# On the Number of Invariant Polynomials of Matrix Commutators

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**Abstract**

We study the possible numbers of nonconstant invariant polynomials of the matrix commutator $XA - AX$, when $X$ varies.

Let $F$ be a field, $A, B \in F^{n\times n}$ and denote by $i(A)$ the number of nonconstant invariant polynomials of $A$.

In [7], it was proved that, if there exists $X$ such that $A + XBX^{-1}$ is nonderogatory (i. e., $i(A+XBX^{-1}) = 1$), then $i(A)+i(B) \leq n+1$; and it was conjectured that the converse is true, under rather slight restrictions on $F$. This conjecture was proved in a theorem [9] that gives necessary and sufficient conditions for the existence of $X$ such that $i(A + XBX^{-1}) \leq t$, where $t$ is a positive integer. Later [11], all the possible values of $i(A + XBX^{-1})$, when $X$ varies, were described, assuming that $F$ is algebraically closed. In general fields, this is an open problem.

The possible numbers of nonconstant invariant polynomials of partially given matrices were also studied in several papers, e. g., [1, 4, 5, 10].

Some properties of the commutator $XA - AX$, when $X$ varies, have already been studied. Suppose that $D$ is a division ring and $A \in D^{n\times n}$. The

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rank of $XA - AX$, when $X$ runs over $D^{n\times n}$, was studied in [2]. The same problem, when $X$ runs over the set of the nonsingular matrices of $D^{n\times n}$, was studied in [8]. The eigenvalues of $XA-AX$, when $X$ runs over $F^{n\times n}$ and also when $X$ runs over the set of the nonsingular matrices of $F^{n\times n}$, were studied in [6].

Given $A \in F^{n\times n}$, the following theorem solves the problem of characterizing the possible values of $i(XA-AX)$, when $X$ varies, assuming that all the irreducible polynomials in $F[x]$ have degree $\leq 2$. In particular, the problem is solved for algebraically closed fields and for the field of real numbers, $\mathbb{R}$. We shall prove Theorem 1 later. Observe that a field $F$ such that all the irreducible polynomials in $F[x]$ have degree $\leq 2$ has to be infinite.

From now on, $A$ denotes an $n \times n$ matrix over $F$, $f_1(x) \mid \cdots \mid f_r(x)$, $r = i(A)$, are the nonconstant invariant polynomials of $A$, and $t \in \{1, \ldots, n\}$. We assume that the invariant polynomials are always monic.

**Theorem 1** *Suppose that $F$ is a field such that all the irreducible polynomials in $F[x]$ have degree $\leq 2$. Let $A \in F^{n\times n}$, $t \in \{1, \ldots, n\}$. The following conditions are equivalent:*

$(a_1)$ *There exists a nonsingular matrix $X \in F^{n\times n}$ such that $i(XA-AX) = t$.*

$(b_1)$ *There exists $X \in F^{n\times n}$ such that $i(XA - AX) = t$.*

$(c_1)$ *One of the following conditions holds:*

- $(i_1)$ *$f_r(x)$ is irreducible of degree 2 and $t$ is even.*
- $(ii_1)$ *$f_r(x)$ is irreducible of degree 2 and $t \leq n/2$.*
- $(iii_1)$ *$f_r(x)$ is not irreducible of degree 2 and $2i(A) \leq n + t$.*

**Corollary 2** *Suppose that $F$ is an algebraically closed field. Then $(a_1)$, $(b_1)$ and the following condition $(c_2)$ are equivalent:*

$(c_2)$ $2i(A) \leq n + t$.

Given a polynomial $f(x) = x^k - a_{k-1}x^{k-1} - \cdots - a_1x - a_0$, denote by $d(f)$ the degree of $f$ and denote by $C(f)$ the companion matrix

$$\left[\begin{array}{c|ccc} 0 & & I_{k-1} & \\ \hline a_0 & a_1 & \cdots & a_{k-1} \end{array}\right].$$

Let $\bar{F}$ be an algebraically closed extension of $F$. Let

$$R_{\bar{F}}(A) = \min_{\lambda \in \bar{F}} \operatorname{rank}(A - \lambda I_n).$$

**Lemma 3** [7] $R_{\bar{F}}(A) = n - i(A)$.

It is well-known that $A$ is nonderogatory if and only if $i(A) = 1$. It follows, from the previous lemma, that $A$ is nonderogatory if and only if $R_{\bar{F}}(A) = n - 1$.

