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Quadratures of Pontryagin extremals for optimal control problems[1]

by

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Abstract: We obtain a method to compute effective first integrals by combining Noether's principle with the Kozlov-Kolesnikov integrability theorem. A sufficient condition for the integrability by quadratures of optimal control problems with controls taking values on open sets is obtained. We illustrate our approach on some problems taken from the literature. An alternative proof of the integrability of the sub-Riemannian nilpotent Lie group of type $(2,3,5)$ is also given.

Keywords: Noether's symmetry theorem, Kozlov-Kolesnikov integrability, integrability by quadratures, optimal control.

1. Introduction

Optimal control problems, with controls taking values on an open set, are now a subject of intensive investigation because of their recent applications to modern technology, like in "smart materials" (Lasiecka, 2004). Geodesics of (sub)-Riemannian manifolds can also be seen as solutions of a class of these problems (Bonnard and Chyba, 2003). Meanwhile, solutions of optimal control problems are closely related with solutions of Hamiltonian equations through the Pontryagin Maximum Principle (Pontryagin et al., 1962). In the literature, Hamiltonian equations are usually classified as integrable or non-integrable. However, it was not always clear in what sense a system is (non)integrable, as the following quotations confirm (Arnold, Kozlov, Neĭshtadt, 1993; Goriely, 2001): Birkhoff comment *"when, however, one attempts to formulate a precise definition of integrability, many possibilities appear, each with a certain intrinsic theoretic interest"*; and the dictum of Poincaré *"a system of differential equations is only more or less integrable"*. The reasons are the existence of three main approaches

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to study integrability (the dynamical system approach through bifurcation theory, the analytic approach using singularity analysis and Painlevé property, and the algebraic approach through differential geometry), which are not always compatible, and the stratification of the phase space in regions that may be integrable with different notions.

In the Control Theory setting, integrability concerns the existence of a foliation as the collection of all maximal integral manifolds of the underlying distribution (see Frobenius and Nagano-Sussmann theorems, Sussmann, 1973). On the other hand, integration by quadratures of a differential equation is the search for representing the solutions by a finite number of algebraic operations, inversion of functions, and calculations of integrals of known functions ("quadratures"); where a precise meaning of integrability defines the allowed operations and the set of known functions. Hence, an optimal control problem is integrable by quadratures if the corresponding Hamiltonian equations are integrable by quadratures. Since an optimal control problem can be, simultaneously, integrable and not integrable by quadratures, and it is frequent to shortcut integration by quadratures to integration, we will adopt the word "solvability" to mean "integrability by quadratures" of optimal control problems.

In the algebraic approach, verification of solvability of an optimal control problem frequently requires the existence of a set of first integrals (or conservation laws) of the true Hamiltonian equations, and an appropriate method of reduction. The process can be divided in three steps: (i) find a sufficient number of first integrals; (ii) verify that such first integrals imply the existence of quadratures; and (iii) apply a method to find the quadratures. The first step can be attained by methods such as the Noether's theorem, finding Casimirs, or even solving the PDE that appear in the definition of variational symmetry. The second step can be accomplished by using theorems such as Bour-Liouville, Liouville-Arnold (abelian case), Mishchenko-Fomenko (nonabelian case), or Kozlov-Kolesnikov. Last step is usually a consequence of the choices on the first and second steps, e.g. Liouville-Poincaré method, Cartan method, or Prykarpastsky method. Notice that step (i), alone, is not enough to solve the problem, since any C^1 function of a set of first integrals is a first integral (Goriely, 2001), meaning that there exists plenty of first integrals that are useless.

In this work, we use the algebraic approach to derive in Section 3 a method for computing effective first integrals (steps (i) and (ii)) for optimal control problems by combining Noether's symmetry theorem with the Kozlov-Kolesnikov integrability theorem (recalled in Section 2). Main result gives a sufficient condition for the solvability of a given optimal control problem (Theorem 4). A key issue is the construction of a system of algebraic equations, whose solutions determine the set of effective first integrals, based on the simple observation that for optimal control problems Noether's theorem usually gives a parametric family of first integrals. The proposed method is applied in Section 4 to concrete optimal control problems from the literature. An alternative proof to

Sachkov (2004) for the solvability of the sub-Riemannian nilpotent Lie group of type $(2,3,5)$ is presented in Section 5.

2. Preliminaries

2.1. The problem

The optimal control problem consists in minimizing a cost functional

$$I\left[x(\cdot),u(\cdot)\right]=\int_a^b \mathcal{L}\left(t,x(t),u(t)\right)dt \tag{1}$$

subject to a control system described by ordinary differential equations

$$\dot{x}(t)=\varphi\left(t,x(t),u(t)\right) \tag{2}$$

together with certain appropriate endpoint conditions. The Lagrangian $\mathcal{L}:[a,b]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ and the velocity vector $\varphi:[a,b]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ are given, and assumed to be smooth: $\mathcal{L}(\cdot,\cdot,\cdot)$, $\varphi(\cdot,\cdot,\cdot)\in C^1$. We are interested in the case where the control set is open: $u(t)\in U\subseteq\mathbb{R}^m$, with U an open set. We denote the problem by (P). In the particular case of $\varphi(t,x,u)=u$, one obtains the fundamental problem of the calculus of variations, which covers all classical mechanics. The choice of the classes $\mathcal{X}$ and $\mathcal{U}$, respectively of the state $x:[a,b]\to\mathbb{R}^n$ and control variables $u:[a,b]\to\mathbb{R}^m$, are important for the problem to be well-defined. We will assume, for simplicity, that $\mathcal{X}=PC^1\left([a,b];\mathbb{R}^n\right)$ and $\mathcal{U}=PC\left([a,b];\mathbb{R}^m\right)$.

The Pontryagin Maximum Principle (Pontryagin et al., 1962) is a necessary optimality condition which can be obtained from a general Lagrange multiplier theorem in spaces of infinite dimension. Introducing the Hamiltonian function

$$H(x,u,\psi,t)=-\mathcal{L}(t,x,u)+\psi\cdot\varphi(t,x,u)\,, \tag{3}$$

where ψ_i, $i=1,\ldots,n$, are the "Lagrange multipliers" or the "generalized momenta", the multiplier theorem asserts that the optimal control problem is equivalent to the maximization of the augmented functional

$$J\left[x(\cdot),u(\cdot),\psi(\cdot)\right]=\int_a^b \left(H\left(x(t),u(t),\psi(t),t\right)-\psi(t)\cdot\dot{x}(t)\right)dt\,.$$

