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# OPTIMAL CONTROL OF NEWTON-TYPE PROBLEMS OF MINIMAL RESISTANCE

**Abstract.** We address Newton-type problems of minimal resistance from an optimal control perspective. It is proven that for Newton-type problems the Pontryagin maximum principle is a necessary and sufficient condition. Solutions are then computed for concrete situations, including the new case when the flux of particles is non-parallel.

In 1686, in his celebrated *Principia Mathematica*, Isaac Newton propounded the problem of determining the profile of a body of revolution, moving along its axis with constant speed, through some resisting medium, which would minimize the total resistance (see [9, 14]). Problems of this kind find application in the building of high-speed and high-altitude flying vehicles, such as in the design of missiles or artificial satellites. Newton has given the correct answer to his problem, in the situation of a "rare" medium of perfectly elastic particles with constant mass and at equal distances from each other, the resisting pressure at a surface point of the body being proportional to the square of the normal component of its velocity, but without explaining how he obtained it. He didn't write, however, "I have a great proof, but no space for it in the margins of this book". A proof "from the Book" was waiting for the Pontryagin maximum principle.

When one writes the resistance force $\mathcal{R}$ associated to Newton's problem,

$$\mathcal{R}\left[\dot{x}(\cdot)\right] = \int_0^T t \, \frac{1}{1+\dot{x}(t)^2} \, \mathrm{d}t \,,$$

one obtains an integral functional of the type of those studied throughout the history of the calculus of variations. However, due to the restrictions on the derivatives of admissible trajectories, $\dot{x}(t) \geq 0$, no satisfactory theory is available within the calculus of variations framework (see [1, 25, 26]). As first noticed by Legendre in 1788 (see [2] and references therein), without such restrictions on the derivatives the problem has no solution (the infimum is zero), since one can obtain arbitrarily small values for the integral resistance $\mathcal{R}\left[\dot{x}(\cdot)\right]$ by choosing a zig-zag function $x(\cdot)$ wildly oscillating, with large derivatives in absolute value. To make the problem physically consistent one must take into account the monotonicity of the profile, and this means, as was first remarked by V.M. Tikhomirov (cf. [1, 24]), that Newton's problem belongs to optimal control:

$$\begin{gathered}\mathcal{R}\left[u(\cdot)\right] = \int_0^T t \, \frac{1}{1+u(t)^2} \, \mathrm{d}t \longrightarrow \min \,, \\ \dot{x}(t) = u(t) \,, \quad u(t) \geq 0 \,, \\ x(0) = 0 \,, \quad x(T) = H \,.\end{gathered} \tag{1}$$

Most part of the literature wrongly assume Newton's problem to be "one of the first applications of the calculus of variations" but, in spite of this, the same literature correctly asserts the birth of the calculus of variations: 1697, the publication date of the solution to the brachystochrone problem, and not 1686, the publication date of the solution to Newton's problem of minimal resistance.

In 1997 H. J. Sussmann and J. C. Willems, in the beautiful paper [23], defended the polemic thesis that the brachystochrone date 1697 marks not only the birth of the calculus of variations but also the birth of optimal control. The truth seems to be deeper: optimal control was born in 1686, before the calculus of variations, with Newton's problem of minimal resistance. The restriction on the control $u(\cdot)$, which appear in Newton's problem (1), is a common ingredient of the optimal control problems. Such constraints appears naturally in practical engineering control problems, and are treated with the Pontryagin maximum principle – the central result of optimal control theory, first conjectured by L. S. Pontryagin, and then proved, in the late 1950's, by him and his collaborators, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko [22]. In an optimal control problem, the control functions take values on a set which is, in general, not a vector space. This is precisely what happens in Newton's classical problem (1), and the reason why Newton's problem must be classified as an optimal control problem, and not as a problem of the calculus of variations.

Newton's problem has been widely studied, and the literature about it is extensive. The main difficulty is that of existence [8]: the Lagrangian $L(t, u) = \frac{t}{1+u^2}$ associated to Newton's problem (1) is neither coercive nor convex, and Tonelli's direct method (see [10]) fails. In order to prove existence, several different classes of admissible functions have been proposed. The question is now usually treated with the help of relaxation techniques (see [5]), although direct arguments are also possible (see [16, 17]). As we shall prove (§2), for the Newton-type problems, the existence of a minimizer follows easily from the Pontryagin maximum principle: one can show that the Pontryagin extremals are, for such problems, absolute minimizers (cf. Theorem 2).

Several extensions of Newton's problem have been considered in recent years. This revival of interest in Newton's problem, and in the study of many variations around it, has been motivated by the paper [9] of G. Buttazzo and B. Kawohl. Recent results on Newton-type problems include: bodies without rotational symmetry (nonsymmetric cases) [4, 7, 15]; unbounded body (resistance per unit area) with one-impact assumption [12]; bodies with rotational symmetry and one-impact assumption, but not convex [11]; friction between particles and body (non-elastic collisions) [13]; bodies with prescribed volume [3]; multiple collisions allowed [16, 17]; unbounded body and multiple collisions allowed [20]. More recently, Newton-type problems have been related with problems of mass transportation [18, 19]. For a good survey on mass optimization problems and open problems, we refer the reader to [6].

Here we consider convex $d$-dimensional bodies of revolution with Height $H$ and radius of maximal cross section $T$, and treat them using an optimal control approach. We will not be restricted to two-dimensional or three-dimensional bodies, considering bodies of arbitrary dimension $d \geq 2$. We also introduce a different point of view. For us the body does not move, and the particles are the ones who move. The body is situated in a flux of infinitesimal particles, the flux being invariant with respect to translations and rotations around the symmetry axis of the body. This new point of view is, in our opinion, physically more realistic. Newton has considered the particles with no temperature (not moving). When the particles have temperature, they move, and the flux of particles is not necessarily falling vertically downwards the body, as considered by Newton. We will be considering new interesting situations with a non-parallel flux of particles. We obtain complete solution to this class of Newton-type problems, by showing that, under some physically relevant assumptions on the Lagrangian, a control is an absolute minimizing control for the problem if, and only if, it is a Pontryagin extremal control. Thus, for the Newton-type problems we are dealing with, Pontryagin maximum principle holds not only as a necessary optimality condition, but also as a sufficient condition. As very special situations, one obtains the solution found by Newton himself (§3.2), and solutions to Newton's problem in higher-dimensions (§3.2).