**Lemma 4** [2, 8] *Suppose that either $F \neq \{0,1\}$ or $n \neq 2$. Let $\rho \in \{0, \ldots, n\}$. The following statements are equivalent:*

($a_4$) *There exists a nonsingular matrix $X \in F^{n\times n}$ such that* $\operatorname{rank}(XA - AX) = \rho$.

($b_4$) *There exists $X \in F^{n\times n}$ such that* $\operatorname{rank}(XA - AX) = \rho$.

($c_4$) *One of the following conditions holds:*

  ($i_4$) *$f_r(x)$ is irreducible of degree 2 and $\rho$ is even.*

  ($ii_4$) *$f_r(x)$ is irreducible of degree $\geq 3$ and $\rho \neq 1$.*

  ($iii_4$) *$f_r(x)$ is not irreducible of degree $\geq 2$ and $\rho \leq 2R_{\bar{F}}(A)$.*

**Remark 1** *In the original papers, Lemma 4 was established, with a slightly different statement, for arbitrary division rings. More precisely, [2] gives a necessary and sufficient condition for ($b_4$) and [8] gives a necessary and sufficient condition for ($a_4$).*

**Lemma 5** *[6] Let $c_1, \ldots, c_n$ be elements of $F$ such that $c_1 + \cdots + c_n = 0$. If $2i(A) \leq n$ and $d(f_r) \geq 3$, then there exists a nonsingular matrix $X \in F^{n\times n}$ such that $XA - AX$ has eigenvalues $c_1, \ldots, c_n$.*

If $A' \in F^{n\times n}$ is similar to $A$ and $Z \in F^{n\times n}$ is a nonsingular matrix such that $A' = Z^{-1}AZ$, then $R_{\bar{F}}(A) = R_{\bar{F}}(A')$ and, for every $X \in F^{n\times n}$, $XA - AX$ and $(Z^{-1}XZ)A' - A'(Z^{-1}XZ)$ are similar. Therefore, in the proofs of the following lemmas and in the proof of Theorem 1, we can replace $A$ by any similar matrix and we shall assume, without loss of generality, that

$$A = C(f_1) \oplus \cdots \oplus C(f_r). \tag{1}$$

Assume that $A$ has the form (1). Then, for every $\lambda \in \bar{F}$, $\mathrm{rank}(A-\lambda I_n) \geq n-r$. The equality holds if and only if $\lambda$ is a root of $f_1(x)$. Note that this argument gives a proof for Lemma 3. The eigenvalues of $A$ are the roots of $f_1(x)\cdots f_r(x)$. We call *primary* eigenvalues of $A$ to the roots of $f_1(x)$.

If $A$ has a primary eigenvalue $\lambda \in F$, then $R_{\bar{F}}(A) = R_{\bar{F}}(A-\lambda I_n)$ and, for every $X \in F^{n\times n}$, $XA - AX = X(A-\lambda I_n) - (A-\lambda I_n)X$. Therefore, in the proofs of the following lemmas and in the proof of Theorem 1, if $A$ has a primary eigenvalue in $F$, we shall also assume, without loss of generality, that 0 is a primary eigenvalue of $A$. Note that, in this case, the first column of $C(f_i)$, $i \in \{1,\ldots,r\}$, is equal to zero.

**Corollary 6** *Suppose that $F$ is an infinite field. Let $c \in F$. If $n \geq 4$, $2i(A) \leq n$ and $d(f_r) \geq 3$, then there exists a nonsingular matrix $X \in F^{n\times n}$ such that $XA - AX$ is nonderogatory and $c$ is eigenvalue of $XA - AX$.*

Proof. As $F$ is infinite and $n \geq 4$, there exist distinct elements $c_1,\ldots,c_n \in F$ such that $c_1 = c$ and $c_1+\cdots+c_n = 0$. According to Lemma 5, there exists a nonsingular matrix $X \in F^{n\times n}$ such that $XA-AX$ has eigenvalues $c_1,\ldots,c_n$. Clearly, $XA - AX$ is nonderogatory. ∎

The following corollary is also easy to prove.

