tikzcd}$$

This functor α is indeed faithful and obviously no pair of morphisms with reversed and distinct domain and codomain go to the identity 0 in B.

PROPOSITION 3.10. *A functor $\alpha : A \to B$ belongs to $\mathcal{E}'$ if and only if it is a full functor surjective on objects.*

PROOF. We will first prove the only if part, through claims 1 and 2 below, which state respectively that if a functor α is not surjective on objects or if it is not full then it cannot belong to $\mathcal{E}'$.

In what follows 0, 1, and 2 are the usual discrete categories, and **2** is the category with two objects and just one arrow not the identity.

Claim 1. If the functor α is not surjective on objects then it does not belong to $\mathcal{E}'$.

Indeed, we have

the object b in B is not in the image of A under $\alpha \Rightarrow$

the empty functor $u : 0 \to 1$ is the following pullback of α

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\ 1 \arrow[r, "\varphi"] & B \end{tikzcd}$$

along the functor $\varphi : 1 \to B$ such that $\varphi(*) = b \Rightarrow$

the reflection $I(u) : 0 \to I(1)$ is no isomorphism $\Rightarrow$

the functor u does not belong to $\mathcal{E} \Rightarrow$

the functor α does not belong to $\mathcal{E}'$.

Claim 2. If the functor α is not full then it does not belong to $\mathcal{E}'$.

Consider the pullback diagram

$$\begin{tikzcd} \mathbf{2} \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\ \mathbf{2} \arrow[r, "\varphi"] & B \end{tikzcd}$$

where φ is the functor which takes the unique non-identity morphism $(0,1) : 0 \to 1$ of $\mathbf{2}$ to a morphism $g : \alpha(a) \to \alpha(a')$ in B.

We then observe:

the morphism $g : \alpha(a) \to \alpha(a')$ in B is not in the image of $Hom_A(a, a')$

under the map induced by $\alpha \Rightarrow$

(since $Hom_{\mathbf{2}}(0,1) = \{(0,1)\}$ and $Hom_{\mathbf{2}\times_B A}((0,a),(1,a'))$ is empty)

the reflection $I(u) : I(\mathbf{2} \times_B A) \to I(\mathbf{2})$ is no isomorphism of orders $\Rightarrow$

the functor u does not belong to $\mathcal{E} \Rightarrow$

the functor α does not belong to $\mathcal{E}'$.

We are going to do now the if part of the proof.

Suppose that the following pullback u of α along some $\gamma : C \to B$

$$\begin{tikzcd} C \times_B A \arrow[r] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\ C \arrow[r, "\gamma"] & B \end{tikzcd}$$

does not belong to $\mathcal{E}$, i.e., $I(u) : I(C \times_B A) \to I(C)$ is no isomorphism of orders:

the functor α is surjective on objects $\Rightarrow$

(since u is the pullback of α along γ)

the functor $u : C \times_B A \to C$ is surjective on objects $\Rightarrow$

the reflection $I(u) : I(C \times_B A) \to I(C)$ is surjective on objects $\Rightarrow$

(since, by hypothesis, $I(u)$ is *not* an isomorphism of orders!)

there are two objects $[(c, a)]$ and $[(c', a')]$ in $I(C \times_B A)$ such that the morphism

$[(c, a)] \leq [(c', a')]$ is *not* in $I(C \times_B A)$ and the morphism $[c] \leq [c']$ is in $I(C) \Rightarrow$

there are two objects (c, a) and (c', a') in $C \times_B A$ such that

$Hom_{C \times_B A}((c, a), (c', a'))$ is empty and $Hom_C(c, c')$ is nonempty $\Rightarrow$

there is some morphism $f : c \to c'$ in C such that the morphism

$\gamma(f) : \alpha(a) \to \alpha(a')$ in B is not in the image of $Hom_A(a, a')$

under the map induced by $\alpha \Rightarrow$

the functor α is not full.

□

Remark that the following proposition 3.11 is an immediate consequence of Lemma 2.13. Nevertheless, we give in addition a proof in which we actually produce the monotone-light factorization for a general functor α.

PROPOSITION 3.11. *$(\mathcal{E}', \mathcal{M}^*)$ is a factorization system on* **Cat**.

PROOF. As follows from Proposition 1.17, we have to check that every functor $\alpha : A \to B$ has a factorization $\alpha = m^* \cdot e'$, with $e' : A \to C$ in $\mathcal{E}'$ and $m^* : C \to B$ in $\mathcal{M}^*$.

Consider first the following equivalence relation on the objects of A: two objects a and a' are related, $a \sim a'$, if and only if there are two morphisms $f : a \to a'$ and $f' : a' \to a$ in A (with reversed domain and codomain) such that $\alpha(f) = 1_{\alpha(a)} = \alpha(f')$.

Using the previous equivalence relation on the objects of A, define a new equivalence relation on the morphisms of A: two morphisms f and g in A are related, $f \sim g$, if and only if $dom(f) \sim dom(g)$ and $cod(f) \sim cod(g)$[2] and $\alpha(f) = \alpha(g)$.

Now, let C be the category such that:

(1) its objects are the equivalence classes $[a]$ of the former equivalence relation;
(2) its morphisms are the equivalence classes $[f]$ of the latter equivalence relation;

[2]Where *dom* and *cod* stand of course for *domain* and *codomain*, respectively.

(3) its domain and codomain maps are displayed as, for an arbitrary $f : a \to a'$, $dom([f]) = [a]$, $cod([f]) = [a']$;

(4) its identity morphisms $1_{[a]}$ are given by $[1_a]$;

(5) its composition law is displayed as $[g]\cdot[f] = [g \cdot h \cdot f]$, for any $h : cod(f) \to dom(g)$ such that $\alpha(h)$ is an identity morphism.

Finally, let $e' : A \to C$ and $m^* : C \to B$ be the functors displayed as

$$e'(a) = [a],\ e'(f) = [f]$$

and

$$m^*([a]) = \alpha(a),\ m^*([f]) = \alpha(f).$$

By simple inspection we conclude that:

(1) $\alpha = m^* \cdot e'$;

(2) e' is a full functor surjective on objects;

(3) m^* is a faithful functor.

So, in order to complete the proof, it is only left to show that there cannot be two morphisms $[g]$ and $[g']$ in C with reversed and distinct domain and codomain such that their common image under m^* is an identity morphism.

Indeed, we have

there are two morphisms $[g]$ and $[g']$ in C such that $dom([g]) = cod([g'])$ and

$dom([g']) = cod([g])$ and $m^*([g]) = 1 = m^*([g']) \Rightarrow$

there are two morphisms $f : a \to a'$ and $f' : a' \to a$ in A such that

$[f] = [g]$ and $[f'] = [g']$ and $m^*([f]) = 1 = m^*([f']) \Rightarrow$

there are two morphisms $f : a \to a'$ and $f' : a' \to a$ in A such that

$f \sim g$ and $f' \sim g'$ and $\alpha(f) = 1 = \alpha(f') \Rightarrow$ (since $a \sim a'$ and $f \sim g$)

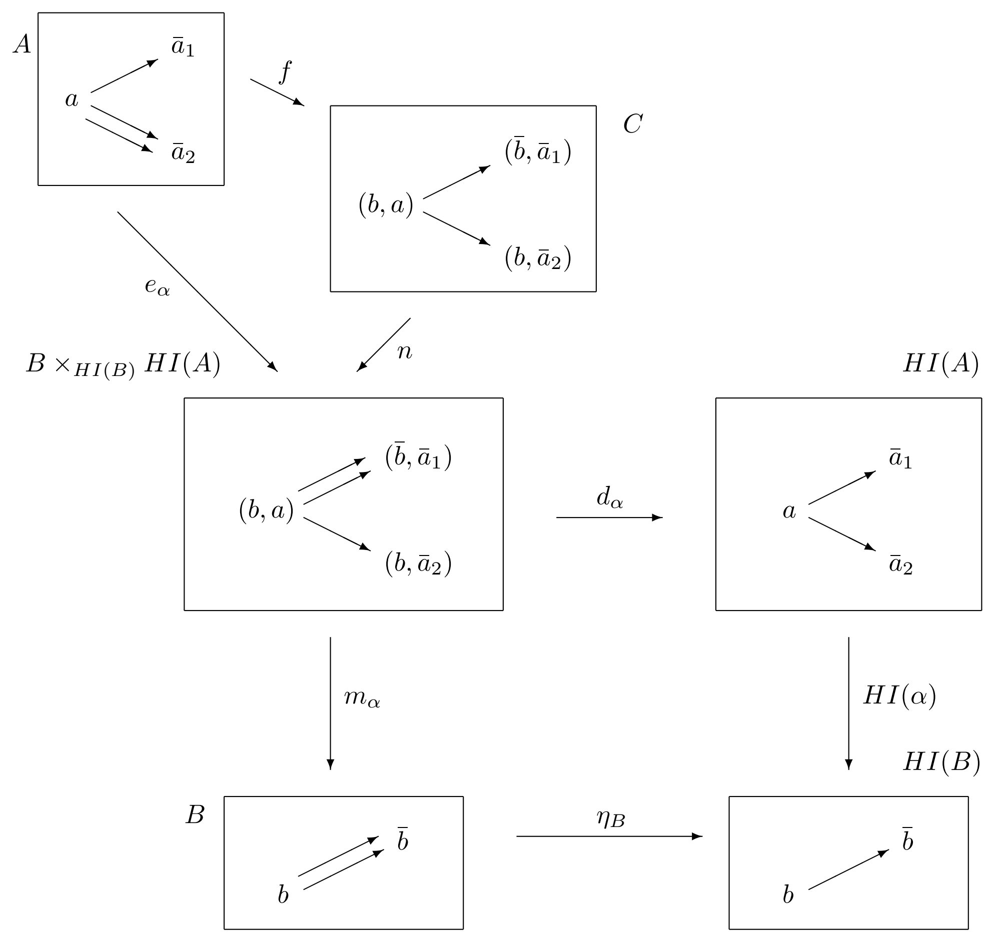
$dom(g) \sim cod(g) \Rightarrow$

$cod([g]) = dom([g])$.

□

Example 3.12. The factorizations of functors from and to one-object categories, i.e., a monoid homomorphism or in particular a group homomorpism, with respect to the reflection of categories into orders are analogous to the ones with respect to the reflection into preorders (cf. Example 2.18); that is, each $(\mathcal{E}, \mathcal{M})$-factorization is the trivial one, and every $(\mathcal{E}', \mathcal{M}^*)$-factorization is the usual epi-mono factorization of an homomorphism.

