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A fractional calculus of variations for multiple integrals with application to vibrating string

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We introduce a fractional theory of the calculus of variations for multiple integrals. Our approach uses the recent notions of Riemann–Liouville fractional derivatives and integrals in the sense of Jumarie. The main results provide fractional versions of the theorems of Green and Gauss, fractional Euler–Lagrange equations, and fractional natural boundary conditions. As an application we discuss the fractional equation of motion of a vibrating string. © *2010 American Institute of Physics.* [doi:10.1063/1.3319559]

I. INTRODUCTION

The fractional calculus (FC) is one of the most interdisciplinary fields of mathematics, with many applications in physics and engineering. The history of FC goes back more than three centuries, when in 1695 the derivative of the order of $\alpha=1/2$ was described by Leibniz. Since then, many different forms of fractional operators were introduced: the Grunwald–Letnikov, Riemann–Liouville, Riesz, and Caputo fractional derivatives,[31,35,39] and the more recent notions of Klimek,[32] Cresson,[13] and Jumarie.[29,28,26,30]

FC is nowadays the realm of physicists and mathematicians, who investigate the usefulness of such noninteger order derivatives and integrals in different areas of physics and mathematics.[11,23,31] It is a successful tool for describing complex quantum field dynamical systems, dissipation, and long-range phenomena that cannot be well illustrated using ordinary differential and integral operators.[19,23,32,38] Applications of FC are found, e.g., in classical and quantum mechanics, field theories, variational calculus, and optimal control.[20,22,26]

Although FC is an old mathematical discipline, the fractional vector calculus is at the very beginning. We mention the recent paper,[42] where some fractional versions of the classical results of Green, Stokes, and Gauss are obtained via Riemann–Liouville and Caputo fractional operators. For the purposes of a multidimensional fractional calculus of variations (FCVs), the Jumarie fractional integral and derivative seems, however, to be more appropriate.

The FCV started in 1996 with the work of Riewe.[38] Riewe formulated the problem of the calculus of variations with fractional derivatives and obtained the respective Euler–Lagrange equations, combining both conservative and nonconservative cases. Nowadays, the FCV is a subject under strong research. Different definitions for fractional derivatives and integrals are used, depending on the purpose under study. Investigations cover problems depending on Riemann–Liouville fractional derivatives (see, e.g., Refs. 5, 19, and 21), the Caputo fractional derivative (see, e.g., Refs. 1, 7, and 8), the symmetric fractional derivative (see, e.g., Ref. 32), the

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Jumarie fractional derivative (see, e.g., Refs. 3 and 26), and others.[2,13,20] For applications of the FCV, we refer the reader to Refs. 17, 16, 19, 26, 32, 36, 37, and 40. Although the literature of FCV is already vast, much remains to be done.

Knowing the importance and relevance of multidimensional problems of the calculus of variations in physics and engineering,[43] it is at a first view surprising that a multidimensional FCV is a completely open research area. We are only aware of some preliminary results presented in Ref. 19, where it is claimed that an appropriate fractional variational theory involving multiple integrals would have important consequences in mechanical problems involving dissipative systems with infinitely many degrees of freedom, but where a formal theory for that is missing. There is, however, a good reason for such omission in literature: most of the best well-known fractional operators are not suitable for a generalization of the FCV to the multidimensional case due to lack of good properties, e.g., an appropriate Leibniz rule.

The main aim of the present work is to introduce a FCV for multiple integrals. For that we make use of the recent Jumarie fractional integral and derivative,[29,28,30] extending such notions to the multidimensional case. The main advantage of using Jumarie's approach lies in the following facts: the Leibniz rule for the Jumarie fractional derivative is equal to the standard one and, as we show, the fractional generalization of some fundamental multidimensional theorems of calculus is possible. We mention that Jumarie's approach is also useful for the one-dimensional FCV, as recently shown in Ref. 3 (see also Ref. 27).

The plan of the paper is as follows. In Sec. II some basic formulas of Jumarie's FC are briefly reviewed. Then, in Sec. III the differential and integral vector operators are introduced, and fractional Green's, Gauss's, and Stokes' theorems formulated. Section IV is devoted to the study of problems of FCVs with multiple integrals. Our main results provide Euler–Lagrange necessary optimality type conditions for such problems (Theorems 3 and 5) as well as natural boundary conditions (Theorem 4). We end with Sec. V of applications and future perspectives.

