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Symbolic computation of variational symmetries in optimal control[1]

by

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Abstract: We use a computer algebra system to compute, in an efficient way, optimal control variational symmetries up to a gauge term. The symmetries are then used to obtain families of Noether's first integrals, possibly in the presence of nonconservative external forces. As an application, we obtain eight independent first integrals for a sub-Riemannian nilpotent problem $(2, 3, 5, 8)$.

Keywords: variational symmetries, gauge term, nonconservative forces, computer algebra systems, Noether's theorem, first integrals, optimal control.

1. Introduction

The concept of variational symmetry entered into optimal control in the 1970s (Djukic, 1973). Variational symmetries, which keep an optimal control problem invariant, are described mathematically in terms of a group of parameter transformations: two transformations performed one after another may be replaced by one transformation of the same family; there exists an identity transformation; to each transformation there exists an inverse one. Variational symmetries are very useful in optimal control, but, unfortunately, their study is not easy, requiring lengthy and cumbersome calculations (Torres, 2004).

Recently, there has been an interest in the application of Computer Algebra Systems to the study of control systems, and collections of symbolic tools are being developed to help in the analysis and solution of complex problems. The first computer algebra package for computing the variational symmetries in the

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calculus of variations was given in Gouveia, Torres (2005a); then extended to the more general setting of optimal control in Gouveia, Torres (2005b).

In this work we provide a new Maple package for the automatic computation of variational symmetries and respective Noether's first integrals in the calculus of variations and optimal control. The present package generalize the previous results in Gouveia, Torres (2005b) by introducing two new possibilities: (i) invariance symmetries up to a gauge term (Torres, 2002); (ii) presence of nonconservative external forces (Frederico, Torres, 2007). Moreover, the efficiency in computing the variational symmetries is largely improved when we compare the running times with the ones in Gouveia, Torres (2005b). With the improvements in the efficiency of the package, we are now able, for the first time in the literature, to obtain eight independent first integrals for the nilpotent problem $(2, 3, 5, 8)$ of sub-Riemannian geometry.

2. Nonconservative forces

Without loss of generality, we consider the optimal control problem in Lagrange form: to minimize an integral functional

$$I[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] = \int_a^b L(t, \mathbf{x}(t), \mathbf{u}(t))\, \mathrm{d}t \tag{1}$$

subject to a control system described by a system of ordinary differential equations of the form

$$\dot{\mathbf{x}}(t) = \boldsymbol{\varphi}(t, \mathbf{x}(t), \mathbf{u}(t))\,, \tag{2}$$

together with appropriate boundary conditions, not relevant for the present study (the results of the paper are valid for arbitrary boundary conditions). The Lagrangian $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and the velocity vector $\boldsymbol{\varphi} \colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are assumed to be continuously differentiable functions with respect to all their arguments. The controls $\mathbf{u} : [a, b] \to \Omega \subseteq \mathbb{R}^m$ are piecewise continuous functions taking values on an open set Ω; the state variables $\mathbf{x} \colon [a, b] \to \mathbb{R}^n$ are continuously differentiable functions.

The resolution of optimal control problems usually goes by identifying the Pontryagin extremals (Pontryagin et al., 1962). In presence of nonconservative external forces $\mathbf{F} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ the Pontryagin Maximum Principle (PMP) takes the following form (Frederico, Torres, 2007).

Theorem 1 (PMP under a nonconservative force $\mathbf{F}$) *If $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ is a solution of the optimal control problem (1)-(2) under the presence of a nonconservative force $\mathbf{F}(t, \mathbf{x}, \mathbf{u})$, then there exists a non-vanishing pair $(\psi_0, \boldsymbol{\psi}(\cdot))$, where $\psi_0 \leq 0$ is a constant and $\boldsymbol{\psi}(\cdot)$ an n-vectorial piecewise C^1-smooth function with domain $[a, b]$, such that the quadruple $(\mathbf{x}(\cdot), \mathbf{u}(\cdot), \psi_0, \boldsymbol{\psi}(\cdot))$ satisfies the following conditions almost everywhere in $[a, b]$:*

(i) *the nonconservative Hamiltonian system*

$$\begin{cases} \dot{\mathbf{x}}(t)^{\mathrm{T}} = \frac{\partial H}{\partial \boldsymbol{\psi}}(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) \,, \\ \dot{\boldsymbol{\psi}}(t)^{\mathrm{T}} = -\frac{\partial H}{\partial \mathbf{x}}(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) + \mathbf{F}(t, \mathbf{x}(t), \mathbf{u}(t))^{\mathrm{T}} \,; \end{cases} \tag{3}$$

(ii) *the maximality condition*

$$H(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) = \max_{\mathbf{v} \in \Omega} H(t, \mathbf{x}(t), \mathbf{v}, \psi_0, \boldsymbol{\psi}(t)) \,; \tag{4}$$

where the Hamiltonian H is defined by

$$H(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \psi_0 L(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\psi}^{\mathrm{T}} \cdot \boldsymbol{\varphi}(t, \mathbf{x}, \mathbf{u}) \,. \tag{5}$$

REMARK 1 *The right-hand side of the equations of the nonconservative Hamiltonian system (3) represents a row-vector. First equation in (3) is nothing more than the control system (2); the second equation is known as the* nonconservative adjoint system.

DEFINITION 1 *A quadruple* $(\mathbf{x}(\cdot), \mathbf{u}(\cdot), \psi_0, \boldsymbol{\psi}(\cdot))$*, satisfying Theorem 1, is said to be a* nonconservative extremal. *A nonconservative extremal is said to be* normal *when* $\psi_0 \neq 0$, abnormal *when* $\psi_0 = 0$.

REMARK 2 *Since we are assuming* Ω *to be an open set, the maximality condition* (4) *implies the* stationary condition

$$\frac{\partial H}{\partial \mathbf{u}}(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) = \mathbf{0} \,, \quad t \in [a, b] \,. \tag{6}$$

3. Invariance up to a gauge term

Let $\mathbf{h}^s : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ be a one-parameter group of $\mathbb{C}^1$ transformations of the form

$$\mathbf{h}^s(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) =$$
$$(h_t^s(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}), \mathbf{h}_{\mathbf{x}}^s(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}), \mathbf{h}_{\mathbf{u}}^s(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}), \mathbf{h}_{\boldsymbol{\psi}}^s(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi})) \,. \tag{7}$$

Without loss of generality, we assume that the identity transformation of the group (7) is obtained when the parameter s is zero:

$$h_t^0(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = t, \; \mathbf{h}_{\mathbf{x}}^0(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \mathbf{x},$$
$$\mathbf{h}_{\mathbf{u}}^0(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \mathbf{u}, \; \mathbf{h}_{\boldsymbol{\psi}}^0(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \boldsymbol{\psi}.$$

Associated with a one-parameter group of transformations (7), we introduce its *infinitesimal generators*:

$$\begin{aligned} T(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) &= \left.\frac{\partial}{\partial s} h_t^s\right|_{s=0}, \; \mathbf{X}(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \left.\frac{\partial}{\partial s} \mathbf{h}_{\mathbf{x}}^s\right|_{s=0}, \\ \mathbf{U}(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) &= \left.\frac{\partial}{\partial s} \mathbf{h}_{\mathbf{u}}^s\right|_{s=0}, \; \boldsymbol{\Psi}(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \left.\frac{\partial}{\partial s} \mathbf{h}_{\boldsymbol{\psi}}^s\right|_{s=0}. \end{aligned} \tag{8}$$

