Universidade de Aveiro Departamento de Matemática
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Cristiana João Soares da Silva

Regularização e Pontos Conjugados Bang-bang no Controlo Óptimo

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Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Doutor Delfim Fernando Marado Torres, Professor Associado do Departamento de Matemática da Universidade de Aveiro, e sob a co-orientação científica do Doutor Emmanuel Trélat, Professor no Laboratório MAPMO (Matemática e Aplicações, Física Matemática de Orléans) da Universidade de Orléans, França.

Thesis submitted to the University of Aveiro in fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics, under the supervision of Dr. Delfim Fernando Marado Torres, Associate Professor at the Department of Mathematics of the University of Aveiro, and co-supervision of Dr. Emmanuel Trélat, Professor at the Laboratory MAPMO (Mathematic and Applications, Mathematical Physics of Orléans) of the University of Orléans, France.

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o júri

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palavras-chave

Controlo óptimo, princípio do máximo de Pontryagin, problemas de tempo mínimo, controlos bang-bang, procedimentos de regularização, método de tiro simples, tempos conjugados.

resumo

Consideramos o problema de controlo óptimo de tempo mínimo para sistemas de controlo mono-entrada e controlo afim num espaço de dimensão finita com condições inicial e final fixas, onde o controlo escalar toma valores num intervalo fechado. Quando aplicamos o método de tiro a este problema, vários obstáculos podem surgir uma vez que a função de tiro não é diferenciável quando o controlo é *bang-bang*. No caso *bang-bang* os tempos conjugados são teoricamente bem definidos para este tipo de sistemas de controlo, contudo os algoritmos computacionais directos disponíveis são de difícil aplicação. Por outro lado, no caso suave o conceito teórico e prático de tempos conjugados é bem conhecido, e ferramentas computacionais eficazes estão disponíveis.

Propomos um procedimento de regularização para o qual as soluções do problema de tempo mínimo correspondente dependem de um parâmetro real positivo suficientemente pequeno e são definidas por funções suaves em relação à variável tempo, facilitando a aplicação do método de tiro simples. Provamos, sob hipóteses convenientes, a convergência forte das soluções do problema regularizado para a solução do problema inicial, quando o parâmetro real tende para zero. A determinação de tempos conjugados das trajectórias localmente óptimas do problema regularizado enquadra-se na teoria suave conhecida. Provamos, sob hipóteses adequadas, a convergência do primeiro tempo conjugado do problema regularizado para o primeiro tempo conjugado do problema inicial *bang-bang*, quando o parâmetro real tende para zero. Consequentemente, obtemos um algoritmo eficiente para a computação de tempos conjugados no caso *bang-bang*.

keywords

Optimal control, Pontryagin maximum principle, minimal time problems, bang-bang controls, regularization procedures, single shooting methods, conjugate times.

abstract

In this thesis we consider a minimal time control problem for single-input control-affine systems in finite dimension with fixed initial and final conditions, where the scalar control take values on a closed interval. When applying a shooting method for solving this problem, one may encounter numerical obstacles due to the fact that the shooting function is non smooth whenever the control is bang-bang. For these systems a theoretical concept of conjugate time has been defined in the bang-bang case, however direct algorithms of computation are difficult to apply. Besides, theoretical and practical issues for conjugate time theory are well known in the smooth case, and efficient implementation tools are available.

We propose a regularization procedure for which the solutions of the minimal time problem depend on a small enough real positive parameter and are defined by smooth functions with respect to the time variable, facilitating the application of a single shooting method. Under appropriate assumptions, we prove a strong convergence result of the solutions of the regularized problem towards the solution of the initial problem, when the real parameter tends to zero. The conjugate times computation of the locally optimal trajectories for the regularized problem falls into the standard theory. We prove, under appropriate assumptions, the convergence of the first conjugate time of the regularized problem towards the first conjugate time of the initial bang-bang control problem, when the real parameter tends to zero. As a byproduct, we obtain an efficient algorithmic way to compute conjugate times in the bang-bang case.

2010 Mathematics Subject Classification: 49K21, 49M15, 49N60.

mots-clés

Contrôle optimal, principe du maximum de Pontryagin, problème de temps minimal, contrôle bang-bang, procédures de régularisation, méthode de tir simple, temps conjugué.

résumé

On considère le problème de contrôle optimal de temps minimal pour des systèmes affine et mono-entrée en dimension finie, avec conditions initiales et finales fixées, où le contrôle scalaire prend ses valeurs dans un intervalle fermé. Lors de l'application d'une méthode de tir pour résoudre ce problème, on peut rencontrer des obstacles numériques car la fonction de tir n'est pas lisse lorsque le contrôle est *bang-bang*. Pour ces systèmes, dans le cas bang-bang, un concept théorique de temps conjugué a été défini, toutefois les algorithmes de calcul direct sont difficiles à appliquer. En outre, les questions théoriques et pratiques de la théorie du temps conjugué sont bien connues dans le cas lisse, et des outils efficaces de mise en œuvre sont disponibles.

On propose une procédure de régularisation pour laquelle les solutions du problème de temps minimal dépendent d'un paramètre réel positif suffisamment petit et sont définis par des fonctions lisses en temps, ce qui facilite l'application de la méthode de tir simple. Sous des hypothèses convenables, nous prouvons un résultat de convergence forte des solutions du problème régularisé vers la solution du problème initial, lorsque le paramètre réel tend vers zéro. Le calcul des temps conjugués pour les trajectoires localement optimales du problème régularisé est standard. Nous prouvons, sous des hypothèses appropriées, la convergence du premier temps conjugué du problème régularisé vers le premier temps conjugué du problème de contrôle *bang-bang* initial, quand le paramètre réel tend vers zéro. Ainsi, on obtient une procédure algorithmique efficace pour calculer les temps conjugués dans le cas *bang-bang*.

Contents

Introduction

In this thesis, we investigate the minimal time Optimal Control Problem **(OCP)** for single-input control-affine systems in $\mathbb{R}^n$

$$\dot{x} = X(x) + u_1 Y_1(x),$$

with fixed initial and final times conditions

$$x(0) = \hat{x}_0\,, \quad x(t_f) = \hat{x}_1\,,$$

where X and Y_1 are smooth vector fields, and the control u_1 is a measurable scalar function satisfying the constraint

$$|u_1(t)| \leq 1\,, \quad \forall t \in [0, t_f]$$

with t_f the final time. We develop regularization procedures in order to compute smooth approximations of the above bang-bang control problem, and to compute conjugate times.

The first conjugate time of a trajectory $x(\cdot)$ is the time at which it loses its local optimality. The definition and computation of conjugate points are an important topic in the theory of calculus of variations (see e.g. [13]). In [99] the investigation of the definition and computation of conjugate points for minimal time control problems is based on the study of necessary and/or sufficient second order conditions. In [110], the theory of envelopes and conjugate points is used for the study of the structure of locally optimal bang-bang trajectories for the problem **(OCP)** in $\mathbb{R}^2$ and $\mathbb{R}^3$; these results were generalized in [60]. In [81, 100] first and second order sufficient optimality conditions are derived in terms of a quadratic form Q_t, for a minimal time control problem with control-affine systems. In [100] L^1-local optimality is considered and in [81] strong local optimality. In [5] the authors derive second order sufficient conditions, under the same regularity assumptions as [81], for an optimal control problem in the Mayer form with fixed final time, with control-affine systems and bang-bang optimal controls. In [90] the authors proved the equivalence of the second order sufficient conditions given in [81] with the ones given in [5]. In [95] an analogous quadratic form to the one in [5] is defined, but the sufficient optimality conditions derived are valid for a stronger kind of optimality (state local optimality).

The combination of necessary and sufficient conditions for bang-bang extremals provided in [3,5,81,87,95] allows to relate the local strong optimality status of a trajectory $x(\cdot)$ with the existence of conjugate times. More precisely, if the strict bang-bang Legendre condition holds for a bang-bang extremal trajectory $x(\cdot)$ and the quadratic form Q_t is positive definite on $[0,t]$, then $x(\cdot)$ is locally optimal for problem **(OCP)** in the C^0 topology on $[0,t]$ ([5, 81, 87, 95]). If we assume moreover, that $x(\cdot)$ has a unique extremal lift (up to a multiplicative scalar) $(x(\cdot), p(\cdot), p^0, u_1(\cdot))$, which is moreover normal ($p^0 = -1$) and $x(\cdot)$ is locally optimal in C^0 topology for problem **(OCP)** on $[0,t]$ then Q_t is nonnegative ([3]). Under these assumptions, the times t, $t > 0$, such that the quadratic form Q_t has a trivial kernel are isolated and can only consist of some switching times of the bang-bang extremal control (see [5]); the *first conjugate time* t_c of a bang-bang strong locally optimal trajectory $x(\cdot)$ (starting from $\hat{x}_0$) is then defined by

$$t_c = \sup\{t \mid Q_t \text{ is positive definite}\} = \inf\{t \mid Q_t \text{ is indefinite}\}.$$

The point $x(t_c)$ is called the *first conjugate point* of the trajectory $x(\cdot)$.

Sufficient optimality conditions are developed in [87] (see also [113]) based on the method of characteristics and the theory of extremal fields. Sufficient optimality conditions are given for embedding a reference trajectory into a local field of broken extremals. In [1,4,5,95], using Hamiltonian methods and the extremal field theory, the authors construct, under certain conditions, a non-intersecting field of state extremals that covers a given extremal trajectory $x(\cdot)$. In [5, 61, 87] the authors associate the occurrence of a conjugate point with a *fold point* of the flow of the extremal field, that is, a so-called overlap of the flow near the switching surface.

The computation of conjugate times in the bang-bang case is difficult in practice. In the last years works have been developed on the numerical implementation of second order sufficient optimality conditions (see, e.g., [61, 78, 81] and references cited therein). These procedures allow the characterization of the first conjugate time, for bang-bang optimal control problems with control-affine systems, whenever it exists and is attained at a j^{th} switching time. However, in practice, if j is too large then the numerical computation may become very difficult. Besides, theoretical and practical issues for conjugate time theory are well known in the smooth case (see e.g. [2, 86]), and efficient implementation tools are available (see [15]).

The contributions of this thesis are the following.

We propose a regularization procedure which permits to use the efficient tools of computation of conjugate times in the smooth case provided in [15] for the computation of the first conjugate time of the problem **(OCP)**. The regularization procedure is the following. Let ε be a positive real parameter and let $Y_2, \dots, Y_m$ be $m-1$ arbitrary smooth vector fields on $\mathbb{R}^n$, where $m \geq 2$ is an integer. We consider the minimal time problem $\mathbf{(OCP)}_\varepsilon$ for the control system

$$\dot{x}^\varepsilon(t) = X\left(x^\varepsilon(t)\right) + u_1^\varepsilon(t) Y_1\left(x^\varepsilon(t)\right) + \varepsilon \sum_{i=2}^{m} u_i^\varepsilon(t) Y_i\left(x^\varepsilon(t)\right),$$

under the constraint

$$\sum_{i=1}^{m} (u_i^\varepsilon(t))^2 \leq 1\,,$$

with the fixed boundary conditions $x^\varepsilon(0) = \hat{x}_0$, $x^\varepsilon(t_f) = \hat{x}_1$ of the initial problem **(OCP)**.

In the next theorem we derive nice convergence properties.

Theorem 0.0.1 (cf. Chapter 2, p. 61). *Assume that the problem **(OCP)**[1] has a unique solution $x(\cdot)$, defined on $[0, t_f]$, associated with a control $u_1(\cdot)$ on $[0, t_f]$. Moreover, assume that $x(\cdot)$ has a unique extremal lift (up to a multiplicative scalar), that is moreover normal, and denoted $(x(\cdot), p(\cdot), -1, u_1(\cdot))$.*

*Then, under the assumption $\mathrm{Span}\{Y_i \mid i = 1, \ldots, m\} = I\!R^n$, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, the problem **(OCP)**$_\varepsilon$ has at least one solution $x^\varepsilon(\cdot)$, defined on $[0, t_f^\varepsilon]$ with $t_f^\varepsilon \leq t_f$, associated with a smooth control $u^\varepsilon = (u_1^\varepsilon, \ldots, u_m^\varepsilon)$ satisfying the constraint $\sum_{i=1}^{m} (u_i^\varepsilon(t))^2 \leq 1$, every extremal lift of which is normal. Let $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), -1, u^\varepsilon(\cdot))$ be such a normal extremal lift. Then, as ε tends to 0,*

- *t_f^ε converges to t_f;*
- *$x^\varepsilon(\cdot)$ converges uniformly to $x(\cdot)$, and $p^\varepsilon(\cdot)$ converges uniformly to $p(\cdot)$ on $[0, t_f]$;*
- *$u_1^\varepsilon(\cdot)$ converges weakly to $u_1(\cdot)$ for the weak $L^1(0, t_f)$ topology.*

If the control $u_1(\cdot)$ is moreover bang-bang, i.e., if the (continuous) switching function $\varphi(t) = \langle p(t), Y_1(x(t))\rangle$ does not vanish on any subinterval of $[0, t_f]$, then $u_1^\varepsilon(\cdot)$ converges to $u_1(\cdot)$ and $u_i^\varepsilon(\cdot)$, $i = 2, \ldots, m$, converge to 0 almost everywhere on $[0, t_f]$, and thus in particular for the strong $L^1(0, t_f)$ topology.

We provide an example where the optimal control of the initial system is not bang-bang (it has a singular arc) and for which the almost everywhere convergence fails.

Among the numerous numerical methods that exist to solve optimal control problems, the *shooting methods* consist in solving, via Newton-like methods, the two-point or multi-point boundary value problem arising from the application of the Pontryagin maximum principle. For the minimal time problem **(OCP)**, optimal controls may be discontinuous, and it follows that the shooting function is not smooth on $\mathbb{R}^n$ in general. Actually it may be non differentiable on switching surfaces. This implies two difficulties when using a shooting method. First, if one does not know a priori the structure of the optimal control, then it may be very difficult to initialize properly the shooting method, and in general the iterates of the underlying Newton method will be unable to cross barriers generated by switching surfaces (see e.g. [71]). Second, the numerical computation of the shooting function and of its differential may be

[1] This Theorem remains valid if we consider $x(0) \in M_0$ and $x(1) \in M_1$ where M_0 and M_1 are two compact sets of $\mathbb{R}^n$ (see Chapter 2).

intricate since the shooting function is not continuously differentiable. This observation is one of the possible motivations of the regularization procedure considered in this thesis. Indeed, the shooting functions related to the smooth optimal control problems $\mathbf{(OCP)}_\varepsilon$ are smooth.

From Theorem 0.0.1, under appropriate assumptions, the optimal controls of problem $\mathbf{(OCP)}_\varepsilon$ are smooth, therefore the computation of associated conjugate points $x^\varepsilon(t_c^\varepsilon)$ falls into the standard smooth theory. Our next result asserts the convergence, as ε tends to 0, of t_c^ε towards the conjugate time t_c of the initial bang-bang optimal control problem.

Theorem 0.0.2 (cf. Chapter 3, p. 95)**.** *Assume that the problem ($\boldsymbol{OCP}$) has a unique solution $x(\cdot)$, associated with a bang-bang control $u_1(\cdot)$, on a maximal interval I. Moreover, assume that $x(\cdot)$ has a unique extremal lift (up to a multiplicative scalar), which is moreover normal, and denoted by $(x(\cdot), p(\cdot), -1, u_1(\cdot))$. If the extremal $(x(\cdot), p(\cdot), -1, u_1(\cdot))$ satisfies, moreover, the strict bang-bang Legendre condition on $[0, t_c]$, then the first geometric conjugate time t_c^ε converges to the first conjugate time t_c as ε tends to 0.*

This result permits to use the available efficient implementation procedures for the smooth case, like for instance the free package $COTCOT$[2] (see [15]), to compute conjugate times in the bang-bang case. We claim that when applying the smooth procedures to the regularized procedure, it is not needed to consider very small values of ε to estimate the first conjugate time t_c. Indeed, a conjugate time of a locally bang-bang trajectory can only occur at a switching time and, under our assumptions, switching times are isolated. From Theorem 0.0.2, the first geometric conjugate time t_c^ε converges to t_c, when ε tend to 0. Therefore, as soon as ε is small enough so that t_c^ε is in a (not necessarily so small) neighborhood of some switching time τ_s of the bang-bang trajectory $x(\cdot)$, this means that the bang-bang conjugate time t_c is equal to that switching time τ_s.

This thesis is organized in the following way.

In the first chapter we recall some important definitions and theorems of linear and nonlinear optimal control theory. In Chapter 2 we propose a regularization procedure for bang-bang optimal control problems with single-input control-affine systems and prove, under appropriate assumptions, convergence properties of the optimal solutions of the regularized problem towards the solutions of the initial problem. These convergence results are illustrated in several examples. In Chapter 3 the regularization procedure introduced in Chapter 2 is used and we prove the convergence of the first geometric conjugate time t_c^ε of the regularized problem to the first conjugate time of t_c of the bang-bang optimal trajectory, as ε tends to 0. Several examples are provided where the convergence properties proved in Theorems 0.0.1 and 0.0.2 are illustrated. In Appendix A we recall first and second order sufficient optimality conditions proved in [78–81] and apply them to one of the examples considered in Chapter 3.

[2] Conditions of Order Two, COnjugate Times, http://apo.enseeiht.fr/cotcot/

Chapter 1

Preliminaries on Optimal Control Theory

1.1 Introduction

In this chapter some important definitions and results of the optimal control theory are given. We start with general explanations of the main elements of an optimal control problem and give some motivations for the study of these problems. Section §1.2 gives a brief historical overview of the optimal control theory. In §1.3 we present some important results of the linear optimal control theory and an example of a linear optimal control problem. Some results of the nonlinear optimal control theory are presented in §1.4 together with two examples. For both linear and nonlinear general optimal control problems the Pontryagin maximum principle is formulated and in §1.5 a proof of this theorem is given for a general nonlinear minimal time optimal control problem, using needle-like variations which are needed to derive the main result of Chapter 2 (Theorem 2.5.1). In §1.6 we derive the maximization condition of Pontryagin maximum principle for a minimal time problem using Gamkrelidze's generalized controls.

All of us already tried, in some occasion, to keep in balance a ball on a finger (i.e., solve the problem of the inverted pendulum). However it is much more difficult to keep in balance a double inverted pendulum, that is, a system composed by two balls one over the other, specially if we close our eyes. The control theory allows to do it, if we dispose of a suitable mathematical model that describes the physical process.

The main elements of an optimal control problem are: the mathematical model which relates the state x to the input or control u by a differential system; the *initial point* or state x_0 and a *final point* x_1 or target S; the output of the system which characterize the process, i.e., the *state* of the controlled object at each instant of time; a set of admissible inputs or *controls* which determine the course of the process; the *cost functional* (also called performance index, or objective functional, or effort) that consists of a quantitative criteria for the efficiency of each admissible control; and the length of time t_f required to reach the terminal state.

A control system is a dynamical system, which evolves over time, on which we can work through a command function or a control, and their origin is vast (mechanics, electronic, biology, economy, etc.). Some examples of control systems which can be modeled and treated by the theory of control systems are: a computer that allows the user to perform a series of basic commands; an ecosystem on which we can act promoting a particular situation to achieve a balance; nerve tissues forming a network controlled by the brain processing the stimuli from outside and having an effect on the body; a robot performing a specific task; a car that we can command with the accelerator, brake and wheel; a satellite or a spacecraft.

The control theory analyzes the properties of such systems, with the aim of steering an initial state to a certain final state, eventually respecting certain restrictions. The objective can be also to stabilize the system making it insensitive to some perturbations (*stabilization* problem), or even to compute the optimal solutions for a certain optimization criteria (*optimal control* problem). For the construction of the control system model, we can make use of differential equations, functional integrals, finite differences, partial derivatives, etc. For this reason the control theory is the interconnection of many mathematical areas (see, e.g., [21,38, 39,65,85,106]).

The dynamics of a system define the system possible transformations, occurring in time in a deterministic or random way. An equation is given, or typically a system of differential equations, relating the variables and modeling the dynamics of the system. The examples already given show that the structure and dynamics of a control system may have very different meanings. In particular, the control system can be described by discrete, continuous, or hybrid transformations or, more generally, on a *time scale* or *measure chain* [43,45,72].

Consider a control system whose state at a given moment is represented by a vector. The controls are functions or parameters, usually subject to restrictions, which act on the system in the form of outside forces that affect the dynamics. Given the system of differential equations which models the dynamics of the system, it is then necessary to use the available information and features of the problem to construct the appropriate controls that will enable us to attain our objective. For example, when we travel in our car acting accordingly to the code of the road (at least this is advisable) and we construct the travel plan to reach our destination, there are some restrictions on the trajectory and/or on the controls, which must be taken into consideration.

A control system is called *controllable* if we can steer it (in a finite time) from a given initial state to any final state. Kalman proved in 1949 an important result on controllability which characterizes controllable linear control systems of finite dimension (Theorem 1.3.9). For nonlinear systems the controllability problem is much more difficult and remains an active domain of research.

Once the controllability problem is solved, we may wish to go from an initial state to a final state minimizing or maximizing a specific criteria. In this case we are speaking about an optimal control problem. For example, a driver going from Lisbon to Porto may wish to travel in minimal time, and in that case he will take the highway and spend more money and fuel. Another optimal control problem is obtained if the driver chooses as a criteria spend less money as possible. The solution to this problem implies to chose secondary roads, for free, and he will take a lot more time to his destination (following the internet site `http://www.google.pt/maps` choosing the highway the driver takes 3h from Lisbon to Porto and by the secondary roads 6h45m).

The theory of optimal control is of great importance in aerospace engineering, in particular for conduction problems, aero-assisted transfer orbits, development of recoverable launchers (the financial aspect here is very important) and problems of atmospheric reentry, such as the famous project *Mars Sample Return* from the *European Space Agency* (ESA) which consists in sending a spacecraft to Mars with the objective of bringing to Earth martian samples (Figure 1.1).

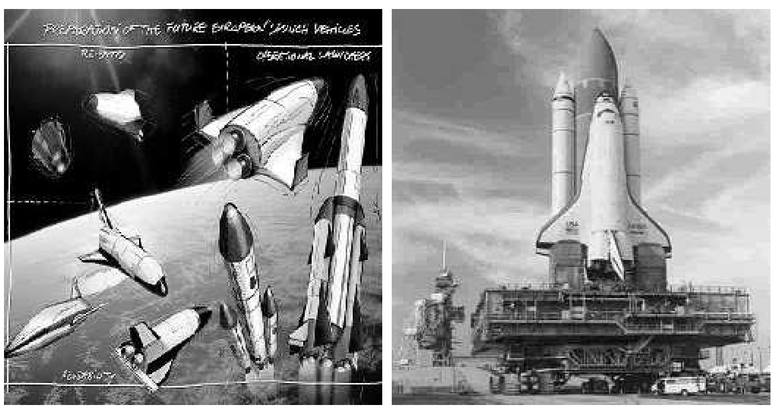

Figure 1.1: Optimal control theory has an important role in the aeroespacial engineering.

1.2 Short historical overview

The calculus of variations was born in the seventeen century with the contribution of Bernoulli, Fermat, Leibniz and Newton. Some mathematicians as H.J. Sussmann and J.C. Willems defend that the origin of optimal control coincides with the birth of calculus of variations, in 1697, date of the publication of the solution of the brachistochrone problem by the mathematician Johann Bernoulli [114]. The brachistochrone problem (in Greek *brakhistos*, "the shortest", and *chronos*, "time") was studied by Galileu in 1638. The aim was to determine the curve between two points on a vertical plane that is covered in the least time by a sphere that starts at the first point A with zero speed and is constrained to move along the curve to the second point B, under the action of constant gravity and assuming no friction (optimal sliding, see Figure 1.2). In contrast to what could be our intuitive first answer, the shortest time path

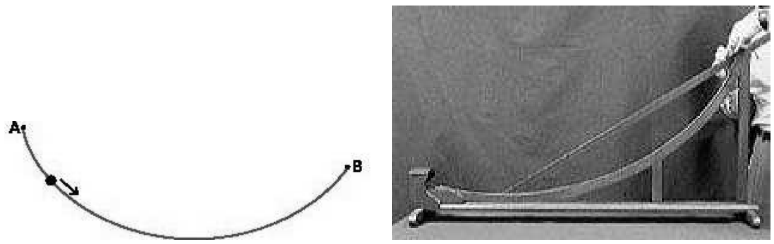

Figure 1.2: Brachistochrone problem.

between two points is not a straight line! Galileo believed (wrongly) that the required curve was an arc of a circle, but he had already noticed that the straight line is not the shortest time path. In 1696, Jean Bernoulli posed the problem as a challenge to the best mathematicians of his time. Jean Bernoulli himself found the solution, as well as his brother Jacques Bernoulli, Newton, Leibniz and the Marquis de l'Hopital. The solution is a cycloid arc starting with a vertical tangent [64, 114]. Skateboarding ramps, as well as the fastest decreases of *aqua-parks*, have the form of cycloid (Figure 1.3).

Figure 1.3: Cycloid arcs lead to fastest decreases and maximal adrenaline.

Some authors go further, remarking that Newton's problem of aerodynamical resistance, proposed and solved by Isaac Newton in 1686, in his *Principia Mathematica*, is a typical optimal control problem (see §1.4.8 and e.g. [102, 118]).

In mathematics, optimal control theory emerged after the Second World War responding to practical needs of engineering, particularly in the field of aeronautics and flight dynamics. The formalization of this theory raised several new questions. For example, the theory of optimal control motivated the introduction of new concepts for generalized solutions in the theory of differential equations and generated new results on the existence of trajectories.

In general, it is considered that the theory of optimal control has emerged in the late fifties in the former Soviet Union, with the formulation and demonstration of the Pontryagin maximum principle by L.S. Pontryagin (Figure 1.4) and his group of collaborators in 1956: V.G. Boltyanskii, R.V. Gamkrelidze and E.F. Mishchenko [96].

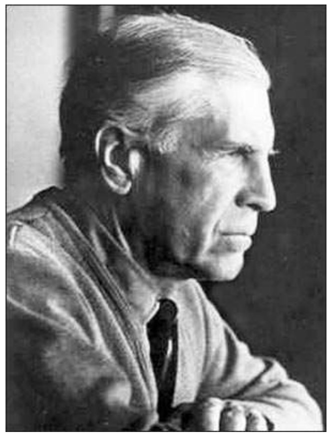

Figure 1.4: Lev Semenovich Pontryagin (3/September/1908 – 3/May/1988)

Pontryagin and his associates introduced an importante point: they generalized the theory of calculus of variations to curves that take values on closed sets (with boundary). The theory of optimal control is closely related to classical mechanics, in particular variational principles (Fermat's principle, Euler-Lagrange, etc.). In fact the maximum principle of Pontryagin generalizes the necessary conditions of Euler-Lagrange and Weierstrass. Some strengths of the new theory was the discovery of the dynamic programming method, the introduction of functional analysis to the theory of optimal systems and the discovery of links between the solutions of an optimal control problem and the results on stability of Lyapunov theory [120, 122]. Later came the foundations of stochastic control and filtering in dynamic systems, game theory, control of partial differential equations and hybrid control systems, which are some among the many areas of current research [2, 106].

1.3 Linear optimal control

The optimal control theory is much more simple when the control system under study is linear. The nonlinear optimal control theory will be recalled in Section 1.4. Even in our days the linear control theory is one of the areas more used in engineering and its applications (see e.g. [8]).

1.3.1 Statement of the problem

Let $\mathcal{M}_{n,p}(\mathbb{R})$ denote the set of matrices with n rows and p columns, with entries in $\mathbb{R}$. Let I be an interval of $\mathbb{R}$; A, B, r three locally integrable mappings on I ($A, B \in L^1_{\text{loc}}$), taking values respectively in $\mathcal{M}_{n,n}(\mathbb{R})$, $\mathcal{M}_{n,m}(\mathbb{R})$ and $\mathcal{M}_{n,1}(\mathbb{R})$. Let Ω be a subset of $\mathbb{R}^m$, and let $x_0 \in \mathbb{R}^n$. We consider the linear control system

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)u(t) + r(t)\,, \quad \forall t \in I\,, \\ &x(0) = x_0\,, \end{aligned} \tag{1.1}$$

where the controls u are mensurable locally bounded mappings over I, taking values on a subset $\Omega \subset \mathbb{R}^m$.

The existence theorem for solutions of differential equations ensures (see e.g. [121, Chapter 11]), for every control u, the existence of a unique, absolutely continuous, solution $x(\cdot) : I \to \mathbb{R}^n$ for the system (1.1). Let $M(\cdot) : I \to \mathcal{M}_{n,n}(\mathbb{R})$ be the fundamental matrix solution of the homogeneous linear system $\dot{x}(t) = A(t)x(t)$, defined by $\dot{M}(t) = A(t)M(t)$, $M(0) = \mathrm{Id}$. Note that if $A(t) = A$ is constant over I, then $M(t) = e^{tA}$. Therefore, the solution $x(\cdot)$ of system (1.1) associated to the control u is given by

$$x(t) = M(t)x_0 + \int_0^t M(t)M(s)^{-1}\left(B(s)u(s) + r(s)\right) ds\,,$$

for every $t \in I$.

This mapping depends on the control u. Therefore, if we change the function u we obtain a different trajectory $t \mapsto x(t)$ in $\mathbb{R}^n$ (see Figure 1.5).

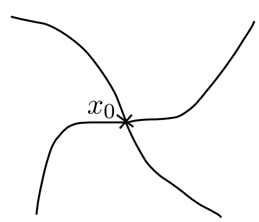

Figure 1.5: The trajectory solution of the control system (1.1) depends on the choice of the control u.

In this context, some questions arise naturally:

(i) Given a point $x_1 \in \mathbb{R}^n$, is there a control u such that the associated trajectory x steers x_0 to x_1 in a finite time t_f? (see Figure 1.6) This is the *controllability problem.*

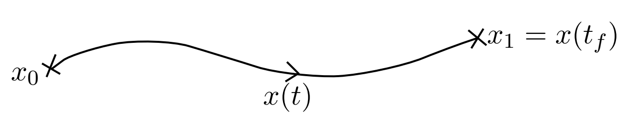

Figure 1.6: Controllability problem.

(ii) If the previous question is satisfied, is there a control whose associated trajectory steers x_0 to x_1 and minimizes a given functional $C(u)$ (See Figure 1.7). It is an *optimal control problem*. The functional $C(u)$ is the optimization criteria, and we call it *cost*. For example if the cost is the transfer time from x_0 to x_1, then we have the so-called *minimal time problem*.

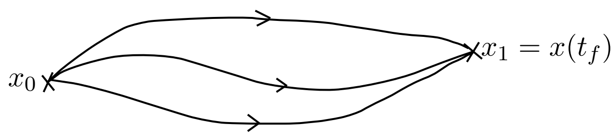

Figure 1.7: Optimal control problem

1.3.2 Controllability: definition and accessible set

Consider the linear control system (1.1). In what follows we introduce a very important set: *the accessible set*, also called *attainable set* or *reachable set* (see e.g. [53, 62]).

Definition 1.3.1. The set of accessible points from x_0 in time $T > 0$ is denoted by $A(x_0, T)$ and defined by

$$A(x_0, T) = \{x_u(T) \mid u \in L^\infty([0, T], \Omega)\},$$

where $x_u(\cdot)$ is the solution of system (1.1) associated to the control u.

In other words, $A(x_0, T)$ is the set of endpoints of the solutions of (1.1) in time T, when the control u varies (see Figure 1.8). We set $A(x_0, 0) = \{x_0\}$.

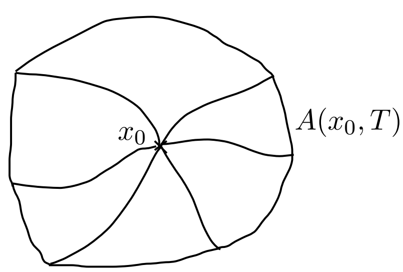

Figure 1.8: Accessible set.

In what follows some properties of the accessible set for linear control systems are given (see, e.g. [62, 121] for the respective proofs).

Theorem 1.3.2. *Consider the linear control system in* $\mathbb{R}^n$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + r(t)$$

where $\Omega \subset \mathbb{R}^m$ *is compact. Let* $T > 0$ *and* $x_0 \in \mathbb{R}^n$. *Then for every* $t \in [0, T]$, $A(x_0, t)$ *is compact, convex and varies continuously with* t *in* $[0, T]$.

Corollary 1.3.3. *If we note by* $A_\Omega(x_0, t)$ *the accessible set starting at* x_0 *in time* t *for controls taking values in* Ω, *then we set*

$$A_\Omega(x_0, t) = A_{\mathrm{Conv}(\Omega)}(x_0, t)\,,$$

where $\mathrm{Conv}(\Omega)$ *is the convex envelope of* Ω. *In particular, we have* $A_{\partial\Omega}(x_0, t) = A_\Omega(x_0, t)$, *where* $\partial\Omega$ *is the boundary of* Ω.

This last result illustrates the *bang-bang principle* (see Theorem 1.3.15). In fact, in many optimal control problems the optimal controls take values always on the boundary $\partial\Omega$ of the control constraint set Ω.

Remark 1.3.4. We observe that if $r = 0$ and $x_0 = 0$, then the solution of $\dot{x} = Ax + Bu$, $x(0) = 0$, is given by

$$x(t) = M(t) \int_0^t M(s)^{-1} B(s) u(s) ds\,,$$

and is linear with respect to u.

This remark lead us to the following proposition.

Proposition 1.3.5. *Suppose that* $r = 0$, $x_0 = 0$ *and* $\Omega = \mathbb{R}^m$. *Then,*

1. $\forall\, t > 0$ $A(0, t)$ *is a vectorial subspace of* $\mathbb{R}^n$. *Moreover,*
2. $\forall t_1, t_2$, *s.t.* $0 < t_1 < t_2$, $A(0, t_1) \subset A(0, t_2)$.

Definition 1.3.6. The set $A(0) = \cup_{t \geq 0} A(0, t)$ is the set of accessible points (at any time) starting at the origin.

Corollary 1.3.7. *The set* $A(0)$ *is a vectorial subspace of* $\mathbb{R}^n$.

The controllability definition for linear control systems follows.

Definition 1.3.8. The control system $\dot{x}(t) = A(t)x(t) + B(t)u(t) + r(t)$ is said to be *controllable in time* T if $A(x_0, T) = \mathbb{R}^n$, that is, for every $x_0, x_1 \in \mathbb{R}^n$, there exists a control u such that the associated trajectory steers x_0 to x_1 in time T (see Figure 1.9).

The following theorem give us a necessary and sufficient condition for controllability, in the case where A and B do not depend of t and there are no constraints on the control ($u(t) \in \mathbb{R}^m$).

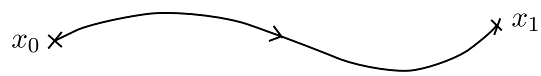

Figure 1.9: Controllability

Theorem 1.3.9 (Kalman condition)**.** *Suppose that $\Omega = \mathbb{R}^m$ (no constraints on the control). The system $\dot{x}(t) = Ax(t) + Bu(t) + r(t)$ is controllable in time T (arbitrary) if and only if the matrix $C = (B, AB, \cdots, A^{n-1}B)$ is of rank n.*

The matrix C is called the *Kalman matrix.*

Remark 1.3.10. The Kalman condition does not depend on T neither on x_0. In other words, if an autonomous linear system is controllable in time T starting at x_0, then is controllable in any time starting at any point.

In Theorem 1.3.9 no constraint on the control is considered. The next theorem is a controllability result when the control is scalar, i.e., $m = 1$, and $u(t) \in \Omega \subset \mathbb{R}$.

Theorem 1.3.11. *Let $b \in \mathbb{R}^n$ and $\Omega \subset \mathbb{R}$ an interval having 0 in its interior. Consider the system $\dot{x}(t) = Ax(t) + bu(t)$, with $u(t) \in \Omega$. Then every point of $\mathbb{R}^n$ can be steered to the origin in finite time if and only if the couple (A, b) satisfies the Kalman condition and the real part of each eigenvalue of A is less or equal than zero.*

1.3.3 Minimal time problem

We start by formalizing, with the help of the accessible set $A(x_0, t)$, the notion of minimal time.

Consider the control system on $\mathbb{R}^n$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + r(t)\,,$$

where the controls u take values in a compact set $\Omega \subset \mathbb{R}^m$ with nonempty interior. Let x_0, x_1 be two points of $\mathbb{R}^n$. Suppose that x_1 is accessible from x_0, i.e., suppose that there exists at least one trajectory steering x_0 to x_1. Between all the trajectories that steer x_0 to x_1 we would like to characterize the one that does it in minimal time $\hat{t}_f$ (see Figure 1.10).

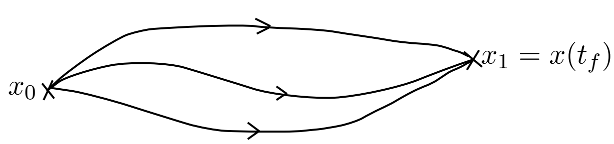

Figure 1.10: Which is the trajectory x with minimal time?

If $\hat{t}_f$ is the minimal time, then for every $t < \hat{t}_f$, $x_1 \notin A(x_0, t)$ (in effect, otherwise x_1 would be accessible from x_0 in a time smaller than $\hat{t}_f$ and $\hat{t}_f$ would not be the minimal time). Therefore,

$$\hat{t}_f = \inf\{t > 0 \,|\, x_1 \in A(x_0, t)\}\,. \tag{1.2}$$

The value of $\hat{t}_f$ is well defined because, from Theorem 1.3.2, $A(x_0, t)$ varies continuously with t, thus $\{t > 0 \,|\, x_1 \in A(x_0, t)\}$ is closed in $\mathbb{R}$. In particular the infimum in (1.2) is a minimum. The time $t = \hat{t}_f$ is the first instant such that $A(x_0, t)$ contains x_1 (see Figure 1.11).

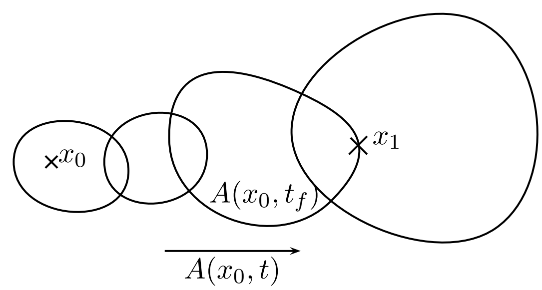

Figure 1.11: Minimal time.

On the other hand, we have

$$x_1 \in \partial A(x_0, \hat{t}_f) = A(x_0, \hat{t}_f) \backslash int\, A(x_0, \hat{t}_f)\,.$$

In fact, if x_1 belongs to the interior of $A(x_0, \hat{t}_f)$, then for $t < \hat{t}_f$ close to $\hat{t}_f$, x_1 also belongs to $A(x_0, t)$ since $A(x_0, t)$ varies continuously with t. This contradicts the fact that $\hat{t}_f$ is minimal time.

The next theorem states that if a minimal time problem with a linear control system in $\mathbb{R}^n$ is controllable then it has at least one solution.

Theorem 1.3.12. *If the point x_1 is accessible from x_0 then there exists a minimal time trajectory steering x_0 to x_1.*

Remark 1.3.13. We can also consider the steering problem to a target that does not reduce to a single point. Therefore, let $(M_1(t))_{0 \leq t \leq t_f}$ be a family of compact subsets of $\mathbb{R}^n$ varying continuously with t. As before, we see that if there exists a control u taking values in Ω steering x_0 to $M_1(t_f)$, then there exists a minimal time control defined on $[0, \hat{t}_f]$ steering x_0 to $M(\hat{t}_f)$.

This remark give us a geometric vision of the notion of minimum time and lead us to the following definition.

Definition 1.3.14. The control u is an *extremal* on $[0, t]$ if the trajectory of system (1.1) associated to u satisfies $x(t) \in \partial A(x_0, t)$.

Every minimal time control is an extremal. The converse does not hold in general.

Optimality condition: maximum principle in the linear case

The next theorem give us a necessary and sufficient condition in order that extremal controls are also optimal controls.

Theorem 1.3.15. *Consider the linear control system*

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + r(t)\,, \\ x(0) = x_0\,, \end{cases}$$

where the domain of control constraints $\Omega \subset I\!R^m$ is compact. Let $t_f > 0$. The control u is an extremal on $[0, t_f]$ if and only if there exists a nontrivial solution $p(t)$ of the equation $\dot{p}(t) = -p(t)A(t)$ such that

$$p(t)B(t)u(t) = \max_{w \in \Omega} p(t)B(t)w \tag{1.3}$$

for every $t \in [0, t_f]$. The row vector $p(t) \in I\!R^n$ is called the adjoint vector.

Remark 1.3.16. In the case of a scalar control, and if moreover $\Omega = [-a, a]$ where $a > 0$, the maximization condition (1.3) implies immediately that $u(t) = a\,\mathrm{sign}\langle p(t), B(t)\rangle$. The function $\varphi(t) = \langle p(t), B(t)\rangle$ is called a *switching function*, and the time t_s at which the extremal control $u(t)$ change its sign is called a *switching time*. It is, in particular, a root of the function φ.

The initial condition $p(0)$ depends on x_1. As this condition is not directly known, the application of Theorem 1.3.15 is mostly done indirectly. Let us see an example.

1.3.4 Example: optimal control of an harmonic oscillator (linear case)

Consider a punctual mass m, forced to move along an axis (Ox), attached to a spring (see Figure 1.12).

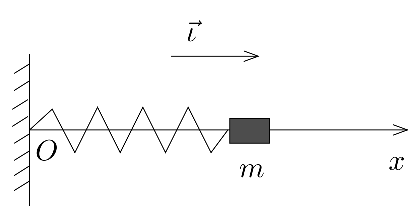

Figure 1.12: A spring

The mass is then drawn towards the origin by a force that is assumed equal to $-k_1(x-l)-k_2(x-l)^3$, where l is the length of the spring at rest, and k_1, k_2 are the coefficients of stiffness. We apply to this mass point an external horizontal force $u(t)\vec{l}$. The laws of physics give us the motion equation

$$m\ddot{x}(t)+k_1(x(t)-l)+k_2(x(t)-l)^3=u(t)\,. \tag{1.4}$$

Moreover we impose a constraint on the external force,

$$|u(t)|\leq 1\,, \quad \forall\, t\,.$$

This means we can not apply any external horizontal force to the point mass: the external force can only take values on the interval $[-1,1]$, reflecting the fact that our power of action is limited.

Assume that the initial position and velocity of the object are, respectively, $x(0)=x_0$ and $\dot{x}(0)=y_0$. The problem consists in driving the point mass to the equilibrium position $x=l$ in *minimal time controlling the external force* $u(t)$ that is applied to this object, and taking into account the *constraint* $|u(t)|\leq 1$. The function u is the *control*.

Problem 1.3.17. Given the initial conditions $x(0)=x_0$ and $\dot{x}(0)=y_0$, the goal is to find a function $u(t)$ which allows the movement of the point mass to its equilibrium position in minimal time.

Mathematic modeling

To simplify the presentation, we will suppose that $m=1\,kg$, $k_1=1\,N.m^{-1}$ and $l=0\,m$ (we pass to $l=0$ by translation). The equation of motion (1.4) is equivalent to the controlled differential system

$$\begin{cases}\dot{x}(t)=y(t)\\ \dot{y}(t)=-x(t)-k_2x(t)^3+u(t)\end{cases} \tag{1.5}$$

with $x(0)=x_0$ and $\dot{x}(0)=y_0$.

Writing (1.5) in matricial notation we have

$$\dot{X}(t)=AX(t)+f(X(t))+Bu(t)\,, \quad X(0)=X_0\,, \tag{1.6}$$

where

$$A=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad B=\begin{pmatrix}0\\ 1\end{pmatrix},$$
$$X=\begin{pmatrix}x\\ y\end{pmatrix}, \quad X_0=\begin{pmatrix}x_0\\ y_0\end{pmatrix}, \quad f(X)=\begin{pmatrix}0\\ -k_2x^3\end{pmatrix}.$$

In this section we are considering linear control systems, therefore we fixe $k_2 = 0$, and we do not take into account the nonlinear conservative effects (in Section 1.4, we consider nonlinear control systems and take $k_2 \neq 0$). If $k_2 = 0$ then $f(X) \equiv 0$ and the control system (1.6) has the form of (1.1) (linear control system). We wish to answer the two following questions.

1. Is there always, for any initial condition $x(0) = x_0$ and $\dot{x}(0) = y_0$, an horizontal exterior force (a control) that allows to move, in finite time t_f, the point mass to its equilibrium position $x(t_f) = 0$ and $\dot{x}(t_f) = 0$?

2. If the answer to the first question is affirmative, which is the force (which is the control) that minimizes the transfer time of the point mass to its equilibrium position?

System controllability

Our system writes in the form

$$\begin{cases} \dot{X} = AX + Bu \\ X(0) = X_0 \end{cases}$$

with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We have then

$$rank\,(B, AB) = rank \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$$

and the eigenvalues of A have zero real part. Therefore, from Theorem 1.3.11, the system is controllable, that is, there exist controls u satisfying the constraint $|u(t)| \leq 1$ such that the associated trajectories steer X_0 to 0. We answered affirmatively to the first question.

This answer corresponds to the physical interpretation of the problem. In fact, if we do not apply an exterior force, that is, if $u = 0$ then the motion equation is $\ddot{x} + x = 0$ and the point mass will continues to oscillate, never stopping, in a finite time, at its equilibrium position. On the other hand, when exterior forces are applied, we tend to dampen the oscillations. The control theory predicts that we can stop the object in a finite time.

Computation of the optimal control

We know that there exist controls that allow to steer the system from X_0 to 0 in finite time. Now we want to compute, concretely, which one of these controls does it in minimal time. To do so we apply the Theorem 1.3.15 and obtain

$$u(t) = \text{sign}\,(\langle p(t), B\rangle)\ ,$$

where $p(t) \in \mathbb{R}^2$ is the solution of $\dot{p} = -pA$. Let $p(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}$. Then, $u(t) = \text{sign }(p_2(t))$ and $\dot{p}_1 = p_2$, $\dot{p}_2 = -p_1$, that is, $\ddot{p}_2 + p_2 = 0$. Thus $p_2(t) = \lambda \cos t + \mu \sin t$. Therefore, the optimal control is piecewise constant in intervals of size π and take alternately the values ± 1.