## 1. Optimal control

The optimal control problem in Lagrange form consists in the minimization of an integral functional

$$J\left[x(\cdot), u(\cdot)\right] = \int_0^T L\left(t, x(t), u(t)\right) \mathrm{d}t \tag{2}$$

among all the solutions of a differential equation

$$x'(t) = \varphi\left(t, x(t), u(t)\right), \quad t \in [0, T] \tag{3}$$

subject to the boundary conditions

$$x(0) = \alpha, \quad x(T) = \beta. \tag{4}$$

The Lagrangian $L$ and the velocity function $\varphi$ are defined on $[a, b] \times \mathbb{R}^n \times \Omega$, where $\Omega \subseteq \mathbb{R}^r$ is called the control set. The main difference between the problems of optimal control and those of the calculus of variations, is that $\Omega$ is in general not an open set. In the case $\varphi(t, x, u) = u$, and $\Omega = \mathbb{R}^n$, one gets the fundamental problem of the calculus of variations. For the Newton problem, we have $n = r = 1$, $\Omega = \mathbb{R}_0^+$, $\varphi(t, x, u) = u$, $\alpha = 0$, $\beta > 0$, and $L(t, x, u) = \frac{t}{1+u^2}$. Typically, $L(t, x, u)$ and $\varphi(t, x, u)$ are continuous with respect to all arguments and have continuous derivatives with respect to $x$; the admissible processes $(x(\cdot), u(\cdot))$ are formed by absolutely continuous state trajectories $x(\cdot)$ and measurable and bounded controls $u(\cdot)$, taking values on the control set $\Omega$ and satisfying (3)-(4).

The Pontryagin maximum principle is a first-order necessary optimality condition, which provides a generalization of the classical Euler-Lagrange equations and Weierstrass condition, to problems in which upper and/or lower bounds are imposed on the control variables.

THEOREM 1 (Pontryagin maximum principle). *Let $(x(\cdot), u(\cdot))$ be a minimizer of the optimal control problem. Then there exists a pair $(\psi_0, \psi(\cdot))$, where $\psi_0 \leq 0$ is a constant and $\psi(\cdot)$ an n-vector absolutely continuous function with domain* $[0, T]$, not all zero, *such that the following holds true for almost all $t$ on the interval* $[0, T]$*:*

**(i)** *the Hamiltonian system*

$$\begin{cases} x'(t) = \frac{\partial \mathcal{H}}{\partial \psi}\left(t, x(t), u(t), \psi_0, \psi(t)\right) , \\ \psi'(t) = -\frac{\partial \mathcal{H}}{\partial x}\left(t, x(t), u(t), \psi_0, \psi(t)\right) ; \end{cases}$$

**(ii)** *the maximality condition*

$$\mathcal{H}\left(t, x(t), u(t), \psi_0, \psi(t)\right) = \max_{u \in \Omega} \mathcal{H}\left(t, x(t), u, \psi_0, \psi(t)\right) ; \tag{5}$$

*where the Hamiltonian $\mathcal{H}$ is defined by*

$$\mathcal{H}(t, x, u, \psi_0, \psi) = \psi_0 L(t, x, u) + \psi \cdot \varphi(t, x, u) .$$

The first equation in the Hamiltonian system is just the control equation (3). The second equation is known as the *adjoint system*.

DEFINITION 1. *A quadruple $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ satisfying the Hamiltonian system and the maximality condition is called a Pontryagin extremal. The control $u(\cdot)$ is said to be an extremal control. The extremals are said to be abnormal when $\psi_0 = 0$ and normal otherwise.*

REMARK 1. If $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ is a Pontryagin extremal, then, for any $\gamma > 0$, $(x(\cdot), u(\cdot), \gamma\, \psi_0, \gamma\, \psi(\cdot))$ is also a Pontryagin extremal. From this simple observation one can consider, without any loss of generality, that $\psi_0 = -1$ in the normal case.

REMARK 2. The fact that Theorem 1 asserts the existence of Hamiltonian multipliers $\psi_0$ and $\psi(\cdot)$ not vanishing simultaneously is of primordial importance: without this condition, all admissible pairs $(x(\cdot), u(\cdot))$ would be Pontryagin extremals.

In some situations, it may happen that functions $L$ and/or $\varphi$ depend upon some parameters $\omega \in W \subseteq \mathbb{R}^k$. In this case, given a control $u(\cdot)$, the corresponding state trajectory $x(\cdot)$ and the cost functional $J$ will in general depend on the choice of the parameters $\omega$. The problem in then to choose the parameters $\tilde{\omega}$ in $W$ for which there exists an admissible pair $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ such that $J\left[\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{\omega}\right] \leq J\left[x(\cdot), u(\cdot), \omega\right]$ for all $\omega \in W$ and corresponding admissible pairs $(x(\cdot), u(\cdot))$. The parameter problem can be reformulated in the format (2)–(3) by considering $\omega$ as a state variable with dynamics $\omega'(t) = 0$ and initial condition $\omega(0) \in W$.

## 2. Optimal control of Newton-type problems

The standard method to solve a problem in optimal control proceeds by first proving that a solution to the problem exists; then assuring the applicability of the Pontryagin maximum principle; and, finally, identifying the Pontryagin extremals (the candidates). Further elimination, if necessary, identifies the minimizer or minimizers of the problem. It is not easy to prove existence for Newton's problem with the classical arguments, because the Lagrangian $L(t,u) = \frac{t}{1+u^2}$ is not coercive and it is not convex with respect to $u$ for $u \geq 0$. Here we will make use of a different approach. We will show, by a simple and direct argument, that for Newton-type problems (6)–(7) the Pontryagin extremals are absolute minimizers. This means that, in order to solve a Newton-type problem, it is enough to identify the Pontryagin extremals (cf. Theorem 2).