Let $\left(\tilde{x}(\cdot),\tilde{u}(\cdot),\tilde{\psi}(\cdot)\right)$ solve the problem, and consider arbitrary C^1-functions $h_1\,,h_3:[a,b]\to\mathbb{R}^n$, $h_1(\cdot)$ vanishing at a and b ($h_1(\cdot)\in C_0^1\left([a,b]\right)$), and arbitrary continuous $h_2:[a,b]\to\mathbb{R}^m$. Let ε be a scalar. By the definition of maximizer, we have

$$J\left[(\tilde{x}+\varepsilon h_1)(\cdot),(\tilde{u}+\varepsilon h_2)(\cdot),(\tilde{\psi}+\varepsilon h_3)(\cdot)\right]\leq J\left[\tilde{x}(\cdot),\tilde{u}(\cdot),\tilde{\psi}(\cdot)\right]\,,$$

and one has the following necessary condition:

$$\frac{d}{d\varepsilon} J\left[(\tilde{x}+\varepsilon h_1)(\cdot),(\tilde{u}+\varepsilon h_2)(\cdot),(\tilde{\psi}+\varepsilon h_3)(\cdot)\right]\Big|_{\varepsilon=0} = 0\,. \tag{4}$$

Differentiating (4) gives

$$\begin{aligned} 0 = \int_a^b \Bigg[&\frac{\partial H}{\partial x}\left(\tilde{x}(t),\tilde{u}(t),\tilde{\psi}(t),t\right)\cdot h_1(t) + \frac{\partial H}{\partial u}\left(\tilde{x}(t),\tilde{u}(t),\tilde{\psi}(t),t\right)\cdot h_2(t) \\ &+ \frac{\partial H}{\partial \psi}\left(\tilde{x}(t),\tilde{u}(t),\tilde{\psi}(t),t\right)\cdot h_3(t) - h_3(t)\cdot\dot{\tilde{x}}(t) - \tilde{\psi}(t)\cdot\dot{h}_1(t)\Bigg]\,dt\,. \end{aligned}$$

Integrating the $\tilde{\psi}(t)\cdot\dot{h}_1(t)$ term by parts, and having in mind that $h_1(a) = h_1(b) = 0$, one derives

$$\begin{aligned} \int_a^b \Bigg[&\left(\frac{\partial H}{\partial x}\left(\tilde{x}(t),\tilde{u}(t),\tilde{\psi}(t),t\right) + \dot{\tilde{\psi}}(t)\right)\cdot h_1(t) + \frac{\partial H}{\partial u}\left(\tilde{x}(t),\tilde{u}(t),\tilde{\psi}(t),t\right)\cdot h_2(t) \\ &+ \left(\frac{\partial H}{\partial \psi}\left(\tilde{x}(t),\tilde{u}(t),\tilde{\psi}(t),t\right) - \dot{\tilde{x}}(t)\right)\cdot h_3(t)\Bigg]\,dt = 0\,. \end{aligned} \tag{5}$$

Note that (5) was obtained for any variation $h_1(\cdot)$, $h_2(\cdot)$, and $h_3(\cdot)$. Choosing $h_1(t) = h_2(t) \equiv 0$, and $h_3(\cdot)$ arbitrary, one obtains the control system (2):

$$\dot{\tilde{x}}(t) = \frac{\partial H}{\partial \psi}\left(\tilde{x}(t),\tilde{u}(t),\tilde{\psi}(t),t\right)\,, \quad t\in[a,b]\,. \tag{6}$$

With $h_1(\cdot)$ arbitrary, and $h_2(t) = h_3(t) \equiv 0$, we obtain the *adjoint system*:

$$\dot{\tilde{\psi}}(t) = -\frac{\partial H}{\partial x}\left(\tilde{x}(t),\tilde{u}(t),\tilde{\psi}(t),t\right)\,, \quad t\in[a,b]\,. \tag{7}$$

Finally, with $h_2(\cdot)$ arbitrary, and $h_1(t) = h_3(t) \equiv 0$, the *stationary condition* is obtained:

$$\frac{\partial H}{\partial u}\left(\tilde{x}(t),\tilde{u}(t),\tilde{\psi}(t),t\right) = 0\,, \quad t\in[a,b]\,. \tag{8}$$

Hence, a necessary optimality condition for $(\tilde{x}(\cdot),\tilde{u}(\cdot))$ to be a minimizer of problem (P) is given by the *Pontryagin Maximum Principle*: there exists $\tilde{\psi}(\cdot)$ such that the 3-tuple $\left(\tilde{x}(\cdot),\tilde{u}(\cdot),\tilde{\psi}(\cdot)\right)$ satisfies all the conditions (6), (7), and (8). Observe that we are interested in the study of normal extremals. Additional extremals, known as abnormal, which correspond to the Lagrangian multiplied by zero in the Hamiltonian (3), may occur in some situations. For instance, in Example 3 abnormal extremals occur but with no consequence in the application of the proposed method, since the abnormal extremals are also normal.

We recall that conditions (6), (7), and (8) imply the equality

$$\frac{d}{dt} H\left(\tilde{x}(t), \tilde{u}(t), \tilde{\psi}(t), t\right) = \frac{\partial H}{\partial t}\left(\tilde{x}(t), \tilde{u}(t), \tilde{\psi}(t), t\right) . \tag{9}$$

We assume, without loss of generality, that there exist at least one $k \in \mathbb{N}_0$ such that

$$\exists\, (x, \psi, t) \in M : \frac{\partial}{\partial u} \frac{d^k}{dt^k} \frac{\partial H}{\partial u}(x, u, \psi, t) \not\equiv 0. \tag{10}$$

Further, we denote by k_u the smallest of such k and by $\bar{u}$ the solution of the equation

$$\frac{d^{k_u}}{dt^{k_u}} \frac{\partial H}{\partial u}(x, u, \psi, t) = 0 \tag{11}$$

with respect to u; giving as *true Hamiltonian* the expression

$$\mathcal{H}(x, \psi, t) = H\left(x, \bar{u}, \psi, t\right). \tag{12}$$

A mapping $F(x, \psi, t)$ is a *first integral* of the Hamiltonian equations with Hamiltonian $\mathcal{H}(x, \psi, t)$ if

$$\frac{\partial F}{\partial t} + \{\mathcal{H}, F\} = 0, \tag{13}$$

where $\{\cdot, \cdot\}$ denotes the canonical Poisson bracket.