EXAMPLE 3.13. The pullback diagram



displays, for a specific morphism $\alpha : A \to B$ of preorders such that $\alpha(\bar{a}_1) = \bar{b}$ and $\alpha(a) = \alpha(\bar{a}_2) = b$, both the $(\mathcal{E}, \mathcal{M})$-factorization $m_\alpha \cdot e_\alpha$ and the $(Conc, Diss)$-factorization $(m_\alpha \cdot n) \cdot f$ (cf. diagrams 1.3 and 1.18).

3.3. Separable and purely inseparable morphisms

PROPOSITION 3.14. *A functor $\alpha : A \to B$ in* **Cat** *is a separable morphism if and only if the following two conditions hold:*

(1) *α is a faithful functor;*

(2) *for every three morphisms $g : a \to a$, $f : a \to a'$ and $f' : a' \to a$ in A, if $\alpha(f) = \alpha(g) = \alpha(f')$ then $a' = a$.*

PROOF. According to Definition 1.26 and Proposition 3.2, one has to show that the maps $Hom_A(a, a') \to Hom_{A\times_B A}((a, a), (a', a'))$ and $[a] \to [(a, a)]$, induced by the following functor δ_α

$$
\begin{tikzcd}
A \arrow[drr, "1_A"] \arrow[dr, "\delta_\alpha"] \arrow[ddr, "1_A"'] & & \\
& A \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\
& A \arrow[r, "\alpha"] & B
\end{tikzcd}
$$

for every two objects a and a' in A with $Hom_A(a, a')$ nonempty, are all bijections if and only if conditions (1) and (2) in the statement hold.

Hence it suffices to check the following three simple claims:

Claim 1. The maps $Hom_A(a, a') \to Hom_{A\times_B A}((a, a), (a', a'))$, induced by δ_α for every two objects a and a' in A, are all surjections if and only if α is a faithful functor.

(Notice that the maps $Hom_A(a, a') \to Hom_{A\times_B A}((a, a), (a', a'))$ are obviously injective, and that if $Hom_A(a, a')$ is empty then so is $Hom_{A\times_B A}((a, a), (a', a')))$

Indeed, we have

the maps $Hom_A(a, a') \to Hom_{A\times_B A}((a, a), (a', a'))$ induced by δ_α

are all surjections $\Leftrightarrow$

for every two morphisms f and g in A with the same domain and codomain,

if $\alpha(f) = \alpha(g)$ then $f = g \Leftrightarrow$

α is a faithful functor.

Claim 2. If condition (2) in the statement holds then the maps $[a] \to [(a, a)]$, induced by δ_α for every object a in A, are all surjections.

(Notice that the maps $[a] \to [(a, a)]$ induced by δ_α are obviously injective)

We have

the map $[a] \to [(a, a)]$ induced by δ_α is *not* a surjection $\Rightarrow$

(if any pair (a', a') with equal components is in $[(a, a)]$ then necessarily

a' is in $[a]$, implying that (a', a') is in the image of the considered map)

there is an object (a', a'') in $A \times_B A$ such that it is in $[(a, a)]$ and $a'' \neq a' \Rightarrow$

there are two morphisms $(f, g) : (a, a) \to (a', a'')$ and $(f', g') : (a', a'') \to (a, a)$

in $A \times_B A$ such that $a'' \neq a' \Rightarrow$

the three morphisms $g \cdot g' : a'' \to a''$, $g \cdot f' : a' \to a''$ and $f \cdot g' : a'' \to a'$ in A

are such that $\alpha(gf') = \alpha(gg') = \alpha(fg')$ and $a'' \neq a'$.

Claim 3. If the maps $[a] \to [(a, a)]$, induced by δ_α for every object a in A, are all surjections then condition (2) in the statement holds.

Indeed, we have

the morphisms $g : a \to a$, $f : a \to a'$ and $f' : a' \to a$ in A are such that

$\alpha(f) = \alpha(g) = \alpha(f')$ and $a' \neq a \Rightarrow$

the morphisms $(f, g) : (a, a) \to (a', a)$ and $(f', g) : (a', a) \to (a, a)$ in $A \times A$

are also morphisms of $A \times_B A$ and $a' \neq a \Rightarrow$

there is an object (a', a) in $A \times_B A$ such that it is in $[(a, a)]$

and $a' \neq a \Rightarrow$

the map $[a] \to [(a, a)]$ induced by δ_α is *not* a surjection.

$\square$

Example 3.15. The following functor $\alpha : A \to B$, which takes the two non-identity morphisms in A to the *non-idempotent* morphism g in $B \cong \mathbb{Z}/2\mathbb{Z}$, is separable:

$$\begin{tikzcd} a_0 \arrow[d, shift right] & & \\ & {} \arrow[r, "\alpha"] & b \arrow[loop right, "g"] \\ a_1 \arrow[u, shift right] & & \end{tikzcd}$$

Notice that, if the two non-identity morphisms in A did go instead to the identity morphism in $B \cong \mathbb{Z}/2\mathbb{Z}$ then α would not be separable. In fact, it would not even be a covering.

Proposition 3.16. *A functor $\alpha : A \to B$ in* **Cat** *is a purely inseparable morphism if and only if, for every two objects a and a' in A, if $\alpha(a) = \alpha(a')$ then there exist three morphisms $h : a \to a$, $f : a \to a'$ and $f' : a' \to a$ in A such that $\alpha(f) = \alpha(h) = \alpha(f')$.*

Proof. According to Definition 1.26, one has to check that the condition in the statement holds for α if and only if conditions (1) and (2) in Proposition 3.4 also hold for the following functor $\delta_\alpha : A \to A \times_B A$

$$\begin{tikzcd} A \arrow[dr, "\delta_\alpha"] \arrow[drr, "1_A"] \arrow[ddr, "1_A"'] & & \\ & A \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\ & A \arrow[r, "\alpha"] & B \end{tikzcd}$$

In fact, as $Hom_{A\times_B A}((a,a),(a',a'))$ being nonempty implies that $Hom_A(a,a')$

is also so, the condition (1) of Proposition 3.4 holds trivially for the functor δ_α.

We then observe:

for every object (a, a') in $A \times_B A$, there exists an object $\bar{a}$ in A such that
(a, a') is in $[(\bar{a}, \bar{a})] \Leftrightarrow$
for every two objects a and a' in A, if $\alpha(a) = \alpha(a')$ then there exists an object
$\bar{a}$ in A and there exist two morphisms $(g, g') : (a, a') \to (\bar{a}, \bar{a})$ and
$(\bar{g}, \bar{g}') : (\bar{a}, \bar{a}) \to (a, a')$ in $A \times_B A \Leftrightarrow$
(just make $f' = \bar{g} \cdot g'$, $f = \bar{g}' \cdot g$ and $h = \bar{g} \cdot g$, noticing that
$\alpha(\bar{g} \cdot g') = \alpha(\bar{g} \cdot g) = \alpha(\bar{g}' \cdot g)$, for the implication downwards;
and make $\bar{a} = a$, $\bar{g} = g = h$, $g' = f'$ and $\bar{g}' = f$ for the implication upwards)
for every two objects a and a' in A, if $\alpha(a) = \alpha(a')$ then there exist
three morphisms $h : a \to a$, $f : a \to a'$ and $f' : a' \to a$ in A, such that
$\alpha(f) = \alpha(h) = \alpha(f')$.

□

The next example shows that the purely inseparable morphisms, with respect to the reflection of categories into orders, are not closed under composition.

Example 3.17. Consider the two functors $\alpha : A \to B$ and $\beta : B \to 1$, such that the former is the inclusion of the discrete category A in B and the latter is the unique functor to the terminal object in **Cat**:

$$\begin{tikzcd}
a & & & a \arrow[dd, shift left] & & & \\
& {} \arrow[r, "\alpha"] & {} & & {} \arrow[r, "\beta"] & {} & c \\
a' & & & a' \arrow[uu, shift left] & & &
\end{tikzcd}$$

It is easy to check that α and β are purely inseparable but its composite $\beta \cdot \alpha$ is not.

3.4. Concordant-dissonant factorization

We are going to consider as before the factorization system $(ExtEpi, Mono)$ on **Cat**, for the definition of concordant-dissonant factorization, now with respect to the reflection of categories into ordered sets.

Hence, according to Proposition 1.35 and Definition 1.36, the pair $(\mathcal{E} \cap ExtEpi, Diss)$

is a factorization system. The following proposition gives an explicit description of the functors in its second component *Diss*.

PROPOSITION 3.18. *A functor $\alpha : A \to B$ in* **Cat** *is dissonant if and only if α is faithful and the following maps induced by α are injections:*

$$[a] \to [\alpha(a)], \text{ for every object } a \text{ in } A.$$

We can also express the latter condition by saying that, for every object a in A, α induces a monomorphism between the full subcategories of A and B determined respectively by $[a]$ and $[\alpha(a)]$.

PROOF. According to Proposition 1.35 and Definition 1.36, the functor $\alpha : A \to B$ is dissonant if and only if the following functor $\langle \alpha, \eta_A \rangle : A \to B \times I(A)$ is monic:

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[d, "\langle \alpha, \eta_A \rangle"] \arrow[dr, "\eta_A"] & \\ B & B \times I(A) \arrow[l] \arrow[r] & I(A) \end{tikzcd}$$

The functor $\langle \alpha, \eta_A \rangle : A \to B \times I(A)$ is displayed as

$$\langle \alpha, \eta_A \rangle(a) = (\alpha(a), [a]).$$

Now it is easy to conclude that $\langle \alpha, \eta_A \rangle$ is a monic functor if and only if α is faithful and induces the injections in the statement.

□

EXAMPLE 3.19. The functor $\alpha : \mathbf{2} \to \mathbb{Z}/2\mathbb{Z}$ from the ordinal number to the integers modulo 2, such that the unique non-identity morphism in $\mathbf{2}$ goes under α to the identity morphism, is dissonant:

$$\begin{tikzcd} 0 \arrow[d] & & \\ 1 & \xrightarrow{\alpha} & 1 \arrow[loop left, "b"] \end{tikzcd}$$

Notice that α is not a trivial covering, since it is not "almost" full.

CONCLUSION 3.20. *Consider the separable functor α of Example 3.15:*

$$\begin{array}{c} a_0 \\ \downarrow\uparrow \\ a_1 \end{array} \xrightarrow{\qquad\alpha\qquad} b \circlearrowleft g$$

It is easy to check that it is not dissonant. In effect, its induced map $[a_0] \to [b]$ *is no injection.*

Hence, we conclude that the class of dissonant morphisms is strictly contained in the class of separable morphisms, $Diss \subset Sep$.