II. PRELIMINARIES

For an introduction to the classical FC, we refer the reader to Refs. 31, 34, 35, and 39. In this section we briefly review the main notions and results from the recent FC proposed by Jumarie.[28,26,30] Let $f:[0,1]\to\mathbb{R}$ be a continuous function and $\alpha\in(0,1)$. The Jumarie fractional derivative of f may be defined by

$$f^{(\alpha)}(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}(f(t)-f(0))dt. \tag{1}$$

One can obtain (1) as a consequence of a more basic definition, a local one, in terms of a fractional finite difference [cf. Eq. (2.2) in Ref. 30],

$$f^{(\alpha)}(x)=\lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f(x+(\alpha-k)h).$$

Note that the Jumarie and the Riemann–Liouville fractional derivatives are equal if $f(0)=0$. The advantage of definition (1) with respect to the classical definition of Riemann–Liouville is that the fractional derivative of a constant is now zero, as desired. An antiderivative of f, called the $(dt)^{\alpha}$ integral of f, is defined by

$$\int_0^x f(t)(dt)^{\alpha}=\alpha\int_0^x (x-t)^{\alpha-1}f(t)dt.$$

The following equalities can be considered as fractional counterparts of the first and the second fundamental theorems of calculus and can be found in Refs. 26 and 30,

$$\frac{d^\alpha}{dx^\alpha}\int_0^x f(t)(dt)^\alpha = \alpha! f(x), \tag{2}$$

$$\int_0^x f^{(\alpha)}(t)(dt)^\alpha = \alpha!(f(x)-f(0)), \tag{3}$$

where $\alpha! := \Gamma(1+\alpha)$. The Leibniz rule for the Jumarie fractional derivative is equal to the standard one,

$$(f(x)g(x))^{(\alpha)} = f^{(\alpha)}(x)g(x) + f(x)g^{(\alpha)}(x).$$

Here, we see another advantage of derivative (1): the fractional derivative of a product is not an infinite sum, in contrast to the Leibniz rule for the Riemann–Liouville fractional derivative (Ref. 35, p. 91).

One can easily generalize the previous definitions and results for functions with a domain $[a,b]$,

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha}(f(t)-f(a))dt$$

and

$$\int_a^x f(t)(dt)^\alpha = \alpha\int_a^x (x-t)^{\alpha-1}f(t)dt.$$

III. FRACTIONAL INTEGRAL THEOREMS

In this section we introduce some useful fractional integral and fractional differential operators. With them we prove fractional versions of the integral theorems of Green, Gauss, and Stokes. Throughout the text we assume that all integrals and derivatives exist.

A. Fractional operators

Let us consider a continuous function $f=f(x_1,\ldots,x_n)$ defined on $R=\Pi_{i=1}^n[a_i,b_i]\subset\mathbb{R}^n$. Let us extend Jumarie's fractional derivative and the $(dt)^\alpha$ integral to functions with n variables. For $x_i \in [a_i,b_i]$, $i=1,\ldots,n$, and $\alpha\in(0,1)$, we define the fractional integral operator as

$${}_{a_i}I^\alpha_{x_i}[i] = \alpha\int_{a_i}^{x_i}(x_i-t)^{\alpha-1}dt.$$

These operators act on f in the following way:

$${}_{a_i}I^\alpha_{x_i}[i]f(x_1,\ldots,x_n) = \alpha\int_{a_i}^{x_i} f(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)(x_i-t)^{\alpha-1}dt, \quad i=1,\ldots,n.$$

Let $\Xi=\{k_1,\ldots,k_s\}$ be an arbitrary nonempty subset of $\{1,\ldots,n\}$. We define the fractional multiple integral operator over the region $R_\Xi=\Pi_{i=1}^s[a_{k_i},x_{k_i}]$ by

$$I^\alpha_{R_\Xi}[k_1,\ldots,k_s] = {}_{a_{k_1}}I^\alpha_{x_{k_1}}[k_1]\cdots {}_{a_{k_s}}I^\alpha_{x_{k_s}}[k_s] = \alpha^s\int_{a_{k_1}}^{x_{k_1}}\cdots\int_{a_{k_s}}^{x_{k_s}}(x_{k_1}-t_{k_1})^{\alpha-1}\cdots(x_{k_s}-t_{k_s})^{\alpha-1}dt_{k_s}\cdots dt_{k_1},$$

which acts on f by

$$I^{\alpha}_{R_{\Xi}}[k_1,\ldots,k_s]f(x_1,\ldots,x_n)=\alpha^s\int_{a_{k_1}}^{x_{k_1}}\cdots\int_{a_{k_s}}^{x_{k_s}}f(\xi_1,\ldots,\xi_n)(x_{k_1}-t_{k_1})^{\alpha-1}\cdots(x_{k_s}-t_{k_s})^{\alpha-1}dt_{k_s}\cdots dt_{k_1},$$

where $\xi_j=t_j$ if $j\in\Xi$, and $\xi_j=x_j$ if $j\notin\Xi$, $j=1,\ldots,n$. The fractional volume integral of f over the whole domain R is given by

$$I^{\alpha}_{R}f=\alpha^n\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}f(t_1,\ldots,t_n)(b_1-t_1)^{\alpha-1}\cdots(b_n-t_n)^{\alpha-1}dt_n\cdots dt_1.$$

