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Two-dimensional Newton's problem of minimal resistance[1]

by

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Abstract: Newton's problem of minimal resistance is one of the first problems of optimal control: it was proposed, and its solution given, by Isaac Newton in his masterful *Principia Mathematica*, in 1686. The problem consists of determining, in dimension three, the shape of an axis-symmetric body, with assigned radius and height, which offers minimum resistance when it is moving in a resistant medium. The problem has a very rich history and is well documented in the literature. Of course, at a first glance, one suspects that the two dimensional case should be well known. Nevertheless, we have looked into numerous references and asked at least as many experts on the problem, and we have not been able to identify a single source. Solution was always plausible to everyone who thought about the problem, and writing it down was always thought not to be worthwhile. Here we show that this is not the case: the two-dimensional problem is richer than the classical one, being, in some sense, more interesting. Novelties include: (i) while in the classical three-dimensional problem only the restricted case makes sense (without restriction on the monotonicity of admissible functions the problem does not admit a local minimum), we prove that in dimension two the unrestricted problem is also well-posed when the ratio of height versus radius of base is greater than a given quantity; (ii) while in three dimensions the (restricted) problem has a unique solution, we show that in the restricted two-dimensional problem the minimizer is not always unique – when the height of the body is less or equal than its base radius, there exists infinitely many minimizing functions.

Keywords: Newton's problem of minimal resistance, dimension two, calculus of variations, optimal control.

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1. Introduction

Newton's aerodynamical problem, in dimension three, is a classic problem (see, e.g., Azevedo do Amaral, 1913; Fraeijs de Veubeke, 1966; Kneser et al., 1913). It consists in joining two given points of the plane by a curve's arc that, while turning around a given axis, generates the body of revolution offering the least resistance when moving in a fluid in the direction of the axis. Newton has considered several hypotheses: that the body moves with constant velocity, and without rotation, on a very rare and homogeneous medium of particles which are all equal; that the axis-symmetric body is inscribed in a cylinder of height H and radius r; that the particles of the medium are infinitesimally small and immovable (there exists no temperature motion of particles); that collisions of the particles with the body are absolutely elastic. Newton has indicated in the *Mathematical principles of natural philosophy* the correct solution to his problem. He has not explained, however: how such solution can be obtained; how the problem is formulated in the language of mathematics. This has been the work of many mathematicians since Newton's time (see, e.g., Bryson, Ho, 1975; Tikhomirov, 1990; Torres, Plakhov, 2006). Extensions of Newton's problem is a topic of current intensive research, with many questions remaining open challenging problems. Recent results, obtained by relaxing Newton's hypotheses, include: non-symmetric bodies (Buttazzo, Kawohl, 1993); one-collision non-convex bodies (Comte, Lachand-Robert, 2001); collisions with friction (Horstmann, Kawohl, Villaggio, 2002); multiple collisions allowed (Plakhov, 2003); temperature noise of particles (Plakhov, Torres, 2004, 2005). Here we are interested in the classical problem, under the classical hypotheses considered by Newton. Our main objective is to study the apparently simpler Newton's problem of minimal resistance for a two-dimensional body moving with constant velocity in a homogeneous rarefied medium of particles. The first work on a two-dimensional Newton-type problem seems to be Plakhov, Torres (2004), where the authors study the problem in a chaotically moving media of particles (in the classical problem particles are immovable). The results in Plakhov, Torres (2004) were later generalized to dimension three, Plakhov, Torres (2005). This paper is motivated by the results in Plakhov, Torres (2005): when one considers temperature motion of particles, the three-dimensional problem admits only two types of solutions; while the two-dimensional case is richer, showing solutions of five distinct types. Here we prove that in the classical framework, with an immovable media of particles, also the two-dimensional case is richer: in certain cases of input of data (height H and radius r of the body) the problem is well-posed (admitting local minima) without imposing the restriction $\dot{y}(x) \geq 0$ on the admissible curves $y(\cdot)$. This is different from the three-dimensional classical problem or the problem in higher-dimensions, where the restriction $\dot{y}(x) \geq 0$ is always necessary for the problem to make sense: without it there exist no strong and no weak local minima for Newton's problem of minimal resistance (see, e.g., Fraeijs de Veubeke, 1966; Silva, 2005). We show that for $H > \frac{\sqrt{3}}{3}r$ the function $\hat{y}(x) = \frac{H}{r}x$ is a local minimum for the unrestricted Newton's problem of minimal resistance in dimension two. In the restricted case, while in dimension three (or higher) the problem has always a unique solution, we prove that infinitely many different minimizers appear in dimension two for $r \geq H$. These simple facts seem to be new in the literature, and never noticed before.

