Connectedness and Compactness on Standard Sets

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Abstract

We present a nonstandard characterization of connected compact sets

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1 Introduction

There is as yet no simple nonstandard characterization of connectedness, and little work has been done in that direction. In [4], Steven Leth presents a sufficient condition for a set $A \subseteq \mathbb{R}^n$ to be connected. As Leth remarked, however, it is not a necessary condition. His hypothesis involves internal polygonal paths joining distinct points. We will work with hyper-finite sets instead of polygonal paths, thus eliminating the implicit local path-connectedness that is present in [4]. We mention also the work of Sérgio Rodrigues [5] characterizing connectedness in nonstandard terms using the monad of a set.

The paper is organized as follows. In section 2 we collect some necessary background for the reader's convenience. In section 3 we present new results about connectedness and compactness on standard sets; we introduce a new concept, the discrete infinitesimal path, which will be used to characterize connected compact sets in metric spaces.

2 Preliminaries

All the sets which come up in classical analysis have nonstandard extensions using a map denoted by "$*$". For example, if $\mathbb{R}$ denotes the set of real numbers, ${}^*\mathbb{R}$ will be its nonstandard extension. This extension contains "ideal elements", like infinitesimals and infinite numbers, but also a copy of the set of real numbers, denoted by the symbol ${}^\sigma\mathbb{R}$. It is not our intention to give a full exposition on this subject, we will just fix notation and present some results needed. For further details, the reader is referred to [2, 3, 6, 8].

Definition 1. *Let $x, y \in {}^*\mathbb{R}$. We say that*

1. *x is infinitesimal if $|x| < \epsilon$, for all positive real number ϵ and we write $x \approx 0$;*
2. *x is finite if, for some positive real number ϵ, $|x| < \epsilon$;*
3. *x is infinite (or infinitely large) if it is not finite, i.e., for any positive real number ϵ, $|x| > \epsilon$; we write $x \approx \infty$;*
4. *x, y are infinitely close if $x - y$ is infinitesimal; we write $x \approx y$.*

In the following, (X, d) is a metric space.

Definition 2. *For $x \in {}^*X$, the monad of x is the subset of ${}^*X$ given by*

$$\mu(x) := \{y \in {}^*X \mid d(x,y) \approx 0\}.$$

As before, the nonstandard extension of X contains a copy of the original set, which we denote by ${}^\sigma X$ (elements of ${}^\sigma X$ are called standard). A point $y \in {}^*X$ is *nearstandard* if there exists some standard $x \in {}^\sigma X$ such that $y \in \mu(x)$; in this case we say that x is the *standard part* of y and write $st(y) = x$. We say that $x, y \in {}^*X$ are *infinitely close*, and write $x \approx y$, if $d(x,y) \approx 0$. If x and y are not infinitely close, we write $x \not\approx y$.

The set of the nearstandard points of ${}^*X$ is

$$ns({}^*X) := \bigcup\{\mu(x) \mid x \in {}^\sigma X\}.$$

Theorem 1. *[3] Let $A \subseteq X$. Then*

1. *A is open if and only if for all $a \in {}^\sigma A$, $\mu(a) \subseteq {}^*A$ holds;*
2. *A is closed if and only if, whenever $a \in {}^*A$ and $a \approx x$ for some $x \in {}^\sigma X$, we have $x \in {}^\sigma A$;*
3. *A is compact if and only if for all $a \in {}^*A$, there is an $x \in {}^\sigma A$ with $a \approx x$;*

In every metric space, monads of distinct standard points are disjoint (see [3]). Therefore, for all $x \in ns({}^*X)$, there exists exactly one element in ${}^\sigma X$, called $st(x)$, infinitely close to x. Hence we have a well-defined function

$$\begin{array}{rccc} st: & ns({}^*X) & \to & {}^\sigma X \\ & x & \mapsto & st(x) \end{array}$$

called the *standard part function.*

Theorem 2. *[3] Let X and Y be two topological spaces and $f : X \to Y$ a function. Then f is continuous if and only if*

$$\forall x \in {}^\sigma X \quad f(\mu(x)) \subseteq \mu(f(x)),$$

or equivalently,

$$\forall x \in {}^\sigma X \, \forall y \in {}^*X \quad [x \approx y \Rightarrow f(x) \approx f(y)].$$

