Universidade de Aveiro Departamento de Matemática
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Continuous Wavelet Transforms on the Unit Sphere

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica de Uwe Kähler, Professor Auxiliar Convidado com Agregação do Departamento de Matemática da Universidade de Aveiro e co-orientação de Paula Cristina Supardo M. M. Cerejeiras, Professora Associada do Departamento de Matemática da Universidade de Aveiro.

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Dedico este trabalho à minha família pelo incansável apoio e compreensão.

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palavras-chave

Grupo de Lorentz, Transformações de Möbius, Girogrupos, Espaços Homogéneos, Secções, Transformadas Contínuas de Onduleta, Onduletas Esféricas Conformes, Anisotropia, Frames.

resumo

Esta tese estuda transformadas contínuas de onduleta na superfície esférica S^{n-1} a partir do seu grupo conforme, o grupo próprio de Lorentz $Spin^+(1,n)$. As transformações envolvidas são rotações do subgrupo Spin(n) e transformações de Möbius da forma $\varphi_a(x)=(x-a)(1+ax)^{-1}$, com a um ponto na bola unitária B^n. A nossa abordagem é desenvolvida a partir da teoria de representações de grupos, de quadrado integrável, em espaços homogéneos. O espaço homogéneo subjacente às transformadas contínuas de onduleta esféricas resulta de uma extensão adequada do espaço homogéneo obtido pela factorização do girogrupo da bola unitária por um dos seus giro-subgrupos. As transformadas contínuas de onduleta são definidas à custa da escolha de secções globais no espaço homogéneo. O caso isotrópico (introduzido por Antoine e Vandergheynst) corresponde às dilatações puras na superfície esférica, enquanto que o caso anisotrópico corresponde às dilatações conformes gerais. O comportamento local dessas dilatações é estudado em detalhe, dando origem à noção de anisotropia local. No final mostramos como construir frames para a superfície esférica S^2 e apresentamos um algoritmo para a reconstrução de sinais esféricos.

keywords

Lorentz group, Möbius transformations, Gyrogroups, Homogeneous spaces, Sections, Spherical Continuous Wavelet Transforms, Spherical Conformlets, Anisotropy, Frames.

abstract

This thesis studies continuous wavelet transforms on the unit sphere S^{n-1} based on its conformal group, the Lorentz group $Spin^+(1,n)$. The transformations involved are rotations of the subgroup Spin(n) and Möbius transformations of the form $\varphi_a(x) = (x-a)(1+ax)^{-1}$, where a is a point of the open unit ball B^n. Our approach is based on the group-theoretical background of square integrable group representations over homogeneous spaces. For the spherical continuous wavelet transforms (SCWT) the underlying homogeneous space results from an appropriate extension of the homogeneous space obtained from the factorization of the gyrogroup of the unit ball by one of its gyro-subgroups. The SCWT arise from the choice of global sections on the homogeneous space. The isotropic case (introduced by Antoine and Vandergheynst) corresponds to pure dilations on the unit sphere while the anisotropic case corresponds to general conformal dilations. The local behaviour of these dilations is investigated in detail giving rise to the notion of local anisotropy. In the end we show how to construct frames on the unit sphere S^2 and we present a numerical algorithm for the reconstruction of spherical signals.

Contents

Introduction

“The real voyage of discovery consists not in seeking new landscapes, but in having new eyes.”
Marcel Proust

Over the last few decades the field of signal processing has weathered a revolution. Techniques that previously dominated the field such as Fourier transforms now have to compete with many other integral transforms and in particular with wavelet transforms.

Wavelet analysis is a particular time or space scale representation of signals which has found a wide range of applications in signal processing, physics and applied mathematics.

The concept of wavelets can be viewed as a synthesis of ideas which originated during the last forty years in engineering (subband coding), physics (coherent states), and pure mathematics (study of Calderón-Zygmund operators).

In one of the papers initiating the study of the continuous wavelet transform on the real line Grossman, Morlet and Paul [43] considered systems $(\psi_{(a,b)})_{(a,b)\in\mathbb{R}^+\times\mathbb{R}}$ arising from a single function $\psi \in L^2(\mathbb{R})$ via $\psi_{(a,b)} = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right)$. They showed that every function $\psi \in L^2(\mathbb{R})$ fulfilling the admissibility condition

$$C_\psi = \int_{\mathbb{R}^+} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty,$$

gives rise to an inversion formula

$$f = C_\psi^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}^+} \langle f, \psi_{(a,b)} \rangle \psi_{(a,b)} \frac{da}{a^2} db$$

to be read in the weak sense. An equivalent formulation of this fact is that the wavelet transform $f \mapsto V_\psi f,\ \ V_\psi f(a,b) = \langle f, \psi_{(a,b)} \rangle$ is an isometry between $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^+ \times \mathbb{R}, \frac{da}{a^2} db)$. The admissibility condition as well as the choice of the measure used in the reconstruction appear to be somewhat obscure until one reads it in group-theoretic terms. The relation to the theory of square integrable group representations was pointed in [43]. Square integrable representations of locally compact groups have important applications in many fields of physics (generalized coherent states, quantization, quantum measurement theory, signal analysis, etc. - see the review paper [3], the book [2] and the rich bibliography therein) and mathematics (the theory of Plancherel measure for locally compact groups [30], wavelet analysis [27] and its generalizations, the theory of localization operators [78], etc.).

The fundamental properties of these representations have been studied originally by Godement, in the case of unimodular groups [40], [41], and by Duflo and Moore [30], Grossmann et al. [43], in the general case.

The wavelet transform is seen to be a special instance of the following construction: Given a strongly continuous, unitary representation π of a locally compact group G on the Hilbert space $\mathcal{H}$ and a vector $\psi \in \mathcal{H}$, we define the coefficient operator

$$V_\psi : \mathcal{H} \ni f \mapsto V_\psi f \in C_b(G), \quad V_\psi f(x) = \langle f, \pi(x)\psi \rangle.$$

Here $C_b(G)$ denotes the space of bounded continuous functions on G. Since we are interested in inversion formulae we consider V_ψ as an operator $\mathcal{H} \to L^2(G)$, with the obvious domain $dom(V_\psi) = \{f \in \mathcal{H} : V_\psi f \in L^2(G)\}$. The vector $\psi \in H$ is admissible whenever $V_\psi : \mathcal{H} \to L^2(G)$ is an isometric embedding and in this case V_ψ is called (generalized) wavelet transform. While the definition itself is rather simpler, the problem of identifying admissible vectors is highly nontrivial, and the question whether these vectors exist for a given representation does not have a simple general answer.