2. Restricted and unrestricted problems

In the classical three dimensional Newton's problem of minimal aerodynamical resistance, the resistance force is given by $R\left[\dot{y}(\cdot)\right] = \int_0^r \frac{x}{1+\dot{y}(x)^2}\,dx$. Minimization of this functional is a typical problem of the calculus of variations. Most part of the old literature wrongly assume the classical Newton's problem to be "one of the first applications of the calculus of variations". The truth, as Legendre first noticed in 1788 (see Belloni, Kawohl, 1997), is that some restrictions on the derivatives of admissible trajectories must be imposed: $\dot{y}(x) \geq 0$, $x \in [0, r]$. The restriction is crucial, because without it there exists no solution, and the problem suffers from Perron's paradox (Young, 1969, §10): since the *a priori* assumption that a solution exists is not fulfilled, it does not make sense to try to find it by applying the necessary optimality conditions. It turns out that, with the necessary restriction, the problem is better considered as an optimal control one (see Tikhomirov, 2002, p. 67, and Torres, Plakhov, 2006). Correct formulation of Newton's problem of minimal resistance in dimension three is (see, e.g., Fraeijs de Veubeke, 1966; Tikhomirov, 1990):

$$
\begin{aligned}
&\mathcal{R}\left[u(\cdot)\right] = \int_0^r \frac{x}{1+u(x)^2}dx \longrightarrow \min\,,\\
&\dot{y}(x) = u(x)\,, \quad u(x) \geq 0\,,\\
&y(0) = 0\,, \quad y(r) = H\,, \quad H > 0\,,
\end{aligned}
$$

where we minimize the resistance $\mathcal{R}$ in the class of continuous functions $y : [0, r] \to \mathbb{R}$ with piecewise continuous derivative. Here we consider Newton's problem of minimal resistance in dimension two (see Torres, Plakhov, 2006):

$$
\begin{aligned}
&R\left[u(\cdot)\right] = \int_0^r \frac{1}{1+u(x)^2}dx \longrightarrow \min\,,\\
&\dot{y}(x) = u(x)\,, \quad u(x) \in \Omega\,,\\
&y(0) = 0\,, \quad y(r) = H\,, \quad H > 0\,.
\end{aligned}
\tag{1}
$$

We consider two cases: (i) the unrestricted problem, where no restriction on the admissible trajectories $y(\cdot)$ other than the boundary conditions $y(0) = 0$, $y(r) = H$ is considered ($\Omega = \mathbb{R}$); (ii) the restricted problem, where the admissible functions must satisfy the restriction $\dot{y}(x) \geq 0$, $x \in [0, r]$ ($\Omega = \mathbb{R}_0^+$). While for the classical three-dimensional problem only the restricted problem admits a minimizer, we prove in Section 4 that the two-dimensional problem (1) is richer:

the unrestricted case also admits a local minimizer when the given height H of the body is big enough. In Section 5 we study the restricted problem. Also in the restricted case the two-dimensional problem is more interesting: if $r \geq H$, then infinitely many different minimizers are possible, while in the classical three-dimensional problem the minimizer is always unique.