Therefore, if (Ins, Sep) *is an inseparable-separable factorization system, it does not coincide with the concordant-dissonant factorization system.*

In fact, in that case we have three distinct factorization systems for categories into orders, since Example 3.8 shows that the separable morphisms are strictly contained in the coverings. A situation opposite to what happened in the reflection of categories into preorders (cf. Conclusion 2.17).

3.5. Normal morphisms

The next Lemma 3.21 has a very straightforward proof, although its characterization of normal morphisms is not the simpler. Actually, its last two items will be simplified in the following Proposition 3.22.

LEMMA 3.21. *A functor $\alpha : A \to B$ in* **Cat** *is a normal morphism if and only if the following four conditions hold:*

(1) *α is a faithful functor;*
(2) *for every two morphisms $f : a \to a'$ and $\bar{f} : \bar{a} \to \bar{a}'$ in A,*
if $\alpha(f) = \alpha(\bar{f})$ then $\alpha(Hom_A(a, a')) = \alpha(Hom_A(\bar{a}, \bar{a}'))$;
(3) *for every four morphisms $\bar{f}, \bar{f}' : \bar{a} \to \bar{a}$, $f : a \to a'$ and $f' : a' \to a$ in A,*
if $\alpha(f) = \alpha(\bar{f})$ and $\alpha(\bar{f}') = \alpha(f')$ then $a' = a$;
(4) *for every object $\bar{a}$ in A and for every two morphisms $f : a \to a'$ and $f' : a' \to a$ in A, if $\alpha(a) = \alpha(\bar{a})$ then there is an object $\bar{a}'$ in A and four morphisms $g : a \to a'$, $g' : a' \to a$, $\bar{f} : \bar{a} \to \bar{a}'$ and $\bar{f}' : \bar{a}' \to \bar{a}$ in A such that $\alpha(g) = \alpha(\bar{f})$ and $\alpha(g') = \alpha(\bar{f}')$.*

Of course that, for instance, $\alpha(Hom_A(a, a'))$ stands for the subset of $Hom_B(\alpha(a), \alpha(a'))$ whose elements are in the image of α.

PROOF. We will show that conditions (1), (2), (3) and (4) hold respectively if and only if the following pullback $u : A \times_B A \to A$

$$
\begin{tikzcd}
A \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\
A \arrow[r, "\alpha"] & B
\end{tikzcd}
$$

of $\alpha : A \to B$ along itself, is faithful, “almost full”, and the maps $[(a, \bar{a})] \to [a]$, induced by u for every object $(a, \bar{a})$ in $A \times_B A$, are all injections and surjections (i.e., bijections). Hence, according to Definition 1.26 and Proposition 3.2, the statement will be proved.

Claim 1. The maps $Hom_{A \times_B A}((a, \bar{a}), (a', \bar{a}')) \to Hom_A(a, a')$, induced by u for every two objects $(a, \bar{a})$ and $(a', \bar{a}')$ in $A \times_B A$, are all *injections* if and only if

(1) holds.

Indeed, we have

the maps $Hom_{A\times_B A}((a,\bar{a}),(a',\bar{a}')) \to Hom_A(a,a')$ induced by u

are all injections $\Leftrightarrow$

(since u is displayed as $u(h,\bar{h}) = h$, for every morphism $(h,\bar{h})$ in $A \times_B A$,

i.e., for any pair $(h,\bar{h})$ of morphisms in A such that $\alpha(h) = \alpha(\bar{h})$)

for any three morphisms $h : a \to a'$ and $\bar{h}_1, \bar{h}_2 : \bar{a} \to \bar{a}'$ in A,

if $\alpha(h) = \alpha(\bar{h}_1)$ and $\alpha(h) = \alpha(\bar{h}_2)$ then $\bar{h}_1 = \bar{h}_2 \Leftrightarrow$

α is a faithful functor.

Claim 2. The maps $Hom_{A\times_B A}((a,\bar{a}),(a',\bar{a}')) \to Hom_A(a,a')$, induced by u for every two objects $(a,\bar{a})$ and $(a',\bar{a}')$ in $A \times_B A$, provided $Hom_{A\times_B A}((a,\bar{a}),(a',\bar{a}'))$ is *nonempty*, are all *surjections* if and only if (2) holds.

We have

the maps $Hom_{A\times_B A}((a,\bar{a}),(a',\bar{a}')) \to Hom_A(a,a')$, induced by u for every two

objects $(a,\bar{a})$ and $(a',\bar{a}')$ in $A \times_B A$, provided $Hom_{A\times_B A}((a,\bar{a}),(a',\bar{a}'))$ is

nonempty, are all surjections $\Leftrightarrow$

for every two morphisms $f : a \to a'$ and $\bar{f} : \bar{a} \to \bar{a}'$ in A such that $\alpha(f) = \alpha(\bar{f})$,

the map $Hom_{A\times_B A}((a,\bar{a}),(a',\bar{a}')) \to Hom_A(a,a')$ induced by u is always a

surjection $\Leftrightarrow$

for every two morphisms $f : a \to a'$ and $\bar{f} : \bar{a} \to \bar{a}'$ in A, if $\alpha(f) = \alpha(\bar{f})$ then

for every morphism h in $Hom_A(a,a')$ there is some morphism $\bar{h}$ in $Hom_A(\bar{a},\bar{a}')$

such that $\alpha(\bar{h}) = \alpha(h) \Leftrightarrow$

for every two morphisms $f : a \to a'$ and $\bar{f} : \bar{a} \to \bar{a}'$ in A,

if $\alpha(f) = \alpha(\bar{f})$ then $\alpha(Hom_A(a,a')) \subseteq \alpha(Hom_A(\bar{a},\bar{a}')) \Leftrightarrow$

(by the symmetry in f and $\bar{f}$!)

for every two morphisms $f : a \to a'$ and $\bar{f} : \bar{a} \to \bar{a}'$ in A,

if $\alpha(f) = \alpha(\bar{f})$ then $\alpha(Hom_A(a,a')) = \alpha(Hom_A(\bar{a},\bar{a}'))$.

Claim 3. The maps $[(\bar{a},a)] \to [\bar{a}]$, induced by u for every object $(\bar{a},a)$ in $A\times_B A$, are all *injections* if and only if (3) holds.

Indeed, we have

the maps $[(\bar{a}, a)] \to [\bar{a}]$ induced by u are all injections $\Leftrightarrow$

for every object $(\bar{a}, a)$ in $A \times_B A$, if $(\bar{a}, a')$ is in $[(\bar{a}, a)]$ then $a' = a \Leftrightarrow$

for every four morphisms $\bar{f}, \bar{f}' : \bar{a} \to \bar{a}$, $f : a \to a'$ and $f' : a' \to a$ in A,

if $\alpha(f) = \alpha(\bar{f})$ and $\alpha(\bar{f}') = \alpha(f')$ then $a' = a$.

Claim 4. The maps $[(a, \bar{a})] \to [a]$, induced by u for every object $(a, \bar{a})$ in $A \times_B A$, are all *surjections* if and only if (4) holds.

We have

the maps $[(a, \bar{a})] \to [a]$ induced by u are all surjections $\Leftrightarrow$

for every object $(a, \bar{a})$ in $A \times_B A$ and a' in $[a]$, there is another object $\bar{a}'$ in A

such that $(a', \bar{a}')$ is in $[(a, \bar{a})] \Leftrightarrow$

for every two objects a and $\bar{a}$ in A such that $\alpha(a) = \alpha(\bar{a})$, and for every two

morphisms $f : a \to a'$ and $f' : a' \to a$ in A, there exist two such morphisms

$(g, \bar{f}) : (a, \bar{a}) \to (a', \bar{a}')$ and $(g', \bar{f}') : (a', \bar{a}') \to (a, \bar{a})$ in $A \times_B A \Leftrightarrow$

for every object $\bar{a}$ in A and for every two morphisms $f : a \to a'$ and $f' : a' \to a$

in A, if $\alpha(a) = \alpha(\bar{a})$ then there is an object $\bar{a}'$ in A and four morphisms $g : a \to a'$,

$g' : a' \to a$, $\bar{f} : \bar{a} \to \bar{a}'$ and $\bar{f}' : \bar{a}' \to \bar{a}$ in A such that

$$\alpha(g) = \alpha(\bar{f}) \ \ \textit{and} \ \ \alpha(g') = \alpha(\bar{f}') \ .$$

$\square$

PROPOSITION 3.22. *A functor $\alpha : A \to B$ in* **Cat** *is a normal morphism if and only if the following four conditions hold:*

(a) *α is a faithful functor;*

(b) *for every two morphisms $f : a \to a'$ and $\bar{f} : \bar{a} \to \bar{a}'$ in A,*

if $\alpha(f) = \alpha(\bar{f})$ then $\alpha(Hom_A(a, a')) = \alpha(Hom_A(\bar{a}, \bar{a}'))$;

(c) *for every two morphisms $f : a \to a'$ and $f' : a' \to a$ in A,*

if $\alpha(f) = 1_{\alpha(a)} = \alpha(f')$ then $a' = a$;

(d) *for every object $\bar{a}$ in A and for every two morphisms $f : a \to a'$ and $f' : a' \to a$ in A, if $\alpha(a) = \alpha(\bar{a})$ then there is an object $\bar{a}'$ in A and two morphisms $\bar{f} : \bar{a} \to \bar{a}'$ and $\bar{f}' : \bar{a}' \to \bar{a}$ in A such that $\alpha(f) = \alpha(\bar{f})$ and $\alpha(f') = \alpha(\bar{f}')$.*

PROOF. According to the previous Lemma 3.21, we have to show that our present conditions (a), (b), (c) and (d) are equivalent to the former conditions (1), (2), (3) and (4).

As the two conditions (1) and (a) are identical, and so are also (2) and (b), the following first two claims assert that the former conditions imply the present ones.

Claim 1. The condition (3) implies (c).

Indeed, we only need to make $\bar{f}' = 1_a = \bar{f}$ in (3) in order to obtain (c).

Claim 2. The conditions (2) and (4) imply (d).

We have

the object $\bar{a}$ in A and the two morphisms $f : a \to a'$ and $f' : a' \to a$ in A are

such that $\alpha(a) = \alpha(\bar{a}) \Rightarrow$ (by condition (4))

there is an object $\bar{a}'$ in A and four morphisms $g : a \to a'$, $g' : a' \to a$,

$\bar{g} : \bar{a} \to \bar{a}'$ and $\bar{g}' : \bar{a}' \to \bar{a}$ in A such that $\alpha(g) = \alpha(\bar{g})$ and $\alpha(g') = \alpha(\bar{g}')$

$\Rightarrow$ (applying the condition (2) twice)

there is an object $\bar{a}'$ in A such that

$\alpha(Hom_A(a, a')) = \alpha(Hom_A(\bar{a}, \bar{a}'))$ and $\alpha(Hom_A(a', a)) = \alpha(Hom_A(\bar{a}', \bar{a})) \Rightarrow$

there is an object $\bar{a}'$ in A and two morphisms $\bar{f} : \bar{a} \to \bar{a}'$, $\bar{f}' : \bar{a}' \to \bar{a}$ in A,

such that $\alpha(f) = \alpha(\bar{f})$ and $\alpha(f') = \alpha(\bar{f}')$.