DEFINITION 2 (Invariance up to a gauge term) *An optimal control problem (1)-(2) is said to be invariant under a one-parameter group of transformations (7) up to a gauge term* $g^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) \in \mathbb{C}^1([a,b],\mathbb{R}^n,\mathbb{R}^m,\mathbb{R},\mathbb{R}^n;\mathbb{R})$*, if for all s sufficiently small and for any subinterval* $[\alpha,\beta] \subseteq [a,b]$ *one has*

$$\int_{\alpha^s}^{\beta^s} \left(H(t^s,\mathbf{x}^s(t^s),\mathbf{u}^s(t^s),\psi_0,\boldsymbol{\psi}^s(t^s)) - \boldsymbol{\psi}^s(t^s)^{\mathrm{T}} \cdot \frac{\mathrm{d}}{\mathrm{d}t^s}\mathbf{x}^s(t^s) \right) \mathrm{d}t^s$$
$$= \int_{\alpha}^{\beta} \Big(H(t,\mathbf{x}(t),\mathbf{u}(t),\psi_0,\boldsymbol{\psi}(t)) - \boldsymbol{\psi}(t)^{\mathrm{T}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)$$
$$+ \frac{\mathrm{d}}{\mathrm{d}t} g^s(t,\mathbf{x}(t),\mathbf{u}(t),\psi_0,\boldsymbol{\psi}(t)) \Big) \mathrm{d}t\,, \tag{9}$$

where $\alpha^s = h_t^s(\alpha,\mathbf{x}(\alpha),\mathbf{u}(\alpha),\psi_0,\boldsymbol{\psi}(\alpha))$*,* $\beta^s = h_t^s(\beta,\mathbf{x}(\beta),\mathbf{u}(\beta),\psi_0,\boldsymbol{\psi}(\beta))$*, and* $(t^s,\mathbf{x}^s,\mathbf{u}^s,\boldsymbol{\psi}^s) = \left(h_t^s,\mathbf{h}_{\mathbf{x}}^s,\mathbf{h}_{\mathbf{u}}^s,\mathbf{h}_{\boldsymbol{\psi}}^s\right)$.

When we write (9) in terms of the generators (8), one gets a necessary and sufficient condition of invariance – see Djukic (1973), Torres (2005).

THEOREM 2 (Necessary and sufficient condition of invariance) *An optimal control problem is* invariant *under* (8) *up to*

$$G(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) = \frac{\mathrm{d}}{\mathrm{d}s} g^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})\Big|_{s=0}$$

or, equivalently, (8) *is a* symmetry *of the problem up to G, if, and only if,*

$$\frac{\partial H}{\partial t}T + \frac{\partial H}{\partial \mathbf{x}} \cdot \mathbf{X} + \frac{\partial H}{\partial \mathbf{u}} \cdot \mathbf{U} + \frac{\partial H}{\partial \boldsymbol{\psi}} \cdot \boldsymbol{\Psi} - \boldsymbol{\Psi}^{\mathrm{T}} \cdot \dot{\mathbf{x}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} + H\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t}\,, \tag{10}$$

with H the Hamiltonian (5)*.*

REMARK 3 *The function* $G(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) = \frac{\mathrm{d}}{\mathrm{d}s} g^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})\big|_{s=0}$ *is also known in the literature as a* gauge term*.*

Proof Transforming the integral on the left-hand side of (9) to the interval $[\alpha,\beta]$, and having in mind that (9) is satisfied for all subintervals $[\alpha,\beta] \subseteq [a,b]$, the invariance condition can be written in the following equivalent form:

$$\left(H(\mathbf{h}^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})) - \mathbf{h}_{\boldsymbol{\psi}}^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})^{\mathrm{T}} \cdot \frac{\frac{\mathrm{d}\mathbf{h}_{\mathbf{x}}^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})}{\mathrm{d}t}}{\frac{\mathrm{d}h_t^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})}{\mathrm{d}t}} \right) \frac{\mathrm{d}h_t^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})}{\mathrm{d}t}$$
$$= H(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} + \frac{\mathrm{d}}{\mathrm{d}t} g^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})\,.$$

Differentiating both sides of the equation with respect to s,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(H(\mathbf{h}^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})) - \mathbf{h}_{\boldsymbol{\psi}}^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})^{\mathrm{T}} \cdot \frac{\frac{\mathrm{d}\mathbf{h}_{\mathbf{x}}^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})}{\mathrm{d}t}}{\frac{\mathrm{d}h_t^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})}{\mathrm{d}t}} \right) \right.$$
$$\left. \times \frac{\mathrm{d}h_t^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})}{\mathrm{d}t} \right] = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}}{\mathrm{d}t} g^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) \right),$$

we obtain the equality

$$\left(H(\mathbf{h}^s) - {\mathbf{h}_{\boldsymbol{\psi}}^s}^{\mathrm{T}} \cdot \frac{\mathrm{d}\mathbf{h}_{\mathbf{x}}^s/\mathrm{d}t}{\mathrm{d}h_t^s/\mathrm{d}t}\right) \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}h_t^s}{\mathrm{d}s} + \left(\frac{\partial H(\mathbf{h}^s)}{\partial h_t^s}\frac{\partial h_t^s}{\partial s} + \frac{\partial H(\mathbf{h}^s)}{\partial \mathbf{h}_{\mathbf{x}}^s} \cdot \frac{\partial \mathbf{h}_{\mathbf{x}}^s}{\partial s}\right.$$
$$+ \frac{\partial H(\mathbf{h}^s)}{\partial \mathbf{h}_{\mathbf{u}}^s} \cdot \frac{\partial \mathbf{h}_{\mathbf{u}}^s}{\partial s} + \frac{\partial H(\mathbf{h}^s)}{\partial \mathbf{h}_{\boldsymbol{\psi}}^s} \cdot \frac{\partial \mathbf{h}_{\boldsymbol{\psi}}^s}{\partial s} - \frac{\mathrm{d}{\mathbf{h}_{\boldsymbol{\psi}}^s}^{\mathrm{T}}}{\mathrm{d}s} \cdot \frac{\mathrm{d}\mathbf{h}_{\mathbf{x}}^s/\mathrm{d}t}{\mathrm{d}h_t^s/\mathrm{d}t}$$
$$\left. - {\mathbf{h}_{\boldsymbol{\psi}}^s}^{\mathrm{T}} \cdot \left(\frac{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{h}_{\mathbf{x}}^s}{\mathrm{d}s}}{\frac{\mathrm{d}h_t^s}{\mathrm{d}t}} - \frac{\frac{\mathrm{d}\mathbf{h}_{\mathbf{x}}^s}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}h_t^s}{\mathrm{d}s}}{\frac{\mathrm{d}h_t^s}{\mathrm{d}t}\frac{\mathrm{d}h_t^s}{\mathrm{d}t}}\right)\right)\frac{\mathrm{d}h_t^s}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}g^s}{\mathrm{d}s}\,.$$