3. General results for both problems

The central result of optimal control theory is the Pontryagin Maximum Principle (Pontryagin, Boltyanskii, Gamkrelidze, Mishchenko, 1962), which gives a generalization of the classical necessary optimality conditions of the calculus of variations. The following results are valid for both restricted and unrestricted problems: respectively $\Omega = \mathbb{R}_0^+$ and $\Omega = \mathbb{R}$ in (1).

THEOREM 1 (Pontryagin Maximum Principle for (1)) *If $(y(\cdot), u(\cdot))$ is a minimizer of problem (1), then there exists a non-zero pair $(\psi_0, \psi(\cdot))$, where $\psi_0 \leq 0$ is a constant and $\psi(\cdot) \in PC^1([0,r];\mathbb{R})$, such that the following conditions are satisfied for almost all x in $[0,r]$:*

(i) the Hamiltonian system

$$\begin{cases} \dot{y}(x) & = \frac{\partial \mathcal{H}}{\partial \psi}(u(x), \psi_0, \psi(x)) \quad \textit{(control equation } \dot{y} = u\textit{)}, \\ \dot{\psi}(x) & = -\frac{\partial \mathcal{H}}{\partial y}(u(x), \psi_0, \psi(x)) \quad \textit{(adjoint system } \dot{\psi} = 0\textit{)}; \end{cases}$$

(ii) the maximality condition

$$\mathcal{H}(u(x), \psi_0, \psi(x)) = \max_{u \in \Omega} \mathcal{H}(u, \psi_0, \psi(x)); \tag{2}$$

where the Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H}(u, \psi_0, \psi) = \psi_0 \frac{1}{1+u^2} + \psi u. \tag{3}$$

The adjoint system asserts that $\psi(x) \equiv c$, with c a constant. From the maximality condition it follows that $\psi_0 \neq 0$ (there are no abnormal extremals for problem (1)).

PROPOSITION 1 *All the Pontryagin extremals $(y(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of problem (1) are normal extremals ($\psi_0 \neq 0$), with $\psi(\cdot)$ a negative constant: $\psi(x) \equiv -\lambda$, $\lambda > 0$, $x \in [0,r]$.*

Proof. The Hamiltonian $\mathcal{H}$ for problem (1), $\mathcal{H}(u, \psi_0, \psi) = \psi_0 \frac{1}{1+u^2} + \psi u$, does not depend on y. Therefore, by the adjoint system we conclude that

$$\dot{\psi}(x) = -\frac{\partial \mathcal{H}}{\partial y}(u(x), \psi_0, \psi(x)) = 0,$$

that is, $\psi(x) \equiv c$, c a constant, for all $x \in [0, r]$. If $c = 0$, then $\psi_0 < 0$ (because one can not have both ψ_0 and ψ zero) and the maximality condition (2) simplifies to

$$\frac{\psi_0}{1+u^2(x)} = \max_{u\in\Omega}\left\{\frac{\psi_0}{1+u^2}\right\}. \tag{4}$$

From (4) we conclude that the maximum is not achieved ($u \to \infty$). Therefore $c \neq 0$. Similarly, for $c > 0$ the maximum

$$\frac{\psi_0}{1+u^2(x)} + cu(x) = \max_{u\in\Omega}\left\{\frac{\psi_0}{1+u^2} + cu\right\}$$

does not exist, and we conclude that $c < 0$. It remains to prove that $\psi_0 \neq 0$. Let us assume $\psi_0 = 0$. Then the maximality condition reads

$$cu(x) = \max_{u\in\Omega}\{cu\}, \quad c < 0. \tag{5}$$

For $\Omega = \mathbb{R}$ the maximum does not exist, and we conclude that $\psi_0 \neq 0$. For $\Omega = \mathbb{R}_0^+$ (5) implies $u(x) \equiv 0$ and $y(x) \equiv w$, w a constant ($\dot{y}(x) = u(x)$). This is not possible, given the boundary conditions $y(0) = 0$ and $y(r) = H$ with $H > 0$. Therefore $\psi_0 \neq 0$: there exist no abnormal Pontryagin extremals. ■

