A STRONG FORM OF ALMOST DIFFERENTIABILITY

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Abstract. We present an uniformization of Reeken's macroscopic differentiability ([5]), discuss its relations to uniform differentiability ([6]) and classical continuous differentiability, prove corresponding Chain Rule, Taylor's Theorem, Mean Value Theorem and Inverse Mapping Theorem. An attempt at comparison with observability ([1, 4]) is made too.

1. A summary

In section 2 we establish the main context and language and also review Stroyan's S-uniform differentiability ([6]) for it not only is important for the matter at hand but also because we see it as a touchstone, or at least a basic paradigm, for other notions either finite or infinite dimensional. In section 3, we establish the uniformization we call **mu-differentibility** (definition 3.3), and explore its relation to S-continuity (theorem 3.6), S-uniform differentiability (theorems 3.8 and 3.13) and classical continuous differentiability (theorem and 3.9 and corollary 3.11). Section 3 also includes a short discussion of mu-differentiability of higher order (theorem 3.12).

In sections 4, 5, 6 and 7 we treat the remaining theorems in the order given in the abstract.

In section 8 we sketch a line along which a comparison of observability and macroscopic differentiability might be studied.

2. Preliminaries

Our presentation is made in a poly-saturated model of Robinson's Nonstandard Analysis, as given for instance in [7] or [2]. Definitions and theorems in this introduction aim at making the article self-contained, at least on what regards terminology.

Unless otherwise specified, E and F are two arbitrary normed spaces with nonstandard extensions ${}^*E$ and ${}^*F$, and U an open subset of E. We begin by presenting some basic notions and theorems.

Definition 2.1. *Let x and y be two vectors of ${}^*E$. We say that*

1. *x is infinitesimal if $|x| < r$ for all positive real numbers r, and we write $x \approx 0$;*
2. *x is finite if, for some positive real number r, $|x| < r$; the set of the finite vectors of ${}^*E$ will be denoted by $fin({}^*E)$;*
3. *x is infinite if it is not finite, and write $x \approx \infty$;*
4. *x and y are infinitely close if $x - y$ is infinitesimal, and we write $x \approx y$;*

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(5) *x is nearstandard if there exists a standard $z \in {}^\sigma E$ with $x \approx z$, and we write $z = st(x)$; in this case we say that z is the standard part of x. The set of the nearstandard vectors of ${}^*E$ will be denoted by $ns({}^*E)$;*
(6) *The monad of x is the set $\mu(x) := \{z \in {}^*E |\ z \approx x\}$.*

The set of infinitesimal vectors is the monad of zero. ${}^*\mathbb{N}_\infty$ denotes the set of infinitely large positive integers, ${}^*\mathbb{N}_\infty = {}^*\mathbb{N} \setminus {}^\sigma\mathbb{N}$. ${}^*\mathbb{Z}^+_\infty, {}^*\mathbb{Z}^-_\infty, {}^*\mathbb{Z}_\infty$, ${}^*\mathbb{R}_\infty$, etc are defined analogously.

Theorem 2.2. *The inclusion $ns({}^*E) \subseteq fin({}^*E)$ holds. Moreover, E is finite dimensional if and only if $ns({}^*E) = fin({}^*E)$.*

In infinite dimensional spaces, finite vectors need not be nearstandard. For example, let $E = l_1(\mathbb{R})$ and take $x = (x_n) \in {}^*l_1(\mathbb{R})$, $(n \in {}^*\mathbb{N})$ where

$$x_n = \begin{cases} 0 & n \neq \omega \\ 1 & n = \omega \end{cases}$$

and $\omega \in {}^*\mathbb{N}_\infty$. Then x is finite ($|x| = 1$) but its distance to any standard element is not infinitesimal.

Theorem 2.3. ***Spillover Principle*** *Let A be an internal subset of ${}^*\mathbb{R}$. If A contains all positive infinitesimal numbers, then A contains a positive standard number.*

Definition 2.4. *Let $f : {}^*U \to {}^*F$ be an internal function. We say that f is S-continuous at $a \in {}^*U$ if $x \approx a$ implies $f(x) \approx f(a)$. If this is true for all $a \in {}^\sigma U$, f is called S-continuous. If it still holds for all $a \in {}^*U$, then we say that f is SU-continuous.*

Theorem 2.5. *A standard function f is continuous (resp. uniformly continuous) if and only if it is S-continuous (resp. SU-continuous).*

For instance, $f(x) = x^2, x \in \mathbb{R}$ is not uniformly continuous since if ω is an infinite hyper-real number, then

$$f\left(\omega + \frac{1}{\omega}\right) = \omega^2 + \frac{1}{\omega^2} + 2 \not\approx \omega^2 = f(\omega).$$

In the following we will denote

$$ns({}^*U) := \{x \in {}^*U \,|\, x \in ns({}^*E) \wedge st(x) \in {}^\sigma U\}$$

Given an internal linear operator $L \in {}^*L(E,F)$, we say that L is finite if $L(fin({}^*E)) \subseteq fin({}^*F)$

Definition 2.6. *Let $f : {}^*U \to {}^*F$ be an internal function and $a \in {}^\sigma U$. We say that f is S-differentiable at a if it satisfies both conditions*

(1) *$f(ns({}^*U)) \subseteq ns({}^*F)$.*
(2) *there exists a finite linear operator $Df_a \in {}^*L(E,F)$ such that, for each $x \approx a$ there exists some $\eta \approx 0$ satisfying*

$$f(x) - f(a) = Df_a(x-a) + |x-a|\eta$$

We say that f is a S-differentiable function if it is S-differentiable at all $a \in {}^\sigma U$. Finally, we say that f is SU-differentiable if the previous condition is true for every $a \in ns({}^*U)$.

Theorem 2.7. *A standard function $f : U \to F$ is differentiable (resp. continuously differentiable) if and only if it is S-differentiable (resp. SU-differentiable).*

Theorem 2.8. *An internal function $f : {}^*U \to {}^*F$ is SU-differentiable if and only if for all $a \in {}^\sigma U$, there exists a finite linear operator $L_a \in {}^*L(E,F)$ such that, whenever $y \approx x \approx a$, there exists an infinitesimal vector η satisfying*

$$f(x) - f(y) = L_a(x-y) + |x-y|\eta.$$

It is well known that a real function of one real variable f is differentiable with derivative f', then it is of class C^1 if and only if

$$f'(x) \;\approx\; \frac{f(x+\eta)-f(x)}{\eta}$$

whenever η is infinitesimal and x is near standard (see [7, 5.7.6]); this idea is already extended in the following theorem (2.9) and was even more extended in [6].