Remark 1 *The restriction for the control set to be open is crucial. For closed control sets U the stationary condition (8) becomes a more general maximality condition $H(\tilde{x}(t), \tilde{u}(t), \tilde{\psi}(t)) = \max_{v \in U} H(\tilde{x}(t), v, \tilde{\psi}(t))$, and for such cases the true Hamiltonian may be discontinuous. Our approach, through symplectic geometry, can only deal with at least C^1 Hamiltonians.*

Remark 2 *If there exists a mapping $G(x, \psi)$ such that $\{\mathcal{H}, G\} = c\mathcal{H}$ for some $c \in \mathbb{R}$, then $F(x, \psi, t) = G(x, \psi) - c\, t\, \mathcal{H}(x, \psi)$ is a (nonautonomous) first integral.*

Remark 3 *In this work we will consider nonautonomous problems, for this reason we define the extended cotangent space of $\mathbb{R}^n$ by $M = \mathbb{R}^n\{x\} \times (\mathbb{R}^n)^*\{\psi\} \times \mathbb{R}\{t\}$. However, notice that a nonautonomous Hamiltonian on $\mathbb{R}^n$ is not significantly different from an autonomous Hamiltonian on $\mathbb{R}^{n+1}$, since a nonautonomous Hamiltonian $H(x, \psi, t)$ with the Hamiltonian system*

$$\dot{x} = \frac{\partial H}{\partial \psi} \quad \textit{and} \quad \dot{\psi} = -\frac{\partial H}{\partial x}$$

can be transformed into an autonomous Hamiltonian $K(x, \psi, \theta, t) = H(x, \psi, t) - \theta$ with Hamiltonian equations

$$\dot{x} = \frac{\partial K}{\partial \psi}, \quad \dot{\psi} = -\frac{\partial K}{\partial x}, \quad \dot{\theta} = \frac{\partial K}{\partial t} \quad \textit{and} \quad \dot{t} = -\frac{\partial K}{\partial \theta}.$$

The reverse procedure partially justifies the statement that an autonomous Hamiltonian is always a first integral for the problem, so the equations of an autonomous Hamiltonian can be dimension-reduced. Furthermore, a system with only one degree of freedom is always integrable.

2.2. Noether's theorem

In 1918 Emmy Noether established the key result to find conservation laws in the calculus of variations (Noether, 1971). We sketch here the standard argument used to derive Noether's theorem and conservation laws in the optimal control setting (see, e.g., Djukic, 1973; Torres, 2002).

Let us consider a one-parameter group of C^1-transformations of the form

$$h_s(x,u,\psi,t) = \left(h_s^x(x,u,\psi,t), h_s^u(x,u,\psi,t), h_s^\psi(x,u,\psi,t), h_s^t(x,u,\psi,t)\right) , \quad (14)$$

where s denotes the independent parameter of the transformations. We require that to the parameter value $s = 0$ correspond the identity transformation:

$$\begin{aligned} h_0(x,u,\psi,t) &= \Big(h_0^x(x,u,\psi,t), h_0^u(x,u,\psi,t), h_0^\psi(x,u,\psi,t), h_0^t(x,u,\psi,t)\Big) \\ &= (x,u,\psi,t) . \end{aligned} \quad (15)$$

Associated to the group of transformations (14) we consider the *infinitesimal generators*

$$\begin{aligned} T(x,u,\psi,t) &= \frac{d}{ds}h_s^t(x,u,\psi,t)\Big|_{s=0} , \quad X(x,u,\psi,t) = \frac{d}{ds}h_s^x(x,u,\psi,t)\Big|_{s=0} , \\ U(x,u,\psi,t) &= \frac{d}{ds}h_s^u(x,u,\psi,t)\Big|_{s=0} , \quad \Psi(x,u,t,\psi) = \frac{d}{ds}h_s^\psi(x,u,\psi,t)\Big|_{s=0} . \end{aligned} \quad (16)$$

DEFINITION 1 *The optimal control problem* (P) *is said to be invariant under a one-parameter group of* C^1*-transformations* (14) *if, and only if,*

$$\frac{d}{ds}\left\{\left[H\left(h_s\left(x(t),u(t),\psi(t),t\right)\right) \right.\right. \\ \left.\left. -h_s^\psi\left(x(t),u(t),\psi(t),t\right)\cdot\frac{\frac{dh_s^x(x(t),u(t),\psi(t),t)}{dt}}{\frac{dh_s^t(x(t),u(t),\psi(t),t)}{dt}}\right]\frac{dh_s^t\left(x(t),u(t),\psi(t),t\right)}{dt}\right\}\Bigg|_{s=0} = 0 , \quad (17)$$

with H *the Hamiltonian* (3).

Having in mind (15), condition (17) is equivalent to

$$\frac{\partial H}{\partial t}T + \frac{\partial H}{\partial x}\cdot X + \frac{\partial H}{\partial u}\cdot U + \frac{\partial H}{\partial \psi}\cdot\Psi - \Psi\cdot\dot{x}(t) - \psi(t)\cdot\frac{d}{dt}X + H\frac{d}{dt}T = 0 , \quad (18)$$

where all functions are evaluated at $(x(t), u(t), \psi(t), t)$ whenever not otherwise indicated. Along a Pontryagin extremal $(x(\cdot), u(\cdot), \psi(\cdot))$ equalities (6), (7), (8), and (9) are in force, and (18) reduces to

$$\frac{dH}{dt}T - \dot{\psi}(t) \cdot X - \psi(t) \cdot \frac{dX}{dt} + H\frac{dT}{dt} = 0 \Leftrightarrow \frac{d}{dt}\left(\psi(t) \cdot X - HT\right) = 0\,. \quad (19)$$

Therefore, we have just proved *Noether's theorem for optimal control problems.*

THEOREM 1 (Noether's Theorem) *If the optimal control problem is invariant under* (14)*, in the sense of Definition 1, then*

$$\psi(t) \cdot X\left(x(t), u(t), \psi(t), t\right) - H\left(x(t), u(t), \psi(t), t\right) T\left(x(t), u(t), \psi(t), t\right) = const \quad (20)$$

($t \in [a, b]$*;* T *and* X *are given according to* (16)*;* H *is the Hamiltonian* (3)*) is a* conservation law*, that is,* (20) *is valid along all the minimizers* $(x(\cdot), u(\cdot))$ *of* (P) *which are Pontryagin extremals.*

2.3. Solvability and reduction

E. Bour and J. Liouville, in the middle of the 19th century, obtained fundamental concepts and results concerning integrability by quadratures of differential equations. Namely, the notion of elementary function: a C^n function that belongs to the set Λ of elementary functions. The set Λ is obtained from rational functions on C^k $(k \in \mathbb{N}_0)$, using a finite number of the following operations: (i) algebraic operations (if $f_1, f_2 \in \Lambda$ then $f_1 \star f_2 \in \Lambda$, where $\star$ is either the addition, subtraction, multiplication, or division); (ii) solutions of algebraic equations with coefficients in Λ; (iii) differentiation; and (iv) exponential and logarithm operations. The set of elementary functions together with the operation of integration (if $f \in \Lambda$ then $\int f(x)\,dx \in \Lambda$) is called the set of Liouvillian functions. Liouville showed that the solution of the equation $\dot{x}(t) = t^\alpha - x^2$ is only Liouvillian for $\alpha = -2$ and $\alpha = 4k/(1-2k)$ $(k \in \mathbb{N})$.