**Corollary 7** *Suppose that $F$ is an infinite field. Let $G$ be a finite subset of $F$. If $2i(A) \leq n$ and $d(f_r) \geq 3$, then there exists a nonsingular matrix $X \in F^{n\times n}$ such that $XA - AX$ is nonderogatory and does not have any eigenvalue in $G$.*

**Lemma 8** *Suppose that $F$ is an infinite field. If $C \in F^{2\times 2}$ is nonderogatory, then the set of the values $\det(XC - CX)$, where $X$ runs over the set of the nonsingular matrices of $F^{2\times 2}$ such that $XC - CX$ is nonderogatory, is infinite.*

Proof. Without loss of generality, suppose that

$$C = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}. \tag{2}$$

Choose $u \in F \setminus \{0,1\}$. For every $v \in F$, let

$$X_v = \begin{bmatrix} u-1 & 0 \\ v & u \end{bmatrix}.$$

Then $X_v$ is nonsingular, $X_vC - CX_v$ is nonderogatory and $\det(X_vC - CX_v) = -v^2 - bv + a$. For every $\lambda \in F$, the quadratic equation, in $v$, $-v^2 - bv + a = \lambda$ has, at most, 2 roots. Therefore $\{-v^2 - bv + a : v \in F\}$ is infinite. ∎

**Lemma 9** *Suppose that $F$ is an infinite field. Let $G$ be a finite subset of $\bar{F}$. If $d(f_1) = \cdots = d(f_r) = 2$, then there exists a nonsingular matrix $X \in F^{n\times n}$ such that $XA - AX$ is nonderogatory and does not have any eigenvalue in $G$.*

Proof. We have $f_1 = \cdots = f_r$. Let $C = C(f_1)$. It follows, from Lemma 8, that there exists an infinite list of nonsingular matrices $X_1, X_2, \ldots,$ in $F^{2\times 2}$, such that $D_i := X_iC - CX_i$ is nonderogatory and $\det D_i \neq \det D_j$, $i, j \in \{1, 2, \ldots\}$, $i \neq j$. As all the matrices $D_i$ have trace equal to zero, it is easy to deduce that, if $i \neq j$, then $D_i$ and $D_j$ do not have a common eigenvalue. Without loss of generality, we may assume that none of the matrices $D_1, \ldots, D_r$ has an eigenvalue in $G$. Then $XA - AX$, where $X = X_1 \oplus \cdots \oplus X_r$, is nonderogatory and does not have any eigenvalue in $G$. ∎

**Lemma 10** *Suppose that $F \neq \{0,1\}$. Let $\rho \in \{2, \ldots, n-1\}$. If $A$ is nonderogatory, then there exists a nonsingular matrix $X \in F^{n\times n}$ such that $XA - AX$ is nilpotent and $R_{\bar{F}}(XA - AX) = \rho$.*

Proof. According to a previous assumption, $A = C(f_1)$. If $\rho$ is even, let $X$ be the $n \times n$ matrix with the principal entries, and the entries $(2k+1, 2k-1)$, $k \in \{1, \ldots, \rho/2\}$, equal to 1.

If $\rho$ is odd, let $e \in F \setminus \{0,1\}$ and let $X$ be the $n \times n$ matrix with the principal entries, and the entries $(2k+1, 2k-1)$, $k \in \{1, \ldots, (\rho-1)/2\}$, equal to 1 and the entry $(\rho+1, \rho-1)$ equal to $e$.

