A nonstandard characterization of regular surfaces

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Abstract

In the present work we approach the study of surfaces using Nonstandard Analysis. To begin with we will give a nonstandard characterization of a surface. Later the tangent space to a surface will be also defined.

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1 Introduction

In order to understand the present work the reader must have some knowledge of Nonstandard Analysis. Specifically, we need to fix some terminology and some facts about continuity and differentiability of functions.

We will begin by presenting a contained exposition of the theory. For further details the reader is referred to [6] or [8].

We will work on a proper extension ${}^*\mathbb{R}^n$ of the Euclidean space $\mathbb{R}^n$. Given two vectors $x, y \in {}^*\mathbb{R}^n$, we say that x is infinitesimal if $|x| < \epsilon$ for all standard $\epsilon \in {}^\sigma\mathbb{R}^+$ and we write $x \approx 0$; x is finite if $|x| < \epsilon$ for some $\epsilon \in {}^\sigma\mathbb{R}^+$; x is infinite if it is not finite and x is infinitely close to y, $x \approx y$, if $x - y$ is infinitesimal.

If y is standard and $x \approx y$, we say that y is the standard part of x, that x is near-standard and we write $y = st(x)$.

The set of finite (resp. near-standard) points of ${}^*\mathbb{R}^n$ is denoted by $fin({}^*\mathbb{R}^n)$ (resp. $ns({}^*\mathbb{R}^n)$).

Given a subset $U \subseteq \mathbb{R}^n$, we say that $a \in ns({}^*U)$ is there exists $st(a)$ and $st(a) \in {}^\sigma U$.

The monad of x, $\mu(x)$ is the set of points in ${}^*\mathbb{R}^n$ infinitely close to x.

In the following, U will be an open subset of $\mathbb{R}^n$.

Definition 1 Let $f : {}^*U \to {}^*\mathbb{R}^m$ be an internal function. We say that f is **S-continuous** if for all $a \in {}^\sigma U$ and $x \in {}^*U$ with $x \approx a$, holds $f(x) \approx f(a)$. If the sentence it is true for all $a \in {}^*U$, f is called **SU-continuous**.

For standard functions, S-continuity is equivalent to continuity and SU-continuity to uniform continuity.

Definition 2 Let $f : {}^*U \to {}^*\mathbb{R}^m$ be an internal function. We say that f is **S-differentiable** if $f(ns({}^*U)) \subseteq ns({}^*\mathbb{R}^m)$ and, for each $a \in {}^\sigma U$, there exists a finite linear operator $Df_a \in {}^*L(\mathbb{R}^n, \mathbb{R}^m)$ such that, for all $x \in {}^*U$, there exists some $\eta \approx 0$ with

$$x \approx a \Rightarrow f(x) - f(a) = Df_a(x - a) + |x - a|\eta. \tag{1.1}$$

The function f is called **SU-differentiable** if the previous condition is still true for all $a \in ns({}^*U)$.

Theorem 1 *[8] A standard function $f : U \to \mathbb{R}^m$ is differentiable (resp. of class C^1) if and only if ${}^*f$ is S-differentiable (resp. SU-differentiable).*

One final result needed: a standard subset $U \subseteq \mathbb{R}^n$ is open iff for all $x \in {}^\sigma U$ and $y \in {}^*\mathbb{R}^n$, if $x \approx y$ then $y \in {}^*U$.

2 Regular Surfaces

In this section we shall present the main result of our work. To start, let us recall the following definition.

Definition 3 Let $S \subseteq \mathbb{R}^3$ be a nonempty set. We say that S is a **regular surface** if for each $P \in S$, there exist an open neighbourhood V of P, an open set U in $\mathbb{R}^2$ and a function $x : U \to V \cap S$ satisfying the following conditions:

1. x is a homeomorphism;
2. x is of class C^1;
3. for each $q \in U$, the differential $Dx_q : \mathbb{R}^2 \to \mathbb{R}^3$ is $1-1$.

The function x is called a **parametrization** of S in P.

As usual, we denote $x_u(q) := \dfrac{\partial x}{\partial u}(q)$ and $x_v(q) := \dfrac{\partial x}{\partial v}(q)$.

Definition 4 If $x : U \to V \cap S$ is a parametrization in $P = x(p)$, we define the **unit normal vector** at each point $Q = x(q) \in x(U)$ by the rule

$$N(Q) := \frac{x_u \times x_v}{|x_u \times x_v|}(q).$$

In [5] is presented a nonstandard characterization of submanifolds in Euclidean spaces. Using that result we will give a characterization of regular surfaces using a field of unit normal vectors on the set.