There is a concrete method that permits not only to verify that solutions of the Hamiltonian system are Liouvillian functions, but also to reduce the system in order to obtain the extremals. We describe it briefly. Let $M = \mathbb{R}^n\{x\} \times (\mathbb{R}^n)^*\{\psi\} \times \mathbb{R}\{t\}$ and $f : M \to \mathbb{R}$ be a first integral of the Hamiltonian system with Hamiltonian $\mathcal{H}$. If $df(q) \neq 0$, then in some neighborhood of the point $q \in M$ there exist symplectic coordinates $(\tilde{x}, \tilde{\psi}, \tilde{\theta}, \tilde{t})$ such that $f(\tilde{x}, \tilde{\psi}, \tilde{t}) = \tilde{\psi}_1$. In these coordinates $\mathcal{H}$ does not depend on $\tilde{x}_1$, therefore if we fix a value $f = \tilde{\psi}_1 = c$, then the Hamiltonian system will only have $n-1$ degrees of freedom. In order to have an effective reduction of dimension by a set of first integrals, this method requires the first integrals to be independent and in involution

$$\{f_i, f_j\} = 0 \ \ \forall\, i, j \in \{1, \ldots, N\}.$$

E. Cartan (1971) extended Liouville's method for the case where the algebra L of first integrals is not commutative, and possible infinite-dimensional. Cartan assumed that the first integrals satisfy the relation

$$\{f_i, f_j\} = \zeta_{ij}(f_1, \ldots, f_N) \quad \forall\, i, j \in \{1, \ldots, N\} \tag{21}$$

for some (nonlinear) functions $\zeta_{ij} : \mathbb{R}^N \to \mathbb{R}$. His method is based on the following theorem.

THEOREM 2 (S.Lie - E.Cartan, Cartan, 1971) *Let $F = (f_1, \ldots, f_N)$. Suppose that the point $c \in \mathbb{R}^N$ is not a critical value of the mapping F and that in its neighborhood the rank of the matrix (ζ_{ij}) is constant. Then, in a small neighborhood $U \subset \mathbb{R}^N$ of c, one can find N independent functions $\phi_j : U \to \mathbb{R}$ such that the functions $\Phi_j = \phi_j \circ F : V \to \mathbb{R}$, where $V = F^{-1}(U)$, satisfy the relations*

$$\{\Phi_1, \Phi_2\} = \cdots = \{\Phi_{2\eta-1}, \Phi_{2\eta}\} = 1,$$

whereas the remaining brackets vanish, and the rank of the matrix (ζ_{ij}) is 2η.

Using Theorem 2 we can lower the order of the system in the following way: the level set $M_c = \{(x, \psi) \in M : \Phi_j(x, \psi) = c_j, 1 \le j \le N\}$, where $c = (c_1, \ldots, c_N)$ satisfies the theorem, is a smooth $(2n-N)$-dimensional submanifold of M. The theorem also implies that there is an action of the commutative group $\mathbb{R}^l$ $(l = N - 2\eta)$ on M_c, generated by the phase flows of the Hamilton's equations with Hamiltonians Φ_j for $j > 2\eta$. Now, thanks to the functional independence of the integrals Φ_j, this action has no fixed points. Hence, if its orbits are compact, then the quotient space $M_{red} = M_c/\mathbb{R}^l$ is a smooth manifold with dimension $2(n - N + \eta)$ endowed with a natural symplectic structure. Let H' denote the restriction of the Hamiltonian H to the level set M_c of the first integrals. Since H' is constant on the orbits of the group $\mathbb{R}^l$, there is a smooth function $H_{red} : M_{red} \to \mathbb{R}$ such that the diagram

$$M_c \xrightarrow{pr} M_{red} \xrightarrow{H_{red}} \mathbb{R} \xleftarrow{H'} M_c$$

commutes. To end, let us observe that one can obtain Liouville's method from Cartan's method by choosing $\zeta_{ij} \equiv 0$ and $\eta = 0$. Locally, they give the same result, however the factorization by Cartan method can be accomplished globally only under more restrictive assumptions.

Kozlov and Kolesnikov (1979) proved an intermediate result, considering that Poisson brackets of first integrals are a linear combination of first integrals. In fact, this result is more suitable for our purposes.

THEOREM 3 (Kozlov-Kolesnikov) *Suppose that the Hamiltonian $\mathcal{H} : \mathbb{R}^n \times (\mathbb{R}^n)^* \times \mathbb{R} \to \mathbb{R}$ has n first integrals $F_1, \ldots, F_n : \mathbb{R}^n \times (\mathbb{R}^n)^* \times \mathbb{R} \to \mathbb{R}$ that* *satisfy relation*

$$\exists\,\xi^{ij} \in \mathbb{R}^n \;\;:\;\; \{F_i, F_j\} = \sum_{s=1}^{n} \xi_s^{ij} F_s \quad \forall i, j \in \{1, \ldots, n\}, \tag{22}$$

where $\xi^{ij} = (\xi_1^{ij}, \ldots, \xi_n^{ij})^T$*. Additionally, assume that*

1. *on the set* $M_f = \{(x, \psi, t) \in M : F_i(x, \psi, t) = r_i, 1 \le i \le n\}$ *the functions* $F_1, \ldots, F_n$ *are independent;*
2. $(r_1, \ldots, r_n)\xi^{ij} = 0$ *for all* $i, j = \{1, \ldots, n\}$*;*
3. *the Lie algebra* L *of linear combination* $\sum_i c_i F_i$*,* $c_i \in \mathbb{R}$*, is solvable.*

Then the solutions of the Hamiltonian system lie on M_f *and can be found by quadratures.*

3. Main results – effective first integrals

In order to use Noether's theorem (Theorem 1) to obtain effective first integrals for the Hamiltonian equations, we need to compute the solutions of the first order partial differential equation (19) considering the optimal control $\bar{u}$, i.e.

$$\begin{aligned} &\frac{d\mathcal{H}(x,\psi,t)}{dt}\mathcal{T}(x,\psi,t) - \dot{\psi}(t)\cdot\mathcal{X}(x,\psi,t) \\ &\qquad\qquad - \psi(t)\cdot\frac{d\mathcal{X}(x,\psi,t)}{dt} + \mathcal{H}(x,\psi,t)\frac{d\mathcal{T}(x,\psi,t)}{dt} = 0, \end{aligned} \tag{23}$$

where $\mathcal{X}(x,\psi,t) \equiv X(x,\bar{u},\psi,t)$ and $\mathcal{T}(x,\psi,t) \equiv T(x,\bar{u},\psi,t)$. A particular solution (if it exists) can be found, e.g. by the well known method of (additive) separation of variables for PDE, assuming

$$\begin{aligned} \mathcal{T}(x,\psi,t) &= \mathcal{T}^0(t) + \mathcal{T}^{x_1}(x_1) + \cdots + \mathcal{T}^{x_n}(x_n) + \mathcal{T}^{\psi_1}(\psi_1) + \cdots + \mathcal{T}^{\psi_n}(\psi_n), \\ \mathcal{X}(x,\psi,t) &= \mathcal{X}^0(t) + \mathcal{X}^{x_1}(x_1) + \cdots + \mathcal{X}^{x_n}(x_n) + \mathcal{X}^{\psi_1}(\psi_1) + \cdots + \mathcal{X}^{\psi_n}(\psi_n). \end{aligned}$$