We proceed to present a nonstandard version of Taylor's theorem. Note that it provides a necessary and sufficient condition for a function to be of class C^k. Denote $SL^h(E,F)$ the symmetric h-linear operators from $E \times \ldots \times E = E^h$ into F.

Theorem 2.9. *Let $f : U \to F$ be a function. Then f is of class C^k if and only if there exist unique maps $L^h_{(.)} : U \to SL^h(E,F)$, $h \in \{1,\ldots,k\}$ such that, whenever $a \in ns({}^*U)$ and $x \approx a$, there is an infinitesimal $\eta \in {}^*F$ satisfying*

$$f(x) = \sum_{h=0}^{k} \frac{1}{h!} L^h_a (x-a)^{(h)} + |x-a|^k \eta.$$

The unique maps L^h are the h-th derivatives of f also denoted $D^h f$.

3. MU-DIFFERENTIABILITY OF AN INTERNAL FUNCTION

In this section treat a new kind of differentiability, we call mu-differentiability. We will see that mu-differentiability contrary to SU-differentiability demands less smoothness on f, but still approaches class C^1, namely when we deal with perturbations of classical functions (see Theorem 3.9 below).

In 1992, M. Reeken defined a new *macroscopic differentiability* (m-differentiability for short). The notion was used essentially for the definition of quasi-manifolds and we know of no developments other than the hereby presented. For standard functions, the m-derivative is the Fréchet derivative, but m-differentiability of internal functions does appear to be more adapted to physics ([5]).

Definition 3.1. *Let $f : {}^*U \to {}^*F$ be an internal function. We say that f is **m-differentiable** at $a \in {}^\sigma U$ if it satisfies both conditions*

1. *$f(ns({}^*U)) \subseteq ns({}^*F)$.*
2. *there exists an infinitesimal $\delta_a \in {}^*\mathbb{R}^+$ and a finite linear operator $Df_a \in {}^*L(E,F)$ such that, for all $x \in {}^*U$, where $\delta_a < |x-a| \approx 0$, there is some $\eta \approx 0$ such that*

$$f(x) - f(a) = Df_a(x-a) + |x-a|\eta$$

The function f is called m-differentiable if it is m-differentiable at all $a \in {}^\sigma U$.

Since $f(ns({}^*U)) \subseteq ns({}^*F)$, it makes sense to define the standard function

$$\begin{array}{rccc} st(f): & {}^\sigma U & \to & {}^\sigma F \\ & x & \mapsto & st(f(x)) \end{array}$$

Let us denote $st(f)$ by $\overline{f}$; this is merely a device to emphasize the fact that standard parts are actually extensions of classical objects, in particular the notation $st(f)$ is bound to hide the fact that $st(f) = {}^*g$, for some classical g.

If g is a standard differentiable function and $\sup_{x\in {}^*U} |f(x)-g(x)| \approx 0$, then f is m-differentiable. Actually, it can be proved that

Theorem 3.2. [5] *If E and F are standard finite dimensional normed spaces, K a standard compact subset of E and $f : {}^*K \to {}^*F$ an internal function, then the following statements are equivalent:*

1. *f is S-continuous and m-differentiable;*
2. *There exists a differentiable standard function $g : K \to F$ such that*

$$\sup_{x\in {}^*K} |f(x)-g(x)| \approx 0.$$

This result played a very important role in the characterization of nonstandard manifolds as presented in [5]: under some conditions, the internal transition functions φ_{ij} are S-continuous, m-differentiable with S-continuous m-derivative if and only if there exist standard C^1 transition functions infinitely close to φ_{ij}.

Here we extend the last result for m-uniformly differentiable functions as well as study other properties of this differentiability. First we introduce the notion of mu-differentiability (short for m-uniformly differentiability).

Definition 3.3. *Let $f : {}^*U \to {}^*F$ be an internal function. We say that f is **mu-differentiable** if*

1. *$f(ns({}^*U)) \subseteq ns({}^*F)$.*
2. *There exists an internal function from *U into ${}^*L(E,F)$, $x \mapsto Df_x$ such that*
   - (a) *when x is near-standard in ${}^*U$, Df_x is a finite map.*
   - (b) *for each $a \in {}^\sigma U$, there exists a positive infinitesimal δ_a for which, when $x, y \approx a \in {}^\sigma U$, some infinitesimal vector η verifies*

$$|x-y| > \delta_a \Rightarrow f(x)-f(y) = Df_x(x-y) + |x-y|\eta.$$

Since $a \in \mu(a)$, every mu-differentiable function is m-differentiable.

For example, let

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \epsilon & \text{if } x = 0 \end{cases} \qquad (1)$$

where ϵ is a positive infinitesimal number. Then f is mu-differentiable and (we can choose) $f'(x) \approx 0$ for every $x \in ns({}^*\mathbb{R})$. In fact, let $a = 0$ (the case $0 \neq a \in {}^\sigma\mathbb{R}$ is obvious) and let $x \approx y \approx 0$ with $|x-y| > \delta_0 := \sqrt{\epsilon}$. Then

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq \frac{\epsilon}{\sqrt{\epsilon}} \approx 0.$$

Observe that f is not S-differentiable (nor SU-differentiable) since

$$\frac{f(\epsilon^2)-f(0)}{\epsilon^2-0} = -\frac{\epsilon}{\epsilon^2}$$

is infinite.

In the next example the choice of δ_a is independent of the choice of a.

Let $[x]$ denote the largest integer less than or equal to x. The function $f(x) = [x]\epsilon$, $x \in {}^*\mathbb{R}$, where ϵ is any positive infinitesimal, is mu-differentiable and $f'(x) = 0$, for every $x \in ns({}^*\mathbb{R})$. In fact, we suppose $a \in {}^\sigma\mathbb{R}$ and choose a positive infinitesimal δ such that ϵ/δ is still infinitesimal (for example, $\delta = \sqrt{\epsilon}$). If $x, y \approx a$ with $|x-y| > \delta$ then

1. if $a \notin \mathbb{Z}$ then $\dfrac{f(x)-f(y)}{x-y} = 0$;
2. if $a \in \mathbb{Z}$ and $x, y \geq a$ or $x, y < a$ then $\dfrac{f(x)-f(y)}{x-y} = 0$;

(3) in the other cases, $\left|\dfrac{f(x)-f(y)}{x-y}\right| \leq \dfrac{\epsilon}{\delta} \approx 0.$

Actually one encompassing δ may be taken in Definition 3.3, i.e., the following holds.

Theorem 3.4. *Let $f : {}^*U \to {}^*F$ be an internal function; f is mu-differentiable if and only if all the following conditions are verified*

(1) $f(ns({}^*U)) \subseteq ns({}^*F)$.