3 Main results

In what follows, (X, d_X) and (Y, d_Y) will denote two metric spaces, and $A \subseteq X$ a nonempty subset. To simplify notation, we will denote both metrics by the same symbol d. Given two points $x, y \in {}^*A$, we define the set (possibly external)

$$\mathcal{P}^{{}^*A}_{x,y} := \{u = (u_n)_{n=1,\ldots,N} \mid N \in {}^*\mathbb{N}, u_1 = x, u_N = y, u_n \in {}^*A$$
$$\text{and } u_n \approx u_{n+1}, \text{ for all } n \in \{1, \ldots, N-1\}\}.$$

We call the hyper-finite sequence $u = (u_n)_{n \in \{1,\ldots N\}}$ a *discrete infinitesimal path* (abbreviation *d.i.p.*) joining x to y in ${}^*A$. We define a binary relation on ${}^*A$ by $x \sim y$ if $\mathcal{P}^{{}^*A}_{x,y}$ is nonempty; it is easy to prove that $\sim$ is an equivalence relation.

We will simply write $\mathcal{P}_{x,y}$ instead of $\mathcal{P}^{{}^*A}_{x,y}$ whenever there is no danger of confusion.

The existence of (standard) discrete paths joining points on connected sets is known. In fact, it can be proved that if A is a connected set and ϵ is a (standard) real, then for all $x, y \in A$, there exists a finite sequence of points, all lying in A,

$$x = u_1, u_2, \ldots, u_n = y,$$

such that the distance between any two successive points in this sequence is less than ϵ.

The next result follows from the compactness of A and the consequent uniform continuity of f.

Theorem 3. *Let $f : X \to Y$ be a function. If f is continuous, then for any subset $A \subseteq X$ satisfying*

$$\forall x, y \in {}^*A\, \exists u \in \mathcal{P}_{x,y} \text{ with } u_n \in ns({}^*X) \text{ and } st(u_n) \in {}^\sigma A, \text{ for all } n \tag{1}$$

the following condition is valid

$$\forall z, w \in {}^*f(A)\, \exists v \in \mathcal{P}_{z,w} \text{ with } v_n \in ns({}^*Y) \text{ and } st(v_n) \in {}^\sigma f(A), \text{ for all } n.$$

Proof. Let A be a set that satisfies condition (1). Given z and w in ${}^*f(A)$, let $z = f(x)$ and $w = f(y)$, for some $x, y \in {}^*A$. Then, there exists $u = (u_n)_{n=1,\ldots,N} \in \mathcal{P}_{x,y}$, such that $u_n \in ns({}^*X)$ and $st(u_n) \in {}^\sigma A$, for all $n = 1, \ldots, N$. Define $v_n := f(u_n)$, for all $n = 1, \ldots, N$. It is easy to see that $v = (v_n)$ satisfies the necessary conditions. □

Theorem 4. *The set A is connected if*

$$\forall x, y \in {}^\sigma A\, \exists u \in \mathcal{P}_{x,y} \text{ with } u_n \in ns({}^*X) \text{ and } st(u_n) \in {}^\sigma A, \text{ for all } n. \tag{2}$$

Proof. Assume that A is not connected. Then A has a subset $B \notin \{\emptyset, A\}$ that is simultaneously relatively open and closed. Pick $x \in {}^\sigma B$, $y \in {}^\sigma(A - B)$ and $u = (u_n)_{n=1,\ldots,N} \in \mathcal{P}_{x,y}$ such that $u_n \in ns({}^*X)$ and $st(u_n) \in {}^\sigma A$, for all n, and define the internal set

$$K := \{n \in \{1, \ldots, N\} \mid u_n \in {}^*B\}.$$

Since K is nonempty (for example, $1 \in K$), it has a maximum. Let $k := \max K$. Since $y \notin {}^*B$ then $k \neq N$. Besides this, $u_k \in {}^*B$ and $u_{k+1} \in {}^*(A - B)$. Since B and $A - B$ are both closed, $st(u_k) \in B$ and $st(u_{k+1}) \in A - B$.