Remark 1 *If $(y(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ is an extremal, then $(y(\cdot), u(\cdot), \gamma\psi_0, \gamma\psi(\cdot))$ is also a Pontryagin extremal, for all $\gamma > 0$. Therefore one can fix, without loss of generality, $\psi_0 = -1$.*

From Proposition 1 and Remark 1 it follows that the Hamiltonian (3) takes the form

$$\mathcal{H}(u) = -\frac{1}{1+u^2} - \lambda u, \quad \lambda > 0. \tag{6}$$

It is not easy to prove the existence of a solution for problem (1) with classical arguments. We will use a different approach. We will show, following Torres, Plakhov (2006), that for problem (1) the Pontryagin extremals are absolute minimizers. This means that to solve problem (1) it is enough to identify its Pontryagin extremals.

Theorem 2 *Pontryagin extremals for problem* (1) *are absolute minimizers.*

Proof. Let $\hat{u}(\cdot)$ be a Pontryagin extremal control for problem (1). We want to prove that

$$\int_0^r \frac{1}{1+u^2(x)}dx \geq \int_0^r \frac{1}{1+\hat{u}^2(x)}dx$$

for any admissible control $u(\cdot)$. Given (6), we conclude from the maximality condition (2) that

$$-\frac{1}{1+\hat{u}^2(x)} - \lambda \hat{u}(x) \geq -\frac{1}{1+u^2(x)} - \lambda u(x) \tag{7}$$

for all $u(\cdot) \in PC\left([0,r],\Omega\right)$. Having in mind that all the admissible processes $(y(\cdot), u(\cdot))$ of (1) satisfy

$$\int_0^r u(x)dx = \int_0^r \dot{y}(x)dx = y(r) - y(0) = H\,,$$

we only need to integrate (7) to conclude that $\hat{u}(\cdot)$ is an absolute control minimizer:

$$\begin{aligned}
&\int_0^r \left(-\frac{1}{1+\hat{u}^2(x)} - \lambda \hat{u}(x)\right) dx \geq \int_0^r \left(-\frac{1}{1+u^2(x)} - \lambda u(x)\right) dx \\
&\Leftrightarrow \int_0^r \frac{1}{1+\hat{u}^2(x)} dx + \lambda \int_0^r \hat{u}(x)dx \leq \int_0^r \frac{1}{1+u^2(x)} dx + \lambda \int_0^r u(x)dx \\
&\Leftrightarrow \int_0^r \frac{1}{1+\hat{u}^2(x)} dx + \lambda H \leq \int_0^r \frac{1}{1+u^2(x)} dx + \lambda H \\
&\Leftrightarrow \int_0^r \frac{1}{1+\hat{u}^2(x)} dx \leq \int_0^r \frac{1}{1+u(x)^2} dx\,.
\end{aligned}$$

■

Roughly speaking, Theorem 2 reduces the infinite dimension optimization problem (1) to the study of a one-dimension maximization problem:

$$\max_{u\in\Omega} \mathcal{H}\left(u\right) = \max_{u\in\Omega} \left\{-\frac{1}{1+u^2} - \lambda u\right\}\,, \quad \lambda > 0\,. \tag{8}$$

4. The unrestricted problem

The following standard result of calculus (see, e.g., Fenske, 2003) will be used in the sequel.