- If $u = -1$, we get the differential system

$$\begin{cases} \dot{x} = y \,, \\ \dot{y} = -x - 1 \,. \end{cases} \tag{1.7}$$

- If $u = +1$, we get

$$\begin{cases} \dot{x} = y \,, \\ \dot{y} = -x + 1 \,. \end{cases} \tag{1.8}$$

The optimal trajectory, steering X_0 to 0, consists in concatenated pieces of solutions of (1.7) and (1.8). The solutions of (1.7) and (1.8) are obtained easily: from equation (1.7) we have $(x+1)^2 + y^2 = const = R^2$ and we conclude that the solution curves of (1.7) are circles centered on $x = -1$ and $y = 0$ of period 2π (in fact, $x(t) = -1 + R\cos t$ and $y(t) = R\sin t$); as solutions of (1.8) we get $x(t) = 1 + R\cos t$ and $y(t) = R\sin t$, i.e., the solutions of (1.8) are circles centered in $x = 1$ and $y = 0$ of period 2π.

The optimal trajectory that steers X_0 to 0 follows alternately an arc of a circle centered in $x = -1$ and $y = 0$ and an arc of a circle centered in $x = 1$ and $y = 0$. The detailed study of the optimal trajectory and its numerical implementation, for every X_0, can be founded in [121]. See also Section 2.6 where the optimal control problem is solved.

1.4 Nonlinear optimal control

We now present some techniques to analyze nonlinear optimal control problems (the proofs of the presented results can be found, for example, in [62, 121]). In particular, we enunciate the Pontryagin maximum principle in a more general form than the one we have seen in Section 1.3. The nonlinear example of the spring will be one of the application examples.

1.4.1 Statement of the problem

From a general point of view, the problem should be presented in a manifold M, but our point of view will be *local* and we work on an open V of $\mathbb{R}^n$ small enough. The general optimal control problem is the following. Consider the control system

$$\dot{x}(t) = f(t, x(t), u(t)) \,, \quad x(t_0) = x_0 \,, \tag{1.9}$$

where f is a mapping of class C^1 [1] from $I \times V \times U$ into $\mathbb{R}^n$, I is an interval of $\mathbb{R}$, V is an open set of $\mathbb{R}^n$, U is an open set of $\mathbb{R}^m$, $(t_0, x_0) \in I \times V$. We suppose that the controls $u(\cdot)$ belong to a subset of $L^\infty_{\text{loc}}(I, \mathbb{R}^m)$.

These hypotheses assure, for every control $u(\cdot)$, the existence and uniqueness of a maximal solution $x_u(\cdot)$ over an interval $J \subset I$, of the Cauchy problem (1.9) (see e.g. [121, Chapter 11]).

In what follows we will consider, without loss of generality, $t_0 = 0$.

Definition 1.4.1. Let $t_f > 0$, $t_f \in I$. A control function $u(\cdot) \in L^\infty([0, t_f], \mathbb{R}^m)$ is said *admissible* on $[0, t_f]$ if the trajectory $x(\cdot)$, solution of (1.9) associated to $u(\cdot)$, is well defined on $[0, t_f]$. The set of admissible controls on $[0, t_f]$ is denoted $\mathcal{U}_{t_f, \mathbb{R}^m}$, and the set of admissible controls on $[0, t_f]$ taking their values in Ω is denoted $\mathcal{U}_{t_f, \Omega}$.

In what follows we will abbreviate the notation for admissible controls taking value in $\mathbb{R}^m$ writing $\mathcal{U}_{t_f}$.

Let f^0 be a function of class C^1 over $I \times V \times U$, and g a continuous function over V. For every control $u(\cdot) \in \mathcal{U}_{t_f}$ we define the cost of the associated trajectory $x_u(\cdot)$ over the interval $[0, t_f]$ by

$$C(t_f, u) = \int_0^{t_f} f^0(t, x_u(t), u(t))dt + g(t_f, x_u(t_f)) .$$

Let M_0 and M_1 be two subsets of V. The optimal control problem is to compute the trajectories $x_u(\cdot)$ solutions of

$$\dot{x}_u(t) = f(t, x_u(t), u(t)) ,$$

such that $x_u(0) \in M_0$, $x_u(t_f) \in M_1$, and minimizing the cost $C(t_f, u)$. We say that the optimal control has *free final time* if the final time t_f is free, otherwise we say that the problem has *fixed final time*.

1.4.2 End-point mapping

Consider for the system (1.9) the following optimal control problem: given a point $x_1 \in \mathbb{R}^n$, find a time t_f and a control u over $[0, t_f]$ such that the trajectory x_u associated to the control u, solution of (1.9), satisfies

$$x_u(0) = x_0 , \quad x_u(t_f) = x_1 .$$

This leads us to the following definition.

[1] F.H. Clarke is the author of the so-called *Nonsmooth Analysis* created in the seventies which allows the study of more general optimal control problems, where the used functions are not necessarily differentiable in the classic sense. For a detailed study on Nonsmooth Analysis see, e.g., [30–33] and the references cited therein.

Definition 1.4.2. Let $t_f > 0$. The *end-point mapping* in time t_f of the control system (1.9) starting in x_0 is the mapping

$$\begin{aligned} E_{t_f} : \mathcal{U}_{t_f} &\longrightarrow \mathbb{R}^n \\ u &\longmapsto x_u(t_f)\,. \end{aligned}$$

In other words, the end-point mapping in time t_f associes to a control u the final point of the trajectory associated to the control u (see Figure 1.13).

Remark 1.4.3. We also can denote the end-point mapping by $E(x_0, t_f, u)$ (see, e.g., Section §1.5).

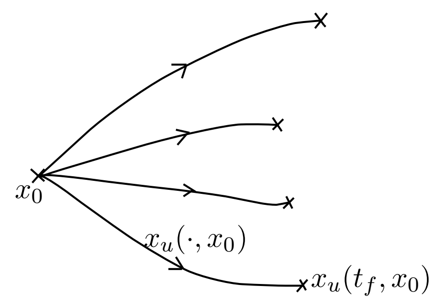

Figure 1.13: End-point mapping.

A very important issue in the theory of optimal control is the study of the map E_{t_f}, describing its image, singularities, regularity, etc. The answer to these questions depends, obviously, on the space $\mathcal{U}_{t_f}$ and on the shape of the system (on the function f). With all the generality we have the following result (see, e.g., [16, 53, 106]).

Proposition 1.4.4. *Consider the system* (1.9) *where f is C^p, $p \geq 1$, and let $\mathcal{U}_{t_f} \subset L^\infty([0, t_f], \mathbb{R}^m)$ be the domain of E_{t_f}, that is, the set of controls whose associated trajectory is well defined over $[0, t_f]$. Then $\mathcal{U}_{t_f}$ is an open set of $L^\infty([0, t_f], \mathbb{R}^m)$, and E_{t_f} is C^p in the L^∞ sense.*

Moreover the Fréchet differential of E_{t_f} at a point $u \in \mathcal{U}_{t_f}$ is given by the linearized system at u in the following way. Let, for every $t \in [0, t_f]$,

$$A(t) = \frac{\partial f}{\partial x}(t, x_u(t), u(t))\,, \quad B(t) = \frac{\partial f}{\partial u}(t, x_u(t), u(t))\,.$$

The linearized control system

$$\begin{aligned} &\dot{y}_v(t) = A(t) y_v(t) + B(t) v(t) \\ &y_v(0) = 0 \end{aligned}$$

is called the linearized system along the trajectory x_u. The Fréchet differential of E_{t_f} at u is then the mapping $dE_{t_f}(u)$ such that, for every $v \in L^\infty([0, t_f], \mathbb{R}^m)$,

$$dE_{t_f}(u) \cdot v = y_v(t_f) = M(t_f) \int_0^{t_f} M^{-1}(s) B(s) v(s) ds$$

where M is the fundamental matrix of the linearized system, i.e., the matricial solution of $\dot{M} = AM$, $M(0) = \mathrm{Id}$.

The previous result can be improved for control-affine systems (see [106, 119]).

Definition 1.4.5. A *control-affine system* is a system of the form

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)) \,,$$

where f_i are vector fields of $\mathbb{R}^n$.

Proposition 1.4.6. *Consider a smooth control-affine system, and let $\mathcal{U}_{t_f}$ be the domain of E_{t_f}. Then $\mathcal{U}_{t_f}$ is an open set of $L^2([0,t_f], \mathbb{R}^m)$, and the end-point mapping E_{t_f} is smooth in the L^2 sense, and is analytic if the vector field are analytic.*

1.4.3 Accessible set and controllability

Definition 1.4.7. *The accessible set in a time t_f* for the system (1.9), denoted by $A(x_0, t_f)$, is the set of all extremities in time t_f of the solutions of the system starting at x_0 in time $t = 0$. In other words, is the image of the end-point mapping in time t_f.

Theorem 1.4.8. *Consider the control system*

$$\dot{x} = f(t, x, u) \,, \quad x(0) = x_0 \,,$$

where the function f is C^1 over $\mathbb{R}^{1+n+m}$, and the controls u belong to the set $\mathcal{U}_{t_f,\Omega}$ of measurable functions taking values in a compact $\Omega \subset \mathbb{R}^m$. We suppose that

- *there exists a positive real b such that the associated trajectory is uniformly bounded by b over $[0, t_f]$, i.e.,*

$$\exists b > 0 \,|\, \forall u \in \mathcal{U} \quad \forall t \in [0, t_f] \quad \|x_u(t)\| \leq b \,, \tag{1.10}$$

- *for every (t, x), the set of velocity vectors*

$$V(t, x) = \{f(t, x, u) \,|\, u \in \Omega\} \tag{1.11}$$

  is convex.

Then the set $A(x_0, t)$ is compact and varies continuously in t over $[0, t_f]$.

Remark 1.4.9. The hypothesis (1.10) is not a consequence of the other hypotheses and is indispensable. In fact, consider the system $\dot{x} = x^2 + u$, $x(0) = 0$, where we suppose that $|u(t)| \leq 1$ and that the final time is $t_f = \frac{\pi}{2}$. Then for every control u constant equal to c, with $0 < c < 1$, the trajectory associated is $x_c(t) = \sqrt{c} \tan \sqrt{c} t$, therefore is well defined over $[0, t_f]$, but when c tends to 1 then $x_c(t_f)$ tends to $+\infty$ (see Figure 1.14). On the other hand it is easy to see that in this example the set of admissible controls, taking values in $[-1, 1]$, is the set of measurable functions such that $u(t) \in [-1, 1[$.

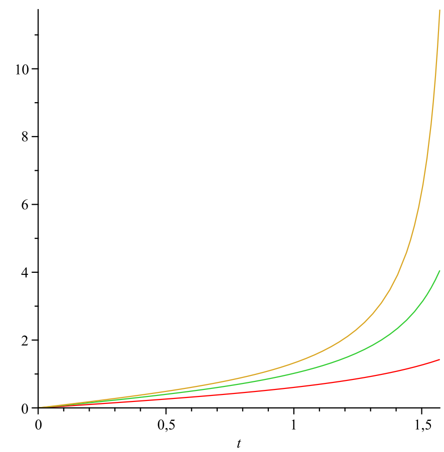

Figure 1.14: Trajectory $x_c(t)$ of example in Remark 1.4.9, for $t \in [0, \frac{\pi}{2}]$ and $c = 0.5; 0.75; 0.9$.

Remark 1.4.10. Analogously, the convexity hypothesis (1.11) is necessary (see [62, Example 2,pag. 244]).

Definition 1.4.11. The system (1.9) is said to be *controllable (in an arbitrary time)* starting at x_0 if

$$\bigcup_{T \geq 0} A(x_0, T) = \mathbb{R}^n .$$

The system (1.9) is said to be *controllable in time* T if $A(x_0, T) = \mathbb{R}^n$.

Arguments based on the implicit function theorem allow to deduce results on *local controllability* of the starting system by the study of the controllability of the linearized system (see, e.g., [62]). For example, we deduce from the controllability theorem in the linear case the following proposition.

Proposition 1.4.12. *Consider the control system* (1.9) *where* $f(x_0, u_0) = 0$. *Let* $A = \frac{\partial f}{\partial x}(x_0, u_0)$ *and* $B = \frac{\partial f}{\partial u}(x_0, u_0)$. *If*

$$rank\,(B|AB|\cdots|A^{n-1}B) = n$$

then the nonlinear system (1.9) *is locally controllable at* x_0.

In general the controllability problem is difficult. Different approaches are possible. Some of them make use of Analysis, others Geometry, others Algebra, etc. The controllability problem is connected, for example, to the question of knowing when a given semi-group acts transitively. There are also some techniques to prove, in some cases, global controllability. One of them, an important one, is called *enlargement technique* (see [53]).

1.4.4 Singular controls

Definition 1.4.13. Let u be a control defined on $[0, t_f]$ such that the associated trajectory x_u starting at $x(0) = x_0$ is defined on $[0, t_f]$. We say that a control u (or the trajectory x_u) is *singular*[2] over $[0, t_f]$ if the Fréchet derivative $dE_{t_f}(u)$ of the end-point mapping at the point u is not surjective. Otherwise we say that u is *regular*.

Proposition 1.4.14. *Let x_0 and t_f be fixed. If u is a regular control, then E_{t_f} is an open map in a neighborhood of u.*

In other words, at a point x_1 accessible in time t_f from x_0 by a regular trajectory $x(\cdot)$, the accessible set $A(x_0, t_f)$ is *locally open*, i.e., is a neighborhood of the point x_1. In particular this implies that the system is *locally controllable* in a neighborhood of the point x_1. We also say *controllability along the trajectory* $x(\cdot)$. The next proposition follows.

Proposition 1.4.15. *If u is a regular control over $[0, t_f]$, then the system is locally controllable along the trajectory associated to that control.*

Corollary 1.4.16. *Let u be a control defined on $[0, t_f]$ such that the associated trajectory x_u starting at $x(0) = x_0$ is defined over $[0, t_f]$ and satisfies at time t_f*

$$x(t_f) \in \partial A(x_0, t_f)\,.$$

Then the control u is singular over $[0, t_f]$.

Remark 1.4.17. The system can be locally controllable along a singular trajectory. This is the case of the scalar system $\dot{x} = u^3$, where the control $u = 0$ is singular.

1.4.5 Existence of optimal trajectories

More than a control problem, we consider also an optimization problem: between all the solutions of the system (1.9) steering 0 to x_1, find a trajectory that minimizes (or maximizes) a certain *cost* function $C(t_f, u)$. Such a trajectory, if it exists, is called *optimal* for that cost. The existence of optimal trajectories depende on the regularity of the system and of the cost. For a general existence theorem see, e.g., [53, 62]. It can also happen that an optimal control does not exist in the class of considered controls, but there exists in a wider space . This question leads us to an important area: the study of regularity of optimal trajectories. An important contribution in this area is given in [34, 36, 123], where a systematic study of the Lipschitizian regularity of the minimizers on the linear optimal control is introduced. General results on the Lipschitizian regularity of minimizing trajectories for nonlinear control systems can be founded in [117].

[2] In this chapter the term "singular" is associated to a geometric control theory definition. On the other hand, please note that, in Chapter 2 "singular control" is associated to control-affine systems when the switching function vanishes on a nontrivial interval.

The following theorem applies to general control systems, eventually, with state constraints.

Theorem 1.4.18. *Consider the control system*

$$\dot{x}(t) = f(t, x(t), u(t)),$$

where f is C^1 from $I\!R^{1+n+m}$ into $I\!R^n$, the controls u take values in a compact $\Omega \subset I\!R^m$, and where there exist, eventually, constraints on the state variable

$$c_1(x(t)) \leq 0, ..., c_r(x(t)) \leq 0 \quad \forall\, 0 \leq t \leq t_f = t(u),$$

where $c_1, ..., c_r$ are continuous functions in $I\!R^n$. Let M_0 and M_1 be two compacts subsets of $I\!R^n$ such that M_1 is accessible from M_0. Let $\mathcal{U}$ be the set of controls taking values in Ω steering M_0 to M_1. Let f^0 be a C^1 function over $I\!R^{1+n+m}$, and g a continuous function over $I\!R^n$. We consider the cost

$$C(u) = \int_0^{t(u)} f^0(t, x(t), u(t))dt + g(t(u), x(t(u))),$$

where $t(u) \geq 0$ is such that $x(t(u)) \in M_1$. We suppose that

- *there exists a positive real b such that every trajectory associated to a control $u \in \mathcal{U}_{t_f}$ is uniformly bounded by b over $[0, t(u)]$, i.e.*

$$\exists b > 0 \,|\, \forall u \in \mathcal{U} \;\; \forall t \in [0, t(u)] \quad \|x_u(t)\| \leq b,$$

- *for every $(t, x) \in I\!R^{1+n}$, the augmented set of velocity vectors*

$$\tilde{V}(t, x) = \{(f^0(t, x, u), f(t, x, u)) \,|\, u \in \Omega\}$$

  is convex.

Then there exists an optimal control u over $[0, t(u)]$ such that the associated trajectory steers M_0 to M_1 in time $t(u)$ with minimal cost.

For an optimal control problem with fixed final time we impose $t(u) = t_f$ (in particular we suppose that the target M_1 is accessible from M_0 in time t_f).

Remark 1.4.19. A more general result can be stated where the sets M_0 and M_1 depend on the time t, as well as the domain of the control constraints (see [62]).

For control-affine systems the following result holds.

Proposition 1.4.20. *Consider the affine system in $I\!R^n$*

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \;\; x(0) = x_0, \; x(t_f) = x_1,$$

with the cost

$$C_{t_f}(u) = \int_0^{t_f} \sum_{i=1}^{m} u_i^2(t)dt\,,$$

where $t_f > 0$ *is fixed and the class* $\mathcal{U}_{t_f}$ *of admissible controls is the subset of* $L^2([0,t_f], I\!R^m)$ *such that*

1. $\forall u \in \mathcal{U} \quad x_u$ *is well defined over* $[0,t_f]$*;*
2. $\exists B_{t_f} \mid \forall u \in \mathcal{U} \quad \forall t \in [0,t_f]\ \|x_u\| \leq B_{t_f}$.

If x_1 *is accessible from* x_0 *in time* t_f*, then there exist an optimal control steering* x_0 *to* x_1*.*

1.4.6 Pontryagin maximum principle

Given an optimal control problem for which existence and regularity conditions are satisfied for the optimal solution, how to find the optimal processes? The answer to this question is given by the well known *Pontryagin Maximum Principle*. For a detailed study on necessary optimality conditions we suggest [30, 105, 121].

We start by showing that a singular trajectory can be parametrized as a projection of a solution of an hamiltonian system subject to a *constraint equation*. Consider the *Hamiltonian* for the control system (1.9):

$$\begin{aligned} H\ :\ \mathbb{R}^n \times \mathbb{R}^n\backslash\{0\} \times \mathbb{R}^m &\to \mathbb{R} \\ (x,p,u) &\mapsto H(x,p,u) = \langle p, f(x,u)\rangle \end{aligned}$$

where $\langle\, ,\, \rangle$ denotes the usual inner product of $\mathbb{R}^n$.

Proposition 1.4.21. *Let* u *be a singular control and* x *a singular trajectory associated to this control on* $[0,t_f]$*. Then, there exists a continuous row vector* $p : [0,t_f] \to I\!R^n\backslash\{0\}$ *such that the following equations are satisfied for almost every* $t \in [0,t_f]$*:*

$$\begin{aligned} &\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), u(t))\,, \\ &\dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t))\,, \\ &\frac{\partial H}{\partial u}(x(t), p(t), u(t)) = 0 \ \ \textit{(constraint equation)} \end{aligned}$$

where H *is the Hamiltonian of the system.*

Proof. By Definition 1.4.13, the pair (x,u) is singular over $[0,t_f]$ if $dE_{t_f}(u)$ is not surjective. Therefore, there exists a row vector $\bar{p} \in \mathbb{R}^n\backslash\{0\}$ such that

$$\forall\, v(\cdot) \in L^\infty([0,t_f])\ \ \langle \bar{p}, dE_{t_f}(u)\cdot v\rangle = \bar{p}\int_0^{t_f} M(t_f)M^{-1}(s)B(s)v(s)ds = 0\,.$$

Thus,

$$\bar{p}M(t_f)M^{-1}(s)B(s) = 0 \text{ for almost every point of } [0, t_f]\,.$$

Let $p(t) = \bar{p}M(t_f)M^{-1}(t)$, $t \in [0, t_f]$. We have that p is a row vector of $\mathbb{R}^n \backslash \{0\}$ and $p(t_f) = \bar{p}$. Differentiating, we get

$$\dot{p}(t) = -p(t)\frac{\partial f}{\partial x}(x(t), u(t))\,.$$

Introducing the Hamiltonian $H(x, p, u) = \langle p, f(x, u)\rangle$ we conclude that

$$\dot{x}(t) = f(x(t), u(t)) = \frac{\partial H}{\partial p}(x(t), p(t), u(t))$$

and

$$\dot{p}(t) = -p(t)\frac{\partial f}{\partial x}(x(t), u(t)) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t))\,.$$

The constraint equation comes from $p(t)B(t) = 0$ because $B(t) = \frac{\partial f}{\partial u}(x(t), u(t))$. □

Definition 1.4.22. The row vector $p : [0, t_f] \to \mathbb{R}^n \backslash \{0\}$ of Proposition1.4.21 is called *adjoint vector* of the system (1.9).

Weak maximum principle

Consider the Lagrange problem given by the control system

$$\dot{x}(t) = f(t, x(t), u(t))\,, \tag{1.12}$$

where the controls $u(\cdot) \in \mathcal{U}_{t_f}$ are defined in $[0, t_f]$ and take values in $\Omega = \mathbb{R}^m$ (there are no restrictions on the values of the control). The associated trajectories must satisfy $x(0) = x_0$ and $x(t_f) = x_1$. The problem consist in minimizing a cost of the form

$$C(u) = \int_0^{t_f} f^0(t, x(t), u(t))dt\,, \tag{1.13}$$

where t_f is fixed.

Associate to the system (1.12) the following *augmented system*

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)) \\ \dot{x}^0(t) &= f^0(t, x(t), u(t)) \end{aligned} \tag{1.14}$$

and use the notation $\tilde{x} = (x, x^0)$ and $\tilde{f} = (f, f^0)$. The problem is reduced to finding a trajectory solution of (1.14) with $\tilde{x}_0 = (x_0, 0)$ and $\tilde{x}_1 = (x_1, x^0(t_f))$ such that the last coordinate $x^0(t_f)$ is minimized.

The set of accessible states starting at $\tilde{x}_0$ for the system (1.14) is $\tilde{A}(\tilde{x}_0, t_f) = \cup_{u(\cdot)}\tilde{x}(t_f, \tilde{x}_0, u)$.

The following Lemma is crucial.

Lemma 1.4.23. *If the control u associated to the control system* (1.12) *is optimal for the cost* (1.13)*, then it is singular on* $[0, t_f]$ *for the augmented system* (1.14).

Proof. Let u be a control and $\tilde{x}$ the associated trajectory, solution of the augmented system (1.14) starting at $\tilde{x}_0 = (x_0, 0)$. If u is optimal for the criteria (1.13), then the point $\tilde{x}(t_f)$ belongs to the boundary of the set $\tilde{A}(\tilde{x}_0, t_f)$. In fact, if that was not the case then there would exist a neighborhood of the point $\tilde{x}(t_f) = (x_1, x^0(t_f))$ in $\tilde{A}(\tilde{x}_0, t_f)$ containing a point $\tilde{y}(t_f)$ solution of system (1.14) and such that $y^0(t_f) < x^0(t_f)$, which contradicts the optimality of the control u (see Figure 1.15). Therefore, by Proposition 1.4.14 the control $\tilde{u}$ is singular for the *augmented system* (1.14). □

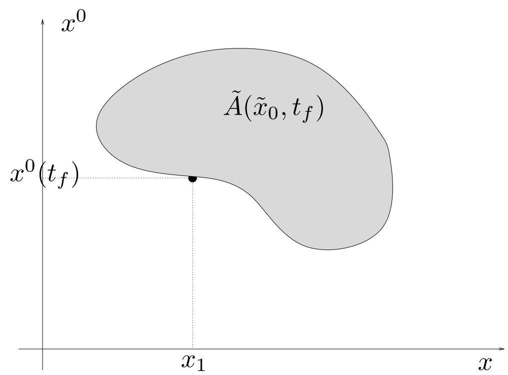

Figure 1.15: If u is optimal, then $\tilde{x}(t_f) \in \partial\tilde{A}(\tilde{x}_0, t_f)$.

Under the assumptions of the previous lemma, following Proposition 1.4.21, there exists a map $\tilde{p} : [0, t_f] \to \mathbb{R}^{n+1}\backslash\{0\}$ such that $(\tilde{x}, \tilde{p}, \tilde{u})$ is solution of the Hamiltonian system

$$\begin{aligned}
&\dot{\tilde{x}}(t) = \frac{\partial \tilde{H}}{\partial \tilde{p}}(t, \tilde{x}(t), \tilde{p}(t), u(t)), \quad \dot{\tilde{p}}(t) = -\frac{\partial \tilde{H}}{\partial \tilde{x}}(t, \tilde{x}(t), \tilde{p}(t), u(t)), \\
&\frac{\partial \tilde{H}}{\partial u}(t, \tilde{x}(t), \tilde{p}(t), u(t)) = 0
\end{aligned}$$

where $\tilde{H}(t, \tilde{x}, \tilde{p}, u) = \langle \tilde{p}, \tilde{f}(t, \tilde{x}, u)\rangle$.

Writing $\tilde{p} = (p, p^0) \in (\mathbb{R}^n \times \mathbb{R})\backslash\{0\}$, where p^0 is called the dual variable of the cost, we get

$$(\dot{p}, \dot{p}^0) = -(p, p^0)\begin{pmatrix} \frac{\partial f}{\partial x} & 0 \\ \frac{\partial f^0}{\partial x} & 0 \end{pmatrix}.$$

In particular, $\dot{p}^0(t) = 0$, that is, p^0 is constant in $[0, t_f]$. As the vector $\tilde{p}(t)$ is defined up to a multiplicative scalar, we chose $p^0 \leq 0$. On the other hand, $\tilde{H} = \langle \tilde{p}, \tilde{f}(t, x, u)\rangle = pf + p^0 f$, thus

$$\frac{\partial \tilde{H}}{\partial u} = 0 = p\frac{\partial f}{\partial u} + p^0\frac{\partial f^0}{\partial u}.$$

We get the following result.

Theorem 1.4.24 (Weak maximum principle – Hestenes's theorem [49]). *If the control u associated to the system* (1.12) *is optimal for the cost* (1.13)*, then there exists a map $p(\cdot)$ absolutely continuous on $[0,t_f]$, taking values in $\mathbb{R}^n$, called adjoint vector, and a real number $p^0 \leq 0$, such that the couple $(p(\cdot),p^0)$ is nontrivial, and the following equations are satisfied for almost every $t \in [0,t_f]$*

$$\begin{aligned}
&\dot{x}(t) = \frac{\partial H}{\partial p}(t,x(t),p(t),p^0,u(t)),\\
&\dot{p}(t) = -\frac{\partial H}{\partial x}(t,x(t),p(t),p^0,u(t)),\\
&\frac{\partial H}{\partial u}(t,x(t),p(t),p^0,u(t)) = 0,
\end{aligned} \tag{1.15}$$

where H is the Hamiltonian

$$H(t,x,p,p^0,u) = \langle p, f(t,x,u)\rangle + p^0 f^0(t,x,u)$$

associated to the system (1.12) *and to the cost* (1.13).

The Theorem 1.4.24 has its origin in the works of Graves of 1933, being firstly obtained by Hestenes in 1950 [49]. It is a particular case of Pontryagin Maximum Principle where no restrictions on the controls are considered (i.e., $u(t) \in \Omega$ with $\Omega = \mathbb{R}^m$).

Pontryagin maximum principle (strong version of Theorem 1.4.24)

The Pontryagin maximum principle is a strong version of Theorem 1.4.24 where restrictions on the values of the controls are allowed ($\Omega \subset \mathbb{R}^m$ can be a closed set). The existence of such restrictions are imposed by applications and change completely the nature of the solutions. The Pontryagin maximum principle is much more difficult to prove than Hestenes's Theorem (see, e.g., [62, 96]).

The general formulation is the following.

Theorem 1.4.25 (Pontryagin maximum principle). *Consider the control system in $\mathbb{R}^n$*

$$\dot{x}(t) = f(t,x(t),u(t)), \tag{1.16}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class C^1 and the controls are bounded mensurable mappings defined on the interval $[0,t_f(u)]$ of $\mathbb{R}^+$ and taking values in $\Omega \subset \mathbb{R}^m$. Let M_0 and M_1 be two subsets of $\mathbb{R}^n$. We denote by $\mathcal{U}_{t(u),\Omega}$ the set of admissible controls u whose associated trajectories steer an initial point of M_0 to a final point of M_1 in time $t(u) < t_f(u)$. For such a control we define the cost of a control u on $[0,t]$ by

$$C(u) = \int_0^t f^0(s,x(s),u(s))ds + g(t,x(t)),$$

where $f^0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are of class C^1, and x is the trajectory solution of (1.16) *associated to the control u.*

We consider the following optimal control problem: determine a trajectory steering M_0 to M_1 and minimizing the cost. The final time t_f can be fixed or not.

If the control $u(\cdot) \in \mathcal{U}_{t_f,\Omega}$ associated to the trajectory $x(\cdot)$ is optimal on $[0, t_f]$, then there exists a mapping $p(\cdot) : [0, t_f] \to \mathbb{R}^n$ absolutely continuous called adjoint vector, and a real number $p^0 \leq 0$, such that the pair $(p(\cdot), p^0)$ is nontrivial, and such that, for almost every $t \in [0, t_f]$,

$$\begin{aligned} \dot{x}(t) &= \frac{\partial H}{\partial p}(t, x(t), p(t), p^0, u(t)) \,, \\ \dot{p}(t) &= -\frac{\partial H}{\partial x}(t, x(t), p(t), p^0, u(t)) \,, \end{aligned} \tag{1.17}$$

where $H(t, x, p, p^0, u) = \langle p, f(t, x, u)\rangle + p^0 f^0(t, x, u)$ is the Hamiltonian of the system and we have the maximization condition almost everywhere on $[0, t_f]$

$$H(t, x(t), p(t), p^0, u(t)) = \max_{w \in \Omega} H(t, x(t), p(t), p^0, w) \,. \tag{1.18}$$

If the final time to steer the target M_1 is not fixed, we have the condition

$$\max_{w \in \Omega} H(t_f, x(t_f), p(t_f), p^0, w) = -p^0 \frac{\partial g}{\partial t}(t_f, x(t_f)) \tag{1.19}$$

at the final time t_f.

If M_0 and M_1 (or just one of these two sets) are manifolds in $\mathbb{R}^n$ having tangent spaces at $x(0) \in M_0$ and $x(t_f) \in M_1$, then the adjoint vector can be constructed in such a way that the transversality conditions hold at both extremities (or at just one of them):

$$p(0) \perp T_{x(0)} M_0 \tag{1.20}$$

and

$$p(t_f) - p^0 \frac{\partial g}{\partial x}(t_f, x(t_f)) \perp T_{x(t_f)} M_1 \,. \tag{1.21}$$

Remark 1.4.26. Under the conditions of Theorem 1.4.25, we have moreover that

$$\frac{d}{dt} H(t, x(t), p(t), p^0, u(t)) = \frac{\partial H}{\partial t}(t, x(t), p(t), p^0, u(t))$$

for almost every $t \in [0, t_f]$. In particular if the augmented system is *autonomous*, i.e., if f and f^0 do not depend on t, then H does not depend on t, and we have

$$\max_{w \in \Omega} H(x(t), p(t), p^0, w) = \text{constant} \quad \forall t \in [0, t_f] \,.$$

Note that this equality is then true everywhere on $[0, t_f]$ (in fact this function of t is Lipschitzian).

Remark 1.4.27. The convention $p^0 \leq 0$ lead us to the *maximum* principle. The convention $p^0 \geq 0$ will lead to the *minimum* principle, i.e., the condition (1.18) will be a minimum condition.

Remark 1.4.28. In the case where $\Omega = \mathbb{R}^m$, i.e., when there are no constraints on the control, the maximization condition (1.18) becomes $\frac{\partial H}{\partial u} = 0$, and we find the weak maximum principle (Theorem 1.4.24).

Definition 1.4.29. An *extremal* for the optimal control problem is a 4-tuple $(x(\cdot), p(\cdot), p^0, u(\cdot))$ solution of the equations (1.17) and (1.18). If $p_0 = 0$, we say that the extremal is *abnormal*, and if $p^0 \neq 0$ the extremal is said to be *normal*.

The designation *abnormal* is historical. We know nowadays that abnormal minimizers are usually "normal" in many optimization problems. For a study on abnormal extremals see, e.g., [7].

Remark 1.4.30. When $\Omega = \mathbb{R}^m$, i.e., when there is no constraint on the control, then the trajectory $x(\cdot)$ associated to the control $u(\cdot)$ is a singular trajectory of the system (1.16) if and only if it is the projection of an abnormal extremal $(x(\cdot), p(\cdot), 0, u(\cdot))$.

This results on the Hamiltonian characterization of singular trajectories (cf. Proposition 1.4.21). Note that once $p^0 = 0$, the trajectories do not depend on the cost. They are intrinsic to the system. The fact that they can be optimal can be explained in the following way: in general, a singular trajectory has a *rigidity* property, i.e., it's the only trajectory joining two extremities, and therefore in particular it is optimal, independently of the chosen optimization criteria.

This relation between abnormal extremals and singular trajectories, for $\Omega = \mathbb{R}^m$, shows very well the difficulty of proving the existence of such trajectories.

Definition 1.4.31. The conditions (1.20) and (1.21) are called *transversality conditions on the adjoint vector*. The condition (1.19) is called *transversality condition on the Hamiltonian.*

Remark 1.4.32. The *minimal time* problem corresponds to the case where $f^0 = 1$ and $g = 0$, or $f^0 = 0$ and $g(t, x) = t$. In these two cases the transversality conditions are the same.

Remark 1.4.33. The transversality condition over the Hamiltonian (1.19) is valid only if the final time t_f to attain the target is not fixed. In this case, if the function g does not depend on time t (which is true, for example, for the Lagrange problem), then condition (1.19) becomes

$$\max_{w \in \Omega} H(t_f, x(t_f), p(t_f), p^0, w) = 0\,,$$

or even, if u is continuous at time t_f,

$$H(t_f, x(t_f), p(t_f), p^0, w) = 0\,.$$

In other words, the Hamiltonian vanishes at the final time.

Moreover, if the augmented system is autonomous, i.e., if f and f^0 do not depend on t, then from Remark 1.4.26 we have

$$\max_{w\in\Omega} H(x(t),p(t),p^0,w) = 0 \quad \forall\, t \in [0,t_f]$$

along any extremal.

The Pontryagin maximum principle is a deep and important result in contemporary Mathematics, with many applications in Physics, Biology, Management, Economy, Social Sciences, Enginery, etc. (see, e.g., [21]). There are other more general versions of the maximum principle for non smooth or hybrid dynamics (see for example [30,111,112] and the references therein).

1.4.7 Example: optimal control of an harmonic oscillator (nonlinear case)

Consider again the example (nonlinear) of the spring, modeled by the control system

$$\begin{aligned}\dot{x}(t) &= y(t)\,,\\ \dot{y}(t) &= -x(t) - 2x(t)^3 + u(t)\,,\end{aligned}$$

where we admit as controls every function $u(\cdot)$ piecewise continuous such that $|u(t)| \leq 1$. The aim is to move the spring from any initial position $(x_0, y_0 = \dot{x}_0)$ to its equilibrium position $(0,0)$ *in minimal time* t_*.

Let us apply the Pontryagin maximum principle to this problem. The Hamiltonian is given by

$$H(x,y,p_x,p_y,p^0,u) = p_x y + p_y(-x - 2x^3 + u) + p^0\,.$$

If (x,y,p_x,p_y,p^0,u) is an extremal, then

$$\dot{p}_x = -\frac{\partial H}{\partial x} = p_y(1+6x^2) \quad\text{and}\quad \dot{p}_y = -\frac{\partial H}{\partial y} = -p_x\,.$$

Notice that since the adjoint vector (p_x,p_y,p^0) should be nontrivial, p_y can not vanish on an interval (otherwise we would also have $p_x = -\dot{p}_y = 0$ and, by the vanishing of the Hamiltonian, we would have also $p^0 = 0$). On the other hand, the maximization condition give us

$$p_y(t)u(t) = \max_{|w|\leq 1} p_y(t)w\,.$$

In particular, the optimal controls are successively equal to ± 1, that is, the *bang-bang* principle holds (see, e.g., [62,65]). Concretely, we can say that

$$u(t) = sign(p_y(t)) \text{ where } p_y \text{ is the solution of } \begin{cases}\ddot{p}_y(t) + p_y(t)(1+6x(t)^2) = 0\\ p_y(t_*) = \cos\alpha,\ \dot{p}_y(t_*) = -\sin\alpha\,,\end{cases}$$

$\alpha \in [0,2\pi[$.

Considering the time inversion ($t \mapsto -t$) our problem is equivalent to the minimal time problem for the system

$$\begin{cases} \dot{x}(t) = -y(t) \\ \dot{y}(t) = x(t) + 2x(t)^3 - sign(p_y(t)) \\ \dot{p}_y(t) = p_x(t) \\ \dot{p}_x(t) = -p_y(t)(1 + 6x(t)^2) \, . \end{cases}$$

Given the initial conditions x_0 and $\dot{x}_0$ (state and initial velocity of the mass), the problem is easily solved. In [121] a resolution of the system is done using the Computer Algebra System (CAS) Maple. For the use of Maple on the calculus of variations and optimal control see, e.g., [47, 64].

1.4.8 Example: Newton's problem of minimal resistance

Newton's problem of minimal resistance is one of the first problems of optimal control: it was proposed, and its solution given, by Isaac Newton in his masterful *Principia Mathematica*, in 1686. The problem consists of determining, in dimension three, the shape of an axis-symmetric body, with assigned radius and height, which offers minimum resistance when it is moving in a resistant medium. The problem has a very rich history and is well documented in the literature (see e.g. [101]).

Newton has indicated in the *Mathematical principles of natural philosophy* the correct solution to his problem (see Figure 1.16). He has not explained, however: how such solution can be obtained; how the problem is formulated in the language of mathematics. This has been the work of many mathematicians since Newton's time (see e.g. [22,115,118]). Extensions of Newton's problem is a topic of current intensive research, with many questions remaining open challenging problems. Recent results, obtained by relaxing Newton's hypotheses, include: non-symmetric bodies [23]; one-collision non-convex bodies [37]; collisions with friction [51]; multiple collisions allowed [92]; temperature noise of particles [93, 94]. Here we are interested in the classical problem, under the classical hypotheses considered by Newton.

Newton's problem of minimal resistance in dimension three

Newton's aerodynamical problem, in dimension three, is a classic problem (see e.g. [11,44,57]). It consists in joining two given points $(0, 0)$ and (T, h) of the plane by a curve's arc that, while turning around a given axis, generate the body of revolution offering the least resistance when moving in a fluid in the direction of the axis.

In the classical three dimensional Newton's problem of minimal aerodynamical resistance, the resistance force is given by $R\left[\dot{x}(\cdot)\right] = \int_0^T \frac{t}{1+\dot{x}(t)^2}\, dt$. Minimization of this functional is a typical problem of the calculus of variations. Most part of the old literature wrongly assume the

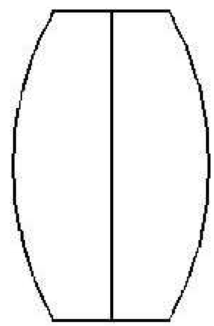

Figure 1.16: Newton's solid.

classical Newton's problem to be "one of the first applications of the calculus of variations". The truth, as Legendre first noticed in 1788 (see [12]), is that some restrictions on the derivatives of admissible trajectories must be imposed: $\dot{x}(t) \geq 0$, $t \in [0,T]$. This restriction is crucial, because without it there exists no solution, and the problem suffers from Perron's paradox [125, §10]: since the *a priori* assumption that a solution exists is not fulfilled, does not make any sense to try to find it by applying necessary optimality conditions. It turns out that, with the necessary restriction, the problem is better considered as an optimal control one (see [116, p. 67] and [118]). Correct formulation of Newton's problem of minimal resistance in dimension three is (cf. e.g. [44,115]):

$$\begin{aligned} &\mathcal{R}\left[u(\cdot)\right] = \int_0^T \frac{t}{1+u(t)^2} dt \longrightarrow \min\,, \\ &\dot{x}(t) = u(t)\,, \quad u(t) \geq 0\,, \\ &x(0) = 0\,, \quad x(T) = h\,, \quad h > 0\,, \end{aligned} \tag{1.22}$$

where we minimize the resistance $\mathcal{R}$ in the class of continuous functions $x : [0,T] \to \mathbb{R}$ with piecewise continuous derivative.

According to Pontryagin Maximum Principle (see Theorem 1.4.25) if $(x(\cdot), u(\cdot))$ is a minimizer of problem (1.22), then there exists a non-zero pair $(p^0, p(\cdot))$, where $p^0 \leq 0$ is a constant and $p(\cdot)$ is an absolutely continuous function on $[0,T]$, such that the following conditions are satisfied for almost all t in $[0,T]$:

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(u(t), p^0, p(t)) = 0 \tag{1.23}$$

$$H(p^0, p(t), u(t)) = \max_{w \geq 0} H(p^0, p(t), w) \tag{1.24}$$

where the Hamiltonian H is defined by

$$H(p^0, p, u) = pu + p^0 \frac{t}{1+u^2}\,. \tag{1.25}$$

The adjoint equation (1.23) asserts that $p(t) \equiv c$, with c a constant. From the maximization condition (1.24) it follows that $p^0 \neq 0$ (there are no abnormal extremals for problem (1.22)).

Proposition 1.4.34. *All the Pontryagin extremals* $\left(x(\cdot), p^0, p(\cdot), u(\cdot)\right)$ *of problem* (1.22) *are normal extremals* ($p^0 \neq 0$)*, with* $p(\cdot)$ *a negative constant:* $p(t) \equiv -\lambda$, $\lambda > 0$, $t \in [0, T]$.

Proof. The Hamiltonian H for problem (1.22), $H\left(p^0, p, u\right) = pu + p^0 \frac{t}{1+u^2}$, does not depend on x. Therefore, by (1.23) we conclude that

$$\dot{p}(t) = -\frac{\partial H}{\partial x}\left(p^0, p(t), u(t)\right) = 0\,,$$

that is, $p(t) \equiv c$, c a constant, for all $t \in [0, T]$. If $c = 0$, then $p^0 < 0$ (because one can not have both p^0 and p zero) and the maximization condition (1.24) simplifies to

$$p^0 \frac{t}{1+u^2(t)} = \max_{w \geq 0}\left\{p^0 \frac{t}{1+w^2}\right\}\,. \tag{1.26}$$

From (1.26) we conclude that the maximum is not achieved ($w \to \infty$). Therefore $c \neq 0$. Similarly, for $c > 0$ the maximum

$$cu(t) + p^0 \frac{t}{1+u^2(t)} = \max_{w \geq 0}\left\{cw + p^0 \frac{t}{1+w^2}\right\}$$

does not exist, and we conclude that $c < 0$. We can fix $p(t) \equiv -\lambda$, with $\lambda \in \mathbb{R}^+$. It remains to prove that $p^0 \neq 0$. If we assume that $p^0 = 0$, then the maximization condition reads

$$-\lambda u(t) = \max_{w \geq 0}\{-\lambda w\}\,, \quad \lambda \in \mathbb{R}^+\,, \tag{1.27}$$

and it follows $u(t) \equiv 0$ and $x(t) \equiv c_2$, c_2 a constant ($\dot{x}(t) = u(t)$). This is not possible, given the boundary conditions $x(0) = 0$ and $x(T) = h$ with $h > 0$. Therefore $p^0 \neq 0$ and we conclude that there exists no abnormal Pontryagin extremals. $\square$

Remark 1.4.35. If $\left(x(\cdot), p^0, p(\cdot), u(\cdot)\right)$ is an extremal, then $\left(x(\cdot), \gamma p^0, \gamma p(\cdot), u(\cdot)\right)$ is also a Pontryagin extremal, for all $\gamma > 0$. Therefore one can fix, without loss of generality, $p^0 = -1$.

From Proposition 1.4.34 and Remark 1.4.35 the Hamiltonian (1.25) takes the form

$$H\left(u\right) = -\lambda u - \frac{t}{1+u^2}\,, \quad \lambda > 0\,. \tag{1.28}$$

For $u > 0$, if follows from the maximization condition, $H(t, u(t)) = \max_{w>0}\left\{-\lambda w - \frac{t}{1+w^2}\right\}$ that

$$\frac{\partial H}{\partial u}(t, u(t)) = 0 \Leftrightarrow -\lambda + \frac{2tu(t)}{(1+u^2(t))^2} = 0 \Leftrightarrow \frac{tu(t)}{(1+u^2(t))^2} = \frac{\lambda}{2}\,,$$

that is,

$$\frac{tu(t)}{(1+u^2(t))^2} = q\,, \quad \text{with}\, q\, \text{a strictly positive constant.} \tag{1.29}$$

The conservation law (1.29) is known as *Newton's differential equation.*

It is not easy to prove the existence of a solution for problem (1.22) with classical arguments. We will use a different approach. We will show, following [118], that for problem (1.22) the Pontryagin extremals are absolute minimizers. This means that to solve problem (1.22) it is enough to identify its Pontryagin extremals.