The construction principle for wavelet transforms had also been studied in mathematical physics, where admissible vectors ψ are called fiducial vectors, systems of the type $\{\pi(x)\psi : x \in G\}$ coherent state systems, and the corresponding inversion formulae, resolutions of the identity, see [2] for more details and references.

The earliest and most prominent examples were the original coherent states obtained by time-frequency shifts of the Gaussian, which were studied in quantum optics. Perelomov [60] discussed the existence of resolutions of the identity in more generality, restricting attention to irreducible representations of unimodular groups. In this setting discrete series representations, i.e., irreducible subrepresentations of the regular representation λ_G of G, turned out to be the right choice. Here, every nonzero vector is admissible up to normalization. Moreover, Perelomov devised a construction which gives rise to resolutions of the identity for a large class of irreducible representations which were not in the discrete series. The idea behind this construction was to replace the group as integration domain by a well chosen quotient, i.e., to construct isometries $\mathcal{H} \hookrightarrow L^2(G/H)$ for a suitable closed subgroup

H. In all these constructions, irreducibility was essential: only the well-definedness and a suitable intertwining property needed to be proved, and Schur's Lemma would provide for the isometry property.

Nevertheless, this is not yet sufficient since many groups of physical relevance, like the Euclidean, Galilei, or Poincaré relativity groups, have no square integrable representations and the standard Gilmore-Perelomov approach cannot be applied. Fortunately, the whole formalism may be extended to the case in which the relevant representation is only square integrable over a homogeneous manifold of the group, i.e., the quotient of the group by some closed subgroup. This will be described in Chapter 1. In this generalization the result, including the coherent states, will depend on the choice of a section in the corresponding fiber bundle. In addition, the familiar resolution of the identity, which is present for any square integrable representation, is now replaced by the resolution of a more complicated operator. In this way, we can construct coherent states for a whole new class of groups, including the relativity groups, and we also obtain a continuous generalization of the notion of frame, familiar in signal processing. Further information and references to the original papers may be found in the review paper [3] and in the book [2].

Wavelets are a powerful signal analysis tool due to the spatial and scale localization encoded in the analysis. The usefulness of such analysis has previously been demonstrated on a large range of physical applications. However, many of these applications are restricted to Euclidean space $\mathbb{R}^n$, where the dimension of the space is often one or two. Nevertheless, data are often defined on other manifolds, such as the 2-sphere. For example, applications where data are defined on the sphere are found in astrophysics (e.g. [45]), geophysics (e.g. [23]), computer vision (e.g. [46]) and quantum chemistry (e.g. [51]).

Fourier analysis on S^2 is standard, but cumbersome, since it amounts to work with expansions in spherical harmonics [57]. While these constitute an orthonormal basis of $L^2(S^2)$, they are not localized at all on the sphere. Actually, there are some specific combinations of spherical harmonics which are well localized, the so-called spherical harmonic kernels ([37], [62]), but then one loses the simplicity of an orthonormal basis.

Alternative solutions have been proposed by several authors. We may refer, for instance, Gabor analysis on the tangent bundle [68]; frequential wavelets based on spherical harmonics [37]; or diffusion methods with a heat equation [14]. Discrete wavelets on the sphere have also been designed, using a multiresolution analysis on S^2. For instance, Haar wavelets on a triangulation of S^2 and refined with the lifting scheme [66]; C^1 wavelets constructed by a factorization of the refinement matrices [76]; or wavelets obtained by radial projection from a polyhedron inscribed in the sphere (typically locally supported spline wavelets on spherical triangulations) [63], [64]. References to the (vast) literature on discrete spherical wavelets may be found in [63], [76] for earlier work and in [55] for recent work.

However, these constructions have some problems, such as an inadequate notion of dilation, the lack of wavelet localization, the excessive rigidity of the wavelets obtained, the lack of directionality, etc.. In this respect, the Continuous Wavelet Transform (CWT) has many advantages: locality is controlled by dilation, the wavelets are easily transported around the sphere by rotations from $SO(3)$ and efficient algorithms are available. Holschneider was the first to build a genuine spherical continuous wavelet transform (SCWT) [49], but his construction involves several assumptions and lacks a geometrical feeling. In particular, it contains a parameter that plays the role of dilations but has to fulfill a number of assumptions and is, therefore, difficult to compute. But it is possible to introduce local dilations in a quite natural way on the sphere if one uses its conformal group, the proper Lorentz group $SO_0(1,3)$. A successful solution was obtained by Antoine and Vandergheynst [7], [8]. In [7], the authors use the Iwasawa decomposition of $SO_0(1,3)$ (or $KAN-$decomposition, where K is the maximal compact subgroup, $A = SO(1,1) \cong \mathbb{R} \cong \mathbb{R}_*^+$ is the subgroup of Lorentz boosts in the $z-$direction and $N \cong \mathbb{C}$ is a two dimensional abelian subgroup) to construct the parameter space $X = SO_0(1,3)/N \cong SO(3) \cdot \mathbb{R}_*^+$, i.e. the product of $SO(3)$ for motions and $\mathbb{R}_*^+$ for dilations on S^2. A generalization of this approach for the $(n-1)-$sphere is presented in [8]. One of the most important results is the correspondence principle between spherical and Euclidean wavelets in the sense that the inverse stereographic projection of a wavelet on the plane gives rise to a wavelet on the sphere [77]. Recently, in [48], Hobson et al. extend the case of isotropic dilations on the 2-sphere to the case of anisotropic dilations defined in two orthogonal directions.

One of the limitations of the SCWT of Antoine and Vandergheynst is that it does not take into account relativistic movements on the sphere. But in many applications such movements are required, e.g. an observer which moves at relativistically velocity with respect to the Earth would see the appearance of the night sky (as modeled by points on the celestial sphere) transformed by means of a Möbius transformation. With the present approach we are able to connect the geometry of conformal transformations on the sphere with wavelet theory, incorporating arbitrary relativistic boosts. Another motivation for this work comes from the case of the plane, where a wide variety of wavelets, such as ridglets, curvelets or shearlets, exists (see e.g. [15], [29], [56]). For future consideration of such transformations on the sphere, it seems to be necessary to incorporate general conformal transformations first. From the physical point of view, we will obtain relativistic coherent states which will be called *spherical conformlets*.