For any value of $\rho$, $XA-AX$ is lower triangular with the principal entries equal to zero, the entries $(i+1,i)$, $i \in \{1,\ldots,\rho\}$, different from zero, and the columns $\rho+1,\ldots,n$ equal to zero. Clearly, $XA-AX$ is nilpotent and $R_{\bar{F}}(XA-AX)=\rho$. ∎

**Lemma 11** *Suppose that $F$ is an infinite field. Let $\rho \in \{0,\ldots,n-1\}$. Condition $(c_4)$ is equivalent to any of the following conditions:*

$(a_{11})$ *There exists a nonsingular matrix $X \in F^{n\times n}$ such that $R_{\bar{F}}(XA-AX)=\operatorname{rank}(XA-AX)=\rho$.*

$(b_{11})$ *There exists $X \in F^{n\times n}$ such that $R_{\bar{F}}(XA-AX)=\operatorname{rank}(XA-AX)=\rho$.*

Proof. We only need to prove that $(c_4)$ implies $(a_{11})$. This proof is by induction on $n$. Suppose that $(c_4)$ is satisfied. Note that, if $\rho=0$, then $(a_{11})$ is trivial. Thus, suppose that $\rho>0$. Then $(c_4)$ implies that $A$ is nonscalar.

Suppose that $\rho=1$. According to Lemma 4, there exists a nonsingular matrix $X \in F^{n\times n}$ such that $\operatorname{rank}(XA-AX)=1$. Then $R_{\bar{F}}(XA-AX)=1$.

We have already proved Lemma 11 when $n \leq 2$.

Suppose that $n \geq 3$ and that $\rho \geq 2$. The case $r=1$ follows, immediately, from Lemma 10. Then, we also suppose that $r \geq 2$.

Case 1. Suppose that $\rho=n-1$.

Subcase 1.1. Suppose that $d(f_r) \geq 3$. As $r \geq 2$, we have $n \geq 4$. From $(c_4)$, it follows that $2i(A) \leq n+1$. If $2i(A) \leq n$, then, according to Corollary 6, there exists a nonsingular matrix $X \in F^{n\times n}$ such that $XA-AX$ is nonderogatory and 0 is eigenvalue of $XA-AX$, that is, $(a_{11})$ is satisfied, with $\rho=n-1$.

Now suppose that $2i(A)=n+1$. Then $f_1(x)$ has degree 1 and $A=[c]\oplus A_0$, where $c \in F$ and $A_0=C(f_2)\oplus\cdots\oplus C(f_r)$. According to Corollary 7, there exists a nonsingular matrix $X_0 \in F^{(n-1)\times(n-1)}$ such that $X_0A_0-A_0X_0$ is nonderogatory and nonsingular. Taking $X=[1]\oplus X_0$, $XA-AX$ is nonderogatory and 0 is eigenvalue of $XA-AX$, that is, $(a_{11})$ is satisfied.

Subcase 1.2. Suppose that $d(f_r)=2$. As $(c_4)$ is satisfied, $f_r(x)$ is reducible and, therefore, $A$ has a primary eigenvalue in $F$. According to a previous assumption, 0 is a primary eigenvalue of $A$.

Note that, if $n$ is even, then $r=n/2$ and $f_1(x)=\cdots=f_r(x)$; and, if $n$ is odd, then $r=(n+1)/2$, $f_1(x)=x$ and $f_2(x)=\cdots=f_r(x)$.

Let $A_1 = C(f_1)$ and $A_2 = C(f_2) \oplus \cdots \oplus C(f_r)$. If $n$ is even, then there exists, as we have already seen, a nonsingular matrix $X_1 \in F^{2\times 2}$ such that $R_{\bar{F}}(X_1A_1 - A_1X_1) = \mathrm{rank}(X_1A_1 - A_1X_1) = 1$. If $n$ is odd, let $X_1 = [1] \in F^{1\times 1}$.

In any case, according to Lemma 9, there exists a nonsingular matrix $X_2 \in F^{n'\times n'}$, where $n'$ is the largest even integer less than $n$, such that $X_2A_2 - A_2X_2$ is nonderogatory and does not have any eigenvalue in common with $X_1A_1 - A_1X_1$.

Let $X = X_1 \oplus X_2$. Then $R_{\bar{F}}(XA - AX) = \mathrm{rank}(XA - AX) = n - 1$.