Since $u_k \approx u_{k+1}$, the point $st(u_k) = st(u_{k+1}) \in {}^\sigma B \cap {}^\sigma(A - B)$, which ends the proof. □

The previous condition is not enough to assert that A is path connected; e.g. take the set

$$\{(x, \sin(1/x)) \mid x > 0\} \cup (\{0\} \times [-1, 1]).$$

However, if A is path connected then condition (2) is satisfied. Indeed, if we fix $x, y \in {}^\sigma A$, then by hypothesis there exists a continuous path $\alpha : [0, 1] \to A$ with $\alpha(0) = x$ and $\alpha(1) = y$. Take an infinite $N \in {}^*\mathbb{N}$ and define $u_n := \alpha(\frac{n}{N})$ for $n \in \{0, \ldots, N\}$. It is easy to check that (u_n) satisfies condition (2).

The converse of Theorem 4 is false in general, however we will obtain a related result.

Theorem 5. *If A is a connected set then for all $x, y \in {}^*A$ the condition $\mathcal{P}_{x,y} \neq \emptyset$ holds.*

Proof. Fix $x, y \in {}^\sigma A$ and $\epsilon \in {}^\sigma\mathbb{R}^+$. Then there exists an ϵ-chain that joins x and y (c.f. [7], pag 120). Therefore

$$\forall x, y \in {}^\sigma A\, \forall \epsilon \in {}^\sigma\mathbb{R}^+\, \exists N \in {}^\sigma\mathbb{N}\, \exists \{u_2, \ldots, u_{N-1}\} \subset {}^\sigma A$$
$$\forall i \in \{1, \ldots, N-1\} \quad d(u_i, u_{i+1}) < \epsilon,$$

where $u_1 := x$ and $u_N := y$. Now, pick two points $x, y \in {}^*A$. By the Transfer Principle, condition holds with $\epsilon \approx 0$. □

Observe that we actually proved that, for all infinitesimal ϵ, there exists $u \in \mathcal{P}_{x,y}$ satisfying $d(u_i, u_{i+1}) < \epsilon$.

Unfortunately, the *d.i.p.* need not to be nearstandard in A, as is shown in the next example.

Let A be the subset of $\mathbb{R}^2$ defined by

$$([0, 1] \times \{0\}) \cup \left\{ \left(\frac{1}{n}, y\right) \mid n \in \mathbb{N}, y \in [0, 1] \right\} \cup \{(0, 0), (0, 1)\}.$$

The set A is connected but there is no *d.i.p.* joining the points $(0, 0)$ to $(0, 1)$ nearstandard in the set.

Corollary 1. *Let A be a compact set. Then A is connected if and only if*

$$\forall x, y \in {}^{\sigma}A \, \exists u \in \mathcal{P}_{x,y} \text{ such that } u_n \in ns({}^{*}X) \text{ and } st(u_n) \in {}^{\sigma}A, \text{ for all } n.$$

Proof. Follows from Theorems 4 and 5 and the fact that, for the nonstandard extension of compact sets, all points are nearstandard on the set. □

In conclusion, we have now a nice characterization of connected compact sets.

Corollary 2. *Let A be a non-empty set. Then A is connected and compact if and only if*

$$\forall x, y \in {}^{*}A \, \exists u \in \mathcal{P}_{x,y} \text{ such that } u_n \in ns({}^{*}X) \text{ and } st(u_n) \in {}^{\sigma}A, \text{ for all } n. \tag{3}$$

Proof. We only need to prove that condition (3) implies the compactness condition. Fix $x \in {}^{*}A$. By condition (3), there exists some $u \in \mathcal{P}_{x,x}$ nearstandard on A. So $u_1 = x \in ns({}^{*}X)$ and $st(x) \in {}^{\sigma}A$. □

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