We begin to show that there are no abnormal extremals for a Newton-type problem.

PROPOSITION 1. *Let $L(t,u)$ be a continuous function satisfying the following conditions:*

$$L(t,u) > \xi \geq 0 \quad \forall\,(t,u) \in \,]0,T] \times \mathbb{R}_0^+\,, \qquad \lim_{u\to+\infty} L(t,u) = \xi \quad \forall\, t \in [0,T]\,. \tag{6}$$

*Then all Pontryagin extremals $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the Newton-type problem*

$$\begin{gathered} J[u(\cdot)] = \int_0^T L\,(t,u(t))\,\mathrm{d}t \longrightarrow \min\,, \\ x'(t) = u(t)\,, \quad u(t) \geq 0\,, \\ x(0) = 0\,, x(T) = \beta \text{ with } \beta > 0\,, \end{gathered} \tag{7}$$

*are normal extremals ($\psi_0 = -1$) with $\psi$ a negative constant ($\psi(t) \equiv -\lambda$, $\lambda > 0$).*

*Proof.* As far as the Hamiltonian does not depend on $x$,

$$\mathcal{H}(t,u,\psi_0,\psi) = \psi_0 L(t,u) + \psi u\,,$$

we conclude from the adjoint system that $\psi(t) \equiv c$, with $c$ a constant. If $c$ is equal to zero, then $\psi_0 < 0$ (they are not allowed to be both zero) and the maximality condition (5) simplifies to

$$\psi_0 L(t,u(t)) = \max_{u\geq 0}\{\psi_0 L(t,u)\}\ .$$

Under the hypotheses (6) the maximum is not achieved ($u \to +\infty$) and we conclude that $c \neq 0$. Similarly, for $c > 0$ the maximum

$$\max_{u\geq 0}\{\psi_0 L(t,u) + cu\}$$

does not exist and one concludes that $c < 0$. It remains to prove that $\psi_0$ is different from zero. Indeed, if $\psi_0 = 0$, (5) reads

$$cu(t) = \max_{u\geq 0}\{cu\}\ ,$$

and follows that $u(t) \equiv 0$ and $x(t) \equiv$ constant. This is not a possibility since $\beta > 0$. The proof is complete. □

Theorem 2 reduces the procedure of solving a Newton-type problem to the computation of Pontryagin extremals.

THEOREM 2. *The control* $\tilde{u}(\cdot)$ *is an absolute minimizing control for the Newton-type problem* (6)–(7)*, i.e.,* $J[\tilde{u}(\cdot)] \leq J[u(\cdot)]$ *for all* $u(\cdot) \in L_\infty\left([0,T],\mathbb{R}_0^+\right)$*, if, and only if, it is an extremal control.*

*Proof.* Theorem 2 is a direct consequence of the maximality condition. From Proposition 1 one can write (5) as

$$-L(t,\tilde{u}(t)) - \lambda\tilde{u}(t) \geq -L(t,u(t)) - \lambda u(t) \tag{8}$$

for all admissible controls $u(\cdot)$ and for almost all $t \in [0,T]$. Having in mind that all admissible processes $(x(\cdot),u(\cdot))$ of the problem (7) satisfy

$$\int_0^T u(t)\mathrm{d}t = \int_0^T x'(t)\mathrm{d}t = \beta\,,$$

it is enough to integrate (8) to obtain the desirable conclusion:

$$\int_0^T L(t,\tilde{u}(t))\mathrm{d}t \leq \int_0^T L(t,u(t))\mathrm{d}t\,.$$

□

The required optimal solutions of the Newton-type problem (6)–(7) are exactly the Pontryagin extremals. This means, essentially, that we have reduced a dynamic optimization problem (a minimization problem in the space of functions) to the static optimization problem given by the maximality condition.

COROLLARY 1. *Finding the solutions for the Newton-type problem* (6)–(7) *amounts to find the minimum of the function* $h(u) = L(t,u) + \lambda u$*,* $t \in [0,T]$*,* $\lambda > 0$*, for* $u \geq 0$*.*

## 3. An application

Consider a $d$-dimensional body of revolution

$$\{(\xi_0,\ \xi) : \xi_0 \in [0,\ H],\ |\xi| < \Phi(\xi_0)\} \subset \mathbb{R}^d,$$

where $d \geq 2$, $\xi = (\xi_1,\ldots,\xi_{d-1})$, $\Phi$ is a non-negative function defined on $[0,\ H]$. Denote by $T$ the radius of maximal cross section of the body, $T = \max_{0\leq\zeta\leq H}\Phi(\zeta)$. Let us assume that the body is convex, then the function $\Phi$ is concave, and there exists $c \in [0,\ H]$ such that $\Phi(\zeta)$ is monotone increasing as $\zeta \leq c$, and monotone decreasing as $\zeta \geq c$.

We suppose that the body is unmovable and is situated in a flux of infinitesimal particles. The flux is invariant with respect to translations and rotations around the

$\xi_0$-axis, which is the symmetry axis of the body. So, the specific pressure of the flux on an infinitesimal element of the body surface depends only on the value of $\Phi'$ at that element. It is convenient to consider, instead of $\Phi$, two functions that are generalized inverses of $\Phi$; denote them by $x_-(t)$ and $H - x_+(t)$. They are defined in the following way: $x_-(t) = 0$ as $t \in [0,\ \Phi(0)]$, and $x_-(t)$ is inverse to the strictly monotone increasing branch of $\Phi$ as $t \in [\Phi(0),\ T]$; $x_+(t) = 0$ as $t \in [0,\ \Phi(H)]$, and $H - x_+(t)$ is inverse to the strictly monotone decreasing branch of $\Phi$ as $t \in [\Phi(H),\ T]$. The obtained functions $x_-$ and $x_+$ are convex, continuous, and monotone increasing, besides $x_-(0) = x_+(0) = 0$, $x_-(T) \le c$, $x_+(T) \le H - c$. In such a representation, the specific pressure is a function of $x'_+$ or of $x'_-$, if the point belongs to the front or to the rear part of surface, respectively; we denote the corresponding functions by $p_+(\cdot)$ and $-p_-(\cdot)$.