The fractional partial derivative operator with respect to the ith variable x_i, $i=1,\ldots,n$, of the order of $\alpha\in(0,1)$, is defined as follows:

$${}_{a_i}D^{\alpha}_{x_i}[i]=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x_i}\int_{a_i}^{x_i}(x_i-t)^{-\alpha}dt,$$

which acts on f by

$$\begin{aligned}{}_{a_i}D^{\alpha}_{x_i}[i]f(x_1,\ldots,x_n)=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x_i}\int_{a_i}^{x_i}&(x_i-t)^{-\alpha}[f(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)\\&-f(x_1,\ldots,x_{i-1},a_i,x_{i+1},\ldots,x_n)]dt,\quad i=1,\ldots,n.\end{aligned}$$

We observe that the Jumarie fractional integral and the Jumarie fractional derivative can be obtained by putting $n=1$,

$${}_aI^{\alpha}_{x}[1]f(x)=\alpha\int_a^x(x-t)^{\alpha-1}f(t)dt=\int_a^x f(t)(dt)^{\alpha}$$

and

$${}_aD^{\alpha}_{x}[1]f(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x(x-t)^{-\alpha}(f(t)-f(a))dt=f^{(\alpha)}(x).$$

Using these notations, formulas (2) and (3) can be presented as

$${}_aD^{\alpha}_{x}[1]\,{}_aI^{\alpha}_{x}[1]f(x)=\alpha!f(x),$$

$${}_aI^{\alpha}_{x}[1]\,{}_aD^{\alpha}_{x}[1]f(x)=\alpha!(f(x)-f(a)).\tag{4}$$

In the two dimensional case, we define the fractional line integral on ∂R, $R=[a,b]\times[c,d]$, by

$$I^{\alpha}_{\partial R}f=I^{\alpha}_{\partial R}[1]f+I^{\alpha}_{\partial R}[2]f,$$

where

$$I^{\alpha}_{\partial R}[1]f={}_aI^{\alpha}_{b}[1][f(b,c)-f(b,d)]=\alpha\int_a^b[f(t,c)-f(t,d)](b-t)^{\alpha-1}dt$$

and

$$I^{\alpha}_{\partial R}[2]f={}_cI^{\alpha}_{d}[2][f(b,d)-f(a,d)]=\alpha\int_c^d[f(b,t)-f(a,t)](d-t)^{\alpha-1}dt.$$

B. Fractional differential vector operations

Let $W_X=[a,x]\times[c,y]\times[e,z]$, $W=[a,b]\times[c,d]\times[e,f]$, and denote (x_1,x_2,x_3) by (x,y,z). We introduce the fractional nabla operator by

$$\nabla^{\alpha}_{W_X} = i\ {}_aD^{\alpha}_x[1] + j\ {}_cD^{\alpha}_y[2] + k\ {}_eD^{\alpha}_z[3],$$

where i, j, and k define a fixed right-handed orthonormal basis. If $f:\mathbb{R}^3\to\mathbb{R}$ is a continuous function, then we define its fractional gradient as

$$\mathrm{Grad}^{\alpha}_{W_X} f = \nabla^{\alpha}_{W_X} f = i\ {}_aD^{\alpha}_x[1]f(x,y,z) + j\ {}_cD^{\alpha}_y[2]f(x,y,z) + k\ {}_eD^{\alpha}_z[3]f(x,y,z).$$

If $F=[F_x,F_y,F_z]:\mathbb{R}^3\to\mathbb{R}^3$ is a continuous vector field, then we define its fractional divergence and fractional curl by

$$\mathrm{Div}^{\alpha}_{W_X} F = \nabla^{\alpha}_{W_X}\circ F = {}_aD^{\alpha}_x[1]F_x(x,y,z) + {}_cD^{\alpha}_y[2]F_y(x,y,z) + {}_eD^{\alpha}_z[3]F_z(x,y,z)$$

and

$$\mathrm{Curl}^{\alpha}_{W_X} F = \nabla^{\alpha}_{W_X}\times F = i({}_cD^{\alpha}_y[2]F_z(x,y,z) - {}_eD^{\alpha}_z[3]F_y(x,y,z)) + j({}_eD^{\alpha}_z[3]F_x(x,y,z) - {}_aD^{\alpha}_x[1]F_z(x,y,z)) + k({}_aD^{\alpha}_x[1]F_y(x,y,z) - {}_cD^{\alpha}_y[2]F_x(x,y,z)).$$

Note that these fractional differential operators are nonlocal. Therefore, the fractional gradient, divergence, and curl depend on the region W_X.