For the same reason stated above, before the first claim, the next two and final claims assert that our present conditions imply the former ones.

Claim 3. The condition (d) implies (4).

In fact, in order to conclude so we only need to take g to be f and g' to be f' in (4).

Claim 4. The conditions (b) and (c) imply (3).

We have

the four morphisms $\bar{f}, \bar{f}' : \bar{a} \to \bar{a}$, $f : a \to a'$ and $f' : a' \to a$ in A

are such that $\alpha(f) = \alpha(\bar{f})$ and $\alpha(\bar{f}') = \alpha(f') \Rightarrow$

(applying the condition (b) twice)
$\alpha(Hom_A(a, a')) = \alpha(Hom_A(\bar{a}, \bar{a})) = \alpha(Hom_A(a', a)) \Rightarrow$

there are two morphisms $g : a \to a'$ and $g' : a' \to a$ in A

such that $\alpha(g') = 1_{\alpha(a)} = \alpha(g) \Rightarrow$ (by condition (c))

$a' = a$.

□

EXAMPLE 3.23. Consider the two functors $\alpha : A \to B$ and $\beta : B \to C$, such that both of them are faithful and $\alpha(a_i) = b_i$, for $i = 0, 1, 2, 3$:

$$
\begin{tikzcd}
a_0 \arrow[d, shift left] \arrow[d, shift right] & & & b_0 \arrow[d, shift left] \arrow[d, shift right] & & & \\
a_1 & & & b_1 & & & c_1 \arrow[dd, shift left] \arrow[dd, shift right] \\
 & {} \arrow[r, "\alpha"] & {} & & {} \arrow[r, "\beta"] & {} & \\
a_2 \arrow[d] & & & b_2 \arrow[d, shift left] \arrow[d, shift right] & & & c_2 \\
a_3 & & & b_3 & & &
\end{tikzcd}
$$

It is easy to check that α and β are normal but that their composite $\beta \cdot \alpha$ is not.

3.6. The reflection of Preord into Ord

The following full reflection is a simplified version of the reflection of categories into orders:

(3.2) $$(I, H, \eta, \epsilon) : \mathbf{Preord} \to \mathbf{Ord},$$

where:

- $H(X)$ is the ordered set X regarded as a preordered set;
- $I(A) = A_0/\sim$ is the ordered set of classes $[a]$ of objects a in A, in which $[a] \leq [a']$ if and only if $a \leq a'$;
- $\eta_A : A \to HI(A)$ associates with each object $a \in A_0$ its equivalence class under $[a]$;
- ϵ is the canonical isomorphism $IH \to 1$.

We are going to list, in the rest of the chapter, with respect to the adjunction 3.2 above, the results and explicit descriptions analogous to the ones already presented for the reflections of categories into orders and preorders.

The fact that this adjunction has stable units and a (non-trivial) monotone-light factorization follows immediately from the same fact for the reflection into orders.

The characterizations of the classes $\mathcal{M}$, $\mathcal{E}$, $\mathcal{M}^*$, $\mathcal{E}'$, Sep, Pin and the class of normal morphisms in **Cat**, with respect to the adjunction 3.2, may be obtained simply by removing, from the same characterizations for the adjunction 3.1, the conditions that become redundant when one just considers preorders. For instance, faithfulness is not needed anymore, since “functors” between preorders are necessarily faithful.

We will also find that the classes of coverings and separable morphisms, which were distinct for the reflection of categories into orders, collapse into each other for the reflection of preorders into orders, although this does not happen with any of the other classes of morphisms studied.

In fact, the reflection of preorders into orders behaves in the same way as the reflection of categories into preorders: the monotone-light factorization coincides with the inseparable-separable and the concordant-dissonant one.

3.6.1. The reflection of Preord into Ord has stable units.

PROPOSITION 3.24. *The adjunction 3.2 has stable units, that is, the functor* $I : \mathbf{Preord} \to \mathbf{Ord}$ *preserves every pullback of the form*

$$\begin{tikzcd}
A \times_{H(X)} B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "\beta"] \\
A \arrow[r, "\alpha"] & H(X)
\end{tikzcd} \quad .$$

Remember that this implies at once that the pair $(\mathcal{E}, \mathcal{M})$ is a factorization system (cf. Propositions 1.9 and 1.11).

3.6.2. Trivial coverings and vertical morphisms.

PROPOSITION 3.25. *A morphism $\alpha : A \to B$ of preorders is a trivial covering if and only if the maps $[a] \to [\alpha(a)]$, induced by α for every object a in A, are bijections.*

EXAMPLE 3.26. The morphism $\alpha : A \to B$ of preorders, such that $\alpha(a_0) = \alpha(a_3) = b_0$, $\alpha(a_1) = b_1$ and $\alpha(a_2) = b_2$, is a trivial covering:

$$\begin{tikzcd}
a_0 \arrow[d] & & & & \\
a_1 \arrow[d, shift left] & {} \arrow[r, "\alpha"] & {} & b_1 \arrow[d, shift left] & b_0 \arrow[l] \arrow[dl] \\
a_2 \arrow[u, shift left] & & & b_2 \arrow[u, shift left] & \\
a_3 \arrow[u] & & & &
\end{tikzcd}$$

Notice that, in order to simplify the diagram, we have omitted two composite morphisms in the preordered set A.

PROPOSITION 3.27. *A morphism $\alpha : A \to B$ of preorders is vertical if and only if the following two conditions hold:*

- *for every two objects a and a' in A, if $\alpha(a) \leq \alpha(a')$ then $a \leq a'$;*
- *for every object b in B, there exists an object a in A such that b is in $[\alpha(a)]$.*

EXAMPLE 3.28. If one applies the reflector $I : \mathbf{Cat} \to \mathbf{Preord}$ in adjunction 2.1 to the Example 3.5, one obtains the *vertical* morphism $I(\alpha) : I(A) \to I(B)$ of preorders, such that $I(\alpha)(a_0) = I(\alpha)(a_1) = b_1$ and $I(\alpha)(a_2) = I(\alpha)(a_3) = b_2$:

$$\begin{tikzcd}
a_0 \arrow[d, shift left] & & \\
a_1 \arrow[u, shift left] \arrow[d] & & b_1 \arrow[d] \\
a_2 \arrow[d, shift right] & \xrightarrow{I(\alpha)} & b_2 \arrow[d, shift right] \\
a_3 \arrow[u, shift right] & & b_3 \arrow[u, shift right]
\end{tikzcd}$$

(Notice that we have omitted in the diagram the composite arrows from a_0 to a_2 and to a_3, from a_1 to a_3, and from b_1 to b_3)

3.6.3. Coverings and stably-vertical morphisms.

PROPOSITION 3.29. *A morphism $\alpha : A \to B$ of preorders is a covering if and only if, for every two objects a and a' in A, if $\alpha(a) = \alpha(a')$ and $a \leq a'$ and $a' \leq a$ then $a' = a$.*

EXAMPLE 3.30. Notice that, if one applies the reflector $I : \mathbf{Cat} \to \mathbf{Preord}$ in adjunction 2.1 to the functor α of Example 3.8, the resulting morphism $I(\alpha)$ of preorders is *not* a covering:

$$\begin{tikzcd} a_0 \arrow[d, shift right] & & \\ a_1 \arrow[u, shift right] & \arrow[r, "I(\alpha)"] & b \end{tikzcd}$$

But the unique morphism of preorders from the ordinal number **2** to the terminal object 1 in **Preord** is a covering:

$$\begin{tikzcd} 0 \arrow[d] & & \\ 1 & \arrow[r] & 0 \end{tikzcd}$$

Proposition 3.31. *A morphism $\alpha : A \to B$ of preorders is stably-vertical if and only if it is surjective on objects and, for every two objects a and a' in A, if $\alpha(a) \leq \alpha(a')$ then $a \leq a'$.*

The following proposition is an immediate consequence of Lemma 2.13, since for each preordered set B there is a monadic extension (E, p) over B in **Preord** (see in the Preamble the section 3.1.1, and Example 1.13).

Nevertheless, in the proof of Proposition 3.11 we produce in particular the monotone-light factorization of a morphism of preorders with respect to the reflection **Preord** $\to$ **Ord**.

Proposition 3.32. *$(\mathcal{E}', \mathcal{M}^*)$ is a factorization system on* **Preord**.

3.6.4. Separable and purely inseparable morphisms.

Proposition 3.33. *A morphism $\alpha : A \to B$ of preorders is separable if and only if it is a covering (cf. Proposition 3.29).*

Proposition 3.34. *A morphism $\alpha : A \to B$ of preorders is purely inseparable if and only if, for every two objects a and a' in A, if $\alpha(a) = \alpha(a')$ then $a \leq a'$ and $a' \leq a$.*

Remark 3.35. Notice that Example 3.17 shows that the purely inseparable morphisms are not closed under composition with respect to the reflection of preorders into orders.

Example 3.36. A morphism $\alpha : A \to 1$ of preorders to the terminal object 1 in **Preord** is purely inseparable if and only if its domain A is a connected equivalence relation, meaning that $a \leq a'$ for every two objects $a, a' \in A$.

As any epimorphism in **Preord** is a surjection on objects, one easily concludes, from the characterizations given above, that

$$\mathcal{E} \cap RegEpi \subseteq \mathcal{E}' \subseteq RegEpi$$

for the reflection of preorders into orders.

Then, we have

$$\mathcal{E}' \subseteq Pin \cap RegEpi \subseteq Ins \subseteq \mathcal{E} \cap RegEpi \subseteq \mathcal{E}'$$

by Lemma 1.30 and Remark 1.28.

Hence, $Pin \cap RegEpi = Ins = \mathcal{E} \cap RegEpi = \mathcal{E}'$, and by Corollary 1.34 we conclude that (Ins, Sep) is a factorization system.

On the other hand, the known existence of an extremal epi-mono factorization system $(ExtEpi, Mono)$ on **Preord** implies by Proposition 1.35 that $\mathcal{E} \cap ExtEpi$ is the left-hand class of a factorization system on **Preord**.