Theorem 1.4.36. *Pontryagin extremals for problem* (1.22) *are absolute minimizers.*

Proof. Let $\hat{u}(\cdot)$ be a Pontryagin extremal control for problem (1.22). We want to prove that

$$\int_0^T \frac{t}{1+u^2(t)}dt \geq \int_0^T \frac{t}{1+\hat{u}^2(t)}dt$$

for any admissible control $u(\cdot)$. Given (1.28), we conclude from the maximization condition (1.24) that

$$-\lambda\hat{u}(t) - \frac{t}{1+\hat{u}^2(t)} \geq -\lambda u(t) - \frac{t}{1+u^2(t)} \tag{1.30}$$

for all piecewise continuous functions $u(\cdot)$ defined in $[0,T]$ satisfying $u(t) \geq 0$. Having in mind that all the admissible processes $(x(\cdot), u(\cdot))$ of (1.22) satisfy

$$\int_0^T u(t)dt = \int_0^T \dot{x}(t)dt = x(T) - x(0) = h\,,$$

we only need to integrate (1.30) to conclude that $\hat{u}(\cdot)$ is an absolute control minimizer:

$$\begin{aligned}
&\int_0^T \left(-\lambda\hat{u}(t) - \frac{t}{1+\hat{u}^2(t)}\right)dt \geq \int_0^T \left(-\lambda u(t) - \frac{t}{1+u^2(t)}\right)dt \\
&\Leftrightarrow \lambda\int_0^T \hat{u}(t)dt + \int_0^T \frac{t}{1+\hat{u}^2(t)}dt \leq \lambda\int_0^T u(t)dt + \int_0^T \frac{t}{1+u^2(t)}dt \\
&\Leftrightarrow \lambda h + \int_0^T \frac{t}{1+\hat{u}^2(t)}dt \leq \lambda h + \int_0^T \frac{t}{1+u^2(t)}dt \\
&\Leftrightarrow \int_0^T \frac{t}{1+\hat{u}^2(t)}dt \leq \int_0^T \frac{t}{1+u(t)^2}dt\,.
\end{aligned}$$

We conclude,

$$R[\hat{u}(\cdot)] \leq R[u(\cdot)]\,,$$

and $\hat{u}(\cdot)$ is a absolute minimizer for Newton's problem of minimal resistance. $\square$

Theorem 1.4.37 (Solution of Newton's problem of minimal resistance)**.** *The solution $\hat{x}(\cdot)$ for Newton's problem of minimal resistance* (1.22) *is given by $\hat{x}(t) = 0$ for $0 \leq t \leq \xi$ and, when $\xi \leq t \leq T$, it is given in the parametric form by*

$$\begin{cases} t(u) = \frac{\lambda}{2}\left(\frac{1}{u} + 2u + u^3\right)\,, \\ x(u) = \frac{\lambda}{2}\left(-\ln u + u^2 + \frac{3}{4}u^4\right) - \frac{7\lambda}{8} \end{cases} \tag{1.31}$$

where the constant λ is defined by the boundary condition $x(T) = h$ and $\xi = 2\lambda$.

Proof. Let $\hat{x}(\cdot)$ be the solution of Newton's problem of minimal resistance (1.22).

The solution, $\hat{x}(\cdot)$, is given by two different conditions: first is a line segment with start point at the origin of the frame of reference tOx and end point at the point $(\xi, 0)$ in the positive semi-axis tt; after the point $(\xi, 0)$, Newton's solution follows the so-called *Newton's curve.*

Let us study in detail each one of the parts of the solution of Newton's problem.

As we have observed, in Newton's problem (1.22) the controls take values in a closed interval of $\mathbb{R}$, thus two cases must be taken in consideration: $u = 0$ and $u > 0$.

When $u = 0$ the solution is given by $x(t) = 0$: if $u(t) = 0$ then, as $u(t) = \dot{x}(t)$, we have that $\dot{x}(t) = 0$, therefore $x(t) = c$, with c a real constant; from the boundary condition $x(0) = 0$ we conclude that $c = 0$. The absolute minimizer (cf. Theorem 1.4.36) starts with the line segment $x(t) = 0$, with $t \in [0, \xi]$ and $0 < \xi < T$ (after some point $(\xi, 0)$, $u > 0$ since $x(T) = h > 0$).

On the other hand, when $u > 0$, we can define in a parametric form the solution of Newton's problem from Newton's differential equation (1.29) (which derives from Pontryagin maximization condition).

From equation (1.29) we can write t as a function of the parameter u, that is,

$$\frac{tu}{(1+u^2)^2} = \frac{\lambda}{2} \Leftrightarrow 2ut = \lambda(1+u^2)^2 \Leftrightarrow t = \frac{\lambda}{2}\left(\frac{1}{u} + 2u + u^3\right) .$$

We define in a parametric form $t(\cdot)$ by

$$t(u) = \frac{\lambda}{2}\left(\frac{1}{u} + 2u + u^3\right) .$$

To define in a parametric form $x(\cdot)$, recall the chain rule

$$\frac{d}{du}x(t(u)) = \frac{dx}{dt}\frac{dt}{du} = u\frac{dt}{du} ,$$

since $\frac{dx}{dt} = u$. Therefore, $x(u) = \int u \frac{dt}{du} du$. We have,

$$\frac{dt}{du}(u) = \frac{\lambda}{2}\left(-\frac{1}{u^2} + 2 + 3u^2\right) ,$$

thus,

$$x(u) = \int \frac{\lambda}{2} u \left(-\frac{1}{u^2} + 2 + 3u^2\right) du = \frac{\lambda}{2}\left(-\ln u + u^2 + \frac{3}{4}u^4\right) + m , \tag{1.32}$$

where m is a constant.

To compute the constant m on the previous equation, we must compute ξ. At $(\xi, 0)$, by continuity of $\hat{x}(\cdot)$, both branches coincide.

Let $\hat{u}(t)$ be the minimizing control of Newton's problem. Then,

$$H(\xi, 0) = H\left(\xi, \hat{u}(\xi)\right) . \tag{1.33}$$

By definition of the Hamiltonian for Newton's problem of minimal resistance, we have

$$H(\xi,0)=-\lambda\times 0-\frac{\xi}{1+0^2}=-\xi \quad\text{and}\quad H(\xi,\hat{u}(\xi))=-\lambda\hat{u}(\xi)-\frac{\xi}{1+(\hat{u}(\xi))^2}\,.$$

Therefore, from (1.33), we have

$$H(\xi,0)=H\left(\xi,\hat{u}(\xi)\right)\Leftrightarrow \xi=\lambda\hat{u}(\xi)+\frac{\xi}{1+(\hat{u})^2}\,. \tag{1.34}$$

On the other hand, $\hat{u}(\xi)$ must satisfy Newton's differential equation (1.29), thus

$$\frac{\xi\hat{u}(\xi)}{\left(1+(\hat{u}(\xi))^2\right)^2}=\frac{\lambda}{2}\,. \tag{1.35}$$

Let us solve equation (1.34) in order to compute the constant λ,

$$\xi=\frac{\xi}{1+(\hat{u}(\xi))^2}+\lambda\hat{u}(\xi)\Leftrightarrow -\frac{\xi}{1+(\hat{u}(\xi))^2}+\xi=\lambda\hat{u}(\xi)\Leftrightarrow \frac{-\xi+\xi\left(1+(\hat{u}(\xi))^2\right)}{1+(\hat{u}(\xi))^2}=\lambda\hat{u}(\xi)$$
$$\Leftrightarrow \frac{\xi\left(\hat{u}(\xi)\right)^2}{1+(\hat{u}(\xi))^2}=\lambda\hat{u}(\xi)\Leftrightarrow \frac{\xi\hat{u}(\xi)}{1+(\hat{u}(\xi))^2}=\lambda\,.$$

That is, the constant λ is given by condition

$$\lambda=\frac{\xi\hat{u}(\xi)}{1+(\hat{u}(\xi))^2}\,. \tag{1.36}$$

Replacing (1.36) into (1.35) we get

$$\frac{\xi\hat{u}(\xi)}{\left(1+(\hat{u}(\xi))^2\right)^2}=\frac{\xi\hat{u}(\xi)}{2\left(1+(\hat{u}(\xi))^2\right)}\Leftrightarrow \hat{u}^2(\xi)=1\,,$$

as $\hat{u}(x)\geq 0$, then $\hat{u}^2(\xi)=1\Rightarrow\hat{u}(\xi)=1$.

As Newton stated in his *Principia*, "the tangent to the graphic at the *break point* is equal to 1". That is, say that at the break point, namely, at the point $(\xi,0)$, the tangent is 1, is equivalent to say that $\hat{u}(\xi)=1$ ($\tan\alpha=1\Leftrightarrow\dot{x}(\xi)=1\Leftrightarrow\hat{u}(\xi)=1$).

Inserting $\hat{u}(\xi)=1$ into equation (1.35) we have $\frac{\xi}{(1+1^2)^2}=\frac{\lambda}{2}$, that is, $\xi=2\lambda$.

We are in condition to determine the constant m of equation (1.32). This is possible if we take into account that at the point $(\xi,0)$, $\hat{u}(\xi)=1$ and $x\left(\hat{u}(\xi)\right)=0$. Then,

$$x\left(\hat{u}(\xi)\right)=0\Leftrightarrow x(1)=0\Leftrightarrow\frac{\lambda}{2}\left(-\ln 1+1+\frac{3}{4}\right)+m=0\Leftrightarrow\frac{7\lambda}{8}=-m\,,$$

that is, $m=-\frac{7\lambda}{8}$.

Finally, we can conclude that in the case $u>0$, the solution of Newton's problem of minimal resistance is given in a parametric form by equations (1.31), as we wanted to prove.

□

The obtained curve from (1.31) is called *Newton's curve.*

Is important to remark the reason why Newton's problem solution starts with $x(t) = 0$ for $t \in [0, \xi]$, $0 < \xi = 2\lambda$, and for $x \in [\xi, T]$ by (1.31). In fact, if Newton's problem solution was given by equations (1.31) for every $t \in [0, T]$ the boundary condition $x(0) = 0$ would not be satisfied.

Let us now see how we can obtain the graphic representation of Newton's problem solution, given a radius and an height.

The first part of the solution is given by $x(t) = 0$ for every $t \in [0, \xi]$, with $\xi = 2\lambda$, and its graphic representation is easily obtained.

With respect to the second part, $t \in [\xi, T]$, in order to represent graphically Newton's curve the value of λ, the break point $(\xi, 0)$ and variation interval of the parameter u must be determined for a radius and an height previously given. In practice, when we compute the value of the constant λ the point $(\xi, 0)$ is automatically determined, because $\xi = 2\lambda$.

The variation interval of the parameter u is given by the inequalities

$$\xi \leq t(u) \leq T \Leftrightarrow \xi \leq \frac{\lambda}{2}\left(\frac{1}{u} + 2u + u^3\right) \leq T\,,$$

that is, as $\xi = 2\lambda$,

$$2\lambda \leq \frac{\lambda}{2}\left(\frac{1}{u} + 2u + u^3\right) \leq T\,.$$

From inequality $2\lambda \leq \frac{\lambda}{2}\left(\frac{1}{u} + 2u + u^3\right)$, we observe that the minimal value taken by the parameter u is 1, independently from the value of the radius and the height of the solid, which, once more, leads us to Newton's statement that *the tangent to the graphic at the break point is equal to 1.* The maximal value taken by the parameter u can be found simultaneously with the constant λ solving the system

$$\begin{cases} t(u) = T \\ \\ x(u) = h \end{cases} \Leftrightarrow \begin{cases} T = \frac{\lambda}{2}\left(\frac{1}{u} + 2u + u^3\right) \\ \\ h = \frac{\lambda}{2}\left(-\ln u + u^2 + \frac{3}{4}u^4\right) - \frac{7\lambda}{8} \end{cases}$$

since the constant λ is computed using the boundary condition $x(T) = h$.

The previous system is easily solved by Maple (see, e.g., [101]), as well as the graphical representation of Newton's problem of minimal resistance. In Figure 1.17 the graphics (obtained with Maple) of Newton's problem solution are given for a fixed radius $T = 1$ and an height $h = 0.5$, $h = 1$, $h = 2$, $h = 5$.

Newton's problem of minimal resistance in dimension two

At first glance, one suspects that the two dimensional case should be well known, in [102] it is shown that the two dimensional problem is more rich than the classical one being, in

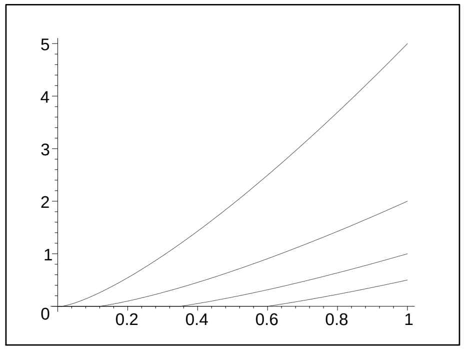

Figure 1.17: Newton's problem solution

some sense, more interesting. The novelties are: (i) while in the classical three-dimensional problem (1.22) only the restricted case makes sense (without restriction on the monotonicity of admissible functions the problem doesn't admit a local minimum), in dimension two the unrestricted problem is also well-posed when the ratio height versus radius of base is greater than a given quantity; (ii) while in three dimensions the (restricted) problem has a unique solution, in the restricted two-dimensional problem the minimizer is not always unique – when the height of the body is less or equal than its base radius, there exists infinitely many minimizing functions.

The formulation of Newton's problem of minimal resistance in dimension two is given by (see [118]):

$$\begin{aligned}
&R\,[u(\cdot)] = \int_0^T \frac{1}{1+u(t)^2}dt \longrightarrow \min\,, \\
&\dot{x}(t) = u(t)\,, \quad u(t) \in \Omega\,, \\
&x(0) = 0\,, \quad x(T) = h\,, \quad h > 0\,.
\end{aligned} \tag{1.37}$$

We consider two cases: (i) unrestricted problem, where no restriction on the admissible trajectories $x(\cdot)$ other than the boundary conditions $x(0) = 0$, $x(T) = h$ is considered ($\Omega = \mathbb{R}$); (ii) restricted problem, where the admissible functions must satisfy the restriction $\dot{x}(t) \geq 0$, $t \in [0,T]$ ($\Omega = \mathbb{R}_0^+$). While for the classical three-dimensional problem only the restricted problem admits a minimizer, the two-dimensional problem (1.37) is more rich: the unrestricted case also admits a minimizer when the given height h of the body is big enough. Also in the restricted case the two-dimensional problem is more interesting: if $T \geq h$, then infinitely many different minimizers are possible, while in the classical three-dimensional problem the minimizer is always unique.

According to Pontryagin Maximum Principle (see Theorem 1.4.25) if $(x(\cdot), u(\cdot))$ is a minimizer of problem (1.37), then there exists a non-zero pair $(p^0, p(\cdot))$, where $p^0 \leq 0$ is a constant and $p(\cdot)$ is an absolutely continuous function on $[0,T]$, such that the following conditions are satisfied for almost all t in $[0,T]$:

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(u(t), p^0, p(t)) = 0$$

$$H(p^0, p(t), u(t)) = \max_{w \in \Omega} H(p^0, p(t), w)\,;$$

where the Hamiltonian H is defined by

$$H(p^0, p, u) = pu + p^0 \frac{1}{1+u^2}\,.$$

Proposition 1.4.38. *All the Pontryagin extremals* $\left(x(\cdot), p^0, p(\cdot), u(\cdot)\right)$ *of problem* (1.37) *are normal extremals ($p^0 \neq 0$), with $p(\cdot)$ a negative constant: $p(t) \equiv -\lambda$, $\lambda > 0$, $t \in [0,T]$.*

Theorem 1.4.39. *Pontryagin extremals for problem* (1.37) *are absolute minimizers.*

The proofs of Proposition 1.4.38 and Theorem 1.4.39 valid for the two dimensional Newton's problem (1.37) are analogous to the proofs of Proposition 1.4.34 and Theorem 1.4.36, respectively, valid for the three dimensional Newton's problem (1.22).

Unrestricted problem ($\Omega = \mathbb{R}$) The following standard result of calculus (see e.g. [42]) will be used in the sequel.

Theorem 1.4.40. *Let $n \geq 2$ and $\Omega \subseteq \mathbb{R}$ be an open set. If $f : \Omega \to \mathbb{R}$ is $n-1$ times differentiable on Ω and n times differentiable at some point $a \in \Omega$ where $f^{(k)}(a) = 0$ for $k = 0, \ldots, n-1$ and $f^{(n)}(a) \neq 0$, then:*

- *either n is even, and $f(\cdot)$ has an extremum at a, that is a maximum in case $f^{(n)}(a) < 0$ and a minimum in case $f^{(n)}(a) > 0$;*
- *or n is odd, and $f(\cdot)$ does not attain a local extremum at a.*

From Theorem 1.4.39 the problem (1.37) can be reduced to the study of the one-dimensional maximization problem:

$$\max_{u \in \Omega} H(u) = \max_{u \in \Omega} \left\{ -\frac{1}{1+u^2} - \lambda u \right\}, \quad \lambda > 0\,. \tag{1.38}$$

We are considering now the unrestricted two-dimensional Newton's problem of minimal resistance, that is, $\Omega = \mathbb{R}$ in (1.37). A necessary (sufficient) condition for u to be a local maximizer for problem (1.38) is given by $H'(u) = 0$ and $H''(u) \leq 0$ ($H''(u) < 0$), where

$$H'(u) = \frac{2u}{(1+u^2)^2} - \lambda\,,$$

$$H''(u) = -2\,\frac{3\,u^2 - 1}{(1+u^2)^3}\,.$$

From the first order condition (maximization condition (1.24)) it follows that

$$\frac{u(t)}{\left(1+u^2(t)\right)^2}=\frac{\lambda}{2}\Leftrightarrow\frac{\dot{x}(t)}{\left(1+\dot{x}^2(t)\right)^2}=\frac{\lambda}{2}\,. \tag{1.39}$$

Using the boundary conditions $x(0)=0$ and $x(T)=h$, we conclude that $x(t)=\frac{h}{T}t$ $(u=\frac{h}{T})$ is a local candidate for the solution of the unrestricted problem ($\lambda=\frac{2T^3h}{(T^2+h^2)^2}$). However, by Theorem 1.4.40, we conclude that such u is a maximizer only when $h>\frac{\sqrt{3}}{3}T$. For $h<\frac{\sqrt{3}}{3}T$ the value $u=\frac{h}{T}$ corresponds to a local minimizer of $H\left(u\right)$ since $H''>0$; for $h=\frac{\sqrt{3}}{3}r$ function $H\left(u\right)$ has neither local maximum nor minimum since $H''\left(\frac{\sqrt{3}}{3}T\right)=0$ and $H'''\left(\frac{\sqrt{3}}{3}T\right)=-\frac{27\sqrt{3}}{16}\neq 0$.

Theorem 1.4.41. *If $h>\frac{\sqrt{3}}{3}T$, then function $x(t)=\frac{h}{T}t$ is a (local) minimum for the unrestricted problem* (1.37)*. For $h\leq\frac{\sqrt{3}}{3}T$ the problem has no solution.*

Remark 1.4.42. The unrestricted problem (1.37) does not admit global minimum. Indeed, let us take, for large values of the parameter a, the control function

$$\tilde{u}(t)=\begin{cases}a & \text{if}\quad 0\leq t\leq\frac{T}{2}+\frac{h}{2a}\\ -a & \text{if}\quad \frac{T}{2}+\frac{h}{2a}\leq t\leq T\,.\end{cases}$$

This gives $R[\tilde{u}(t)]=\frac{T}{1+a^2}$ which vanishes as $a\to\infty$, showing that no global solution can exist.

By the symmetry with respect to the xx axis, the solution to the unrestricted two-dimensional Newton's problem of minimal resistance with $h>\frac{\sqrt{3}}{3}T$ is a triangle, with value for resistance R equal to $\frac{T^3}{T^2+h^2}$.

Restricted problem ($\Omega=\mathbb{R}_0^+$) We now study problem (1.37) with $\Omega=\mathbb{R}_0^+$. In this case the optimal control can take values on the boundary of the admissible set of control values Ω ($u=0$). If the optimal control $u(\cdot)$ is always taking values in the interior of Ω, $u(t)>0$ $\forall$ $t\in[0,T]$, then the optimal solution must satisfy (1.39) and it corresponds to the one found for the unrestricted problem:

$$u(t)=\frac{h}{T}\,,\quad\forall t\in[0,T]\,, \tag{1.40}$$

with resistance

$$R=\frac{T^3}{T^2+h^2}\,. \tag{1.41}$$

We show next that this is solution of the restricted problem only for $h\geq T$: for $h\leq T$ the minimum value for the resistance is $R=T-\frac{h}{2}$.

It is clear, from the boundary conditions $x(0)=0$, $x(T)=h$, $T>0$, $h>0$, that $u(t)=0$, $\forall$ $t\in[0,T]$, is not a possibility: there must exist at least one non-empty subinterval of $[0,T]$

for which $u(t) > 0$ (otherwise $x(t)$ would be constant, and it would be not possible to satisfy simultaneously $x(0) = 0$ and $x(T) = h$). The simplest situations are given by

$$u(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq \xi\,, \\ \frac{h}{T-\xi} & \text{if} \quad \xi \leq t \leq T\,, \end{cases} \tag{1.42}$$

or

$$u(t) = \begin{cases} \frac{h}{\xi} & \text{if} \quad 0 \leq t \leq \xi\,, \\ 0 & \text{if} \quad \xi \leq t \leq T\,. \end{cases} \tag{1.43}$$

We get (1.40) from (1.42) taking $\xi = 0$; (1.40) from (1.43) with $\xi = T$. For (1.42) the resistance is given by $R(\xi) = \xi + \frac{(T-\xi)^3}{(T-\xi)^2+h^2}$, that has a minimum value for $\xi = T - h \geq 0$: $R(T-h) = T - \frac{h}{2}$,

$$u(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq T-h\,, \\ 1 & \text{if} \quad T-h \leq t \leq T\,. \end{cases} \tag{1.44}$$

For $T = h$ (1.44) coincides with (1.40); for $T > h$

$$\left(T - \frac{h}{2}\right) - \left(\frac{T^3}{T^2+h^2}\right) = -\frac{h(T-h)^2}{2(T^2+h^2)} < 0\,,$$

and (1.44) is better than (1.40). Similarly, for (1.43) the resistance is given by

$$R(\xi) = \frac{\xi^3}{\xi^2+h^2} + T - \xi\,, \tag{1.45}$$

that has minimum value for $\xi = h > 0$:

$$u(t) = \begin{cases} 1 & \text{if} \quad 0 \leq t \leq h\,, \\ 0 & \text{if} \quad h \leq t \leq T\,, \end{cases} \tag{1.46}$$

$R(h) = T - \frac{h}{2}$, which coincides with the value for the resistance obtained with (1.44). If one compares directly (1.41) with (1.45) one get the conclusion that (1.40) is better than (1.43) precisely when $T < h$:

$$\frac{T^3}{T^2+h^2} - \left(\frac{\xi^3}{\xi^2+h^2} + T - \xi\right) = \frac{\xi h^2\left(T^2 - T\xi - h^2\right)}{\left[(T-\xi)^2+h^2\right]\left(T^2+h^2\right)}\,, \tag{1.47}$$

and since $-h^2 \leq T^2 - T\xi - h^2 \leq T^2 - h^2$, (1.47) is negative if $T < h$, that is, for $T < h$ (1.40) is better than (1.43). For $T = h$ (1.46) coincide with (1.40), for $T > h$ (1.46) is better than (1.40) and as good as (1.44).

We now show that for $T > h$ it is possible to obtain the resistance value $T - \frac{h}{2}$ from infinitely many other ways, but no better (smaller) value than this quantity. Generic situation is given by

$$u_n(t) = \begin{cases} 0 & \text{if} \quad \xi_{2i} \leq t \leq \xi_{2i+1}\,, \quad i = 0, \ldots, n\,, \\ \frac{\mu_{i+1}-\mu_i}{\xi_{2i+2}-\xi_{2i+1}} & \text{if} \quad \xi_{2i+1} \leq t \leq \xi_{2i+2}\,, \quad i = 0, \ldots, n-1\,, \end{cases} \tag{1.48}$$

where $n \in \mathbb{N}$, $0 = \xi_0 \le \xi_1 \le \cdots \le \xi_{2n+1} = T$, $0 = \mu_0 \le \mu_1 \le \cdots \le \mu_n = h$. We remark that for the simplest case $n = 1$ (1.48) simplifies to

$$u_1(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \xi_1\,, \\ \frac{h}{\xi_2 - \xi_1} & \text{if} \quad \xi_1 \le t \le \xi_2\,, \\ 0 & \text{if} \quad \xi_2 \le t \le T\,, \end{cases}$$

which covers all the previously considered situations: for $\xi_1 = 0$, $\xi_2 = T$ we obtain (1.40); for $\xi_2 = T$ (1.42); and for $\xi_1 = 0$ one obtains (1.43). All Pontryagin control extremals of the restricted problem are of the form (1.48), and by Theorem 1.4.39 also the minimizing controls. The resistance force R_n associated with (1.48) is given by

$$R_n\left(\xi_0, \ldots, \xi_{2n+1}, \mu_0, \ldots, \mu_n\right) = \sum_{i=0}^{n} \left(\xi_{2i+1} - \xi_{2i}\right) + \sum_{i=0}^{n-1} \frac{\left(\xi_{2i+2} - \xi_{2i+1}\right)^3}{\left(\xi_{2i+2} - \xi_{2i+1}\right)^2 + \left(\mu_{i+1} - \mu_i\right)^2}\,. \tag{1.49}$$

It is a simple exercise of calculus to see that function (1.49) has three critical points: two of them not admissible, the third one a minimizer. The first critical point is defined by $\mu_i = 0$, $i = 0, \ldots, n$, which is not admissible given the fact that $\mu_n = h > 0$. The second critical point is given by $\mu_i - \mu_{i-1} = \xi_{2i-1} - \xi_{2i}$, $i = 1, \ldots, n$, which is not admissible since $\mu_i - \mu_{i-1} \ge 0$, $\xi_{2i-1} - \xi_{2i} \le 0$, and $\mu_i = \mu_{i-1}$, $i = 1, \ldots, n$, is not a possibility given $\mu_n = H > \mu_0 = 0$. The third critical point is

$$\mu_i - \mu_{i-1} = \xi_{2i} - \xi_{2i-1}\,, \quad i = 1, \ldots, n\,, \tag{1.50}$$

which is a minimizer for $h \le T$. Thus, all the minimizing controls for the restricted two-dimensional problem with $h \le T$ are of the following form:

$$u_n(t) = \begin{cases} 0 & \text{if} \quad \xi_{2i} \le t \le \xi_{2i+1}\,, \quad i = 0, \ldots, n\,, \\ 1 & \text{if} \quad \xi_{2i+1} \le t \le \xi_{2i+2}\,, \quad i = 0, \ldots, n-1\,, \end{cases} \tag{1.51}$$

$n = 1, 2, \ldots$, $0 = \xi_0 \le \xi_1 \le \cdots \le \xi_{2n+1} = r$. For $u_n(t)$ given by (1.51) the resistance (1.49) reduces to $R_n = T - \frac{h}{2}$, $\forall\, n \in \mathbb{N}$.

Theorem 1.4.43. *The restricted two-dimensional Newton's problem of minimal resistance always admits a solution:*

- *the unique solution associated to control* (1.40)*, when $h > T$;*
- *infinitely many solutions associated to the controls* (1.51)*, when $h \le T$.*

In the case $h > T$ the minimum value for the resistance is $\frac{T^3}{T^2 + h^2}$, otherwise $T - \frac{h}{2}$.

1.5 Proof of the Pontryagin maximum principle for a general minimal time problem

In this section, we recall elements of a standard proof of the maximum principle for a general minimal time problem using needle-like variations (see e.g. [96]). Some definitions and properties of this section will be used in Chapter 2.

Consider a general control system

$$\dot{x}(t) = f(x(t), u(t)),\ x(0) = x_0, \tag{1.52}$$

where $x_0 \in \mathbb{R}^n$ is fixed, $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is smooth, the control u is a bounded measurable function taking its values in a measurable subset Ω of $\mathbb{R}^m$.

Consider the set of admissible controls on $[0, t_f]$, $\mathcal{U}_{t_f,\mathbb{R}^m}$, and the set of admissible controls on $[0, t_f]$ taking their values in Ω, $\mathcal{U}_{t_f,\Omega}$.

The set $\mathcal{U}_{t_f,\mathbb{R}^m}$, endowed with the standard topology of $L^\infty([0, t_f], \mathbb{R}^m)$, is open, and the end-point mapping $E(x_0, t_f, u) = x(t_f)$ is smooth on $\mathcal{U}_{t_f,\mathbb{R}^m}$.

Let $x_1 \in \mathbb{R}^n$. Consider the optimal control problem $(\mathcal{P})$ of determining a trajectory solution of (1.52) steering x_0 to x_1 in minimal time.[3] In other words, this is the problem of minimizing t_f among all admissible controls $u \in L^\infty([0, t_f], \Omega)$ satisfying the constraint $E(x_0, t_f, u) = x_1$.

For every $t \geq 0$, consider the *accessible set* $A_\Omega(x_0, t)$ previously defined as the image of the mapping $E(x_0, t, \cdot) : \mathcal{U}_t \to \mathbb{R}^n$, with the agreement $A_\Omega(x_0, 0) = \{x_0\}$.

Moreover, define

$$A_\Omega(x_0, \leq t) = \bigcup_{0 \leq s \leq t} A_\Omega(x_0, s).$$

The set $A_\Omega(x_0, \leq t)$ coincides with the image of the mapping $E(x_0, \cdot, \cdot) : [0, t] \times \mathcal{U}_t \to \mathbb{R}^n$ (see Figure 1.18).

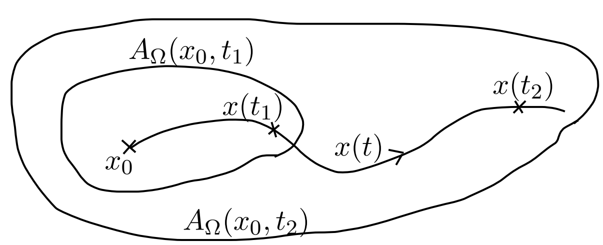

Figure 1.18: Accessible set $A_\Omega(x_0, \leq t)$.

[3] Note that we consider here a problem with fixed extremities, for simplicity of presentation. All what follows however easily extends to the case of initial and final subsets (see e.g. [62]).

Let u be a minimal time control on $[0, t_f]$ for the problem $(\mathcal{P})$, and denote by $x(\cdot)$ the trajectory solution of (1.52) associated to the control u on $[0, t_f]$. Then the point $x_1 = x(t_f)$ belongs to the boundary of $A_\Omega(x_0, \leq t_f)$. This geometric property is at the basis of the proof of the Pontryagin maximum principle (see Figure 1.19).

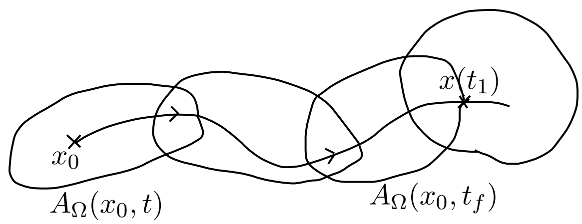

Figure 1.19: $x_1 \in \partial A_\Omega(x_0, t_f)$.

Theorem 1.5.1 (Pontryagin maximum principle)**.** *If the trajectory $x(\cdot)$, associated to a control $u \in \mathcal{U}_{t_f,\Omega}$, is optimal on $[0, t_f]$, then there exists a nonpositive real number p^0 and an absolutely continuous mapping $p(\cdot)$ on $[0, t_f]$, called* adjoint vector*, satisfying $(p(\cdot), p^0) \neq (0, 0)$, such that there holds*

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), p^0, u(t)),$$
$$\dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), p^0, u(t)),$$

almost everywhere on $[0, t_f]$, where $H(x, p, p^0, u) = \langle p, f(x, u)\rangle + p^0$ is the Hamiltonian, and

$$H(x(t), p(t), p^0, u(t)) = \max_{w \in \Omega} H(x(t), p(t), p^0, w)$$

holds almost everywhere on $[0, t_f]$. Moreover, $\max_{w\in\Omega} H(x(t), p(t), p^0, w) = 0$ for every $t \in [0, t_f]$.

We next recall the standard concepts of needle-like variations and of Pontryagin cone which permit to derive a standard proof of the maximum principle.

1.5.1 Needle-like variations

Let $t_1 \in [0, t_f)$ and $u_1 \in \Omega$. For $\eta_1 > 0$ such that $t_1 + \eta_1 \leq t_f$, the needle-like variation $\pi_1 = \{t_1, \eta_1, u_1\}$ of the control u is defined by

$$u_{\pi_1}(t) = \begin{cases} u_1 & \text{if } t \in [t_1, t_1 + \eta_1], \\ u(t) & \text{otherwise} \end{cases}$$

(see Figure 1.20).

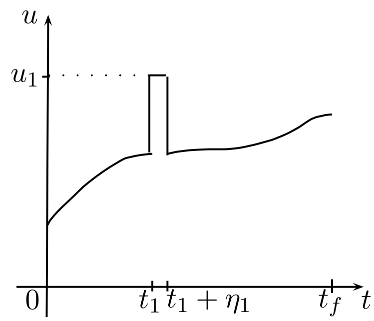

Figure 1.20: Needle variation π_1.

The control u_{π_1} takes its values in Ω. It is not difficult to prove that, if $\eta_1 > 0$ is small enough, then the control u_{π_1} is admissible, i.e., the trajectory $x_{\pi_1}(\cdot)$ associated with u_{π_1} and starting from $x_{\pi_1}(0) = x_0$ is well defined on $[0, t_f]$. Moreover, $x_{\pi_1}(\cdot)$ converges uniformly to $x(\cdot)$ on $[0, t_f]$ whenever η_1 tends to 0.

Recall that t_1 is a Lebesgue point of the function $t \mapsto f(x(t), u(t))$ on $[0, t_f)$ whenever

$$\lim_{h\to 0} \frac{1}{h} \int_{t_1}^{t_1+h} f(x(t), u(t))dt = f(x(t_1), u(t_1)),$$

and that almost every point of $[0, t_f)$ is a Lebesgue point.

Definition 1.5.2. Let t_1 be a Lebesgue point on $[0, t_f)$, let $\eta_1 > 0$ be small enough, and u_{π_1} be a needle-like variation of u, with $\pi_1 = \{t_1, \eta_1, u_1\}$. For every $t \in [t_1, t_f]$, define the *variation vector* $v_{\pi_1}(t)$ as the solution on $[t_1, t_f]$ of the Cauchy problem

$$\dot{v}_{\pi_1}(t) = \frac{\partial f}{\partial x}(x(t), u(t))v_{\pi_1}(t)\,, \tag{1.53}$$

$$v_{\pi_1}(t_1) = f(x(t_1), u_1) - f(x(t_1), u(t_1))\,.$$

Lemma 1.5.3 (see e.g. [96])**.** *Let t_1 be a Lebesgue point on $[0, t_f)$, let $\eta_1 > 0$ be small enough, and u_{π_1} be a needle-like variation of u, with $\pi_1 = \{t_1, \eta_1, u_1\}$. Then,*

$$x_{\pi_1}(t_f) = x(t_f) + \eta_1 v_{\pi_1}(t_f) + \mathrm{o}(\eta_1)\,. \tag{1.54}$$

Proof. By definition of u_{π_1} and x_{π_1}, we have $x_{\pi_1}(t_1) = x(t_1)$. Then

$$x_{\pi_1}(t_f) = x(t_1) + \int_{t_1}^{t_1+\eta_1} f(x_{\pi_1}(t), u_1)dt + \int_{t_1+\eta_1}^{t_f} f(x_{\pi_1}(t), u(t))dt\,.$$

By definition of Lebesgue point, we have

$$\int_{t_1}^{t_1+\eta_1} f(x_{\pi_1}(t), u_1)dt = \eta_1 f(x(t_1), u_1) + o(\eta_1)\,,$$

and

$$\begin{aligned}\int_{t_1+\eta_1}^{t_f} f(x_{\pi_1}(t), u(t))dt &= \int_{t_1}^{t_f} f(x_{\pi_1}(t), u(t))dt - \int_{t_1}^{t_1+\eta_1} f(x_{\pi_1}(t), u(t))dt \\ &= \int_{t_1}^{t_f} f(x_{\pi_1}(t), u(t))dt - \eta_1 f(x(t_1), u(t_1)) + o(\eta_1)\,,\end{aligned}$$

since $x_{\pi_1}(t_1) \to x(t_1)$ when $\eta \to 0$. We deduce that

$$x_{\pi_1}(t_f) = x(t_1) + \eta_1(f(x(t_1), u_1) - f(x(t_1), u(t_1))) + \int_{t_1}^{t_f} f(x_{\pi_1}(t), u(t))dt + o(\eta_1)\,.$$

On the other hand,

$$x(t_f) = x(t_1) + \int_{t_1}^{t_f} f(x(t), u(t))dt$$

thus

$$\frac{x_{\pi_1}(t_f) - x(t_f)}{\eta_1} = v_{\pi_1}(t_1) + \frac{1}{\eta_1}\int_{t_1}^{t_f} (f(x_{\pi_1}(t), u(t)) - f(x(t), u(t)))dt\,.$$

From (1.53) we have

$$v_{\pi_1}(t_f) = v_{\pi_1}(t_1) + \int_{t_1}^{t_f} \frac{\partial f}{\partial x}(x(t), u(t))v_{\pi_1}(t)dt\,.$$

Taking the difference, we easily deduce from Gronwall lemma's that the quotient $\frac{x_{\pi_1}(t_f)-x(t_f)}{\eta_1}$ admits a unique limit when $\eta_1 \to 0$, $\eta_1 > 0$, and this limit is equal to $v_{\pi_1}(t_f)$. □

Remark 1.5.4. The sign of η_1 is important. In fact, for η_1 of an arbitrary sign, if we define the perturbation $\pi_1 = \{t_1, \eta_1, u_1\}$ by

$$u_{\pi_1}(t) = \begin{cases} u_1 & \text{if} \quad t \in [t_1, t_1+\eta_1] \quad \text{and if} \quad \eta_1 > 0\,, \\ u_1 & \text{if} \quad t \in [t_1+\eta_1, t_1] \quad \text{and if} \quad \eta_1 < 0\,, \\ u(t) & \text{otherwise}\,, \end{cases}$$

then

$$x_{\pi_1}(t_f) = x(t_f) + |\eta_1|(f(x(t_1), u_1) - f(x(t_1), u(t_1))) + \int_{t_1}^{t_f} f(x_{\pi_1}(t), u(t))dt\,.$$

In particular, the function $\eta_1 \mapsto x_{\pi_1}(t_f)$ is right and left differentiable when $\eta_1 = 0$, but is not differentiable at this point.

Remark 1.5.5. For every $\alpha > 0$, the variation $\{t_1, \alpha\eta_1, u_1\}$ generates the variation vector αv_{π_1}. It follows that the set of variation vectors at time t is a cone of vertex $x(t)$.

Definition 1.5.6. For every $t \in (0, t_f]$, the *first Pontryagin cone* at time t, denoted $K(t)$, is the smallest closed convex cone containing all variation vectors $v_{\pi_1}(t)$ for all Lebesgue points t_1 such that $0 < t_1 < t$.

An immediate iteration leads to the following generalization of Lemma 1.5.3.

Lemma 1.5.7. *Let $t_1 < t_2 < \cdots < t_p$ be Lebesgue points of the function $t \mapsto f(x(t), u(t))$ on $(0, t_f)$, and $u_1, \ldots, u_p$ be points of Ω. Let $\eta_1, \ldots, \eta_p$ be small enough positive real numbers. Consider the variations $\pi_i = \{t_i, \eta_i, u_i\}$, and denote by $v_{\pi_i}(\cdot)$ the associated variation vectors, defined as above. Define the variation*

$$\pi = \{t_1, \ldots, t_p, \eta_1, \ldots, \eta_p, u_1, \ldots, u_p\}$$

of the control u on $[0, t_f]$ by

$$u_\pi(t) = \begin{cases} u_i & \text{if } t_i \leq t \leq t_i + \eta_i, \quad i = 1, \ldots, p, \\ u(t) & \text{otherwise.} \end{cases} \tag{1.55}$$

Let $x_\pi(\cdot)$ be the solution of (1.52) corresponding to the control u_π on $[0, t_f]$ and such that $x_\pi(0) = x_0$. Then,

$$x_\pi(t_f) = x(t_f) + \sum_{i=1}^{p} \eta_i v_{\pi_i}(t_f) + \mathrm{o}\Big(\sum_{i=1}^{p} \eta_i\Big). \tag{1.56}$$

The variation formula (1.56) shows that every combination with positive coefficients of variation vectors (taken at distinct Lebesgue points) provides the point $x(t) + v_\pi(t)$, where

$$v_\pi(t) = \sum_{i=1}^{p} \eta_i v_{\pi_i}(t), \tag{1.57}$$

which belongs, up to the remainder term, to the accessible set $A_\Omega(x_0, t)$ at time t for the system (1.52) starting from the point x_0. In this sense, the first Pontryagin cone serves as an estimate of the accessible set $A_\Omega(x_0, t)$.

Since we deal with a minimal time problem, we must rather consider the set $A_\Omega(x_0, \leq t)$, which leads to introduce also oriented time variations, as follows. Assume first that $x(\cdot)$ is differentiable at time t_f.[4] Let $\delta > 0$ be small enough; then, with the above notations,

$$x_\pi(t_f - \delta) = x(t_f) + \sum_{i=1}^{p} \eta_i v_{\pi_i}(t_f) - \delta f(x(t_f), u(t_f)) + \mathrm{o}\Big(\delta + \sum_{i=1}^{p} \eta_i\Big). \tag{1.58}$$

[4]This holds true e.g. whenever t_f is a Lebesgue point of the function $t \mapsto f(x(t), u(t))$.

Definition 1.5.8. The cone $K_1(t_f)$ is the smallest closed convex cone containing $K(t_f)$ and the vector $-f(x(t_f), u(t_f))$.

See Figure 1.21 for the convex cone $K_1(t_f)$.

Remark 1.5.9. If $x(\cdot)$ is not differentiable at time t_f, then the above construction is slightly modified, by replacing $f(x(t_f), u(t_f))$ with any closure point of the corresponding difference quotient in an obvious way.

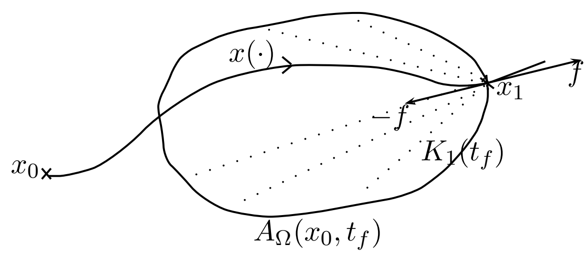

Figure 1.21: Cone $K_1(t_f)$

1.5.2 Conic Implicit Function Theorem

We next provide a *conic implicit function theorem*, which is at the basis of the proof of the maximum principle (see e.g. [2] for a proof).

Recall the following definition of differentiability in the sense of Gâteaux .

Definition 1.5.10. Let E, F be two locally convex topological vector spaces, $f : E \to F$, $x_0 \in E$ and $h \in E$. The Gâteaux derivative $df(x_0) \cdot h$ at x_0 with the direction h is defined as

$$df(x_0) \cdot h = \lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t}$$

if the limit exists.
If the limit exists for all $h \in E$ and it is equal to a linear map $g_{x_0}(h)$, then one says that f is Gâteaux differentiable at x_0 and

$$df(x_0) \cdot h = g_{x_0}(h) \,.$$

Lemma 1.5.11. *Let $C \subset I\!R^m$ be a convex subset of $I\!R^m$ with nonempty interior, of vertex* 0*, and $F : C \to I\!R^n$ be a Lipschitzian mapping such that $F(0) = 0$ and F is differentiable in the sense of Gâteaux at* 0*. Assume that $dF(0) \cdot \mathrm{Cone}(C) = I\!R^n$, where $\mathrm{Cone}(C)$ stands for the (convex) cone generated by elements of C. Then* 0 *belongs to the interior of $F(V \cap C)$, for every neighborhood V of* 0 *in $I\!R^m$.*

1.5.3 Lagrange multipliers and Pontryagin maximum principle

We next restrict the end-point mapping to time and needle-like variations. Let p be a positive integer. Set

$$\mathbb{R}_+^{p+1} = \{(\delta, \eta_1, \ldots, \eta_p) \in \mathbb{R}^{p+1} \mid \delta \geq 0, \eta_1 \geq 0, \ldots, \eta_p \geq 0\}.$$

Let $t_1 < \cdots < t_p$ be Lebesgue points of the function $t \mapsto f(x(t), u(t))$ on $(0, t_f)$, and $u_1, \ldots, u_p$ be points of Ω. Let V be a small neighborhood of 0 in $\mathbb{R}^p$. Define the mapping $F : V \cap \mathbb{R}_+^{p+1} \to \mathbb{R}^n$ by

$$F(\delta, \eta_1, \ldots, \eta_p) = x_\pi(t_f - \delta),$$

where π is the variation $\pi = \{t_1, \ldots, t_p, \eta_1, \ldots, \eta_p, u_1, \ldots, u_p\}$ and $\delta \geq 0$ is small enough so that $t_p < t_f - \delta$. If V is small enough, then F is well defined; moreover this mapping is clearly Lipschitzian, and $F(0) = x(t_f)$. From (1.58), F is Gâteaux differentiable on the conic neighborhood $V \cap \mathbb{R}_+^{p+1}$ of 0.

If the cone $K_1(t_f)$ would coincide with $\mathbb{R}^n$, then there would exist $\delta \geq 0$, an integer p and variations $\pi_i = \{t_i, \eta_i, u_i\}$, $i = 1, \ldots, p$, such that $F_0' \mathbb{R}_+^{p+1} = \mathbb{R}^n$, and then Lemma 1.5.11 would imply that the point $x(t_f)$ would belong to the interior of the accessible set $A_\Omega(x_0, \leq t_f)$, which would raise a contradiction.

Therefore the convex cone $K_1(t_f)$ is not equal to $\mathbb{R}^n$. As a consequence, there exists $\psi \in \mathbb{R}^n \backslash \{0\}$ called *Lagrange multiplier* such that $\langle \psi, v(t_f) \rangle \leq 0$ (see Figure 1.22) for every variation vector $v(t_f) \in K(t_f)$ and $\langle \psi, f(x(t_f), u(t_f)) \rangle \geq 0$ (at least whenever $x(\cdot)$ is differentiable at time t_f; otherwise replace $f(x(t_f), u(t_f))$ with any closure point of the corresponding difference quotient).

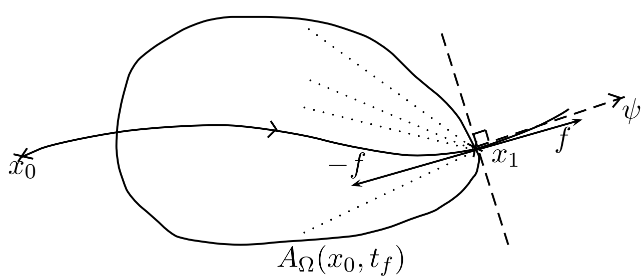

Figure 1.22: $\langle \psi, f(x(t_f), u(t_f)) \rangle \geq 0$

These inequalities then permit to prove the maximum principle (see [96]), according to which the trajectory $x(\cdot)$, associated to the optimal control $u(\cdot)$, is the projection of an *extremal* $(x(\cdot), p(\cdot), p^0, u(\cdot))$ (called extremal lift), where $p^0 \leq 0$ and $p(\cdot) : [0, t_f] \to \mathbb{R}^n$ is a nontrivial absolutely continuous mapping called *adjoint vector*, such that

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), u(t)), \;\; \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t)),$$

almost everywhere on $[0, t_f]$, where $H(x, p, u) = \langle p, f(x, u) \rangle + p^0$ is the Hamiltonian, and $H(x(t), p(t), p^0, u(t)) = M(x(t), p(t), p^0)$ almost everywhere on $[0, t_f]$, where $M(x(t), p(t), p^0) = \max\limits_{w \in \Omega} H(x(t), p(t), p^0, w)$. Moreover, the function $t \mapsto M(x(t), p(t), p^0)$ is identically equal to zero on $t \in [0, t_f]$.