(2) *There exist an internal function from *U into ${}^*L(E,F)$, $x \mapsto Df_x$ and a positive infinitesimal δ such that*
   (a) *when x is near-standard in ${}^*U$, Df_x is a finite map.*
   (b) *when x and y are near-standard in ${}^*U$, some infinitesimal vector η verifies*

$$|x-y| > \delta \Rightarrow f(x) - f(y) = Df_x(x-y) + |x-y|\eta.$$

Proof. It is obvious that the existence of one δ as above implies mu-differentiability. Suppose that f is mu-differentiable as in Definition 3.3 and define

$$\nu := \bigcup_{a \in {}^\sigma U}]0, \delta_a].$$

ν is a union of a family of internal sets whose cardinal does not exceed the cardinal of the contextual classical model of analysis from which the poly-saturated nonstandard model is obtained; let μ denote the monad of zero in ${}^*\mathbb{R}$, so that $\nu \subseteq \mu$. The proof that actually $\nu \subset \mu$ is an easy exercise on poly-saturation. Any infinitesimal $\delta \in \mu \setminus \nu$ may be chosen. □

The following is obvious

Theorem 3.5. *Let f and g be two mu-differentiable functions and $k \in ns({}^*\mathbb{R})$. Then $f+g$ and kf are mu-differentiable.*

We shall prove

Theorem 3.6. *If the function $f : {}^*U \to {}^*F$ is mu-differentiable then*

$$\forall x, y \in ns({}^*U) \quad x \approx y \Rightarrow f(x) \approx f(y),$$

i.e., the function is S-continuous.

Proof. Let us fix $x, y \in ns({}^*U)$ with $x \approx y$ and let $a := st(x)$. Since $x, y \in \mu(a)$, there exist two finite linear operators $Df_x, Df_y \in {}^*L(E,F)$ such that, for all $z \in \mu(a)$

- $|x-z| > \delta_a \Rightarrow f(x) - f(z) = Df_x(x-z) + |x-z|\eta_1$,
- $|y-z| > \delta_a \Rightarrow f(y) - f(z) = Df_y(y-z) + |y-z|\eta_2$,

with $\eta_1 \approx \eta_2 \approx 0$. Choose any $z \in \mu(a)$ with $\min\{|x-z|, |y-z|\} > \delta_a$. Then

$$f(x) - f(z) \approx 0 \approx f(y) - f(z)$$

so that $f(x) \approx f(y)$, which concludes the proof. □

Remark 3.7. m-differentiability of a function does not imply S-continuity. Let

$$\begin{array}{rccl} f: & {}^*]-1,1[& \longrightarrow & {}^*\mathbb{R} \\ & x & \mapsto & \begin{cases} 0 & \text{if} \quad x \neq \epsilon \\ 1 & \text{if} \quad x = \epsilon \end{cases} \end{array}$$

where ϵ is a positive infinitesimal number. Then f is m-differentiable at $x = 0$ (take $\delta_0 \geq \epsilon$) but it is not S-continuous.

The next theorem shows that derivatives of mu-differentiable functions are S-continuous:

Theorem 3.8. *Let f be a mu-differentiable function, $x, y \in ns({}^*U)$ with $x \approx y$. Then for all $d \in {}^*E$ with $|d| = 1$, $Df_x(d) \approx Df_y(d)$.*

Proof. Let $a = st(x)$ and $d \in {}^*E$ with $|d| = 1$. We will divide the proof in two cases. The first part of our proof is inspired by Stroyan's argument in the proof of [6, Proposition (2.4)].

First Case: $|x - y| > \delta_a$

Let $\epsilon := \sqrt{|x - y|}$ and $z := \epsilon d + x = \epsilon\left(d + \frac{x-y}{\epsilon}\right) + y$. Since

(1) $0 \approx |x - y| > \delta_a$;
(2) $0 \approx |z - x| = \epsilon > \delta_a$;
(3) $0 \approx |z - y| \geq \epsilon(1 - \epsilon) > \delta_a$;

the following hold for some η_i:

(1) $f(x) - f(y) = Df_y(x - y) + \epsilon\eta_1$, $\eta_1 \approx 0$;
(2) $f(z) - f(x) = \epsilon Df_x(d) + \epsilon\eta_2$, $\eta_2 \approx 0$;
(3) $f(z) - f(y) = \epsilon Df_y(d) + Df_y(x - y) + \epsilon\eta_3$, $\eta_3 \approx 0$.

So we conclude that, for some infinitesimal η,

$$Df_y(x - y) + \epsilon\eta_1 = f(x) - f(y) = \epsilon(Df_y(d) - Df_x(d)) + Df_y(x - y) + \epsilon\eta$$

and thus $Df_x(d) \approx Df_y(d)$.

Second Case: $|x - y| \leq \delta_a$

Let $w \in {}^*U$ be such that

$$0 \approx |x - w| > \delta_a \quad \& \quad 0 \approx |y - w| > \delta_a$$

Similarly to the first case, one can prove that for all $d \in {}^*E$ with $|d| = 1$

$$Df_x(d) \approx Df_w(d) \approx Df_y(d).$$

□

We now present the main result of this chapter. It extends Theorem 3.2 for mu-differentiable functions. As one might expect, in this case, the internal function is infinitely close to a C^1 standard function.

Theorem 3.9. *Let $f : {}^*U \to {}^*F$ be an internal function. Then:*

(1) *If F is a finite dimensional space and f is a mu-differentiable function, then $\overline{f} : U \to F$ is a C^1 function and $D\overline{f}_a = st(Df_a)$ for $a \in {}^\sigma U$. Furthermore, if E is also finite dimensional then*

$$\forall a \in {}^\sigma U\, \exists \eta_0 \approx 0\, \forall x \approx a \quad |f(x) - \overline{f}(x)| \leq \eta_0.$$

(2) *If there exists a C^1 standard function $g : U \to F$ with*

$$\forall a \in {}^\sigma U\, \exists \eta_0 \approx 0\, \forall x \approx a \quad |f(x) - g(x)| \leq \eta_0,$$

then f is mu-differentiable. Moreover, $g = \overline{f}$.