Finally, choosing $s = 0$, we express the condition in terms of the infinitesimal generators (8) and the function $G(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \frac{\mathrm{d}}{\mathrm{d}s} g^s(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi})\big|_{s=0}$:

$$\left(H - \boldsymbol{\psi}^{\mathrm{T}} \cdot \dot{\mathbf{x}}\right) \frac{\mathrm{d}T}{\mathrm{d}t} + \left(\frac{\partial H}{\partial t} T + \frac{\partial H}{\partial \mathbf{x}} \cdot \mathbf{X} + \frac{\partial H}{\partial \mathbf{u}} \cdot \mathbf{U} + \frac{\partial H}{\partial \boldsymbol{\psi}} \cdot \boldsymbol{\Psi} - \boldsymbol{\Psi}^{\mathrm{T}} \cdot \dot{\mathbf{x}}\right.$$
$$\left. - \boldsymbol{\psi}^{\mathrm{T}} \cdot \left(\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} - \dot{\mathbf{x}}\frac{\mathrm{d}T}{\mathrm{d}t}\right)\right) = \frac{\mathrm{d}G}{\mathrm{d}t}\,. \qquad \blacksquare$$

4. Nonconservative Noether's theorem

Emmy Noether was the first who established the relation between the existence of invariance transformations of the problems and the existence of conservation laws – first integrals of the Euler-Lagrange or Hamiltonian equations (Noether, 1918). A generalization of the classical result of E. Noether for the nonconservative calculus of variations was recently given by Fu and Chen (2003); then extended to the more general setting of optimal control by Frederico and Torres (2007).

Using (3), together with the stationary condition (6), one can deduce that along the nonconservative Pontryagin extremals (Definition 1), the total derivative of the Hamiltonian with respect to the independent variable t is equal to its partial derivative plus the scalar product of the velocity vector with the resultant nonconservative forces $\mathbf{F}$ (Frederico, Torres, 2007):

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) = \frac{\partial}{\partial t} H(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) + \dot{\mathbf{x}}(t)^{\mathrm{T}} \cdot \mathbf{F}(t, \mathbf{x}(t), \mathbf{u}(t))\,. \tag{11}$$

Using this fact, the nonconservative optimal control version of E. Noether's theorem is easily obtained from the necessary and sufficient invariance condition (10), restricting attention to the quadruples $(\mathbf{x}(\cdot), \mathbf{u}(\cdot), \psi_0, \boldsymbol{\psi}(\cdot))$ that satisfy the nonconservative Hamiltonian system (3) and the maximality condition (4): along the extremals, equalities (3), (6), and (11) permit to simplify (10) to the form

$$\begin{aligned}&\left(\frac{\mathrm{d}H}{\mathrm{d}t} - \dot{\mathbf{x}}^{\mathrm{T}} \cdot \mathbf{F}\right) T + \left(\mathbf{F}^{\mathrm{T}} - \dot{\boldsymbol{\psi}}^{\mathrm{T}}\right) \cdot \mathbf{X} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} + H \frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t} \\ &\Leftrightarrow \frac{\mathrm{d}H}{\mathrm{d}t} T + H \frac{\mathrm{d}T}{\mathrm{d}t} - \dot{\boldsymbol{\psi}}^{\mathrm{T}} \cdot \mathbf{X} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} - \frac{\mathrm{d}G}{\mathrm{d}t} - \left(\dot{\mathbf{x}}^{\mathrm{T}} T - \mathbf{X}^{\mathrm{T}}\right) \cdot \mathbf{F} = 0 \\ &\Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t} \left(HT - \boldsymbol{\psi}^{\mathrm{T}} \cdot \mathbf{X} - G - \int \left(\dot{\mathbf{x}}^{\mathrm{T}} T - \mathbf{X}^{\mathrm{T}}\right) \cdot \mathbf{F} \, \mathrm{d}t \right) = 0\,.\end{aligned}$$

This means that $HT - \boldsymbol{\psi}^{\mathrm{T}} \cdot \mathbf{X} - G - \int\left(\dot{\mathbf{x}}^{\mathrm{T}} T - \mathbf{X}^{\mathrm{T}}\right) \cdot \mathbf{F} \, \mathrm{d}t$ is a first integral whenever the optimal control problem under consideration admits a symmetry (8) up to the gauge term G:

THEOREM 3 (Nonconservative Optimal Control version of Noether's Principle) *If the infinitesimal generators (8) constitute a symmetry of the optimal control problem (1)-(2) under the presence of nonconservative forces with the resultant vector* $\mathbf{F}(t, \mathbf{x}, \mathbf{u})$*, then*

$$\begin{aligned}\int \Big(\dot{\mathbf{x}}(t)^{\mathrm{T}} T(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) - \mathbf{X}(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t))^{\mathrm{T}}\Big) \cdot \mathbf{F}(t, \mathbf{x}(t), \mathbf{u}(t)) \mathrm{d}t \\ + \boldsymbol{\psi}(t)^{\mathrm{T}} \cdot \mathbf{X}(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) + G(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) \\ - H(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t))\, T(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) = const \qquad (12)\end{aligned}$$

is a conservation law, i.e., condition (12) *holds for all* t *in* $[a, b]$ *and for every nonconservative extremal* $(\mathbf{x}(\cdot), \mathbf{u}(\cdot), \psi_0, \boldsymbol{\psi}(\cdot))$ *of the problem.*

5. Computation of symmetries up to a gauge term

The main problem in obtaining Noether's conservation laws (in applying Theorem 3) resides in the determination of the symmetries and respective gauge terms. If n effective first integrals exist (Rocha, Torres, 2006), then the optimal control problem is integrable, and classical results allow the integration of the equations of motion.

Here we propose an algorithm for determining the infinitesimal generators (8) and the gauge terms G, which define a variational symmetry. Let us assume, for the moment, that the optimal controls are C^1 functions (in §7 we will drop this restrictive assumption, just by assuming that T, $\mathbf{X}$, and G do not depend on the control variables). The key point to compute symmetries consists in generalizing the method used in Gouveia, Torres, (2005b, §3) to the nonconservative and gauge-invariant cases. The idea is simple: when we substitute the Hamiltonian H and its partial derivatives in the invariance identity (10), then the condition becomes a polynomial in $\dot{\mathbf{x}}$, $\dot{\mathbf{u}}$ and $\dot{\boldsymbol{\psi}}$, and one can equal the coefficients of the polynomial to zero. Thus, given an optimal control problem (1)-(2), defined by a Lagrangian L and a velocity vector $\boldsymbol{\varphi}$, we determine the infinitesimal generators T, $\mathbf{X}$, $\mathbf{U}$ and $\boldsymbol{\Psi}$ and the gauge term G, which define a symmetry for the problem, by the following method: (i) we define the respective Hamiltonian (5); (ii) we substitute H and its partial derivatives into (10); (iii) expanding the total derivatives