The relation between the Lagrange multiplier ψ and $p(\cdot)$, p^0 is

$$\psi = p(t_f) \quad \text{and} \quad p^0 = -\max_{w \in \Omega} \langle \psi, f(x(t_f), w) \rangle. \tag{1.59}$$

In particular, the Lagrange multiplier ψ is unique (up to a multiplicative scalar) if and only if the trajectory $x(\cdot)$ admits a unique extremal lift (up to a multiplicative scalar).

In the case of a normal extremal, i.e., $p^0 < 0$, since the Lagrange multiplier is defined up to a multiplicative scalar, it is usual to normalize it so that $p^0 = -1$.

Remark 1.5.12. The trajectory $x(\cdot)$ has an abnormal extremal lift $(x(\cdot), p(\cdot), 0, u(\cdot))$ on $[0, t_f]$ if and only if there exists a unit vector $\psi \in \mathbb{R}^n$ such that $\langle \psi, v \rangle \leq 0$ for every $v \in K(t_f)$ and $\max\limits_{w \in \Omega} \langle \psi, f(x(t_f), w) \rangle = 0$. In that case, one has $p(t_f) = \psi$, up to a multiplicative scalar.

Definition 1.5.13. The *first extended Pontryagin cone* $\tilde{K}(t)$ along $x(\cdot)$ is the smallest closed convex cone containing $K_1(t)$ and $f(x(t), u(t))$ (at least whenever $x(\cdot)$ is differentiable at time t; otherwise replace $f(x(t), u(t))$ with any closure point of the corresponding difference quotient).

Note that $x(\cdot)$ does not admit any abnormal extremal lift on $[0, t_f]$ if and only if $\tilde{K}(t_f) = \mathbb{R}^n$.

The following remark easily follows from the above considerations.

Remark 1.5.14. For the optimal trajectory $x(\cdot)$, the following statements are equivalent:

- The trajectory $x(\cdot)$ has a unique extremal lift (up to a multiplicative scalar); moreover, the extremal lift is normal.
- $K_1(t_f)$ is a half-space and $\tilde{K}(t_f) = \mathbb{R}^n$.
- $K(t_f)$ is a half-space and $\max\limits_{w \in \Omega} \langle \psi, f(x(t_f), w) \rangle > 0$.

This remark permits to translate the assumptions of the main result of Chapter 2 (Theorem 2.5.1) into geometric considerations.

1.6 Generalized controls

Following Gamkrelidze arguments in [46], we can expand the class of admissible controls introducing the generalized controls.

1.6.1 Generalized control definition

Let μ_t, $t \in \mathbb{R}$ be a family of Radon measures on $\mathbb{R}^m$ that depend on the parameter $t \in \mathbb{R}$ and $g(t,u)$ a continuous (scalar- or vector-valued) function of its arguments $t \in \mathbb{R}$ and $u \in \mathbb{R}^m$ with a compact support in u for every fixed $t \in \mathbb{R}$ (the support can depend on t).

Definition 1.6.1. Integrating $g(t,u)$ with respect to μ_t, we obtain the following function of t:

$$h(t) = \int_{\mathbb{R}^m} g(t,u)\, d\mu_t(u) = \int_{\mathbb{R}^m} g(t,u)\, d\mu_t\,, \quad t \in \mathbb{R}\,.$$

If the function $h(t)$ is Lebesgue measurable for an arbitrary $g(t,u)$ of this type, then we say that the family μ_t, $t \in \mathbb{R}$, is *weakly measurable* (with respect to t).

Definition 1.6.2. If there exists a compact set $K \subset \mathbb{R}^m$ that does not depend on $t \in \mathbb{R}$ and is such that the measures μ_t are concentrated on K for almost all $t \in \mathbb{R}$ (in the sense of the Lebesgue measure on $\mathbb{R}$), then the family μ_t, $t \in \mathbb{R}$, is said to be *finite*.

The result of the integration of a continuous function $g(t,u)$ with respect to a measure μ_t can be denoted by

$$\langle \mu_t, g(t,u) \rangle = \int_{\mathbb{R}^m} g(t,u)\, d\mu_t\,.$$

An admissible control taking values in a subset of $\mathbb{R}^m$, $u(t) \in \mathcal{U}_U$, can be considered as a family of Dirac measures (a Dirac measure is a unit, positive measure concentrated at a point) on $\mathbb{R}^m$ that depend on time $t \in \mathbb{R}$. Indeed, the value $u(t)$ of the control at the time t, corresponds to the unit, positive measure $\delta_{u(t)}$ which is concentrated at the point $u(t) \in U$ and acts on an arbitrary continuous function $g(t,u)$ in accordance with the formula

$$\langle \delta_{u(t)}, g(t,u) \rangle = \int_{\mathbb{R}^m} g(t,u)\, d\delta_{u(t)} = g(t, u(t))\,.$$

The family of measures $\delta_{u(t)}$ is finite and weakly measurable.

Conversely, if we assume that $\delta_{v(t)}$, $t \in \mathbb{R}$ is an arbitrary, weakly measurable finite family of Dirac measures, where the measure $\delta_{v(t)}$ is concentrated at the point $v(t) \in U$ at the time t, then the function $v(t)$, $t \in \mathbb{R}$, is essentially bounded. Setting $g(t,u) = u$, we obtain the measurable function

$$\langle \delta_{v(t)}, u \rangle = v(t) \in U\,.$$

Thus, we have established a natural correspondence between admissible controls $u(t) \in \mathcal{U}_U$ and weakly measurable and finite families of Dirac measures $\delta_{u(t)}$, $t \in \mathbb{R}$, concentrated on the set $U \subset \mathbb{R}^m$.

Definition 1.6.3. Any weakly measurable and finite family of *probability measures*, i.e., unit, positive, Radon measures μ_t with $t \in \mathbb{R}$ that are concentrated on the set $U \subset \mathbb{R}^m$, is said to be a *generalized control*.

We denote the set of all generalized controls by $\mathcal{M}_U$ and call it the *class of generalized controls*. Subsequently, μ_t with $t \in \mathbb{R}$ will always denote a generalized control. Moreover, we have $\mathcal{U}_U \subset \mathcal{M}_U$.

Remark 1.6.4. The reason for taking a probability measure, and not an arbitrary Radon measure in the definition of a generalized control, is that only families of probability measures have the property that makes them useful in control problems and that is expressed in Gamkrelidze's approximation lemma (see [46, Chapter 3]).

1.6.2 Minimal time problem

Consider the minimal time problem $(\mathcal{P})$ that consists in finding a control $u(\cdot) \in \mathcal{U}_U$ such that the associated trajectory $x(\cdot)$ is solution of the control system

$$\dot{x}(t) = f(x(t), u(t)) \tag{1.60}$$

with $u(t) \in U$ and where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function and has continuous derivative with respect to x, and steers the point $x_0 = x(0)$ to $x_1 = x(t_f)$ in minimal time t_f.

Substituting a generalized control μ_t for u on the control system (1.60) we obtain the following differential equation

$$\dot{x} = \langle \mu_t, f(x,u) \rangle = \int_{\mathbb{R}^m} f(x,u)\, d\mu_t\,, \tag{1.61}$$

which is analogous to equation (1.60). If the initial condition $x(0) = x_0$ is given, then the equation obtained is equivalent to the integral equation

$$x(t) = x_0 + \int_0^{t_f} \langle \mu_s, f(x(s),u) \rangle ds\,,$$

which has a uniquely determined solution defined on a neighborhood of the point $t = 0$ (see [46, Chapter 4]).

The minimal time optimal control problem $(\mathcal{P}_G)$ consists in finding a generalized control $\mu_t \in \mathcal{M}_U$ such that the associated trajectory is solution of the differential equation (1.61) and steers $x_0 = x(0)$ to $x_1 = x(t_f)$ in minimal time t_f. The problem $(\mathcal{P}_G)$ will also be called the *convex optimal problem* which corresponds to the optimal problem $(\mathcal{P})$.

Remark 1.6.5. The set of all generalized controls $\mathcal{M}_U$ and the set of right-hand-sides of equation (1.61), $\mu_t \in \mathcal{M}_U$ are convex. In particular, the set of all possible phase velocities of the control system (1.61), with fixed t and x, is also convex in $\mathbb{R}^n$.

1.6.3 Variation of generalized controls and Pontryagin maximization condition

Let $\tilde{\mu}_t$ be an arbitrary generalized control, and let $\tilde{x}(t)$, $t_0 \leq t \leq t_f$, be a trajectory of the equation

$$\dot{x} = \langle \tilde{\mu}_t,\, f(x,u) \rangle = F(x)\,. \tag{1.62}$$

The function $F(x)$ is defined on the entire space $\mathbb{R}^n$, continuously differentiable with respect to x and bounded on any compact set $K \subset \mathbb{R}^n$ (see [46]).

Definition 1.6.6. Any difference

$$\delta\mu_t = \mu_t - \tilde{\mu}_t\,, \quad \mu_t \in \mathcal{M}_U\,,$$

will be called a *variation* or a *perturbation* of the generalized control $\tilde{\mu}_t$.

The set of all variations of the control $\tilde{\mu}_t$ will be denoted by $\delta\mathcal{M}_{\tilde{\mu}_t}$. The set $\delta\mathcal{M}_{\tilde{\mu}_t}$ is convex (see [46] for an intensive and complete study).

Definition 1.6.7. We shall say that a sequence of generalized controls $\mu_t^{(i)}$ *converges weakly* to a generalized control* μ_t as $i \to \infty$ if we have

$$\int_{\mathbb{R}} \langle \mu_t^{(i)}, g(t,u)\rangle\, dt \to \int_{\mathbb{R}} \langle \mu_t, g(t,u)\rangle\, dt\,, \quad (i \to \infty)$$

for an arbitrary continuous function $g(t,u)$ with compact support.

Let $\mu \in \mathcal{M}_U$ and define the *end-point mapping*

$$\begin{aligned} E_{x_0,t_f}(\mu)\,:\, \mathcal{M}_U &\longrightarrow \mathbb{R}^n \\ \mu &\longmapsto x(t_f) \end{aligned}$$

where x is solution of $\dot{x} = \langle \mu(t), f(x(t),u)\rangle$ with $x(0) = x_0$.

Proposition 1.6.8. *[46, Chapter 5] The end-point mapping E_{x_0,t_f} is Gâteaux differentiable for the weak* topology and*

$$dE_{x_0,t_f}(\mu)\cdot\delta\mu = M(t_f)\int_0^{t_f} M^{-1}(s)\langle \delta\mu, f(x(s),u)\rangle ds\,, \;\; \forall \delta\mu \in \mathcal{M}_U. \tag{1.63}$$

Pontryagin maximization condition In what follows we derive the maximization condition of Pontryagin maximum principle for the minimal time problem $(\mathcal{P}_G)$.

Let $(x(t),\mu_t)$ be optimal for the problem $(\mathcal{P}_G)$, then $(x(t),\mu_t)$ is singular for the augmented system (see Lemma 1.4.23)

$$\begin{cases} \dot{x}(t) = \langle \mu_t, f(x(t),u)\rangle \\ \dot{x}^0(t) = \langle \mu_t, f^0(x(t),u)\rangle\,. \end{cases}$$

By the conic implicit function theorem (Theorem 1.5.11)

$$\begin{aligned} dE_{x_0,t_f}\,:\, Cone(\Omega-\mu) &\longrightarrow \mathbb{R}^n \\ \delta\mu &\longmapsto \delta x(t_f) \end{aligned}$$

is not surjective. Therefore, there exists $\psi \in \mathbb{R}^n \backslash \{0\}$ such that

$$\psi \cdot dE_{x_0,t_f}(\mu) \cdot \delta\mu \leq 0\,.$$

From Proposition 1.6.8, $\int_0^{t_f} \psi M(t_f) M(t)^{-1} \langle \delta\mu,\, f(x(t), u) \rangle \, dt \leq 0$.

Let us denote $p(t) = \psi M(t_f) M(t)^{-1}$. Then,

$$p(t) \langle \delta\mu_t,\, f(x(t), u) \rangle \leq 0$$

holds almost everywhere on $[0, t_f]$, for every Lebesgue point t and for every $\delta\mu_t$ of $Cone(\Omega - \mu)$.

In particular, if $\delta\mu_t = \delta_v - \delta_{\mu(t)}$ (where $v \in \mathcal{U}_U$), then

$$p(t) \cdot (f(x(t), v) - f(x(t), u(t))) \leq 0\,. \tag{1.64}$$

Therefore

$$\forall v \in \mathcal{U}_U,\ p(t) \cdot f(x(t), v) \leq p(t) \cdot f(x(t), u(t))\,, \tag{1.65}$$

but

$$p(t) \cdot f(x(t), v) = H\left(x(t), p(t), v\right) \text{ and } p(t) \cdot f(x(t), u(t)) = H\left(x(t), p(t), u(t)\right)\,. \tag{1.66}$$

It follows that

$$H\left(x(t), p(t), u(t)\right) = \max_{v \in \Omega} H\left(x(t), p(t), v\right)\,. \tag{1.67}$$

Chapter 2

Smooth regularization of bang-bang optimal control problems

2.1 Introduction

In this chapter we consider the minimal time control problem for a single-input control-affine system $\dot{x} = X(x) + u_1 Y_1(x)$ in $\mathbb{R}^n$, where the scalar control $u_1(\cdot)$ satisfies the constraint $|u_1(t)| \leq 1$, for every $t \in [0, t_f]$.

We propose the following smoothing procedure. For $\varepsilon > 0$ small and $Y_1, \ldots, Y_m$ arbitrary given vector fields, we consider the minimal time problem for the control system $\dot{x} = X(x) + u_1^\varepsilon Y_1(x) + \varepsilon \sum_{i=2}^{m} u_i^\varepsilon Y_i(x)$, where the scalar controls $u_i^\varepsilon(\cdot)$, $i = 1, \ldots, m$, with $m \geq 2$, satisfy the constraint $\sum_{i=1}^{m} (u_i^\varepsilon(t))^2 \leq 1$.

One of the possible motivations for this regularization procedure is the use of shooting methods. Among the numerous numerical methods that exist to solve optimal control problems, the *shooting methods* consist in solving, via Newton-like methods, the two-point or multi-point boundary value problem arising from the application of the Pontryagin maximum principle. More precisely, a Newton method is applied in order to compute a zero of the *shooting function* associated to the problem (see e.g. [109]).

For the initial problem, optimal controls may be discontinuous, and it follows that the shooting function is not smooth on $\mathbb{R}^n$ in general. Actually it may be non differentiable on switching surfaces. This implies two difficulties when using a shooting method. First, if one does not know a priori the structure of the optimal control, then it may be very difficult to initialize properly the shooting method, and in general the iterates of the underlying Newton method will be unable to cross barriers generated by switching surfaces (see e.g. [71]). Second, the numerical computation of the shooting function and of its differential may be intricate since the shooting function is not continuously differentiable. However, the shooting function related to the proposed regularized optimal control problem is smooth.

In the main result of this chapter (Section §2.5, Theorem 2.5.1) we prove, under appropriate assumptions, that the optimal controls of the latter system, depending on ε, are smooth functions of t, and converge weakly to the optimal control of the initial system; moreover the associated trajectories converge uniformly. If the optimal control of the initial system is moreover bang-bang, then the convergence of the regularized control holds almost everywhere; this property may however fail whenever the bang-bang property does not hold.

In Section §2.6 examples and counterexamples are provided which illustrate Theorem 2.5.1.

2.2 Statement of the problem

Consider the single-input control-affine system in $\mathbb{R}^n$

$$\dot{x} = X(x) + u_1 Y_1(x), \tag{2.1}$$

where X and Y_1 are smooth vector fields, and the control u_1 is a measurable scalar function satisfying the constraint

$$|u_1(t)| \leq 1\,, \quad \forall t \in [0, t_f]. \tag{2.2}$$

Let M_0 and M_1 be two compact subsets of $\mathbb{R}^n$. Assume that M_1 is reachable from M_0, that is, there exist a time $T > 0$ and a control function $u_1(\cdot) \in L^\infty(0,T)$ satisfying the constraint (2.2), such that the trajectory $x(\cdot)$, solution of (2.1) with $x(0) \in M_0$, satisfies $x(T) \in M_1$.

We consider the optimal control problem **(OCP)** of determining, among all solutions of (2.1)–(2.2) steering M_0 to M_1 in minimal time.

2.3 Pontryagin extremals

Assume that the subset M_1 is reachable from M_0; it follows that the optimal control problem **(OCP)** admits a solution $x(\cdot)$, associated to a control $u_1(\cdot)$, on $[0, t_f]$, where $t_f > 0$ is the minimal time (see e.g. [26, Chapter 9] for optimal control existence theorems).

According to the Pontryagin maximum principle (see [96] and Chapter 1),there exist a real number $p^0 \leq 0$ and a nontrivial absolutely continuous mapping $p(\cdot) : [0, t_f] \to \mathbb{R}^n$, called *adjoint vector*, with $(p(\cdot), p^0) \neq 0$ and such that

$$\begin{aligned} \dot{p}(t) &= -\frac{\partial H}{\partial x}(x(t), p(t), p^0, u(t)) \\ &= -\left\langle p(t), \frac{\partial X}{\partial x}(x(t)) \right\rangle - u_1(t) \left\langle p(t), \frac{\partial Y_1}{\partial x}(x(t)) \right\rangle \end{aligned} \tag{2.3}$$

where the function $H(x, p, p^0, u) = \langle p, X + uY_1(x) \rangle + p^0$ is called the Hamiltonian, and the maximization condition

$$H(x(t), p(t), p^0, u(t)) = \max_{|w| \leq 1} H(x(t), p(t), p^0, w) \tag{2.4}$$

holds almost everywhere on $[0, t_f]$. Moreover, $\max_{|w|\leq 1} H(x(t), p(t), p^0, w) = 0$ for every $t \in [0, t_f]$. It follows from (2.4) that

$$u_1(t) = \operatorname{sign}\langle p(t), Y_1(x(t))\rangle \tag{2.5}$$

for almost every t, provided the (continuous) switching function $\varphi(t) = \langle p(t), Y_1(x(t))\rangle$ does not vanish on any subinterval of $[0, t_f]$. In that case, $u_1(t)$ only depends on $x(t)$ and on the adjoint vector, and it follows from (2.3) that the extremal $(x(\cdot), p(\cdot), p^0, u_1(\cdot))$ is completely determined by the initial adjoint vector $p(0)$. The case where the switching function may vanish on a subinterval I is related to singular trajectories[1]. In that case, derivating the relation $\langle p(t), Y_1(x(t))\rangle = 0$ on I leads to $\langle p(t), [X, Y_1](x(t))\rangle = 0$ on I, and a second derivation leads to $\langle p(t), [X, [X, Y_1]](x(t))\rangle + u_1(t)\langle p(t), [Y_1, [X, Y_1]](x(t))\rangle = 0$ on I, which permits, under generic assumptions on the vector fields X and Y_1 (see [27–29] for genericity results related to singular trajectories), to compute the singular control $u_1(\cdot)$ on I. Under such generic assumptions, the extremal $(x(\cdot), p(\cdot), p^0, u_1(\cdot))$ is still completely determined by the initial adjoint vector.

Note that, since $x(\cdot)$ is optimal on $[0, t_f]$, and since the control system under study is autonomous, it follows that $x(\cdot)$ is solution of the optimal control problem of steering the system (2.1)–(2.2) from $x_0 = x(0)$ to $x(t)$ in minimal time.

2.4 Regularization procedure

Let ε be a positive real parameter and let $Y_2, \dots, Y_m$ be $m-1$ arbitrary smooth vector fields on $\mathbb{R}^n$, where $m \geq 2$ is an integer. Consider the control-affine system

$$\dot{x}^\varepsilon(t) = X(x^\varepsilon(t)) + u_1^\varepsilon(t) Y_1(x^\varepsilon(t)) + \varepsilon \sum_{i=2}^{m} u_i^\varepsilon(t) Y_i(x^\varepsilon(t)), \tag{2.6}$$

where the control $u^\varepsilon(t) = (u_1^\varepsilon(t), \dots, u_m^\varepsilon(t))$ satisfies the constraint

$$\sum_{i=1}^{m} (u_i^\varepsilon(t))^2 \leq 1. \tag{2.7}$$

Consider the optimal control problem $\mathbf{(OCP)}_\varepsilon$ of determining a trajectory $x^\varepsilon(\cdot)$, solution of (2.6)–(2.7) on $[0, t_f^\varepsilon]$, such that $x^\varepsilon(0) \in M_0$ and $x^\varepsilon(t_f^\varepsilon) \in M_1$, and minimizing the time of transfer t_f^ε. The parameter ε is viewed as a penalization parameter, and it is expected that any solution $x^\varepsilon(\cdot)$ of $\mathbf{(OCP)}_\varepsilon$ tends to a solution $x(\cdot)$ of **(OCP)** as ε tends to zero. It is our aim to derive such a result.

According to the Pontryagin maximum principle, any optimal solution $x^\varepsilon(\cdot)$ of $\mathbf{(OCP)}_\varepsilon$, associated with controls $(u_1^\varepsilon, \dots, u_m^\varepsilon)$ satisfying the constraint (2.7), is the projection of an

[1]Recall that here the term "singular" has a different meaning from the one used in Chapter 1 (see page 23).

extremal $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), p^{0\varepsilon}, u^\varepsilon(\cdot))$ such that

$$\begin{aligned}\dot{p}^\varepsilon(t) &= -\frac{\partial H^\varepsilon}{\partial x}(x^\varepsilon(t), p^\varepsilon(t), p^{0\varepsilon}, u^\varepsilon(t)) \\ &= -\left\langle p^\varepsilon(t), \frac{\partial X}{\partial x}(x^\varepsilon(t))\right\rangle - u_1^\varepsilon(t)\left\langle p^\varepsilon(t), \frac{\partial Y_1}{\partial x}(x^\varepsilon(t))\right\rangle \\ &\quad - \varepsilon \sum_{i=2}^m u_i^\varepsilon(t)\left\langle p^\varepsilon(t), \frac{\partial Y_i}{\partial x}(x^\varepsilon(t))\right\rangle\end{aligned} \tag{2.8}$$

where $H^\varepsilon(x,p,p^0,u) = \langle p, X(x) + u_1 Y_1(x) + \varepsilon \sum_{i=2}^m u_i Y_i(x)\rangle + p^0$ is the Hamiltonian, and

$$H(x^\varepsilon(t), p^\varepsilon(t), p^{0\varepsilon}, u^\varepsilon(t)) = \max_{\sum_{i=1}^m w_i^2 \leq 1} H(x^\varepsilon(t), p^\varepsilon(t), p^{0\varepsilon}, w) \tag{2.9}$$

almost everywhere on $[0, t_f^\varepsilon]$. Moreover, the maximized Hamiltonian is equal to 0 on $[0, t_f^\varepsilon]$. The maximization condition (2.9) turns into

$$\begin{aligned}&u_1^\varepsilon(t)\langle p^\varepsilon(t), Y_1(x^\varepsilon(t))\rangle + \varepsilon \sum_{i=2}^m u_i^\varepsilon(t)\langle p^\varepsilon(t), Y_i(x^\varepsilon(t))\rangle \\ &= \max_{\sum_{i=1}^m w_i^2 \leq 1}\left(w_1\langle p^\varepsilon(t), Y_1(x^\varepsilon(t))\rangle + \varepsilon \sum_{i=2}^m w_i \langle p^\varepsilon(t), Y_i(x^\varepsilon(t))\rangle\right),\end{aligned} \tag{2.10}$$

and two cases may occur: either the maximum is attained in the interior of the domain, or it is attained on the boundary. In the first case, there must hold $\langle p^\varepsilon(t), Y_i(x^\varepsilon(t))\rangle = 0$, for every $i \in \{1, \ldots, m\}$; in particular, if the m functions $t \mapsto \langle p^\varepsilon(t), Y_i(x^\varepsilon(t))\rangle$, $i = 1, \ldots, m$, do not vanish simultaneously, then the maximum is attained on the boundary of the domain. Throughout this thesis, we make the following assumption.

Assumption 2.4.1. *The integer m and the vector fields $Y_2, \ldots, Y_m$ are chosen such that*

$$\mathrm{Span}\{Y_i \mid i = 1, \ldots, m\} = \mathbb{R}^n.$$

Under this assumption, the maximization condition (2.10) yields

$$\begin{aligned}u_1^\varepsilon(t) &= \frac{\langle p^\varepsilon(t), Y_1(x^\varepsilon(t))\rangle}{\sqrt{\langle p^\varepsilon(t), Y_1(x^\varepsilon(t))\rangle^2 + \varepsilon^2 \sum_{i=2}^m \langle p^\varepsilon(t), Y_i(x^\varepsilon(t))\rangle^2}}, \\ u_i^\varepsilon(t) &= \frac{\varepsilon\langle p^\varepsilon(t), Y_i(x^\varepsilon(t))\rangle}{\sqrt{\langle p^\varepsilon(t), Y_1(x^\varepsilon(t))\rangle^2 + \varepsilon^2 \sum_{i=2}^m \langle p^\varepsilon(t), Y_i(x^\varepsilon(t))\rangle^2}}, \quad i = 2, \ldots, m,\end{aligned} \tag{2.11}$$

for almost every $t \in [0, t_f^\varepsilon]$, and moreover the control functions $u_i^\varepsilon(\cdot)$, $i = 1, \ldots, m$ are smooth functions of t (so that the above formula holds actually for every $t \in [0, t_f^\varepsilon]$). Indeed, to prove this fact, it suffices to prove that the functions $t \mapsto \langle p^\varepsilon(t), Y_i(x^\varepsilon(t))\rangle$, $i = 1, \ldots, m$ do not vanish simultaneously. The argument goes by contradiction: if these functions would vanish simultaneously, then, using the Assumption 2.4.1, this would imply that $p^\varepsilon(t) = 0$ for some t; combined with the fact that the maximized Hamiltonian is equal to zero along any extremal, it would follow that $p^{0\varepsilon} = 0$, and this would raise a contradiction since the adjoint vector $(p^\varepsilon(\cdot), p^{0\varepsilon})$ of the maximum principle must be nontrivial.

From (2.11), it is expected that $u_1^\varepsilon(\cdot)$ converges to $u_1(\cdot)$ and $u_i^\varepsilon(\cdot)$, $i = 2, \ldots, m$, tend to zero, in some topology to specify. This fact is derived rigorously in the next section.

2.5 Convergence results

The main result of this chapter is the following theorem.

Theorem 2.5.1. *Assume that the problem* ***(OCP)*** *has a unique solution $x(\cdot)$, defined on $[0, t_f]$, associated with a control $u_1(\cdot)$ on $[0, t_f]$. Moreover, assume that $x(\cdot)$ has a unique extremal lift (up to a multiplicative scalar), that is moreover normal, and denoted by $(x(\cdot), p(\cdot), -1, u_1(\cdot))$.*

Then, under the Assumption 2.4.1, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, the problem ***(OCP)$_\varepsilon$*** *has at least one solution $x^\varepsilon(\cdot)$, defined on $[0, t_f^\varepsilon]$ with $t_f^\varepsilon \leq t_f$, associated with a smooth control $u^\varepsilon = (u_1^\varepsilon, \ldots, u_m^\varepsilon)$ satisfying the constraint (2.7), every extremal lift of which is normal. Let $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), -1, u^\varepsilon(\cdot))$ be such a normal extremal lift. Then, as ε tends to 0,*

- *t_f^ε converges to t_f;*
- *$x^\varepsilon(\cdot)$ converges uniformly[2] to $x(\cdot)$, and $p^\varepsilon(\cdot)$ converges uniformly to $p(\cdot)$ on $[0, t_f]$;*
- *$u_1^\varepsilon(\cdot)$ converges weakly[3] to $u_1(\cdot)$ for the weak $L^1(0, t_f)$ topology.*

If the control u_1 is moreover bang-bang, i.e., if the (continuous) switching function $\varphi(t) = \langle p(t), Y_1(x(t))\rangle$ does not vanish on any subinterval of $[0, t_f]$, then $u_1^\varepsilon(\cdot)$ converges to $u_1(\cdot)$ and $u_i^\varepsilon(\cdot)$, $i = 2, \ldots, m$, converge to 0 almost everywhere on $[0, t_f]$, and thus in particular for the strong $L^1(0, t_f)$ topology.

Remark 2.5.2. We provide in Section §2.6 examples with numerical simulations in order to illustrate Theorem 2.5.1. The first example is the Rayleigh problem, on which the minimal time trajectory is bang-bang, and almost everywhere convergence of the regularized control can be observed in agreement with our main result. Our second example involves a singular arc and we prove and observe that oscillations appear, so that the regularized control weakly converges, but fails to converge almost everywhere.

[2] We consider any continuous extension of $x^\varepsilon(\cdot)$ on $[0, t_f]$.

[3] It means that $\int_0^{t_f} u_1^\varepsilon(t) g(t) dt \to \int_0^{t_f} u_1(t) g(t) dt$ as $\varepsilon \to 0$, for every $g \in L^1(0, t_f)$, and where the function $u_1^\varepsilon(\cdot)$ is extended continuously on $[0, t_f]$.

Remark 2.5.3. It is assumed that the problem **(OCP)** has a unique solution $x(\cdot)$, having a unique extremal lift that is normal. Such an assumption holds true whenever the minimum time function (the value function of the optimal control problem) enjoys differentiability properties (see e.g. [9, 35] for a precise relationship, see also [24, 97, 98, 108] for results on the size of the set where the value function is differentiable).

If one removes these uniqueness assumptions, then the following result still holds, provided that every extremal lift of every solution of **(OCP)** is normal. Consider the topological spaces $\mathcal{X} = C^0([0,t_f],\mathbb{R}^n)$, endowed with the uniform convergence topology, and $\mathcal{Y} = L^\infty(0,t_f;[-1,1])$, endowed with the weak star topology. In the following statement, the space $\mathcal{X}\times\mathcal{X}\times\mathcal{Y}$ is endowed with the resulting product topology. For every $\varepsilon \in (0,\varepsilon_0)$, let $x^\varepsilon(\cdot)$ be a solution of $\mathbf{(OCP)}_\varepsilon$, and let $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), -1, u^\varepsilon(\cdot))$ be a (normal) extremal lift of $x^\varepsilon(\cdot)$. Then, every closure point in $\mathcal{X}\times\mathcal{X}\times\mathcal{Y}$ of the family of triples $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), u_1^\varepsilon(\cdot))$ is a triple $(\bar{x}(\cdot), \bar{p}(\cdot), \bar{u}_1(\cdot))$, where $\bar{x}(\cdot)$ is an optimal solution of **(OCP)**, associated with the control $\bar{u}_1(\cdot)$, having as a normal extremal lift the 4-tuple $(\bar{x}(\cdot), \bar{p}(\cdot), -1, \bar{u}_1(\cdot))$. The rest of the statement of Theorem 2.5.1 still holds with an obvious adaptation in terms of closure points.

Remark 2.5.4. When applying a shooting method to the problem $\mathbf{(OCP)}_\varepsilon$, one is not ensured to determine an optimal solution, but only an extremal solution that is not necessarily optimal.[4] Notice however that the arguments of the proof of Theorem 2.5.1 permit to prove the following statement. Assume that there is no abnormal extremal among the set of extremals obtained by applying the Pontryagin maximum principle to the problem **(OCP)**; then, for $\varepsilon > 0$ small enough, every extremal solution of $\mathbf{(OCP)}_\varepsilon$ is normal, and, using the notations of the previous remark, every closure point of such extremal solutions is a normal extremal solution of **(OCP)**.

Remark 2.5.5. There is a large literature dealing with optimal control problems depending on some parameters, involving state, control or mixed constraints, using a stability and sensitivity analysis in order to investigate the dependence of the optimal solution with respect to parameters (see e.g. [40, 48, 56, 66, 67, 74, 76, 81, 82, 84] and references therein). In the sensitivity approach, under second order sufficient conditions, results are derived that prove that the solutions of the parametrized problems, as well as the associated Lagrange multipliers, are Lipschitz continuous or directionally differentiable functions of the parameter. We stress however that Theorem 2.5.1 cannot be derived from these former works. Indeed, in these references, the results rely on second order sufficient conditions and certain regularity conditions on the initial problem. In our work we do not assume any second order sufficient condition; our approach is different from the usual sensitivity analysis and is rather, in some sense, a topological approach.

[4] This fact is well known, due to the fact that the Pontryagin maximum principle is only a first order necessary condition for optimality; sufficient conditions do exist but this is outside the scope of this Chapter (see Section §3.1.3 for sufficient conditions).

In what follows several lemmas will be proved. The proof of Theorem 2.5.1 follows from Lemmas 2.5.6–2.5.18.

From now on, assume that all assumptions of Theorem 2.5.1 hold. We denote the end-point mapping for the system (2.6) by

$$E(\varepsilon, x_0, t_f, u^\varepsilon) = x^\varepsilon(t_f),$$

where $x^\varepsilon(\cdot)$ is the solution of (2.6) associated with the control $u^\varepsilon(\cdot) = (u_1^\varepsilon(\cdot), \dots, u_m^\varepsilon(\cdot))$ and such that $x^\varepsilon(0) = x_0$. By extension, the end-point mapping for the system (2.1) corresponds to $\varepsilon = 0$,

$$E(0, x_0, t_f, (u_1, 0, \dots, 0)) = x(t_f),$$

where $x(\cdot)$ is the solution of (2.1) associated with the control $u_1(\cdot)$ and such that $x(0) = x_0$. It will be also denoted $E(x_0, t_f, u_1) = E(0, x_0, t_f, (u_1, 0, \dots, 0)) = x(t_f)$.

In the sequel, we denote by $u_1(\cdot)$ the minimal time control steering the system (2.1) from M_0 to M_1 in time t_f.

We first derive the following existence result.

Lemma 2.5.6. *For every $\varepsilon > 0$,*[5] *the problem **(OCP)**$_\varepsilon$ admits at least one solution $x^\varepsilon(\cdot)$, associated with a control $u^\varepsilon(\cdot) = (u_1^\varepsilon(\cdot), \dots, u_m^\varepsilon(\cdot))$ satisfying the constraint* (2.7) *on $[0, t_f^\varepsilon]$. Moreover, $0 \le t_f^\varepsilon \le t_f$.*

Proof. Knowing that the constrained minimization problem

$$\begin{cases} \min t_f \\ |u_1| \le 1, \ E(0, x_0, t_f, (u_1, 0, \dots, 0)) = x_1 \\ x_0 \in M_0, \ x_1 \in M_1 \end{cases}$$

has a solution, it is our aim to prove that the problem

$$\begin{cases} \min t_f^\varepsilon \\ u^\varepsilon = (u_1^\varepsilon, \dots, u_m^\varepsilon), \ \displaystyle\sum_{i=1}^m (u_i^\varepsilon)^2 \le 1, \ E(\varepsilon, x_0, t_f^\varepsilon, u^\varepsilon) = x_1 \\ x_0 \in M_0, \ x_1 \in M_1 \end{cases}$$

has a solution, for every $\varepsilon > 0$. First of all, we claim that, for every $\varepsilon > 0$, the subset M_1 is reachable from the subset M_0, i.e., it is possible to solve the equation

$$E(\varepsilon, x_0, t_f^\varepsilon, u^\varepsilon) = x_1$$

with a control $u^\varepsilon = (u_1^\varepsilon, \dots, u_m^\varepsilon)$ satisfying the constraint $\sum_{i=1}^m (u_i^\varepsilon)^2 \le 1$, and with some $x_0 \in M_0$ and $x_1 \in M_1$. Indeed, if $u_i^\varepsilon = 0$, $i = 2, \dots, m$, then the system (2.6) coincides with

[5] Note that ε is not needed to be small.

the system (2.1), and it suffices to choose $u_1^\varepsilon = u_1$ and the corresponding initial and final points. The existence of a minimal time control steering the system (2.6) from M_0 to M_1 is then a standard fact to derive for such a control-affine system (see e.g. [26, Chapter 9], and note that M_0 and M_1 are compact). Moreover, the minimal time t_f^ε for the problem $\mathbf{(OCP)}_\varepsilon$ is less or equal than the minimal time t_f for the initial problem. □

As explained in Section 2.4, for $\epsilon > 0$ fixed, and with Assumption 2.4.1 satisfied, it follows from the Pontryagin maximum principle applied to $\mathbf{(OCP)}_\varepsilon$ that $x^\varepsilon(\cdot)$ is the projection of an extremal $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), p^{0\varepsilon}, u^\varepsilon(\cdot))$ such that

$$\begin{aligned}\dot{p}^\varepsilon(t) = &- \left\langle p^\varepsilon(t), \frac{\partial X}{\partial x}(x^\varepsilon(t)) \right\rangle - u_1^\varepsilon(t) \left\langle p^\varepsilon(t), \frac{\partial Y_1}{\partial x}(x^\varepsilon(t)) \right\rangle \\ &- \varepsilon \sum_{i=2}^m u_i^\varepsilon(t) \left\langle p^\varepsilon(t), \frac{\partial Y_i}{\partial x}(x^\varepsilon(t)) \right\rangle\end{aligned}$$

and

$$\begin{aligned}u_1^\varepsilon(t) &= \frac{\langle p^\varepsilon(t), Y_1(x^\varepsilon(t))\rangle}{\sqrt{\langle p^\varepsilon(t), Y_1(x^\varepsilon(t))\rangle^2 + \varepsilon^2 \sum_{i=2}^m \langle p^\varepsilon(t), Y_i(x^\varepsilon(t))\rangle^2}}, \\ u_i^\varepsilon(t) &= \frac{\varepsilon\langle p^\varepsilon(t), Y_i(x^\varepsilon(t))\rangle}{\sqrt{\langle p^\varepsilon(t), Y_1(x^\varepsilon(t))\rangle^2 + \varepsilon^2 \sum_{i=2}^m \langle p^\varepsilon(t), Y_i(x^\varepsilon(t))\rangle^2}}, \quad i = 2, \ldots, m.\end{aligned}$$

We stress the fact that the controls u_i^ε, $i = 1, \ldots, m$, are continuous functions of t.

Lemma 2.5.7. *If $\varepsilon > 0$ tends to 0, then t_f^ε converges to t_f, $u_1^\varepsilon(\cdot)$ converges to $u_1(\cdot)$ in $L^\infty(0, t_f)$ for the weak star topology, and $x^\varepsilon(\cdot)$ converges to $x(\cdot)$ uniformly on $[0, t_f]$.*

Proof. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive real numbers converging to 0 as n tends to $+\infty$. From Lemma 2.5.6, $0 \leq t_f^{\varepsilon_n} \leq t_f$, hence, up to a subsequence, $(t_f^{\varepsilon_n})_{n\in\mathbb{N}}$ converges to some $T \geq 0$ such that $T \leq t_f$. By definition, the sequence of controls $(u_1^{\varepsilon_n}(\cdot))_{n\in\mathbb{N}}$ is bounded in $L^\infty(0, t_f)$ (with the agreement that the function $u_1^{\varepsilon_n}(\cdot)$ is extended on $(t_f^{\varepsilon_n}, t_f]$ e.g. by 0). Therefore, up to subsequence, it converges weakly to some control $\bar{u}_1(\cdot) \in L^\infty(0, t_f)$ for the weak star topology. In particular, it converges weakly to $\bar{u}_1(\cdot) \in L^2(0, t_f)$ for the weak topology of $L^2(0, t_f)$. The limit control $\bar{u}_1(\cdot)$ satisfies $|\bar{u}_1(t)| \leq 1$ almost everywhere on $[0, t_f]$. To prove this fact, consider the set

$$\mathcal{V} = \{g \in L^2(0, t_f) \ \mid \ |g(t)| \leq 1 \text{ almost everywhere on } [0, t_f]\}.$$

For every integer n, $u_1^{\varepsilon_n}(\cdot) \in \mathcal{V}$; moreover $\mathcal{V}$ is a convex closed (for the strong topology) subset of $L^2(0, t_f)$, and hence is a convex closed (for the weak topology) subset of $L^2(0, t_f)$. It follows that $\bar{u}_1 \in \mathcal{V}$.

Since M_0 and M_1 are compact, it follows that, up to a subsequence, $x^{\varepsilon_n}(0)$ converges to some $\bar{x}_0 \in M_0$, and $x^{\varepsilon_n}(t_f^{\varepsilon_n})$ converges to some $\bar{x}_1 \in M_1$.

Let $\bar{x}(\cdot)$ denote the solution of the system (2.1), associated with the control $\bar{u}_1(\cdot)$ on $[0, T]$, and such that $\bar{x}(0) = \bar{x}_0$. Since the control systems under consideration are control-affine, it is not difficult to prove that the weak convergence of controls implies the uniform convergence of corresponding trajectories (see [119] for details). In particular, it follows that $\bar{x}(T) = \bar{x}_1$.

Therefore, we have proved that the control $\bar{u}$ on $[0, T]$ steers the system (2.1) from M_0 to M_1 in time T. Since $T \leq t_f$ and the problem **(OCP)** has a unique solution, we infer that $T = t_f$, $\bar{u}_1 = u_1$ and $\bar{x}(\cdot) = x(\cdot)$.

To conclude, it suffices to remark that the above reasoning proves that $(t_f, u_1(\cdot), x(\cdot))$ is the unique closure point of $(t_f^{\varepsilon_n}, u_1^{\varepsilon_n}(\cdot), x^{\varepsilon_n}(\cdot))$, where $(\varepsilon_n)_{n\in\mathbb{N}}$ is any sequence of positive real numbers converging to 0. ☐

Remark 2.5.8. If one does not assume the uniqueness of the optimal solution of **(OCP)**, then the following statement still holds. If $\varepsilon > 0$ tends to 0, then t_f^ε still converges to the minimal time t_f, the family $(u_1^\varepsilon(\cdot))_\varepsilon$ has a closure point $\bar{u}_1(\cdot)$ in $L^\infty(0, t_f)$ for the weak star topology, and the family $(x^\varepsilon(\cdot))_\varepsilon$ has a closure point $\bar{x}(\cdot)$ in $C^0([0, t_f], \mathbb{R}^n)$ for the uniform convergence topology, where $\bar{x}(\cdot)$ is the solution of the system (2.1) corresponding to the control $\bar{u}_1(\cdot)$ on $[0, t_f]$, such that $\bar{x}(0) \in M_0$ and $\bar{x}(t_f) \in M_1$. This means that $\bar{x}(\cdot)$ is another possible solution of (**OCP**).

In other words, every closure point of a family of solutions of $\mathbf{(OCP)}_\varepsilon$ is a solution of (**OCP**).

The next lemma will serve as a technical tool to derive Lemma 2.5.10.

Lemma 2.5.9. *Let $T > 0$, and let $(g_\varepsilon)_{\varepsilon>0}$ be a family of continuous functions on $[0, T]$ converging weakly to some $g \in L^2(0, T)$ as ε tends to 0, for the weak topology of $L^2(0, T)$. Then, for every $t \in (0, T)$, there exists a family $(t_\varepsilon)_{\varepsilon>0}$ of points of $[t, T)$ such that $t_\varepsilon \to t$ and $g_\varepsilon(t_\varepsilon) \to g(t)$ as $\varepsilon \to 0$.*

Proof. First of all, note that, since g_ε converges weakly to g on $[0, T]$, its restriction to any subinterval of $[0, T]$ converges weakly, as well, to the corresponding restriction of g. Let us prove that, for every $\beta > 0$, for every $\alpha > 0$ (small enough so that $t + \alpha \leq T$), there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, there exists $t_\varepsilon \in [t, t + \alpha]$ such that $|g_\varepsilon(t_\varepsilon) - g(t)| \leq \beta$. The proof goes by contradiction. Assume that there exist $\beta > 0$ and $\alpha > 0$ such that, for every integer n, there exists $\varepsilon_n \in (0, 1/n)$ such that, for every $\tau \in [t, t + \alpha]$, there holds $|g_{\varepsilon_n}(\tau) - g(t)| \geq \beta$. Since g_{ε_n} is continuous, it follows that either $g_{\varepsilon_n}(\tau) \geq g(t) + \beta$ for every $\tau \in [t, t+\alpha]$, or $g_{\varepsilon_n}(\tau) \leq g(t) - \beta$ for every $\tau \in [t, t+\alpha]$. This inequality contradicts the weak convergence of the restriction of g_{ε_n} to $[t, t + \alpha]$ towards the restriction of g to $[t, t + \alpha]$. ☐

In what follows, we denote by $K(t)$, $K_1(t)$, $\tilde{K}(t)$, the Pontryagin cones along the trajectory $x(\cdot)$ solution of **(OCP)**, defined as in the previous Section §1.5. Similarly, for every $\varepsilon > 0$, we denote by $K^\varepsilon(t)$, $K_1^\varepsilon(t)$, $\tilde{K}^\varepsilon(t)$ the Pontryagin cones along the trajectory $x^\varepsilon(\cdot)$, which is a solution of $\mathbf{(OCP)}_\varepsilon$.

Lemma 2.5.10. *For every $v \in K(t_f)$, for every $\varepsilon > 0$, there exists $v^\varepsilon \in K^\varepsilon(t_f^\varepsilon)$ such that v^ε converges to v as ε tends to 0.*

Proof. By construction of $K(t_f)$, it suffices to prove the lemma for a single needle-like variation. Assume that $v = v_\pi(t_f)$, where the variation vector $v_\pi(\cdot)$ is the solution on $[t_1, t_f]$ of the Cauchy problem

$$\begin{aligned} \dot{v}_\pi(t) &= \left(\frac{\partial X}{\partial x}(x(t)) + u_1(t)\frac{\partial Y_1}{\partial x}(x(t)) \right) v_\pi(t) \\ v_\pi(t_1) &= (\bar{u}_1 - u_1(t_1))Y_1(x(t_1)), \end{aligned} \tag{2.12}$$

where t_1 is a Lebesgue point of $[0, t_f)$, $\bar{u}_1 \in [-1, 1]$, and the needle-like variation $\pi = \{t_1, \eta_1, \bar{u}_1\}$ of the control u_1 is defined by

$$u_{1,\pi}(t) = \begin{cases} \bar{u}_1 & \text{if } t \in [t_1, t_1 + \eta_1], \\ u_1(t) & \text{otherwise.} \end{cases}$$

For every $\varepsilon > 0$, consider the control $u^\varepsilon = (u_1^\varepsilon, \dots, u_m^\varepsilon)$ of Lemma 2.5.6, solution of $\mathbf{(OCP)}_\varepsilon$. It satisfies the constraint $\sum_{i=1}^m (u_i^\varepsilon)^2 \leq 1$. From Lemma 2.5.7, the continuous control function u_1^ε converges weakly to u_1 in $L^2(0, t_f)$. It then follows from Lemma 2.5.9 that, for every $\varepsilon > 0$, there exists $t_\varepsilon \geq t_1$ such that $t_\varepsilon \to t_1$ and $u_1^\varepsilon(t_\varepsilon) \to u_1(t_1)$ as $\varepsilon \to 0$.