Proof. (1) Suppose that F is a finite dimensional normed space and f is mu-differentiable. We will begin by proving that $\overline{f}$ is classically differentiable at $a \in {}^\sigma U$ with derivative $x \mapsto st(Df_a(x))$.

$$\forall \eta \in {}^\sigma\mathbb{R}^+\, \exists \epsilon \in {}^\sigma\mathbb{R}^+\, \forall h \in {}^\sigma E \quad 0 < |h| < \epsilon \Rightarrow \frac{|\overline{f}(a + h) - \overline{f}(a) - st(Df_a(h))|}{|h|} < \eta.$$

Fix $\eta \in {}^\sigma\mathbb{R}^+$ and let

$$A := \Big\{\epsilon \in {}^*\mathbb{R}^+ \,|\, \epsilon \leq \delta_a \vee [\forall h \in {}^*E$$

$$\delta_a < |h| < \epsilon \Rightarrow \frac{|f(a + h) - f(a) - Df_a(h)|}{|h|} < \frac{\eta}{2}\Big]\Big\}.$$

Since A is an internal set and contains all positive infinitesimal numbers, by the Spillover Principle there exists $\epsilon \in {}^\sigma\mathbb{R}^+$ such that $\epsilon \in A$. Choose now $h \in {}^\sigma E$ with $0 < |h| < \epsilon$. As h is standard, $\delta_a < |h| < \epsilon$; therefore

$$\frac{|f(a+h) - f(a) - Df_a(h)|}{|h|} < \frac{\eta}{2}.$$

Taking standard parts one gets

$$\frac{|\overline{f}(a+h) - \overline{f}(a) - st(Df_a(h))|}{|h|} < \eta.$$

So $\overline{f}$ is differentiable and $D\overline{f}_a = st(Df_a)$ for $a \in {}^\sigma U$.

Next we will prove that the function $x \mapsto D\overline{f}_x$ is classically continuous, *i.e.*,

$$\forall a \in {}^\sigma U\, \forall \eta \in {}^\sigma\mathbb{R}^+\, \exists \epsilon \in {}^\sigma\mathbb{R}^+\, \forall x \in {}^\sigma U\, \forall d \in {}^\sigma E$$
$$[|x-a| < \epsilon \wedge |d| = 1] \Rightarrow |D\overline{f}_x(d) - D\overline{f}_a(d)| < \eta.$$

Choose any $a \in {}^\sigma U$ and $\eta \in {}^\sigma\mathbb{R}^+$ and let

$$B := \Big\{\epsilon \in {}^*\mathbb{R}^+ \,|\, \forall x \in {}^*U\, \forall d \in {}^*E$$
$$[|x-a| < \epsilon \wedge |d| = 1] \Rightarrow |Df_x(d) - Df_a(d)| < \frac{\eta}{2}\Big\}.$$

Again the internal set B contains all positive infinitesimals. In fact, if $0 < \epsilon \approx 0$, for any $x \in {}^*U$ and $d \in {}^*E$ with $|d| = 1$ and $|x-a| < \epsilon$, by Theorem 3.8, one has $Df_x(d) \approx Df_a(d)$ and so

$$|Df_x(d) - Df_a(d)| < \frac{\eta}{2}.$$

So B must contain a positive standard ϵ. Choose now $x \in {}^\sigma U$ and $d \in {}^\sigma E$ satisfying $|d| = 1$ and $|x-a| < \epsilon$; hence

$$|Df_x(d) - Df_a(d)| < \frac{\eta}{2},$$

which implies

$$|D\overline{f}_x(d) - D\overline{f}_a(d)| < \eta,$$

proving that $\overline{f}$ is a C^1 function.

Assume now that E is finite dimensional. Observe that for $a \in {}^\sigma U$ and $x \approx a$, both $f(x) \approx f(a)$, by theorem 3.6, and $\overline{f}(x) \approx \overline{f}(a)$, as we just saw, therefore

$$f(x) - \overline{f}(x) \approx f(a) - \overline{f}(a) = f(a) - st(f(a)) \approx 0.$$

Therefore $f(x) \approx \overline{f}(x)$ for every $x \in ns({}^*U)$.

Moreover, for every $a \in {}^\sigma U$, we can choose $n \in {}^\sigma\mathbb{N}$ such that $B_{2/n}(a) \subseteq U$. So, if we define K as being the closed ball $\overline{B}_{1/n}(a)$, we have

$$a \in K \subseteq U.$$

Let $y \in {}^*K$. Since K is compact, $st(y)$ belong to ${}^\sigma K \subseteq {}^\sigma U$. Define

$$\eta_0 := \sup_{y \in {}^*K} |f(y) - \overline{f}(y)|.$$

It is easy to verify that $\eta_0 \approx 0$, which ends the proof of *1*.

(2) Let $g \in C^1(U, F)$. Fix any $a \in {}^\sigma U$ and let $\delta_a := \sqrt{\eta_0}$. Choose any $x, y \in \mu(a)$ with $\delta_a < |x-y|$. Since g is continuously differentiable, there exists a finite linear operator Dg_x which satisfies the condition

$$g(x) - g(y) = Dg_x(x-y) + |x-y|\eta$$

for some $\eta \approx 0$.

For $\epsilon_1 := g(x) - f(x)$ and $\epsilon_2 := g(y) - f(y)$, it is true that $\max\{|\epsilon_1|, |\epsilon_2|\} \leq \eta_0$ and

$$f(x) - f(y) = Dg_x(x - y) + |x - y|\eta + \epsilon_2 - \epsilon_1.$$

Furthermore, we also have

$$\frac{|\epsilon_1 - \epsilon_2|}{|x - y|} \leq \frac{|\epsilon_1| + |\epsilon_2|}{|x - y|} \leq \frac{2\eta_0}{\sqrt{\eta_0}} \approx 0.$$

To see that $g = \overline{f}$, note that both are standard functions and for every $a \in {}^\sigma U$, $g(a) = \overline{f}(a)$.

□

Remark 3.10. The previous theorem is false if we replace mu-differentiability by SU-differentiability. Of course *1* still holds since SU-differentiability is a stronger condition, but *2* may fail. For example, suppose $g(x) = 0, x \in \mathbb{R}$ and $f(x) = 0$, if $x \in {}^*\mathbb{R}\backslash\{0\}$ and $f(0) = \epsilon$ with $0 \neq \epsilon \in \mu(0)$. Then g is a standard C^1 function infinitely close to f but f is not SU-differentiable.

It is easy to prove that

Corollary 3.11. *For a standard function $f : U \to F$, the following conditions are equivalent:*

1. *f is of class C^1;*
2. *f is mu-differentiable.*

A mu-differentiable function $f : {}^*U \to {}^*F$, internal, by definition, has an internal derivative $x \mapsto Df_x \in^* L(E, F)$, determined up to an infinitesimal map ([6]) called the mu-derivative of f. As $L(E, F)$ is still a standard normed space, we may define higher-order derivatives. We say that f is **twice mu-differentiable** provided f and $Df_{(\cdot)}$ are both mu-differentiable.