Theorem 2 *[5] A standard subset $M^m \subseteq \mathbb{R}^n$ with $n \in {}^\sigma\mathbb{N}$ is a C^1-submanifold iff there exists a standard tangent plane map $T : M \to G(m,n)$ into the set of affine m-planes such that, for every near-standard point $P \in ns({}^*M)$,*

1. *$P \in T(P)$;*
2. *the ortogonal projection $\pi_P : {}^*M \to T(P)$ is an infinitesimal bijection;*
3. *if ${}^*M \ni Q \approx P$ then $\frac{|Q-\pi_P(Q)|}{|Q-P|} \approx 0$, i.e., the angle between the secant line through P and Q and the plane $T(P)$ is infinitesimal.*

We present now our result:

Theorem 3 *Let $S \subseteq \mathbb{R}^3$ be a nonempty set. Then S is a regular surface iff for each $P \in ns({}^*S)$, there exist a standard neighbourhood ${}^*V$ of P and a standard continuous function $N : V \cap S \to \mathbb{R}^3$ such that:*

1. *for all $Q \in V \cap S$, $|N(Q)| = 1$;*
2. *for all $Q, R \in {}^*V \cap {}^*S$ with $Q \neq R$,*
$$R \approx Q \Rightarrow N(Q) \cdot \frac{Q-R}{|Q-R|} \approx 0;$$
3. *If $T(P)$ is the plane containing P and orthogonal to $N(P)$, then*
$$\mu(P) \cap T(P) \subseteq \pi_P(\mu(P) \cap {}^*S)$$
where $\pi_P : {}^*\mathbb{R}^3 \to T(P)$ is the orthogonal projection.

Proof. We begin by assuming that S is a regular surface and let us fix $P \in ns({}^*S)$. Choose a standard neighbourhood V of $st(P)$ and a parametrization $x : U \to V \cap S$ in P. Define $N : V \cap S \to \mathbb{R}^3$ as the unit normal vector function at $x(U)$. It is easy to see that conditions 1 and 2 are satisfied. About condition 3, observe that $T(P)$ is the tangent plane to the surface at P, and by Theorem 2, condition 2, the proof follows.

To prove the reverse, we will prove that there exists a standard function $T : S \to G(2,3)$, (where $G(2,3)$ denotes the set of planes in $\mathbb{R}^3$) such that, for each $P \in ns({}^*S)$, we have:

1. $P \in T(P)$;
2. the orthogonal projection $\pi_P : {}^*S \to T(P)$ is an infinitesimal bijection in the sense that:
    (a) if $R, R' \in {}^*S$ with $R \approx R' \approx P$ and $\pi_P(R) = \pi_P(R')$, then $R = R'$;

(b) if $Q \in T(P)$ and $Q \approx P$, then there exists $R \in {}^*S$ with $R \approx P$ and $\pi_p(R) = Q$;

3. If ${}^*S \ni Q \approx P$ then $\frac{|Q-\pi_P(Q)|}{|Q-P|} \approx 0$.

Since it is a local problem, we will define a standard function $T : V \cap S \to G(2,3)$, where ${}^*V$ is a neighbourhood of P. First, choose a continuous function $u_1 : V \cap S \to \mathbb{R}^3$ such that $u_1(Q) \cdot N(Q) = 0$ and $|u_1(Q)| = 1$, for all $Q \in V \cap S$. Define $u_2 : V \cap S \to \mathbb{R}^3$ by $u_2(Q) = u_1(Q) \times N(Q)$ and let

$$\begin{array}{rccl} T: & V \cap S & \to & G(2,3) \\ & Q & \mapsto & \{Q + \lambda_1 u_1(Q) + \lambda_2 u_2(Q) \mid \lambda_1, \lambda_2 \in \mathbb{R}\} \end{array}$$

Clearly, $P \in T(P)$.

Suppose now that there exist $R, R' \in {}^*S$ with $R \approx R' \approx P$, $\pi_P(R) = \pi_P(R')$ but $R \neq R'$. Thus

$$P + ((R-P) \cdot u_1(P)) \cdot u_1(P) + ((R-P) \cdot u_2(P)) \cdot u_2(P) =$$

$$= P + ((R'-P) \cdot u_1(P)).u_1(P) + ((R'-P) \cdot u_2(P)) \cdot u_2(P) \Leftrightarrow \tag{2.1}$$

$$\Leftrightarrow \left\{ \begin{array}{l} (R-R') \cdot u_1(P) = 0 \\ (R-R') \cdot u_2(P) = 0 \end{array} \right. .$$

So we may conclude that

$$\frac{R-R'}{|R-R'|} = \pm N(P). \tag{2.2}$$

Multiplying both members by $N(R)$, we get

$$N(R) \cdot \frac{R-R'}{|R-R'|} = \pm N(R) \cdot N(P). \tag{2.3}$$

Moreover, the first member of this equation is infinitesimal and the second member is infinitely close to ± 1 (a contradiction). So the function is $1-1$. For the second part, it follows from condition 3.