The pressure on an element $d^{d-1}s$ of the front part of surface is $dp_+ = p_+(x'_+)d^{d-1}s$. The projection of the pressure vector to the $\xi_0$-axis equals $dp_0 = dp_+/\sqrt{1 + x'^2_+}$, and the projection of the surface element to $\mathbb{R}^{d-1}_{\xi_1 \ldots \xi_{d-1}}$ has area $d^{d-1}\xi = d^{d-1}s/\sqrt{1 + x'^2_+}$. Thus, the $\xi_0$-projection of pressure corresponding to the element $d^{d-1}\xi$ is $dp_0 = p_+(x'_+)d^{d-1}\xi$. Passing to polar coordinates and integrating over the ball $\{|\xi| < T\}$, one obtains the resistance $\mathcal{R}_+$ of the front part of body to the flux:

$$\mathcal{R}_+[x_+(\cdot)] = \upsilon_{d-1} \int_0^T p_+(x'_+(t))\, dt^{d-1},$$

here $\upsilon_{d-1}$ stands for the volume of $(d-1)$-dimensional unit ball. Similarly, the resistance of the rear part of body to the flux (which is positive) equals $-\mathcal{R}_-[x_-(\cdot)]$, where

$$\mathcal{R}_-[x_-(\cdot)] = \upsilon_{d-1} \int_0^T p_-(x'_-(t))\, dt^{d-1}.$$

So, the resistance of body to the flux is $\mathcal{R}[x_+(\cdot), x_-(\cdot)] = \mathcal{R}_+[x_+(\cdot)] + \mathcal{R}_-[x_-(\cdot)]$.

It is required to minimize $\mathcal{R}[x_+(\cdot), x_-(\cdot)]$ over all pairs $(x_+(\cdot),\ x_-(\cdot))$ of convex monotone increasing functions defined on $[0,\ T]$, provided $x_\pm$ take values in $[0,\ \beta_\pm]$, where $\beta_- = c$, $\beta_+ = H - c$, $T$ and $H$ are fixed, and $c$ varies between 0 and $H$.

We are acting as follows. First we fix the sign "+" or "−", minimize $\mathcal{R}_\pm$ over monotone increasing functions $x : [0,\ T] \mapsto [0,\ \beta_\pm]$, with $\beta_\pm$ fixed, and verify that among all the solutions, the convex one is unique; denote it by $x_\pm^{\beta_\pm}$. Then we minimize the sum $\mathcal{R}_+[x_+^{\beta_+}(\cdot)] + \mathcal{R}_-[x_-^{\beta_-}(\cdot)]$ over all positive $\beta^+$ and $\beta^-$ such that $\beta^+ + \beta^- = H$.

### 3.1. Solving the problem in general case

In what follows, we assume that the functions $p_+$ and $p_-$ satisfy the following conditions:

(i) $p_\pm \in C^1[0,\ +\infty)$;

(ii) there exist $\lim_{u \to +\infty} p_\pm(u)$;

(iii) $p'_\pm(0) = \lim_{u\to+\infty} p'_\pm(u) = 0$;

(iv) for some $\bar{u}_\pm > 0$, $p'_\pm$ is strictly monotone decreasing on $[0, \bar{u}_\pm]$, and strictly monotone increasing on $[\bar{u}_\pm, +\infty)$.

For simplicity, we further put $\upsilon_{d-1} = 1$. Let us fix the sign "+" or "−", and introduce shorthand notations

$$p_\pm = p, \quad \beta_\pm = \beta, \quad \mathcal{R}_\pm = \mathcal{R}, \quad x_\pm^{\beta_\pm} = x^\beta.$$

PROPOSITION 2. *There exists a unique solution $u^0$ of the problem*

$$\frac{p(0) - p(u)}{u} \to \max,$$

*besides $u^0 > \bar{u}$.*

*Proof.* Denote $q(u) = p(0) - p(u)$ and $B = \sup_{u>0}(q(u)/u)$. From (i)–(iv) it follows that the function $q(u)/u$, $u > 0$ is continuous, positive, and satisfies the relations $\lim_{u\to0+}(q(u)/u) = \lim_{u\to+\infty}(q(u)/u) = 0$, hence $0 < B < +\infty$, and there exists a value $u^0 > 0$ such that $q(u^0)/u^0 = B$. Obviously, at $u = u^0$ one has $(q(u)/u)' = 0$, hence $q'(u^0) = q(u^0)/u^0$. At some $\theta \in (0, 1)$ one has $q(u^0)/u^0 = q'(\theta u^0)$, hence $q'(u^0) = q'(\theta u^0)$. This implies that $q'$ is not strictly monotone on $[0, u^0]$; thus, by virtue of (iv), $u^0 > \bar{u}$.