Case 2. Suppose that $\rho < n - 1$. Let $\delta = d(f_1)$, $\rho_1 = \min\{\delta - 1, \rho\}$, $\rho_2 = \rho - \rho_1$, $A_1 = C(f_1)$, $A_2 = C(f_2) \oplus \cdots \oplus C(f_r)$. Then $R_{\bar{F}}(A_1) = \delta - 1$ and $R_{\bar{F}}(A_2) = R_{\bar{F}}(A) - \delta + 1$. If $\delta = 1$, we have $2R_{\bar{F}}(A_2) = 2R_{\bar{F}}(A) \geq \rho = \rho_2$. If $\delta > 1$ and $\rho_1 = \delta - 1$, we have $2R_{\bar{F}}(A_2) \geq n - \delta > \rho - \delta + 1 = \rho_2$. If $\rho_1 = \rho$, then $\rho_2 = 0$ and we also have $2R_{\bar{F}}(A_2) \geq \rho_2$.

If the induction assumption can be used, then there exist nonsingular matrices $X_1 \in F^{\delta\times\delta}$, $X_2 \in F^{(n-\delta)\times(n-\delta)}$ such that $R_{\bar{F}}(X_1A_1 - A_1X_1) = \mathrm{rank}(X_1A_1 - A_1X_1) = \rho_1$ and $R_{\bar{F}}(X_2A_2 - A_2X_2) = \mathrm{rank}(X_2A_2 - A_2X_2) = \rho_2$. Let $X = X_1 \oplus X_2$. Clearly, $R_{\bar{F}}(XA - AX) = \mathrm{rank}(XA - AX) = \rho$.

Now suppose that the induction assumption cannot be used in the previous argument. Then one, at least, of the following conditions is satisfied:

($a'_{11}$) $f_1(x)$ is irreducible and $\rho_1 = 1$.

($b'_{11}$) $f_r(x)$ is irreducible and $\rho_2 = 1$.

Subcase 2.1. Suppose that ($a'_{11}$) is satisfied.

Subcase 2.1.1. Suppose that $\rho < n - 2$. Note that, if $\rho_1 = \rho$, then ($c_4$) implies that $f_r(x)$ is reducible; and that, if $\rho_1 = \delta - 1 < \rho$ and $\rho$ is odd, then ($c_4$) also implies that $f_r(x)$ is reducible. In any situation, according to the induction assumption, there exists a nonsingular matrix $X_2 \in F^{(n-\delta)\times(n-\delta)}$ such that $R_{\bar{F}}(X_2A_2 - A_2X_2) = \mathrm{rank}(X_2A_2 - A_2X_2) = \rho$. Let $X = I_\delta \oplus X_2$. Then $R_{\bar{F}}(XA - AX) = \mathrm{rank}(XA - AX) = \rho$.

Subcase 2.1.2. Suppose that $\rho = n - 2$. Then $1 = \rho_1 = \delta - 1 < \rho$. According to Corollary 7 or Lemma 9, there exists a nonsingular matrix $X_2 \in F^{(n-2)\times(n-2)}$ such that $X_2A_2 - A_2X_2$ is nonderogatory and 0 is not eigenvalue of $X_2A_2 - A_2X_2$. Let $X = I_\delta \oplus X_2$. Then $R_{\bar{F}}(XA - AX) = \mathrm{rank}(XA - AX) = \rho$.

Subcase 2.2. Suppose that ($a'_{11}$) is false and that ($b'_{11}$) is satisfied. Then $1 < \rho_1 = \delta - 1 < \rho$.

Firstly, suppose that $\rho_1 > 2$. According to the induction assumption, there exist nonsingular matrices $X_1 \in F^{\delta\times\delta}$ and $X_2 \in F^{(n-\delta)\times(n-\delta)}$ such that $R_{\bar{F}}(X_1A_1 - A_1X_1) = \text{rank}(X_1A_1 - A_1X_1) = \rho - 2$ and $R_{\bar{F}}(X_2A_2 - A_2X_2) = \text{rank}(X_2A_2 - A_2X_2) = 2$. Let $X = X_1 \oplus X_2$. Then $R_{\bar{F}}(XA - AX) = \text{rank}(XA - AX) = \rho$.