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{\partial T}{\partial t} + \frac{\partial T}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial T}{\partial \mathbf{u}} \cdot \dot{\mathbf{u}} + \frac{\partial T}{\partial \boldsymbol{\psi}} \cdot \dot{\boldsymbol{\psi}}, \\ \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} &= \frac{\partial \mathbf{X}}{\partial t} + \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial \mathbf{X}}{\partial \mathbf{u}} \cdot \dot{\mathbf{u}} + \frac{\partial \mathbf{X}}{\partial \boldsymbol{\psi}} \cdot \dot{\boldsymbol{\psi}}, \\ \frac{\mathrm{d}G}{\mathrm{d}t} &= \frac{\partial G}{\partial t} + \frac{\partial G}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial G}{\partial \mathbf{u}} \cdot \dot{\mathbf{u}} + \frac{\partial G}{\partial \boldsymbol{\psi}} \cdot \dot{\boldsymbol{\psi}},\end{aligned} \tag{13}$$

we write equation (10) as a polynomial

$$\begin{aligned}&A(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) + B(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) \cdot \dot{\mathbf{x}} + C(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) \cdot \dot{\mathbf{u}} \\ &+D(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) \cdot \dot{\boldsymbol{\psi}} = 0\end{aligned} \tag{14}$$

in the $2n+m$ derivatives $\dot{\mathbf{x}}$, $\dot{\mathbf{u}}$ and $\dot{\boldsymbol{\psi}}$:

$$\begin{aligned}&\left(\frac{\partial H}{\partial t} T + \frac{\partial H}{\partial \mathbf{x}} \cdot \mathbf{X} + \frac{\partial H}{\partial \mathbf{u}} \cdot \mathbf{U} + \frac{\partial H}{\partial \boldsymbol{\psi}} \cdot \boldsymbol{\Psi} + H\frac{\partial T}{\partial t} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial t} - \frac{\partial G}{\partial t}\right) \\ &+ \left(-\boldsymbol{\Psi}^{\mathrm{T}} + H\frac{\partial T}{\partial \mathbf{x}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{x}} - \frac{\partial G}{\partial \mathbf{x}}\right) \cdot \dot{\mathbf{x}} + \left(H\frac{\partial T}{\partial \mathbf{u}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{u}} - \frac{\partial G}{\partial \mathbf{u}}\right) \cdot \dot{\mathbf{u}} \\ &+ \left(H\frac{\partial T}{\partial \boldsymbol{\psi}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial \boldsymbol{\psi}} - \frac{\partial G}{\partial \boldsymbol{\psi}}\right) \cdot \dot{\boldsymbol{\psi}} = 0\,.\end{aligned} \tag{15}$$

The terms in (15), which involve derivatives with respect to vectors, are expanded in row-vectors or in matrices, depending, respectively, if the function is a scalar or a vectorial one. For example,

$$\begin{aligned}\frac{\partial T}{\partial \mathbf{x}} &= \left[\frac{\partial T}{\partial x_1}\ \frac{\partial T}{\partial x_2}\ \cdots\ \frac{\partial T}{\partial x_n}\right], \\ \frac{\partial \mathbf{X}}{\partial \boldsymbol{\psi}} &= \left[\frac{\partial \mathbf{X}}{\partial \psi_1}\ \frac{\partial \mathbf{X}}{\partial \psi_2}\ \cdots\ \frac{\partial \mathbf{X}}{\partial \psi_n}\right] = \begin{bmatrix} \frac{\partial X_1}{\partial \psi_1} & \frac{\partial X_1}{\partial \psi_2} & \cdots & \frac{\partial X_1}{\partial \psi_n} \\ \frac{\partial X_2}{\partial \psi_1} & \frac{\partial X_2}{\partial \psi_2} & \cdots & \frac{\partial X_2}{\partial \psi_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial X_n}{\partial \psi_1} & \frac{\partial X_n}{\partial \psi_2} & \cdots & \frac{\partial X_n}{\partial \psi_n} \end{bmatrix}.\end{aligned}$$

Equation (15) is a differential equation in the $2n+m+2$ unknown functions T, $X_1, \ldots, X_n$, $U_1, \ldots, U_m$, $\Psi_1, \ldots, \Psi_n$ and G. This equation must hold for all $\dot{x}_1, \ldots, \dot{x}_n$, $\dot{u}_1, \ldots, \dot{u}_n$, $\dot{\psi}_1, \ldots, \dot{\psi}_n$, and therefore the coefficients A, B, C and

D of polynomial (14) must vanish, that is,

$$\begin{cases} \dfrac{\partial H}{\partial t} T + \dfrac{\partial H}{\partial \mathbf{x}} \cdot \mathbf{X} + \dfrac{\partial H}{\partial \mathbf{u}} \cdot \mathbf{U} + \dfrac{\partial H}{\partial \boldsymbol{\psi}} \cdot \boldsymbol{\Psi} + H \dfrac{\partial T}{\partial t} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \dfrac{\partial \mathbf{X}}{\partial t} - \dfrac{\partial G}{\partial t} = 0\,, \\ -\boldsymbol{\Psi}^{\mathrm{T}} + H \dfrac{\partial T}{\partial \mathbf{x}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \dfrac{\partial \mathbf{X}}{\partial \mathbf{x}} - \dfrac{\partial G}{\partial \mathbf{x}} = \mathbf{0}\,, \\ H \dfrac{\partial T}{\partial \mathbf{u}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \dfrac{\partial \mathbf{X}}{\partial \mathbf{u}} - \dfrac{\partial G}{\partial \mathbf{u}} = \mathbf{0}\,, \\ H \dfrac{\partial T}{\partial \boldsymbol{\psi}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \dfrac{\partial \mathbf{X}}{\partial \boldsymbol{\psi}} - \dfrac{\partial G}{\partial \boldsymbol{\psi}} = \mathbf{0}\,. \end{cases} \tag{16}$$

The system of equations (16), obtained from (15), is a system of $2n + m + 1$ partial differential equations with $2n + m + 2$ unknown functions; so, in general, there exists not a unique symmetry but a family of such symmetries. The system (16) becomes even more under-determined when one assumes, as in Section 7, that T, $\mathbf{X}$, and G do not depend on the control variables $\mathbf{u}$. Although a system of partial differential equations, solving (16) is possible, because the system is of the first order and linear with respect to the unknown functions and their derivatives. We solve the system of PDEs by the method of (additive) separation of variables, as explained in Cheb-Terrab, von Bulow (1995). Following Cheb-Terrab, von Bulow (1995), the generators are replaced by the sum of unknown functions, one for each variable. For example, $T(t, x_1, x_2, \psi_1, \psi_2) = T_1(t) + T_2(x_1) + T_3(x_2) + T_4(\psi_1) + T_5(\psi_2)$. When dealing with optimal control problems with several state and control variables, the number of calculations is big enough, and the help of the computer is more than welcome. We define a Maple procedure *Symmetry* that does all the cumbersome calculations for us. The procedure receives, as input, the Lagrangian and the velocity vector; and returns, as output, a family of symmetries $(T, \mathbf{X}, \mathbf{U}, \boldsymbol{\Psi})$ and, if necessary, the respective gauge term G. We remark that since system (16) is homogeneous, we always have, as trivial solution, $(T, \mathbf{X}, \mathbf{U}, \boldsymbol{\Psi}) = \mathbf{0}$.

6. The computer algebra package

We obtain Noether conservation laws, in an automatic way, through two steps: (i) with our procedure *Symmetry* we obtain the variational symmetries and respective gauge terms; (ii) using the obtained symmetries, gauge terms, and nonconservative forces as input to procedure *Noether*, we obtain the correspondent conservation laws. In Section 8 we give several examples, not covered by the previous results in Gouveia, Torres (2005a, b), illustrating the whole process. Given the limit on the maximum number of pages of the paper, we do not provide the Maple definitions for the procedures *Symmetry* and *Noether* here. The complete Maple package can be freely obtained from `http://www.mat.ua.pt/delfim/maple.htm` together with an online help database for the Maple system.