For every $\varepsilon > 0$, consider the needle-like variation $\pi^\varepsilon = \{t_1^\varepsilon, \eta_1, (\bar{u}_1, 0, \dots, 0)\}$ of the control $(u_1^\varepsilon, \dots, u_m^\varepsilon)$ defined, for $i = 2, \dots, m$, by[6]

$$u_{1,\pi^\varepsilon}^\varepsilon(t) = \begin{cases} \bar{u}_1 & \text{if } t \in [t_1^\varepsilon, t_1^\varepsilon + \eta_1], \\ u_1^\varepsilon(t) & \text{otherwise,} \end{cases}$$

and

$$u_{i,\pi^\varepsilon}^\varepsilon(t) = \begin{cases} 0 & \text{if } t \in [t_1^\varepsilon, t_1^\varepsilon + \eta_1], \\ u_i^\varepsilon(t) & \text{otherwise} \end{cases}$$

Let the variation vector $v_{\pi^\varepsilon}(\cdot)$ be the solution on $[t_1^\varepsilon, t_f^\varepsilon]$ of the Cauchy problem

$$\begin{aligned} \dot{v}_{\pi^\varepsilon}(t) &= \left(\frac{\partial X}{\partial x}(x^\varepsilon(t)) + u_1^\varepsilon(t)\frac{\partial Y_1}{\partial x}(x^\varepsilon(t)) + \varepsilon \sum_{i=2}^m u_i^\varepsilon(t)\frac{\partial Y_i}{\partial x}(x^\varepsilon(t)) \right) v_{\pi^\varepsilon}(t) \\ v_{\pi^\varepsilon}(t_1^\varepsilon) &= (\bar{u}_1 - u_1^\varepsilon(t_1^\varepsilon))Y_1(x^\varepsilon(t_1^\varepsilon)) - \varepsilon \sum_{i=2}^m u_i^\varepsilon(t_1^\varepsilon)Y_i(x^\varepsilon(t_1^\varepsilon)). \end{aligned} \tag{2.13}$$

From Lemma 2.5.7, t_f^ε converges to t_f, $u_1^\varepsilon(\cdot)$ converges weakly to $u_1(\cdot)$, $x^\varepsilon(\cdot)$ converges uniformly to $x(\cdot)$; moreover, $\varepsilon u_i^\varepsilon(\cdot)$ converges weakly to 0, $\varepsilon u_i^\varepsilon(t_1^\varepsilon)$ converges to 0, for $i = 2, \dots, m$, and $u_1^\varepsilon(t_1)$ converges to $u_1(t_1)$. As in the proof of Lemma 2.5.7, we infer the uniform convergence of $v_\pi^\varepsilon(\cdot)$ to $v_\pi(\cdot)$ (see [119] for details), and the conclusion follows. □

[6] Note that t_1^ε is a Lebesgue point of the function $t \mapsto X(x^\varepsilon(t)) + u_1^\varepsilon(t)Y_1(x^\varepsilon(t)) + \varepsilon \sum_{i=2}^m u_i^\varepsilon(t)Y_i(x^\varepsilon(t))$ since the controls u_i^ε are continuous functions of t.

The next lemma will be useful in the proof of Lemma 2.5.12.

Lemma 2.5.11. *Let m be a positive integer, g be a continuous function on $\mathbb{R}\times\mathbb{R}^m$, and C be a compact subset of $\mathbb{R}^m$. For every $\varepsilon>0$, set $M(\varepsilon)=\max_{u\in C} g(\varepsilon,u)$, and $M=\max_{u\in C} g(0,u)$. Then, $M(\varepsilon)$ tends to M as ε tends to 0.*

Proof. For every $\varepsilon>0$, let $u_\varepsilon\in C$ such that $M(\varepsilon)=g(\varepsilon,u_\varepsilon)$, and let $u\in C$ such that $M=g(0,u)$. Note that u_ε does not necessarily converge to u, however we will prove that $M(\varepsilon)$ tends to M, as ε tends to 0. Let $u_0\in C$ be a closure point of the family $(u_\varepsilon)_{\varepsilon>0}$. Then, by definition of M, one has $g(0,u_0)\leq M$. On the other hand, since g is continuous, $g(\varepsilon,u)$ tends to $g(0,u)=M$ as ε tends to 0. By definition, $g(\varepsilon,u)\leq M(\varepsilon)=g(\varepsilon,u_\varepsilon)$ for every $\varepsilon>0$. Therefore, passing to the limit, one gets $M\leq g(0,u_0)$. It follows that $M=g(0,u_0)$. We have thus proved that the (bounded) family $(M(\varepsilon))_{\varepsilon>0}$ of real numbers has a unique closure point, which is M. The conclusion follows. $\square$

Lemma 2.5.12. *There exists $\varepsilon_0>0$ such that, for every $\varepsilon\in(0,\varepsilon_0)$, every extremal lift $(x^\varepsilon(\cdot),p^\varepsilon(\cdot),p^{0\varepsilon},u^\varepsilon(\cdot))$ of any solution $x^\varepsilon(\cdot)$ of $\boldsymbol{(OCP)}_\varepsilon$ is normal.*

Proof. We argue by contradiction. Assume that, for every integer n, there exist $\varepsilon_n\in(0,1/n)$ and a solution $x^{\varepsilon_n}(\cdot)$ of $\mathbf{(OCP)}_{\varepsilon_n}$ having an abnormal extremal lift $(x^{\varepsilon_n}(\cdot),p^{\varepsilon_n}(\cdot),0,u^{\varepsilon_n}(\cdot))$. Set $\psi^{\varepsilon_n}=p^{\varepsilon_n}(t_f^{\varepsilon_n})$, for every integer n. Then, from Remark 1.5.12, one has

$$\langle\psi^{\varepsilon_n},v^{\varepsilon_n}\rangle\leq 0,$$

for every $v^{\varepsilon_n}\in K^{\varepsilon_n}(t_f^{\varepsilon_n})$, and

$$M(\varepsilon_n)=\max_{\sum_{i=1}^m w_i^2\leq 1}\Big(\Big\langle\psi^{\varepsilon_n},X(x^{\varepsilon_n}(t_f^{\varepsilon_n}))\Big\rangle+w_1\Big\langle\psi^{\varepsilon_n},Y_1(x^{\varepsilon_n}(t_f^{\varepsilon_n}))\Big\rangle$$
$$+\varepsilon_n\sum_{i=2}^m w_i\Big\langle\psi^{\varepsilon_n},Y_i(x^{\varepsilon_n}(t_f^{\varepsilon_n}))\Big\rangle\Big)=0,$$

for every integer n. Since the final adjoint vector $(p^{\varepsilon_n}(t_f^{\varepsilon_n}),p^{0\,\varepsilon_n})$ is defined up to a multiplicative scalar, and $p^{0\,\varepsilon_n}=0$, we assume that ψ^{ε_n} is a unit vector. Then, up to a subsequence, the sequence $(\psi^{\varepsilon_n})_{n\in\mathbb{N}}$ converges to some unit vector ψ. Using Lemmas 2.5.7, 2.5.10 and 2.5.11, we infer that

$$\langle\psi,v\rangle\leq 0,$$

for every $v\in K(t_f)$, and

$$M=\max_{|w_1|\leq 1}\left(\langle\psi,X(x(t_f))\rangle+w_1\langle\psi,Y_1(x(t_f))\rangle\right)=0.$$

It then follows from Remark 1.5.12 that the trajectory $x(\cdot)$ has an abnormal extremal lift. This is a contradiction since, by assumption, $x(\cdot)$ has a unique extremal lift, which is moreover normal. $\square$

Remark 2.5.13. If we remove the assumption that the optimal trajectory $x(\cdot)$ has a unique extremal lift, which is moreover normal, then Lemma 2.5.12 still holds provided that every extremal lift of $x(\cdot)$ is normal.

With the notations of Lemma 2.5.12, from now on we normalize the adjoint vector so that $p^{0\,\varepsilon} = -1$, for every $\varepsilon \in (0, \varepsilon_0)$.

Lemma 2.5.14. *In the setting of Lemma 2.5.12, the set of all possible $p^\varepsilon(t_f^\varepsilon)$, with $\varepsilon \in (0, \varepsilon_0)$, is bounded.*

Proof. The proof goes by contradiction. Assume that there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers converging to 0 such that $\|p^{\varepsilon_n}(t_f^{\varepsilon_n})\|$ tends to $+\infty$. Since the sequence $\left(\frac{p^{\varepsilon_n}(t_f^{\varepsilon_n})}{\|p^{\varepsilon_n}(t_f^{\varepsilon_n})\|}\right)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^n$, up to a subsequence it converges to some unit vector ψ. Using the Lagrange multipliers property and (1.59), there holds

$$\langle p^{\varepsilon_n}(t_f^{\varepsilon_n}), v^{\varepsilon_n}\rangle \leq 0,$$

for every $v^{\varepsilon_n} \in K^{\varepsilon_n}(t_f^{\varepsilon_n})$, and

$$\max_{\sum_{i=1}^m w_i^2 \leq 1} \Bigg(\left\langle p^{\varepsilon_n}(t_f^{\varepsilon_n}), X(x^{\varepsilon_n}(t_f^{\varepsilon_n}))\right\rangle + w_1 \left\langle p^{\varepsilon_n}(t_f^{\varepsilon_n}), Y_1(x^{\varepsilon_n}(t_f^{\varepsilon_n}))\right\rangle \\ + \varepsilon_n \sum_{i=2}^m w_i \left\langle p^{\varepsilon_n}(t_f^{\varepsilon_n}), Y_i(x^{\varepsilon_n}(t_f^{\varepsilon_n}))\right\rangle \Bigg) = 1,$$

for every integer n. Dividing by $\|p^{\varepsilon_n}(t_f^{\varepsilon_n})\|$, and passing to the limit, using Lemmas 2.5.7, 2.5.10 and 2.5.11, and Remark 1.5.12, the same reasoning as in the proof of the previous lemma yields that the trajectory $x(\cdot)$ has an abnormal extremal lift, which is a contradiction. $\square$

Remark 2.5.15. Remark 2.5.13 applies as well to Lemma 2.5.14.

Lemma 2.5.16. *For every $\varepsilon \in (0, \varepsilon_0)$, let $x^\varepsilon(\cdot)$ be a solution of* $\boldsymbol{(OCP)}_\varepsilon$*, and let $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), -1, u^\varepsilon(\cdot))$ be a (normal) extremal lift of $x^\varepsilon(\cdot)$. Then $p^\varepsilon(\cdot)$ converges uniformly*[7] *to $p(\cdot)$ on $[0, t_f]$ as ε tends to 0, where $(x(\cdot), p(\cdot), -1, u(\cdot))$ is the unique (normal) extremal lift of $x(\cdot)$.*

Proof. For every $\varepsilon > 0$, set $\psi^\varepsilon = p^\varepsilon(t_f^\varepsilon)$. The adjoint equation of the Pontryagin Maximum Principle is

$$\dot{p}^\varepsilon(t) = -\left\langle p^\varepsilon(t), \frac{\partial X}{\partial x}(x^\varepsilon(t))\right\rangle - u_1^\varepsilon(t)\left\langle p^\varepsilon(t), \frac{\partial Y_1}{\partial x}(x^\varepsilon(t))\right\rangle \\ - \varepsilon \sum_{i=2}^m u_i^\varepsilon(t) \left\langle p^\varepsilon(t), \frac{\partial Y_i}{\partial x}(x^\varepsilon(t))\right\rangle,$$

with $p^\varepsilon(t_f^\varepsilon) = \psi^\varepsilon$. Moreover, there holds

$$\langle \psi^\varepsilon, v^\varepsilon \rangle \leq 0,$$

[7] We consider any continuous extension of $p^\varepsilon(\cdot)$ on $[0, t_f]$.

for every $v^\varepsilon \in K^\varepsilon(t_f^\varepsilon)$, and

$$\max_{\sum_{i=1}^m w_i^2 \le 1} \Big(\langle \psi^\varepsilon, X(x^\varepsilon(t_f^\varepsilon)) \rangle + w_1 \langle \psi^\varepsilon, Y_1(x^\varepsilon(t_f^\varepsilon)) \rangle + \varepsilon \sum_{i=2}^m w_i \langle \psi^\varepsilon, Y_i(x^\varepsilon(t_f^\varepsilon)) \rangle \Big) = 1.$$

From Lemma 2.5.14, the family of all ψ^ε, $0 < \varepsilon < \varepsilon_0$, is bounded. Let ψ be a closure point of that family, and $(\varepsilon_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers converging to 0 such that ψ^{ε_n} tends to ψ. Using Lemma 2.5.7, and as in the proof of this lemma, we infer that the sequence $(p^{\varepsilon_n}(\cdot))_{n\in\mathbb{N}}$ converges uniformly to the solution $z(\cdot)$ of the Cauchy problem

$$\dot z(t) = -\left\langle z(t), \frac{\partial X}{\partial x}(x(t)) \right\rangle - u_1(t) \left\langle z(t), \frac{\partial Y_1}{\partial x}(x(t)) \right\rangle, \quad z(t_f) = \psi.$$

Moreover, passing to the limit as in the proof of Lemma 2.5.14

$$\langle \psi, v \rangle \le 0,$$

for every $v \in K(t_f)$, and

$$\max_{|w_1| \le 1} \left(\langle \psi, X(x(t_f)) \rangle + w_1 \langle \psi, Y_1(x(t_f)) \rangle \right) = 1.$$

It follows that $(x(\cdot), z(\cdot), -1, u_1(\cdot))$ is an extremal lift of $x(\cdot)$, and from the uniqueness assumption we infer that $z(\cdot) = p(\cdot)$. The conclusion follows. □

Remark 2.5.17. If one removes the assumptions of uniqueness of the solution of **(OCP)** and uniqueness of the extremal lift, then the following result still holds, provided that every extremal lift of every solution of **(OCP)** is normal. Consider the topological spaces $\mathcal{X} = C^0([0,t_f], \mathbb{R}^n)$, endowed with the uniform convergence topology, and $\mathcal{Y} = L^\infty(0, t_f; [-1,1])$, endowed with the weak star topology. In the following statement, the space $\mathcal{X} \times \mathcal{X} \times \mathcal{Y}$ is endowed with the resulting product topology. For every $\varepsilon \in (0, \varepsilon_0)$, let $x^\varepsilon(\cdot)$ be a solution of $\mathbf{(OCP)}_\varepsilon$, and let $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), -1, u^\varepsilon(\cdot))$ be a (normal) extremal lift of $x^\varepsilon(\cdot)$. Then, every closure point of the family $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), u_1^\varepsilon(\cdot))$ in $\mathcal{X} \times \mathcal{X} \times \mathcal{Y}$ is a triple $(\bar x(\cdot), \bar p(\cdot), \bar u_1(\cdot))$, where $\bar x(\cdot)$ is an optimal solution of **(OCP)**, associated with the control $\bar u_1(\cdot)$, having as a normal extremal lift the 4-tuple $(\bar x(\cdot), \bar p(\cdot), -1, \bar u_1(\cdot))$. This statement indeed follows from Remarks 2.5.8, 2.5.13 and 2.5.15.

Lemma 2.5.18. *If the control u_1 is moreover bang-bang, i.e., if the (continuous) switching function $\varphi(t) = \langle p(t), Y_1(x(t)) \rangle$ does not vanish on any subinterval of $[0, t_f]$, then $u_1^\varepsilon(\cdot)$ converges to $u_1(\cdot)$ and $u_i^\varepsilon(\cdot)$, $i = 2, \dots, m$, converge to 0 almost everywhere on $[0, t_f]$, and thus in particular for the strong $L^1(0, t_f)$ topology.*

Proof. Using the expression (2.11) of the controls u_1^ε and u_i^ε, $i = 2, \dots, m$, the expression (2.5) of the control u_1, and from Lemmas 2.5.7 and 2.5.16, it is clear that $u_1^\varepsilon(t)$ converges to $u_1(t)$ and $u_i^\varepsilon(t)$, $i = 2, \dots, m$, converge to 0 as ε tends to 0, for almost every $t \in [0, t_f]$. Since the controls are bounded by 1, the strong L^1 convergence follows from the dominated convergence theorem (see e.g. [20]). □

This last lemma ends the proof of Theorem 2.5.1.

Remark 2.5.19. Assumption 2.4.1 requires that $m \geq n$. One may however wish to choose $m = 2$, i.e., to add only one new vector field Y_2, in the regularization procedure. In that case, the Assumption 2.4.1 does not hold whenever $n > 3$, and then two problems may occur: first, in the maximization condition (2.10) the maximum is not necessarily obtained at the boundary, i.e., the expressions (2.11) do not necessarily hold, and second, the controls $u_i^\varepsilon(\cdot)$, $i = 1, \dots, m$ are not necessarily continuous (the continuity is used in a crucial way in the proof of our main result). These two problems are however not likely to occur, in what follows we provide some comments on the generic validity of (2.11) and on the smoothness of the regularized controls, in the case $m = 2$.

Let $m = 2$, that is, consider only one arbitrary additional smooth vector field Y_2. For $\varepsilon > 0$ fixed, the maximization condition from the Pontryagin maximum principle applied to the problem $\mathbf{(OCP)}_\varepsilon$ is

$$\begin{aligned}
&u_1^\varepsilon(t)\langle p^\varepsilon(t), Y_1(x^\varepsilon(t))\rangle + \varepsilon u_2^\varepsilon(t)\langle p^\varepsilon(t), Y_2(x^\varepsilon(t))\rangle \\
&= \max_{w_1^2+w_2^2\leq 1} \left(w_1\langle p^\varepsilon(t), Y_1(x^\varepsilon(t))\rangle + \varepsilon w_2\langle p^\varepsilon(t), Y_2(x^\varepsilon(t))\rangle\right)
\end{aligned}$$

almost everywhere on $[0, t_f^\varepsilon]$. There are two cases: either the maximum is attained in the interior of the domain, or it is attained at the boundary. The proof of our main result requires this maximum to be attained at the boundary (see (2.11)), and the corresponding controls to be continuous. This fact depends on the choice of the vector field Y_2.

A simple example where this holds true is the case $Y_2 = X$. In that case it is indeed possible to ensure that both functions $t \mapsto \langle p^\varepsilon(t), Y_1(x^\varepsilon(t))\rangle$ and $t \mapsto \langle p^\varepsilon(t), Y_2(x^\varepsilon(t))\rangle$ do not vanish simultaneously for $\varepsilon > 0$ small enough (and this implies the desired conclusion). To prove this assertion, we argue by contradiction and assume that, for every $n \in \mathbb{N}$, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0 and a sequence $(t^{\varepsilon_n})_{n\in\mathbb{N}}$ such that $\langle p^{\varepsilon_n}(t^{\varepsilon_n}), X(x^{\varepsilon_n}(t^{\varepsilon_n}))\rangle = \langle p^{\varepsilon_n}(t^{\varepsilon_n}), Y_2(x^{\varepsilon_n}(t^{\varepsilon_n}))\rangle = 0$. Combined with the fact that the Hamiltonian is constant along any extremal, and vanishes at the final time, these equalities imply that $p^{0\varepsilon_n} = 0$. This contradicts the conclusion of Lemma 2.5.12.

More generally, and although such a statement may be nontrivial to derive, we conjecture that this fact holds true for generic vector fields Y_2 (see [27–29] for such genericity statements). Note that, for generic triples of vector fields (X, Y_1, Y_2), this fact holds true. Indeed, to derive this statement it suffices to combine the fact that any totally singular minimizing trajectory must satisfy the Goh condition (see [2] and [15, Theorem 1.9] for details) and the fact that, for generic (in the strong sense of Whitney) triplets of vector fields (X, Y_1, Y_2), the associated control-affine system does not admit nontrivial Goh singular trajectories (see [29, Corollary 2.7]).

2.6 Examples

2.6.1 The Rayleigh minimal time control problem

To illustrate our results, we consider the minimal time control problem for the Rayleigh control system described in [76],

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -x_1(t) + x_2(t)(1.4 - 0.14x_2(t)^2) + u_1(t), \end{aligned} \tag{2.14}$$

with initial and final conditions

$$x_1(0) = x_2(0) = -5,\ x_1(t_f) = x_2(t_f) = 0, \tag{2.15}$$

and the control constraint

$$|u_1(t)| \leq 4\,, \quad \forall t \in [0, t_f]. \tag{2.16}$$

According to the Pontryagin maximum principle, any optimal solution $x(\cdot)$ of (2.14)–(2.16) is the projection of an extremal $(x(\cdot), p(\cdot), p^0, u_1(\cdot))$ such that

$$\begin{aligned} \dot{p}_1(t) &= p_2(t) \\ \dot{p}_2(t) &= -p_1(t) - p_2(t)\left(1.4 - 0.42x_2(t)^2\right) \end{aligned}$$

and the maximization condition $p_2(t)u_1(t) = \max_{|w|\leq 4}(p_2(t)w)$ holds almost everywhere on $[0, t_f]$. It is easy to see that $p_2(\cdot)$ cannot vanish on some subinterval, and it follows that the optimal control $u_1(\cdot)$ is bang-bang, equal to $u_1(t) = 4\,\mathrm{sign}(p_2(t))$.

Applying a shooting method to problem (2.14)–(2.16) (with $p^0 = -1$), we determine the initial adjoint vector $p(0) \simeq (0.12234128; 0.08265161)$, and observe that the trajectory has two switching times $\tau_1 \simeq 1.12050659$ and $\tau_2 \simeq 3.31004697$ on $[0, t_f]$, that is, $u_1(\cdot)$ is given by

$$u_1(t) = \begin{cases} +4 & \text{for } 0 \leq t \leq \tau_1 \\ -4 & \text{for } \tau_1 \leq t \leq \tau_2 \\ +4 & \text{for } \tau_2 \leq t \leq t_f\,, \end{cases}$$

with a final time $t_f \simeq 3.66817338$ (see Figures 2.1–2.4). Furthermore, $x(\cdot)$ is the unique minimal time solution and has a unique extremal lift (up to a multiplicative scalar), which is moreover normal (see [76]).

We propose the regularized control system

$$\begin{aligned} \dot{x}_1^\varepsilon(t) &= x_2^\varepsilon(t) + \varepsilon u_2^\varepsilon(t)\,, \\ \dot{x}_2^\varepsilon(t) &= -x_1^\varepsilon(t) + x_2^\varepsilon(t)(1.4 - 0.14x_2^\varepsilon(t)^2) + u_1^\varepsilon(t), \end{aligned} \tag{2.17}$$

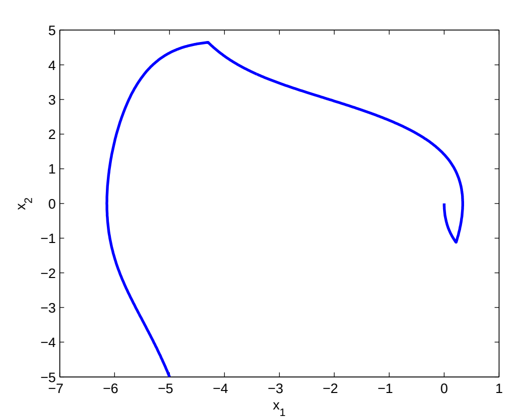

Figure 2.1: Optimal trajectory

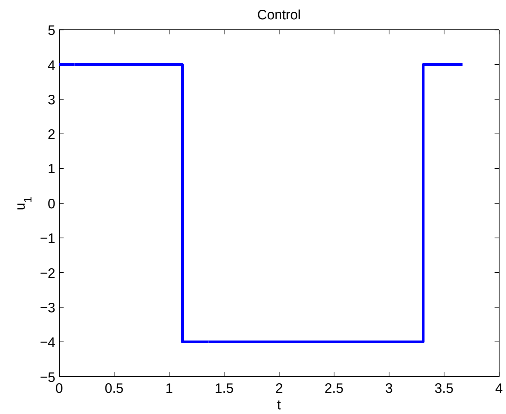

Figure 2.2: Optimal control

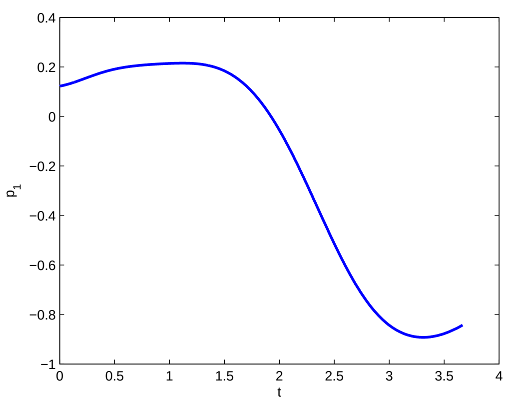

Figure 2.3: Adjoint vector p_1

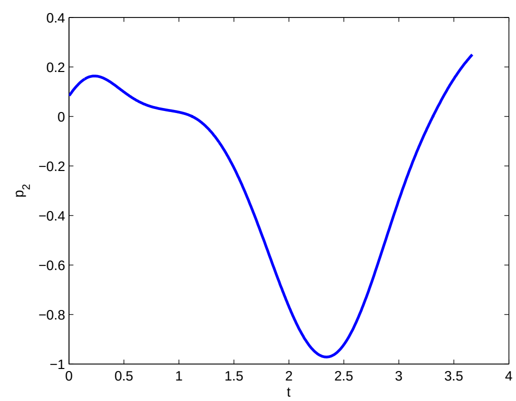

Figure 2.4: Adjoint vector p_2

with the same initial and final conditions, and where the control $u^\varepsilon(\cdot) = (u_1^\varepsilon(\cdot), u_2^\varepsilon(\cdot))$ satisfies the constraint

$$(u_1^\varepsilon(t))^2 + (u_2^\varepsilon(t))^2 \leq 16. \tag{2.18}$$

Any optimal solution $x^\varepsilon(\cdot)$ of (2.15), (2.17), (2.18) is the projection of an extremal $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), p^{0\varepsilon}, u^\varepsilon(\cdot))$ such that

$$\begin{aligned} \dot{p}_1^\varepsilon(t) &= p_2^\varepsilon(t) \\ \dot{p}_2^\varepsilon(t) &= -p_1^\varepsilon(t) - p_2^\varepsilon(t)\left(1.4 - 0.42 x_2^\varepsilon(t)^2\right) . \end{aligned}$$

The Assumption 2.4.1 is verified, and the controls that satisfy the Pontryagin maximization condition (2.10) are given by

$$u_1^\varepsilon(t) = \frac{4 p_2^\varepsilon(t)}{\sqrt{(p_2^\varepsilon(t))^2 + \varepsilon^2 (p_1^\varepsilon(t))^2}}, \quad u_2^\varepsilon(t) = \frac{4 \varepsilon p_1^\varepsilon(t)}{\sqrt{(p_2^\varepsilon(t))^2 + \varepsilon^2 (p_1^\varepsilon(t))^2}}. \tag{2.19}$$

All assumptions of Theorem 2.5.1 are satisfied.

Applying a shooting method to the problem (2.15), (2.17), (2.18), we determine the optimal trajectory of the regularized problem, and we indeed observe the expected convergence of $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), -1, u^\varepsilon)$ towards $(x(\cdot), p(\cdot), -1, u_1)$, as ε tends to 0, in agreement with Theorem 2.5.1 (see Figures 2.5, 2.6 and 2.7). In this example, the minimal time control solution of (2.14)–(2.16) is bang-bang, and we indeed observe, on the numerical simulations, the almost everywhere convergence of the regularized control.

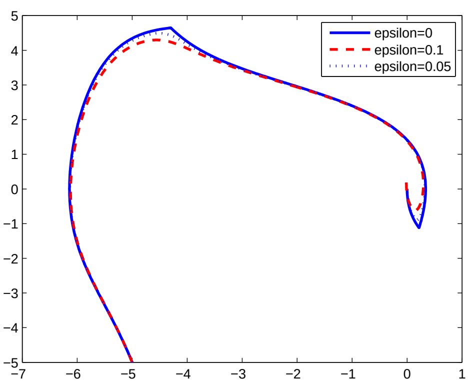

Figure 2.5: Trajectory

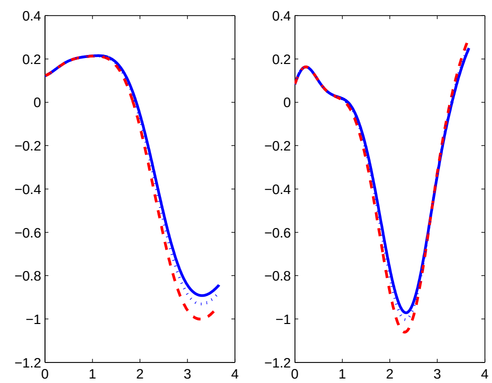

Figure 2.6: Adjoint vector

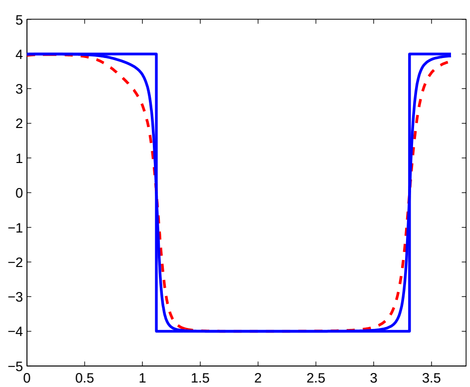

Figure 2.7: Control

2.6.2 Minimal time optimal control problem involving a singular arc

In the example provided in this subsection, the minimal time control $u_1(\cdot)$ is singular. It is then not expected a priori that the regularized control $u_1^\varepsilon(\cdot)$ converges almost everywhere to $u_1(\cdot)$ along the singular arc. Our main result only asserts a weak convergence property along this arc. In the example presented below, the regularized control $u_1^\varepsilon(\cdot)$ converges weakly to $u_1(\cdot)$ but not almost everywhere. We then provide some numerical simulations, on which we indeed observe that the almost everywhere convergence property fails along the singular arc, and we observe an oscillating property, which is a typical feature of weak convergence.

Consider the minimal time control problem for the system

$$\begin{aligned} \dot{x}_1(t) &= 1 - x_2(t)^2, \\ \dot{x}_2(t) &= u_1(t), \end{aligned} \tag{2.20}$$

with initial and final conditions

$$x_1(0) = x_2(0) = 0, \quad x_1(t_f) = 1,\ x_2(t_f) = 0, \tag{2.21}$$

and the control constraint

$$|u_1(t)| \leq 1\,, \quad \forall t \in [0, t_f]. \tag{2.22}$$

It is clear that the solution of this optimal control problem is unique, and is provided by the singular control $u_1(t) = 0$, for every $t \in [0, t_f]$, with $t_f = 1$. The corresponding trajectory is given by $x_1(t) = t$ and $x_2(t) = 0$.

We claim that this optimal trajectory has a unique extremal lift (up to a multiplicative scalar), which is moreover normal. Indeed, denoting by $p = (p_1, p_2)$ the adjoint vector, the Hamiltonian of the above optimal control problem is $H = p_1(1 - x_2^2) + p_2 u_1 + p^0$, and the differential equations of the adjoint vector are $\dot{p}_1 = 0$, $\dot{p}_2 = 2x_2 p_1$. Since $x_2(t) = 0$, it follows that the adjoint vector of any extremal lift of the optimal trajectory is constant. Moreover, the Hamiltonian vanishes at the final time, and hence there must hold $p_1(t) + p^0 = 0$, for every $t \in [0, t_f]$. Since the singular control $u_1(t) = 0$ is optimal and belongs to the interior of the domain of constraint (2.22), the maximization condition yields $\frac{\partial H}{\partial u_1} = 0$, and thus, $p_2(t) = 0$ for every $t \in [0, t_f]$. Then, since the adjoint vector is nontrivial, p^0 cannot be equal to 0, and up to a multiplicative scalar we assume that $p^0 = -1$. The assertion is thus proved, and the unique (normal) extremal lift is given by $(x_1(t), x_2(t), p_1(t), p_2(t), p^0, u_1(t)) = (t, 0, 1, 0, -1, 0)$.

We propose the following regularization of the problem (2.20)–(2.22). Let $g(\cdot)$ and $h(\cdot)$ be smooth functions, to be chosen; consider the minimal time control problem for the system

$$\begin{aligned} \dot{x}_1^\varepsilon(t) &= 1 - x_2^\varepsilon(t)^2 + \varepsilon u_2^\varepsilon(t) g(x_1^\varepsilon(t)), \\ \dot{x}_2^\varepsilon(t) &= u_1^\varepsilon(t) + \varepsilon u_2^\varepsilon(t) h(x_1^\varepsilon(t)), \end{aligned} \tag{2.23}$$

with initial and final conditions

$$x_1^\varepsilon(0) = x_2^\varepsilon(0) = 0, \quad x_1^\varepsilon(t_f^\varepsilon) = 1,\ x_2^\varepsilon(t_f^\varepsilon) = 0, \tag{2.24}$$

and the control constraint

$$(u_1^\varepsilon(t))^2 + (u_2^\varepsilon(t))^2 \leq 1\,, \quad \forall t \in [0, t_f^\varepsilon]. \tag{2.25}$$

Since the function g to be chosen below vanishes at some points, the Assumption 2.4.1 does not hold everywhere. We claim however that, if the function g may only vanish on a subset of zero measure, and if $\varepsilon > 0$ is small enough, then the formula (2.11) holds, and the regularized controls are continuous, so that we are in the framework of Theorem 2.5.1.

Indeed, the Hamiltonian of this regularized optimal control problem is

$$H = p_1^\varepsilon(1 - (x_2^\varepsilon)^2) + p_2^\varepsilon u_1^\varepsilon + \varepsilon u_2^\varepsilon(p_1^\varepsilon g(x_1^\varepsilon) + p_2^\varepsilon h(x_1^\varepsilon)) + p^{0\varepsilon},$$

and the adjoint equations are

$$\begin{aligned} \dot{p}_1^\varepsilon(t) &= -\varepsilon u_2^\varepsilon(t)(p_1^\varepsilon(t) g'(x_1^\varepsilon(t)) + p_2^\varepsilon(t) h'(x_1^\varepsilon(t))), \\ \dot{p}_2^\varepsilon(t) &= 2x_2^\varepsilon(t)p_1^\varepsilon(t). \end{aligned}$$

It is not difficult to see that, for $\varepsilon > 0$ small enough, the optimal trajectory must be such that $\dot{x}_1^\varepsilon(t) > 0$; hence, $x_1^\varepsilon(\cdot)$ is an increasing function of t. Now, argue by contradiction, and assume that the optimal control takes its values in the interior of the domain (2.25), for $t \in I$, where I is a subset of $[0, t_f^\varepsilon]$ of positive measure. Then, the maximization condition yields $\frac{\partial H}{\partial u_1^\varepsilon} = \frac{\partial H}{\partial u_2^\varepsilon} = 0$, and hence $p_2^\varepsilon(t) = 0$ and $p_1^\varepsilon(t)g(x_1^\varepsilon(t)) + p_2^\varepsilon(t)h(x_1^\varepsilon(t)) = 0$, for $t \in I$. It follows that $p_1^\varepsilon(t)g(x_1^\varepsilon(t)) = 0$, for $t \in I$. Since the function g may only vanish on a subset of zero measure, and since $x_1^\varepsilon(\cdot)$ is increasing, it follows that there exists $t_1 \in I$ such that $g(x_1^\varepsilon(t_1)) \neq 0$, and therefore $p_1^\varepsilon(t_1) = p_2^\varepsilon(t_1) = 0$. Since the Hamiltonian vanishes almost everywhere, this yields moreover $p^{0\varepsilon} = 0$, which is a contradiction.

Therefore, under the above assumption on g, the formula (2.11) holds, and the optimal controls are given by

$$\begin{aligned} u_1^\varepsilon(t) &= \frac{p_2^\varepsilon(t)}{\sqrt{p_2^\varepsilon(t)^2 + \varepsilon^2 \left(p_1^\varepsilon(t)g(x_1^\varepsilon(t)) + p_2^\varepsilon(t)h(x_1^\varepsilon(t))\right)^2}}, \\ u_2^\varepsilon(t) &= \frac{\varepsilon \left(p_1^\varepsilon(t)g(x_1^\varepsilon(t)) + p_2^\varepsilon(t)h(x_1^\varepsilon(t))\right)}{\sqrt{p_2^\varepsilon(t)^2 + \varepsilon^2 \left(p_1^\varepsilon(t)g(x_1^\varepsilon(t)) + p_2^\varepsilon(t)h(x_1^\varepsilon(t))\right)^2}}, \end{aligned} \tag{2.26}$$

for almost every $t \in [0, t_f^\varepsilon]$.

Let us prove that the controls $u_1^\varepsilon(\cdot)$ and $u_2^\varepsilon(\cdot)$ are smooth functions of t. For this purpose, we prove hereafter that the function $p_2^\varepsilon(\cdot)$ does not vanish on any subset of positive measure. Argue by contradiction and assume that there exists a subset I of $[0, t_f^\varepsilon]$ on which $p_2^\varepsilon(\cdot)$ vanishes. Then, on one part, (2.26) implies that $u_1^\varepsilon(t) = 0$ and $u_2^\varepsilon(t) = \text{sign}(p_1^\varepsilon(t)g(x_1^\varepsilon(t)) + p_2^\varepsilon(t)h(x_1^\varepsilon(t)))$, for almost every $t \in I$. On the other part, using the adjoint equations, we have $x_2^\varepsilon(t)p_1^\varepsilon(t) = 0$, for $t \in I$. The scalar $p_1^\varepsilon(t)$ cannot vanish, for any $t \in I$; indeed otherwise there would hold $p_1^\varepsilon(t) = p_2^\varepsilon(t) = 0$, and since the Hamiltonian vanishes, it would follow that $p^{0\varepsilon} = 0$, which is a contradiction with the normality of the extremal lift (see Lemma 2.5.12). Hence, $x_2^\varepsilon(t) = 0$ for $t \in I$, and thus, by differentiation, $u_1^\varepsilon(t) + \varepsilon u_2^\varepsilon(t) = 0$. This contradicts the equalities $u_1^\varepsilon(t) = 0$ and $u_2^\varepsilon(t) = \text{sign}(p_1^\varepsilon(t)g(x_1^\varepsilon(t)) + p_2^\varepsilon(t)h(x_1^\varepsilon(t)))$.

From Theorem 2.5.1, we can assert that, as ε tends to 0,

- $x_1^\varepsilon(\cdot)$ (resp., $x_2^\varepsilon(\cdot)$) converges uniformly to $x_1(\cdot)$ (resp., $x_2(\cdot)$) on $[0,1]$,
- $p_1^\varepsilon(\cdot)$ (resp., $p_2^\varepsilon(\cdot)$) converges uniformly to $p_1(\cdot) \equiv 1$ (resp., $p_2(\cdot) \equiv 0$),
- $u_1^\varepsilon(\cdot)$ converges weakly to $u_1(\cdot) \equiv 0$.

Let us next prove that, for certain choices of the functions $g(\cdot)$ and $h(\cdot)$, the regularized control $u_1^\varepsilon(\cdot)$ does not converge almost everywhere to $u_1(\cdot)$. We choose a smooth function $g(\cdot)$ defined on $\mathbb{R}$ that is strongly oscillating in the neighborhood of $1/2$, for instance,

$$g(x) = h(x) \sin \frac{1}{x - 1/2},$$

and a flat function h so that g is indeed smooth, for instance,

$$h(x) = \exp\left(\frac{-1}{(x-1/2)^2}\right).$$

If ε is small enough, then $x_1^\varepsilon(t)$ is close to t, $p_1^\varepsilon(t)$ is close to 1, $p_2^\varepsilon(t)$ is close to 0, and hence the sign of $u_2^\varepsilon(t)$, that is equal to the sign of

$$h(x_1^\varepsilon(t))\left(p_1^\varepsilon(t) \sin \frac{1}{x_1^\varepsilon(t) - 1/2} + p_2^\varepsilon(t)\right)$$

is close to the sign of $\sin \frac{1}{t-1/2}$. Therefore, the control $u_2^\varepsilon(\cdot)$ strongly oscillates between -1 and 1 for t close to $1/2$. Since $u_1^\varepsilon(\cdot)$ and $u_2^\varepsilon(\cdot)$ are continuous and satisfy $(u_1^\varepsilon(t))^2 + (u_2^\varepsilon(t))^2 = 1$, for every $t \in [0,1]$, it follows that the control $u_1^\varepsilon(\cdot)$ strongly oscillates as well between -1 and 1 for t close to $1/2$.

This oscillation feature is similar to what happens with chattering controls, and illustrates the fact that $u_1^\varepsilon(\cdot)$ weakly converges to $u_1(\cdot) = 0$ as ε tends to 0, but does not converge almost everywhere.

Numerical simulations lead to Figures 2.8 and 2.9, on which we can observe the oscillating properties of the regularized controls. Note that these numerical simulations are difficult to obtain with the above function h, because of its flatness. First of all, in our numerical simulations we rather choose the function $h(x) = (x - 1/2)^3$, that is not so flat, but for which the system is however not smooth (but this does not change anything to the result). Second, it is difficult to make converge the shooting method for small values of ε, and we had to make use of a continuation method, starting with a large value of ε and decreasing that value step by step.

2.6.3 The harmonic oscillator problem (linear case)

This example was considered in Section §1.4.7. Here we propose to solve the harmonic oscillator problem (in the linear case) using a single shooting method. We illustrate the convergence result of Theorem 2.5.1 for this minimal time problem.

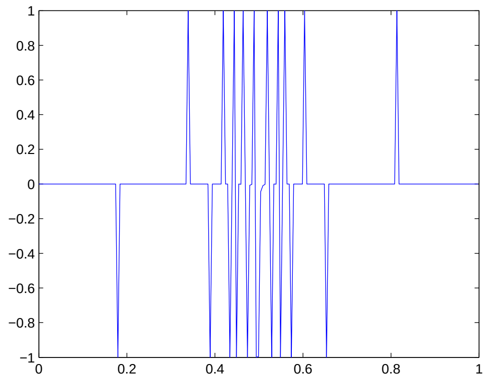

Figure 2.8: Control u_1^ϵ ($\varepsilon = 0.01$)

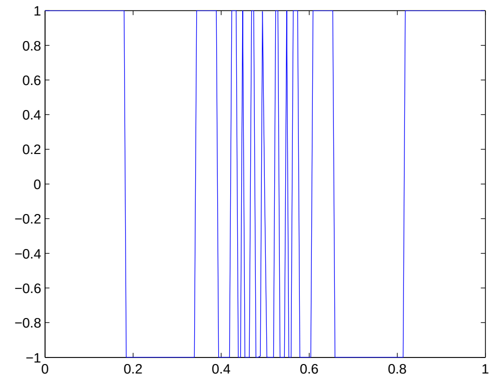

Figure 2.9: Control u_2^ϵ ($\varepsilon = 0.01$)

Consider the minimal time control problem for the system

$$\begin{cases} \dot{x}(t) = y(t)\,, \\ \dot{y}(t) = -x(t) + u_1(t)\,, \end{cases} \tag{2.27}$$

with initial and final conditions

$$\begin{aligned} &x(0) = 3\,, \quad y(0) = 1\,, \\ &x(t_f) = 0\,, \quad y(t_f) = 0\,, \end{aligned} \tag{2.28}$$

and the control constraint

$$|u_1(t)| \leq 1\,, \quad \forall\, t \in [0, t_f]\,. \tag{2.29}$$

We propose the regularized control system

$$\begin{cases} \dot{x}^\varepsilon(t) = y^\varepsilon(t) + \varepsilon u_2^\varepsilon(t)\,, \\ \dot{y}^\varepsilon(t) = -x^\varepsilon(t) + u_1^\varepsilon(t)\,, \end{cases}$$

with the same initial conditions, and where the control $u^\varepsilon(\cdot) = (u_1^\varepsilon(\cdot), u_2^\varepsilon(\cdot))$ satisfies the constraint

$$(u_1^\varepsilon(t))^2 + (u_2^\varepsilon(t))^2 \leq 1\,, \quad \forall t \in [0, t_f^\varepsilon]\,.$$

All assumptions of Theorem 2.5.1 are satisfied (the minimal time problem (2.27)–(2.29) has a unique solution $(x(\cdot), y(\cdot))$, defined on $[0, t_f]$, associated with a control $u_1(\cdot)$ on $[0, t_f]$. And $(x(\cdot), y(\cdot))$ has a unique extremal lift (up to a multiplicative scalar), that is moreover normal). In Figures 2.10 and 2.11 we can observe the optimal trajectory and optimal bang-bang control with minimal time is $t_f \simeq 5.202346$. The adjoint vector (p_x, p_y) associated to the optimal trajectory is represented in Figures 2.12 and 2.13.