Recursively, f is k-times mu-differentiable if f is mu-differentiable and there exist mu-differentiable functions $Df_{(\cdot)}$, ...,$D^{k-1}f_{(\cdot)}$ such that all $D^j f_{(\cdot)}$ are a mu-derivative of $D^{j-1}f_{(\cdot)}$, $j = 1, \ldots, k - 1$.

Theorem 3.12. *Let $f : {}^*U \to {}^*F$ be an internal function. Then:*

(1) *If F is a finite dimensional space and f is k-times mu-differentiable, then $\overline{f} : U \to F$ is a C^k function and for each $a \in {}^\sigma U$, $D^j\overline{f}_a = st(D^j f_a)$ for $j = 1, 2, \ldots, k$. Furthermore, if E is also finite dimensional,*

$$\forall a \in {}^\sigma U\, \exists \eta_0 \approx 0\, \forall x \approx a \quad |f(x) - \overline{f}(x)| \leq \eta_0$$

and

$$\forall j \in \{1, 2, \ldots, k - 1\}\, \forall a \in {}^\sigma U\, \exists \eta_j \approx 0\, \forall x \approx a \quad |D^j f_x - D^j\overline{f}_x| \leq \eta_j.$$

(2) *If there exists a C^k standard function $g : U \to F$ with*

$$\forall a \in {}^\sigma U\, \exists \eta_0 \approx 0\, \forall x \approx a \quad |f(x) - g(x)| \leq \eta_0$$

and

$$\forall j \in \{1, 2, \ldots, k - 1\}\, \forall a \in {}^\sigma U\, \exists \eta_j \approx 0\, \forall x \approx a \quad |D^j f_x - D^j g_x| \leq \eta_j$$

then f is k-times mu-differentiable. Moreover, $g = \overline{f}$.

Proof. The proof is by induction on k as follows:

For $k = 1$: it was proved in Theorem 3.9 that *1* and *2* hold.

For $k \Rightarrow k + 1$:

We will begin by proving that *1* holds. Assume then that f is $(k+1)$-times mu-differentiable. By hypothesis of induction, $\overline{f}$ is of class C^k and satisfies the other conditions of *1*. Since

$$\begin{array}{rccc} D^k f_{(\cdot)}: & {}^*U & \to & {}^*L^k(E,F) \\ & x & \mapsto & D^k f_x \end{array}$$

is still mu-differentiable, its standard part

$$\begin{array}{rccc} st(D^k f_{(\cdot)}): & {}^\sigma U & \to & {}^\sigma L^k(E,F) \\ & x & \mapsto & st(D^k f_x) \end{array}$$

is of class C^1 and, for every $a \in {}^\sigma U$, $Dst(D^k f_a) = st(D(D^k f_a))$. But since, when a is standard, $st(D^k f_a) = D^k \overline{f}_a$,

- $D^k \overline{f}_{(\cdot)}$ is also of class C^1 and so $\overline{f}$ is of class C^{k+1};
- $D^{k+1}\overline{f}_a = st(D^{k+1} f_a)$.

Furthermore, for $a \in {}^\sigma U$ and $x \approx a$,

$$D^k f_x \approx D^k f_a \approx D^k \overline{f}_a \approx D^k \overline{f}_x.$$

Similarly, as in the proof of Theorem 3.9, we can prove that there exists an infinitesimal number η_k for which holds

$$|D^k f_x - D^k \overline{f}_x| \le \eta_k$$

whenever $x \approx a$ and E is a finite dimensional normed space, which ends the first part of the proof.

To prove *2*, assume that g is a C^{k+1} satisfying the conditions in *2*. Then f is k-times mu-differentiable. Besides this, $D^k g_{(\cdot)}$ is a C^1 function and

$$\forall a \in {}^\sigma U \,\exists \eta_k \approx 0 \,\forall x \approx a \quad |D^k f_x - D^k g_x| \le \eta_k.$$

By Theorem 3.9, $D^k f_{(\cdot)}$ is mu-differentiable and so f is $(k+1)$-times mu-differentiable. □

From the previous result one can see that there exist functions k-times mu-differentiable which are not k-times SU-differentiable. For example, let f be the function defined in (1), pag. 4. Since f is infinitely close to g, where $g(x) := 0, x \in \mathbb{R}$, and g is of class C^k, then f is k-times mu-differentiable yet is not SU-differentiable.

The next theorem establishes a relation between mu-differentiability and a condition similar to SU-differentiability (see Definition 2.6).

Theorem 3.13. *For every mu-differentiable function $f : {}^*U \to {}^*F$ we have*

$$\forall x \in ns({}^*U)\,\exists \delta_x \approx 0\,\exists Df_x \in {}^*L(E,F)\,\forall y \in {}^*U\,\exists \eta \approx 0 \tag{2}$$

$$|Df_x| \text{ is finite } \wedge [\delta_x < |x-y| \approx 0 \Rightarrow f(x) - f(y) = Df_x(x-y) + |x-y|\eta].$$

Proof. For any $x \in ns({}^*U)$, define $a := st(x)$ and $\delta_x := \delta_a$. The proof follows easily. □

The reverse of Theorem 3.13 is false, as shown in the following example.

Let f be the real valued function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Since f is not continuously differentiable, it can not be mu-differentiable. But it satisfies condition (2). Indeed, if $x \approx 0$ (the other cases are obvious), for $\delta_x := |x|$ and $y \in {}^*\mathbb{R}$ with $0 \approx |x-y| > \delta_x$, we get

$$\frac{f(x)-f(y)}{x-y} = \frac{x^2}{x-y}\left(\sin\frac{1}{x} - \sin\frac{1}{y}\right) + \frac{x^2-y^2}{x-y}\sin\frac{1}{y} \approx 0$$

since
$$\left|\frac{x^2}{x-y}\right| \leq \frac{x^2}{|x|} \approx 0 \quad \& \quad \frac{x^2-y^2}{x-y} = x+y \approx 0.$$

As a consequence of the continuity of the derivative, we have (compare with Theorem 2.8)

Theorem 3.14. *Let $f : {}^*U \to {}^*F$ be an internal function. Then conditions 1 and 2 are equivalent:*

(1) *f is mu-differentiable.*
(2) (a) *$f(ns({}^*U)) \subseteq ns({}^*F)$.*
 (b)
$$\forall a \in {}^\sigma U \, \exists \delta_a \approx 0 \, \exists Df_a \in {}^*L(E,F) \, \forall x,y \in \mu(a)$$
$$|Df_a| \text{ is finite } \wedge [|x-y| > \delta_a \Rightarrow f(x)-f(y) = Df_a(x-y) + |x-y|\eta]$$
$$\text{for some } \eta \approx 0;$$