Novelties of the procedures *Symmetry* and *Noether* with respect to the previous versions in Gouveia, Torres (2005a, b) are: (i) capacity of procedure *Symmetry* to cover invariance symmetries up to a gauge term, according with Sections 3 and 5; (ii) improvements of efficiency – see Section 7; (iii) capacity of procedure *Noether* to consider problems of the calculus of variations and optimal control under nonconservative external forces, according to Section 4; (iv) improvement of the usage of the procedures by introduction of several optional parameters, as illustrated in Section 8. Moreover, a new Maple procedure called *PMP* was added, which implements Theorem 1, according to Section 2.[1] The procedure *PMP* is very useful in practice, when dealing with concrete problems of the calculus of variations and optimal control – see Section 8. The input to the procedure is: the Lagrangian L and the velocity vector $\boldsymbol{\varphi}$, that define the optimal control problem (1)-(2) and the respective Hamiltonian H; the nonconservative external forces (if present); and several useful optional arguments which define the output. The output of *PMP* is either (depending on the optional parameters): the (nonconservative) extremals; the equations of the (nonconservative) Hamiltonian system and stationary condition; or, alternatively, the Hamiltonian. We refer the reader to the Examples in Section 8 for a general overview on the usage of the developed Maple procedures; to the annotated Maple worksheet available at http://www.mat.ua.pt/delfim/maple.htm, with all the definitions of the package, detailed documentation, and many other examples not given here, for more details. The reader is free to experiment with the Maple package in order to determine variational symmetries and Noether conservation laws on his/her own problems.

7. Efficiency, comparison with previous results

The high number of dependences that the infinitesimal generators may present, affect, excessively, the efficiency of the method described in Section 5, namely for problems with a large number of state and control variables. In order to quantify this effect, we measured the computing running times of our procedure *Symmetry* for different dependences of the infinitesimal generators (8), with a large set of optimal control problems: the ten problems considered in Gouveia, Torres (2005b; Sections 4 and 5) (examples 4.1–4.6 and 5.1–5.4), together with twelve new problems. Three of these new problems are given in Section 8, the complete set of problems being available as a Maple worksheet, as mentioned in Section 6. All the computational processing was carried out with the Maple 10 Computer Algebra System on a 1.4GHz Pentium Centrino with 512MB of RAM. In Gouveia, Torres (2005b), the maximum number of dependences for each generator, as indicated in (8), is always considered. We denote here such situation by *D1*. In the *D1* case, and as noticed in Gouveia, Torres (2005b),

[1] In the software Cotcot, available from http://www.n7.fr/apo/cotcot/, the tool Adifor for automatic differentiation in Fortran is also used to generate, in the conservative case, the equations of the Pontryagin maximum principle (Bonnard, Caillau, Trélat, 2005).

the involved computational effort is sometimes very high: the computing times increase exponentially with the dimension of the problem. This is particularly well illustrated with the following problems of sub-Riemannian geometry: the nilpotent problem $(2,3)$, with three state variables, requires a total computing time of one minute (Gouveia, Torres, 2005b, Example 4.5); problem $(2,3,5)$, with five state variables, requires thirty minutes (Gouveia, Torres, 2005b, Example 4.6); the problem $(2,3,5,8)$, with eight state variables, was not studied in Gouveia, Torres (2005b), and thought to be out of its capacities. We compute here its symmetries in Example 3, with the present Maple package, with forty one minutes of computing time; while the method in Gouveia, Torres (2005b) requires, approximately, thirty times this value: twenty hours of computing time are needed.[2]

The computing running times largely depend on the numbers n and m, respectively the number of state and control variables: besides directly influencing the number of dependences of the unknown functions (infinitesimal generators), they determine the amount of those functions and the number of partial differential equations that must be solved in order to find the variational symmetries. Without considering the gauge term, we come across a system of $m + 2n + 1$ partial differential equations and $m+2n+1$ unknown functions, each one of the unknown functions being dependent of $m + 2n + 1$ variables. We address here the following question: is there some way to simplify the process of obtaining the variational symmetries?

Although knowing that the complexity of the method is intimately related with the values n and m, that are fixed with a given optimal control problem, we get, even so, a quite satisfactory answer to the question. Analyzing the results from the test set of problems, we verify that, in spite of considering the maximum number of dependences (*D1*), the infinitesimal generators obtained through the procedure *Symmetry* are, nevertheless, almost always, dependent functions of a quite reduced number of variables. When we restrict ourselves to the dependences $T(t)$, $\mathbf{X}(t,\mathbf{x})$, $\mathbf{U}(\mathbf{u},\boldsymbol{\psi})$, $\boldsymbol{\Psi}(\boldsymbol{\psi})$ – that we identify as *D2* – we are able to cover the totality of the twenty two considered problems in our study. If in the formulation of the system of PDEs (16) we only enter with these dependences, besides the obvious reduction of the number of dependences of the unknown functions, we reduce the number of equations to less than half: from $m+2n+1$ to $n+1$. In agreement with the simulations done, the efficiency of the procedure *Symmetry* increases significantly with this new group of dependences (*D2*). For instance, for the problem $(2,3,5)$ of sub-Riemannian geometry (Gouveia, Torres, 2005b, Example 4.6), a problem with two controls and five state variables, the running time passed from half an hour to less than one and a half minute. We have also considered another more simplified set of dependences, denoted by *D3*: $T(t)$, $\mathbf{X}(t,\mathbf{x})$, $\mathbf{U}(t,\mathbf{u})$, $\boldsymbol{\Psi}(t,\boldsymbol{\psi})$. With it, it is now possible to obtain the

[2] We believe that the forty minutes of computing time can still be diminished by using a programming language closer to machine, for instance using Adifor: http://www-unix.mcs.anl.gov/autodiff/ADIFOR.

symmetries of the sub-Riemannian nilpotent problem $(2,3,5,8)$ (Example 3), in less than 45 minutes; and it is still possible to obtain the same conservation laws for all the twenty two studied problems (in three of the problems, Gouveia, Torres, 2005b, Examples 4.4, 5.2 and 5.3), the generators were different, since the more general generators $\mathbf{U}$ depend on the variables $\boldsymbol{\psi}$, but the correspondent Noether conservation laws (12) are exactly the same since they only depend on the generators T and $\boldsymbol{X}$). Finally, we repeated the study for a more restricted group of dependences (*D4*): $T(t)$, $\mathbf{X}(\mathbf{x})$, $\mathbf{U}(\mathbf{u})$, $\mathbf{\Psi}(\boldsymbol{\psi})$. As expected, the time of processing suffered an additional reduction (for the $(2,3,5,8)$ problem the running time passed from $44'16''$ to $28'21''$), but, in this case, not the entire family of conservation laws for the problems are obtained. For four of the problems – Example 4.3 in Gouveia, Torres (2005b), Examples 2 and 3 in the Maple worksheet, and Example 1 here – only particular cases of the complete family of conservation laws are obtained.

To summarize the influence that the different dependences of the generators have on the efficiency of the procedure *Symmetry*, we give in Table 1 the running times for computing the variational symmetries of the three problems of sub-Riemannian geometry already mentioned: Gouveia, Torres (2005b, Examples 4.5 and 4.6) and Example 3. All the three problems have two control variables and the same Lagrangian, but a different number of state variables, respectively, 3, 5, and 8.