And it is very easy to check that $Conc := \mathcal{E} \cap ExtEpi = \mathcal{E} \cap RegEpi$.

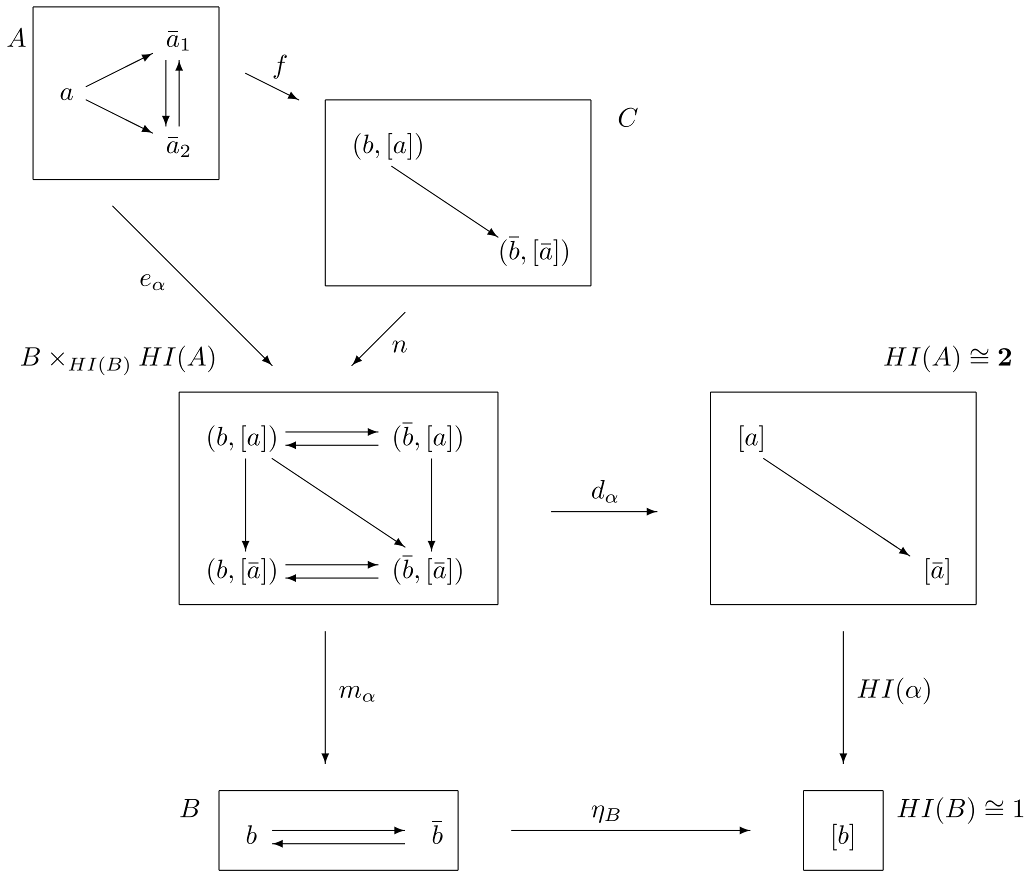
Therefore, we conclude that our monotone-light factorization, besides being inseparable-separable as just seen, is in addition concordant-dissonant:

Conclusion 3.37. *(Ins, Sep) is a factorization system.*

It is in fact the monotone-light factorization of Proposition 3.32. And it is also a concordant-dissonant factorization:

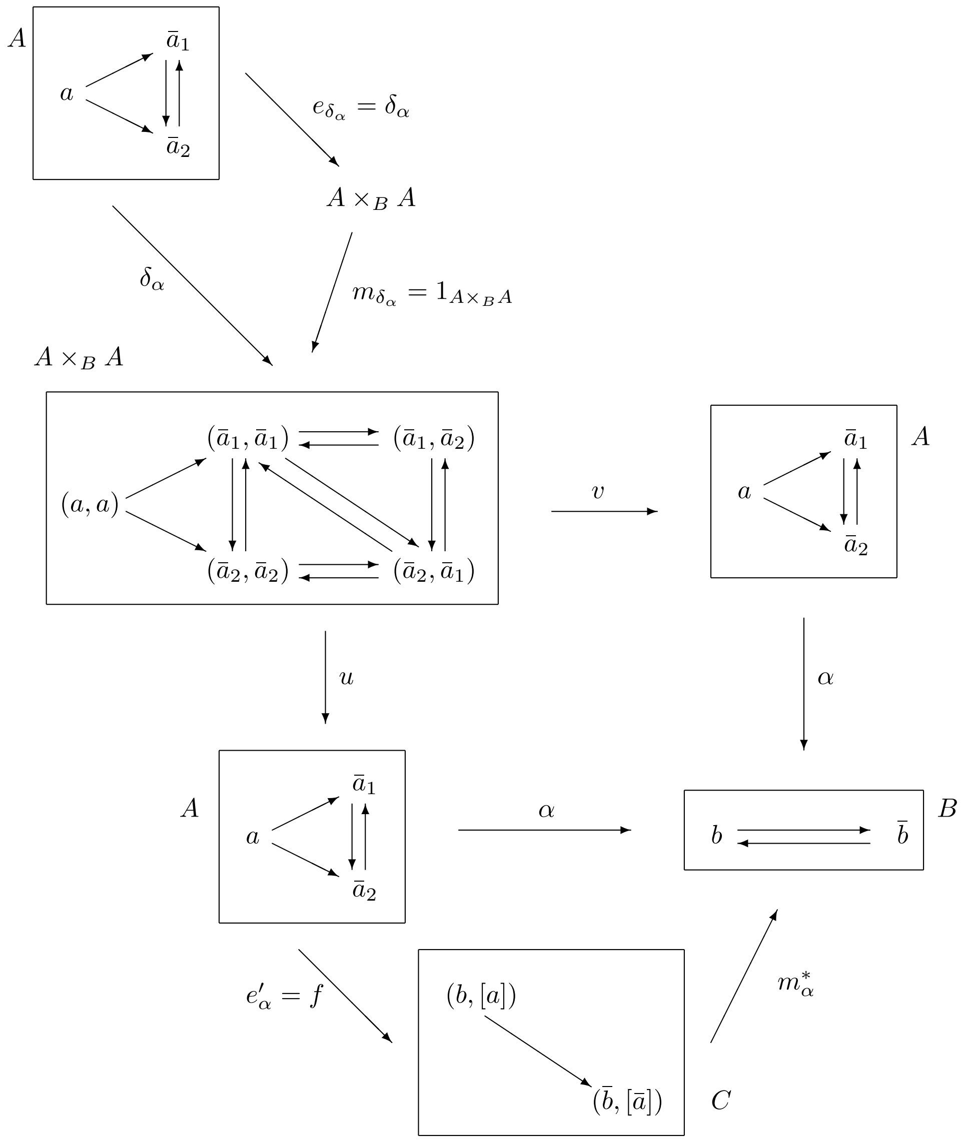
$$(\mathcal{E}', \mathcal{M}^*) = (Ins, Sep) = (Conc, Diss).$$

Example 3.38. In the pullback diagram



it is given the $(\mathcal{E},\mathcal{M})$-factorization $m_\alpha \cdot e_\alpha$ and the $(Conc, Diss)$-factorization $(m_\alpha \cdot n) \cdot f$ of a specific morphism $\alpha : A \to B$ of preorders (cf. diagrams 1.3 and 1.18), such that $\alpha(a) = b$, $\alpha(\bar{a}_i) = \bar{b}$, for $i = 1, 2$.

As concluded before, the $(Conc, Diss)$-factorization $\alpha = (m_\alpha \cdot n) \cdot f$ is also its (Ins, Sep)-factorization, and so it can be obtained using the diagram 1.17:



Remark that, as in this example the morphism $\delta_\alpha : A \to A \times_B A$ of preorders is vertical, e'_α is simply the coequalizer of the kernel-pair (u, v) of $\alpha : A \to B$.

Notice also that, in order to simplify the diagram, we have omitted four morphisms in the preorder $A \times_B A$, the ones corresponding to $(a, a) \leq (\bar{a}_1, \bar{a}_2)$, $(a, a) \leq (\bar{a}_2, \bar{a}_1)$, $(\bar{a}_1, \bar{a}_2) \leq (\bar{a}_2, \bar{a}_2)$ and $(\bar{a}_2, \bar{a}_2) \leq (\bar{a}_1, \bar{a}_2)$.

3.6.5. Normal morphisms.

PROPOSITION 3.39. *A morphism $\alpha : A \to B$ of preorders is normal if and only if the following two conditions hold:*

- *for every two objects a and a' in A,*
  if $\alpha(a) = \alpha(a')$ and $a \leq a'$ and $a' \leq a$ then $a' = a$ (it is a covering!);
- *for every three objects a, a' and $\bar{a}$ in A, if $\alpha(a) = \alpha(\bar{a})$ and $a \leq a'$ and $a' \leq a$ then there exists an object $\bar{a}'$ in A such that $\alpha(a') = \alpha(\bar{a}')$ and $\bar{a} \leq \bar{a}'$ and $\bar{a}' \leq \bar{a}$.*

EXAMPLE 3.40. The morphism $\alpha : A \to B$ of preorders, such that $\alpha(a_1) = b_1$ and $\alpha(a_2) = b_2$, is normal:

$$
\begin{tikzcd}
a_1 \arrow[d, shift left] & & b_1 \arrow[d, shift left] \arrow[dr, shift left] & \\
a_2 \arrow[u, shift left] \arrow[r, phantom, ""] & {} \arrow[r, "\alpha"] & b_2 \arrow[u, shift left] \arrow[r, shift left] & b_0 \arrow[ul, shift left] \arrow[l, shift left]
\end{tikzcd}
$$

Notice that α is not a trivial covering, since the map $[a_1] \to [b_1]$ it induces is not a bijection, and that it is in addition dissonant.

EXAMPLE 3.41. The morphism $\alpha : A \to B$ of preorders, such that $\alpha(a_1) = \alpha(\bar{a}_1) = b_1$ and $\alpha(a_2) = b_2$, is a covering which is not normal:

$$
\begin{tikzcd}
\bar{a}_1 & a_1 \arrow[d, shift left] & & b_1 \arrow[d, shift left] \\
& a_2 \arrow[u, shift left] & {} \arrow[r, "\alpha"] & b_2 \arrow[u, shift left]
\end{tikzcd}
$$

EXAMPLE 3.42. Consider the two functors $\alpha : A \to B$ and $\beta : B \to C$, such that $\alpha(a_i) = b_i$ for $i = 0, 1, 2$, and $\alpha(b_i) = \alpha(b_{i+2}) = \alpha(c_{i+1})$ for $i = 0, 1$:

$$
\begin{tikzcd}
a_0 \arrow[d, shift right] & & b_0 \arrow[d, shift right] & & \\
a_1 \arrow[u, shift right] & & b_1 \arrow[u, shift right] & & c_1 \arrow[dd, shift right] \\
 & {} \arrow[r, "\alpha"] & {} & {} \arrow[r, "\beta"] & {} \\
a_2 & & b_2 \arrow[d, shift right] & & c_2 \arrow[uu, shift right] \\
 & & b_3 \arrow[u, shift right] & &
\end{tikzcd}
$$

It is easy to check that α and β are normal but that their composite $\beta \cdot \alpha$ is not.