THEOREM 3 *Let $n \geq 2$ and $\Omega \subseteq \mathbb{R}$ be an open set. If $f : \Omega \to \mathbb{R}$ is $n-1$ times differentiable on Ω and n times differentiable at some point $a \in \Omega$ where $f^{(k)}(a) = 0$ for $k = 0, \ldots, n-1$ and $f^{(n)}(a) \neq 0$, then:*

- *either n is even, and $f(\cdot)$ has an extremum at a, that is a maximum in case $f^{(n)}(a) < 0$ and a minimum in case $f^{(n)}(a) > 0$;*
- *or n is odd, and $f(\cdot)$ does not attain a local extremum at a.*

We are considering now the unrestricted two-dimensional Newton's problem of minimal resistance, that is, $\Omega = \mathbb{R}$ in (1). A necessary (sufficient) condition for

u to be a local maximizer for problem (8) is given by $\mathcal{H}'(u) = 0$ and $\mathcal{H}''(u) \leq 0$ ($\mathcal{H}''(u) < 0$), where

$$\mathcal{H}'(u) = \frac{2u}{\left(1+u^2\right)^2} - \lambda\,,$$
$$\mathcal{H}''(u) = -2\,\frac{3\,u^2-1}{\left(1+u^2\right)^3}\,.$$

From the first order condition (maximality condition (2)) it follows that

$$\frac{u(x)}{\left(1+u^2(x)\right)^2} = \frac{\lambda}{2} \Leftrightarrow \frac{\dot{y}(x)}{\left(1+\dot{y}^2(x)\right)^2} = \frac{\lambda}{2}\,. \tag{9}$$

Using the boundary conditions $y(0) = 0$ and $y(r) = H$, we conclude that $y(x) = \frac{H}{r}x$ ($u = \frac{H}{r}$) is a local candidate for the solution of the unrestricted problem ($\lambda = \frac{2r^3H}{(r^2+H^2)^2}$). However, by Theorem 3, we conclude that such u is a maximizer only when $H > \frac{\sqrt{3}}{3}r$. For $H < \frac{\sqrt{3}}{3}r$ the value $u = \frac{H}{r}$ corresponds to a local minimizer of $\mathcal{H}(u)$ since $\mathcal{H}'' > 0$; for $H = \frac{\sqrt{3}}{3}r$ function $\mathcal{H}(u)$ has neither local maximum nor minimum since $\mathcal{H}''\left(\frac{\sqrt{3}}{3}r\right) = 0$ and $\mathcal{H}'''\left(\frac{\sqrt{3}}{3}r\right) = -\frac{27\sqrt{3}}{16} \neq 0$.

THEOREM 4 *If $H > \frac{\sqrt{3}}{3}r$, then function $y(x) = \frac{H}{r}x$ is a (local) minimum for the unrestricted problem* (1)*. For $H \leq \frac{\sqrt{3}}{3}r$ the problem has no solution.*

REMARK 2 *The unrestricted problem* (1) *does not admit global minimum. Take indeed, for large values of the parameter a, the control function*

$$\tilde{u}(x) = \begin{cases} a & \textit{if} \quad 0 \leq x \leq \frac{r}{2} + \frac{H}{2a} \\ -a & \textit{if} \quad \frac{r}{2} + \frac{H}{2a} < x \leq r\,. \end{cases}$$

This gives $R[\tilde{u}(\cdot)] = \frac{r}{1+a^2}$ which vanishes as $a \to +\infty$, showing that no global solution can exist.

By the symmetry with respect to the yy axis, a local solution to the unrestricted two-dimensional Newton's problem of minimal resistance with $H > \frac{\sqrt{3}}{3}r$ is a triangle, with value for resistance R equal to $\frac{r^3}{r^2+H^2}$.

5. The restricted problem

We now study problem (1) with $\Omega = \mathbb{R}_0^+$. In this case the optimal control can take values on the boundary of the admissible set of control values Ω ($u = 0$). If the optimal control $u(\cdot)$ is always taking values in the interior of Ω, $u(x) > 0$ $\forall\, x \in [0, r]$, then the optimal solution must satisfy (9) and it corresponds to the one found in Section 4:

$$u(x) = \frac{H}{r}\,, \quad \forall x \in [0, r]\,, \tag{10}$$

with resistance

$$R = \frac{r^3}{r^2 + H^2} \,. \tag{11}$$

We show next that this is a solution of the restricted problem only for $H \geq r$: for $H \leq r$ the minimum value for the resistance is $R = r - \frac{H}{2}$.