Proof. Let us fix $a \in {}^\sigma U$ and $0 < \delta_a \approx 0$ satisfying
$$\forall x,y \in \mu(a) \quad |x-y| > \delta_a \Rightarrow \frac{f(x)-f(y)}{|x-y|} \approx Df_x\left(\frac{x-y}{|x-y|}\right).$$
By Theorem 3.8 it follows that
$$Df_x\left(\frac{x-y}{|x-y|}\right) \approx Df_a\left(\frac{x-y}{|x-y|}\right)$$
which proves that *1⇒2.*

To prove the converse, let $a \in {}^\sigma U$ and δ_a as in *2(a)* Then, given $x \in \mu(a)$, define $Df_x := Df_a$. The proof follows. □

Theorem 3.15. *If $f : {}^*U \to {}^*F$ is a mu-differentiable function, then for all standard $a \in {}^\sigma U$, there exists a positive $\delta \approx 0$ such that, for all $d \in {}^*E$ with $|d| = 1$, there exists $k \in fin({}^*F)$ for which*
$$\forall x \in {}^*U \quad x \approx a \Rightarrow \frac{f(x+\delta d)-f(x)}{\delta} \approx k$$
holds.

Proof. Fix $a \in {}^\sigma U$ and define $\delta := 2\delta_a$. Fix an unit vector d and let $k := Df_a(d)$. Then for $x \approx a$
$$\frac{f(x+\delta d)-f(x)}{\delta} \approx Df_x(d) \approx Df_a(d) = k.$$
□

4. The Chain Rule

Mu-differentiable functions are m-differentiable, but not conversely thus making the latter a weaker notion, which nevertheless still verifies the chain rule below; actually the proof is generalizable to mu-differentiable functions (theorem 4.3).

Theorem 4.1. ***Chain Rule*** *Let g and f be two m-differentiable functions at a and $g(a)$, respectively, where a and $g(a)$ are two standard vectors. In addition, if Dg_a is invertible and $\left\|(Dg_a)^{-1}\right\|$ is finite, then $f \circ g$ is m-differentiable at a and $D(f \circ g)_a = Df_{g(a)} \circ Dg_a$.*

Proof. Define $\delta = \max\{\delta_a, 2\delta_{g(a)}\left\|(Dg_a)^{-1}\right\|\}$ and choose x with $\delta < |x-a| \approx 0$.

Since $0 \approx |x-a| > \delta_a$ then $g(x) \approx g(a)$. On the other hand, for some $\eta_1 \approx 0$,
$$|g(x)-g(a)| = |Dg_a(x-a) + |x-a|\eta_1|$$
$$= |x-a|\left|Dg_a\left(\frac{x-a}{|x-a|}\right)+\eta_1\right| > 2\delta_{g(a)}|(Dg_a)^{-1}|\left|Dg_a\left(\frac{x-a}{|x-a|}\right)+\eta_1\right| \geq$$

$$2\delta_{g(a)}\left|(Dg_a)^{-1}\left(Dg_a\left(\frac{x-a}{|x-a|}\right)+\eta_1\right)\right| = 2\delta_{g(a)}\left|\frac{x-a}{|x-a|}+(Dg_a)^{-1}(\eta_1)\right| > \delta_{g(a)}.$$

So we conclude that $\delta_{g(a)} < |g(x)-g(a)| \approx 0$. Hence there exists $\eta_2 \approx 0$ such that

$$\begin{aligned} f(g(x))-f(g(a)) &= Df_{g(a)}(g(x)-g(a))+|g(x)-g(a)|\eta_2 \\ &= Df_{g(a)}(Dg_a(x-a)+|x-a|\eta_1)+|Dg_a(x-a)+|x-a|\eta_1|\eta_2 \\ &= Df_{g(a)}Dg_a(x-a)+|x-a|\left(Df_{g(a)}(\eta_1)+\left|Dg_a\left(\frac{x-a}{|x-a|}\right)+\eta_1\right|\eta_2\right) \end{aligned}$$

with

$$Df_{g(a)}(\eta_1)+\left|Dg_a\left(\frac{x-a}{|x-a|}\right)+\eta_1\right|\eta_2 \approx 0.$$

□

Remark 4.2. Suppose that g and f are two m-differentiable functions at a and $g(a)$, respectively. This is not sufficient to guarantee that $f\circ g$ is also m-differentiable at a, as it will be shown in the following example.

Let ϵ be a positive infinitesimal,

$$\begin{array}{rccc} g: & {}^*\mathbb{R} & \to & {}^*\mathbb{R} \\ & x & \mapsto & \epsilon x \end{array}$$

and

$$\begin{array}{rccc} f: & {}^*\mathbb{R} & \to & {}^*\mathbb{R} \\ & x & \mapsto & \left\{\begin{array}{lll} 1 & \text{if} & 0<x<\epsilon \\ 0 & \text{if} & x\le 0 \vee x\ge\epsilon \end{array}\right. \end{array}.$$

It is easy to verify that g is m-differentiable at $x=0$ and f is m-differentiable at $g(0)=0$. But

$$\begin{array}{rccc} f\circ g: & {}^*\mathbb{R} & \to & {}^*\mathbb{R} \\ & x & \mapsto & \left\{\begin{array}{lll} 1 & \text{if} & 0<x<1 \\ 0 & \text{if} & x\le 0 \vee x\ge 1 \end{array}\right. \end{array}$$

is not m-differentiable at $x=0$.

Theorem 4.3. ***Chain Rule II*** *Let g and f be two mu-differentiable functions. If Dg_x is invertible and $\left\|(Dg_x)^{-1}\right\|$ is finite, whenever x is nearstandard, then $f\circ g$ is mu-differentiable and $D(f\circ g)_x = Df_{g(x)}\circ Dg_x$.*

Proof. Sketch of proof: To make it simple, denote δ_f and δ_g the infinitesimals as in Theorem 3.4 (with obvious meanings). Given a nearstandard x, let $\delta := \max\{\delta_g, 2\delta_f\left\|(Dg_x)^{-1}\right\|\}$. Replacing a by x and x by y (where $y\approx x$) in the proof of Theorem 4.1, the proof follows. □

5. Taylor's Theorem

We can now formulate Taylor's Theorem for a mu-differentiable function defined on finite dimensional spaces. We will prove two different versions of this theorem; the first Taylor's expansion is made with internal functions and the second with standard functions.