For $F:\mathbb{R}^3\to\mathbb{R}^3$ and $f,g:\mathbb{R}^3\to\mathbb{R}$, it is easy to check the following relations:

(i) $\mathrm{Div}^{\alpha}_{W_X}(fF)=f\ \mathrm{Div}^{\alpha}_{W_X} F+F\circ\mathrm{Grad}^{\alpha}_{W_X} f$,
(ii) $\mathrm{Curl}^{\alpha}_{W_X}(\mathrm{Grad}^{\alpha}_{W_X} f)=[0,0,0]$,
(iii) $\mathrm{Div}^{\alpha}_{W_X}(\mathrm{Curl}^{\alpha}_{W_X} F)=0$,
(iv) $\mathrm{Grad}^{\alpha}_{W_X}(fg)=g\ \mathrm{Grad}^{\alpha}_{W_X} f+f\ \mathrm{Grad}^{\alpha}_{W_X} g$, and
(v) $\mathrm{Div}^{\alpha}_{W_X}(\mathrm{Grad}^{\alpha}_{W_X} f)={}_aD^{\alpha}_x[1]{}_aD^{\alpha}_x[1]f+{}_cD^{\alpha}_y[2]{}_cD^{\alpha}_y[2]f+{}_eD^{\alpha}_z[3]{}_eD^{\alpha}_z[3]f$.

In general, let us recall that $(D^{\alpha}_{W_X})^2\neq D^{2\alpha}_{W_X}$ (see Ref. 30).

A fractional flux of the vector field F across ∂W is a fractional oriented surface integral of the field such that

$$(I^{\alpha}_{\partial W},F) = I^{\alpha}_{\partial W}[2,3]F_x(x,y,z) + I^{\alpha}_{\partial W}[1,3]F_y(x,y,z) + I^{\alpha}_{\partial W}[1,2]F_z(x,y,z),$$

where

$$I^{\alpha}_{\partial W}[1,2]f(x,y,z) = {}_aI^{\alpha}_b[1]{}_cI^{\alpha}_d[2][f(b,d,f) - f(b,d,e)],$$

$$I^{\alpha}_{\partial W}[1,3]f(x,y,z) = {}_aI^{\alpha}_b[1]{}_eI^{\alpha}_f[3][f(b,d,f) - f(b,c,f)],$$

and

$$I^{\alpha}_{\partial W}[2,3]f(x,y,z) = {}_cI^{\alpha}_d[2]{}_eI^{\alpha}_f[3][f(b,d,f) - f(a,d,f)].$$

C. Fractional theorems of Green, Gauss, and Stokes

We now formulate the fractional formulae of Green, Gauss, and Stokes. Analogous results via Caputo fractional derivatives and Riemann–Liouville fractional integrals were obtained by Tarasov in Ref. 42.

Theorem 1: *(Fractional Green's theorem for a rectangle) Let f and g be two continuous functions whose domains contain $R=[a,b]\times[c,d]\subset\mathbb{R}^2$. Then,*

$$I^\alpha_{\partial R}[1]f + I^\alpha_{\partial R}[2]g = \frac{1}{\alpha!} I^\alpha_R[{}_aD^\alpha_b[1]g - {}_cD^\alpha_d[2]f].$$

Proof: We have

$$I^\alpha_{\partial R}[1]f + I^\alpha_{\partial R}[2]g = {}_aI^\alpha_b[1][f(b,c) - f(b,d)] + {}_cI^\alpha_d[2][g(b,d) - g(a,d)].$$

By Eq. (4),

$$f(b,c) - f(b,d) = -\frac{1}{\alpha!} {}_cI^\alpha_d[2]{}_cD^\alpha_d[2]f(b,d),$$

$$g(b,d) - g(a,d) = \frac{1}{\alpha!} {}_aI^\alpha_b[1]{}_aD^\alpha_b[1]g(b,d).$$

Therefore,

$$\begin{aligned} I^\alpha_{\partial R}[1]f + I^\alpha_{\partial R}[2]g &= -{}_aI^\alpha_b[1]\frac{1}{\alpha!} {}_cI^\alpha_d[2]{}_cD^\alpha_d[2]f(b,d) + {}_cI^\alpha_d[2]\frac{1}{\alpha!} {}_aI^\alpha_b[1]{}_aD^\alpha_b[1]g(b,d) \\ &= \frac{1}{\alpha!} I^\alpha_R[{}_aD^\alpha_b[1]g - {}_cD^\alpha_d[2]f]. \end{aligned}$$

□

Theorem 2: *(Fractional Gauss's theorem for a parallelepiped) Let $F=(F_x,F_y,F_z)$ be a continuous vector field in a domain that contains $W=[a,b]\times[c,d]\times[e,f]$. If the boundary of W is a closed surface ∂W, then*

$$(I^\alpha_{\partial W}, F) = \frac{1}{\alpha!} I^\alpha_W \,\mathrm{Div}^\alpha_W F. \tag{5}$$