Now suppose that $\rho_1 = 2$. According to Corollary 7, there exists a nonsingular matrix $X_1 \in F^{3\times 3}$ such that $X_1A_1 - A_1X_1$ is nonderogatory and 0 is not eigenvalue of $X_1A_1 - A_1X_1$. Let $X = X_1 \oplus I_{n-3}$. Then $R_{\bar{F}}(XA - AX) = \text{rank}(XA - AX) = 3 = \rho$. ∎

Now suppose that $C \in F^{2\times 2}$ is a matrix of the form (2) and that the characteristic polynomial of $C$ is irreducible. According to [3], $K = F[C]$ is a field and, according to [2], there exists $B \in F^{2\times 2}$ such that $U = BC - CB$ is nonsingular and

$$\{XC - CX \mid X \in F^{2\times 2}\} = UK = KU.$$

Let $S = [\, S_{i,j} \,] \in \bar{F}^{2p\times 2p}$, where the blocks $S_{i,j}$ are of size $2 \times 2$. We shall say that $S$ is a $\star$-matrix if $S_{i,j} \in UK$, whenever $i \leq j$.

**Lemma 12** *With the previous notation, suppose that* $b \neq 0$. *Let* $S = [\, S_{i,j} \,] \in \bar{F}^{2p\times 2p}$, $S_{i,j} \in \bar{F}^{2\times 2}$, *be a* $\star$*-matrix and* $\lambda \in \bar{F} \setminus \{0\}$. *Then* $\text{rank}(S - \lambda I_{2p}) \geq p$.

Proof. By induction on $p$. Firstly, note that a commutator $XC - CX$ is not scalar, unless it is the zero matrix. Therefore, the lemma is true when $p = 1$.

Suppose that $p \geq 2$. Suppose that $S_{1,j} = 0$, for every $j \in \{2, \ldots, p\}$. Considering $S$ as a $p \times p$ matrix with entries in $\bar{F}^{2\times 2}$, let $S_0$ be the principal submatrix obtained from $S$ by deleting the first row and the first column. Using the induction assumption,

$$\text{rank}(S - \lambda I_{2p}) \geq \text{rank}(S_{1,1} - \lambda I_2) + \text{rank}(S_0 - \lambda I_{2p-2}) \geq p.$$

Now suppose that $S_{1,j} \neq 0$, for some $j \in \{2, \ldots, p\}$. Choose the maximum $v \in \{2, \ldots, p\}$ such that $S_{1,v} \neq 0$. Consider $S - \lambda I_{2p}$ as a $p \times p$ matrix with entries in $\bar{F}^{2\times 2}$. For each $w \in \{2, \ldots, v-1\}$, subtract the $v$th column multiplied by $S_{1,v}^{-1}S_{1,w}$ from the $w$th column. It is not hard to see that the matrix obtained has the form $S' - \lambda I_{2p}$, where $S' = [S'_{i,j}]$, $S'_{i,j} \in \bar{F}^{2\times 2}$, is a $\star$-matrix. Clearly, $\text{rank}(S - \lambda I_{2p}) = \text{rank}(S' - \lambda I_{2p})$. If $p = 2$, then rank$(S' - \lambda I_4) \geq \operatorname{rank} S'_{1,2} = 2$. Suppose that $p \geq 3$. Considering $S'$ as a matrix with entries in $\bar{F}^{2\times 2}$, let $S_0$ be the principal submatrix obtained from $S'$ by deleting the first and the $v$th rows and columns. Using the induction assumption,

$$\operatorname{rank}(S' - \lambda I_{2p}) \geq \operatorname{rank}(S_0 - \lambda I_{2p-4}) + \operatorname{rank} S'_{1,v} \geq p. \quad \blacksquare$$

**Lemma 13** *Suppose that $f_r(x)$ is irreducible of degree 2. If $\rho < n/2$ and $\rho$ is odd, then there is no matrix $X \in F^{n\times n}$ such that $R_{\bar{F}}(XA - AX) = \rho$.*

Proof. Suppose that $R_{\bar{F}}(XA - AX) = \rho$, where $X \in F^{n\times n}$, $\rho < n/2$ and $\rho$ is odd. Take $\lambda \in \bar{F}$ such that rank$(XA - AX - \lambda I_n) = \rho$. As $\rho < n/2$, it can be deduced that $\lambda \in F$.