This dissertation is organized in five chapters. In Chapter 1 we consider some theoretical aspects about locally compact groups, we present the basic harmonic analysis of square integrable group representations and we give two examples: time-frequency analysis based on the Heisenberg group and the continuous wavelet transform on $\mathbb{R}$ and $\mathbb{R}^n$ based on the affine linear and similitude groups, respectively. In the end, we present the whole formalism of square integrable representations over a homogeneous manifold of the group, which we will apply in our construction.

In Chapter 2 we study the algebraic structure of the proper Lorentz group $\text{Spin}^+(1,n)$ in the context of Clifford algebras. The main results of this chapter are the decomposition of $\text{Spin}^+(1,n)$ into the gyrosemidirect product of the group $\text{Spin}(n)$ and the gyrogroup of the unit ball $(B^n, \oplus)$, the factorizations of $(B^n, \oplus)$ together with the theorem on duality relations between orbits and sections, the relation between $(B^3, \oplus)$ and $SL(2, \mathbb{C})$, and the relation between $(B^n, \oplus)$ and $SL(2, \Gamma(n) \cup \{0\})$, where $\Gamma(n)$ is the Clifford group in $\mathbb{R}^n$.

We know from Physics [38] that the set of all relativistically admissible velocities in the theory of the special relativity of Einstein is a ball in $\mathbb{R}^3$ with radius c, the speed of light. It is a homogeneous ball and a bounded symmetric domain with respect to the group of projective transformations. Relativistic dynamics is described by elements of the Lie algebra of this group. In our case, the unit ball $(B^n, \oplus)$ corresponds to the normalization of the relativistic ball and it encodes all the necessary information for the definition of dilations on the unit sphere. We will see that the group $\text{Spin}^+(1,n)$, together with its Cartan decomposition, constitutes a very rich and powerful tool for the description of (isotropic and anisotropic) spherical continuous wavelet transforms with a nice geometric description.

In Chapter 3 we construct spherical continuous wavelet transforms arising from sections of the Lorentz group $\text{Spin}^+(1,n)$, generalizing the approach due to Antoine and Vandergheynst ([7], [8]). The results are obtained first for the unit sphere S^2 and later generalized to the $(n-1)-$sphere S^{n-1} using Clifford algebra tools. The generalized SCWT incorporate arbitrary relativistic boosts. To measure the deviation of an arbitrary global left section with respect to the fundamental section on the unit ball, we introduce the concept of $p-$deviation of a section, using the generating function of the respective section. Finally, the covariance properties of our transforms are investigated.

In Chapter 4 we investigate the properties of spherical caps under the action of the Möbius transformation φ_a. This provides us with an understanding of the role of the parameter

$a \in B^n$ in the subsequent dilation. This study is of foremost importance for the local control of the SCWT. Our investigations lead us to the development of new concepts, like the concept of local dilation around the North Pole, zonal surfaces, admissible global and local sections and local anisotropy of a left (global or local) section.

Finally, in Chapter 5 we construct spherical frames for the unit sphere S^2 by discretization of the SCWT. First, we devote our attention to half-continuous spherical frames and we generalize Proposition 17 of [11]. Then, using the results obtained in Chapter 4 and by a convenient discretization of the dilation and rotation parameters we propose an algorithm for the reconstruction of spherical signals. Academic numerical examples are given, based on the spherical Mexican wavelet. We finish this chapter showing the sparsity of the Gramm matrix and some of its properties.

Chapter 1

Harmonic analysis on locally compact groups

In this chapter we discuss the preliminaries that are necessary for the good understanding of our work. We discuss the existence of left and right Haar measures on locally compact groups and the existence of quasi-invariant measures on homogeneous spaces obtained from the factorization of a group by a subgroup. In Section 1.2 we present some basic facts about the theory of square integrable group representations and its connection with the theory of coherent states in Physics is shown in Section 1.4. Two examples are explored in Section 1.3, namely, the short time Fourier transform, that is on the basis of time-frequency analysis, and the continuous wavelet transforms on $\mathbb{R}$ and $\mathbb{R}^n, n \geq 2$.

1.1 Locally compact groups

While we can define many types of integral transforms, it turns out that many useful ones arise as a result of actions of locally compact groups.

Definition 1.1.1 *A topological group G is a group that is also a topological space such that the multiplication $(g_1, g_2) \mapsto g_1 g_2$ and the inversion $g \mapsto g^{-1}$ in G are continuous functions.*

Definition 1.1.2 *A locally compact group G is a topological group such that the underlying topology is Hausdorff and locally compact.*

A topological space is locally compact if every point in the space has a compact neighborhood. There are topological groups that are not Hausdorff.

Let us define the left and right translation operators for functions on G, by

$$L_{g_1}f(g_2) := f(g_1^{-1}g_2) \quad \text{and} \quad R_{g_1}f(g_2) := f(g_2g_1).$$

A Borel measure μ on G is a non-negative countably additive function defined on the $\sigma-$algebra of Borel sets on G taking finite values on compact subsets, and it is called regular if for any $\epsilon > 0$ and any Borel subset A, there exist an open subset U and a closed subset F such that $F \subset A \subset U$ and $\mu(U - F) < \epsilon$. We will assume from now on that all Borel measures are meant to be regular.

Definition 1.1.3 *A Borel measure μ is called left (right) invariant if $\mu(gA) = \mu(A)$ ($\mu(Ag) = \mu(A)$) for any Borel subset A of G and for any $g \in G$.*

By the Riesz representation Theorem [65], which gives a bijection between the set of all regular Borel measures over G, $M(G)$, and the set of positive linear functionals on $C_c(G)$, it follows that μ is left invariant if and only if for any $f \in C_c(G)$ and for any $h \in G$

$$\int f(hg)d\mu(g) = \int f(g)d\mu(g).$$

Theorem 1.1.4 *([53]) There exists a left (resp. right) invariant non-zero Borel measure on G, which is unique up to a multiplicative constant.*

These invariant measures are also called left and right Haar measures. If a right Haar measure on G is also left invariant, then G is called unimodular. In general, a right Haar measure is not left invariant. To measure the deviation, we define the modular function.

Definition 1.1.5 *Let μ be a left Haar measure on G. For each $g \in G$, let μ_g be the measure defined by $\mu_g(A) := \mu(Ag)$. The modular function Δ_G is defined by the relation $\mu_g(A) = \Delta_G(g^{-1})\mu(A)$ for all Borel subset A.*

The definition of Δ_G is independent of μ and $\Delta_G \equiv 1$ if and only if G is unimodular.