In particular, we will consider that each independent component of $\mathcal{T}$ and $\mathcal{X}$ have a polynomial structure with degree $\le p_d$, hence

$$\mathcal{T}(x,\psi,t) = \sum_{\mu=0}^{p_d}\left(C_0^{\mathcal{T}}(\mu)t^\mu + \sum_\nu C_\nu^{\mathcal{T}}(\mu)(q_\nu)^\mu\right), \tag{24}$$

$$\mathcal{X}_i(x,\psi,t) = \sum_{\mu=0}^{p_d}\left(C_0^{\mathcal{X}}(i,\mu)t^\mu + \sum_\nu C_\nu^{\mathcal{X}}(i,\mu)(q_\nu)^\mu\right), \tag{25}$$

for some constants $C_0^{\mathcal{T}}(\mu), C_\nu^{\mathcal{T}}(\mu), C_0^{\mathcal{X}}(i,\mu), C_\nu^{\mathcal{X}}(i,\mu) \in \mathbb{R}$ and $\mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_n)$.

Now, since equation (23) has to be valid for every extremal $(x(t), \psi(t), t) \in M$, this particular choice for the structure of solutions will transform the PDE problem for $\mathcal{T}, \mathcal{X}$ into an algebraic system of equations for the constants $C_0^{\mathcal{T}}(\mu)$, $C_\nu^{\mathcal{T}}(\mu)$, $C_0^{\mathcal{X}}(i,\mu)$, $C_\nu^{\mathcal{X}}(i,\mu) \in \mathbb{R}$. The algebraic system is under-determined, because we have one equation for two unknowns, $\mathcal{T}$ and $\mathcal{X}$. Therefore, if such system has a nontrivial solution, we have a family of first integrals. Let us define the mapping $\digamma : \mathbb{R}^m \to C(M, \mathbb{R})$ by

$$\digamma(\lambda)(x, \psi, t) = \psi \cdot \mathcal{X} - \mathcal{H}\mathcal{T} \tag{26}$$

for $\lambda \in \mathbb{R}^m$. $\digamma$ is a linear mapping, whereas $\mathcal{T}$ and $\mathcal{X}$ are linear with respect to the constants, the PDE (23) is a linear first order equation (superposition of solutions is a solution), and equation (20) is a linear functional combination of $\mathcal{T}$ and $\mathcal{X}$. We resume our statements in the following lemma.

LEMMA 1 *Assume that equation (23) has nontrivial solutions $\mathcal{T}(x, \psi, t)$ and $\mathcal{X}(x, \psi, t)$ of the form (24) and (25), respectively. Then the mapping $\digamma$ is a linear m-parametric family of first integrals with respect to $\lambda \in \mathbb{R}^m$, i.e. it depends on $m \in \mathbb{N}_0$ arbitrary constants $\lambda_1, \ldots, \lambda_m$ and*

$$\forall \lambda \in \mathbb{R}^m : \frac{\partial \digamma(\lambda)}{\partial t} + \{\mathcal{H}, \digamma(\lambda)\} = 0. \tag{27}$$

Considering the previous lemma, it makes sense to assume that $\digamma$ has the following structure

$$\digamma(\lambda)(x, \psi, t) = \sum_{k=1}^{m} \digamma_k(x, \psi, t)\lambda_k, \tag{28}$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$. We call $\digamma_k$ the components of the family of first integrals $\digamma(\lambda)$.

Let n be the dimension of the phase space ($x \in \mathbb{R}^n$), m the number of parameters on the family $\digamma(\lambda)$, and $m \geq n$. The existence of n effective first integrals (in the sense of Kozlov-Kolesnikov) will be related to the existence of a nontrivial solution of a system of algebraic equations involving the components of $\digamma$ and their canonical Poisson brackets. Consider the following system of algebraic equations

$$(\lambda^i)^T A(x, \psi, t)\, \lambda^j = (\xi^{ij})^T \begin{bmatrix} (\lambda^1)^T \\ \vdots \\ (\lambda^n)^T \end{bmatrix} b(x, \psi, t), \quad i < j \in \{1, \ldots, n\}, \tag{29}$$

where $\lambda^1, \ldots, \lambda^n \in \mathbb{R}^m$ and $\xi^{ij} \in \mathbb{R}^n$,

$$A(x, \psi, t) = \left(\{\digamma_p, \digamma_q\}\right)_{p,q=1}^{m} \quad \text{and} \quad b(x, \psi, t) = \left[\digamma_1, \ldots, \digamma_m\right]^T.$$

By a solution of the system of equations (29) we mean a set of constant vectors $(\lambda^1, \ldots, \lambda^n)$, $(\xi^{12}, \ldots, \xi^{(n-1)n})$ that satisfies the system. A nontrivial solution is a solution for which all λ^k ($k = 1, \ldots, n$) are different. From a nontrivial solution, we have the set of first integrals $\{\digamma(\lambda^1), \ldots, \digamma(\lambda^n)\}$. Comparing with relation (21), ζ_{ij} are linear functions. Therefore, the space of linear combinations $L = span\{\digamma(\lambda^1), \ldots, \digamma(\lambda^n)\}$ forms a noncommutative but finite-dimensional Lie algebra, where the first integrals define a basis and the coordinates of ξ^{ij} are the structure constants.

PROPOSITION 1 *If there exists a nontrivial solution* $(\lambda^1, \ldots, \lambda^n)$, $(\xi^{12}, \ldots, \xi^{(n-1)n})$ *to the system (29), then the set of first integrals* $\{\digamma(\lambda^1), \ldots, \digamma(\lambda^n)\}$ *satisfy relation (22).*

Proof. The set $\{\digamma(\lambda^1), \ldots, \digamma(\lambda^n)\}$ is a set of first integrals of the Hamiltonian $\mathcal{H}$, since $\digamma(c_1, \ldots, c_m)(x, \psi, t)$ is a first integral for any $(c_1, \ldots, c_m) \in \mathbb{R}^m$; by Theorem 1 and Lemma 1. Relation (22) is satisfied, using the definition of $\digamma$ (28), properties of the bracket, and relation (29). For $i < j \in \{1, \ldots, n\}$, we have:

$$\{\digamma(\lambda^i), \digamma(\lambda^j)\} = \left\{\sum_{p=1}^{m} \digamma_p \lambda_p^i, \sum_{q=1}^{m} \digamma_q \lambda_q^j\right\} = \sum_{p=1}^{m}\sum_{q=1}^{m} \lambda_p^i \lambda_q^j \{\digamma_p, \digamma_q\} = (\lambda^i)^T A(x, \psi, t) \lambda^j$$
$$= (\xi^{ij})^T \begin{bmatrix} (\lambda^1)^T \\ \vdots \\ (\lambda^m)^T \end{bmatrix} b(x, \psi, t) = (\xi^{ij})^T \begin{bmatrix} \digamma(\lambda^1) \\ \vdots \\ \digamma(\lambda^m) \end{bmatrix} = \sum_{s=1}^{n} \xi_s^{ij} \digamma(\lambda^s).$$