REMARK 3.43. Notice that by the Examples 3.23 and 3.42 normal morphisms are not closed under composition with respect to any of the three reflections **Cat** $\rightarrow$ **Preord** $\rightarrow$ **Ord**.

CHAPTER 4

The reflection of Cat into Set

As in the previous chapters we will always refer (unless stated otherwise) to a certain reflection of the category of all (small) categories. This time it is the reflection into the category of all (small) sets, which is presented right after the following definition of *connected component* of a category.

DEFINITION 4.1. An object A in **Cat** is connected if and only if it is not the empty category and for every two objects a and a' in A, there exists a zigzag of morphisms $a \to\leftarrow \cdots \to\leftarrow a'$ between a and a'.[1]

Hence, every small category A is a disjoint union of connected (small) categories, called the connected components of A.[2]

Consider the adjunction

$$(I, H, \eta, \epsilon) : \mathbf{Cat} \to \mathbf{Set}, \tag{4.1}$$

where:

- $H(X)$ is the set X regarded as a (discrete) category;
- $I(A)$ is the set of connected components of A;
- $\eta_A : A \to HI(A)$ associates with each object a in A the connected component $[a]$ to which it belongs;
- ϵ is the canonical isomorphism $IH \to 1$.

4.1. Preamble

At this chapter we will show that, although the reflection 4.1 has stable units, it does not have an associated monotone-light factorization system.

In fact, one of the conditions of Lemma 2.13 fails in this case: for each category B there is no effective descent morphism $p : E \to B$ with E a set.

In [**6**] another example of an adjunction of the kind $\mathbf{Fam}(\mathbf{Conn}(\mathbb{C})) \to \mathbf{Set}$

[1]In fact this definition is a special case of the following more general one: an object C in a category $\mathbb{C}$ with coproducts and finite limits is said to be connected if the functor $Hom(C, -) : \mathbb{C} \to \mathbf{Set}$ preserves coproducts.

[2]I.e., **Cat** is locally connected: a category $\mathbb{C}$ is said to be locally connected if every object in $\mathbb{C}$ is a coproduct of connected objects.

having stable units but not a monotone-light factorization system is presented.[3]

The coverings for this reflection of categories into sets are the functors which are simultaneously discrete fibrations and discrete opfibrations (cf. Definition 4.8). This result was obtained some years ago by Steven Lack. However, not having appropriate reference, we included below our own proof.

From this characterization, one easily checks that the effective descent morphisms given in section 2.1.1 in the Preamble of chapter 2, which are the projection of sufficiently many copies of the ordinal number **4**, are not coverings (cf. Lemma 4.9).

That is, they are no more normal morphisms, as they were for the reflections of categories into orders and preorders, since they do not split over themselves. And so, we cannot use the Theorem 7.1 in [**8**] to prove that the coverings constitute the right-hand side of a certain factorization system $(\mathcal{E}^*, \mathcal{M}^*)$, as we did for the other reflections.

Nevertheless, Theorem 6.8 in the same reference [**8**] shows that such a factorization system $(\mathcal{E}^*, \mathcal{M}^*)$ does exist, with the class $\mathcal{E}^*$ strictly containing the stably-vertical morphisms $\mathcal{E}'$, since the category **Set** is *locally bounded* in the sense of [**8**].

We also give explicit descriptions of stably-vertical, separable, purely inseparable and dissonant functors.

Of course that the latter do constitute the right-hand of a concordant-dissonant factorization system (in our sense, cf. Definition 1.36), in which the left-hand is the class of extremal epis which induce a bijection on connected components.

But what is more interesting is the need for the new notion "*full on zigzags*" in order to characterize the stably-vertical functors (see Definition 4.11).

Notice that the class of separable functors contains strictly the coverings, whereas in the other reflections studied at the precedent chapters it was the class of coverings that included the separable morphisms.

Remark finally that it remains an open question if there is an inseparable-separable factorization system for the reflection of categories into sets (remember that it is also so with the reflection into orders).

4.2. Stabilization fails

4.2.1. The reflection of Cat into Set has stable units.

The proof of next Lemma 4.2 will not be given here since it is trivial. Anyway, concerning it and the all subject behind it, we refer the reader to [**2**, §6].

That lemma and the following will be useful in proving that the reflection 4.1 has stable units, at the subsequent Proposition 4.4.

LEMMA 4.2. *Let* **Fam**(**Conn**(**Cat**)) *denote the category whose objects are the families* $(A_i)_{i\in I(A)}$ *of connected objects of* **Cat**, *and whose morphisms consist in a map* $I(\alpha) : I(A) \to I(B)$ *and an* $I(A)$*-indexed family of morphisms* $\alpha_i : A_i \to$

[3] Our section 4.2 has the same title as section 4 there.

$B_{I(\alpha)(i)}$.

Then, the coproduct functor **Fam**(**Conn**(**Cat**)) $\rightarrow$ **Cat** *which associates with each family of connected categories its coproduct in* **Cat** *is an equivalence of categories.*

LEMMA 4.3. *If A and B are two connected objects in* **Cat** *then their product $A \times B$ is also connected.*

PROOF. Indeed, we have:

(a, b) and (a', b') are objects in $A \times B \Rightarrow$

(since A and B are connected)

there exist two zigzags of morphisms joining respectively a to a' in A

and b to b' in $B \Rightarrow$

(by adding to the shorter zigzag sufficient identity arrows!)

there exist two zigzags of morphisms with the same length

joining respectively a to a' in A and b to b' in B:

$$a = a_1 \rightarrow a_2 \leftarrow \cdots \leftarrow a_n = a'$$

$$b = b_1 \rightarrow b_2 \leftarrow \cdots \leftarrow b_n = b'$$

$\Rightarrow$

there exists a zigzag joining (a, b) to (a', b') in $A \times B$:

$$(a, b) = (a_1, b_1) \rightarrow (a_2, b_2) \leftarrow \cdots \leftarrow (a_n, b_n) = (a', b')$$

$\square$

PROPOSITION 4.4. *The adjunction 4.1 has stable units, that is, the functor I :* **Cat** $\rightarrow$ **Set** *preserves every pullback of the form*

$$\begin{tikzcd} A \times_{H(X)} B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & H(X) \end{tikzcd} \quad .$$

PROOF. The equivalence of Lemma 4.2 associates respectively to the categories A, B, $H(X)$ and $A \times_{H(X)} B$ the families of connected categories $(A_i)_{i \in I(A)}$, $(B_j)_{j \in I(B)}$, $(H(X)_k)_{k \in IH(X)}$ and $(A \times_{H(X)} B)_{l \in I(A \times_{H(X)} B)}$.

Therefore, the functor I preserves the pullback in the statement if and only if I preserves every pullback of the form

$$\begin{tikzcd}
A_i \times_{H(X)_k} B_j \arrow[r] \arrow[d] & B_j \arrow[d, "\beta_j"] \\
A_i \arrow[r, "\alpha_i"] & H(X)_k
\end{tikzcd} \tag{4.2}$$

where of course $I(\alpha)(i) = k = I(\beta)(j)$.

Hence, as $I(A_i)$, $I(B_j)$ and $I(H(X)_k)$ are one-point sets, the functor I preserves the pullback above if and only if $I(A_i \times_{H(X)_k} B_j)$ is one-point set, i.e., $A_i \times_{H(X)_k} B_j$ is connected.

In fact, $H(X)_k$ is a terminal object in **Cat** since $H(X)$ is a discrete category, and so $A_i \times_{H(X)_k} B_j$ is the cartesian product $A_i \times B_j$ which by Lemma 4.3 is connected.[4]

□

4.2.2. Trivial coverings, vertical morphisms, coverings and stably-vertical morphisms for the reflection of Cat into Set.

We will give now explicit descriptions of trivial coverings, vertical morphisms, coverings and stably-vertical morphisms.

The next two propositions, characterizing trivial coverings and vertical morphisms, follow trivially from the respective definitions.[5]

PROPOSITION 4.5. *A functor $\alpha : A \to B$ is a trivial covering if and only if each α_i in the presentation of α as the family $(\alpha_i : A_i \to B_{I(\alpha)(i)})$ of morphisms in* **Conn**(**Cat**)*, is an isomorphism.*

PROPOSITION 4.6. *A functor $\alpha : A \to B$ is vertical if and only if it induces a bijection between the connected components of A and B.*

We are going now to pave the way, through the following small Lemmas 4.7 and 4.9, and Definition 4.8, for the characterization of coverings in Proposition 4.10, which says that a functor is a covering if and only if it is simultaneously a discrete fibration and a discrete opfibration.

The Lemma 4.9 follows trivially from the Definition 4.8, and so we do not present its obvious proof.

LEMMA 4.7. *A functor $\alpha : A \to B$ in* **Cat** *is a covering with respect to a certain admissible reflection of* **Cat** *into one of its full subcategories, if and only if each*

[4]Notice that, for any category $\mathbb{C}$ with coproducts and finite limits, such that every object is a coproduct of connected objects and its terminal object is connected, the reflection associated to the functor $I : \mathbb{C} \to \mathbf{Set}$ has stable units if the product of two connected objects is always connected. In fact, the proof is completely analogous to the one just given for $I : \mathbf{Cat} \to \mathbf{Set}$.

[5]These two results are special instances of the obvious more general ones applying to the case mentioned in the previous footnote.

following pullback of α

$$\begin{tikzcd} \mathbf{4} \times_B A \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\ \mathbf{4} \arrow[r] & B \end{tikzcd}$$

along any functor from the ordinal number $\mathbf{4}$ *is always a trivial covering with respect to that same reflection.*

Proof. The if part of the proof follows easily from the fact that the monadic extension (E, p) of any object B in **Cat** may be choosen to be the obvious projection from the coproduct $E = \mathbf{Cat}(\mathbf{4}, B) \cdot \mathbf{4}$ of sufficiently many copies of the ordinal number $\mathbf{4}$, one copy for each triple of composable morphisms in B.