Proposition 1.1.6 *([53]) The modular function Δ has the following properties:*

(a) *$\Delta_G : G \to \mathbb{R}^+$ is a continuous group homomorphism;*

(b) *$\Delta_G(g) = 1$ for g in any compact subgroup of G;*

(c) *$\mu_l(g^{-1})$ and $\Delta_G(g)\mu_l(g)$ are right Haar measures and are equal;*

(d) *$\mu_r(g^{-1})$ and $\Delta_G(g)^{-1}\mu_r(g)$ are left Haar measures and are equal;*

(e) $\mu_r(gA) = \Delta_G(g)\mu_r(A)$ *for any right Haar measure on* G *and all Borel subset* A.

Let H be a subgroup of G and $X = H\backslash G$. We can ask when does there exists a natural measure class on the homogeneous space X.

Definition 1.1.7 *Let* μ *be a Borel measure on* X. *For all Borel subsets* A *and all* $g \in G$

(i) μ *is called invariant if* $\mu(Ag) = \mu(A)$;

(ii) μ *is called semi-invariant (with a character* $\chi : G \to \mathbb{R}^+$*) if* $\mu(Ag) = \chi(g)\mu(A)$;

(iii) μ *is called quasi-invariant if* $\mu(Ag) = 0 \Leftrightarrow \mu(A) = 0$.

In general, $H\backslash G$ does not admit a semi-invariant measure. The condition can be reformulated considering the modular function.

Theorem 1.1.8 **(i)** *There exists a semi-invariant measure on* $H\backslash G$ *with character* χ *if and only if* χ *is an extension of* $\Delta_G\Delta_H^{-1} : H \to \mathbb{R}^+$. *Hence there exists a semi-invariant measure* $H\backslash G$ *if and only if* $\Delta_G\Delta_H^{-1} : H \to \mathbb{R}^+$ *can be extended to* $G \to \mathbb{R}^+$.

(ii) $H\backslash G$ *has an invariant measure (i.e. a semi-invariant measure with trivial character) if and only if* $\Delta_G|_H = \Delta_H$.

Corollary 1.1.9 *([53]) Let* G *and* H *be both unimodular. Let* $d\mu(g)$ *and* $d\mu(h)$ *be Haar measures on* G *and* H *respectively. Then there exists a unique invariant measure* μ *on* $H\backslash G$ *such that*

$$\int_G f(g)d\mu(g) = \int_{H\backslash G}\int_H f(hg)d\mu(h)\, d\mu([g]).$$

Quasi-invariant measures always exist on X. In fact, there exists a unique, up to equivalence, quasi-invariant measure on X. Let μ be a quasi-invariant measure on X and $\mu_g(A) = \mu(gA)$, with A a Borel set on X. The Radon-Nikodym derivative of μ_g with respect to μ, $\lambda(g, x) = \frac{d\mu_g(x)}{d\mu(x)}$, is then a cocycle, i.e. $\lambda : G \times X \to \mathbb{R}^+$, with the properties

$$\begin{cases} \lambda(g_1g_2, x) = \lambda(g_1, x)\lambda(g_2, g_1^{-1}x), \\ \lambda(e, x) = 1, \end{cases} \tag{1.1}$$

where the above relations hold for almost all $g_1, g_2 \in G$, and almost all $x \in X$. We note that all the measures $\mu_g, g \in G$, belong to the same measure class, i.e., they all have the same measure-zero sets.

Convolution of functions on a locally compact group is a well-defined operation that shares many properties with its well-known Euclidean counterpart. On the real line, convolution is an operation that quantifies the amount of overlap generated when a function $f(x)$ is shifted over another function $g(x)$. The convolution of these two functions is denoted by $f * g$ and given by the following integral

$$(f * g)(y) = \int_{-\infty}^{+\infty} f(y)g(x-y)dx.$$

Convolution has a large number of applications in signal processing including the important Convolution Theorem. Another use of Convolution Theorem is to apply a filter to any image defined on the plane. Similar applications arise on other domains such as the sphere and the definition of convolution can be extended to such domains in a number of ways.

We will present the general definition of a convolution over some locally compact group G equipped with a left Haar measure μ.

Definition 1.1.10 *(Group convolution) Let G be a locally compact group with left Haar measure $d\mu$, normalized to 1, and let $f, h : G \to \mathbb{C}$ be two measurable functions such that $f, h \in L^2(G, d\mu)$. The convolution product of f and h is defined a.e. by the integral*

$$(f * h)(g_1) = \int_G f(g_1 g_2)h(g_2^{-1})d\mu(g_2) = \int_G f(g_2)h(g_2^{-1}g_1)d\mu(g_2). \tag{1.2}$$

One of the most important properties of the convolution integral is the regularizing effect on L^p elements.

For $f, g \in L^1_{loc}(G)$ the convolution is given by

$$(f * h)(g_1) = \int_G f(g_2)h(g_2^{-1}g_1)d\mu(g_2) = \int_G f(g_2)L_{g_2}h(g_1)d\mu(g_2) \tag{1.3}$$

whenever this integral is well-defined for almost all $g_1 \in G$. It may also be expressed by (see [36], p. 51)

$$(f * h)(g_1) = \int_G h(g_2)f(g_1 g_2^{-1})\Delta(g_2^{-1})d\mu(g_2) = \int_G h(g_2)R_{g_2^{-1}}f(g_1)\Delta(g_2^{-1})d\mu(g_2). \tag{1.4}$$

The convolution is commutative if and only if the group G is commutative. Young's Theorem [36] states that

$$||f * h||_{L^p(G)} \leq ||f||_{L^1(G)}||h||_{L^p(G)} \quad \text{for all } f \in L^1(G), h \in L^p(G),\ 1 \leq p \leq \infty.$$

1.2 Square integrable group representations

Let us recall some basic facts from the theory of group representations. A unitary representation π of G is a strongly continuous homomorphism of G into the group $U(\mathcal{H})$ of unitary operators on a Hilbert space $\mathcal{H}$. This means that π satisfies

$$\pi(g_1g_2) = \pi(g_1)\pi(g_2), \qquad \pi(e) = Id,$$

and the mapping $g \mapsto \pi(g)f$ from G into $\mathcal{H}$ is continuous for all $f \in \mathcal{H}$. We remark that strong continuity is already implied by the less restrictive condition of weak continuity, i.e. for all $f, h \in \mathcal{H}$ the mapping $g \mapsto \langle f, \pi(g)h\rangle_{\mathcal{H}}$ is continuous (see [36]).