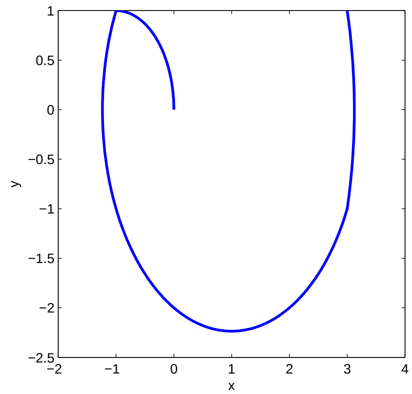

Figure 2.10: Optimal trajectory for (2.27)–(2.29)

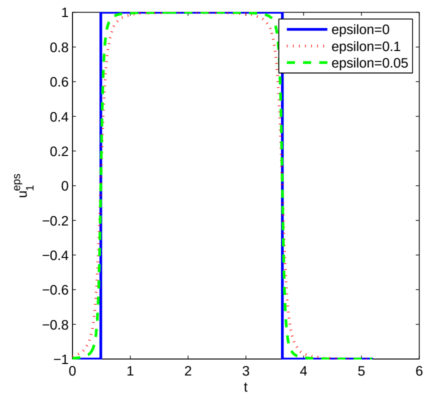

Figure 2.11: Optimal control for (2.27)–(2.29)

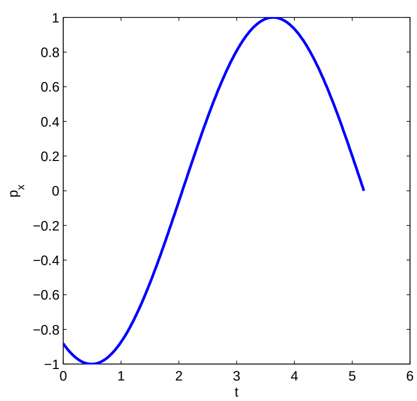

Figure 2.12: p_x for (2.27)–(2.29)

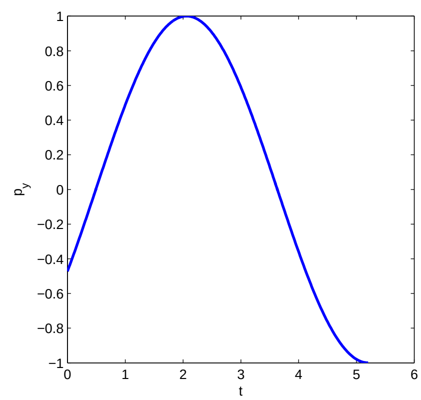

Figure 2.13: p_y for (2.27)–(2.29)

Applying a shooting method to the regularized problem we observe, like in the first example of this chapter, the convergence of the trajectories, the adjoint vectors and the optimal controls towards the optimal trajectory, adjoint vector and optimal control of the minimal time problem problem (2.27)–(2.29), respectively, as ε tends to 0 (see Figures 2.14–2.17 for $\varepsilon = 0.2$ and $\varepsilon = 0.5$ and Figures 2.18–2.21 for $\varepsilon = 0.1$ and $\varepsilon = 0.05$). We report on Table 2.1 the values of the final time t_f^ε of the optimal trajectory $\hat{x}^\varepsilon(\cdot)$, for different values of ε. We observe that, as expected, t_f^ε converges to $t_f \simeq 5.202346$ as ε tends to 0.

ε	t_f^ε
0.1	5.140856...
0.05	5.183549...
0.001	5.202331...

Table 2.1: Values of t_f^ε

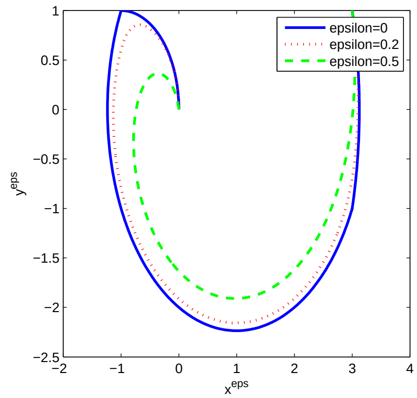

Figure 2.14: $(x^\varepsilon, y^\varepsilon)$

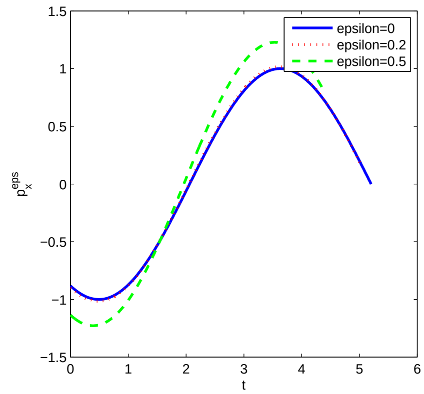

Figure 2.15: p_x^ε

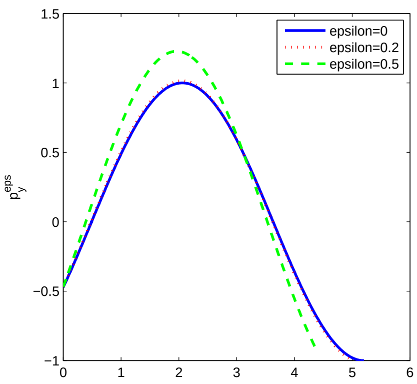

Figure 2.16: p_y^ε

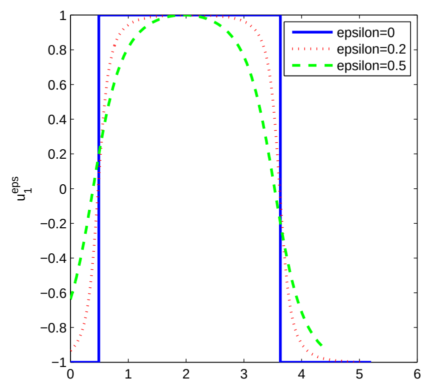

Figure 2.17: u_1^ε

2.6.4 Minimal time control of a Van der Pol oscillator

This optimal control problem (see e.g. [81]) consist in minimizing the final time t_f subject to the control system

$$\begin{cases} \dot{x}_1(t) = x_2(t)\,, \\ \dot{x}_2(t) = -x_1(t) + x_2(t)\left(1 - x_1^2(t)\right) + u_1(t) \end{cases}$$

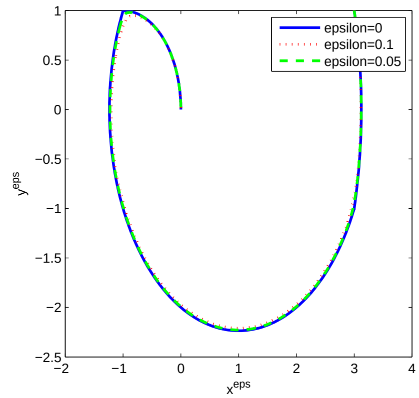

Figure 2.18: $(x^\varepsilon, y^\varepsilon)$

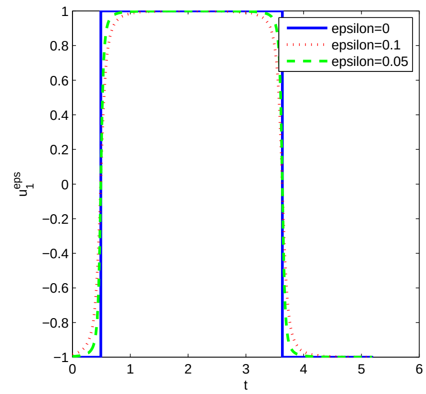

Figure 2.19: u_1^ε

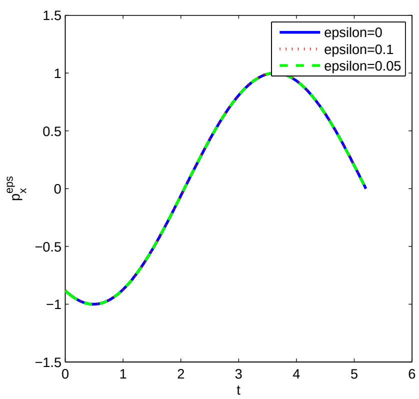

Figure 2.20: p_x^ε

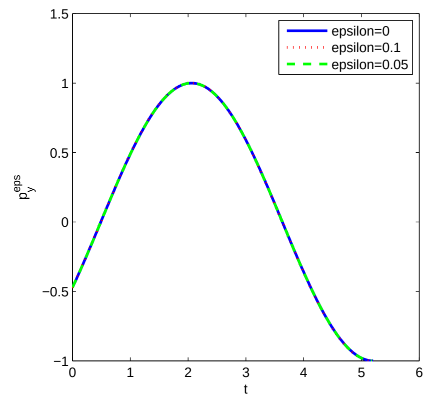

Figure 2.21: p_y^ε

with the initial and final conditions

$$x_1(0) = -0.4\,, \quad x_2(0) = 0.6\,,$$
$$x_1(t_f) = 0.6\,, \quad x_2(t_f) = 0.4\,,$$

and the control constraint

$$|u_1(t)| \leq 1\,, \quad \forall\, t \in [0, t_f]\,.$$

We propose the regularized control system

$$\begin{cases} \dot{x}_1^\varepsilon(t) = x_2^\varepsilon(t) + \varepsilon u_2^\varepsilon(t)\,, \\ \dot{x}_2^\varepsilon(t) = -x_1^\varepsilon(t) + x_2^\varepsilon(t)\left(1 - (x_1^\varepsilon(t))^2\right) + u_1^\varepsilon(t) \end{cases}$$

with the same initial conditions, and where the control $u^\varepsilon(\cdot) = (u_1^\varepsilon(\cdot), u_2^\varepsilon(\cdot))$ satisfies the constraint

$$(u_1^\varepsilon(t))^2 + (u_2^\varepsilon(t))^2 \leq 1\,, \quad \forall\, t \in [0, t_f^\varepsilon]\,.$$

Analogously to examples in Sections §2.6.1 and §2.6.3 the assumptions and the convergence results of Theorem 2.5.1 are verified (see Figures 2.22–2.25).

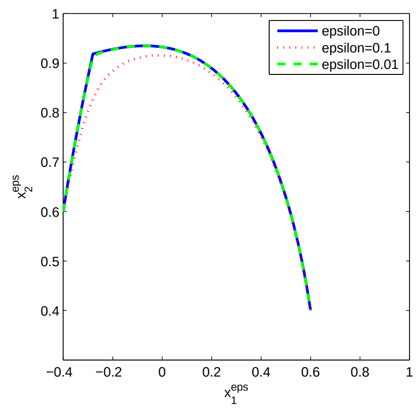

Figure 2.22: Trajectory $(x_1^\varepsilon(\cdot), x_2^\varepsilon(\cdot))$

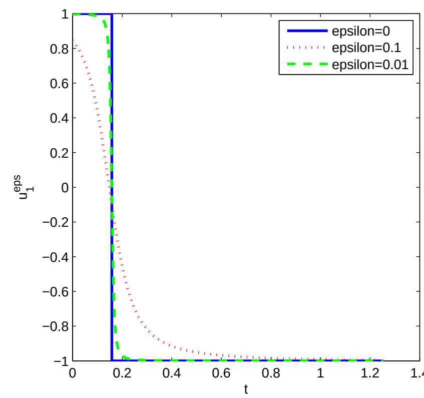

Figure 2.23: Control $u_1^\varepsilon(\cdot)$

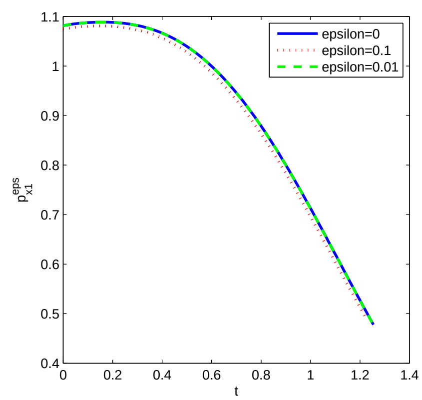

Figure 2.24: Adjoint vector $p_{x_1}^\varepsilon$

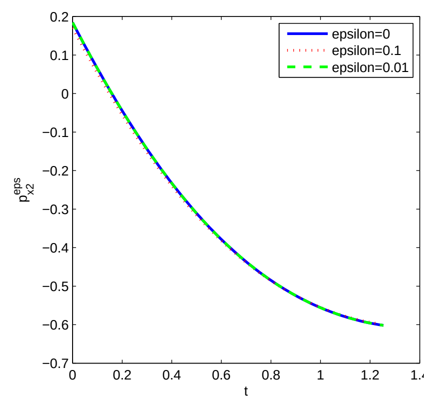

Figure 2.25: Adjoint vector $p_{x_2}^\varepsilon$

Chapter 3

Asymptotic approach on conjugate points for bang-bang control problems

3.1 Introduction

In this chapter we focus on the problem of determining an efficient procedure to compute the first conjugate time t_c for the minimal time control problem considered in Chapter 2, for single-input control-affine systems $\dot{x} = X(x) + u_1 Y_1(x)$ in $\mathbb{R}^n$ with fixed initial and final time conditions $x(0) = \hat{x}_0$, $x(t_f) = \hat{x}_1$, and where the scalar control u_1 satisfies the constraint $|u_1(t)| \leq 1$, for every $t \in [0, t_f]$. For these systems a theoretical concept of conjugate time t_c has been defined in e.g. [5, 81, 87, 95] in the bang-bang case, however direct algorithms of computation are difficult to apply. Besides, theoretical and practical issues for conjugate time theory are well known in the smooth case (see e.g. [2, 86]), and efficient implementation tools are available (see [15]). The first conjugate time along an extremal is the time at which the extremal loses its local optimality. We use the asymptotic approach developed in Chapter 2 which consists in adding new smooth vector fields $Y_2, \ldots, Y_m$ and a small parameter $\varepsilon > 0$, so as to come up with the minimal time problem $\mathbf{(OCP)}_\varepsilon$ for the system $\dot{x} = X(x) + u_1^\varepsilon Y_1(x) + \varepsilon \sum_{i=2}^m u_i^\varepsilon Y_i(x)$, under the control constraint $\sum_{i=1}^m (u_i^\varepsilon(t))^2 \leq 1$, with the same boundary conditions as the initial problem, and investigate the convergence properties of conjugate times. From Theorem 2.5.1, under appropriate assumptions, the optimal controls of the latter problem, depending on ε, are smooth functions of t, and the theoretical and practical results for the conjugate time theory that are well known in the smooth case can be applied to the regularized problem. In our main result (Section §3.2, Theorem 3.2.1)) we prove that the first conjugate time t_c^ε of regularized problem converges to the first conjugate time t_c of the initial problem, when ε tends to 0. We thus get as a byproduct an efficient way to compute conjugate times in the bang-bang case.

In Section §3.1.3 we consider the bang-bang case and recall two different approaches to derive second order necessary and/or sufficient conditions for strong local optimality and their relation with the existence of conjugate times. In Section §3.1.4 we recall the regularization procedure introduced in Section §2.4 of Chapter 2. In Section §3.1.5 we recall a sufficient optimality conditions in the smooth case and the concept of geometric conjugate time. These two sections are very important for the formulation and prove of our main result (Theorem 3.2.1) in Section §3.2. In Section §3.3 we provide two examples to illustrate the main results of this thesis (Theorems 2.5.1 and 3.2.1).

3.1.1 Statement of the problem

Consider the single-input control-affine system in $\mathbb{R}^n$

$$\dot{x} = X(x) + u_1 Y_1(x), \tag{3.1}$$

where X and Y_1 are smooth vector fields, and the control u_1 is a measurable scalar function satisfying the constraint

$$|u_1(t)| \leq 1\,, \quad \forall t \in [0, t_f]. \tag{3.2}$$

Let $\hat{x}_0$ and $\hat{x}_1$ be two points of $\mathbb{R}^n$. Assume that $\hat{x}_1$ is reachable from $\hat{x}_0$, that is, there exists a time $T > 0$ and a control function $u_1(\cdot) \in L^\infty(0, T)$ satisfying the constraint (3.2), such that the trajectory $x(\cdot)$, solution of (3.1) with $x(0) = \hat{x}_0$, satisfies $x(T) = \hat{x}_1$.

We consider the optimal control problem **(OCP)** of determining a solution $\hat{x}(\cdot)$ associated to a control $\hat{u}_1(\cdot)$, on $[0, t_f]$, satisfying (3.1)–(3.2) and steering $\hat{x}_0$ to $\hat{x}_1$ in minimal time t_f. We assume that such a solution $\hat{x}(\cdot)$ for **(OCP)** exists.[1]

3.1.2 Bang-bang Pontryagin extremals

Recalling Section §2.3 we know that, following the Pontryagin maximum principle (see [96]), there exists an absolutely continuous mapping $\hat{p}(\cdot) : [0, t_f] \to \mathbb{R}^n$, called *adjoint vector*, and a real number $p^0 \leq 0$, with $(\hat{p}(\cdot), p^0) \neq (0, 0)$, such that

$$\begin{aligned} \dot{\hat{p}}(t) &= -\frac{\partial H}{\partial x}(\hat{x}(t), \hat{p}(t), p^0, \hat{u}_1(t)) \\ &= -\left\langle \hat{p}(t), \frac{\partial X}{\partial x}(\hat{x}(t)) \right\rangle - \hat{u}_1(t) \left\langle \hat{p}(t), \frac{\partial Y_1}{\partial x}(\hat{x}(t)) \right\rangle \end{aligned} \tag{3.3}$$

where the function $H(x, p, p^0, u_1) = \langle p, X(x) + u_1 Y_1(x) \rangle + p^0$ is called the *Hamiltonian*, and the maximization condition

$$H(\hat{x}(t), \hat{p}(t), p^0, \hat{u}_1(t)) = \max_{|w| \leq 1} H(\hat{x}(t), \hat{p}(t), p^0, w) \tag{3.4}$$

holds almost everywhere on $[0, t_f]$. Moreover, $\max_{|w| \leq 1} H(\hat{x}(t), \hat{p}(t), p^0, w) = 0$ for every $t \in [0, t_f]$. It follows from (3.4) that

$$\hat{u}_1(t) = \operatorname{sign}\langle \hat{p}(t), Y_1(\hat{x}(t)) \rangle \tag{3.5}$$

[1] See e.g. [26] for existence results of optimal solutions.

for almost every t, provided that the (continuous) switching function

$$\varphi_1(t) = \langle \hat{p}(t), Y_1(\hat{x}(t)) \rangle$$

does not vanish on any subinterval of $[0, t_f]$.[2] Such an extremal $(\hat{x}(\cdot), \hat{p}(\cdot), p^0, \hat{u}_1(\cdot))$ is then completely determined by the initial adjoint vector $\hat{p}(0)$. This extremal is a priori defined on the time interval $[0, t_f]$, but since it is completely determined by the differential system (3.1)–(3.3) and its initial condition, it may be extended forward on a maximal time interval I of $[0, +\infty)$, containing $[0, t_f]$. In this way, we consider the trajectory $\hat{x}(\cdot)$ on this maximal interval I.

Note that, since $\hat{x}(\cdot)$ is optimal on $[0, t_f]$, and since the control system under study is autonomous, it follows that $\hat{x}(\cdot)$ is as well optimal for the problem of steering the system (3.1) from $\hat{x}(0) = \hat{x}_0$ to $\hat{x}(t)$, for every $t \in (0, t_f]$.

Assumption 3.1.1. *We assume that the extremal $(\hat{x}(\cdot), \hat{p}(\cdot), p^0, \hat{u}_1(\cdot))$ is bang-bang on the interval I, that is, the switching function φ_1 does not vanish on any subinterval of I.*

Denote by $\hat{\tau}_1, \dots, \hat{\tau}_s, \ldots$ the zeros of φ_1 on I (possibly in infinite number).

Assumption 3.1.2. *We assume moreover that the extremal $(\hat{x}(\cdot), \hat{p}(\cdot), p^0, \hat{u}_1(\cdot))$ satisfies the* strict bang-bang Legendre condition*, that is,*

$$\dot{\varphi}_1(\hat{\tau}_j) = \frac{d}{dt} \langle \hat{p}(t), Y_1(\hat{x}(t)) \rangle \Big|_{t=\hat{\tau}_j} \neq 0,$$

for every $\hat{\tau}_j$ with $j = 1, \ldots, s$.

The Assumption 3.1.2 implies that the times $\hat{\tau}_1, \dots, \hat{\tau}_s$ are isolated and are in finite number on every compact subinterval of I. In particular, we assume that there are exactly s switching times on the interval $[0, t_f]$, such that $0 < \hat{\tau}_1 < \ldots < \hat{\tau}_s < t_f$. Moreover, the Assumption 3.1.2 implies that each $\hat{\tau}_1, \dots, \hat{\tau}_s$ is a switching time of the control and hence the control is given by

$$\hat{u}_1(t) = \begin{cases} 1 & \text{if } \varphi_1(t) > 0, \\ -1 & \text{if } \varphi_1(t) < 0, \end{cases}$$

for every $t \in I$.

Definition 3.1.3. Let $T > 0$, $T \in I$. The trajectory $\hat{x}(\cdot)$ is said to be *locally minimal time on* $[0, T]$ *in C^0 topology* if there exist a neighborhood W of the trajectory $\hat{x}(\cdot)$ in $\mathbb{R}^n$ and a real number $\eta > 0$ such that, for every trajectory $y(\cdot)$ that is solution of (3.1), contained in W, associated with a control v on $[0, T + \eta]$ satisfying the constraint (3.2), satisfying $y(0) = \hat{x}_0$ and $y(t_1) = \hat{x}(T)$ with $t_1 \in [0, T + \eta]$, there holds $t_1 \geq T$.

[2] The case where the switching function may vanish on a subinterval is related to singular trajectories, and is outside of the scope of this chapter where we focus on the bang-bang case.

The C^0 local optimality is also called *strong local optimality.* The notion of global optimality is defined similarly, with $W = \mathbb{R}^n$ and $\eta = +\infty$.

The Pontryagin maximum principle mentioned formerly is a necessary first order condition for optimality; conversely, extremals are not necessarily locally optimal, and there have been many works on high order necessary optimality conditions (see e.g. [18]) and on sufficient (first and second order) optimality conditions detailed in the next section.

3.1.3 Second order optimality conditions and bang-bang conjugate times

Consider the extremal $(\hat{x}(\cdot), \hat{p}(\cdot), p^0, \hat{u}_1(\cdot))$ of the problem **(OCP)** introduced previously.

Definition 3.1.4. The *cut time* $t_{\mathrm{cut}}(\hat{x}_0)$ is defined as the first positive time of I beyond which the trajectory $\hat{x}(\cdot)$ loses its global optimality status for the problem of steering the system (3.1)–(3.2) from $\hat{x}_0$ to $\hat{x}_1$ in minimal time, with the agreement that $t_{\mathrm{cut}}(\hat{x}_0) = +\infty$ whenever $\hat{x}(\cdot)$ is globally optimal on every interval $[0, T]$, $T > 0$, $T \in I$. The point $\hat{x}(t_{\mathrm{cut}}(\hat{x}_0))$ is called a *cut point.*

Whereas such a global optimality status is difficult to characterize, the local optimality status of a trajectory may be characterized using the concept of conjugate time, that is, the time at which the optimal trajectory $\hat{x}(\cdot)$ loses its local optimality. We next recall well known facts on first conjugate times of solutions $\hat{x}(\cdot)$ of the optimal control problem **(OCP)** associated to bang-bang controls $\hat{u}_1(\cdot)$.

The definition and computation of conjugate points are an important topic in the theory of calculus of variations (see e.g. [13]). In [99] the investigation of the definition and computation of conjugate points for minimal time control problems is based on the study of second order conditions. In fact, second order necessary and/or sufficient conditions are crucial for study of the first conjugate time of the problem **(OCP)**. In [110], the theory of envelopes and conjugate points is used for the study of the structure of locally optimal bang-bang trajectories for the problem **(OCP)** in $\mathbb{R}^2$ and $\mathbb{R}^3$; these results were generalized in [60].

Second order optimality conditions

When the optimal control problem has a nonlinear control system and the extremal controls are continuous, the literature on first and/or second order sufficient conditions is vast; see e.g. [14, 41, 68, 75, 77, 79, 83, 86, 126] and references therein. In this case numerical procedures are available to test second order sufficient conditions; see e.g. [10, 70, 77]. For second order necessary and/or sufficient conditions of optimal control problems with nonlinear control systems and discontinuous controls see e.g. [89] and references therein.

We will next focus on second order necessary and/or sufficient optimality conditions for optimal control problems with affine-control systems and bang-bang optimal controls.

In [100] a minimal time control problem for control-affine systems is considered and first and second order sufficient optimality conditions are derived, for bang-bang Pontryagin extremal controls which are L^1-locally optimal. In [81] the same optimal control problem is studied and the authors provide sufficient conditions for strong local optimality and develop numerical methods to test the positive definiteness of a specific quadratic form. In both papers [100] and [81], the sufficient optimality conditions are expressed in terms of quadratic forms, however although the same critical subspace is used, the quadratic form in [100] is a lower bound for the one in [81]. In fact, the second order sufficient optimality condition in [81] is always fulfilled whenever the corresponding condition in [100] is.

In [78, 81] optimization methods are given to test second order sufficient optimality conditions for optimal control problems with bounded scalar controls [81], and vector-valued controls [78].

In [5] the authors derive second order sufficient conditions, under the same regularity assumptions as [81], for an optimal control problem in the Mayer form with fixed final time, with affine-control systems and bang-bang optimal controls. In [90] the authors showed that, in certain cases, the second order sufficient conditions given in [81] are equivalent to the ones in [5]. In the cases where the equivalence holds, the results obtained in [90] extend those in [5] to the problem of free final time, with mixed initial and terminal conditions of equality and inequality type. The detailed proofs of the main results in [90] are given in [91]. In [5] a finite-dimensional subproblem is considered which consists in moving the switching times and a second variation is defined as a certain quadratic form associated to this subproblem; then, finding a conjugate time consists in testing the positivity of that quadratic form. The authors prove that this can only happen at a switching time.

In [95] the minimal time problem for control-affine systems is studied. An analogous quadratic form to the one in [5] is defined, but the kind of optimality studied is a stronger one (state local optimality).

Quadratic forms

As mentioned above the quadratic forms defined in [5,81] are equivalent (see [90,91]), although the way they are defined is different. In this chapter we only give a brief sketch of a possible procedure to define the quadratic form (see Appendix A where the quadratic form deduced in [81] is recalled).

Let $F(t;\tau_1,...,\tau_s) = x(t;\tau_1,...,\tau_s)$ be the mapping associated with the finite-dimensional problem corresponding to **(OCP)** that consists in moving the switching times $\tau_1,\dots,\tau_s$ in a neighborhood of the reference switching times $\hat{\tau}_1,\dots,\hat{\tau}_s$ (see [5, 78, 90, 91, 95]), where $x(t;\tau_1,...,\tau_s)$ is the trajectory solution of (3.1), on $[0,t]$, with $x(0) = \hat{x}_0$, associated to the bang-bang control $u_1(\cdot)$ with switching times $\tau_1,...,\tau_s$ and such that it coincides with the reference trajectory $\hat{x}(\cdot)$ whenever $\tau_i = \hat{\tau}_i$ for every i. Note that the trajectory $x(\cdot;\tau_1,...,\tau_s)$

is not the projection of an extremal whenever $\tau_i \neq \hat{\tau}_i$. The mapping F is well defined for t in a neighborhood of t_f and τ_i in a neighborhood of $\hat{\tau}_i$ for every i, and is the composition of smooth mappings, therefore is differentiable. Denoting $\tau = (\tau_1, ..., \tau_s)$, one has

$$\frac{\partial F}{\partial \tau}(t; \tau_1, ..., \tau_s) = \begin{pmatrix} \frac{\partial x_1}{\partial \tau_1}(\cdot) & \dots & \frac{\partial x_1}{\partial \tau_s}(\cdot) \\ \vdots & \vdots & \vdots \\ \frac{\partial x_n}{\partial \tau_1}(\cdot) & \dots & \frac{\partial x_n}{\partial \tau_s}(\cdot) \end{pmatrix},$$

and

$$\frac{\partial F}{\partial t}(t; \tau_1, ..., \tau_s) = \dot{x}(t; \tau_1, ..., \tau_s).$$

Since $\hat{x}(\cdot)$ is optimal, it follows that

$$\text{rank}\left(\frac{\partial F}{\partial \tau}(t; \hat{\tau}_1, ..., \hat{\tau}_s)\right) \leq n - 1.$$

Indeed, otherwise, if $\text{rank}\left(\frac{\partial F}{\partial \tau}(t; \hat{\tau}_1, ..., \hat{\tau}_s)\right) = n$ then F would be a local submersion, which contradicts the optimality of $\hat{x}(\cdot)$. Therefore, there exists a multiplier $\psi_t \in \mathbb{R}^n \backslash \{0\}$ such that $\psi_t \cdot \frac{\partial F}{\partial \tau}(t; \hat{\tau}_1, ..., \hat{\tau}_s) = 0$. Denote by Q_t the intrinsic second derivative of the mapping F, defined by

$$Q_t = \psi_t \cdot \frac{\partial^2 F}{\partial \tau^2}(t; \hat{\tau}_1, ..., \hat{\tau}_s)\Big|_{\ker \frac{\partial F}{\partial \tau}(t; \hat{\tau}_1, ..., \hat{\tau}_s)}. \tag{3.6}$$

Explicit formulas of Q_t are given in [3,5,81,95]; in particular formulas in terms of Lie brackets of the vector fields can be derived.

The next theorem, combination of several known results, provides a necessary and/or sufficient condition for strong local optimality.

Theorem 3.1.5 ([3,5,81,87,95]). *Let $(\hat{x}(\cdot), \hat{p}(\cdot), p^0, \hat{u}_1(\cdot))$ be a bang-bang extremal for* ***(OCP)*** *defined on a maximal time interval I of $[0, +\infty)$ containing $[0, t_f]$. If this extremal satisfies the strict bang-bang Legendre condition on I (see Assumption 3.1.2), then for every $t \in I$, the following holds:*

- *If the quadratic form Q_t is positive definite then $\hat{x}(\cdot)$ is a local minimizer in the C^0 topology on $[0, t]$.*
- *Assume moreover that $\hat{x}(\cdot)$ has a unique extremal lift (up to a multiplicative scalar) $(\hat{x}(\cdot), \hat{p}(\cdot), p^0, \hat{u}_1(\cdot))$, which is moreover normal ($p^0 = -1$). If $\hat{x}(\cdot)$ is locally optimal in the C^0 topology on $[0, t]$ then Q_t given by (3.6) is nonnegative.*

Remark 3.1.6. Under the assumptions of the Theorem 3.1.5, the set

$$\{t > 0 \mid Q_t \, \text{has a nontrivial kernel}\}$$

is discrete and can only consist of some switching times (see [5]). This remark permits to define the notion of first conjugate time.

Definition 3.1.7. The *first conjugate time* t_c of $\hat{x}(\cdot)$ is defined by

$$t_c = \sup\{t \mid Q_t \text{ is positive definite}\} = \inf\{t \mid Q_t \text{ is indefinite}\}.$$

The point $\hat{x}(t_c)$ is called the *first conjugate point* of the trajectory $\hat{x}(\cdot)$.

Remark 3.1.8. A conjugate time can only occur at a switching time.

Extremal field approach

Sufficient optimality conditions for a general optimal control problem are provided in [87] (see also [5, 95]) with a different point of view than the one recalled in the previous paragraph. In [87] the authors study local optimality conditions for both continuous and piecewise continuous (including bang-bang) controls. The sufficient conditions developed in that article are based on the method of characteristics and the theory of extremal fields. Sufficient optimality conditions are given for embedding a reference trajectory into a local field of broken extremals.[3] The occurrence of a conjugate point is related with a so-called overlap of the flow near the switching surface. Second order sufficient optimality conditions stated in [87] have been tested numerically for bang-bang control problems; see e.g. [61]. See also [113] where sufficient optimality conditions for bang-bang controls based on the extremal field approach are studied.

In [1,4,5], using Hamiltonian methods and the extremal field theory, it is constructed, under certain conditions, a non-intersecting field of state extremals[4] that covers a given extremal trajectory $\hat{x}(\cdot)$. In [5] the authors associate the occurrence of a conjugate point with a fold point of the flow of the extremal field. We next recall the Hamiltonian approach presented in [5, 95].

For every $z_0 = (x_0, p_0) \in \mathbb{R}^n \times \mathbb{R}^n$, let $z(\cdot, z_0) = (x(\cdot, z_0), p(\cdot, z_0))$ denote the solution of the system of equations (3.1) and (3.3), with the control (3.5), such that $z(0, z_0) = z_0$. The *exponential mapping* is then defined by

$$\exp(t, z_0) = x(t, z_0).$$

In **(OCP)** as in the problems considered in [5] and [95] the initial point is not free ($\hat{x}_0$ is a fixed point of $\mathbb{R}^n$). To apply the Hamiltonian approach presented in [5, 95], we consider a C^2 function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that $\alpha'(\hat{x}_0) = \hat{p}_0$, where $\alpha'(x_0)$ denotes $\frac{d\alpha}{dx}(x_0)$ and $\hat{p}_0 = \hat{p}(0)$. The function α represents a penalization on the initial point $\hat{x}_0$ and a new finite-dimensional subproblem is considered, with free initial point $\alpha(\hat{x}_0)$, that consists in moving the switching times and minimizing $\alpha(\hat{x}_0) + t_f$.

[3] Broken extremals are associated to piecewise continuous controls.

[4] By non-intersecting extremals we mean that for any fixed $t \in (0, t_c)$ and any extremal trajectories $x(\cdot)$, $y(\cdot)$ with initial points x_0, y_0, respectively, with x_0, y_0 close to $\hat{x}_0$, we have $x(t) \neq y(t)$.

The existence of a function α in the previous conditions was proved in [50]. Moreover, in [95] the authors proved that if the quadratic form (3.6) is positive definite, then the quadratic form associated to the finite-dimensional subproblem of moving the switching times with free initial point is also positive definite.

Let $\mathcal{O}$ be a neighborhood of the initial point $\hat{x}_0$. Let $x_0 \in \mathcal{O}$; define the *switching time functions* $\tau_j : \mathcal{O} \to \mathbb{R}$ with

$$\tau_0(x_0) = 0 \text{ and } \tau_j(\hat{x}_0) = \hat{\tau}_j\,, \;\; j = 1, ..., s,$$

such that

$$\varphi_1(\tau_j(x_0)) = \langle p(\tau_j(x_0), x_0, \alpha'(x_0)), Y_1(x(\tau_j(x_0), x_0, \alpha'(x_0)))\rangle = 0\,, \;\; j = 1, ..., s\,.$$

In other words, $\tau_j(x_0)$ is the j^{th}-switching time of the extremal $x(\cdot, x_0, \alpha'(x_0)), p(\cdot, x_0, \alpha'(x_0))$ starting from $(x_0, \alpha'(x_0))$, with x_0 close to $\hat{x}_0$.

Since $\hat{x}(\cdot)$ is a minimal time trajectory, there holds $\max\limits_{|w|\leq 1} H(\hat{x}_0, \hat{p}_0, p^0, w) = 0$. Consider the set

$$X = \{x_0 \in \mathcal{O} \,|\, \max_{|w|\leq 1} H(x_0, \alpha'(x_0), p^0, w) = 0\}.$$

We claim that X is a $(n-1)$-dimensional manifold.[5] Indeed, consider the map

$$\begin{aligned} G \,:\, \mathcal{O} &\to \mathbb{R} \\ x_0 &\mapsto G(x_0) = \max_{|w|\leq 1} H(x_0, \alpha'(x_0), p^0, w) \end{aligned} \tag{3.7}$$

and the vector field $h_1(x_0) = X(x_0) + u_1 Y_1(x_0)$ that defines the extremal trajectory $x(\cdot)$ on the interval $[0, \tau_1(x_0))$, associated to the value u_1 that satisfies the maximization condition (3.4) on the referred interval. Proving that X is a $(n-1)$-dimensional manifold amounts to prove that, for every function $\alpha \in C^2$ such that $\alpha'(x_0) = p_0$, there holds $dG(x_0) \neq 0$ before the first conjugate time t_c. The second variation formula given in [95, p. 275, equation (12)] taken at $(\delta x, \varepsilon) = (h_1(x_0), -1, 0, ..., 0)$ is equal to, after some simplifications, $dG(x_0) \cdot h_1(x_0)$. Since the second variation is positive definite on $(0, t_c)$ then $dG(x_0) \cdot h_1(x_0) \neq 0$ before t_c. The claim is proved.

Define the j^{th} switching surface Σ_j, for $j = 1, ..., s$, as the image of the mapping

$$x_0 \mapsto \exp(\tau_j(x_0), x_0, \alpha'(x_0))\,,$$

where $x_0 \in X$.

Remark 3.1.9. If the strict bang-bang Legendre condition holds, then the flow associated to the maximized Hamiltonian crosses the switching surface Σ_j at the instant $\hat{\tau}_j$ transversally, for $j = 1, ..., s$ (see [5]).

[5] The argument that follows is due to L. Poggiolini.

Theorem 3.1.10 ([5,80,81,87]). *Let $(\hat{x}(\cdot), \hat{p}(\cdot), p^0, \hat{u}_1(\cdot))$ be a bang-bang extremal for **(OCP)** that satisfies the strict bang-bang Legendre condition on $[0, t_c)$, with $t_c < +\infty$. The trajectory $\hat{x}(\cdot)$ is strong locally optimal if and only if there exists a function $\alpha \in C^2$ with $\alpha'(\hat{x}_0) = \hat{p}_0$ such that:*

- *the trajectory $\hat{x}(\cdot)$ can be embedded into the field of non-intersecting (broken) extremals $(t, x_0) \mapsto \exp(t, x_0, \alpha'(x_0))$ where $x_0 \in \mathcal{O}$;*

- *this field of extremals crosses the switching surfaces Σ_j transversally, for $j = 1, ..., s$, and for $j = 1, ..., s+1$, with $\tau_{s+1}(\hat{x}_0) = t_c$, the mapping*

$$\begin{aligned}(\tau_{j-1}(x_0), \tau_j(x_0)) \times X &\longrightarrow I\!R^n \\ (t, x_0) &\longmapsto \exp(t, x_0, \alpha'(x_0))\end{aligned}$$

is of rank n.

Remark 3.1.11. In the conditions of Theorem 3.1.10, at the first conjugate point $\hat{x}(t_c)$, the flow of the extremal field reflects off the switching surface, causing an overlap of the flow near this surface (see Figure 3.1 - switching surface Σ_{s+1}, and see [61, 87]).

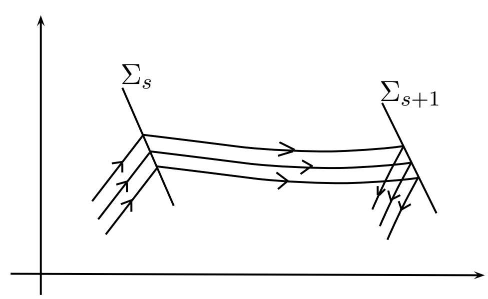

Figure 3.1: Field of extremals

Remark 3.1.12. Let $f_j(x_0) = X(x_0) + u_j Y_1(x_0)$, for $j = 1, ..., s+2$ and $x_0 \in \mathcal{O}$, be the vector fields that define the extremal trajectory $x(\cdot)$ on $(\tau_{j-1}(x_0), \tau_j(x_0))$, with $\tau_{s+1}(\hat{x}_0) = t_c$ and where u_j is the value (1 or -1) of the control that satisfies the maximization condition (3.4) in each respective interval. If we take $x_0 \in X$ and $j = 1, ..., s+1$, then for $(t, x_0) \in (\tau_{j-1}, \tau_j) \times X$

$$\det \big(\exp(t, x_0, \alpha'(x_0)), f_j(x_0)\big)$$

has constant sign (see [95]).

Moreover, the determinants

$$\det \left(\frac{d}{dx_0} \exp(t, x_0, \alpha'(x_0)) \Big|_{(t,x_0) \in (\tau_s(x_0), \tau_{s+1}(x_0)) \times X}, f_{s+1}(x_0) \right)$$

and

$$\det\left(\frac{d}{dx_0}\exp(t,x_0,\alpha'(x_0))\Big|_{(t,x_0)\in(\tau_{s+1}(x_0),\tau_{s+2}(x_0))\times X}, f_{s+2}(x_0)\right)$$

have different signs (see [95]).

The computation of conjugate times in the bang-bang case is difficult in practice. In the last few years several methods have been developed for the numerical implementation of second order sufficient optimality conditions (see, e.g., [78, 81] and references cited therein). These numerical procedures allow the computation of the first conjugate time, for bang-bang optimal control problems with affine-control systems, whenever it exists and is attained at a j^{th} switching time. Besides, in the smooth case, efficient tools are available; see e.g. [15]. We next propose a regularization procedure which allows the use of these tools for the computation of the first conjugate time for the problem **(OCP)**. However, in practice, if j is too large then the numerical computation of the first conjugate time may become very difficult either using the methods for bang-bang or smooth controls.

3.1.4 Regularization procedure

Recall the regularization procedure introduced in Section §2.4 of Chapter 2.

Let ε be a positive real parameter and let $Y_2,\ldots,Y_m$ be $m-1$ arbitrary smooth vector fields on $\mathbb{R}^n$, where $m\geq 2$ is an integer. Consider the control-affine system

$$\dot{x}^{\varepsilon}(t)=X\left(x^{\varepsilon}(t)\right)+u_1^{\varepsilon}(t)Y_1\left(x^{\varepsilon}(t)\right)+\varepsilon\sum_{i=2}^{m}u_i^{\varepsilon}(t)Y_i\left(x^{\varepsilon}(t)\right), \tag{3.8}$$

where the control $u^{\varepsilon}(t)=(u_1^{\varepsilon}(t),\ldots,u_m^{\varepsilon}(t))$ satisfies the constraint

$$\sum_{i=1}^{m}\left(u_i^{\varepsilon}(t)\right)^2\leq 1. \tag{3.9}$$

Consider the optimal control problem $\mathbf{(OCP)}_{\varepsilon}$ of determining a trajectory $x^{\varepsilon}(\cdot)$, solution of (3.8)–(3.9) on $[0,t_f^{\varepsilon}]$, such that $x^{\varepsilon}(0)=\hat{x}_0$ and $x^{\varepsilon}(t_f^{\varepsilon})=\hat{x}_1$, and minimizing the time of transfer t_f^{ε}. The parameter ε is viewed as a penalization parameter. The existence of at least one solution for $\mathbf{(OCP)}_{\varepsilon}$ is proved in Lemma 2.5.6 (Chapter 2).

In Theorem 2.5.1 (Section §2.5 of Chapter 2) we prove that if the problem **(OCP)** has a unique solution $\hat{x}(\cdot)$, defined on $[0,t_f]$, associated with a bang-bang control $\hat{u}_1(\cdot)$ on $[0,t_f]$, and if, moreover, $\hat{x}(\cdot)$ has a unique extremal lift (up to a multiplicative scalar), which is moreover normal, denoted $(\hat{x}(\cdot),\hat{p}(\cdot),-1,\hat{u}_1(\cdot))$, then, under the Assumption 2.4.1, the optimal controls of $\mathbf{(OCP)}_{\varepsilon}$ are smooth functions of t and converge almost everywhere on $[0,t_f]$ to the optimal control of **(OCP)**. Moreover, the associated trajectories $\hat{x}^{\varepsilon}(\cdot)$ and adjoint vectors $\hat{p}^{\varepsilon}(\cdot)$ converge uniformly to $\hat{x}(\cdot)$ and $\hat{p}(\cdot)$, respectively, on $[0,t_f]$, when ε tends to 0.

Remark 3.1.13. This result remains true if we extend forward the interval $[0,t_f]$ on an interval $[0,T]$ for $T\in I$, where I is a maximal time interval of $[0,+\infty)$ containing $[0,t_f]$.

3.1.5 Conjugate times in the smooth case

We recall how to define the concept of first conjugate time for the smooth optimal control problem $\mathbf{(OCP)}_\varepsilon$. A first possible definition of conjugate times is in terms of a quadratic form, which is the second order intrinsic derivative of the *end-point mapping* defined by $E(\varepsilon, t_f^\varepsilon, \hat{x}_0, u^\varepsilon) = x^\varepsilon(t_f^\varepsilon)$ where $t \mapsto x^\varepsilon(\varepsilon, t, \hat{x}_0, u^\varepsilon)$ is the trajectory solution of (3.8), associated to the control u^ε, such that $x^\varepsilon(\varepsilon, 0, \hat{x}_0, u^\varepsilon) = \hat{x}_0$. Testing a conjugate time amounts to testing the positivity of that quadratic form. However, this definition requires a corank one assumption, and we will rather use a geometric concept of conjugate time, defined below. We refer the reader to [15] for a survey on that theory and to [2] for extensive explanations and for the more general Morse index theory.

Geometric conjugate time

Definition 3.1.14. Let $x_0 \in \mathcal{O}$. The point $x^\varepsilon(t_c^\varepsilon)$ is *geometrically conjugate* to $x^\varepsilon(0)$ if and only if the mapping $x_0 \mapsto \exp^\varepsilon(t_c^\varepsilon, x_0, \alpha'(x_0))$ is not immersive, that is,

$$\det\left(\frac{d}{dx_0}\exp^\varepsilon(t_c^\varepsilon, x_0, \alpha'(x_0))\right) = 0.$$

The time t_c^ε is called a *geometric conjugate time.*

Remark 3.1.15. Given an extremal $(\hat{x}^\varepsilon(\cdot), \hat{p}^\varepsilon(\cdot), p^{0\varepsilon}, u^\varepsilon(\cdot))$, the notion of geometric conjugate time coincides with the notion of conjugate time defined in terms of quadratic form, provided the following assumptions hold:

- the strong Legendre condition holds along the extremal, that is, there exists $\gamma > 0$ such that
$$\frac{\partial^2 H}{\partial u^2}(\hat{x}^\varepsilon(\cdot), \hat{p}^\varepsilon(\cdot), p^{0\varepsilon}, u_1^\varepsilon(\cdot)) \cdot (v, v) \leq -\gamma \|v\|^2,$$
for every $v \in \mathbb{R}^m$;
- the control u^ε is of corank one on every subinterval (assumption of *strong regularity*, see [99]).

Moreover, in that case the first conjugate time t_c^ε characterizes the optimality status of the extremal: the trajectory $\hat{x}^\varepsilon(\cdot)$ is strongly locally optimal on $[0, t]$, for every $t < t_c^\varepsilon$; for $t > t_c^\varepsilon$, the trajectory $\hat{x}^\varepsilon(\cdot)$ is not locally optimal on $[0, t]$ (see, e.g., [2, 15, 99]).

Remark 3.1.16. None of the two assumptions of the previous remark will be made for the extremal $(\hat{x}^\varepsilon(\cdot), \hat{p}^\varepsilon(\cdot), p^{0\varepsilon}, \hat{u}^\varepsilon(\cdot))$. In fact, our aim is to prove that the first geometric conjugate time t_c^ε converges to the first conjugate time t_c of the bang-bang case, when ε tends to 0. This result, derived in Theorem 3.2.1 (Section §3.2), will permit to use as well in the bang-bang case the available efficient implementation procedures that exist in the smooth case, like for instance the free package *COTCOT*[6] (see [15]).

[6] Conditions of Order Two, COnjugate Times, http://apo.enseeiht.fr/cotcot/

For normal extremals $(x^\varepsilon(\cdot), p^\varepsilon(\cdot), -1, u^\varepsilon(\cdot))$ that satisfy the strong Legendre condition, the absence of conjugate points is a sufficient condition for local optimality (see e.g. [2]). This sufficient optimality condition will be expressed using the extremal field approach.

Extremal field approach

From Theorem 2.5.1 every extremal lift of the problem $\textbf{(OCP)}_\varepsilon$ is normal ($p^{0\varepsilon} = -1$). Analogously to the bang-bang case, the aim is to construct a family of extremals containing the reference normal extremal $(\hat{x}^\varepsilon(\cdot), \hat{p}^\varepsilon(\cdot), -1, \hat{u}^\varepsilon(\cdot))$, sharing nice non-intersection properties[7] before the first conjugate time.