It remains to prove that the value $u^0$, solving the equation $q(u)/u = B$, is unique. Suppose that $q(u^0)/u^0 = q(u^1)/u^1 = B$, $u^0 < u^1$. Then $q(u^0) = u^0\, q'(u^0)$, $q(u^1) = u^1\, q'(u^1)$. At some $u \in (u^0, u^1)$, one has $q(u^1) - q(u^0) = q'(u)\,(u^1 - u^0)$; this implies that $q'(u)\,(u^1 - u^0) = u^1\, q'(u^1) - u^0\, q'(u^0)$, hence

$$u^0\,(q'(u^0) - q'(u)) + u^1\,(q'(u) - q'(u^1)) = 0. \tag{9}$$

One has $u^0 > \bar{u}$, hence $q'$ is strictly monotone decreasing as $u \geq u^0$, so both terms in (9) are positive. The obtained contradiction proves the proposition. □

Let us denote

$$B = \frac{p(0) - p(u)}{u} = -p'(u).$$

PROPOSITION 3. *(a) As $\lambda\, t^{2-d} > B$, the unique solution of the problem*

$$t^{d-2}\, p(u) + \lambda u \to \min; \tag{10}$$

*is $u = 0$.*

*(b) As $\lambda\, t^{2-d} = B$, there are two solutions: $u = 0$ and $u = u^0$.*

*(c) As $\lambda\, t^{2-d} < B$, the solution $\tilde{u}$ is unique, besides $\tilde{u} > u^0$, and $p'(\tilde{u}) = -\lambda t^{2-d}$.*

*Proof.* (a) and (b) are obvious; let us prove (c). Denote $\tilde{\lambda} := \lambda\, t^{2-d}$. By definition of $B$, for $0 < u < u^0$ one has

$$\frac{p(0) - p(u)}{u} < B = \frac{p(0) - p(u^0)}{u^0}\,,$$
$$p(u^0) + Bu^0 = p(0) < p(u) + Bu\,,$$

hence

$$p(u) - p(u^0) > B\,(u^0 - u) > \tilde{\lambda}\,(u^0 - u),$$

and thus,

$$p(u) + \tilde{\lambda}u > p(u^0) + \lambda u^0.$$

On the other hand, one has $B = -p'(u^0)$, therefore

$$p'(u^0) + \tilde{\lambda} < 0;$$

moreover the function $p(u) + \tilde{\lambda}u$ is convex on $[u^0,\ +\infty)$ and tends to infinity as $u \to +\infty$. All this implies that the solution $\tilde{u}$ of (10) is unique, satisfies the equation $p'(\tilde{u})+\tilde{\lambda} = 0$, and $\tilde{u} > u^0$. □

From Corollary 1 we know that if $x^\beta(\cdot)$ is a solution of the minimization problem $\mathcal{R}[x(\cdot)] \to \min,\ x : [0,\ T] \to [0,\ \beta]$, then for some $\lambda$, the values $u = {x^\beta}'(t)$, $t \in [0,\ T]$ satisfy the equation (10). According to propositions 2 and 3, one should distinguish between three cases: (a) if $\lambda t^{2-d} > B$, then $u = 0$; (b) if $\lambda t^{2-d} = B$, then $u = 0$ or $u = u^0$; (c) if $\lambda t^{2-d} < B$, then $u > u^0$, and $p'(u) = -\lambda t^{2-d}$.

Consider two different cases: $d = 2$ (two-dimensional problem) and $d \geq 3$ (the problem in three or more dimensions).

### Two-dimensional problem ($d = 2$)

If $\lambda > B$, the unique solution of (10) is $u = 0$, hence $x^\beta \equiv 0$. This implies that $\beta = 0$.

If $\lambda = B$, there are two solutions: $u = 0$ and $u = u^0$, therefore any absolutely continuous function $x(\cdot),\ x(0) = 0,\ x(T) = \beta,$ such that $x'(t)$ takes the values 0 and $u^0$, minimizes $\mathcal{R}$. A convex solution $x^\beta$ has monotone increasing derivative, hence for some $t_0 \in [0,\ T]$, ${x^\beta}'(t) = 0$ as $t \in [0,\ t_0]$, and ${x^\beta}'(t) = u^0$ as $t \in [t_0,\ T]$. Thus,

$$x^\beta(t) = \begin{cases} 0 & \text{as } t \in [0,\ t_0] \\ u^0\,(t - t_0) & \text{as } t \in [t_0,\ T]. \end{cases} \tag{11}$$

Taking into account that $x^\beta(T) = \beta$, one concludes that $\beta/T \leq u^0$ and $t_0 = T - \beta/u^0$.

If $\lambda < B$, there is a unique solution $\tilde{u}$, hence $x^\beta(t) = \tilde{u}t$, and $\beta/T = \tilde{u} > u^0$.

Summarizing, one gets that

(i) as $\beta/T < u^0$, the convex solution $x^\beta(t)$ is given by (11);

(ii) as $\beta/T \geq u^0$, $x^\beta(t) = \beta\, t/T$.

As $\beta/T < u^0$, one has

$$\mathcal{R}[x^\beta(\cdot)] = \int_0^{t_0} p(0)\,dt + \int_{t_0}^{T} p(u^0)\,dt,$$

and taking into account that $t_0 = T - \beta/u^0$ and $(p(0) - p(u^0))/u^0 = B$, one gets

$$\mathcal{R}[x^\beta(\cdot)] = T\ p(0) - \beta\, B.$$

As $\beta/T \geq u^0$, one has $\mathcal{R}[x^\beta(\cdot)] = T\ p(\beta/T)$. Introduce the function

$$\bar{p}(u) = \begin{cases} p(0) - B\,u, & \text{if } 0 \leq u \leq u^0, \\ p(u), & \text{if } u \geq u^0, \end{cases}$$

then

$$\mathcal{R}[x^\beta(\cdot)] = T\ \bar{p}(\beta/T).$$

Thus, the minimization problem $\mathcal{R}_+[x_+^{\beta_+}(\cdot)] + \mathcal{R}_-[x_-^{\beta_-}(\cdot)] \to \min$ is reduced to the problem

$$p_h(z) = \bar{p}_+(z) + \bar{p}_-(h - z) \to \min, \qquad 0 \leq z \leq h, \tag{12}$$

where $h = H/T$. The introduced functions $\bar{p}_\pm(u)$ are continuously differentiable on $[0, +\infty)$, and

$$\bar{p}'_\pm(u) = \begin{cases} -B_\pm & \text{if } 0 \leq u \leq u^0_\pm, \\ p'_\pm(u) & \text{if } u > u^0_\pm. \end{cases}$$

Using that $u^0_\pm > \bar{u}_\pm$, one concludes that $\bar{p}'_\pm(u)$ is monotone increasing, hence $p'_h(z)$, $0 \leq z \leq h$ is also monotone increasing.