Table 1. Running times of procedure *Symmetry* for three problems of sub-Riemannian geometry (Gouveia, Torres, 2005b, Examples 4.5, 4.6, and Example 3 here), with different dependences of the infinitesimal generators: *D1* – $[T(t,\mathbf{x},\mathbf{u},\psi)$, $\mathbf{X}(t,\mathbf{x},\mathbf{u},\psi)$, $\mathbf{U}(t,\mathbf{x},\mathbf{u},\psi)$, $\mathbf{\Psi}(t,\mathbf{x},\mathbf{u},\psi)]$; *D2* – $[T(t)$, $\mathbf{X}(t,\mathbf{x})$, $\mathbf{U}(\mathbf{u},\psi)$, $\mathbf{\Psi}(\psi)]$; *D3* – $[T(t)$, $\mathbf{X}(t,\mathbf{x})$, $\mathbf{U}(t,\mathbf{u})$, $\mathbf{\Psi}(t,\psi)]$; *D4* – $[T(t)$, $\mathbf{X}(\mathbf{x})$, $\mathbf{U}(\mathbf{u})$, $\mathbf{\Psi}(\psi)]$.

Dependences	Number of PDEs*	Problem (2, 3)	Problem (2, 3, 5)	Problem (2, 3, 5, 8)
D1	$m+2n+1$	$1'04''$	$30'34''$	$20h07'12''$
D2	$n+1$	$5''$	$1'26''$	$51'28''$
D3	$n+1$	$4''$	$1'09''$	$44'16''$
D4	$n+1$	$2''$	$38''$	$28'21''$

* n = number of state variables; m = number of control variables.

We verify that of the four sets of studied generators, just with *D4* it was not possible to obtain, with full generality, the totality of Noether's conservation laws for the twenty two considered problems. The set of generators *D3* ($T(t)$, $\mathbf{X}(t,\mathbf{x})$, $\mathbf{U}(t,\mathbf{u})$, $\mathbf{\Psi}(t,\boldsymbol{\psi})$) gives the best compromise: it presents the best running times, between the generators that give the complete family of variational symmetries and Noether conservation laws for the problems we have studied; running times are much better than the ones obtained with the generators *D1*. We recommend the user to try configuration *D3* first on his/her own optimal control problems. Considering t and $\mathbf{x}$ for the dependences of the gauge term – $G(t,\mathbf{x})$ – the system of PDEs that we have to solve, in order to find the variational symmetries, takes the form (see (16))

$$\begin{cases} \frac{\partial H}{\partial t}T + \frac{\partial H}{\partial \mathbf{x}} \cdot \mathbf{X} + \frac{\partial H}{\partial \mathbf{u}} \cdot \mathbf{U} + \frac{\partial H}{\partial \boldsymbol{\psi}} \cdot \boldsymbol{\Psi} + H\frac{\partial T}{\partial t} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial t} - \frac{\partial G}{\partial t} = 0\,, \\ \boldsymbol{\Psi}^{\mathrm{T}} + \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{x}} + \frac{\partial G}{\partial \mathbf{x}} = \mathbf{0}\,. \end{cases} \tag{17}$$

Our present procedure *Symmetry* computes, by default, the variational symmetries as defined by *D3*, and with a gauge term $G(t,\mathbf{x})$: by default *Symmetry* solves system (17). Through optional parameters, it is possible to find the variational symmetries for other generators and gauge terms: in order to use all the dependences (*D1*) one must use option `alldep`; to use a minimum of dependences (*D4*) one uses option `mindep`. We remark that with the class of generators *D3*, T and $\mathbf{X}$ are not functions of $\mathbf{u}$, and there is no need to assume the control variables $\mathbf{u}$ to be smooth functions (see (13)).

Table 2 shows the computing running times needed to obtain all the variational symmetries of the problems in Gouveia, Torres (2005b, Sections 4 and 5), by using the default version of procedure *Symmetry* we give here (generators *D3*); and by using the version in Gouveia, Torres (2005b), which is a particular case of our present procedure – see Section 8 for examples not covered by the previous methods in Gouveia, Torres (2005b) – obtained using option `alldep`, that is, generators *D1*. The time needed to compute the variational symmetries for the $(2,3,5)$ problem (Example 4.6 in Gouveia, Torres, 2005b) decreased from thirty minutes to one.

Table 2. Running times of procedure *Symmetry* for all the problems of Gouveia, Torres, 2005b, with the generator sets *D1* (the only possibility in Gouveia, Torres, 2005b) – $[T(t,\mathbf{x},\mathbf{u},\psi)$, $\mathbf{X}(t,\mathbf{x},\mathbf{u},\psi)$, $\mathbf{U}(t,\mathbf{x},\mathbf{u},\psi)$, $\boldsymbol{\Psi}(t,\mathbf{x},\mathbf{u},\psi)]$, and *D3* – $[T(t)$, $\mathbf{X}(t,\mathbf{x})$, $\mathbf{U}(t,\mathbf{u})$, $\boldsymbol{\Psi}(t,\psi)]$.

	4.1	4.2	4.3	4.4	4.5	4.6	5.1	5.2	5.3	5.4
D1	2″	1′13″	2′44″	6′41″	1′04″	30′34″	8″	17″	6′42″	1″
D3	0″	5″	11″	18″	4″	1′09″	0″	3″	16″	0″

The use of generators with a smaller number of dependences leads to a drastic reduction of the computing running times. For the studied problems, the use of generators *D3* permits to obtain the same results while decreasing the total processing times for about 4% of the ones verified in Gouveia, Torres, 2005b (generators *D1*).

8. Examples of the new possibilities

In order to show the functionality and the use of the new procedures, we apply our Maple package to three concrete optimal control problems which are not covered by the previous results in Gouveia, Torres (2005a,b). All the examples were solved with Maple version 10 on a 1.4GHz 512MB RAM Pentium Centrino. The running time of procedure *Symmetry* is indicated, for each example, in the format min'sec". All the other Maple commands run instantaneously.

8.1. Variational symmetries up to a gauge term

We begin with a very simple example of the classical calculus of variations. We recall that for the fundamental problem of the calculus of variations there are no abnormal extremals, so one can choose $\psi_0 = -1$ (we use option `noabn` of our Maple package).

EXAMPLE 1 *(0'00") Let us consider the following scalar problem of the calculus of variations (*$n = m = 1$*):*

$$\int_a^b (u(t))^2 \,\mathrm{d}t \longrightarrow \min\,,$$
$$\dot{x}(t) = u(t)\,.$$

In this case $L = u^2$ *and* $\varphi = u$*. First we obtain the variational symmetries of the problem (Maple procedure* Symmetry*) up to a gauge term (parameter* ***gauge****).*

```
> S := Symmetry(u^2,u,t,x,u,showt,gauge);
```

$$S := \Big[T = 2\,C_2 t + C_6,\ X = \frac{1}{2}\,\frac{C_3 t}{\psi_0} + C_2 x(t) + C_4,\ U = \frac{1}{2}\,\frac{C_3}{\psi_0} - u(t) C_2, \\ \Psi = -\psi(t) C_2 - C_3,\ GAUGE = C_3 x(t) + C_5\Big].$$

Noether conservation laws are obtained through Theorem 3 (Maple procedure Noether*) with the generators and the gauge term just obtained.*

```
> CL := Noether(u^2,u,t,x,u,S,showt,noabn,H);
```

$$CL := \left(-\frac{1}{2}\,C_3 t + C_2 x(t) + C_4\right)\psi(t) - H\,(2\,C_2 t + C_6) + C_3 x(t) + C_5 = const$$

The Hamiltonian H*, which appears in the above family of conservation laws, is given by* (5)*:*

```
> H := PMP(u^2,u,t,x,u, evalH,showt,noabn);
```

$$H := -u(t)^2 + u(t)\psi(t).$$

This is a very simple problem, just used to illustrate, in the simplest possible way, our Maple *procedures. In this case it is an easy exercise to obtain the extremals by direct application of the Pontryagin Maximum Principle or the Euler-Lagrange equations,*

```
> extremals := PMP(u^2,u,t,x,u,showt,noabn);
```

$$extremals := \left\{\psi(t) = K_2,\ x(t) = \frac{1}{2}K_2\,t + K_1,\ u(t) = \frac{1}{2}K_2\right\}$$

and one can validate the obtained conservation laws by applying the definition of conservation law: by definition, the obtained family of conservation laws must hold along all the extremals of the problem.

```
> subs(extremals,CL);
```

$$K_2\,C_2K_1 + K_2\,C_4 - \frac{1}{4}K_2^2C_6 + C_3K_1 + C_5 = const.$$

8.2. Presence of nonconservative forces

We consider now a problem of the calculus of variations under the action of a nonconservative force. The problem is borrowed from Djukic, Strauss (1980, Section 4).