■

PROPOSITION 2 *Assume that the set of first integrals* $\{\digamma(\lambda^1), \ldots, \digamma(\lambda^n)\}$ *satisfy relation (22). Let* $\mathcal{S} = \{(a, b, p, q, i, j) \in \{1, \ldots, n\}^4 \times \{1 \ldots, m\}^2 : a < b, p < q, a < p, i < j\}$. *If*

$$\xi_i^{ab} \xi_j^{pq} = \xi_i^{pq} \xi_j^{ab} \quad \forall\, (a, b, p, q, i, j) \in \mathcal{S}, \tag{30}$$

then the Lie algebra L *of linear combination* $\sum_s c_s \digamma(\lambda^s)$, $c_s \in \mathbb{R}$, *is solvable.*

Proof. Let $L^0 \equiv L$. We recall that a Lie algebra L is solvable if the descent series is nilpotent, i.e.

$$\exists \bar{k} \in \mathbb{N} : L^{\bar{k}} \equiv 0 \quad \text{where} \quad L^k = [L^{k-1}, L^{k-1}]. \tag{31}$$

For our purpose it will be enough to consider $\bar{k} = 2$. Let us observe that the Liouville method (first integrals in involution) is the case $\bar{k} = 1$. Hence, for $k = 1$ and using relation (22),

$$\begin{aligned} \left\{\sum_a \alpha_a \digamma(\lambda^a), \sum_b \beta_b \digamma(\lambda^b)\right\} &= \sum_{a<b} (\alpha_a \beta_b - \beta_a \alpha_b) \{\digamma(\lambda^a), \digamma(\lambda^b)\} \\ &= \sum_i \sum_{a<b} (\alpha_a \beta_b - \beta_a \alpha_b) \xi_i^{ab} \digamma(\lambda^i), \end{aligned}$$

and, for $k = 2$,

$$\left\{\sum_i \sum_{a<b} (\alpha_a\beta_b - \beta_a\alpha_b)\xi_i^{ab} F(\lambda^i), \sum_j \sum_{p<q} (\alpha_p\beta_q - \beta_p\alpha_q)\xi_j^{pq} F(\lambda^j)\right\}$$
$$= \sum_s \sum_{i<j} \left(\left(\sum_{a<b} (\alpha_a\beta_b - \beta_a\alpha_b)\xi_i^{ab} \right) \left(\sum_{p<q} (\alpha_p\beta_q - \beta_p\alpha_q)\xi_j^{pq} \right) \right.$$
$$\left. - \left(\sum_{p<q} (\alpha_p\beta_q - \beta_p\alpha_q)\xi_i^{pq} \right) \left(\sum_{a<b} (\alpha_a\beta_b - \beta_a\alpha_b)\xi_j^{ab} \right) \right) \xi_s^{ij} F(\lambda^s)$$
$$= \sum_s \sum_{i<j} \sum_{a<b} \sum_{p<q} (\alpha_a\beta_b - \beta_a\alpha_b)(\alpha_p\beta_q - \beta_p\alpha_q) \left(\xi_i^{ab}\xi_j^{pq} - \xi_i^{pq}\xi_j^{ab} \right) \xi_s^{ij} F(\lambda^s).$$

■

We are now in a condition to present the main result of the paper: a practical method to find effective first integrals for optimal control problems. Theorem 4 is a direct consequence of Propositions 1 and 2, and the Kozlov-Kolesnikov theorem.

THEOREM 4 *Assume that the optimal control problem (1)-(2) has a solution, and there exists an m-parametric family of first integrals F, given by Lemma 1, with the form*

$$F(\lambda)(x,\psi,t) = \sum_{k=1}^{m} F_k(x,\psi,t)\lambda_k,$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$. Let $\mathcal{S} = \{(a,b,p,q,i,j) \in \{1,\ldots,n\}^4 \times \{1\ldots,m\}^2 : a<b, p<q, a<p, i<j\}$, $A(x,\psi,t) = (\{F_p, F_q\})_{p,q=1}^m$, $\Lambda = \left[(\lambda^1)^T, \ldots, (\lambda^n)^T\right]^T \in M_{n\times m}$, and $b(x,\psi,t) = [F_1, \ldots, F_m]^T$.

If *there exists a solution $(\lambda^1, \ldots, \lambda^n, \xi^{12}, \ldots, \xi^{(n-1)n}, r_1, \ldots, r_n)$ (with $\lambda^i \in \mathbb{R}^m$, $\xi^{ij} \in \mathbb{R}^n$, $r_i \in \mathbb{R}$ and $i<j \in \{1,\ldots,n\}$) of the algebraic system of equations*

$$\begin{cases} (\lambda^i)^T A(x,\psi,t)\, \lambda^j - (\xi^{ij})^T \Lambda\, b(x,\psi,t) = 0, & \text{for } \forall\, i<j \in \{1,\ldots,n\}, \\ \xi_i^{ab}\xi_j^{pq} - \xi_i^{pq}\xi_j^{ab} = 0, & \text{for } \forall\, (a,b,p,q,i,j) \in \mathcal{S}, \\ \sum_{s=1}^n r_s \xi_s^{ij} = 0, & \text{for } \forall\, i<j \in \{1,\ldots,n\}, \end{cases}$$

and

$$\operatorname{rank}\left[\nabla_{(x,\psi)} F(\lambda^1), \ldots, \nabla_{(x,\psi)} F(\lambda^n)\right] = n \tag{32}$$

on the manifold $M_F = \{\alpha \in M : F(\lambda^i)(\alpha) = r_i\}$, ***then*** *the optimal control problem (1)-(2) is solvable on M_F.*

Although simple, the arguments behind Lemma 1 and Theorem 4 give a powerful method that can be applied with success to several problems of optimal control.

4. Illustrative examples

We now present three interesting applications, many others can be chosen from the literature. Families of first integrals (Lemma 1) were obtained using the *Maple* package described in Gouveia, Torres and Rocha (2006).