It is clear, from the boundary conditions $y(0) = 0$, $y(r) = H$, $r > 0$, $H > 0$, that $u(x) = 0$, $\forall\ x \in [0,r]$, is not a possibility: there must exist at least one non-empty subinterval of $[0,r]$ for which $u(x) > 0$ (otherwise $y(x)$ would be constant, and it would be not possible to satisfy simultaneously $y(0) = 0$ and $y(r) = H$). The simplest situations are given by

$$u(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq \xi \,, \\ \frac{H}{r-\xi} & \text{if} \quad \xi \leq x \leq r \,, \end{cases} \tag{12}$$

or

$$u(x) = \begin{cases} \frac{H}{\xi} & \text{if} \quad 0 \leq x \leq \xi \,, \\ 0 & \text{if} \quad \xi \leq x \leq r \,. \end{cases} \tag{13}$$

We get (10) from (12) taking $\xi = 0$; (10) from (13) with $\xi = r$. For (12) the resistance is given by $R(\xi) = \xi + \frac{(r-\xi)^3}{(r-\xi)^2+H^2}$, that has a minimum value for $\xi = r - H \geq 0$: $R(r-H) = r - \frac{H}{2}$,

$$u(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq r - H \,, \\ 1 & \text{if} \quad r - H \leq x \leq r \,. \end{cases} \tag{14}$$

For $r = H$ (14) coincides with (10); for $r > H$

$$\left(r - \frac{H}{2}\right) - \left(\frac{r^3}{r^2 + H^2}\right) = -\frac{H(r-H)^2}{2(r^2+H^2)} < 0 \,,$$

and (14) is better than (10). Similarly, for (13) the resistance is given by

$$R(\xi) = \frac{\xi^3}{\xi^2 + H^2} + r - \xi \,, \tag{15}$$

that has minimum value for $\xi = H > 0$:

$$u(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq H \,, \\ 0 & \text{if} \quad H \leq x \leq r \,, \end{cases} \tag{16}$$

$R(H) = r - \frac{H}{2}$, which coincides with the value for the resistance obtained with (14). If one compares directly (11) with (15) one gets the conclusion that (10) is better than (13) precisely when $r < H$:

$$\frac{r^3}{r^2 + H^2} - \left(\frac{\xi^3}{\xi^2 + H^2} + r - \xi\right) = \frac{\xi H^2 \left(r^2 - r\xi - H^2\right)}{\left[(r-\xi)^2 + H^2\right]\left(r^2 + H^2\right)} \,, \tag{17}$$

and since $-H^2 \leq r^2 - r\xi - H^2 \leq r^2 - H^2$, (17) is negative if $r < H$, that is, for $r < H$, (10) is better than (13). For $r = H$, (16) coincides with (10), for $r > H$, (16) is better than (10) and as good as (14).

We now show that for $r > H$ it is possible to obtain the resistance value $r - \frac{H}{2}$ in infinitely many other ways, but no better (no less value) than this quantity. Generic situation is given by

$$u_n(x) = \begin{cases} 0 & \text{if} \quad \xi_{2i} \leq x \leq \xi_{2i+1}\,, \quad i = 0, \ldots, n\,, \\ \frac{\mu_{i+1}-\mu_i}{\xi_{2i+2}-\xi_{2i+1}} & \text{if} \quad \xi_{2i+1} \leq x \leq \xi_{2i+2}\,, \quad i = 0, \ldots, n-1\,, \end{cases} \tag{18}$$

where $n \in \mathbb{N}$, $0 = \xi_0 \leq \xi_1 \leq \cdots \leq \xi_{2n+1} = r$, $0 = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_n = H$. We remark that for the simplest case $n = 1$ (18) simplifies to