Example 2 *($n = 1$, $m = 2$, 0'01") The problem is defined by the Lagrangian $L(q, \dot{q}, \ddot{q}) = \frac{1}{2}\ddot{q}(t)^2 + \frac{1}{2}a\dot{q}(t)^2 + \frac{1}{2}bq(t)^2$, and presence of the nonconservative force $f(t) = \mu\,\dot{q}(t) + \frac{\mu^2}{a^2}\ddot{q}(t) - 2\,\frac{\mu}{a}\dddot{q}(t)$ which depends on higher-order derivatives (a, b, and μ are constants).*

```
> PDEtools[declare](prime=t);
```

derivatives with respect to t of functions of one variable will now be displayed with '

```
> L := u^2/2+a*v^2/2+b*q^2/2;
> phi := [v,u];
> f := mu*v+mu^2/a^2*u-2*mu/a*z(t);
```

$$\begin{aligned} L &:= \frac{1}{2}u^2 + \frac{1}{2}av^2 + \frac{1}{2}bq^2 \\ \varphi &:= [v,\ u] \\ f &:= \mu\,v + \frac{\mu^2u}{a^2} - 2\,\frac{\mu\,z(t)}{a} \end{aligned}$$

```
> S := Symmetry(L, phi, t, [q,v], u);
```

$$S := [T = C_1,\ X_1 = 0,\ X_2 = 0,\ U = 0,\ \Psi_1 = 0,\ \Psi_2 = 0]$$

```
> CL := Noether(L, phi, t, [q,v], u, S, ncf=[f,0], noabn);
```

$$CL := -\left(-\frac{1}{2}\,u(t)^2 - \frac{1}{2}\,a\,v(t)^2 - \frac{1}{2}\,b\,q(t)^2 + \psi_1(t)v(t) + \psi_2(t)u(t)\right)C_1 + \int C_1 q' \left(\mu\,v(t) + \frac{\mu^2 u(t)}{a^2} - 2\,\frac{\mu\,z(t)}{a}\right) dt = const.$$

The multipliers $\psi_1(t)$ and $\psi_2(t)$ are obtained using the adjoint system and the stationary condition, as given by Theorem 1.

```
> sys := PMP(L, phi, t, [q,v], u, noabn, evalSyst, ncf=[f,0], showt);
```

$$sys := \left[\{q' = v(t), v' = u(t)\}, \left\{-\psi_1{}' = -\mu\,v(t) - \frac{\mu^2 u(t)}{a^2} + 2\,\frac{\mu\,z(t)}{a} - bq(t), -\psi_2{}' = -av(t) + \psi_1(t)\right\}, \{-u(t) + \psi_2(t) = 0\}\right]$$

```
> dsolve({sys[2][2],sys[3][]},{psi[1](t),psi[2](t)});
```

$$\{\psi_2(t) = u(t),\ \psi_1(t) = -u' + av(t)\}.$$

With substitutions

```
> subs(%, z(t)=diff(u(t),t), u(t)=diff(v(t),t), v(t)=diff(q(t),t),
                                                        C[1]=1, CL);
```

$$-\frac{1}{2}\,q''^2 + \frac{1}{2}\,a q'^2 + \frac{1}{2}\,b\,q(t)^2 - \left(-q''' + a q'\right) q' + \int q' \left(\mu\,q' + \frac{\mu^2 q''}{a^2} - 2\,\frac{\mu\,q'''}{a}\right) dt = const$$

one obtains the conservation law, Djukic, Strauss (1980, Section 4). We remark that the conclusion is nontrivial, and difficult to obtain without Noether's principle.

8.3. The sub-Riemannian nilpotent case $(2, 3, 5, 8)$

We finish the section by applying our Maple package to one important problem: the study of sub-Riemannian geodesics. The reader, interested in the study of symmetries of flat distributions of sub-Riemannian geometry, is referred to Sachkov (2004). Here we use a formulation of the nilpotent problem $(2, 3, 5, 8)$ which is obtained using the results of Rocha (2004).

Example 3 *(44'16") The problem can be defined in the following way:*

$$\frac{1}{2}\int_a^b \left(u_1(t)^2 + u_2(t)^2\right) \mathrm{d}t \longrightarrow \min, \quad \begin{cases} \dot{x}_1(t) = u_1(t), \\ \dot{x}_2(t) = u_2(t), \\ \dot{x}_3(t) = u_2(t)x_1(t), \\ \dot{x}_4(t) = \frac{1}{2}\,u_2(t)x_1(t)^2, \\ \dot{x}_5(t) = u_2(t)x_1(t)x_2(t), \\ \dot{x}_6(t) = \frac{1}{6}\,u_2(t)x_1(t)^3, \\ \dot{x}_7(t) = \frac{1}{2}\,u_2(t)x_1(t)^2 x_2(t), \\ \dot{x}_8(t) = \frac{1}{2}\,u_2(t)x_1(t)x_2(t)^2. \end{cases}$$

The integrability of the problem is still an open question, Rocha, Torres (2006), Sachkov (2004), but eight independent conservation laws can be determined with our present Maple package.

```
> L := 1/2*(u[1]^2+u[2]^2);
> phi:=[u[1], u[2], u[2]*x[1], (u[2]/2)*x[1]^2, u[2]*x[1]*x[2],
        (u[2]/6)*x[1]^3, (u[2]/2)*x[1]^2*x[2], (u[2]/2)*x[1]*x[2]^2];
> XX := [x[i]$i=1..8];
> UU := [u[1],u[2]];
```

$$L := \frac{1}{2}{u_1}^2 + \frac{1}{2}{u_2}^2$$

$$\varphi := \left[u_1,\ u_2,\ u_2x_1, \frac{1}{2}u_2{x_1}^2,\ u_2x_1x_2, \frac{1}{6}u_2{x_1}^3, \frac{1}{2}u_2{x_1}^2x_2, \frac{1}{2}u_2x_1{x_2}^2\right]$$