Finally, the angle between the plane $T(P)$ and the straight line PQ is infinitesimal because

$$N(P) \cdot \frac{Q-P}{|Q-P|} \approx 0 \tag{2.4}$$

and $N(P)$ is orthogonal to $T(P)$.

■

Let us note that it is also true that

$$\pi_P(\mu(P) \cap {}^*S) \subseteq \mu(P) \cap T(P) \tag{2.5}$$

because if $Q \in {}^*S$ with $Q \approx P$, the continuity of π_P implies that

$$\pi_P(Q) \approx \pi_P(P) = P \in T(P). \tag{2.6}$$

We will now present a new definition of tangent space to a surface. We think that this definition is more intuitive than the classical one and, in a certain way, it is the geometric idea of the tangent space that we keep.

Definition 5 Let $P \in S$ be a point and $V \in \mathbb{R}^3$ a vector. We say that V is **tangent** to the surface at P if there exist $Q \in {}^*S$ with $Q \approx P$ and $k \in {}^*\mathbb{R}$ such that $k\overrightarrow{PQ} \in ns({}^*\mathbb{R}^3)$ and $V = st(k\overrightarrow{PQ})$.

Let $x : U \to V \cap S$ be a parametrization in P and fix $Q \in {}^*S$ with $P \approx Q$. Since V is open, $Q \in {}^*x(U)$ and so $P = x(p)$ and $Q = x(q)$, for some $p, q \in {}^*U$. By the continuity of x^{-1}, $p \approx q$. Consequently,

$$\overrightarrow{PQ} = x(q) - x(p) = Dx_p\,(q - p) + |q - p|\eta, \tag{2.7}$$

for some $\eta \approx 0$. Thus

$$k\overrightarrow{PQ} = k|q - p| \left(Dx_p \left(\frac{q - p}{|q - p|} \right) + \eta \right). \tag{2.8}$$

Observe that, if $u \in {}^*\mathbb{R}^2$ is an unit vector, then $Dx_p\,(u) \not\approx 0$ (if not, we would have

$$st(Dx_p(u)) = 0 \Leftrightarrow Dx_p\,(st(u)) = 0 \tag{2.9}$$

and $st(u) \neq 0$, a contradiction). So, if $k\overrightarrow{PQ} \in ns({}^*\mathbb{R}^3)$, then $k|q - p| \in fin({}^*\mathbb{R})$ and so

$$k\overrightarrow{PQ} \approx Dx_p\,(k(q - p)). \tag{2.10}$$

Definition 6 The set of tangent vectors to a surface S at P is called the **tangent plane** to S at P and denoted by $T_P S$.

Theorem 4 *It is true that* $T_P S = Dx_p\,(\mathbb{R}^2)$.

Proof. Let $V \in T_P S$ be a vector. Then

$$V = st(k\overrightarrow{PQ}) = Dx_p\,(st(k(q - p))), \tag{2.11}$$

and therefore $V \in Dx_p\,(\mathbb{R}^2)$.

To prove the reverse, if $V = Dx_p\,(u)$, for some $u \in \mathbb{R}^2$, let $q := p + \epsilon u$, with $0 < \epsilon \approx 0$.

Then

$$x(q) - x(p) = Dx_p\,(\epsilon u) + \epsilon|u|\eta, \tag{2.12}$$

for some $\eta \approx 0$, which implies that

$$\frac{x(q) - x(p)}{\epsilon} \approx Dx_p(u). \tag{2.13}$$

Define $Q = x(q)$ and $k = 1/\epsilon$, and therefore $V = st\left(k\overrightarrow{PQ}\right)$. ∎

Theorem 5 *Let U be an open subset of $\mathbb{R}^n$, $p \in U$, $m > n$ and $x : U \to \mathbb{R}^m$ an injective C^1 function. Let $\{p_0, \ldots, p_n\} \subseteq {}^*U$ be a set such that:*

1. *$p_i \neq p_j$ for $i \neq j$ and $0 \leq i, j \leq n$;*
2. *$p_0 \approx \ldots \approx p_n \approx p$;*
3. *the vectors $\left\{st\left(\frac{p_1 - p_0}{|p_1 - p_0|}\right), \ldots, st\left(\frac{p_n - p_0}{|p_n - p_0|}\right)\right\}$ are linearly independents;*
4. *the vectors $\left\{\frac{\partial x}{\partial u_1}(p), \ldots, \frac{\partial x}{\partial u_n}(p)\right\}$ are also linearly independents.*