From now and until the end of subsection 3.1, we shall assume that $p'_+(u) < p'_-(u)$, $u \geq 0$, hence $B_+ > B_-$. Denote by $u_*$ a positive value such that $\bar{p}'_+(u_*) = -B_-$. This value is unique, and $u_* > u^0_+$. Consider four cases:
1) $0 < h < u^0_+$; 2) $u^0_+ \leq h \leq u_*$; 3) $u_* < h < u_* + u^0_-$; and 4) $h \geq u_* + u^0_-$.

In the cases 1) and 2) one has

$$p'_h(z) \leq p'_h(h) = \bar{p}'_+(h) + B_- \leq 0 \quad \text{as} \quad 0 \leq z \leq h,$$

hence the minimum of (12) is achieved at $z = h$. Therefore, the optimal value of $\beta_-$ is zero, so $x_-^{\beta_-} \equiv 0$.

In the case 1) one has $\beta_+/T = h < u^0_+$, hence $x_+^{\beta_+}(\cdot)$ is given by (11), with $t_0 = T\,(1 - h/u^0_+)$. So, the optimal body is a trapezium.

In the case 2) one has $x_+^{\beta_+}(t) = ht$, hence the optimal body is an isosceles triangle.

In the cases 3) and 4), one has $\bar{p}'_+(h) > -B_- > -B_+ = \bar{p}'_+(u^0_+)$, hence $h > u^0_+$. Further, one has

$$p'_h(h) = \bar{p}'_+(h) - B_- > 0;$$

on the other hand,

$$p_h'(u_+^0) = \bar{p}_+'(u_+^0) - \bar{p}_-'(h - u_+^0) \le -B_+ + B_- < 0.$$

It follows that the minimum of $p_h$ is achieved at an interior point of $[u_+^0,\ h]$, so the optimal value of $\beta_-$ satisfies the relation $u_+^0 < \beta_+/T < h$, and $x_+^{\beta_+}(t) = t\,\beta_+/T$.

In the case 3), denoting $\tilde{h} = \max\{0,\ h - u_-^0\}$, one has $\tilde{h} < u_*$, hence

$$p_h'(\tilde{h}) = \bar{p}_+'(\tilde{h}) - \bar{p}_-'(h - \tilde{h}) \le \bar{p}_+'(\tilde{h}) + B_- < 0,$$

hence the minimum of $p_h$ is reached at an interior point of $[\tilde{h},\ h]$, thus $0 < \beta_-/T < h - \tilde{h} \le u_-$, and

$$x_-^{\beta_-}(t) = \begin{cases} 0 & \text{if } t \in [0,\ T - \beta_-/u_-^0] \\ u_-^0\,(t - T + \beta_-/u_-^0) & \text{if } t \in [T - \beta_-/u_-^0,\ T]. \end{cases}$$

The optimal body here is the union of a triangle and a trapezium turned over.

In the case 4), one has $p_h'(h-u_-^0) = \bar{p}_+'(h-u_-) + B_- \ge 0$, hence the minimum of $p_h$ is reached at a point of $[u_+^0,\ h-u_-^0)$. Thus, $\beta_-/T > u_-^0$, and $x_-^{\beta_-}(t) = t\,\beta_-/T$. The optimal body is a union of two isosceles triangles with common base.

**Problem in three or more dimensions ($d \ge 3$)**

Here we additionally assume that $p_\pm \in C^2[0,\ +\infty)$ and $p_+''(u) > 0$ as $u > u_+^0$.

Denote $\omega = 1/(d-2)$ and $t_0 = (\lambda/B)^\omega$. As $0 \le t < t_0$, the unique solution of (10) is $u = 0$, hence $x^\beta(t) = 0$. As $t_0 < t \le T$, the solution $u$ satisfies the relation

$$t^{d-2}\,p'(u) + \lambda = 0,$$

and $u > \tilde{u}$.

If $T \le t_0$, one has $x^\beta \equiv 0$ and $\beta = 0$. Let, now, $t_0 < T$; using that $p'$ is negative, continuous, and strictly monotone increasing on $[u^0,\ +\infty)$, one concludes that $x^{\beta\,\prime}(t) = u$ is also continuous, and is strictly monotone increasing on $[t_0,\ T)]$ from $x^{\beta\,\prime}(t_0+) = u^0$ to the value $U$ defined from the relation

$$T^{d-2}\,p'(U) + \lambda = 0, \quad U > u^0.$$

Thus, $x^\beta(\cdot)$ is convex; moreover, as $t_0 \le t \le T$, $x^\beta$ can be represented as a function of $u \in [u^0,\ U]$. Using that $x^{\beta\,\prime}(t) = u$ and that

$$t = \frac{\lambda^\omega}{|p'(u)|^\omega}, \tag{13}$$

one gets

$$\frac{dx^\beta}{du} = \frac{dx^\beta}{dt}\frac{dt}{du} = u\,\lambda^\omega\,\frac{d}{du}\left(\frac{1}{|p'(u)|^\omega}\right),$$

hence

$$x^\beta = \lambda^\omega \int_{u^0}^{u} \nu\, d\left(\frac{1}{|p'(\nu)|^\omega}\right);$$

using that $|p'(u^0)| = B$, one obtains

$$x^\beta = \lambda^\omega \left(\frac{u}{|p'(u)|^\omega} - \frac{u^0}{B^\omega} - \int_{u^0}^{u} \frac{d\nu}{|p'(\nu)|^\omega}\right).$$