As for the only if part of the proof it is sufficient to observe that the ordinal number $\mathbf{4}$ is projective with respect to monadic extensions, i.e., $\mathbf{Cat}(\mathbf{4}, p)$ is a surjection in **Set** for every monadic extension (E, p). Notice that the trivial coverings in the statement are pullback-stable, since the reflection is supposed to be admissible (cf. Lemma 1.8).

□

Definition 4.8. A functor $\alpha : A \to B$ in **Cat** is said to be

(1) a discrete fibration, if the diagram in **Set**

$$\begin{tikzcd} \{\text{Morphisms in } A\} \arrow[r, "\text{codomain}"] \arrow[d] & \{\text{Objects in } A\} \arrow[d] \\ \{\text{Morphisms in } B\} \arrow[r, "\text{codomain}"] & \{\text{Objects in } B\} \end{tikzcd} \tag{4.3}$$

(where the vertical arrows display the functor α) is a pullback;

(2) a discrete opfibration if the diagram in **Set**

$$\begin{tikzcd} \{\text{Morphisms in } A\} \arrow[r, "\text{domain}"] \arrow[d] & \{\text{Objects in } A\} \arrow[d] \\ \{\text{Morphisms in } B\} \arrow[r, "\text{domain}"] & \{\text{Objects in } B\} \end{tikzcd} \tag{4.4}$$

(where the vertical arrows display the functor α) is a pullback.

Lemma 4.9. *A functor $\alpha : A \to B$ in* **Cat** *is both a discrete fibration and a discrete opfibration if and only if, for every object a in A and every morphism $g : b \to b'$ in B, the following two conditions hold:*

(a) *if $\alpha(a) = b$ then there exists in A a unique morphism f such that its domain is a and $\alpha(f) = g$;*

(b) *if $\alpha(a) = b'$ then there exists in A a unique morphism f' such that its codomain is a and $\alpha(f') = g$.*

Proposition 4.10. *For any functor $\alpha : A \to B$ in* **Cat** *the following three conditions are equivalent:*

(1) *the functor α is a covering;*

(2) *every following pullback of α*

$$\begin{tikzcd} \mathbf{2} \times_B A \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\ \mathbf{2} \arrow[r] & B \end{tikzcd}$$

along any functor from the ordinal number **2** *is a trivial covering;*

(3) *the functor α is both a discrete fibration and a discrete opfibration.*

Proof. In order to show that the conditions 1, 2 and 3 in the statement are equivalent, we will assert the chain of implications 1⇒2⇒3 ⇒1.

Claim 1. The condition 1 implies the condition 2.

As every functor β from **2** to B can be factorized through **4**, we may form the diagram

$$\begin{tikzcd} C \arrow[r, "v'"] \arrow[d, "u'"] & \mathbf{4} \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\ \mathbf{2} \arrow[r] & \mathbf{4} \arrow[r] & B \end{tikzcd} \tag{4.5}$$

wherein both squares are pullbacks, and the composite of the two unnamed functors at the bottom is $\beta : \mathbf{2} \to B$.

Then, we have

the functor α is a covering $\Rightarrow$ (by Lemma 4.7)

the functor u is a trivial covering $\Rightarrow$

(the trivial coverings are pullback-stable since the reflection has stable units)

the functor u' is a trivial covering.

Claim 2. The condition 2 implies the condition 3.

We are going to prove this claim in two steps (a) and (b), corresponding respectively to the two conditions that Lemma 4.9 states to be equivalent to condition 3.

For any object a in A and every morphism $g : b \to b'$ in B we have:

(a)

$\alpha(a) = b \Rightarrow$

the ordered pair $(0, a)$ is an object in the pullback $\mathbf{2} \times_B A$ of α

along the functor which sends the unique non-identity morphism

$(0, 1) : 0 \to 1$ in the ordinal number $\mathbf{2}$, to $g : b \to b'$ in $B \Rightarrow$

(since by condition 2 the projection $\mathbf{2} \times_B A \to \mathbf{2}$ is a trivial covering,

and according to the explicit description of trivial coverings given

in Proposition 4.5)

the connected component $[(0, a)]$ of $\mathbf{2} \times_B A$ is isomorphic to $\mathbf{2}$ ($[(0, a)] \cong \mathbf{2}$) $\Rightarrow$

there exists in A a unique morphism $f : a \to a'$ such that

its domain is a and $\alpha(f) = g$;

(b)

$\alpha(a) = b' \Rightarrow$

the ordered pair $(1, a)$ is an object in the pullback $\mathbf{2} \times_B A$ of α

along the functor which sends the unique non-identity morphism

$(0, 1) : 0 \to 1$ in the ordinal number $\mathbf{2}$, to $g : b \to b'$ in $B \Rightarrow$

(since by condition 2 the projection $\mathbf{2} \times_B A \to \mathbf{2}$ is a trivial covering,

and according to the explicit description of trivial coverings given

in Proposition 4.5)

the connected component $[(1, a)]$ of $\mathbf{2} \times_B A$ is isomorphic to $\mathbf{2}$ ($[(1, a)] \cong \mathbf{2}$) $\Rightarrow$

there exists in A a unique morphism $f' : a' \to a$ such that

its codomain is a and $\alpha(f') = g$.

Claim 3. The condition 3 implies the condition 1.

Indeed, for any functor $\alpha : A \to B$ which is both a discrete fibration and a discrete opfibration, we have

> $(b_0, g_1, b_1, g_2, b_2, g_3, b_3)$ is a chain of three composable morphisms
>
> $g_i : b_{i-1} \to b_i$ in B, such that at least one of its objects b_i is in
>
> the image of $\alpha : A \to B \Rightarrow$
>
> (by the existence assertions given at (a) and (b) of Lemma 4.9)
>
> there exists a chain $(a_0, f_1, a_1, f_2, a_2, f_3, a_3)$ of three composable morphisms
>
> $f_i : a_{i-1} \to a_i$ in A, such that $\alpha(f_i) = g_i$ for $i = 1, 2, 3 \Rightarrow$
>
> (by the uniqueness assertions given at (a) and (b) of Lemma 4.9)
>
> the connected component $[(0, a_0)]$ of $\mathbf{4} \times_B A$, which is the pullback of α along
>
> the functor $\beta : \mathbf{4} \to B$ corresponding to the above chain of three
>
> composable morphisms, is isomorphic to $\mathbf{4}$ ($[(0, a_0)] \cong \mathbf{4}$),

and so we proved that any pullback of any discrete fibration and opfibration $\alpha : A \to B$ along every functor from $\mathbf{4}$ to B is a trivial covering. Notice that, in case none of the objects in the image of $\beta : \mathbf{4} \to B$ is in the image of $\alpha : A \to B$ the pullback $\mathbf{4} \times_B A$ of α along β is the empty category.

Hence, according to Lemma 4.7, any functor which is both a discrete fibration and a discrete opfibration is also a covering.

□

DEFINITION 4.11. We will say that a functor $\alpha : A \to B$ is "full on zigzags" if, for any pair of objects a and a' in A and any zigzag $\bar{g} = (g_1, g_2, ..., g_m)$ from $\alpha(a)$ to $\alpha(a')$ in B, there exists a zigzag $\bar{f} = (f_1, f_2, ..., f_n)$ from a to a' in A from which we can obtain the zigzag $\bar{g}$ by just erasing some identity arrows in the sequence $\alpha(\bar{f}) = (\alpha(f_1), \alpha(f_2), ..., \alpha(f_n))$.

Remark that requiring only that each morphism in B is the image under α of a zigzag in A, in the above sense, i.e., up to some identity arrows, is not sufficient for α to be "full on zigzags".

EXAMPLE 4.12. Consider the functor α from the category $a \leftarrow b \to c$, with three objects and only two non-identity arrows, to the ordinal number $\mathbf{2} = 0 \to 1$, such that $\alpha(a) = 0 = \alpha(b)$ and $\alpha(c) = 1$.

One easily concludes that although α is not a full functor it is nevertheless "full on zigzags", in the sense of previous Definition 4.11.

PROPOSITION 4.13. *A functor $\alpha : A \to B$ in* **Cat** *is stably-vertical if and only if it is "full on zigzags" and surjective on objects.*

PROOF. We have:

a functor $\alpha : A \to B$ is stably-vertical $\Leftrightarrow$

(according to the above definitions of stably-vertical and vertical morphisms)

for every pullback u of α, $I(u)$ is an isomorphism $\Leftrightarrow$

$I(\alpha)$ is an isomorphism and I preserves every pullback of α

(see diagram 2.8) $\Leftrightarrow$

$I(\alpha)$ is an isomorphism and I preserves every pullback

$$\begin{tikzcd} C \times_{B_{I(\alpha)(i)}} A_i \arrow[r] \arrow[d] & A_i \arrow[d, "\alpha_i"] \\ C \arrow[r, "\gamma"] & B_{I(\alpha)(i)} \end{tikzcd} \tag{4.6}$$

of each α_i in the presentation of α as the family $(\alpha_i : A_i \to B_{I(\alpha)(i)})$

of morphisms in $\mathbf{Conn}(\mathbf{Cat})$ $\Leftrightarrow$

$I(\alpha)$ is an isomorphism and if C is connected then $C \times_{B_{I(\alpha)(i)}} A_i$ is connected,

for every pullback square as above.

If α is "full on zigzags" and surjective on objects then one easily checks that the statement immediately above, which was seen to be equivalent to α being a stably-vertical morphism, is true.

So, in order to complete the proof we have to show the converse conditional statement.

Suppose first that α is not surjective on objects: if the object b in $B_{I(\alpha)(i)}$ is not in the image of α_i then the empty functor $u : 0 \to 1$ (notice that 0 is not connected!) is the pullback of α_i along the functor $\varphi : 1 \to B_{I(\alpha)(i)}$ such that $\varphi(*) = b$.

Suppose at last that α is not "full on zigzags": take a zigzag $\bar{g} = (g_1, g_2, ..., g_m)$ from $\alpha(a)$ to $\alpha(a')$ in $B_{I(\alpha)(i)}$ which is *not* up to some identity arrows the image of any zigzag from a to a' in A_i, in the sense of Definition 4.11.

Then, consider the category $Z(\bar{g})$ consisting of just one zigzag

$$z(\bar{g}) = (z(g_1), z(g_2), ..., z(g_m))$$

from $z(\alpha(a))$ to $z(\alpha(a'))$, with the same length as $\bar{g}$, and whose arrows have the same directions as the corresponding arrows in $\bar{g}$, but in which all objects are distinct (the sequence $z(\bar{g})$ keeps therefore just the "zigzag" structure of $\bar{g}$, leaving out all other information about the nature of the morphisms in $\bar{g}$).