For every $z_0 = (x_0, p_0) \in \mathbb{R}^n \times \mathbb{R}^n$, let $z^\varepsilon(\cdot, z_0) = (x^\varepsilon(\cdot, z_0), p^\varepsilon(\cdot, z_0))$ be the solution of the system of equations (3.8) and (2.8), with the controls (2.11), such that $z^\varepsilon(0, z_0) = z_0$. The exponential mapping associated to $\textbf{(OCP)}_\varepsilon$ is defined by

$$\exp^\varepsilon(t, z_0) = x^\varepsilon(t, z_0).$$

Let $x_0 \in \mathcal{O}$ and $\alpha^\varepsilon : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function such that $\alpha^{\varepsilon\prime}(x_0) = p^\varepsilon(0)$, and such that the family of functions (α^ε) converges to the function α associated with the problem **(OCP)** in C^2 topology, as ε tends to 0. As in the bang-bang case, define

$$X^\varepsilon = \{x_0 \in \mathcal{O} \mid \max_{\sum_{i=1}^m w_i^2 \le 1} H^\varepsilon(x_0, \alpha^{\varepsilon\prime}(x_0), -1, w^\varepsilon) = 0\}.$$

For $\varepsilon > 0$ small enough, X^ε is a $(n-1)$-dimensional manifold. Indeed, let G^ε be defined on $\mathcal{O}$ by $G^\varepsilon(x_0) = \max_{\sum_{i=1}^m w_i^2 \le 1} H^\varepsilon(x_0, \alpha'(x_0), -1, w^\varepsilon)$. It follows from Theorem 2.5.1 that G^ε converges to G (3.7) (defined in Section 3.1.3) as ε goes to 0, and therefore, for $\alpha \in C^2$ such that $\alpha'(x_0) = p_0$, there holds $dG^\varepsilon(x_0) \neq 0$, since $dG(x_0) \neq 0$.

Theorem 3.1.17 ([2])**.** *If the normal extremal $(\hat{x}^\varepsilon(\cdot), \hat{p}^\varepsilon(\cdot), -1, \hat{u}^\varepsilon(\cdot))$ satisfies the strong Legendre condition and, moreover, can be embedded into the family of extremals $\exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0))$ such that the mapping*

$$\begin{array}{rcl} (0, t_c^\varepsilon) \times X^\varepsilon & \to & \mathbb{R}^n \\ (t, x_0) & \mapsto & \exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0)) \end{array}$$

is of rank n, then $(\hat{x}^\varepsilon(\cdot), \hat{p}^\varepsilon(\cdot), -1, \hat{u}^\varepsilon(\cdot))$ is a local minimum in C^0 topology for the problem $\textbf{(OCP)}_\varepsilon$.

Remark 3.1.18. The typical behavior of the flow of the extremal field at the first conjugate point is a fold point (see Figure 3.2, and see [2, 54]).

[7] By nice non-intersection properties we mean a non-intersecting field of extremals (cf. footnote in page 89).

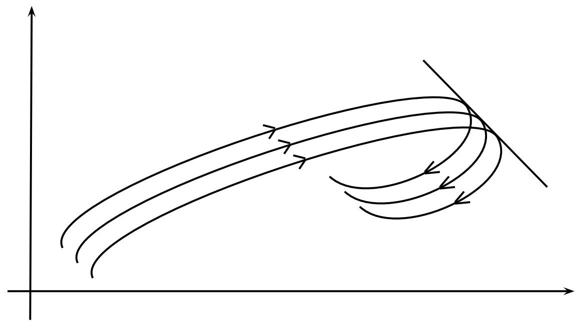

Figure 3.2: Field of extremals in the smooth case

Remark 3.1.19. If one considers $x_0 \in X^\varepsilon$, then $x^\varepsilon(t_c^\varepsilon)$ is geometrically conjugate to $x^\varepsilon(0)$ if and only if

$$\det \left(\frac{d}{dx_0} \exp^\varepsilon(t_c^\varepsilon, x_0, \alpha^{\varepsilon\prime}(x_0))_{|X^\varepsilon}, f^\varepsilon(x_0) \right) = 0\,,$$

where $f^\varepsilon(x_0) = X(x^\varepsilon(x_0)) + \varepsilon \sum_{i=1}^{m} u_i^\varepsilon(x_0, \alpha^{\varepsilon\prime}(x_0))$ and $u_i^\varepsilon(x_0, \alpha^{\varepsilon\prime}(x_0))$ are smooth functions that satisfy the maximization condition (2.10).

Remark 3.1.20. Note that, as long as the minimum time function is differentiable at the point $\hat{x}^\varepsilon(t)$, the optimal trajectory $\hat{x}^\varepsilon(\cdot)$ can be embedded into a non-intersecting extremal field.

Remark 3.1.21. To derive a necessary optimality condition, a corank one assumption is required for the extremal $(\hat{x}^\varepsilon(\cdot), \hat{p}^\varepsilon(\cdot), p^{0\varepsilon}, \hat{u}^\varepsilon(\cdot))$ (see [15]).

3.2 Convergence results

We first recall the context. Let $\hat{x}(\cdot)$ denote the strong locally optimal trajectory of (**OCP**), corresponding to the control $\hat{u}_1$ on $[0, t_f]$. In particular, t_f is the minimal time so that $\hat{x}(0) = \hat{x}_0$ and $\hat{x}(t_f) = \hat{x}_1$. We extend $\hat{x}(\cdot)$ on a maximal interval $I \subset [0, +\infty)$ containing $[0, t_f]$, and denote by t_c its first conjugate time. For every $\varepsilon > 0$, let $\hat{x}^\varepsilon(\cdot)$ denote an optimal trajectory solution of $(\mathbf{OCP})_\varepsilon$, corresponding to a control $\hat{u}^\varepsilon = (\hat{u}_1^\varepsilon, \dots, \hat{u}_m^\varepsilon)$ on $[0, t_f^\varepsilon]$. In particular, t_f^ε is the minimal time so that $\hat{x}^\varepsilon(0) = \hat{x}_0$ and $\hat{x}^\varepsilon(t_f^\varepsilon) = \hat{x}_1$. We extend $\hat{x}^\varepsilon(\cdot)$ on a maximal interval of $[0, +\infty)$ containing $[0, t_f^\varepsilon]$, and denote by t_c^ε its first geometrically conjugate time.

The main theorem of this chapter is the following theorem.

Theorem 3.2.1. *Assume that the problem (**OCP**) has a unique solution $\hat{x}(\cdot)$, associated with a bang-bang control $\hat{u}_1(\cdot)$, on a maximal interval I. Moreover, assume that $\hat{x}(\cdot)$ has a unique extremal lift (up to a multiplicative scalar), which is moreover normal, and denoted by*

$(\hat{x}(\cdot), \hat{p}(\cdot), -1, \hat{u}_1(\cdot))$. *If the extremal* $(\hat{x}(\cdot), \hat{p}(\cdot), -1, \hat{u}_1(\cdot))$ *satisfies, moreover, the strict bang-bang Legendre condition on* $[0, t_c]$*, then the first geometric conjugate time* t_c^ε *converges to the first conjugate time* t_c *as* ε *tends to 0.*

Remark 3.2.2. Let t_{cut} denote the cut time along the extremal $(\hat{x}(\cdot), \hat{p}(\cdot), p^0, \hat{u}(\cdot))$. Analogously to the bang-bang case, we can define the *cut time* $t_{\text{cut}}^\varepsilon$ of the optimal trajectory $\hat{x}^\varepsilon(\cdot)$ for the problem $(\mathbf{OCP})_\varepsilon$ as the first time at which $\hat{x}^\varepsilon(\cdot)$ loses its optimality. We claim that, under the assumptions of Theorem 2.5.1, there holds $\limsup_{\varepsilon \to 0} t_{\text{cut}}^\varepsilon \leq t_{\text{cut}}$.

The next proposition is the key result to derive Theorem 3.2.1.

Proposition 3.2.3. *Let* $\mathcal{O}$ *be a neighborhood of* $\hat{x}_0$ *and* $x_0 \in \mathcal{O}$*. The exponential mapping* $(t, x_0) \mapsto \exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0))$ *converges to* $(t, x_0) \mapsto \exp(t, x_0, \alpha'(x_0))$ *piecewise in* C^1 *topology on* $I \times \mathcal{O}$*, with* $\tau_{s+1}(\hat{x}_0) = t_c$*, as* ε *tends to 0. More precisely, on every compact subinterval of* $(\tau_{j-1}(x_0), \tau_j(x_0)) \times \mathcal{O}$*, with* $(\tau_{j-1}(x_0), \tau_j(x_0)) \subset I$ *and* $j \in \mathbb{N}$*, the mapping* $(t, x_0) \mapsto \exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0))$ *converges to* $(t, x_0) \mapsto \exp(t, x_0, \alpha'(x_0))$ *uniformly in the* C^1 *topology.*

Proof. In what follows, when it is convenient, we simplify the notation and write $\exp(t, x_0)$ or $x(t, x_0)$ (respectively, $\exp^\varepsilon(t, x_0)$ or $x^\varepsilon(t, x_0)$) for $\exp(t, x_0, \alpha'(x_0))$ (respectively, for $\exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0))$).

Let $\varepsilon > 0$ be small enough. For $x_0 \in \mathcal{O}$, consider the function

$$\varphi_1(\varepsilon, t, x_0) = \langle p(\varepsilon, t, x_0), Y_1(x(\varepsilon, t, x_0)) \rangle.$$

For $(\varepsilon, t, x_0) = (0, \hat{\tau}_j, x_0)$, by definition of the switching time, one has $\varphi_1(0, \hat{\tau}_j, x_0) = 0$, and by the strict bang-bang Legendre condition, $\frac{\partial \varphi_1}{\partial t}(0, \hat{\tau}_j, x_0) \neq 0$. By the implicit function theorem there exists a neighborhood $(-\varepsilon_0, \varepsilon_0)$ of $0 \in \mathbb{R}$, such that for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, there exists a C^1 function $\tau_j^\varepsilon(x_0) = \tau_j^\varepsilon(\varepsilon, x_0)$, with $j = 1, ..., s$, satisfying $\varphi_1(\varepsilon, \tau_j^\varepsilon(x_0)) = 0$ and such that, as ε tends to 0, $\tau_j^\varepsilon(x_0)$ converges to $\tau_j(x_0)$, and $\frac{\partial \tau_j^\varepsilon}{\partial x_0}(x_0)$ converges to $\frac{\partial \tau_j}{\partial x_0}(x_0)$.

Analogously to the definition of switching time function of an extremal trajectory $x(\cdot)$, we have thus defined some functions $\tau_j^\varepsilon(\cdot) : \mathcal{O} \to \mathbb{R}$, that are however not switching functions.

Lemma 3.2.4. *The mapping* $(t, x_0) \mapsto \exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0))$ *converges to* $(t, x_0) \mapsto \exp(t, x_0, \alpha'(x_0))$ *uniformly in the* C^1 *topology on* $J \times \mathcal{O}$*, where* J *is any compact subinterval of* $[0, \tau_1(x_0))$*, as* ε *tends to 0.*

Proof. Let J be a compact subinterval of $[0, \tau_1(x_0))$. The uniform C^0 convergence on $J \times \mathcal{O}$ of the mapping $(t, x_0) \mapsto \exp^\varepsilon(t, x_0)$ to $(t, x_0) \mapsto \exp(t, x_0)$, as ε tends to 0, is a direct consequence of Theorem 2.5.1. We have

$$\frac{\partial \exp^\varepsilon}{\partial t}(t, x_0) = \dot{x}^\varepsilon(t, x_0)$$

where $\dot{x}^\varepsilon(t, x_0)$ is given by (3.8). From Theorem 2.5.1, $\dot{x}^\varepsilon(t, x_0)$ converges to $\dot{x}(t, x_0) = \frac{d \exp}{dt}(t, x_0)$ as ε tends to 0. On the other hand,

$$\frac{d}{dx_0} \exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0)) = \frac{\partial \exp^\varepsilon}{\partial x_0}(t, x_0, \alpha^{\varepsilon\prime}(x_0)) + \frac{\partial \exp^\varepsilon}{\partial p_0}(t, x_0, \alpha^{\varepsilon\prime}(x_0)) \alpha^{\varepsilon\prime\prime}(x_0),$$

where $\frac{\partial \exp^\varepsilon}{\partial x_0}(t, x_0, \alpha^{\varepsilon\prime}(x_0))$, and $\frac{\partial \exp^\varepsilon}{\partial p_0}(t, x_0, \alpha^{\varepsilon\prime}(x_0))$ are solutions of the linearized system associated with the Hamiltonian system, for the problem $\mathbf{(OCP)}_\varepsilon$ on $[0,t]$, given by

$$\begin{aligned}
\dot{x}^\varepsilon(t) &= X\left(x^\varepsilon(t)\right) + u_1^\varepsilon(t) Y_1\left(x^\varepsilon(t)\right) + \varepsilon \sum_{i=2}^m u_i^\varepsilon(t) Y_i\left(x^\varepsilon(t)\right) \\
\dot{p}^\varepsilon(t) &= -\left\langle p^\varepsilon(t), \frac{\partial X}{\partial x}(x^\varepsilon(t))\right\rangle - u_1^\varepsilon(t)\left\langle p^\varepsilon(t), \frac{\partial Y_1}{\partial x}(x^\varepsilon(t))\right\rangle \\
&\quad - \varepsilon \sum_{i=2}^m u_i^\varepsilon(t)\left\langle p^\varepsilon(t), \frac{\partial Y_i}{\partial x}(x^\varepsilon(t))\right\rangle .
\end{aligned}$$

From Theorem 2.5.1, $(x^\varepsilon(\cdot), p^\varepsilon(\cdot))$ converges uniformly to the solution of the Hamiltonian system associated with the problem $\mathbf{(OCP)}$ as ε tends to 0. This convergence clearly holds as well for the solutions of the linearized system associated with the Hamiltonian system for $\mathbf{(OCP)}_\varepsilon$; therefore, as ε tends to 0, $\frac{\partial \exp^\varepsilon}{\partial x_0}(t, x_0, \alpha^{\varepsilon\prime}(x_0))$ (respectively, $\frac{\partial \exp^\varepsilon}{\partial p_0}(t, x_0, \alpha^{\varepsilon\prime}(x_0))$) converges to $\frac{\partial \exp}{\partial x_0}(t, x_0, \alpha'(x_0))$ (respectively, $\frac{\partial \exp}{\partial p_0}(t, x_0, \alpha'(x_0))$) uniformly on $[0,t]$. □

In what follows, the notation $\tau_j^+(x_0)$ (resp. $\tau_j^-(x_0)$) stands for the right limit (resp. the left limit). For $x_0 \in \mathcal{O}$ and $j = 1, ..., s$, we call the *jump* of $\frac{\partial \exp}{\partial x_0}(t, x_0)$ at $\tau_j(x_0)$ the difference

$$\frac{\partial \exp}{\partial x_0}(\tau_j^+(x_0), x_0) - \frac{\partial \exp}{\partial x_0}(\tau_j^-(x_0), x_0),$$

which is, according to [87, Equation 3.10, p. 123], given by

$$\begin{aligned}
&\frac{\partial \exp}{\partial x_0}(\tau_j^+(x_0), x_0) - \frac{\partial \exp}{\partial x_0}(\tau_j^-(x_0), x_0) \\
&= \Big(u_1(\tau_j^+(x_0), x_0) - u_1(\tau_j^-(x_0), x_0)\Big) Y_1(x(\tau_1(x_0), x_0)) \frac{\partial \tau_j}{\partial x_0}(x_0) \\
&= \Big(\operatorname{sign}(\varphi_1(\tau_j^+)) - \operatorname{sign}(\varphi_1(\tau_j^-))\Big) Y_1(x(\tau_j(x_0), x_0)) \frac{\partial \tau_j}{\partial x_0}(x_0) .
\end{aligned} \tag{3.10}$$

Due to this jump condition one cannot expect to get a C^1 convergence result on the whole interval. We will next estimate the difference

$$\frac{\partial \exp^\varepsilon}{\partial x_0}(\tau_j^\varepsilon(x_0) + \eta, x_0) - \frac{\partial \exp^\varepsilon}{\partial x_0}(\tau_j^\varepsilon(x_0) - \eta, x_0), \tag{3.11}$$

for $\eta > 0$ small, and show that it converges to (3.10), whenever ε tends to 0, and then η tends to 0.

Lemma 3.2.5. *There holds*

$$\begin{aligned}
\lim_{\eta\to 0}\lim_{\varepsilon\to 0} &\left(\frac{\partial \exp^\varepsilon}{\partial x_0}(\tau_1^\varepsilon(x_0) + \eta, x_0) - \frac{\partial \exp^\varepsilon}{\partial x_0}(\tau_1^\varepsilon(x_0) - \eta, x_0)\right) \\
&= \frac{\partial \exp}{\partial x_0}(\tau_1^+(x_0), x_0) - \frac{\partial \exp}{\partial x_0}(\tau_1^-(x_0), x_0).
\end{aligned} \tag{3.12}$$

Proof. One has

$$\frac{\partial}{\partial t}\left(\frac{\partial x^\varepsilon}{\partial x_0}(t,x_0)\right) = \Big(\frac{\partial X}{\partial x_0}(x^\varepsilon(t,x_0)) + u_1^\varepsilon(t,x_0)\frac{\partial Y_1}{\partial x_0}(x^\varepsilon(t,x_0)) + \varepsilon\sum_{i=2}^m u_i^\varepsilon(t,x_0)\frac{\partial Y_i}{\partial x_0}(x^\varepsilon(t,x_0))\Big)\frac{\partial x^\varepsilon}{\partial x_0}(t,x_0) + Y_1(x^\varepsilon(t,x_0))\frac{\partial u_1^\varepsilon}{\partial x_0}(t,x_0) + \varepsilon\sum_{i=2}^m Y_i(x^\varepsilon(t,x_0))\frac{\partial u_i^\varepsilon}{\partial x_0}(t,x_0).$$

It follows that

$$\frac{\partial x^\varepsilon}{\partial x_0}(\tau_1^\varepsilon(x_0)+\eta,x_0) - \frac{\partial x^\varepsilon}{\partial x_0}(\tau_1^\varepsilon(x_0)-\eta,x_0) = \int_{\tau_1^\varepsilon(x_0)-\eta}^{\tau_1^\varepsilon(x_0)+\eta}\Big(\frac{\partial X}{\partial x_0}(x^\varepsilon(t,x_0)) + u_1^\varepsilon(t,x_0)\frac{\partial Y_1}{\partial x_0}(x^\varepsilon(t,x_0)) + \varepsilon\sum_{i=2}^m u_i^\varepsilon(t,x_0)\frac{\partial Y_i}{\partial x_0}(x^\varepsilon(t,x_0))\Big)\frac{\partial x^\varepsilon}{\partial x_0}(t,x_0)dt + \int_{\tau_1^\varepsilon(x_0)-\eta}^{\tau_1^\varepsilon(x_0)+\eta} Y_1(x^\varepsilon(t,x_0))\frac{\partial u_1^\varepsilon}{\partial x_0}(t,x_0)dt + \int_{\tau_1^\varepsilon(x_0)-\eta}^{\tau_1^\varepsilon(x_0)+\eta}\varepsilon\sum_{i=2}^m Y_i(x^\varepsilon(t,x_0))\frac{\partial u_i^\varepsilon}{\partial x_0}(t,x_0)dt\,.$$

It is easy to see that the limit when η tends to zero of the limit when ε tends to zero of the first and third term of the right-hand side of the last equation is equal to zero. Only the limit term

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}\int_{\tau_1^\varepsilon(x_0)-\eta}^{\tau_1^\varepsilon(x_0)+\eta} Y_1(x^\varepsilon(t,x_0))\frac{\partial u_1^\varepsilon}{\partial x_0}(t,x_0)dt$$

deserves a special attention. Let us denote

$$\varphi_i^\varepsilon(t,x_0) = \langle p^\varepsilon(t,x_0), Y_i(x^\varepsilon(t,x_0))\rangle,\ \ i=1,...,m.$$

From (2.11), we compute easily

$$\frac{\partial u_1^\varepsilon}{\partial x_0}(t,x_0) = \frac{\varepsilon^2\left(\frac{\partial \varphi_1^\varepsilon}{\partial x_0}(t,x_0)\sum_{i=2}^m \varphi_i^\varepsilon(t,x_0)^2 - \varphi_1^\varepsilon(t,x_0)\sum_{i=2}^m \varphi_i^\varepsilon(t,x_0)\frac{\partial \varphi_i^\varepsilon}{\partial x_0}(t,x_0)\right)}{\left(\varphi_1^\varepsilon(t,x_0)^2 + \varepsilon^2\sum_{i=2}^m \varphi_i^\varepsilon(t,x_0)^2\right)^{3/2}}.$$

We will consider asymptotic expansions of these quantities around $\tau_1^\varepsilon(x_0)$. Since $\varphi_1^\varepsilon(\tau_1^\varepsilon(x_0),x_0) = 0$ for every x_0, it follows that

$$\frac{\partial \varphi_1^\varepsilon}{\partial x_0}(\tau_1^\varepsilon(x_0),x_0) = -\frac{\partial \varphi_1^\varepsilon}{\partial t}(\tau_1^\varepsilon(x_0),x_0)\frac{\partial \tau_1^\varepsilon}{\partial x_0}(x_0).$$

In what follows, denote $\tau_1^\varepsilon = (\tau_1^\varepsilon(x_0), x_0)$. One has

$$
\begin{aligned}
&\int_{\tau_1^\varepsilon-\eta}^{\tau_1^\varepsilon+\eta} Y_1^\varepsilon(x^\epsilon(t,x_0))\frac{\partial u_1^\varepsilon}{\partial x_0}(t,x_0)dt \\
&= \int_{\tau_1^\varepsilon-\eta}^{\tau_1^\varepsilon+\eta} \left(Y_1(x^\varepsilon(\tau_1^\varepsilon)) + O(t-\tau_1^\varepsilon)\right)\cdot \\
&\Bigg[\frac{\varepsilon^2\left(\frac{\partial\varphi_1^\varepsilon}{\partial x_0}(\tau_1^\varepsilon)+O(t-\tau_1^\varepsilon)\right)\sum_{i=2}^m\left(\varphi_i^\varepsilon(\tau_1^\varepsilon)+O(t-\tau_1^\varepsilon)\right)^2}{\left(\left(\frac{\partial\varphi_1^\varepsilon}{\partial t}(\tau_1^\varepsilon)(t-\tau_1^\varepsilon)+o(t-\tau_1^\varepsilon)\right)^2+\varepsilon^2\sum_{i=2}^m\left(\varphi_i^\varepsilon(\tau_1^\varepsilon)+\dfrac{\partial\varphi_i^\varepsilon}{\partial t}(\tau_1^\varepsilon)(t-\tau_1^\varepsilon)+o(t-\tau_1^\varepsilon)\right)^2\right)^{3/2}} \\
&-\frac{\varepsilon^2\left(\varphi_1^\varepsilon(\tau_1^\varepsilon)+O(t-\tau_1^\varepsilon)\right)\sum_{i=2}^m\left(\varphi_i^\varepsilon(\tau_1^\varepsilon)+O(t-\tau_1^\varepsilon)\right)\left(\dfrac{\partial\varphi_i^\varepsilon}{\partial x_0}(\tau_1^\varepsilon)+O(t-\tau_1^\varepsilon)\right)}{\left(\left(\frac{\partial\varphi_1^\varepsilon}{\partial t}(\tau_1^\varepsilon)(t-\tau_1^\varepsilon)+o(t-\tau_1^\varepsilon)\right)^2+\varepsilon^2\sum_{i=2}^m\left(\varphi_i^\varepsilon(\tau_1^\varepsilon)+\dfrac{\partial\varphi_i^\varepsilon}{\partial t}(\tau_1^\varepsilon)(t-\tau_1^\varepsilon)+o(t-\tau_1^\varepsilon)\right)^2\right)^{3/2}}\Bigg]dt
\end{aligned}
$$

and simplifying the last expression (the terms of order $O((t-\tau_1^\varepsilon)^k)$ and $o((t-\tau_1^\varepsilon)^l)$, with $k=2,3$ and $l=1,2,3$, are omitted) we get

$$
\begin{aligned}
&\int_{\tau_1^\varepsilon-\eta}^{\tau_1^\varepsilon+\eta} Y_1^\varepsilon(x^\epsilon(t,x_0))\frac{\partial u_1^\varepsilon}{\partial x_0}(t,x_0)dt \\
&= \int_{\tau_1^\varepsilon-\eta}^{\tau_1^\varepsilon+\eta} \left(Y_1(x^\varepsilon(\tau_1^\varepsilon))\right)\frac{-\varepsilon^2\frac{\partial\varphi_1^\varepsilon}{\partial t}(\tau_1^\varepsilon)N_1}{\left(\left(\left(\frac{\partial\varphi_1^\varepsilon}{\partial t}(\tau_1^\varepsilon)\right)^2+\varepsilon^2N_2\right)(t-\tau_1^\varepsilon)^2+\varepsilon^2N_3(t-\tau_1^\varepsilon)+\varepsilon^2N_1\right)^{3/2}}\frac{\partial\tau_1^\varepsilon}{\partial x_0}(x_0) \\
&+\frac{\varepsilon^2(M_1-M_2)O(t-\tau_1^\varepsilon)-\varepsilon^2\frac{\partial\varphi_1^\varepsilon}{\partial t}(\tau_1^\varepsilon)N_1\frac{\partial\tau_1^\varepsilon}{\partial x_0}(x_0)O(t-\tau_1^\varepsilon)}{\left(\left(\left(\frac{\partial\varphi_1^\varepsilon}{\partial t}(\tau_1^\varepsilon)\right)^2+\varepsilon^2N_2\right)(t-\tau_1^\varepsilon)^2+\varepsilon^2N_3(t-\tau_1^\varepsilon)+\varepsilon^2N_1\right)^{3/2}}\,dt\,,
\end{aligned}
$$

where

$$
\begin{aligned}
&N_1=\sum_{i=2}^m\left(\varphi_i^\varepsilon(\tau_1^\varepsilon)\right)^2,\;\; N_2=\sum_{i=2}^m\left(\frac{\partial\varphi_i^\varepsilon}{\partial t}(\tau_1^\varepsilon)\right)^2,\;\; N_3=2\sum_{i=2}^m\varphi_i^\varepsilon(\tau_1^\varepsilon)\frac{\partial\varphi_i^\varepsilon}{\partial t}(\tau_1^\varepsilon),\\
&M_1=2\frac{\partial\varphi_1^\varepsilon}{\partial x_0}(\tau_1^\varepsilon)\sum_{i=2}^m\varphi_i^\varepsilon(\tau_1^\varepsilon)+\sum_{i=2}^m\left(\varphi_i^\varepsilon(\tau_1^\varepsilon)\right)^2,\;\; M_2=\sum_{i=2}^m\left(\frac{\partial\varphi_i^\varepsilon}{\partial x_0}(\tau_1^\varepsilon)\right)^2.
\end{aligned}
$$

Notice that the denominator never vanishes, since by Assumption 2.4.1 the functions $(t,x_0)\mapsto\varphi_i(t,x_0)$, $i=1,\ldots,m$ do not vanish simultaneously.

The limit when η tends to zero of the limit when ε tends to zero, of the first and second term of the right-hand side of the last equality are respectively equal to

$$
\left(\mathrm{sign}(\varphi_1(\tau_1^+))-\mathrm{sign}(\varphi_1(\tau_1^-))\right)Y_1(x(\tau_1(x_0),x_0))\frac{\partial\tau_1}{\partial x_0}(x_0)\quad\text{and}\quad 0\,.
$$

Since

$$\lim_{\varepsilon\to 0}\frac{\partial\tau_1^\varepsilon}{\partial x_0}(x_0)=\frac{\partial\tau_1}{\partial x_0}(x_0),$$

it follows that

$$\begin{aligned}\lim_{\eta\to 0}\lim_{\varepsilon\to 0}&\left(\frac{\partial x^\varepsilon}{\partial x_0}(\tau_1^\varepsilon(x_0)+\eta,x_0)-\frac{\partial x^\varepsilon}{\partial x_0}(\tau_1^\varepsilon(x_0)-\eta,x_0)\right)\\&=\big(\mathrm{sign}(\varphi_1(\tau_1^+))-\mathrm{sign}(\varphi_1(\tau_1^-))\big)\,Y_1(x(\tau_1(x_0),x_0))\frac{\partial\tau_1}{\partial x_0}(x_0),\end{aligned}$$

and the lemma follows. □

A similar lemma holds for $\frac{\partial\exp}{\partial p_0}$. This result permits to extend the convergence result beyond the first switching time; the extension of Lemma 3.2.4 to every further interval (τ_{j-1},τ_j) is then straightforward. This proves the proposition. □

We are now in a position to prove Theorem 3.2.1. From Theorem 3.1.10, the trajectory $\hat{x}(\cdot)$ can be embedded into the field of extremals $x_0\mapsto\exp(t,x_0,\alpha'(x_0))$ with $x_0\in\mathcal{O}$ and the mapping

$$\begin{aligned}(0,t_c)\times X&\to\mathbb{R}^n\\(t,x_0)&\mapsto\exp(t,x_0,\alpha'(x_0))\end{aligned}$$

is of rank n, where $X=\{x_0\in\mathcal{O}\mid\max_{|w|\le 1}H(x_0,\alpha'(x_0),p^0,w)=0\}$, $\mathcal{O}$ is a neighborhood of $\hat{x}_0$, and t_c is the first conjugate time of $\hat{x}(\cdot)$.

From Remark 3.1.12, the determinants

$$\det\left(\frac{d}{dx_0}\exp(t,x_0,\alpha'(x_0))\Big|_{(t,x_0)\in(\tau_s(x_0),\tau_{s+1}(x_0))\times X},f_{s+1}(x_0)\right)$$

and

$$\det\left(\frac{d}{dx_0}\exp(t,x_0,\alpha'(x_0))\Big|_{(t,x_0)\in(\tau_{s+1}(x_0),\tau_{s+2}(x_0))\times X},f_{s+2}(x_0)\right)$$

have different signs, with $\tau_{s+1}(\hat{x}_0)=t_c$.

By Definition 3.1.14, the point $x^\varepsilon(\tau_c^\varepsilon(x_0))$ is geometrically conjugate to $x^\varepsilon(0)=x_0$, with $x_0\in X^\varepsilon$, if and only if

$$\det\left(\frac{d}{dx_0}\exp^\varepsilon(t,x_0,\alpha^{\varepsilon\prime}(x_0)),f^\varepsilon(x_0)_{|x_0\in X^\varepsilon}\right)=0$$

for $t=\tau_c^\varepsilon(x_0)$. Let $x_0\in X^\varepsilon$. We have

$$\begin{aligned}\frac{\partial\exp^\varepsilon}{\partial x_0}(\tau^\varepsilon(x_0),x_0,\alpha^{\varepsilon\prime}(x_0))&=\frac{\partial\exp^\varepsilon}{\partial t}(\tau^\varepsilon(x_0),x_0,\alpha^{\varepsilon\prime}(x_0))\frac{\partial\tau^\varepsilon}{\partial x_0}(x_0)\\&\quad+\frac{\partial\exp^\varepsilon}{\partial x_0}(\tau^\varepsilon(x_0),x_0,\alpha^{\varepsilon\prime}(x_0))\\&\quad+\frac{\partial\exp^\varepsilon}{\partial p_0}(\tau^\varepsilon(x_0),x_0,\alpha^{\varepsilon\prime}(x_0))\alpha^{\varepsilon\prime\prime}(x_0).\end{aligned}$$

Since $\frac{\partial \exp^\varepsilon}{\partial t}(\tau^\varepsilon(x_0), x_0, \alpha^{\varepsilon\prime}(x_0)) = \dot{x}^\varepsilon(x_0) = f^\varepsilon(x_0)$, there holds,

$$\det\left(\frac{\partial \exp^\varepsilon}{\partial t}(\tau^\varepsilon(x_0), x_0, \alpha^{\varepsilon\prime}(x_0))\frac{\partial \tau^\varepsilon}{\partial x_0}(x_0), f^\varepsilon(x_0)\right) = 0.$$

Thus, it follows that

$$\begin{aligned}
&\det\left(\frac{d}{dx_0}\exp^\varepsilon(\tau^\varepsilon(x_0), x_0, \alpha^{\varepsilon\prime}(x_0)), f^\varepsilon(x_0)\right)\\
&= \det\left(\frac{\partial \exp^\varepsilon}{\partial x_0}(\tau^\varepsilon(x_0), x_0, \alpha^{\varepsilon\prime}(x_0)) + \frac{\partial \exp^\varepsilon}{\partial p_0}(\tau^\varepsilon(x_0), x_0, \alpha^{\varepsilon\prime}(x_0))\alpha^{\varepsilon\prime\prime}(x_0), f^\varepsilon(x_0)\right)\\
&= \det\left(\frac{d}{dx_0}\exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0)), f^\varepsilon(x_0)\right)
\end{aligned}$$

for $t = \tau^\varepsilon(x_0)$. By Proposition 3.2.3, on every compact subinterval of $(\tau_{j-1}(x_0), \tau_j(x_0))$, the mapping $(t, x_0) \mapsto \exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0))$ converges to $(t, x_0) \mapsto \exp(t, x_0, \alpha'(x_0))$ uniformly in C^1 topology, therefore the determinants

$$\det\left(\frac{d}{dx_0}\exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0))\Big|_{(t,x_0)\in(\tau_s^\varepsilon(x_0), \tau_{s+1}^\varepsilon(x_0))\times X^\varepsilon}, f^\varepsilon(x_0)\right)$$

and

$$\det\left(\frac{d}{dx_0}\exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0))\Big|_{(t,x_0)\in(\tau_{s+1}^\varepsilon(x_0), \tau_{s+2}^\varepsilon(x_0))\times X^\varepsilon}, f^\varepsilon(x_0)\right)$$

have different signs before and after $\tau_{s+1}^\varepsilon(x_0)$. Therefore, by continuity, the function $t \mapsto \det\left(\frac{d}{dx_0}\exp^\varepsilon(t, x_0, \alpha^{\varepsilon\prime}(x_0)), f^\varepsilon(x_0)\right)$ vanishes for some time, close to $\tau_{s+1}^\varepsilon(x_0)$. By Definition 3.1.14, this time $t_c^\varepsilon(x_0)$ is a geometrically conjugate time, and when ε tends to 0, $t_c^\varepsilon(\hat{x}_0)$ converges to the bang-bang conjugate time $t_c = \tau_{s+1}(\hat{x}_0)$. This ends the proof of the Theorem 3.2.1.

3.3 Examples

In this section we illustrate Theorem 3.2.1 with two examples of minimal time control problems.

3.3.1 First example: Rayleigh minimal time control problem

We consider the minimal time control problem for the Rayleigh control system (see e.g. [76,81]),

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t),\\
\dot{x}_2(t) &= -x_1(t) + x_2(t)(1.4 - 0.14x_2(t)^2) + u_1(t),
\end{aligned} \tag{3.13}$$

with the control constraint

$$|u_1(t)| \leq 4, \quad \forall t \in [0, t_f] \tag{3.14}$$

and with boundary conditions given by

$$x_1(0) = -4,\ x_2(0) = -3,\ x_1(t_f) = x_2(t_f) = 0\,. \tag{3.15}$$

According to the Pontryagin maximum principle, any optimal solution $\hat{x}(\cdot)$ of (3.13)–(3.15) is the projection of an extremal $(\hat{x}(\cdot), \hat{p}(\cdot), p^0, \hat{u}_1(\cdot))$ such that

$$\begin{aligned}\dot{\hat{p}}_1(t) &= \hat{p}_2(t)\\ \dot{\hat{p}}_2(t) &= -\hat{p}_1(t) - \hat{p}_2(t)\left(1.4 - 0.42\hat{x}_2(t)^2\right)\end{aligned} \tag{3.16}$$

and the maximization condition $\hat{p}_2(t)\hat{u}_1(t) = \max_{|w|\le 4}\left(\hat{p}_2(t)w\right)$ holds almost everywhere on $[0, t_f]$. It is easy to see that $\hat{p}_2(\cdot)$ cannot vanish on some subinterval, and it follows that the optimal control $\hat{u}_1(\cdot)$ is bang-bang, equal to $\hat{u}_1(t) = 4\,\mathrm{sign}(\hat{p}_2(t))$. Applying a shooting method to problem (3.13)–(3.15) (with $p^0 = -1$), we determine the initial adjoint vector $\hat{p}(0) \simeq (0.53095052; 0.34206485)$, and observe that the trajectory has only one switching time $\hat{\tau}_1 \simeq 0.57613128$ on $[0, t_f]$, that is, $\hat{u}_1(\cdot)$ is given by

$$\hat{u}_1(t) = \begin{cases} +4 & \text{for } \ 0 \le t \le \hat{\tau}_1 \\ -4 & \text{for } \ \hat{\tau}_1 \le t \le t_f, \end{cases}$$

with a final time $t_f \simeq 2.97812917$ (see Figures 3.3 and 3.4). Furthermore, $\hat{x}(\cdot)$ is the unique minimal time solution and has a unique extremal lift (up to a multiplicative scalar), which is moreover normal.

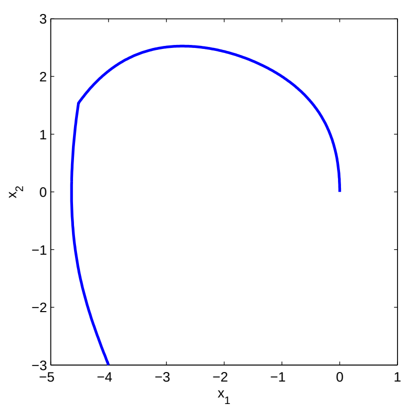

Figure 3.3: Optimal trajectory

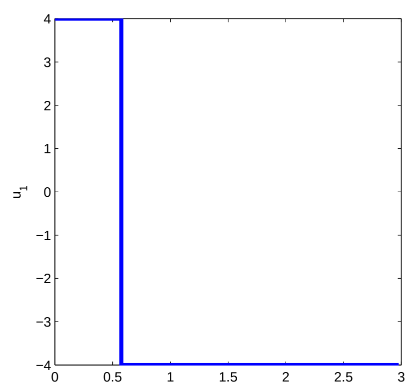

Figure 3.4: Optimal control

Prolongating the trajectory $\hat{x}(\cdot)$ to the interval $[0, 4]$, we observe a second switching time at $\hat{\tau}_2 \simeq 3.14750955$.

Notice that the second-order sufficient conditions of [78–81] are satisfied before $\hat{\tau}_2$, confirming the local optimality status of the trajectory, but are no longer satisfied beyond this second switching time; we can thus expect the trajectory not to be locally optimal beyond $\hat{\tau}_2$ (see Appendix A). To investigate this optimality status we use the extremal field approach.

From Theorem 3.1.10 and Remark 3.1.11, the first conjugate point $\hat{x}(t_c)$ is an overlap point of the extremal field emanating from the horizontal one-dimensional manifold $X = \{x_0 \in \mathcal{O} \mid \max_{|w|\leq 1} H(x_0, \alpha'(x_0), -1, w) = 0\}$. In practice, the function α is not known, and we rather use the field of extremals emanating from the vertical manifold $X_p = \{p_0 \in \mathcal{O}_p \mid \max_{|w|\leq 1} H(\hat{x}_0, p_0, -1, w) = 0\}$ (see [15,95]), where $\mathcal{O}_p$ is a neighborhood of the initial value of the adjoint vector $\hat{p}(0)$. The characterization in terms of fold point still holds for this vertical manifold (see [95]). We observe on Figures 3.5 and 3.6 that this field of extremals reflects off the switching surface at the second switching time; the point $\hat{x}(\hat{\tau}_2)$ is a fold point and the first conjugate time is equal to the second switching time, $t_c = \hat{\tau}_2 \simeq 3.14750955$.

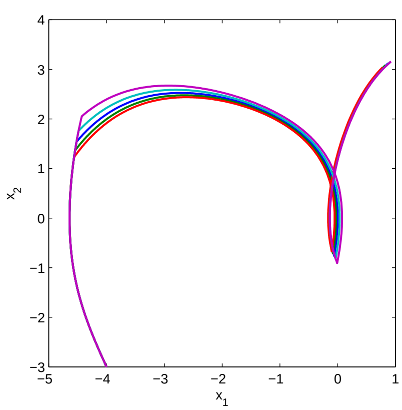

Figure 3.5: Extremal field for $t \in [0,4]$

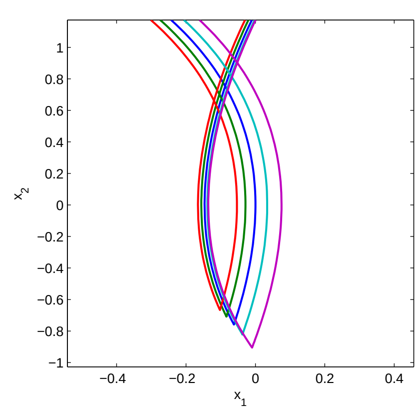

Figure 3.6: Overlap of the flow

We next propose a regularization procedure, for which we compute the first geometric conjugate time t_c^ε and check that it indeed converges to the first conjugate time t_c of the bang-bang case as ε tends to 0.

We consider the regularized control system

$$\begin{aligned} \dot{x}_1^\varepsilon(t) &= x_2^\varepsilon(t) + \varepsilon u_2^\varepsilon(t), \\ \dot{x}_2^\varepsilon(t) &= -x_1^\varepsilon(t) + x_2^\varepsilon(t)(1.4 - 0.14 x_2^\varepsilon(t)^2) + u_1^\varepsilon(t), \end{aligned} \tag{3.17}$$

with the boundary conditions (3.15), and where the control $u^\varepsilon(\cdot) = (u_1^\varepsilon(\cdot), u_2^\varepsilon(\cdot))$ satisfies the constraint

$$(u_1^\varepsilon(t))^2 + (u_2^\varepsilon(t))^2 \leq 16\,, \quad \forall t \in [0, t_f^\varepsilon]. \tag{3.18}$$

Any optimal solution $\hat{x}^\varepsilon(\cdot)$ of (3.15), (3.17) and (3.17) is the projection of an extremal

$(\hat{x}^\varepsilon(\cdot), \hat{p}^\varepsilon(\cdot), p^{0\varepsilon}, \hat{u}^\varepsilon(\cdot))$ such that

$$\dot{\hat{p}}_1^\varepsilon(t) = \hat{p}_2^\varepsilon(t)$$
$$\dot{\hat{p}}_2^\varepsilon(t) = -\hat{p}_1^\varepsilon(t) - \hat{p}_2^\varepsilon(t)\left(1.4 - 0.42\hat{x}_2^\varepsilon(t)^2\right).$$

The Assumption 2.4.1 is verified, and the controls that satisfy the Pontryagin maximization condition (2.10) are given by

$$\hat{u}_1^\varepsilon(t) = \frac{4\hat{p}_2^\varepsilon(t)}{\sqrt{(\hat{p}_2^\varepsilon(t))^2 + \varepsilon^2(\hat{p}_1^\varepsilon(t))^2}}, \quad \hat{u}_2^\varepsilon(t) = \frac{4\varepsilon\hat{p}_1^\varepsilon(t)}{\sqrt{(\hat{p}_2^\varepsilon(t))^2 + \varepsilon^2(\hat{p}_1^\varepsilon(t))^2}}. \tag{3.19}$$

Applying a shooting method to this problem, we determine the optimal trajectory of the regularized problem, and we indeed observe the expected convergence of $(\hat{x}^\varepsilon(\cdot), \hat{p}^\varepsilon(\cdot), -1, \hat{u}^\varepsilon)$ towards $(\hat{x}(\cdot), \hat{p}(\cdot), -1, \hat{u}_1)$, as ε tends to 0, in agreement with Theorem 2.5.1 (see Figures 3.7–3.9).