$$u_1(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq \xi_1\,, \\ \frac{H}{\xi_2-\xi_1} & \text{if} \quad \xi_1 \leq x \leq \xi_2\,, \\ 0 & \text{if} \quad \xi_2 \leq x \leq r\,, \end{cases}$$

which covers all the previously considered situations: for $\xi_1 = 0$, $\xi_2 = r$ we obtain (10); for $\xi_2 = r$ (12); and for $\xi_1 = 0$ one obtains (13). All Pontryagin control extremals of the restricted problem are of the form (18), and by Theorem 2 also the minimizing controls. The resistance force R_n associated with (18) is given by

$$\begin{aligned} &R_n\left(\xi_0, \ldots, \xi_{2n+1}, \mu_0, \ldots, \mu_n\right) \\ &\qquad = \sum_{i=0}^{n} \left(\xi_{2i+1} - \xi_{2i}\right) + \sum_{i=0}^{n-1} \frac{\left(\xi_{2i+2} - \xi_{2i+1}\right)^3}{\left(\xi_{2i+2} - \xi_{2i+1}\right)^2 + \left(\mu_{i+1} - \mu_i\right)^2}\,. \end{aligned} \tag{19}$$

It is a simple exercise of calculus to see that function (19) has three critical points: two of them not admissible, the third one a minimizer. The first critical point is defined by $\mu_i = 0$, $i = 0, \ldots, n$, which is not admissible given the fact that $\mu_n = H > 0$. The second critical point is given by $\mu_i - \mu_{i-1} = \xi_{2i-1} - \xi_{2i}$, $i = 1, \ldots, n$, which is not admissible since $\mu_i - \mu_{i-1} \geq 0$, $\xi_{2i-1} - \xi_{2i} \leq 0$, and $\mu_i = \mu_{i-1}$, $i = 1, \ldots, n$, is not a possibility given $\mu_n = H > \mu_0 = 0$. The third critical point is

$$\mu_i - \mu_{i-1} = \xi_{2i} - \xi_{2i-1}\,, \quad i = 1, \ldots, n\,, \tag{20}$$

which is a minimizer for $H \leq r$. Thus, all the minimizing controls for the restricted two-dimensional problem with $H \leq r$ are of the following form:

$$u_n(x) = \begin{cases} 0 & \text{if} \quad \xi_{2i} \leq x \leq \xi_{2i+1}\,, \quad i = 0, \ldots, n\,, \\ 1 & \text{if} \quad \xi_{2i+1} \leq x \leq \xi_{2i+2}\,, \quad i = 0, \ldots, n-1\,, \end{cases} \tag{21}$$

$n = 1, 2, \ldots$, $0 = \xi_0 \leq \xi_1 \leq \cdots \leq \xi_{2n+1} = r$. For $u_n(x)$ given by (21) the resistance (19) reduces to $R_n = r - \frac{H}{2}$, $\forall\, n \in \mathbb{N}$.

THEOREM 5 *The restricted two-dimensional Newton's problem of minimal resistance admit always a solution:*

- *the unique solution associated to control* (10)*, when* $H > r$*;*
- *infinitely many solutions associated to the controls* (21)*, when* $H \leq r$*.*

In the case $H > r$ *the minimum value for the resistance is* $\frac{r^3}{r^2+H^2}$*, otherwise* $r - \frac{H}{2}$*.*

6. Conclusion

Newton's classical problem of minimal resistance offer two interesting situations to be studied: the problem in dimension two; and the problem in dimension d, d being a real number greater or equal than three. While second situation is well studied in the literature, and well understood, the first one has been ignored. In the classical three-dimensional Newton's problem of minimal resistance, only the problem with restriction $u(x) = \dot{y}(x) \geq 0$ makes sense (without the restriction the problem has no local minimum). In the two-dimensional case, we have proved that the unrestricted case is also a well defined problem when $H > \frac{\sqrt{3}}{3}r$, the minimum value for the resistance being $\frac{r^3}{r^2+H^2}$. The local minimizer is a triangle. The two-dimensional problem with restriction $u(x) = \dot{y}(x) \geq 0$ has always a solution: a unique solution (a triangle) when $H > r$, with value for resistance equal to the unrestricted case; infinitely many alternative solutions for $r \geq H$, the minimal aerodynamical resistance being $r - \frac{H}{2}$.