A closed subspace W of $\mathcal{H}$ is called invariant if $\pi(g)W \subset W$ for all $g \in G$ and π is called irreducible if the only trivial invariant subspaces are $\{0\}$ and $\mathcal{H}$.

Definition 1.2.1 *A representation π is called **square integrable** if π is irreducible and if there exists a non-zero vector $\psi \in \mathcal{H}$ such that*

$$\int_G |\langle \psi, \pi(g)\psi\rangle_{\mathcal{H}}|^2 d\mu(g) < \infty.$$

Such a vector ψ is called admissible.

We associate to π the voice transform or wavelet transform defined by

$$V_\psi f(g) := \langle f, \pi(g)\psi\rangle_{\mathcal{H}}, \quad f, \psi \in \mathcal{H}, g \in G. \tag{1.5}$$

The inner products on the right hand side of (1.5) are also called matrix coefficients of π. From (1.5) we can derive the covariance property of the wavelet transform

$$V_\psi(\pi(g_1)f)(g_2) = \langle \pi(g_1)f, \pi(g_2)\psi\rangle_{\mathcal{H}} = \left\langle f, \pi(g_1^{-1}g_2)\psi\right\rangle_{\mathcal{H}} = L_{g_1}V_\psi f(g_2). \tag{1.6}$$

This means that the wavelet transform intertwines the representation π and the left regular representation L_g.

The theorem due to Duflo and Moore [31] about square integrable group representations is very important for the theory of wavelets. This result was rediscovered by Grossmann, Morlet and Paul [43].

Theorem 1.2.2 *Let π be a square integrable representation of a locally compact group G on the Hilbert space $\mathcal{H}$. Then there exists an unique self-adjoint, positive and dense operator K on $\mathcal{H}$ that satisfies the following properties:*

(a) *$\mathcal{D}(K) = \{\psi \in \mathcal{H}, \psi$ is admissible $\}$, where $\mathcal{D}(K)$ denotes the domain of K;*

(b) *If ψ is admissible then $V_\psi f \in L^2(G)$ for all $f \in \mathcal{H}$;*

(c) *For ψ_1, ψ_2 admissibles and $f_1, f_2 \in \mathcal{H}$ we have the orthogonality relation*

$$\int_G V_{\psi_1} f_1(g) \overline{V_{\psi_2} f_2(g)}\, d\mu(g) = \langle f_1, f_2 \rangle_{\mathcal{H}} \langle K\psi_2, K\psi_1 \rangle_{\mathcal{H}}; \tag{1.7}$$

(d) *It holds $K = S^{-1/2}$ for some self-adjoint, positive, densely defined operator S that satisfies $\pi(g) S \pi(g)^{-1} = \Delta(g)^{-1} S$.*

Furthermore, if G is unimodular then $K = cId$ for some constant $c > 0$ and, hence, any non-zero vector in $\mathcal{H}$ is admissible.

The proof of Theorem 1.2.2 relies on the following extension of Schur's Lemma (see Theorem 1, [31]).

Lemma 1.2.3 *Let π be an irreducible representation of G and assume that χ is a character of G, i.e. $\chi \in C(G)$ with $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$. Let T be a densely defined closed operator in $\mathcal{H}$ that satisfies*

$$\pi(g) T \pi(g)^{-1} = \chi(g) T, \quad \text{for all } g \in G. \tag{1.8}$$

If T' is another operator satisfying (1.8) then $T' = cT$ for some constant $c > 0$.

As an immediate consequence of Theorem 1.2.2 we obtain an inversion formula and reproducing formula for the wavelet transform, see also [43].

Corollary 1.2.4 *Let π be a square-integrable representation of G in $\mathcal{H}$.*

(a) *(Inversion formula) Let $\psi, h \in \mathcal{D}(K)$ with $\langle K\psi, Kh \rangle_{\mathcal{H}} = 1$. Then it holds*

$$f = \int_G V_\psi f(g)\, \pi(g) h\, d\mu(g) \quad \text{for all } f \in \mathcal{H}, \tag{1.9}$$

to be read in a weak sense.

(b) *For $\psi_1, \psi_2 \in \mathcal{D}(K)$, $f_1, f_2 \in \mathcal{H}$, it holds*

$$V_{\psi_1} f_1 * V_{\psi_2} f_2 = \langle K\psi_1, K f_2 \rangle_{\mathcal{H}} V_{\psi_2} f_1. \tag{1.10}$$

(c) *(Reproducing formula) Let $\psi \in \mathcal{D}(K)$ with $||K\psi|| = 1$. Then it holds*

$$V_\psi f = V_\psi f * V_\psi \psi \quad \text{for all } f \in \mathcal{H}.$$

(d) *Suppose $\psi \in \mathcal{D}(K)$ with $||K\psi|| = 1$. Then the mapping $P : L^2(G) \to L^2(G)$, $F \mapsto F * V_\psi\psi$ is an orthogonal projection from $L^2(G)$ onto the image of V_ψ.*

Proof: (a) Let $f, h_1 \in \mathcal{H}$. By the orthogonality relation (Theorem 1.2.2 (c)) it holds

$$\begin{aligned}
\int_G V_\psi f(g) \langle \pi(g)h, h_1\rangle_{\mathcal{H}} \, d\mu(g) &= \int_G V_\psi f(g)\overline{V_h h_1(g)} d\mu(g) \\
&= \langle f, h_1\rangle_{\mathcal{H}} \langle Kh, K\psi\rangle_{\mathcal{H}} \\
&= \langle f, h_1\rangle_{\mathcal{H}} .
\end{aligned}$$

Since $h_1 \in \mathcal{H}$ was arbitrary this equation is exactly the weak definition of (1.9).
For (b), again by Theorem 1.2.2 (c) and (1.6) we have

$$\begin{aligned}
V_{\psi_1} f_1 * V_{\psi_2} f_2(g_1) &= \int_G V_{\psi_1} f_1(g_2) V_{\psi_2} f_2(g_2^{-1} g_1) d\mu(g_2) \\
&= \int_G V_{\psi_1} f_1(g_2) \overline{V_{f_2}(\pi(g_1)\psi_2)(g_2)} d\mu(g_2) \\
&= \langle K\psi_1, Kf_2\rangle_{\mathcal{H}} \langle f_1, \pi(g_1)\psi_2\rangle_{\mathcal{H}} \\
&= \langle K\psi_1, Kf_2\rangle_{\mathcal{H}} V_{\psi_2} f_1(g_1).
\end{aligned}$$