EXAMPLE 1 Let us show the integrability by quadratures (solvability) of the following optimal control problem

$$\frac{1}{2}\int_a^b u_1(t)^2 + u_2(t)^2\,dt \to \min\,, \qquad \begin{cases} \dot{x}_1(t) = u_1(t)\cos(x_3(t)) \\ \dot{x}_2(t) = u_1(t)\sin(x_3(t)) \\ \dot{x}_3(t) = u_2(t) \end{cases}$$

which is known as the Dubin's model for the kinematics of a car (Martin, Murray, Rouchon, 2002, pp. 750–751). The true Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left(\left[\cos\left(x_3(t)\right)\psi_1(t) + \sin\left(x_3(t)\right)\psi_2(t)\right]^2 + \left[\psi_3(t)\right]^2\right).$$

It is clear that the problem has three trivial first integrals (f.i.) $\{\mathcal{H}, \psi_1, \psi_2\}$ in involution. Notice that an autonomous Hamiltonian is always a f.i. by Remark 3, and the other f.i. follow from Remark 2. However, by computing $\digamma(\lambda)$ due to Lemma 1 and by applying Theorem 4, we obtain the trivial f.i. and an extra effective f.i. $F = -\psi_1 x_2 + \psi_2 x_1 + \psi_3$. Therefore, the set $\{\psi_1, \psi_2, F\}$ can also be used to prove the solvability of the problem.

EXAMPLE 2 An interesting variation of the previous problem is the model of a car with one-trailer (Fuka and Susta, 1992), parameterized by constants $(a, b, c) \in \mathbb{R}$,

$$\int_a^b u_1^2 + u_2^2\,dt \to \min\,, \qquad \begin{cases} \dot{x}_1 = u_1\cos(x_3) \\ \dot{x}_2 = u_1\sin(x_3) \\ \dot{x}_3 = \frac{1}{c}u_1\tan(u_2) \\ \dot{x}_4 = \frac{1}{b}u_1\left(\frac{a}{c}\tan(u_2)\cos(x_3 - x_4) - \sin(x_3 - x_4)\right) \end{cases}.$$

The necessary and sufficient condition of invariance is satisfied with the following generators $\{\mathcal{T} = C_2, \mathcal{X}_1 = -C_1 x_2 + C_4, \mathcal{X}_2 = C_1 x_1 + C_3, \mathcal{X}_3 = C_1, \mathcal{X}_4 = C_1\}$. It follows that, for $C = (C_1, C_2, C_3, C_4) \in \mathbb{R}^4$,

$$\digamma(C)(x, \psi, t) = (C_4 - C_1 x_2)\,\psi_1 + (C_3 + C_1 x_1)\,\psi_2 + C_1\psi_3(t) + C_1\psi_4 - C_2\mathcal{H}.$$

Therefore, a possible solution of the algebraic system of Theorem 4 is $\lambda^1 = (1, 0, 0, 0)$, $\lambda^2 = (0, 0, 1, 0)$, $\lambda^3 = (0, 0, 0, 1)$, $\lambda^4 = (0, 1, 0, 0)$, $\xi^{12} = (0, 0, -1, 0)$, $\xi^{13} = (0, 1, 0, 0)$, $\xi^{23} = (0, 0, 0, 0)$, $\xi^{i4} = (0, 0, 0, 0)$ for $i = 1, 2, 3$, and $r = (c_1, 0, 0, c_4)$ for any $c_1, c_4 \in \mathbb{R}$. The set of effective first integrals is

$$\{\digamma(\lambda^1) = -\psi_1 x_2 + \psi_2 x_1 + \psi_3 + \psi_4, \digamma(\lambda^2) = \psi_2, \digamma(\lambda^3) = \psi_1, \digamma(\lambda^4) = \mathcal{H}\}.$$

Comparing with the last example, which is linear in control, this problem is not only nonlinear in control as it has a very complicated true Hamiltonian. However, the set of first integrals is just an extension of the previous one, where the only change is $F(\lambda^1) = F + \psi_4$.

Example 3 We now consider the so-called flat Martinet problem (Bonnard, Chyba, Trélat, 1998):

$$\int_a^b u_1^2 + u_2^2 \, dt \to \min\,, \qquad \begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = \dfrac{u_2}{1+\alpha x_1}\,, \\ \dot{x}_3 = x_2^2 u_1 \end{cases} \qquad \alpha \in \mathbb{R}\,.$$

For $\alpha = 0$ the problem is clearly integrable by using the trivial set of first integrals $\{\mathcal{H}, \psi_1, \psi_3\}$ in involution (remarks 2 and 3). In the $\alpha \neq 0$ case, one has the invariance-generators $\big\{X_2 = 0, \Psi_2 = 0, T = 2\,\lambda^1 t + \lambda^3, \Psi_1 = -\lambda^1\psi_1,\ U_1 = -\lambda^1 u_1, \Psi_3 = -\lambda^1\psi_3, U_2 = -\lambda^1 u_2, X_3 = \lambda^1 x_3 + \lambda^2, X_1 = \lambda^1(\alpha^{-1} + x_1)\big\}$, which, after solving the algebraic system of Theorem 4 gives the following set of effective first integrals

$$\left\{F_1 = \mathcal{H}, F_2 = \left(\frac{1}{\alpha} + x_1\right)\psi_1 + x_3\psi_3 - 2t\mathcal{H}, F_3 = \psi_3\right\}.$$

This example shows that the method not only generates mappings $F(x,\psi)$ verifying $\{F, \mathcal{H}\} = 0$, but also satisfying the more relaxed condition $\{F, \mathcal{H}\} = c\mathcal{H}$ for some $c \in \mathbb{R}$ (see Remark 1). We observe that a nonautonomous first integral (F_2) is required to prove integrability of the problem, in spite of the fact that the problem is autonomous.

5. The sub-Riemannian nilpotent case $(2, 3, 5)$

The sub-Riemannian (SR) problem is to characterize geodesics in some n dimensional SR-manifold M, i.e. to find absolutely continuous curves $t \mapsto q(t) \in M$, $0 \le t \le T$, minimizing the length

$$l(q) = \int_0^T \ < \dot{q}(t), \dot{q}(t) >^{\frac{1}{2}} \ dt$$

such that $\dot{q}(t) \in \Delta(q(t))\backslash\{0\}$ a.e. t, where Δ is a constant rank $m \le n$ distribution with a (degenerate) Riemannian metric g on Δ, and $< \cdot, \cdot >$ is the scalar product induced by g. The SR-problem can be locally formulated as an optimal control problem (Bonnard and Chyba, 2003): let (U, q) be a chart on which Δ is generated by an orthogonal basis $\{X_1, \ldots, X_m\}$, then the SR-problem (U, Δ, g) is equivalent to

$$\frac{1}{2}\int_0^T \left(\sum_{i=1}^m u_i^2(t)\right) dt \longrightarrow \min\,, \quad \dot{q}(t) = \sum_{i=1}^m u_i(t)\, X_i(q(t)).$$