Proof: The result follows by direct transformations:

$$\begin{aligned} (I^\alpha_{\partial W}, F) &= I^\alpha_{\partial W}[2,3]F_x + I^\alpha_{\partial W}[1,3]F_y + I^\alpha_{\partial W}[1,2]F_z = {}_cI^\alpha_d[2]{}_eI^\alpha_f[3](F_x(b,d,f) - F_x(a,d,f)) \\ &\quad + {}_aI^\alpha_b[1]{}_eI^\alpha_f[3](F_y(b,d,f) - F_y(b,c,f)) + {}_aI^\alpha_b[1]{}_cI^\alpha_d[2](F_z(b,d,f) - F_z(b,d,e)) \\ &= \frac{1}{\alpha!} {}_aI^\alpha_b[1]{}_cI^\alpha_d[2]{}_eI^\alpha_f[3]({}_aD^\alpha_b[1]F_x(b,d,f) + {}_cD^\alpha_d[2]F_y(b,d,f) + {}_eD^\alpha_f[3]F_z(b,d,f)) \\ &= \frac{1}{\alpha!} I^\alpha_W({}_aD^\alpha_b[1]F_x + {}_cD^\alpha_d[2]F_y + {}_eD^\alpha_f[3]F_z) = \frac{1}{\alpha!} I^\alpha_W \,\mathrm{Div}^\alpha_W F. \end{aligned}$$

□

Let S be an open, oriented, and nonintersecting surface bounded by a simple and closed curve ∂S. Let $F=[F_x,F_y,F_z]$ be a continuous vector field. Divide up S by sectionally curves into N subregions $S_1,S_2,\ldots,S_N$. Assume that for small enough subregions, each S_j can be approximated by a plane rectangle A_j bounded by curves $C_1,C_2,\ldots,C_N$. Apply Green's theorem to each individual rectangle A_j. Then, summing over the subregions,

$$\sum_j \frac{1}{\alpha!} I^\alpha_{A_j}(\nabla^\alpha_{A_j} \times F) = \sum_j I^\alpha_{\partial A_j} F.$$

Furthermore, letting $N \to \infty$

$$\sum_j \frac{1}{\alpha!} I^{\alpha}_{A_j}(\nabla^{\alpha}_{A_j} \times F) \to \frac{1}{\alpha!}(I^{\alpha}_S, \mathrm{Curl}^{\alpha}_S F),$$

while

$$\sum_j I^{\alpha}_{\partial A_j} F \to I^{\alpha}_{\partial S} F.$$

We conclude with the fractional Stokes formula

$$\frac{1}{\alpha!}(I^{\alpha}_S, \mathrm{Curl}^{\alpha}_S F) = I^{\alpha}_{\partial S} F.$$

IV. FCVs WITH MULTIPLE INTEGRALS

Consider a function $w=w(x,y)$ with two variables. Assume that the domain of w contains the rectangle $R=[a,b]\times[c,d]$ and that w is continuous on R. We introduce the variational functional defined by

$$J(w) = I^{\alpha}_R L(x,y,w(x,y),{}_aD^{\alpha}_x[1]w(x,y),{}_cD^{\alpha}_y[2]w(x,y)) := \alpha^2 \int_a^b \int_c^d L(x,y,w,{}_aD^{\alpha}_x[1]w,{}_cD^{\alpha}_y[2]w)(b - x)^{\alpha-1}(d-y)^{\alpha-1} dy dx. \tag{6}$$

We assume that the Lagrangian L is at least of class C^1. Observe that, using the notation of the $(dt)^{\alpha}$ integral as presented in Ref. 30, (6) can be written as

$$J(w) = \int_a^b \int_c^d L(x,y,w(x,y),{}_aD^{\alpha}_x[1]w(x,y),{}_cD^{\alpha}_y[2]w(x,y))(dy)^{\alpha}(dx)^{\alpha}. \tag{7}$$

Consider the following FCV problem, which we address as problem (P).

Problem (P): Minimize (or maximize) functional J defined by (7) with respect to the set of continuous functions $w(x,y)$ such that $w|_{\partial R}=\varphi(x,y)$ for some given function φ.

The continuous functions $w(x,y)$ that assume the prescribed values $w|_{\partial R}=\varphi(x,y)$ at all points of the boundary curve of R are said to be admissible. In order to prove necessary optimality conditions for problem (P), we use a two dimensional analog of fractional integration by parts. Lemma 1 provides the necessary fractional rule.