Theorem 5.1. ***Taylor's Theorem*** *Let E and F be two standard finite dimensional spaces, $U\subset E$ a standard open set and $f:{}^*U\to{}^*F$ an internal function k-times mu-differentiable, for some $k\in{}^\sigma\mathbb{N}$. Then,*

(1) *for every* $x \in ns({}^*U)$, *there exists* $\epsilon \approx 0$ *such that, whenever* $y \in {}^*U$ *with* $\epsilon < |y-x| \approx 0$, *there exists* $\eta \approx 0$ *satisfying*

$$f(y) = f(x) + Df_x(y-x) + \frac{1}{2!}D^2 f_x(y-x)^{(2)} + ... + \frac{1}{k!}D^k f_x(y-x)^{(k)} + |y-x|^k \eta.$$

(2) *for every* $x \in ns({}^*U)$, *there exists* $\epsilon \approx 0$ *such that, whenever* $y \in {}^*U$ *with* $\epsilon < |y-x| \approx 0$, *there exists* $\eta \approx 0$ *satisfying*

$$f(y) = \overline{f}(x) + D\overline{f}_x(y-x) + \frac{1}{2!}D^2 \overline{f}_x(y-x)^{(2)} + ...$$
$$+\frac{1}{k!}D^k \overline{f}_x(y-x)^{(k)} + |y-x|^k \eta.$$

Proof. (1) Let us begin by fixing $x \in ns({}^*U)$ and let $a := st(x) \in {}^\sigma U$. By Theorem 3.12, we know that $\overline{f}$ is of class C^k,

$$\exists \eta_0 \approx 0 \, \forall y \approx a \quad |f(y) - \overline{f}(y)| \leq \eta_0$$

and for each $j = 1, 2, \ldots, k-1$,

$$\exists \eta_j \approx 0 \, \forall y \approx a \quad \sup_{d_i \in {}^*E, |d_i|=1} |D^j f_y(d_1, ..., d_j) - D^j \overline{f}_y(d_1, ..., d_j)| \leq \eta_j.$$

Define $\epsilon = \max\{\eta_0^{\frac{1}{k+1}}, \eta_1^{\frac{1}{k}}, ..., \eta_{k-1}^{\frac{1}{2}}\}$ and take $y \in {}^*U$ with $\epsilon < |y-x| \approx 0$. Define the finite sequence $(\epsilon_i)_{i=-1,\ldots,k-1}$ by

- $f(y) = \overline{f}(y) + \epsilon_{-1}$,
- $f(x) = \overline{f}(x) + \epsilon_0$,
- $Df_x(y-x) = D\overline{f}_x(y-x) + |y-x|\epsilon_1$,
- $D^2 f_x(y-x)^{(2)} = D^2 \overline{f}_x(y-x)^{(2)} + |y-x|^2 \epsilon_2$,
- ...
- $D^{k-1} f_x(y-x)^{(k-1)} = D^{k-1} \overline{f}_x(y-x)^{(k-1)} + |y-x|^{k-1} \epsilon_{k-1}$.

Furthermore, since the maps $x \mapsto D^k f_x$ and $x \mapsto D^k st(f)_x$ are both S-continuous, we also have

$$D^k f_x \left(\frac{y-x}{|y-x|}\right)^{(k)} \approx D^k f_a \left(\frac{y-x}{|y-x|}\right)^{(k)} \approx$$
$$D^k \overline{f}_a \left(\frac{y-x}{|y-x|}\right)^{(k)} \approx D^k \overline{f}_x \left(\frac{y-x}{|y-x|}\right)^{(k)},$$

so there exists $\epsilon_k \approx 0$ with

$$D^k f_x(y-x)^{(k)} = D^k \overline{f}_x(y-x)^{(k)} + |y-x|^k \epsilon_k.$$

Using the fact that $\overline{f}$ is a C^k function, one has

$$\overline{f}(y) = \overline{f}(x) + D\overline{f}_x(y-x) + \frac{1}{2!}D^2 \overline{f}_x(y-x)^{(2)} + ...$$
$$+\frac{1}{k!}D^k \overline{f}_x(y-x)^{(k)} + |y-x|^k \eta,$$

that is

$$f(y) = f(x) + Df_x(y-x) + \frac{1}{2!}D^2 f_x(y-x)^{(2)} + ... + \frac{1}{k!}D^k f_x(y-x)^{(k)} + |y-x|^k \eta$$
$$+\epsilon_{-1} - \epsilon_0 - |y-x|\epsilon_1 - |y-x|^2 \epsilon_2 - ... - |y-x|^{k-1}\epsilon_{k-1} - |y-x|^k \epsilon_k.$$

If

$$\epsilon_{-1} - \epsilon_0 - |y-x|\epsilon_1 - |y-x|^2 \epsilon_2 - ... - |y-x|^{k-1}\epsilon_{k-1} = |y-x|^k \eta_1,$$

then η_1 is infinitesimal since

$$|\eta_1| \leq \frac{|\epsilon_{-1}|}{|y-x|^k} + \frac{|\epsilon_0|}{|y-x|^k} + \frac{|\epsilon_1|}{|y-x|^{k-1}} + \frac{|\epsilon_2|}{|y-x|^{k-2}} + \ldots + \frac{|\epsilon_{k-1}|}{|y-x|}$$

$$\leq \frac{\eta_0}{\eta_0^{\frac{k}{k+1}}} + \frac{\eta_0}{\eta_0^{\frac{k}{k+1}}} + \frac{\eta_1}{\eta_1^{\frac{k-1}{k}}} + \frac{\eta_2}{\eta_2^{\frac{k-2}{k-1}}} + ... + \frac{\eta_{k-1}}{\eta_{k-1}^{\frac{1}{2}}} \approx 0.$$

(2) Analogously, if we take $\epsilon := \eta_0^{\frac{1}{k+1}}$, the result follows.

□

6. The Mean Value Theorem

We give now a Mean Value Theorem for mu-differentiable functions.

Theorem 6.1. ***Mean Value Theorem*** *Let U be a standard open convex subset of E and $f : {}^*U \to {}^*\mathbb{R}$ an internal mu-differentiable function. Take δ as given by Theorem 3.4. Then, for all $x, y \in ns({}^*U)$ with $|x - y| > \delta$,*

$$\exists c \in [x, y] \quad f(x) - f(y) = Df_c(x - y) + |x - y|\eta$$

for some $\eta \approx 0$.

Proof. If $x \approx y$, it is clear. If not, define a hyper-finite sequence $\{x_n \mid n \in \{1, \ldots, N+1\}\}$ by the formula

$$x_n := x + (n-1)\frac{y - x}{N},$$

where $N \in {}^*\mathbb{N}_\infty$ and $N < \dfrac{|y - x|}{\delta} \approx \infty$.