In particular, substituting $u = U$, one has

$$\lambda^\omega \left(\frac{U}{|p'(U)|^\omega} - \frac{u^0}{B^\omega} - \int_{u^0}^{U} \frac{d\nu}{|p'(\nu)|^\omega}\right) = \beta. \tag{14}$$

Introduce the function

$$g(u) = \int_0^u \frac{d\nu}{|\bar{p}'(\nu)|^\omega}.$$

Using that $|\bar{p}'(\nu)| = B$ as $0 \le \nu \le u^0$, and $\bar{p}'(\nu) = p'(\nu)$ as $\nu \ge u^0$, one gets

$$g(U) = \frac{u^0}{B^\omega} + \int_{u^0}^{U} \frac{d\nu}{|p'(\nu)|^\omega},$$

and using that

$$T = \frac{\lambda^\omega}{|p'(U)|^\omega},$$

from (14) one gets

$$\frac{\beta}{T} = U - |p'(U)|^\omega\, g(U). \tag{15}$$

The minimal resistance equals

$$\mathcal{R}[x^\beta(\cdot)] = \int_0^T p(u(t))\, dt^{d-1} = p(0)\, t_0^{d-1} + \int_{t_0}^{T} p(u(t))\, dt^{d-1}.$$

Using that $u(T) = U,\ u(t_0) = u^0,\ |p'(u^0)| = B,\ p(0) - p(u^0) = B\, u^0,\ t_0 = \lambda^\omega / B^\omega$, and also the formula (13), one finds

$$\begin{aligned}\mathcal{R}[x^\beta(\cdot)] &= \lambda^{1+\omega} \left\{ \frac{p(0)}{B^{1+\omega}} + \frac{p(U)}{|p'(U)|^{1+\omega}} - \frac{p(u^0)}{B^{1+\omega}} - \int_{u^0}^{U} \frac{dp(u)}{|p'(u)|^{1+\omega}} \right\} \\ &= \lambda^{1+\omega} \left\{ \frac{u^0}{B^\omega} + \frac{p(U)}{|p'(U)|^{1+\omega}} + \int_{u^0}^{U} \frac{du}{|p'(u)|^\omega} \right\}.\end{aligned}$$

This implies

$$\frac{\mathcal{R}[x^\beta(\cdot)]}{T^{d-1}} = p(U) + |p'(U)|^{1+\omega}\, g(U). \tag{16}$$

Denote $U_+ = z_+,\ U_- = z_-$. Using (15) and (16), one comes to the following problem of conditional minimum

$$r(z_-, z_+) := \\ p_+(z_+) + |p'_+(z_+)|^{1+\omega}\, g_+(z_+) \ +\ p_-(z_-) + |p'_-(z_-)|^{1+\omega}\, g_-(z_-) \to \min,$$

under the conditions

$$(17)\quad z_- - |p'_-(z_-)|^{\omega}\, g_-(z_-) \,+\, z_+ - |p'_-(z_+)|^{\omega}\, g_+(z_+) = h, \quad z_- \ge u^0,\ \ z_+ \ge u^0.$$

From (17), taking into account that $|p'_\pm(z_\pm)|^{\omega}\, g'_\pm(z_\pm) = 1$, one obtains that $z_+$ is a differentiable function of $z_-$, and

$$\frac{dz_+}{dz_-} = -\frac{|p'_-(z_-)|^{\omega-1}\, p''_-(z_-)\, g_-(z_-)}{|p'_+(z_+)|^{\omega-1}\, p''_+(z_+)\, g_+(z_+)}.$$

Now,

$$\begin{aligned}\frac{d}{dz_-}\, r(z_-, z_+(z_-)) = -\frac{dz_+}{dz_-} &\cdot (1+\omega)\, |p'_+(z_+)|^{\omega}\, p''_+(z_+)\, g_+(z_+) \\ &- (1+\omega)\, |p'_-(z_-)|^{\omega}\, p''_-(z_-)\, g_-(z_-) \\ = (1+\omega)\, |p'_-(z_-)|^{\omega-1}\, &p''_-(z_-)\, g_-(z_-) \cdot (p'_-(z_-) - p'_+(z_+)).\end{aligned}$$

Note that $z_+$ is a monotone decreasing function of $z_-$, hence the function $p'_-(z_-) - p'_+(z_+(z_-))$ is monotone increasing as $z_- \ge u^0,\ z_+(z_-) \ge u^0$.

Recall that $u_*$ is the value satisfying $\bar{p}'_+(u_*) = -B_-$. Denote

$$h_* = u_* - B_-^{\omega}\, g_+(u_*).$$

Consider two cases.

1) $h \le h_*$. One has $z_+(u^0) \le u_*$, hence $p'_-(u^0) - p'_+(z_+(u^0)) \ge -B_- - p'_+(u_*) = 0$. It follows that as $z_- > u^0$, $p'_-(z_-) - p'_+(z_+(z_-)) > 0$, so the minimum of $r(z_-, z_+)$is attained at $z_- = 0$.

2) $h > h_*$. One has $z_+(u^0) > u_*$, hence $p'_-(u^0) - p'_+(z_+(u^0)) < 0$. On the other hand, as $\tilde{z}_- = z_+(\tilde{z}_-)$, one has $p'_-(\tilde{z}_-) - p'_+(\tilde{z}_-) < 0$, hence at some $z_- \in (u^0,\ \tilde{z}_-)$, $p'_-(z_-) = p'_+(z_+(z_-))$, and so, the minimum of resistance is attained.

### 3.2. Examples

We have given in §3.1 complete description of the solutions to the formulated Newton-type problem. We now consider, for illustration purposes, various particular cases of the problem. All the calculations can easily be done with the help of a computer algebra system. We have used Maple to implement a procedure which, given functions $p_+(\cdot)$ and $p_-(\cdot)$ and the values for $T$ and $H$, gives the optimal shape for the respective problem.