Lemma 1: Let F, G, and h be continuous functions whose domains contain R. If $h\equiv 0$ on ∂R, then

$$\int_a^b \int_c^d [G(x,y)\,{}_aD^{\alpha}_x[1]h(x,y) - F(x,y)\,{}_cD^{\alpha}_y[2]h(x,y)](b-x)^{\alpha-1}(d-y)^{\alpha-1} dy dx$$
$$= -\int_a^b \int_c^d [({}_aD^{\alpha}_x[1]G(x,y) - {}_cD^{\alpha}_y[2]F(x,y))h(x,y)](b-x)^{\alpha-1}(d-y)^{\alpha-1} dy dx.$$

Proof: By choosing $f=F\cdot h$ and $g=G\cdot h$ in Green's formula, we obtain

$$I^{\alpha}_{\partial R}[1](Fh) + I^{\alpha}_{\partial R}[2](Gh) = \frac{1}{\alpha!} I^{\alpha}_R[{}_aD^{\alpha}_b[1]G\cdot h + G\cdot {}_aD^{\alpha}_b[1]h - {}_cD^{\alpha}_d[2]F\cdot h - F\cdot {}_cD^{\alpha}_d[2]h],$$

which is equivalent to

$$\frac{1}{\alpha!}I_R^\alpha[G\cdot {}_aD_b^\alpha[1]h - F\cdot {}_cD_d^\alpha[2]h] = I_{\partial R}^\alpha[1](Fh) + I_{\partial R}^\alpha[2](Gh) - \frac{1}{\alpha!}I_R^\alpha[({}_aD_b^\alpha[1]G - {}_cD_d^\alpha[2]F)h].$$

In addition, since $h\equiv 0$ on ∂R, we deduce that

$$I_R^\alpha[G\cdot {}_aD_b^\alpha[1]h - F\cdot {}_cD_d^\alpha[2]h] = -I_R^\alpha[({}_aD_b^\alpha[1]G - {}_cD_d^\alpha[2]F)h].$$

The lemma is proved. □

Theorem 3: *(Fractional Euler–Lagrange equation) Let w be a solution to problem (P). Then, w is a solution of the fractional partial differential equation*

$$\partial_3 L - {}_aD_x^\alpha[1]\partial_4 L - {}_cD_y^\alpha[2]\partial_5 L = 0, \tag{8}$$

where by $\partial_i L$, $i=1,\ldots,5$, *we denote the usual partial derivative of* $L(\cdot,\cdot,\cdot,\cdot,\cdot)$ *with respect to its ith argument.*

Proof: Let h be a continuous function on R such that $h\equiv 0$ on ∂R and consider an admissible variation $w+\epsilon h$ for ϵ taking values on a sufficient small neighborhood of zero. Let

$$j(\epsilon) = J(w+\epsilon h).$$

Then, $j'(0)=0$, i.e.,

$$\alpha^2\int_a^b\int_c^d (\partial_3 Lh + \partial_4 L_a D_x^\alpha[1]h + \partial_5 L_c D_y^\alpha[2]h)(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx = 0.$$

Using Lemma 1, we obtain

$$\alpha^2\int_a^b\int_c^d (\partial_3 L - {}_aD_x^\alpha[1]\partial_4 L - {}_cD_y^\alpha[2]\partial_5 L)h(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx = 0.$$

Since h is an arbitrary function, by the fundamental lemma of calculus of variations, we deduce Eq. (8). □

Let us consider now the situation where we do not impose admissible functions w to be of fixed values on ∂R.

Problem (P′) *:* Minimize (or maximize) J among the set of all continuous curves w whose domain contains R.

Theorem 4: *(Fractional natural boundary conditions) Let w be a solution to problem* (P′) *. Then, w is a solution of the fractional differential equation (8) and satisfies the following equations:*

(1) $\partial_4 L(a,y,w(a,y),{}_aD_a^\alpha[1]w(a,y),{}_cD_y^\alpha[2]w(a,y))=0$ *for all* $y\in[c,d]$;
(2) $\partial_4 L(b,y,w(b,y),{}_aD_b^\alpha[1]w(b,y),{}_cD_y^\alpha[2]w(b,y))=0$ *for all* $y\in[c,d]$;
(3) $\partial_5 L(x,c,w(x,c),{}_aD_x^\alpha[1]w(x,c),{}_cD_c^\alpha[2]w(x,c))=0$ *for all* $x\in[a,b]$; and
(4) $\partial_5 L(x,d,w(x,d),{}_aD_x^\alpha[1]w(x,d),{}_cD_d^\alpha[2]w(x,d))=0$ *for all* $x\in[a,b]$.