The assertion in (c) follows as an immediate consequence.
(d) Let $F \in L^2(G)$. Define $f = \int_G F(g)\pi(g)\psi \, d\mu(g) \in \mathcal{H}$ to be understood in a weak sense. We obtain

$$V_\psi f(g_1) = \int_G F(g_2) \langle \pi(g_2)\psi, \pi(g_1)\psi\rangle_{\mathcal{H}} \, d\mu(g_2) = F * V_\psi\psi(g_1).$$

Together with (a) we conclude that P is surjective onto the image of $\mathcal{H}$ under V_ψ. Further, if $F = V_\psi f$ for some $f \in \mathcal{H}$ then by (c) it holds $F = F * V_\psi f$ and thus P is a projection. Finally, if $F_1, F_2 \in L^2(G)$ then it follows from $V_\psi\psi(g^{-1}) = \overline{V_\psi\psi(g)}$ that

$$\begin{aligned}
\langle F_1 * V_\psi\psi, F_2\rangle_{L^2(G)} &= \int_G \int_G F_1(g_2) V_\psi\psi(g_2^{-1} g_1) d\mu(g_2) \overline{F_2(g_1)} d\mu(g_1) \\
&= \int_G F_1(g_2) \overline{\int_G V_\psi\psi(g_1^{-1} g_2) F_2(g_1) d\mu(g_1)} d\mu(g_2) \\
&= \langle F_1, F_2 * V_\psi\psi\rangle_{L^2(G)} .
\end{aligned}$$

Hence, P is orthogonal.

∎

1.3 Time-frequency and wavelet analysis

The basic object in time-frequency analysis is the Fourier transform. For $f \in L^1(\mathbb{R}^n)$ it is defined by

$$\mathcal{F}(f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,\xi\rangle}dx, \quad \xi \in \mathbb{R}^n.$$

The Plancherel Theorem states that $\mathcal{F}$ extends to a unitary transform on $L^2(\mathbb{R}^n)$ and

$$\left\langle \widehat{f}, \widehat{g} \right\rangle_{L^2(\mathbb{R}^n)} = \langle f, g\rangle_{L^2(\mathbb{R}^n)}, \quad \text{for all } f, g \in L^2(\mathbb{R}^n). \tag{1.11}$$

In particular we obtain Parseval's relation:

$$||\widehat{f}||_{L^2(\mathbb{R}^n)} = ||f||_{L^2(\mathbb{R}^n)}$$

and the inversion of the Fourier transform by its adjoint. We have $\mathcal{F}^{-1} = \mathcal{F}^*$ with

$$\mathcal{F}^{-1}\widehat{f}(x) = f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}d\xi. \tag{1.12}$$

Now we will consider examples that illustrate how the short time Fourier transform (sometimes also called windowed Fourier transform or Gabor transform) and the continuous wavelet transform fit into the setting of square-integrable group representations.

The (reduced) Heisenberg group is topologically the set $\mathbb{H}_n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denotes the torus. The group law on $\mathbb{H}_n$ is given by

$$(b, w, \tau)(b', w', \tau') = (b + b', w + w', \tau\tau' e^{\pi i(\langle b',w\rangle - \langle b,w'\rangle)}).$$

The Heisenberg group is unimodular and has Haar measure

$$\int_{\mathbb{H}_n} f(g)d\mu(g) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^1 f(b, w, e^{2\pi it})\, dt\, dw\, db.$$

The Schrödinger representation ρ is an unitary representation of the Heisenberg group acting on $L^2(\mathbb{R}^n)$ by

$$\rho(b, w, \tau)f(x) = \tau e^{\pi i\langle b,w\rangle}T_b M_w f(x) = \tau e^{-\pi i\langle b,w\rangle}M_w T_b f(x),$$

where T_b and M_w are the translation and modulation operators defined by

$$T_b f(x) := f(x - b) \quad \text{and} \quad M_w f(x) := e^{2\pi i\langle x,w\rangle}f(x), \quad b, \omega \in \mathbb{R}^n.$$

To check that this is in fact a representation we use the commutation relation

$$T_b M_\omega = e^{-2\pi i\langle b,w\rangle}M_\omega T_b.$$

Theorem 1.3.1 *([42], p.182) The Schrödinger representation is a square-integrable representation of the Heisenberg group.*

The corresponding voice transform is

$$\begin{aligned} V_\psi f(b,w,\tau) &= \langle f, \rho(b,\omega,\tau)\psi\rangle_{L^2(\mathbb{R}^n)} = \overline{\tau}\int_{\mathbb{R}^n} f(x)\overline{e^{-\pi i\langle b,w\rangle}M_w T_b \psi(x)}dx \\ &= \overline{\tau}e^{\pi i\langle b,w\rangle}\underbrace{\int_{\mathbb{R}^n} f(x)\overline{\psi(x-b)}e^{-2\pi i\langle x,w\rangle}dx}_{(I)}. \end{aligned} \tag{1.13}$$

where the integral (I) is the short time Fourier transform ($\mathrm{STFT}_\psi f(b,w)$). Thinking of ψ as a window function localized at $b=0$ we may interpret $\mathrm{STFT}_\psi f(b,w)$ as cutting out the parts of f around b by multiplying with the translated window $\overline{T_b\psi}$ followed by an application of the Fourier transform. Thus, one may think of $\mathrm{STFT}_\psi f(b,w)$ as the amplitude of the frequency w at space position b. However, several uncertainty principles state that this interpretation does not hold in a pointwise sense, see e.g. Chapter 3.3 in [42]. Usually one choose windows ψ that have good localization properties in both space and Fourier domain. Since Gaussians are the minimizing functions of Heisenberg's uncertainty principle, they are often suggested as window functions.

The STFT satisfies the orthogonality relation

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} STFT_{\psi_1} f_1(b,w)\overline{STFT_{\psi_2} f_2(b,w)}\,db\,dw = \langle f_1, f_2\rangle\langle \psi_2,\psi_1\rangle .$$

As a consequence of the orthogonality relation we may invert the STFT as follows. Suppose that ψ, h are non-zero functions of $L^2(\mathbb{R}^n)$ with $\langle\psi,h\rangle \neq 0$. Then

$$f = \frac{1}{\langle\psi,h\rangle}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} STFT_\psi f(b,w) M_w T_b h\ db\,dw$$

for all $f\in L^2(\mathbb{R}^n)$ in a weak sense. For more details about the short time Fourier transform see [42].

The next example involves the full affine linear group G_{aff} used in the construction of wavelets on the real line. It consists of elements of the form $G_{aff} = \{(a,b)|b\in\mathbb{R}, a\neq 0\}$ with the group law $(a,b)(a',b') = (aa', b+ab')$.