According to [6], the eigenvalues of $XA - AX$ can be joined in pairs so that the sum of the two eigenvalues of each pair is 0. Therefore $2\lambda = 0$. Assuming that $F$ has characteristic different from 2, we have $\lambda = 0$ and rank$(XA - AX) = \rho$, what contradicts Lemma 4.

Suppose that the companion matrix of $f_r(x)$ has the form (2).

Suppose that $b \neq 0$. As $A$ has the form $C \oplus \cdots \oplus C$, $XA - AX$ is a $\star$-matrix. According Lemmas 4 and 12, the equality rank$(XA - AX - \lambda I_n) = \rho$ is impossible.

Finally, suppose that $F$ has characteristic 2 and that $b = 0$. Take

$$Y = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \in F^{n\times n}.$$

Then $YA - AY = \lambda I_n$ and $(X - Y)A - A(X - Y) = XA - AX - \lambda I_n$ has rank $\rho$, what contradicts Lemma 4. $\blacksquare$

Bearing in mind Lemma 3, it is clear that Theorem 1 follows, immediately, from the following lemma.

**Lemma 14** *Let $F$ be a field such that all the irreducible polynomials in $F[x]$ have degree $\leq 2$. Let $A \in F^{n\times n}$, $\rho \in \{0, \ldots, n-1\}$.*

*The following statements are equivalent:*

($a_{14}$) *There exists a nonsingular matrix $X \in F^{n\times n}$ such that $R_{\bar{F}}(XA - AX) = \rho$.*

($b_{14}$) *There exists $X \in F^{n\times n}$ such that $R_{\bar{F}}(XA - AX) = \rho$.*

$(c_{14})$ *One of the following conditions holds:*

$(i_{14})$ $f_r(x)$ *is irreducible of degree* 2 *and* $\rho$ *is even.*

$(ii_{14})$ $f_r(x)$ *is irreducible of degree* 2 *and* $\rho \geq n/2$.

$(iii_{14})$ $f_r(x)$ *is not irreducible of degree* 2 *and* $\rho \leq 2R_{\bar{F}}(A)$.

Proof. Suppose that $(b_{14})$ is satisfied. Suppose that $f_r(x)$ is irreducible of degree 2. According to Lemma 13, $\rho$ is even or $\rho \geq n/2$. Now suppose that $f_r(x)$ is not irreducible of degree 2. Then, using Lemma 4,

$$\rho = R_{\bar{F}}(XA - AX) \leq \text{rank}(XA - AX) \leq 2R_{\bar{F}}(A).$$

Conversely, suppose that $(c_{14})$ is satisfied, in order to prove $(a_{14})$. If one of the conditions $(i_{14})$, $(iii_{14})$ is satisfied, then, according to Lemma 11, $(a_{14})$ holds. Now suppose that $f_r(x)$ is irreducible of degree 2 and $\rho \geq n/2$. Let $s := n - \rho \leq n/2 = r$. We have $A = C \oplus \cdots \oplus C$, where $C = C(f_1) = \cdots = C(f_r)$. Let $Y \in F^{2\times 2}$ be a nonsingular matrix such that $YC - CY$ is nonderogatory. Let $\lambda_1, \lambda_2 \in \bar{F}$ be the eigenvalues of $YC-CY$. Let $X_1 = Y \oplus \cdots \oplus Y \in F^{2s\times 2s}$.

If $s = r$, let $X = X_1$.

If $s < r$, let $A_1 = C(f_1) \oplus \cdots \oplus C(f_s)$, $A_2 = C(f_{s+1}) \oplus \cdots \oplus C(f_r)$. According to Lemma 9, there exists a nonsingular matrix $X_2 \in F^{2s\times 2s}$ such that $X_2A_2 - A_2X_2$ is nonderogatory and does not have any eigenvalue in $\{\lambda_1, \lambda_2\}$. Let $X = X_1 \oplus X_2$.

For any value of $s$, it is not hard to deduce that

$$\rho = R_{\bar{F}}(XA - AX) = \text{rank}(XA - AX - \lambda_1 I_n). \quad \blacksquare$$

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