Proof: Proceeding as in the proof of Theorem 3 (see also Lemma 1), we obtain

$$\begin{aligned}0 &= \alpha^2\int_a^b\int_c^d (\partial_3 Lh + \partial_4 L\ {}_aD_x^\alpha[1]h + \partial_5 L\ {}_cD_y^\alpha[2]h)(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx = \alpha^2\int_a^b\int_c^d (\partial_3 L \\ &\quad - {}_aD_x^\alpha[1]\partial_4 L - {}_cD_y^\alpha[2]\partial_5 L)h(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx + \alpha! I_{\partial R}^\alpha[2](\partial_4 Lh) - \alpha! I_{\partial R}^\alpha[1](\partial_5 Lh),\end{aligned} \tag{9}$$

where h is an arbitrary continuous function. In particular, the above equation holds for $h\equiv 0$ on ∂R. For such h the second member of (9) vanishes and by the fundamental lemma of the calculus of variations we deduce Eq. (8). With this result, Eq. (9) takes the form

$$
\begin{aligned}
0 = &\int_c^d \partial_4 L(b,y,w(b,y),{}_aD_b^\alpha[1]w(b,y),{}_cD_y^\alpha[2]w(b,y))h(b,y)(d-y)^{\alpha-1}dy \\
&- \int_c^d \partial_4 L(a,y,w(a,y),{}_aD_a^\alpha[1]w(a,y),{}_cD_y^\alpha[2]w(a,y))h(a,y)(d-y)^{\alpha-1}dy \\
&- \int_a^b \partial_5 L(x,c,w(x,c),{}_aD_x^\alpha[1]w(x,c),{}_cD_c^\alpha[2]w(x,c))h(x,c)(b-x)^{\alpha-1}dx \\
&+ \int_a^b \partial_5 L(x,d,w(x,d),{}_aD_x^\alpha[1]w(x,d),{}_cD_d^\alpha[2]w(x,d))h(x,d)(b-x)^{\alpha-1}dx.
\end{aligned} \tag{10}
$$

Since h is an arbitrary function, we can consider the subclass of functions for which $h\equiv 0$ on

$$([a,b]\times\{c\}) \cup ([a,b]\times\{d\}) \cup (\{b\}\times[c,d]).$$

For such h, Eq. (10) reduces to

$$0 = \int_c^d \partial_4 L(a,y,w(a,y),{}_aD_a^\alpha[1]w(a,y),{}_cD_y^\alpha[2]w(a,y))h(a,y)(d-y)^{\alpha-1}dy.$$

By the fundamental lemma of calculus of variations, we obtain

$$\partial_4 L(a,y,w(a,y),{}_aD_a^\alpha[1]w(a,y),{}_cD_y^\alpha[2]w(a,y)) = 0 \quad \text{for all } y \in [c,d].$$

The other natural boundary conditions are proved similarly by appropriate choices of h. □

We can generalize Lemma 1 and Theorem 3 to the three dimensional case in the following way.

Lemma 2: Let A, B, C, and η be continuous functions whose domains contain the parallelepiped W. If $\eta\equiv 0$ on ∂W, then

$$I_W^\alpha(A\cdot{}_aD_b^\alpha[1]\eta + B\cdot{}_cD_d^\alpha[2]\eta + C\cdot{}_eD_f^\alpha[3]\eta) = -I_W^\alpha([{}_aD_b^\alpha[1]A + {}_cD_d^\alpha[2]B + {}_eD_f^\alpha[3]C]\eta). \tag{11}$$

Proof: By choosing $F_x=\eta A$, $F_y=\eta B$, and $F_z=\eta C$ in (5), we obtain the three dimensional analog of integrating by parts,

$$
\begin{aligned}
I_W^\alpha(A\cdot{}_aD_b^\alpha[1]\eta + B\cdot{}_cD_d^\alpha[2]\eta + C\cdot{}_eD_f^\alpha[3]\eta) = &-I_W^\alpha([{}_aD_b^\alpha[1]A + {}_cD_d^\alpha[2]B + {}_eD_f^\alpha[3]C]\eta) \\
&+ \alpha!(I_{\partial W}^\alpha,[\eta A, \eta B, \eta C]).
\end{aligned}
$$

In addition, if we assume that $\eta\equiv 0$ on ∂W, we have formula (11). □

Theorem 5: *(Fractional Euler–Lagrange equation for triple integrals) Let $w=w(x,y,z)$ be a continuous function whose domain contains $W=[a,b]\times[c,d]\times[e,f]$. Consider the functional*

$$
\begin{aligned}
J(w) &= I_W^\alpha L(x,y,z,w(x,y,z),{}_aD_x^\alpha[1]w(x,y,z),{}_cD_y^\alpha[2]w(x,y,z),{}_eD_z^\alpha[3]w(x,y,z)) \\
&= \int_a^b\int_c^d\int_e^f L(x,y,z,w,{}_aD_x^\alpha[1]w,{}_cD_y^\alpha[2]w,{}_eD_z^\alpha[3]w)(dz)^\alpha(dy)^\alpha(dx)^\alpha
\end{aligned}
$$

defined on the set of continuous curves such that their values on ∂W take prescribed values. Let L be at least of class C^1. If w is a minimizer (or maximizer) of J, then w satisfies the fractional partial differential equation,

$$\partial_4 L - {}_aD_b^\alpha[1]\partial_5 L - {}_cD_d^\alpha[2]\partial_6 L - {}_eD_f^\alpha[3]\partial_7 L = 0.$$