$$XX := [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$$

$$UU := [u_1,\ u_2].$$

```
> Symmetry(L, phi, t, XX, UU);
```

$$\Big[T = C_1t + C_7,\ X_1 = \frac{1}{2}C_1x_1,\ X_2 = C_2 + \frac{1}{2}C_1x_2,\ X_3 = C_1x_3 + C_8,$$
$$X_4 = \frac{3}{2}C_1x_4 + C_6,\ X_5 = C_2x_3 + \frac{3}{2}C_1x_5 + C_3,\ X_6 = 2\,C_1x_6 + C_5,$$
$$X_7 = C_2x_4 + 2\,C_1x_7 + C_9,\ X_8 = C_2x_5 + 2\,C_1x_8 + C_4,\ U_1 = -\frac{1}{2}u_1C_1,$$
$$U_2 = -\frac{1}{2}C_1u_2,\ \Psi_1 = -\frac{1}{2}C_1\psi_1, \Psi_2 = -\frac{1}{2}C_1\psi_2,\ \Psi_3 = -\psi_3C_1 - C_2\psi_5,$$
$$\Psi_4 = -\frac{3}{2}\psi_4C_1 - C_2\psi_7,\ \Psi_5 = -\frac{3}{2}C_1\psi_5 - C_2\psi_8,\ \Psi_6 = -2\,C_1\psi_6,$$
$$\Psi_7 = -2\,C_1\psi_7,\ \Psi_8 = -2\,C_1\psi_8\Big]$$

```
> CL := Noether(L, phi, t, XX, UU, %, H);
```

$$CL := \frac{1}{2}C_1x_1\psi_1 + \left(C_2 + \frac{1}{2}C_1x_2\right)\psi_2 + (C_1x_3 + C_8)\,\psi_3 + \left(\frac{3}{2}C_1x_4 + C_6\right)\psi_4$$
$$+ \left(C_2x_3 + \frac{3}{2}C_1x_5 + C_3\right)\psi_5 + (2\,C_1x_6 + C_5)\,\psi_6 + (C_2x_4 + 2\,C_1x_7 + C_9)\,\psi_7$$
$$+ (C_2x_5 + 2\,C_1x_8 + C_4)\,\psi_8 - H\,(C_1t + C_7) = const$$

The Hamiltonian is given by

```
> Hamilt := PMP(L, phi, t, XX, UU, noabn, evalH);
```

$$Hamilt := -\frac{1}{2}{u_1}^2 - \frac{1}{2}{u_2}^2 + \psi_1u_1 + \psi_2u_2 + \psi_3u_2x_1 + \frac{1}{2}\psi_4u_2{x_1}^2 + \psi_5u_2x_1x_2$$
$$+ \frac{1}{6}u_2{x_1}^3\psi_6 + \frac{1}{2}u_2{x_1}^2x_2\psi_7 + \frac{1}{2}u_2x_1{x_2}^2\psi_8$$

and the extremal controls are obtained through the stationary condition.

```
> PMP(L,phi,t, XX, UU, noabn, evalSyst)[3];
```

$$\left\{-u_2 + \psi_2 + \psi_3 x_1 + \frac{1}{2}\psi_4 {x_1}^2 + \psi_5 x_1 x_2 + \frac{1}{6}{x_1}^3\psi_6 + \frac{1}{2}{x_1}^2 x_2 \psi_7 + \frac{1}{2} x_1 {x_2}^2 \psi_8 = 0,\right.$$
$$\left. -u_1 + \psi_1 = 0\right\}$$

```
> solve(%,{u[1],u[2]});
```

$$\left\{u_1 = \psi_1,\, u_2 = \psi_5 x_1 x_2 + \psi_2 + \psi_3 x_1 + \frac{1}{2}\psi_4 {x_1}^2 + \frac{1}{6}{x_1}^3\psi_6 + \frac{1}{2}{x_1}^2 x_2\psi_7 + \frac{1}{2}x_1{x_2}^2\psi_8\right\}$$

```
> H = expand(subs(%, Hamilt));
```

$$H = \frac{1}{2}\psi_2 x_1 {x_2}^2\psi_8 + \psi_5 x_1 x_2 \psi_2 + \psi_5 {x_1}^2 x_2 \psi_3 + \frac{1}{2}\psi_2\psi_4{x_1}^2 + \frac{1}{2}\psi_3{x_1}^3\psi_4$$
$$+\frac{1}{2}{\psi_5}^2{x_1}^2{x_2}^2 + \frac{1}{6}\psi_2{x_1}^3\psi_6 + \frac{1}{8}{x_1}^2{x_2}^4{\psi_8}^2 + \frac{1}{8}{x_1}^4{x_2}^2{\psi_7}^2 + \frac{1}{12}\psi_4{x_1}^5\psi_6$$
$$+\frac{1}{2}{\psi_3}^2{x_1}^2 + \frac{1}{72}{x_1}^6{\psi_6}^2 + \frac{1}{2}{\psi_2}^2 + \frac{1}{2}{\psi_1}^2 + \frac{1}{8}{\psi_4}^2{x_1}^4 + \frac{1}{6}\psi_5{x_1}^4 x_2\psi_6 + \frac{1}{2}\psi_3{x_1}^2{x_2}^2\psi_8$$
$$+\frac{1}{4}\psi_4{x_1}^3{x_2}^2\psi_8 + \frac{1}{4}\psi_4{x_1}^4 x_2\psi_7 + \psi_2\psi_3 x_1 + \frac{1}{4}{x_1}^3{x_2}^3\psi_7\psi_8 + \frac{1}{12}{x_1}^5\psi_6 x_2\psi_7$$
$$+\frac{1}{12}{x_1}^4\psi_6{x_2}^2\psi_8 + \frac{1}{2}\psi_2{x_1}^2 x_2\psi_7 + \frac{1}{2}\psi_5{x_1}^3 x_2\psi_4 + \frac{1}{2}\psi_5{x_1}^2{x_2}^3\psi_8 + \frac{1}{2}\psi_5{x_1}^3{x_2}^2\psi_7$$
$$+\frac{1}{2}\psi_3{x_1}^3 x_2\psi_7 + \frac{1}{6}\psi_3{x_1}^4\psi_6$$

Now, the eight conservation laws, we are looking for, are easily obtained:

```
> subs(C[8]= 1, seq(C[i]=0,i=1..9), CL);
> subs(C[6]= 1, seq(C[i]=0,i=1..9), CL);
> subs(C[3]= 1, seq(C[i]=0,i=1..9), CL);
> subs(C[5]= 1, seq(C[i]=0,i=1..9), CL);
> subs(C[9]= 1, seq(C[i]=0,i=1..9), CL);
> subs(C[4]= 1, seq(C[i]=0,i=1..9), CL);
> subs(C[2]= 1, seq(C[i]=0,i=1..9), CL);
> subs(C[7]=-1, seq(C[i]=0,i=1..9), CL);
```

$$\psi_3 = const$$
$$\psi_4 = const$$
$$\psi_5 = const$$
$$\psi_6 = const$$
$$\psi_7 = const$$
$$\psi_8 = const$$
$$\psi_2 + x_3\psi_5 + x_4\psi_7 + x_5\psi_8 = const$$
$$H = const$$

Given the results of Rocha (2004), one can say that the sub-Riemannian nilpotent Lie group of type $(2, 3, 5, 8)$ has seven trivial first integrals: the Hamiltonian H; and the multipliers ψ_3, ψ_4, ψ_5, ψ_6, ψ_7, ψ_8. Together with the nontrivial first integral $\psi_2 + x_3\psi_5 + x_4\psi_7 + x_5\psi_8$, here first obtained, it is possible to prove that the system is integrable. This is nontrivial since Liouville theorem does not apply: the set of first integrals is not involutive (for instance, Poisson bracket between ψ_3 and $\psi_2 + x_3\psi_5 + x_4\psi_7 + x_5\psi_8$ is not zero). This question is under study and will be addressed in a forthcoming publication.

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