The pullback of α_i along the obvious projection of $Z(\bar{g})$ into $B_{I(\alpha)(i)}$ is obviously not connected, since by hypothesis there cannot be no zigzag between $(z(\alpha(a)), a)$ and $(z(\alpha(a')), a')$ in $Z(\bar{g}) \times_{B_{I(\alpha)(i)}} A_i$. In effect, if there was such a zigzag then the image under α_i of its projection into A_i would be up to some identity arrows the zigzag $\bar{g}$.

□

Consider for instance the inclusion $\iota : \mathbf{1} \to \mathbf{2}$ of the ordinal number $\mathbf{1}$ into the ordinal number $\mathbf{2}$, taking the only object in $\mathbf{1}$ to the object 0 in $\mathbf{2}(= 0 \to 1)$. Using the characterizations given in Proposition 4.13 and Lemma 4.9 one easily checks that the functor ι does not have a $(\mathcal{E}', \mathcal{M}^*)$-factorization. Therefore:

CONCLUSION 4.14. *The pair $(\mathcal{E}', \mathcal{M}^*)$ is not a factorization system on* **Cat**.

We could also arrive to the conclusion above by using a proposition in [**6**].

There it was stated that, for the factorization system associated with a reflection of the kind $\mathbf{Fam}(\mathbf{Conn}(\mathbb{C})) \to \mathbf{Set}$, if there exists a pullback diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & E \end{tikzcd} \tag{4.7}$$

in $\mathbf{Fam}(\mathbf{Conn}(\mathbb{C}))$, with 0 denoting the initial object (the empty family), U and V non-initial, and E connected projective, then the factorization system is not locally stable (cf. Theorem 1.18).

Now, in our case, if we make U and V the terminal object in **Cat** and E the ordinal number $\mathbf{2}$, it is obvious that we have such a pullback diagram 4.7.

4.3. Separable and purely inseparable morphisms

PROPOSITION 4.15. *A functor $\alpha : A \to B$ in* **Cat** *is separable if and only if, for every two morphisms $f : a \to a'$ and $\bar{f} : \bar{a} \to \bar{a}'$ in A, if $\alpha(f) = \alpha(\bar{f})$ and either its domains or its codomains coincide ($a = \bar{a}$ or $a' = \bar{a}'$) then $f = \bar{f}$.*

Notice that this characterization corresponds to the removal from Lemma 4.9 of the existence demands, and to ask only for uniqueness.

PROOF. We have to show that the following functor $\delta_\alpha : A \to A \times_B A$

$$\begin{tikzcd} A \arrow[dr, "\delta_\alpha"] \arrow[drr, bend left, "1_A"] \arrow[ddr, bend right, "1_A"'] & & \\ & A \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\ & A \arrow[r, "\alpha"] & B \end{tikzcd}$$

displayed for any object a as $\delta_\alpha(a) = (a, a)$, belongs to $\mathcal{M}$ if and only if the condition given in the statement above holds.

We already know that $\delta_\alpha : A \to A \times_B A$ belongs to $\mathcal{M}$ if and only if, for every object a in A,the functor induced by δ_α from the connected component $[a]$ of A into the connected component $[(a, a)]$ of $A \times_B A$ is an iso ($[a] \cong [(a, a)]$). That is, if the functor induced by δ_α from $[a]$ to $[(a, a)]$ is fully faithful and a bijection on objects.

As $\delta_\alpha : A \to A \times_B A$ is obviously injective on objects and faithful, we only need to show that, for every object a in A, the functor induced by δ_α from $[a]$ to $[(a, a)]$ is full and a surjection on objects.

Indeed, we have

for every object a in A,

the functor induced by δ_α from $[a]$ to $[(a, a)]$,

displayed for a morphism f in $[a]$ as $\delta_\alpha(f) = (f, f)$, is full $\Leftrightarrow$

(i) the functor $\alpha : A \to B$ is faithful,

and

for every object a in A,

the functor induced by δ_α from $[a]$ to $[(a, a)]$,

displayed for an object a in $[a]$ as $\delta_\alpha(a) = (a, a)$, is a surjection on objects $\Leftrightarrow$

(ii) for any triple of objects a, a' and a'' in A, such that $\alpha(a') = \alpha(a'')$ and $a' \neq a''$, the hom-sets $Hom_{A\times_B A}((a, a), (a', a''))$ and $Hom_{A\times_B A}((a', a''), (a, a))$ are empty.

Now it is easy to check that conditions (i) and (ii), proved above to be equivalent to α being a separable functor, are also equivalent to the condition in the statement. $\square$

PROPOSITION 4.16. *A functor $\alpha : A \to B$ in* **Cat** *is purely inseparable if and only if, for every two objects a and a' in A, if $\alpha(a) = \alpha(a')$ then there exist two zigzags of the same length in A, $\bar{h} = (h_1, h_2, ..., h_n)$ between a and a, and $\bar{f} = (f_1, f_2, ..., f_n)$ between a and a', such that*

$$\alpha(\bar{h}) = (\alpha(h_1), \alpha(h_2), ..., \alpha(h_n)) = (\alpha(f_1), \alpha(f_2), ..., \alpha(f_n)) = \alpha(\bar{f}).$$

PROOF. According to Definition 1.26 and Proposition 4.6, we have to show that the following functor $\delta_\alpha : A \to A \times_B A$

$$\begin{tikzcd}
A \arrow[dr, "\delta_\alpha"] \arrow[drr, bend left, "1_A"] \arrow[ddr, bend right, "1_A"'] & & \\
& A \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\
& A \arrow[r, "\alpha"] & B
\end{tikzcd}$$

induces a bijection between the connected components of A and $A \times_B A$.

The fact that δ_α induces an injection between the connected components holds trivially, since if there is no zigzag between the objects a and a' in A there can be obviously no zigzag between $\delta_\alpha(a) = (a, a)$ and $\delta_\alpha(a') = (a', a')$ in $A \times_B A$.

We then observe:

the functor $\delta_\alpha : A \to A \times_B A$ induces a bijection between

the connected components of A and $A \times_B A \Leftrightarrow$

for every two objects a and a' in A, if $\alpha(a) = \alpha(a')$ then

there exists an object $\bar{a}$ in A such that

there exists a zigzag between (a, a') and $(\bar{a}, \bar{a})$ in $A \times_B A \Leftrightarrow$

for every two objects a and a' in A, if $\alpha(a) = \alpha(a')$ then

there exists a zigzag between (a, a') and (a, a) in $A \times_B A \Leftrightarrow$

for every two objects a and a' in A, if $\alpha(a) = \alpha(a')$ then

there exist two zigzags of the same length in A,

$\bar{h} = (h_1, h_2, ..., h_n)$ between a and a, and

$\bar{f} = (f_1, f_2, ..., f_n)$ between a and a', such that

$$\alpha(\bar{h}) = (\alpha(h_1), \alpha(h_2), ..., \alpha(h_n)) = (\alpha(f_1), \alpha(f_2), ..., \alpha(f_n)) = \alpha(\bar{f}).$$

□

Example 4.17. Consider the two functors $\alpha : A \to B$ and $\beta : B \to C$, such that $\alpha(a) = b' = \alpha(a')$, $\alpha(\bar{a}) = \bar{b}$, $\alpha(\bar{a}') = \bar{b}'$, $\alpha(a'') = b''$, $\beta(\bar{b}) = \bar{c}$, $\beta(\bar{b}') = \bar{c}'$ and $\beta(b') = c' = \beta(b'')$:

$$\begin{tikzcd}
a & & & & & & & & & \\
& \bar{a} \arrow[ul] \arrow[dl] \arrow[l, phantom, "\Downarrow"] & & & & \bar{b} \arrow[dl] & & & & \bar{c} \arrow[dl] \\
a' & & {} \arrow[r, "\alpha"] & {} & b' & & {} \arrow[r, "\beta"] & {} & c' & \\
& \bar{a}' \arrow[ul] \arrow[dl] & & & & \bar{b}' \arrow[ul] \arrow[dl] \arrow[l, phantom, "\Uparrow"] & & & & \bar{c}' \arrow[ul] \\
a'' & & & & b'' & & & & &
\end{tikzcd}$$

It is easy to check that α and β are purely inseparable but its composite $\beta \cdot \alpha$ is not.

Moreover, these three functors α, β and $\beta \cdot \alpha$ between orders are all regular epis but none of them is stably-vertical, and $\beta \cdot \alpha$ is inseparable, as it is easy to check.

CONCLUSION 4.18. *The information given by the last example 4.17 allows us to conclude that the classes* Pin *and* $Pin \cap RegEpi$ *can never be part of a factorization system, since they are not closed under composition, and that the following inclusions are all strict (cf. Remark 1.28 and Lemma 1.30):*

$$\mathcal{E}' \subset Pin \cap RegEpi \subset Ins$$

4.4. Concordant-dissonant factorization

We are going again to consider the factorization system $(ExtEpi, Mono)$ on **Cat**, for the definition of concordant-dissonant factorization, now with respect to the reflection of categories into sets.

Hence, according to Proposition 1.35 and Definition 1.36, the pair $(\mathcal{E} \cap ExtEpi, Diss)$ is a factorization system. The following proposition gives a characterization of the functors in its second component $Diss$.

PROPOSITION 4.19. *A functor* $\alpha : A \to B$ *in* **Cat** *is dissonant if and only if each* α_i *in the presentation of* α *as the family* $(\alpha_i : A_i \to B_{I(\alpha)(i)})$ *of morphisms in* **Conn**(**Cat**)*, is a monomorphism.*

We will also express this by saying that, for every object a *in* A*,* α *induces a monomorphism between the connected components* $[a]$ *and* $[\alpha(a)]$ *of* A *and* B*, respectively.*

PROOF. According to Proposition 1.35 and Definition 1.36, the functor $\alpha : A \to B$ is dissonant if and only if the following functor $\langle \alpha, \eta_A \rangle : A \to B \times I(A)$ is monic:

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[d, "\langle \alpha, \eta_A \rangle"] \arrow[dr, "\eta_A"] & \\ B & B \times I(A) \arrow[l] \arrow[r] & I(A) \end{tikzcd}$$

The functor $\langle \alpha, \eta_A \rangle : A \to B \times I(A)$ is displayed as

$$\langle \alpha, \eta_A \rangle(a) = (\alpha(a), [a]).$$

Now it is easy to conclude that $\langle \alpha, \eta_A \rangle$ is a monic functor if and only if every restriction of $\alpha : A \to B$ to any connected component of A is always a monomorphism.

□

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