Proof: A proof can be done similar to the proof of Theorem 3, where instead of using Lemma 1, we apply Lemma 2. □

V. APPLICATIONS AND POSSIBLE EXTENSIONS

In classical mechanics, functionals that depend on functions of two or more variables arise in a natural way, e.g., in mechanical problems involving systems with infinitely many degrees of freedom (string, membranes, etc.). Let us consider a flexible elastic string stretched under constant tension τ along the x axis with its end points fixed at $x=0$ and $x=L$. Let us denote the transverse displacement of the particle at time t, $t_1 \leq t \leq t_2$, whose equilibrium position is characterized by its distance x from the end of the string at $x=0$ by the function $w=w(x,t)$. Thus, $w(x,t)$, with $0 \leq x \leq L$, describes the shape of the string during the course of the vibration. Assume a distribution of mass along the string of density $\sigma=\sigma(x)$. Then, the function that describes the actual motion of the string is one that renders

$$J(w) = \frac{1}{2}\int_{t_1}^{t_2}\int_0^L (\sigma w_t^2 - \tau w_x^2)dxdt$$

an extremum with respect to functions $w(x,t)$, which describes the actual configuration at $t=t_1$ and $t=t_2$ and which vanishes, for all t, at $x=0$ and $x=L$ (see Ref. 43, p. 95, for more details).

We discuss the description of the motion of the string within the framework of the fractional differential calculus. One may assume that, due to some constraints of physical nature, the dynamics does not depend on the usual partial derivatives but on some fractional derivatives ${}_0D_x^\alpha[1]w$ and ${}_{t_1}D_t^\alpha[2]w$. For example, we can assume that there is some coarse-graining phenomenon (see details in Refs. 27 and 25). In this condition, one is entitled to assume again that the actual motion of the system, according to the principle of Hamilton, is such as to render the action function

$$J(w) = \tfrac{1}{2}I_R^\alpha(\sigma({}_{t_1}D_t^\alpha[2]w)^2 - \tau({}_0D_x^\alpha[1]w)^2),$$

where $R=[0,L]\times[t_1,t_2]$, an extremum. Note that we recover the classical problem of the vibrating string when $\alpha \to 1^-$. Applying Theorem 3, we obtain the fractional equation of motion for the vibrating string,

$${}_0D_x^\alpha[1]{}_0D_1^\alpha[1]w = \frac{\sigma}{\tau}{}_{t_1}D_t^\alpha[2]{}_{t_1}D_2^\alpha[2]w.$$

This equation becomes the classical equation of the vibrating string (cf., e.g., Ref. 43, p. 97) if $\alpha \to 1^-$.

We remark that the fractional operators are nonlocal, therefore they are suitable for constructing models possessing memory effect. In the above example, we discussed the application of the fractional differential calculus to the vibrating string. We started with a variational formulation of the physical process in which we modify the Lagrangian density by replacing integer order derivatives with fractional ones. Then, the action integral in the sense of Hamilton was minimized and the governing equation of the physical process was obtained in terms of fractional derivatives. Similarly, many others physical fields can be derived from a suitably defined action functional. This gives several possible applications of the FCVs with multiple integrals as was introduced in this paper, e.g., in describing nonlocal properties of physical systems in mechanics (see, e.g., Refs. 10, 11, 32, 36, and 41) or electrodynamics (see, e.g., Refs. 9 and 42).

We end with some open problems for further investigations. It has been recognized that FC is useful in the study of scaling in physical systems.[14,15] In particular, there is a direct connection between local fractional differentiability properties and the dimensions of Holder exponents of nowhere differentiable functions, which provide a powerful tool to analyze the behavior of irregular signals and functions on a fractal set.[4,33] FC appear naturally, e.g., when working with fractal sets and coarse-graining spaces,[25,24] and fractal patterns of deformation and vibration in porous media and heterogeneous materials.[12] The importance of vibrating strings to the FC has been given in Ref. 18, where it is shown that a fractional Brownian motion can be identified with a string. The usefulness of our fractional theory of the calculus of variations with multiple integrals in physics, to deal with fractal and coarse-graining spaces, porous media, and Brownian motions, is a question to be studied. It should be possible to prove the theorems obtained in this work for a general form of domains and boundaries and to develop a FCVs with multiple integrals in terms of other type of fractional operators. An interesting open question consists of generalizing the fractional Noether-type theorems obtained in Refs. 6, 21, and 22 to the case of several independent variables.

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