Then

$$f(x) - f(y) = \sum_{n=1}^{N} (f(x_n) - f(x_{n+1})) =$$

$$= \sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1}) + \sum_{n=1}^{N} |x_n - x_{n+1}|\eta_n.$$

If

$$\sum_{n=1}^{N} |x_n - x_{n+1}|\eta_n = |y - x|\eta,$$

for some η, then $\eta \approx 0$. Indeed, by the convexity property of the norm

$$|\eta| \leq \frac{\sum_{n=1}^{N} |x_n - x_{n+1}||\eta_n|}{|y - x|} = \frac{\sum_{n=1}^{N} |x_n - x_{n+1}||\eta_n|}{\sum_{n=1}^{N} |x_n - x_{n+1}|} \leq \max_{n \in \{1, \ldots, N\}} \{|\eta_n|\} \approx 0.$$

We will prove now that there exists $c \in [x, y]$ such that

$$Df_c\left(\frac{x - y}{|x - y|}\right) \approx \frac{\sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1})}{|x - y|}.$$

Letting $d := \dfrac{x - y}{|x - y|}$, it is true that

$$\frac{\sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1})}{|x - y|} = \frac{\sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1})}{\sum_{n=1}^{N} |x_n - x_{n+1}|} = \frac{\sum_{n=1}^{N} Df_{x_n}(d)}{N}.$$

Choosing $m, M \in \{x_1, ..., x_N\}$ with

$$Df_m(d) = \min_{1 \leq n \leq N} Df_{x_n}(d) \quad \& \quad Df_M(d) = \max_{1 \leq n \leq N} Df_{x_n}(d),$$

we get

$$Df_m(d) \leq \frac{\sum_{n=1}^{N} Df_{x_n}(d)}{N} \leq Df_M(d).$$

So, there exists $c \in [m, M] \subseteq [x, y]$ with

$$Df_c(d) \approx \frac{\sum_{n=1}^{N} Df_{x_n}(d)}{N}.$$

□

We can formulate Theorem 6.1 for functions taking values in a normed space:

Theorem 6.2. *Let U be a standard open convex subset of E and $f : {}^*U \to {}^*F$ an internal mu-differentiable function. Take δ as given by Theorem 3.4. Then, for all $x, y \in ns({}^*U)$ with $|x-y| > \delta$,*

$$\exists c \in [x,y] \quad |f(x) - f(y)| \leq |Df_c(x-y)| + |x-y|\eta$$

for some $\eta \approx 0$.

Proof. Following the proof of Theorem 6.1, it is true that:
For some $\eta_1, ..., \eta_N \approx 0$,

$$|f(x) - f(y)| = \left| \sum_{n=1}^{N} (f(x_n) - f(x_{n+1})) \right|$$

$$\leq \sum_{n=1}^{N} |f(x_n) - f(x_{n+1})| = \sum_{n=1}^{N} |Df_{x_n}(x_n - x_{n+1}) + |x_n - x_{n+1}|\eta_n|$$

$$\leq \sum_{n=1}^{N} |Df_{x_n}(x_n - x_{n+1})| + \sum_{n=1}^{N} |x_n - x_{n+1}| \cdot |\eta_n|.$$

Again, choose $m, M \in \{x_1, ..., x_N\}$ with

$$|Df_m(d)| = \min_{1 \leq n \leq N} |Df_{x_n}(d)| \quad \& \quad |Df_M(d)| = \max_{1 \leq n \leq N} |Df_{x_n}(d)|.$$

Since

$$|Df_m(d)| \leq \frac{\sum_{n=1}^{N} |Df_{x_n}(d)|}{N} \leq |Df_M(d)|$$

there exists $c \in [x,y]$ with

$$|Df_c(d)| \approx \frac{\sum_{n=1}^{N} |Df_{x_n}(d)|}{N}.$$

□

7. The Inverse Mapping Theorem

A full Inverse Mapping Theorem is not expected. In fact, take for example the C^1 function $g(x) = x$. By Theorem 3.9, any internal function infinitely close to g is mu-differentiable. So the $1-1$ condition may easily fail. Nevertheless, we have some form of injectivity as the next theorem states.

Theorem 7.1. ***Inverse Mapping Theorem*** *Let $f : {}^*U \to {}^*F$ be an internal mu-differentiable function. Assume that, for a certain $a \in {}^\sigma U$, Df_a is invertible and $\|(Df_a)^{-1}\|$ is finite. Then there exists a standard neighborhood ${}^*V$ of a such that f is 1-to-1 on the standard elements of ${}^*V$, i.e.,*

$$\forall x, y \in {}^\sigma V \quad x \neq y \Rightarrow f(x) \neq f(y).$$

Proof. Let

$$A := \left\{ \epsilon \in {}^*\mathbb{R}^+ \,|\, \forall x, y \in B_\epsilon(a) \quad |x-y| > \delta_a \Rightarrow f(x) \neq f(y) \right\}.$$

Then A contains all positive infinitesimal numbers since, for $0 < \epsilon \approx 0$ and $x, y \in B_\epsilon(a)$ with $|x-y| > \delta_a$, by Theorem 3.14,

$$\frac{f(x) - f(y)}{|x-y|} \approx Df_a\left(\frac{x-y}{|x-y|}\right).$$

But

$$1 = \left|(Df_a)^{-1} Df_a \left(\frac{x-y}{|x-y|}\right)\right| \leq \|(Df_a)^{-1}\| \left|Df_a \left(\frac{x-y}{|x-y|}\right)\right|.$$

Consequently,

$$\left|Df_a \left(\frac{x-y}{|x-y|}\right)\right| \geq \frac{1}{\|(Df_a)^{-1}\|} \not\approx 0.$$

Therefore $f(x) \neq f(y)$. Using the Spillover Principle we can guarantee the existence of $\epsilon \in {}^{\sigma}\mathbb{R}$ with $\epsilon \in A$. Define $V := B_\epsilon(a)$ and take two standard elements of ${}^*V$ with $x \neq y$. Since the distance between two distinct standard vectors is always greater than any infinitesimal number, one obtains $f(x) \neq f(y)$. □

Remark 7.2. With the previous conditions we can not conclude that f is 1-to-1 on ${}^*V$. In fact, consider

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ \epsilon & \text{if } x = 0 \end{cases}$$

where ϵ is any non-zero infinitesimal number. This function is mu-differentiable (it is infinitely close to $g(x) = x$) but it is never injective in any standard neighborhood of zero.

8. A note on observable functions

Harthong defined, and together with Reder treated **observable functions** (see [1, 4]). Although that concept was presented and treated in the context of Internal Set Theory, there might a bridge between observable and observation functions on the one hand and mu-differentiable functions on the other hand, in that *a function might be "strongly observable" if and only if its anti-derivative is m-differentiable.*

9. Acknowledgements

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