**Non-parallel flux of particles**

Let us consider the two-dimensional case ($d = 2$). As proved in §3.1, there exist four possible cases. To illustrate this we choose, as an example, the pressure of the front part of the surface to be $p_+ = \frac{1}{1+u^2} + 0.5$; the pressure on the rear part given by $p_- = \frac{0.5}{1+u^2} - 0.5$; the radius $T$ of the maximal cross section of the body to be two ($T = 2$); and then we choose different values for the height $H$ of the body. Applying the formulas given in §3.1 one obtains that for $H = 1$ the solution is a trapezium (Fig. 1); for $H = 2$ a triangle (Fig. 2); for $H = 4$ the union of a triangle and a trapezium turned over (Fig. 3); and for $H = 6$ the union of two triangles with common base (Fig. 4).

We remark that in Newton's problem one has $p_+ = \frac{1}{1+u^2}$ and $p_- = 0$ (parallel flux), and only the first two situations occur: solution to Newton's two-dimensional problem is either a trapezium or a triangle.

The two-dimensional problem under a non-parallel flux of particles with density of distribution over velocities circular gaussian, with biased mean, is studied in [21].

**Newton's classical problem**

We now obtain the well-known Newton's solution. For that we fix $d = 3$, $p_+(u) = 1/(1+u^2)$, and $p_-(u) = 0$. Applying the method described in §3.1, after some algebra one obtains $\bar{u}_+ = 1/\sqrt{3}$, $u^0 = 1$, $B_+ = 1/2$, $\beta = H$, and the optimal solution $x(t)$ is given in parametric form by

$$x = \frac{\lambda}{2}\left(\frac{3u^4}{4} + u^2 - \ln u - \frac{7}{4}\right),$$

$$t = \frac{\lambda}{2}\left(u^3 + 2u + \frac{1}{u}\right), \quad 1 \le u \le U,$$

all in agreement with classical formulas. Expressing the formulas with respect to $U$ and $T$ one obtains:

$$\lambda = \frac{2TU}{\left(1+U^2\right)^2}, \quad t_0 = \frac{4TU}{\left(1+U^2\right)^2}, \quad \beta = \frac{TU\left(-7 + 4U^2 + 3U^4 - 4\ln(U)\right)}{4\left(1+U^2\right)^2},$$

$$t = \frac{TU\left(1+u^2\right)^2}{u\left(1+U^2\right)^2}, \quad x = \frac{TU\left(-7 + 4u^2 + 3u^4 - 4\ln(u)\right)}{4\left(1+U^2\right)^2},$$

$$\mathcal{R}_+ = \frac{T^2\left(17U^2 + 2 + 10U^4 + 3U^6 + 4\ln(U)U^2\right)}{4\left(1+U^2\right)^4}.$$

In this case $\mathcal{R}_- = 0$.

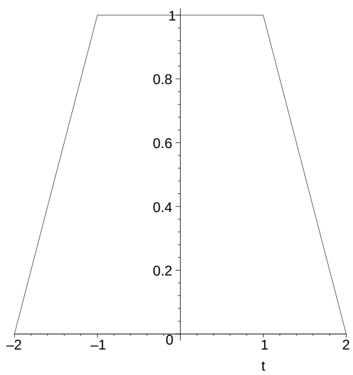


Figure 1: $H = 1$

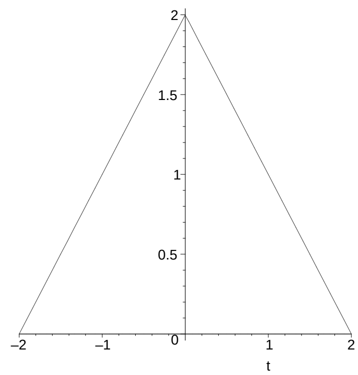


Figure 2: $H = 2$

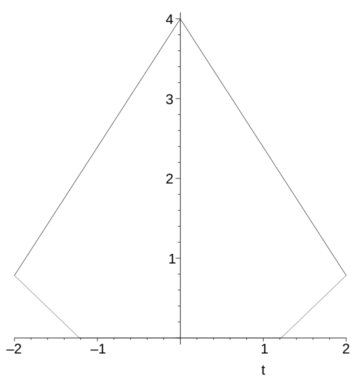


Figure 3: $H = 4$

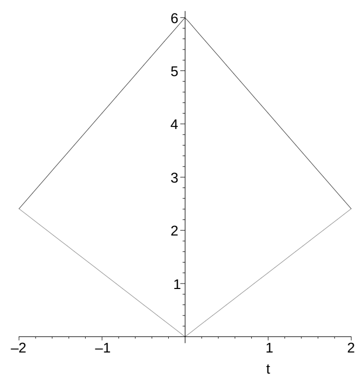


Figure 4: $H = 6$

Solutions of the two-dimensional Newton-type problem with $p_+ = \frac{1}{1+u^2} + 0.5$, $p_- = \frac{0.5}{1+u^2} - 0.5$ (non-parallel flux of particles), $T = 2$, and different values for the height $H$ of the body.

**Newton's problem in higher dimensions**

Our approach to Newton's problem is valid for an arbitrary $d \geq 2$. For example, for $d = 4$ (problem in dimension four) one gets:

$$\lambda = \frac{2\,T^2 U}{\left(1+U^2\right)^2}\,, \quad t_0 = \sqrt{\frac{4T^2U}{\left(1+U^2\right)^2}}\,, \quad \beta = \frac{T\left(-5\,U + 3\,U^3 + 2\sqrt{U}\right)}{5\left(1+U^2\right)}\,,$$

$$t = T\sqrt{\frac{U}{u}}\left(\frac{1+u^2}{1+U^2}\right), \quad x = \frac{T\,\sqrt{U}\left(-5\,\sqrt{u} + 3\,u^{5/2} + 2\right)}{5\left(1+U^2\right)}\,,$$

$$\mathcal{R}_+ = \frac{T^2\left(1+3U^2\right)}{2\left(1+U^2\right)^2}\,.$$

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