The nilpotent case $(2,3,5)$ is the instance where $m = 2$, $n = 5$, and the Lie algebra generated by X_1 and X_2 is a complete nilpotent Lie algebra of nildegree 3. In other words, if $[\cdot,\cdot]$ denotes the Lie bracket of vector fields, $X_3 = [X_1, X_2]$, $X_4 = [X_1, X_3]$ and $X_5 = [X_2, X_3]$, the SR-problem is nilpotent of type $(2,3,5)$ if $X_i(0) = c_i \frac{\partial}{\partial x_i}$ for some $c_i \neq 0 \in \mathbb{R}$ and $i \in \{1,\ldots,5\}$; which gives $\dim(\{X_1, X_2\}) = 2$, $\dim(\{X_1, X_2, X_3\}) = 3$ and $\dim(\{X_1, X_2, X_3, X_4, X_5\}) = 5$. The true Hamiltonian is then given by $\mathcal{H}(q,\psi) = \frac{1}{2}\sum_{i=1}^{m} h_i^2(t)$, where $h_i(t) =< \psi(t), X_i(t) >$ for $i \in \{1,\ldots,m\}$, and the Poincaré system is a system of equations on T^*U, given by completing the set $\{X_1,\ldots,X_m\}$ to form a smooth basis of TU. Such vector fields are obtained by extending $h_i(t) =< \psi(t), X_i(t) >$ to $i \in \{1,\ldots,n\}$ and computing $\dot{h}_i = \sum_{i=1}^{m}\{h_i, h_j\}\, h_j$, where $\{\cdot,\cdot\}$ is the n bracket.

Y. Sachkov (2004) proved that the optimal control associated with the sub-Riemannian nilpotent case $(2,3,5)$ is solvable, by obtaining three first integrals and using them to reduce the Hamiltonian system to the differential equation $\ddot{\theta}(t) = c_1\cos(\theta(t)) + c_2\sin(\theta(t))$ for $c_1, c_2, \theta(t) \in \mathbb{R}$. Hence, the solvability is obtained from previous works showing that such differential equation is integrable by quadratures using Jacobian Elliptic Functions. The autonomous Hamiltonian $\mathcal{H}$ is one of the first integrals used, and the other two, h_4 and h_5, are obtained as a direct consequence of the fact that the Lie algebra is nilpotent, since $\{h_i, h_j\} =< \psi, [X_i, X_j](q) >$ imply that $\dot{h}_4(t) = 0$ and $\dot{h}_5(t) = 0$ along solutions.

With the method presented in this work, we can obtain enough first integrals to directly prove the solvability of the problem. In fact, we will consider a more general problem, parameterized by constants $\alpha, \beta \in \{0,1\}$. Consider the SR nilpotent case $(2,3,3+\alpha+\beta)$ with local generators for Δ given by

$$X_1 = \frac{\partial}{\partial x_1} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + \frac{\alpha}{2}{x_1}^2\frac{\partial}{\partial x_4} + \beta x_1 x_2 \frac{\partial}{\partial x_5}.$$

The distribution Δ is known as the nilpotent Heisenberg distribution ($\alpha = 0$ and $\beta = 0$), the nilpotent Engel distribution ($\alpha = 1$ and $\beta = 0$), or the Cartan distribution ($\alpha = 1$ and $\beta = 1$). The Pontryagin maximum principle gives the true Hamiltonian

$$\mathcal{H}_{\alpha,\beta} = \frac{1}{2}\left[\psi_1^2 + (\psi_2 + x_1\psi_3 + \frac{\alpha}{2}x_1^2\psi_4 + \beta x_1 x_2 \psi_5)^2\right].$$

This class of problems admit the following set of generators (see Gouveia, Torres, Rocha, 2006):

$$\Big\{\Psi_5 = -\frac{3}{4}\lambda^1\psi_5, \Psi_1 = -\frac{1}{2}\lambda^1\psi_1, \Psi_2 = -\frac{1}{2}\psi_2\lambda^1, \Psi_3 = -\lambda^1\psi_3 - \lambda^2\psi_5,$$

$$T = \lambda^1 t + \lambda^4, \Psi_4 = -\frac{3}{2}\lambda^1\psi_4, X_1 = \frac{1}{2}\lambda^1 x_1, U_1 = -\frac{1}{2}\lambda^1 u_1, X_2 = \frac{1}{2}\lambda^1 x_2 + \frac{1}{\beta}\lambda^2,$$

$$U_2 = \frac{1}{2}\lambda^1 u_2, X_5 = \lambda^2 x_3 + \frac{3}{2}\lambda^1 x_5 + \lambda^3, X_4 = \frac{3}{2}\lambda^1 x_4 + \lambda^5, X_3 = \lambda^1 x_3 + \lambda^6 \Big\}.$$

Computing $F(\lambda)$ on Lemma 1 and finding solutions for Theorem 4, we have the effective first integrals (not in involution)

$$\{\mathcal{H}_{\alpha,\beta},\ \psi_2 + \beta\psi_5 x_3,\ \psi_3,\ \alpha\psi_4,\ \beta\psi_5\}.$$

LEMMA 2 *The sub-Riemannian nilpotent cases* $(2,3)$ *(*$\alpha = 0$ *and* $\beta = 0$*),* $(2,3,4)$ *(*$\alpha = 1$ *and* $\beta = 0$*), and* $(2,3,5)$ *(*$\alpha = 1$ *and* $\beta = 1$*), are integrable by quadratures.*

It is not difficult to find a first integral F whereas the set $\{\mathcal{H},\ F,\ \psi_3,\ \psi_4,\ \psi_5\}$ is involutive. Such first integral should satisfy the relations $\{\mathcal{H}, F\} = 0$ and $\{F, \psi_i\} = 0$ for $i \in \{3,4,5\}$. If we consider *a priori* that F does not depend on x_3, x_4 or x_5, then last condition is trivially verified. Hence, it just remains to solve $\{\mathcal{H}, F(x_1, x_2, \psi_1, \ldots, \psi_5)\} = 0$, which by a direct calculation gives the first integral

$$F = -\psi_1\psi_5 + \psi_2\psi_4 - (\psi_3 + \frac{1}{2}\psi_5 x_2)x_2\psi_5.$$

Therefore, the solutions of the sub-Riemannian nilpotent case $(2,3,5)$ are Liouvillian, using the first integrals $\{\mathcal{H},\ F,\ \psi_3,\ \psi_4,\ \psi_5\}$ in involution.

Although the present method can be applied to other hard problems, such as the sub-Riemannian nilpotent cases $(2,3,5,8)$ or $(2,3,5,8,14)$, for which the solvability is still unknown, because of its complexity (8 and 14 effective first integrals are needed, respectively), their study is left for a forthcoming publication. Here we just notice that the Maple package described in Gouveia, Torres, Rocha (2006) is unable to find a sufficiently rich family of first integrals for the case $(2,3,5,8,14)$. Therefore, they need to be found by other theoretical procedure.

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