This group is non-unimodular, the left Haar measure is $d\mu_l = a^{-2}da\,db$ and the right Haar measure is $d\mu_r = a^{-1}da\,db$.

An unitary representation U acting in $L^2(\mathbb{R})$, consisting of translations and dilations, is given by

$$U(a,b)f(x) = \frac{1}{\sqrt{|a|}}f\left(\frac{x-b}{a}\right), a\neq 0, b\in\mathbb{R} \tag{1.14}$$

or, in the Fourier domain,

$$(\widehat{U(a,b)}f)(\xi) = \sqrt{|a|}e^{-i\xi b}\widehat{f}(a\xi).$$

Theorem 1.3.2 *The representation U is square-integrable and a function $\psi \in L^2(\mathbb{R}, dx)$ is said to be admissible (called a wavelet) if and only if it satisfies the admissibility condition*

$$C_\psi = 2\pi \int_{-\infty}^{+\infty} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|} d\xi < +\infty. \tag{1.15}$$

If ψ is regular enough the admissibility condition (1.15) is equivalent to a zero mean condition:

$$\int_{\mathbb{R}} \psi(x)dx = 0.$$

In most of the cases we choose only positive dilations. Therefore we restrict to the connected subgroup of G_{aff}, called G^+ or the "$ax + b$" group:

$$G^+ = \{(a, b) : a > 0, b \in \mathbb{R}\}.$$

The representation U restricted to G^+ splits into the direct sum of two unitary inequivalent, square integrable representations, $U_\pm$, acting in the two Hardy spaces $H^2_+(\mathbb{R})$ and $H^2_-(\mathbb{R})$ defined by

$$H^2_\pm(\mathbb{R}) = \{f \in L^2(R)|\widehat{f}(\xi) = 0, \xi \gtrless 0\}.$$

With these restrictions, the admissibility condition (1.15) becomes

$$C^+_\psi = 2\pi \int_0^{+\infty} \frac{|\widehat{\psi}(\xi)|^2}{\xi} d\xi$$

over $H^2_+(\mathbb{R})$ (and similarly over $H^2_-(\mathbb{R})$). The wavelet analysis on $L^2(\mathbb{R}, dx)$ is possible if it is imposed either a strict equality of wavelet contributions of H^2_+ and H^2_- (see [27]):

$$0 < C^+_\psi = C^-_\psi < +\infty.$$

Given a signal $f \in L^2(\mathbb{R})$, and a wavelet $\psi \in L^2(\mathbb{R})$ the corresponding continuous wavelet transform is defined by

$$\begin{aligned}
(W_\psi f)(a, b) &= \frac{1}{\sqrt{C_\psi}} \langle f, U(a, b)\psi\rangle_{L^2(\mathbb{R})} \\
&= \frac{1}{\sqrt{C_\psi}} |a|^{-1/2} \int_{\mathbb{R}} f(x)\overline{\psi\left(\frac{x-b}{a}\right)} dx. \\
&= \frac{1}{\sqrt{C_\psi}} |a|^{-1/2} \int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{\psi}(a\xi)} e^{-2\pi i b\xi} d\xi.
\end{aligned}$$

$W_\psi : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}^2, a^{-2}da\, db)$ is an isometry between $L^2(\mathbb{R}, dx)$ and a closed subspace of $L^2(\mathbb{R}^2, a^{-2}da\, db)$.

A function $f \in L^2(\mathbb{R}, dx)$ can be recovered from its wavelet transform via the resolution of the identity as follows.

Proposition 1.3.3 *For all $f, h \in L^2(\mathbb{R})$, it follows*

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (W_\psi f)(a,b)\overline{(W_\psi h)(a,b)}\, db\, \frac{da}{a^2} = \langle f, h \rangle . \tag{1.16}$$

For the sake of completeness we give a proof of this result.

Proof: It holds

$$(\widehat{U(a,b)}\psi)(\xi) = M_{-b}D_{a^{-1}}\widehat{\psi}(\xi) = e^{-2\pi i b\xi}|a|^{1/2}\widehat{\psi}(a\xi).$$

Using Plancherel's Theorem (1.11) we obtain

$$\begin{aligned} W_\psi f(a,b) &= \frac{1}{\sqrt{C_\psi}} \langle f, U(a,b)\psi \rangle = \frac{1}{\sqrt{C_\psi}} \left\langle \widehat{f}, M_{-b}D_{a^{-1}}\widehat{\psi} \right\rangle \\ &= \frac{1}{\sqrt{C_\psi}} \mathcal{F}^{-1}(\widehat{f}\, \overline{D_{a^{-1}}\widehat{\psi}})(b). \end{aligned}$$

Applying Plancherel's Theorem once more together with Fubini's Theorem we realize that

$$\begin{aligned} &\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (W_\psi f)(a,b)\overline{(W_\psi h)(a,b)}\, db\, \frac{da}{a^2} \\ &= \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{\mathbb{R}} \widehat{f}(\xi)\, \overline{\widehat{h}(\xi)} D_{a^{-1}}\widehat{\psi}(\xi)\, \overline{D_{a^{-1}}\widehat{\psi}(\xi)}\, d\xi\, \frac{da}{a^2} \\ &= \frac{1}{C_\psi} \int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{h}(\xi)} d\xi \int_{-\infty}^{+\infty} |\widehat{\psi}(a\xi)|^2 \frac{da}{|a|} \\ &= \frac{1}{C_\psi} \int_{\mathbb{R}} f(x)\overline{h(x)} dx \int_{-\infty}^{+\infty} \frac{|\widehat{\psi}(\zeta)|^2}{|\zeta|} d\zeta \\ &= \langle f, h \rangle , \end{aligned}$$

where we made the change of variables $\zeta = a\xi$.

■

It is now clear why we have to impose the admissibility condition (1.15). If C_ψ were infinity, then the resolution of the identity (1.16) would not hold. Formula (1.16) can be read as

$$f = C_\psi^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (W_\psi f)(a,b) U(a,b)\psi\, db\, \frac{da}{a^2}, \tag{1.17}$$

with the convergence of the integral considered in the weak sense.

The reconstruction formula (1.17) yields a decomposition of the signal f into a linear superposition of the wavelets $U(a,b)\psi$ with coefficients $(W_\psi f)(a,b)$. It follows that W_ψ acts as a local filter, both in position and in scale: it selects the part of the signal that is concentrated around the position b and at scale a. This implies that its efficiency increases with frequency since the CWT is a singularity detector (or a mathematical microscope).