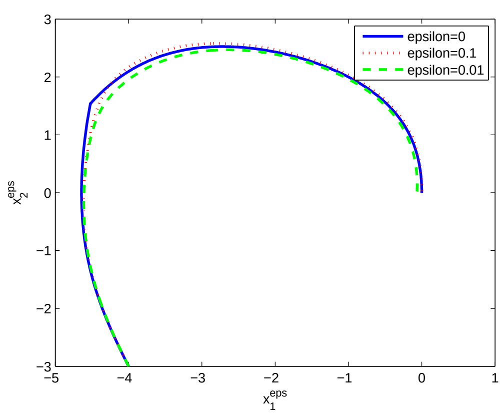

Figure 3.7: Trajectory

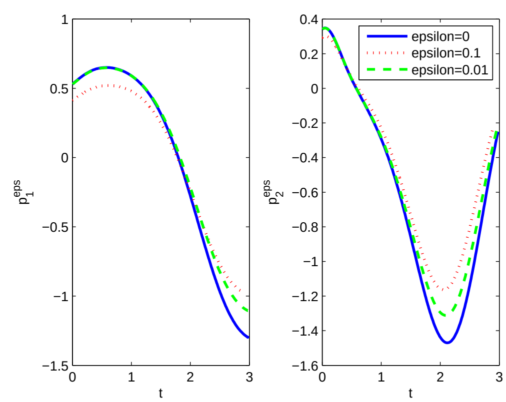

Figure 3.8: Adjoint vector

The optimal controls (3.19) are smooth functions of t, therefore the algorithms presented in [15] to compute the first conjugate time along a smooth extremal curve can be applied. Here we will apply the test for conjugate times explained in [15] when the final time is free and the extremal is normal. Let us briefly recall this test. The maximized Hamiltonian writes as

$$H_r^\varepsilon(\hat{x}^\varepsilon, \hat{p}^\varepsilon) = \hat{p}_1^\varepsilon \left(\hat{x}_2^\varepsilon + \frac{4\varepsilon^2 \hat{p}_1^\varepsilon}{\sqrt{(\hat{p}_2^\varepsilon)^2 + \varepsilon^2(\hat{p}_1^\varepsilon)^2}} \right) + \hat{p}_2^\varepsilon \left(-\hat{x}_1^\varepsilon + \hat{x}_2^\varepsilon(1.4 - 0.14(\hat{x}_2^\varepsilon)^2) + \frac{4\hat{p}_2^\varepsilon}{\sqrt{(\hat{p}_2^\varepsilon)^2 + \varepsilon^2(\hat{p}_1^\varepsilon)^2}} \right) - 1.$$

The aim is to compute the solution $Z^\varepsilon(\cdot) = (\delta x_1^\varepsilon(\cdot), \delta x_2^\varepsilon(\cdot), \delta p_1^\varepsilon(\cdot), \delta p_2^\varepsilon(\cdot))^T$ of the so-called

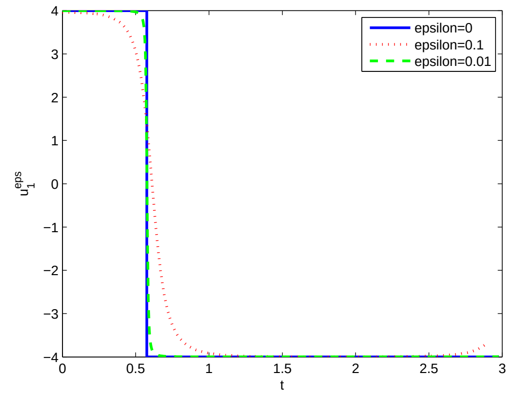

Figure 3.9: Control

variational system $\dot{Z}^{\varepsilon}(t) = V(t)Z^{\varepsilon}(t)$ along the extremal $(\hat{x}^{\varepsilon}(\cdot), \hat{p}^{\varepsilon}(\cdot))$, where

$$V(t) = \begin{pmatrix} \frac{\partial^2 H_r^{\varepsilon}}{\partial x \partial p}(\hat{x}^{\varepsilon}(t), \hat{p}^{\varepsilon}(t)) & \frac{\partial^2 H_r^{\varepsilon}}{\partial p^2}(\hat{x}^{\varepsilon}(t), \hat{p}^{\varepsilon}(t)) \\ -\frac{\partial^2 H_r^{\varepsilon}}{\partial x^2}(\hat{x}^{\varepsilon}(t), \hat{p}^{\varepsilon}(t)) & -\frac{\partial^2 H_r^{\varepsilon}}{\partial x \partial p}(\hat{x}^{\varepsilon}(t), \hat{p}^{\varepsilon}(t)) \end{pmatrix}$$

with initial conditions $(\delta x_1^{\varepsilon}(0), \delta x_2^{\varepsilon}(0)) = (0,0)$ and $(\delta p_1^{\varepsilon}(0), \delta p_2^{\varepsilon}(0))$ such that the scalar product $\langle (f_1^{\varepsilon}(0), f_2^{\varepsilon}(0)), (\delta p_1^{\varepsilon}(0), \delta p_2^{\varepsilon}(0)) \rangle$ is equal to 0, where $(f_1^{\varepsilon}, f_2^{\varepsilon})$ is the dynamics, given by

$$\begin{cases} f_1^{\varepsilon}(t) = x_2^{\varepsilon}(t) + \frac{4\varepsilon^2 p_1^{\varepsilon}(t)}{\sqrt{(p_2^{\varepsilon}(t))^2 + \varepsilon^2 (p_1^{\varepsilon}(t))^2}}, \\ f_2^{\varepsilon}(t) = -x_1^{\varepsilon}(t) + x_2^{\varepsilon}(t)(1.4 - 0.14 x_2^{\varepsilon}(t)^2) + \frac{4 p_2^{\varepsilon}(t)}{\sqrt{(p_2^{\varepsilon}(t))^2 + \varepsilon^2 (p_1^{\varepsilon}(t))^2}}. \end{cases}$$

The first geometric conjugate time is then the first positive zero of the function

$$t \mapsto \det\left(\delta x_1^{\varepsilon}(t)\ \delta x_2^{\varepsilon}(t), f_1^{\varepsilon}(t)\ f_2^{\varepsilon}(t)\right)$$

(see Figure 3.10).

We report on Table 3.3.1 the values of the first geometric conjugate time of the optimal trajectory $\hat{x}^{\varepsilon}(\cdot)$, for different values of ε. We observe that, as expected, t_c^{ε} converges to $t_c \simeq 3.14750955$ as ε tends to 0.

Another possible test (see [15]) is to compute numerically solutions

$$Z_i(\cdot) = (\delta x_{1i}^{\varepsilon}(\cdot), \delta x_{2i}^{\varepsilon}(\cdot), \delta p_{1i}^{\varepsilon}(\cdot), \delta p_{2i}^{\varepsilon}(\cdot)), \quad i = 1, 2,$$

of the variational system considered previously, with initial conditions $(\delta p_{11}^{\varepsilon}(0), \delta p_{21}^{\varepsilon}(0)) = (1,0)$ and $(\delta p_{12}^{\varepsilon}(0), \delta p_{22}^{\varepsilon}(0)) = (0,1)$, and then to compute the rank of the matrix

$$J^{\varepsilon}(t) = \begin{pmatrix} \delta x_{11}^{\varepsilon}(t) & \delta x_{21}^{\varepsilon}(t) \\ \delta x_{12}^{\varepsilon}(t) & \delta x_{22}^{\varepsilon}(t) \end{pmatrix}.$$

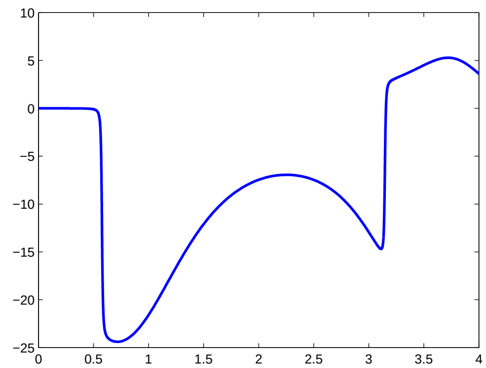

Figure 3.10: $\det\left(\delta x_1^\varepsilon(t)\ \delta x_2^\varepsilon(t), f_1^\varepsilon(t)\ f_2^\varepsilon(t)\right)$, $\varepsilon = 0.01$

ε	t_c^ε
0.1	3.26735859
0.01	3.1559626
0.001	3.14844987
0.0001	3.14760515

Table 3.1: Values of t_c^ε

This rank must be equal to 1 outside a conjugate time, and 0 at a conjugate time. In order to compute it, we use a singular value decomposition of $J^\varepsilon(t)$; then, a conjugate time occurs whenever the first singular value of $J^\varepsilon(t)$ vanishes (see Figure 3.11).

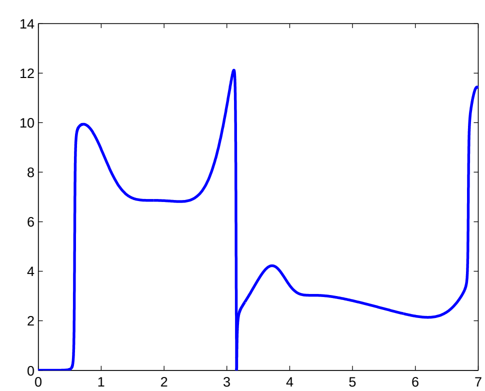

Figure 3.11: First singular value of $J^\varepsilon(t)$ ($\varepsilon = 0.01$)

In this first example, the first conjugate time t_c of the optimal bang-bang trajectory $\hat{x}(\cdot)$ coincides with the second switching time. We next provide an example where the first conjugate time is equal to the third switching time.

3.3.2 Second example

Consider the minimal time control problem for the control system

$$\begin{aligned} \dot{x}_1(t) &= \sin(x_2(t)), \\ \dot{x}_2(t) &= -\sin(x_1(t)) + u_1(t), \end{aligned} \tag{3.20}$$

with the control constraint

$$|u_1(t)| \leq 1\,, \quad \forall t \in [0, t_f]\,, \tag{3.21}$$

and with the boundary conditions

$$x_1(0) = x_2(0) = 0,\ x_1(t_f) = 2.9,\ x_2(t_f) = 0.1. \tag{3.22}$$

From the Pontryagin maximum principle, any optimal solution $\hat{x}(\cdot)$ of (3.20)–(3.22) is the projection of an extremal $(\hat{x}(\cdot), \hat{p}(\cdot), p^0, \hat{u}_1(\cdot))$ such that

$$\begin{aligned} \dot{\hat{p}}_1(t) &= \hat{p}_2(t)\cos(\hat{x}_1(t)), \\ \dot{\hat{p}}_2(t) &= -\hat{p}_1(t)\cos(\hat{x}_2(t)), \end{aligned}$$

and the maximization condition $\hat{p}_2(t)\hat{u}_1(t) = \max_{|w|\leq 1}(\hat{p}_2(t)w)$ must hold almost everywhere on $[0, t_f]$. It is easy to see that $\hat{p}_2(\cdot)$ cannot vanish on some subinterval, and it follows that the optimal control $\hat{u}_1(\cdot)$ is bang-bang, equal to $\hat{u}_1(t) = \mathrm{sign}(\hat{p}_2(t))$. Applying a shooting method to problem (3.20)–(3.22) (with $p^0 = -1$),we determine the initial adjoint vector $\hat{p}(0) = (-0.5, 1)$, and observe that the trajectory has one switching time $\hat{\tau}_1 \simeq 3.26174615$ on $[0, t_f]$, that is, $\hat{u}_1(\cdot)$ is given by

$$\hat{u}_1(t) = \begin{cases} +1 & \text{for } 0 \leq t \leq \hat{\tau}_1, \\ -1 & \text{for } \hat{\tau}_1 \leq t \leq t_f, \end{cases}$$

with a final time $t_f \simeq 4.07756604$ (see Figures 3.12 and 3.13). Furthermore, $\hat{x}(\cdot)$ is the unique minimal time solution and has a unique extremal lift (up to a multiplicative scalar), which is moreover normal.

Prolongating the trajectory $\hat{x}(\cdot)$ to the interval $[0, 11]$, we observe a second switching time at $\hat{\tau}_2 \simeq 6.21787838$, and a third one at $\hat{\tau}_3 \simeq 10.46930198$. Considering as in the previous example the extremal field emanating from the vertical manifold, we observe on Figures 3.14 and 3.15 that the extremal field crosses transversally the second switching surface, but reflects off the third switching surface, and it follows from Theorem 3.1.10 that the first conjugate time t_c is equal to $\hat{\tau}_3$.

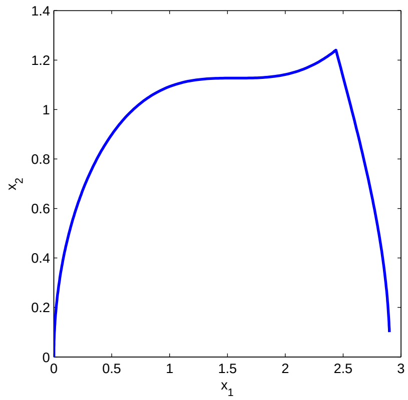

Figure 3.12: Optimal trajectory

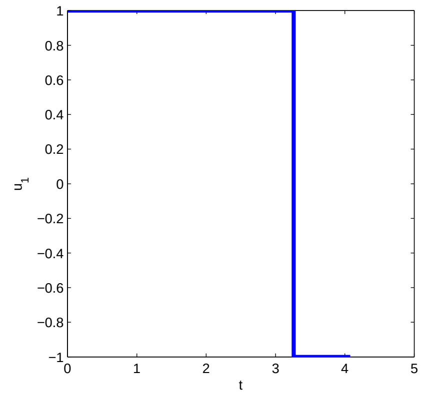

Figure 3.13: Optimal control

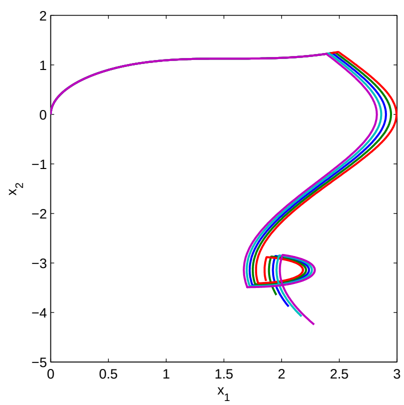

Figure 3.14: Extremal field, $t \in [0, 11]$

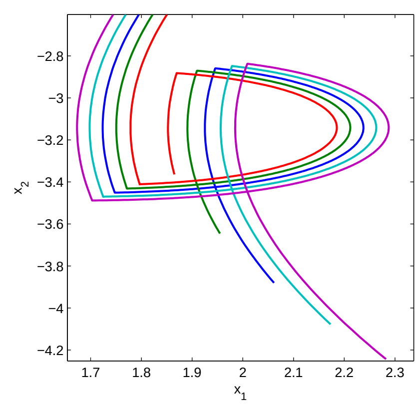

Figure 3.15: Zoom on the overlap of the flow at the third switching time

We propose the following regularization. Consider the control system

$$\begin{aligned} \dot{x}_1^\varepsilon(t) &= \sin(x_2^\varepsilon(t)) + \varepsilon u_2^\varepsilon(t), \\ \dot{x}_2^\varepsilon(t) &= -\sin(x_1^\varepsilon(t)) + u_1^\varepsilon(t), \end{aligned} \tag{3.23}$$

with the control constraint

$$(u_1^\varepsilon(t))^2 + (u_2^\varepsilon(t))^2 \leq 1\,, \quad \forall t \in [0, t_f^\varepsilon]\,, \tag{3.24}$$

and the initial and final conditions (3.22). Any optimal solution $\hat{x}^\varepsilon(\cdot)$ of (3.22)–(3.24) is the projection of an extremal $(\hat{x}^\varepsilon(\cdot), \hat{p}^\varepsilon(\cdot), p^{0^\varepsilon}, \hat{u}^\varepsilon(\cdot))$ such that

$$\begin{aligned} \dot{\hat{p}}_1^\varepsilon(t) &= \hat{p}_2^\varepsilon(t)\cos(\hat{x}_1^\varepsilon(t)), \\ \dot{\hat{p}}_2^\varepsilon(t) &= -\hat{p}_1^\varepsilon(t)\cos(\hat{x}_2^\varepsilon(t)), \end{aligned}$$

and the maximization condition implies that the extremal controls are given by

$$\hat{u}_1^\varepsilon(t) = \frac{\hat{p}_2^\varepsilon(t)}{\sqrt{(\hat{p}_2^\varepsilon(t))^2 + \varepsilon^2(\hat{p}_1^\varepsilon(t))^2}}, \quad \hat{u}_2^\varepsilon(t) = \frac{\varepsilon \hat{p}_1^\varepsilon(t)}{\sqrt{(\hat{p}_2^\varepsilon(t))^2 + \varepsilon^2(\hat{p}_1^\varepsilon(t))^2}}. \tag{3.25}$$

Applying a shooting method to this problem, we determine the optimal trajectory of the regularized problem, and we indeed observe the expected convergence of $(\hat{x}^\varepsilon(\cdot), \hat{p}^\varepsilon(\cdot), -1, \hat{u}^\varepsilon)$ towards $(\hat{x}(\cdot), \hat{p}(\cdot), -1, \hat{u}_1)$, as ε tends to 0, in agreement with Theorem 2.5.1 (see Figures 3.16–3.18).

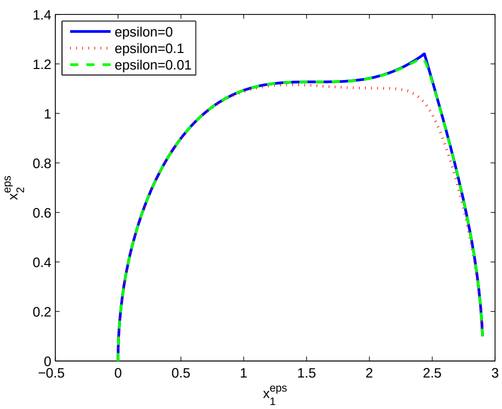

Figure 3.16: Trajectory

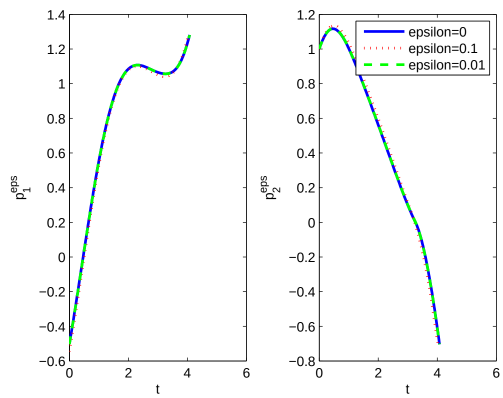

Figure 3.17: Adjoint vector

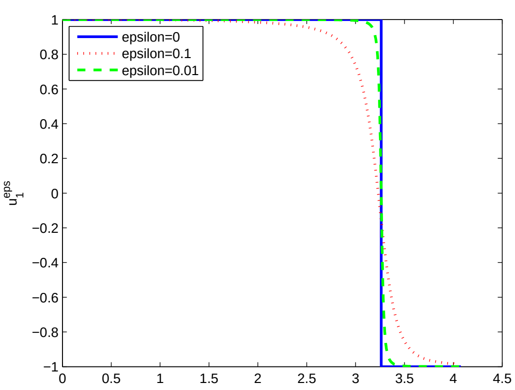

Figure 3.18: Control

As in the previous example, the controls (3.25) are smooth functions of t, and we apply the algorithm described in [15], computing as before the determinant $\det\left(\delta x_1^\varepsilon(t)\ \delta x_2^\varepsilon(t), f_1^\varepsilon(t)\ f_2^\varepsilon(t)\right)$ (see Figure 3.19). We report on Table 3.3.2 the values of the first geometric conjugate time

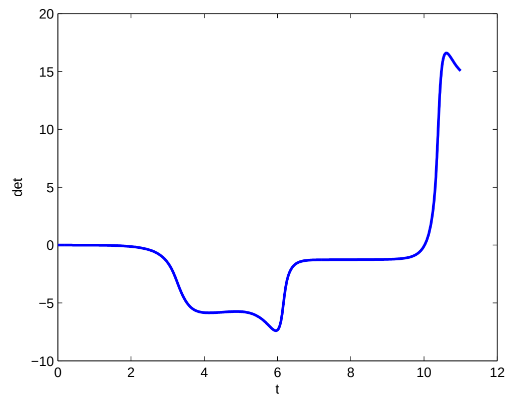

Figure 3.19: $\det\left(\delta x_1^\varepsilon(t)\ \delta x_2^\varepsilon(t), f_1^\varepsilon(t)\ f_2^\varepsilon(t)\right)$, $\varepsilon = 0.1$

of the optimal trajectory $\hat{x}^\varepsilon(\cdot)$, for different values of ε. We observe that, as expected, t_c^ε converges to t_c as ε tends to 0.

ε	t_c^ε
0.1	10.01593283
0.01	10.3164905
0.001	10.41858121
0.0001	10.45291892
0.00001	10.46419119

Table 3.2: Values of t_c^ε

Remark 3.3.1. We observe on both previous examples that it is not needed to consider very small values of ε to estimate the first conjugate time t_c. Indeed, a conjugate time of a locally bang-bang trajectory can only occur at a switching time (see Remark 3.1.8) and, under our assumptions, switching times are isolated (see Remark 3.1.6). From Theorem 3.2.1, the first geometric conjugate time t_c^ε converges to t_c, when ε tend to 0. Therefore, as soon as ε is small enough so that t_c^ε is in a (not necessarily so small) neighborhood of some switching time $\hat{\tau}_s$ of the bang-bang trajectory $\hat{x}(\cdot)$, this means that the bang-bang conjugate time t_c is equal to that switching time $\hat{\tau}_s$.

Conclusion and open problems

In this PhD thesis we focused on the problem of determining an efficient procedure to compute the first conjugate time t_c for the minimal time problem for single-input control-affine systems $\dot{x} = X(x) + u_1 Y_1(x)$ in $\mathbb{R}^n$ with the control constraint $|u_1(t)| \leq 1$, for every $t \in [0, t_f]$.

We proposed a smoothing procedure which consists in adding new smooth vector fields $Y_2, \ldots, Y_m$ and a small parameter $\varepsilon > 0$, so as to come up with the minimal time problem for the system $\dot{x} = X(x) + u_1^\varepsilon Y_1(x) + \varepsilon \sum_{i=2}^m u_i^\varepsilon Y_i(x)$, under the control constraint $\sum_{i=1}^m (u_i^\varepsilon(t))^2 \leq 1$, with the same boundary conditions as the initial problem. We proved, under appropriate assumptions, that the optimal controls of the latter problem, depending on ε, are smooth functions of t, and converge weakly to the optimal control of the initial system; moreover the associated trajectories converge uniformly. If the optimal control of the initial system is moreover bang-bang, then the convergence of the regularized control holds almost everywhere; this property may however fail whenever the bang-bang property does not hold. We provided examples and counterexamples to illustrate our result. Moreover, we proved that the first geometric conjugate time of regularized problem converges to the first conjugate time initial problem, when ε tends to 0. This convergence result, allowed us to use theoretical and practical results for the conjugate time theory that are well known in the smooth case and apply them to the regularized problem in order to compute, consequently, conjugate times of the initial bang-bang problem. Note that our results still hold if the control-affine system is considered on a manifold (in this work we considered $\mathbb{R}^n$ for the sake of simplicity).

An open question is to extend the results proved in Chapters 2 and 3 to general nonlinear control systems. In our point of view, this extension seems difficult, because it may be not obvious to generalize the nice expression (2.11) (see Chapter 2, Section §2.4) to more general situations and, on the other hand, Lemma 2.5.6 does not hold a priori for general control systems, moreover, it is not clear how to derive Lemma 2.5.7 and the subsequent results. Although, it would be interesting if we could extend our results to multi-input control-affine systems $\dot{x} = X(x) + \sum_{i=1}^p u_i Y_i(x)$ in $\mathbb{R}^n$, where $u = (u_1, ..., u_p) \in L^\infty([0, t_f], \Delta)$ and Δ is a polyhedron (see [95]), or a convex polyhedron (see [81]), or a convex compact polyhedron (see [100]) of $\mathbb{R}^p$. For $p > 1$, it would be interesting to consider the case where multiple switching times may occur, that is, when at least two control functions switch at the same time. Another open question concerns the generalization to general cost functions.

Appendix A

First and second order sufficient optimality conditions in the normal case

First and second order necessary and/or sufficient optimality conditions have a crucial role in the study of first conjugate times for bang-bang minimal time optimal control problems with control-affine systems. In [3, 5, 81, 95, 100] first and second order necessary and/or sufficient optimality conditions are given in terms of a quadratic form Q_t. As we recalled in Section §3.1.3 of Chapter 3 the quadratic form in [100] is a lower bound for the one given in [81], and in certain cases the quadratic form in [81] is equivalent to the one in [5] (see [90]). In [95] an analogous quadratic form to the one in [5] is defined. Here we recall the first and second order sufficient optimality conditions given in [81] and apply them to the Rayleigh minimal time control problem with fixed initial and final conditions (see Sections §2.6.1 and §3.3.1).

In [78–81] sufficient optimality conditions are provided for a minimal time problem for multi-input control-affine systems in $\mathbb{R}^n$

$$\dot{x} = X(t,x) + \sum_{i=1}^{p} u_i Y_i(t,x)$$

with fixed initial and final conditions

$$x(0) = x_0\,, \quad x(t_f) = x_1\,,$$

and where $u = (u_1, ..., u_p) \in L^\infty([0,t_f],\Delta)$ and Δ is a convex polyhedron of $\mathbb{R}^p$.

Here we will formulate the first and second order sufficient optimality conditions given in [78–81, 86] for the optimal control problem **(OCP)** considered in Chapters 2 and 3. The optimal control problem **(OCP)** consists of determining a solution $x(\cdot)$ associated to a control $u_1(\cdot)$, on $[0,t_f]$, satisfying the single-input control-affine system in $\mathbb{R}^n$

$$\dot{x} = X(x) + u_1 Y_1(x), \tag{A.1}$$

where X and Y_1 are smooth vector fields, the constraint

$$|u_1(t)| \leq 1\,, \quad \forall t \in [0, t_f]\,,$$

and steering $x_0 = x(0)$ to $x_1 = x(t_f)$ in minimal time t_f.

From Pontryagin maximum principle, there exists a non trivial absolutely continuous mapping $p(\cdot) : [0, t_f] \to \mathbb{R}^n$ (adjoint vector) and a real number $p^0 \leq 0$, with $(p(\cdot), p^0) \neq (0, 0)$, such that

$$\dot{p}(t) = -\left\langle p(t), \frac{\partial X}{\partial x}(x(t))\right\rangle - u_1(t)\left\langle p(t), \frac{\partial Y_1}{\partial x}(x(t))\right\rangle \tag{A.2}$$

where the Hamiltonian function is given by

$$H(x, p, p^0, u_1) = \langle p, f(x, u_1)\rangle = \langle p, X(x) + u_1 Y_1(x)\rangle + p^0\,,$$

and the maximization condition

$$H(x(t), p(t), p^0, u_1(t)) = \max_{|w|\leq 1} H(x(t), p(t), p^0, w) \tag{A.3}$$

holds almost everywhere on $[0, t_f]$. Moreover, $\max_{|w|\leq 1} H(x(t), p(t), p^0, w) = 0$ for every $t \in [0, t_f]$.

It follows from (A.3) that

$$u_1(t) = \operatorname{sign}\langle p(t), Y_1(x(t))\rangle$$

for almost every t, provided that the (continuous) switching function

$$\varphi_1(t) = \langle p(t), Y_1(x(t))\rangle$$

does not vanish on any subinterval of $[0, t_f]$.

Here we will only consider the case where the Pontryagin extremal $(x(\cdot), p(\cdot), p^0, u_1(\cdot))$ is normal ($p^0 = -1$). The abnormal case is also considered in [78–81, 86].

The extremal $(x(\cdot), p(\cdot), -1, u_1(\cdot))$ may be extended forward on a maximal time interval I of $[0, +\infty)$, containing $[0, t_f]$ (see Section §3.1.2, Chapter 3). Let the Assumption 3.1.1 hold, that is assume that the extremal $(x(\cdot), p(\cdot), -1, u_1(\cdot))$ is *bang-bang* on the interval I, i.e., the switching function φ_1 does not vanish on any subinterval of I. Let $\tau_1, \ldots, \tau_s$ be the switching times of the bang-bang trajectory $x(\cdot)$, that is, $\tau_1, \ldots, \tau_s$ are zeros of φ_1 on I, such that $0 < \tau_1 < \ldots < \tau_s$. There holds

$$u_1(t) = \begin{cases} 1 & \text{if } \ \varphi_1(t) > 0, \\ -1 & \text{if } \ \varphi_1(t) < 0, \end{cases}$$

for every $t \in I$.

For $j = 1, ..., s$, let $u_1(\tau_j^-) = u_1(\tau_j - 0)$ and $u_1(\tau_j^+) = u_1(\tau_j + 0)$ be, respectively, the left-hand and the right-hand side of the control $u_1(t)$ at τ_j.

Critical subspace. Let us now introduce the critical subspace $\mathcal{K}$.

Denote by $P_\theta C^1([0,t_f],\mathbb{R}^n)$ be the space of piecewise continuous functions

$$\bar{x}(\cdot)\,:\,[0,t_f]\to\mathbb{R}^n$$

that are continuously differentiable on each interval of the set $[0,t_f]\backslash\theta$, where $\theta=\{\tau_1,...,\tau_s\}$ is the set of switching times. Putting

$$\bar{z}=(\bar{t}_f,\xi,\bar{x})\ \text{ with } \bar{t}_f\in\mathbb{R}\,,\ \ \xi=(\xi_1,...,\xi_s)\in\mathbb{R}^s\,,\ \ \bar{x}\in P_\theta C^1([0,t_f],\mathbb{R}^n)\,,$$

we have

$$\bar{z}\in\mathcal{Z}(\theta)=\mathbb{R}\times\mathbb{R}^s\times P_\theta C^1([0,t_f],\mathbb{R}^n)\,.$$

Let $\mathcal{K}$ be the set of all $\bar{z}\in\mathcal{Z}(\theta)$ satisfying the following conditions

$$\dot{\bar{x}}(t)=f_x(x(t),u_1(t))\bar{x}(t)\,,\ \ \bar{x}(\tau_k^+)-\bar{x}(\tau_j^-)=\Big(\dot{x}(\tau_j^+)-\dot{x}(\tau_j^-)\Big)\,\xi_j\,,\ \ j=1,...,s\,,$$
$$\bar{x}(0)=0\,,\ \ \bar{x}(t_f)=0\,.$$

The set $\mathcal{K}$ is a finite-dimensional subspace of $\mathcal{Z}(\theta)$ and is called the *critical subspace*.

There holds $\bar{x}(t)\equiv 0$ on $[0,\tau_1)$ and $(\tau_s,t_f]$. Thus, $\bar{x}(\tau_1^-)=\bar{x}(\tau_s^+)=0$, for all $\bar{z}\in\mathcal{K}$.

Consider the variational (linearized) system

$$\dot{y}=f_x(t)y$$

and for each $j=1,...,s$, define the vector functions $y^j(t)$ as the solutions of the system

$$\dot{y}=f_x(t)y\,,\ \ y(\tau_j)=(\dot{x}(\tau_j^+)-\dot{x}(\tau_j^-))\,,\ \ t\in[\tau_j,t_f]\,.$$

For $t<\tau_j$ put $y^j(t)=0$ which yields $y^j(\tau_j^+)-y^j(\tau_j^-)=\dot{x}(\tau_j^+)-\dot{x}(\tau_j^-)$. Denote by $x(t,\tau_1,...,\tau_s)$ the solution of (A.1) associated to the bang-bang optimal control with switching times $\tau_1,...,\tau_s$. The derivatives of the trajectories $x(t,\tau_1,...,\tau_s)$ with respect to the switching times are given by

$$\frac{\partial x}{\partial t_j}(t,\tau_1,...,\tau_s)=-y^j(t)\ \text{ for }\ t\geq t_j\,,\ \ j=1,...,s\,.$$

Proposition A.0.2. *[78–81, 86] Assume that one of the following conditions are satisfied (for $p^0=-1$):*

(a) the s vectors $y^j(t_f)=-\frac{\partial x}{\partial t_j}(t_f)$, $j=1,...s$, are linearly independent[1]*,*

(b) the bang-bang control has one switching time, i.e., $s=1$.

Then the critical subspace is $\mathcal{K}=\{0\}$.

[1]If the abnormal case is considered then another condition that implies $\mathcal{K}=\{0\}$ is *the $s+1$ vectors $y^j(t_f)=-\frac{\partial x}{\partial t_j}(t_f)$, $j=1,...,s$, $\dot{x}(t_f)$, are linearly independent,*

Quadratic form. Let $(x(\cdot), p(\cdot), -1, u_1(\cdot))$ be a Pontryagin extremal, and $\bar{z} \in \mathcal{Z}$. Define

$$Q_t(p, \bar{z}) = \sum_{j=1}^{s} \left(\left(-\dot{\varphi}(\tau_j)(u_1(\tau_j^+) - u_1(\tau_j^-)) \right) \xi_j^2 + 2 \left(\frac{\partial H}{\partial x}(\tau_j^+) - \frac{\partial H}{\partial x}(\tau_j^+) \right) \frac{1}{2}(\bar{x}(\tau_j^-) + \bar{x}(\tau_j^-))\xi_j \right) \\ + \int_{\tau_1}^{\tau_s} \langle \frac{\partial^2 H}{\partial x^2}(t)\bar{x}(t), \bar{x}(t) \rangle dt \,. \tag{A.4}$$

A stronger version of the next theorem is given in [81, 86] where the abnormal case is also considered.

Theorem A.0.3. *[78–81, 86] Let $(x(\cdot), p(\cdot), -1, u_1(\cdot))$ be a normal extremal for the problem (**OCP**) on $[0, t_f]$, such that*

(a) $u_1(\cdot)$ is a bang-bang control, that is, the Assumption 3.1.1 holds;

(b) the strict bang-bang Legendre condition holds, that is, $\dot{\varphi}(\tau_j) \neq 0$ for $j = 1, ..., s$ (see Chapter 3);

(c) $\max_p Q_t(p, \bar{z}) > 0 \quad \forall \bar{z} \in \mathcal{K} \backslash \{0\}$.

Then $(x(\cdot), u_1(\cdot))$ is a strong local minimum.[2]

This theorem provides a second order sufficient condition for strong local optimality.

Remark A.0.4. If $\mathcal{K} = \{0\}$ then the condition (c) is automatically fulfilled. Therefore, the property $\mathcal{K} = \{0\}$ is a first order sufficient condition for strong local optimality.

Remark A.0.5. If there exists a vector $p(\cdot)$ solution of (A.1)–(A.2) such that

$$Q_t(p, \bar{z}) > 0 \quad \forall \bar{z} \in \mathcal{K} \backslash \{0\} \,,$$

then the condition (c) is satisfied.

The next theorem follows from Proposition A.0.2 and Theorem A.0.3 and it provides a sufficient condition for bang-bang control with one switching time.

Theorem A.0.6. *[78–81, 86] Let $(x(\cdot), p(\cdot), -1, u_1(\cdot))$ be a normal extremal for the problem (**OCP**) on $[0, t_f]$, such that*

(a) $u_1(\cdot)$ is a bang-bang control with one switching point;

(b) $-\dot{\varphi}(\tau_1) \left(u_1(\tau_1^+) - u_1(\tau_1^-) \right) < 0$.

Then $(x(\cdot), u_1(\cdot))$ is a strong local minimum.[3]

[2] In fact, $(x(\cdot), u_1(\cdot))$ is a strict strong local minimum.
[3] In fact, $(x(\cdot), u_1(\cdot))$ is a strict strong local minimum.

For the case of two switching times, assume that $\dot{x}(\tau_1^+)-\dot{x}(\tau_1^-) \neq 0$ and $\dot{x}(\tau_2^+)-\dot{x}(\tau_2^-) \neq 0$. This imply that $y^1(t_f)=0$ and $y^2(t_f)=0$ where y^1 (respectively y^2) is the solution of

$$\dot{y} = f_x(t)y\,, \;\; y(\tau_1) = \dot{x}(\tau_1^+) - \dot{x}(\tau_1^-)\,, \;\; t \in [\tau_1, t_f]$$

(respectively, $\dot{y} = f_x(t)y\,, \;\; y(\tau_2) = \dot{x}(\tau_2^+) - \dot{x}(\tau_2^-)\,, \;\; t \in [\tau_2, t_f]$). From the superposition principle for linear ordinary differential equations there holds

$$\bar{x}(t) = \sum_{j=1}^{2} y^j(t)\xi_j\,,$$

therefore,

$$0 = \bar{x}(t_f) = y^1(t_f)\xi_1 + y^2(t_f)\xi_2\,. \tag{A.5}$$

Assume, furthermore that $\mathcal{K} \neq \{0\}$. Then from (A.5) the nonzero vectors $y^1(t_f)$ and $y^2(t_f)$ are collinear, i.e.,

$$y^2(t_f) = \alpha y^1(t_f) \tag{A.6}$$

for some $\alpha \neq 0$. The functions $y^1(t)$ and $y^2(t)$ are continuous solutions of the system $\dot{y} = f_x(t)y$ in $(\tau_2, t_f]$, thus the relation $y^2(t) = \alpha y^1(t)$ is valid for all $t \in (\tau_2, t_f]$. In particular, $y^2(\tau_2+0) = \alpha y^1(\tau_2)$ and thus

$$\dot{x}(\tau_2^+) - \dot{x}(\tau_2^-) = \alpha y^1(\tau_2)$$

which is equivalent to (A.6). From (A.5) and (A.6) there holds

$$\xi_2 = -\frac{1}{\alpha}\xi_1\,.$$

Using the previous formulas and $\frac{\partial H}{\partial x}(\tau_j^+) - \frac{\partial H}{\partial x}(\tau_j^-) = -\big(\dot{p}(\tau_1^+) - \dot{p}(\tau_1^-)\big)$, $j=1,2$, in the quadratic form (A.4) we have

$$Q = \rho\xi_1^2$$

where

$$\begin{aligned}\rho = &\big(-\varphi(\tau_1)\left(u_1(\tau_1^+) - u_1(\tau_1^+)\right) - (\dot{p}(\tau_1^+) - \dot{p}(\tau_1^-))(\dot{x}(\tau_1^+) - \dot{x}(\tau_1^-))\big) \\ &+ \frac{1}{\alpha^2}\left(-\varphi(\tau_2)\left(u_1(\tau_2^+) - u_1(\tau_2^+)\right) + (\dot{p}(\tau_2^+) - \dot{p}(\tau_2^-))(\dot{x}(\tau_2^+) - \dot{x}(\tau_2^-))\right) + \int_{t_1}^{t_2} \langle \frac{\partial^2 H}{\partial x^2} y^1, y^1 \rangle dt\,.\end{aligned} \tag{A.7}$$

Proposition A.0.7. *[78–81, 86] Let $(x(\cdot), p(\cdot), -1, u_1(\cdot))$ be a normal extremal for the problem (**OCP**) on $[0, t_f]$. Assume that $u_1(\cdot)$ has two switching times, $\dot{x}(\tau_1^+) - \dot{x}(\tau_1^-) \neq 0$, $\dot{x}(\tau_2^+) - \dot{x}(\tau_2^-) \neq 0$, and $y^2(t_f) = \alpha y^1(t_f)$ with some factor α. Then the condition of positive definiteness of Q on $\mathcal{K}$ is equivalent to the inequality $\rho < 0$, where ρ is defined by* (A.7).

A.1 Example: Rayleigh minimal time control problem

Consider the Rayleigh minimal time control problem considered in Section 3.3.1 in Chapter 3, for the control system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -x_1(t) + x_2(t)(1.4 - 0.14x_2(t)^2) + u_1(t),\end{aligned} \tag{A.8}$$

with the control constraint

$$|u_1(t)| \leq 4\,, \quad \forall t \in [0, t_f] \tag{A.9}$$

and with boundary conditions given by

$$x_1(0) = x_2(0) = x_0,\; x_1(t_f) = x_2(t_f) = x_1\,.$$

In [81] the authors consider Rayleigh minimal time control problem with the boundary conditions

$$x_1(0) = x_2(0) = -5\,,\;\; x_1(t_f) = x_2(t_f) = 0$$

and verified that Proposition A.0.7 is satisfied for the trajectory $x(\cdot)$ associated to the control

$$u(t) = \begin{cases} +4 \text{ for } 0 \leq t \leq \tau_1 \\ -4 \text{ for } \tau_1 \leq t \leq \tau_2 \\ +4 \text{ for } \tau_2 \leq t \leq t_f \end{cases}$$

where $\tau_1 \simeq 1.12$, $\tau \simeq 3.31$ are the switching times and $t_f \simeq 3.668$ is the minimal time (see Section §2.6.1 of Chapter 2).

Here we will consider the boundary conditions considered in Section 3.3.1 of Chapter 3, given by

$$x_1(0) = -4,\; x_2(0) = -3,\; x_1(t_f) = x_2(t_f) = 0\,. \tag{A.10}$$

According to the Pontryagin maximum principle, any optimal solution $x(\cdot)$ of (A.8)–(A.9), (A.10) is the projection of an extremal $(x(\cdot), p(\cdot), p^0, u_1(\cdot))$ such that

$$\begin{aligned}&-\frac{\partial H}{\partial x_1}(t) = \dot{p}_1(t) = p_2(t)\\ &-\frac{\partial H}{\partial x_2}(t)\dot{p}_2(t) = -p_1(t) - p_2(t)\left(1.4 - 0.42x_2(t)^2\right)\end{aligned}$$

and the maximization condition $p_2(t)u_1(t) = \max_{|w|\leq 4}(p_2(t)w)$ holds almost everywhere on $[0, t_f]$. The optimal control $u_1(\cdot)$ is bang-bang, equal to $u_1(t) = 4\,\mathrm{sign}(p_2(t))$.

Recall Section §3.3.1 where we applied a shooting method to problem (A.8)–(A.10) (with $p^0 = -1$), and determined the initial adjoint vector $p(0) \simeq (0.53095052; 0.34206485)$. We observe that the trajectory has only one switching time $\tau_1 \simeq 0.57613094$ on $[0, t_f]$, that is, $u_1(\cdot)$ is given by

$$u_1(t) = \begin{cases} +4 & \text{for } 0 \leq t \leq \tau_1 \\ -4 & \text{for } \tau_1 \leq t \leq t_f , \end{cases}$$

with a final time $t_f \simeq 2.97812917$ (see Figures 3.3 and 3.4, Chapter 3).

We will now apply the sufficient optimality condition Theorem A.0.6 and verify that this trajectory is optimal.

Integrating the system

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) + x_2(t)(1.4 - 0.42x_2^2(t)) + 4 \\ \dot{p}_1(t) = p_2(t) \\ \dot{p}_2(t) = -p_1(t) - p_2(t)(1.4 - 0.42x_2^2(t)) \end{cases} \tag{A.11}$$

in the interval $[0, \tau_1]$ (with $u_1(t) = +4$ and initial conditions $(x_1(0), x_2(0)) = (-4, -3)$ and $(p_1(0), p_2(0)) \simeq (0.53095052; 0.34206485))$ we have $(p_1(\tau_1), p_2(\tau_1)) \simeq (0.6504275; 0)$. Therefore,

$$-\dot{\varphi}(\tau_1) = -\dot{p}_2(\tau_1) = p_1(\tau_1) \simeq 0.6504275$$

and

$$-\dot{\varphi}(\tau_1)\left(u_1(\tau_1^+) - u_1(\tau_1^-)\right) \simeq 0.6504275 \cdot (-8) \simeq -5.20342003 < 0\,.$$

And from Theorem A.0.6 the trajectory $x(\cdot)$ associated to the control $u_1(\cdot)$ with one switching time $\tau_1 \simeq 0.5761$ and final time $t_f \simeq 2.9781$, is strong locally optimal on $[0, t_f]$.

Prolongating the trajectory $\hat{x}(\cdot)$ to the interval $[0, 4]$, we observe a second switching time at $\hat{\tau}_2 \simeq 3.1475101$. Let us apply the Proposition A.0.7 to the trajectory $x(\cdot)$ with two switching times.

For $j = 1, 2$ define the vector functions $y^j \in \mathbb{R}^n$ solution of the system

$$\begin{aligned} \dot{y}_1^j(t) &= y_2^j(t) \\ \dot{y}_2^j(t) &= -y_1^j(t) + (1.4 - 0.42x_2^2(t))y_2^j(t) \end{aligned}$$

with

$$\begin{aligned} \left(y_1^1(\tau_1), y_2^1(\tau_1)\right) &= \left(\dot{x}_1(\tau_1^+) - \dot{x}_1(\tau_1^-), \dot{x}_2(\tau_1^+) - \dot{x}_2(\tau_1^-)\right) \\ &= (0, -8)\,, \quad \text{for } t \in [\tau_1, t_f]\,, \end{aligned}$$

and

$$\begin{aligned} \left(y_1^2(\tau_1), y_2^2(\tau_1)\right) &= \left(\dot{x}_1(\tau_2^+) - \dot{x}_1(\tau_2^-), \dot{x}_2(\tau_2^+) - \dot{x}_2(\tau_2^-)\right) \\ &= (0, 8)\,, \quad \text{for } t \in [\tau_2, t_f]\,, \end{aligned}$$

Let us see if the vectors $(y_1^1(\tau_2), y_2^1(\tau_2))$ and $(y_1^2(\tau_2), y_2^2(\tau_2))$ are collinear, with $(y_1^2(\tau_2), y_2^2(\tau_2)) = (0, 8)$. To compute $(y_1^1(\tau_2), y_2^1(\tau_2))$ we integrate the system

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) + x_2(t)(1.4 - 0.42x_2^2(t)) - 4 \\ \dot{p}_1(t) = p_2(t) \\ \dot{p}_2(t) = -p_1(t) - p_2(t)(1.4 - 0.42x_2^2(t)) \\ \dot{y}_1^1(t) = y_2^1(t) \\ \dot{y}_2^1(t) = -y_1^1(t) + y_2^1(t)(1.4 - 0.42x_2^2(t)) \end{cases}$$

in the interval $[\tau_1, \tau_2]$ where $u_1(t) = -4$, and with initial conditions $(y_1^1(\tau_1), y_2^1(\tau_1)) = (0, -8)$ and $(x_1(\tau_1), x_2(\tau_1)) \simeq (-4.52075342; 1.53745036)$, $(p_1(\tau_1), p_2(\tau_1)) \simeq (0.6504275; 0)$ follows from integrating the system (A.11) in the interval $[0, \tau_1]$ (with $u_1(t) = +4$). We have $(y_1^1(\tau_2), y_2^1(\tau_2)) \simeq (0; 10.73906251)$. The vectors are indeed collinear, since $y^2(\tau_2) = \alpha y^1(\tau_2)$ with $\alpha \simeq 0.74494398$. We can proceed and compute ρ given by equation (A.7),

$$\begin{aligned} \rho &= \left(-\varphi(\tau_1)\left(u_1(\tau_1^+) - u_1(\tau_1^+)\right) - (\dot{p}(\tau_1^+) - \dot{p}(\tau_1^-))(\dot{x}(\tau_1^+) - \dot{x}(\tau_1^-))\right) \\ &\quad + \frac{1}{\alpha^2}\left(-\varphi(\tau_2)\left(u_1(\tau_2^+) - u_1(\tau_2^+)\right) + (\dot{p}(\tau_2^+) - \dot{p}(\tau_2^-))(\dot{x}(\tau_2^+) - \dot{x}(\tau_2^-))\right) + \int_{t_1}^{t_2} \langle \frac{\partial^2 H}{\partial x^2} y^1, y^1 \rangle dt \\ &= -5.20342003 + \frac{1}{0.74494398^2}\left(-\varphi(\tau_2)\left(u_1(\tau_2^+) - u_1(\tau_2^+)\right)\right) + \int_{t_1}^{t_2} \langle \frac{\partial^2 H}{\partial x^2} y^1, y^1 \rangle dt\,. \end{aligned}$$

We have

$$-\varphi(\tau_2)\left(u_1(\tau_2^+) - u_1(\tau_2^+)\right) = -\dot{p}_2(\tau_2)\cdot(4+4) = p_1(\tau_2)\cdot 8 \simeq -1.31854472\cdot 8 \simeq -10.54835772 < 0\,,$$

and

$$\frac{\partial^2 H}{\partial x^2}(t) = \begin{bmatrix} 0 & 0 \\ 0 & -0.84p_2(t)x_2(t) \end{bmatrix}.$$

Therefore,

$$\int_{\tau_1}^{\tau_2} -0.84p_2(t)x_2(t)(y_2^1(t))^2 dt = 27.66812969492819$$

and

$$\rho = -5.20342003874014 - \frac{10.54835772187732}{0.74494398284381^2} + 27.66812969492819 = 3.45665745601652 > 0\,.$$

The Proposition A.0.7 is not satisfied, although we can not assure that the trajectory is not longer locally optimal beyond $\hat{\tau}_2$. We confirmed this, using the extremal field approach, in Section §3.3.1 of Chapter 3.

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