Define, for $1 \leq i \leq n$,

$$v_i := x(p_i) - x(p_0),$$

$$\Pi_0 := \{x(p_0) + \lambda_1 v_1 + \ldots + \lambda_n v_n \mid \lambda_1, \ldots, \lambda_n \in {}^*\mathbb{R}\}$$

and

$$\Pi := \{x(p) + \lambda_1 \frac{\partial x}{\partial u_1}(p) + \ldots + \lambda_n \frac{\partial x}{\partial u_n}(p) \mid \lambda_1, \ldots, \lambda_n \in \mathbb{R}\}.$$

If $a \in fin(\Pi_0)$ then $st(a) \in \Pi$.

Proof. Let $W := \langle v_1, \ldots, v_n \rangle \subseteq {}^*\mathbb{R}^m$ and $k := dim(W) \leq n$. Assume, without any loss of generality, that $\{v_1, \ldots, v_k\}$ is a basis of W, $k \leq n$. Let $\{b_1, \ldots, b_{m-k}\}$ be an orthonormal basis of $W^\perp$. For $j \in \{1, \ldots, m-k\}$, define the functions $f_j : {}^*U \to {}^*\mathbb{R}$ by

$$f_j(u) = (x(u) - x(p_0)) \cdot b_j,\ u \in {}^*U.$$

It is obvious that for each $1 \leq j \leq m-k$, f_j is SU-differentiable. Moreover, since

$$f_j(p_0) = f_j(p_1) = \ldots = f_j(p_n) = 0, \tag{2.14}$$

then for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m-k\}$, there is $\eta_{ij} \approx 0$ such that

$$D(f_j)_{p_0}(p_i - p_0) + |p_i - p_0|\eta_{ij} = 0 \Leftrightarrow Dx_{p_0}\left(\frac{p_i - p_0}{|p_i - p_0|}\right) \cdot b_j \approx 0. \tag{2.15}$$

Taking the standard parts of both members we get

$$Dx_p\left(st\frac{p_i - p_0}{|p_i - p_0|}\right) \cdot st(b_j) = 0. \tag{2.16}$$

Let $st(W) := \{st(v) \mid v \in fin(W)\}$, then $st(W)$ is a linear subspace of $\mathbb{R}^m$ and $dim(st(W)) = k$ (*cf.* [3]). Similarly, $st(W^\perp)$ is a linear space and $st(W^\perp) = \langle st(b_1), \dots, st(b_{m-k})\rangle$. Note that, for $i = 1, \dots, n$, the vectors $st\left(\frac{p_i - p_0}{|p_i - p_0|}\right)$ are linearly independents and since Dx_p is an injective linear operator, the vectors $Dx_p\left(st\left(\frac{p_i - p_0}{|p_i - p_0|}\right)\right)$, for $i = 1, \dots n$ are also linearly independents. Moreover, from

$$Dx_p\left(st\left(\frac{p_i - p_0}{|p_i - p_0|}\right)\right) \in \langle \frac{\partial x}{\partial u_1}(p), \dots, \frac{\partial x}{\partial u_n}(p)\rangle, \tag{2.17}$$

it follows that

$$\langle Dx_p\left(st\left(\frac{p_1 - p_0}{|p_1 - p_0|}\right)\right), \dots, Dx_p\left(st\left(\frac{p_n - p_0}{|p_n - p_0|}\right)\right)\rangle = \langle \frac{\partial x}{\partial u_1}(p), \dots, \frac{\partial x}{\partial u_n}(p)\rangle. \tag{2.18}$$

But

$$\frac{v_i}{|p_i - p_0|} \approx Dx_{p_0}\left(\frac{p_i - p_0}{|p_i - p_0|}\right) \tag{2.19}$$

which implies

$$st\left(\frac{v_i}{|p_i - p_0|}\right) = Dx_p\left(st\left(\frac{p_i - p_0}{|p_i - p_0|}\right)\right), \tag{2.20}$$

and so

$$\left\{\frac{\partial x}{\partial u_i}(p) \mid i \in \{1, \dots, n\}\right\} \subseteq st(W). \tag{2.21}$$

So we conclude that $k = n$ and $\{v_1, \dots, v_n\}$ are linearly independents.

Consequently, if $a \in fin(\Pi_0)$, then for all $j \in \{1, \dots, m-n\}$ we have

$$(a - x(p_0)) \cdot b_j = 0 \Rightarrow (st(a) - x(p)) \cdot st(b_j) = 0 \Rightarrow st(a) \in \Pi. \tag{2.22}$$

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