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References

AZEVEDO DO AMARAL, I.M. (1913) Note sur la solution finie d'un problème de Newton. *Ann. Ac. Pol. Porto* **8**, 207–209.

BELLONI, M. and KAWOHL, B. (1997) A paper of Legendre revisited. *Forum Mathematicum* **9**, 655–668.

BRYSON, A.E. and HO, YU CHI (1975) *Applied Optimal Control.* Hemisphere Publishing Corp. Washington, D.C.

BUTTAZZO, G. and KAWOHL, B. (1993) On Newton's problem of minimal resistance. *The Mathematical Intelligencer* **15** (4), 7–12.

COMTE, M. and LACHAND-ROBERT, T. (2001) Newton's problem of the body of minimal resistance under a single-impact assumption. *Calc. Var. Partial Differential Equations* **12** (2), 173–211.

Fenske, C.C. (2003) Extrema in case of several variables. *The Mathematical Intelligencer* **25** (1), 49–51.

Fraeijs de Veubeke, B. (1966) Le problème de Newton du solide de révolution présentant une traînée minimum. *Acad. Roy. Belg. Bull. Cl. Sci.* **52** (5), 171–182.

Horstmann, D., Kawohl, B. and Villaggio, P. (2002) Newton's aerodynamic problem in the presence of friction. *NoDEA Nonlinear Differential Equations Appl.* **9** (3), 295–307.

Kneser, A., Zermelo, E., Hahn, H. and Lecat, M. (1913) Problème de Newton et questions analogues – Surfaces propulsives. *Encyclopédie des sciences mathématiques pures et appliquées,* Édition Franaise, Tome II, **6** (1), *Calcul des variations*, Paris: Gauthier Villars, Leipzig: B. G. Teubner, 243–250.

Plakhov, A.Yu. (2003) Newton's problem of the body of minimal aerodynamic resistance. *Doklady of the Russian Academy of Sciences* **390** (3), 1–4.

Plakhov, A.Yu. and Torres, D.F.M. (2004) Two-dimensional problems of minimal resistance in a medium of positive temperature. *Proceedings of the 6th Portuguese Conference on Automatic Control - Controlo*, 488–493.

Plakhov, A.Yu. and Torres, D.F.M. (2005) Newton's aerodynamic problem in media of chaotically moving particles. *Sbornik: Mathematics* **196** (6), 885–933.

Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V. and Mishchenko, E.F. (1962) *The Mathematical Theory of Optimal Processes.* Interscience Publishers John Wiley & Sons, Inc. New York-London.

Silva, C.J. (2005) Abordagens do Cálculo das Variações e Controlo Óptimo ao Problema de Newton de Resistência Mínima, M.Sc. thesis (supervisor: Delfim F. M. Torres), Univ. of Aveiro, Portugal.

Tikhomirov, V.M. (1990) Stories about maxima and minima. *American Mathematical Society*, Providence, RI.

Tikhomirov, V.M. (2002) Extremal problems - past and present. In: *The Teaching of Mathematics* **2**, 59–69.

Torres, D.F.M. and Plakhov, A.Yu. (2006) Optimal control of Newton-type problems of minimal resistance. *Rend. Semin. Mat. Univ. Politec. Torino* **64** (1), 79–95.

Young, L.C. (1969) *Lectures on the Calculus of Variations and Optimal Control Theory.* W.B. Saunders Co., Philadelphia.