The generalization of the wavelet transform to the multi-dimensional case ($n \geq 2$) is done using the similitude group of $\mathbb{R}^n$, $SIM(n) = \mathbb{R}^n \rtimes (\mathbb{R}_*^+ \times SO(n))$, consisting of the semidirect product $\rtimes$ of translations with dilations and rotations. Here, $\mathbb{R}_*^+ = (0, \infty)$ denotes the multiplicative group of positive real numbers.

The similitude group has left Haar measure

$$\int_G f(g)\, d\mu(g) = \int_{SO(n)} \int_{\mathbb{R}^n} \int_{\mathbb{R}_*^+} f(b,a,R) \frac{da}{a^{n+1}} db\, dR$$

and modular function $\Delta(b,a,R) = a^{-n}$.

We define the following unitary operators acting on $L^2(\mathbb{R}^n)$ by

$$\begin{aligned} T_b f(x) &= f(x-b), \quad b \in \mathbb{R}^n \\ D_a f(x) &= a^{-n/2} f(a^{-1}x), \quad a \in \mathbb{R}_*^+ \\ U_R f(x) &= f(R^{-1}x), \quad R \in SO(n) \end{aligned}$$

and we associate a unitary representation of $SIM(n)$ on $L^2(\mathbb{R}^n)$ given by

$$\pi(b,a,R) f(x) = a^{-n/2} f(R^{-1}a^{-1}(x-b)) = T_b D_a U_R f(x). \tag{1.18}$$

Theorem 1.3.4 *(Theorem 14.2.1 in [2]) The representation π of the similitude group $SIM(n)$ defined in (1.18) is square integrable. A function $\psi \in L^2(\mathbb{R}^n, dx)$ is admissible if and only it satisfies the condition*

$$C_\psi = \int_{\mathbb{R}^n} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|^n} d\xi < \infty.$$

For an admissible function $\psi \in L^2(\mathbb{R}^n)$, the CWT of $f \in L^2(\mathbb{R}^n)$ is defined by

$$\begin{aligned} (W_\psi f)(b,a,R) &= \langle f, T_b D_a U_R \psi \rangle \\ &= a^{-n/2} \int_{\mathbb{R}^n} f(x) \overline{\psi\left(R^{-1}\left(\frac{x-b}{a}\right)\right)} dx, \end{aligned}$$

for $b \in \mathbb{R}^n, a \in (0, \infty), R \in SO(n)$. The CWT analyzes a given function at different positions in space, various scales and in different orientations. When the wavelet ψ is axially symmetric, i.e. $SO(n-1)$ invariant, we can replace everywhere $SO(n)$ by $SO(n)/SO(n-1) \simeq S^{n-1}$, the unit sphere in $\mathbb{R}^n$. This simplifies the parameter space of the CWT. Clearly, if ψ is a radial function then the CWT does not depend on $R \in SO(n)$ and we obtain the fully isotropic case. Sometimes the terminology isotropic wavelet transform is used for this case. Other properties of the CWT on the multi-dimensional case can be seen e.g. in Chapter 14 of [2].

We conclude this section by giving one example of a multi-dimensional wavelet, the Mexican hat or Marr wavelet. It is a real, rotation invariant wavelet, given by the Laplacian of a Gaussian:

$$\begin{aligned} \psi(x) &= -\Delta \exp(-\frac{1}{2}|x|^2), \quad \Delta = \partial^2_{x_1} + \partial^2_{x_2} + \ldots + \partial^2_{x_n} \\ &= (n - |x|^2) \exp(-\frac{1}{2}|x|^2). \end{aligned}$$

1.4 Coherent states

Coherent states were invented by Schrödinger in 1926, in the context of the quantum harmonic oscillator. They were reinvented in 1960 in the context of quantum optics and since there various generalizations of the concept of coherent states were proposed. The key to the generalization of the notion of a coherent state was the observation, by Perelomov [61], that the construction of canonical coherent states could be reformulated as a problem in group representation theory, associated to the Weyl-Heisenberg group. In particular, most of the interesting properties of those canonical coherent states derive from the square integrability of that representation. This fact them immediately leads to the definition of coherent states associated with any square integrable group representation. For a good overview of the subject we refer to the textbooks of Perelomov [60], and Ali, Antoine and Gazeau [2], which also contain a large number of references.

As we saw in the previous section, an interesting example of such coherent states is given by the theory of wavelets. Nevertheless, the theory of square integrable representations is not sufficient, since many groups of physical relevance have no square integrable representations. However, the whole formalism may be extended to the case in which the relevant representation is only square integrable over a homogeneous manifold of the group, i.e. the quotient of the group by some closed subgroup.

We will describe this general formalism that is on the basis of the construction of spherical wavelets associated to the proper Lorentz group $\mathrm{Spin}^+(1, n)$.

Let G be a locally compact group and π a strongly continuous, irreducible, unitary representation of G on a Hilbert space $\mathcal{H}$. We consider the homogeneous space $X = G/H$, where H is a closed subgroup of G. Because π is not directly defined on X, it is necessary to embed X in G. This is realized by using the canonical fiber bundle structure of G with projection $\Pi : G \to X$. Let $\sigma : X \to G$ be a Borel section of this fiber bundle, i.e. $\Pi \circ \sigma(x) = x$, for all $x \in X$. In general the section cannot be chosen to be continuous but it is always possible to choose it measurable or even continuous on some dense open subset of X. The action of an element $g \in G$ on $\sigma(x)$ for $x \in X$ can be written as $g\sigma(x) = \sigma(gx)h(g, x)$ for some element $h(g, x) \in H$.

Let a quasi-invariant measure μ on X and a section σ be given. It is possible to construct another quasi-invariant measure μ_σ, which in a sense is the standard quasi-invariant measure for the chosen section. It is defined using a strict cocycle by

$$d\mu_\sigma(x) = \lambda(\sigma(x), x)d\mu(x).$$

This is the measure used in the general definition of coherent states (see [2] or [4]).

We will denote by $\langle\cdot,\cdot\rangle_X$ the L^2-inner product on X defined by

$$\langle F_1, F_2\rangle_X = \int_X F_1(x)\overline{F_2(x)}\, d\mu_\sigma(x)$$

whenever the integral is defined.

A unitary representation π of G is called *square-integrable modulo* (H, σ) if there exists a function $\psi \in \mathcal{H}$ such that the self-adjoint operator $A_\sigma : \mathcal{H} \to \mathcal{H}$ (dependent on σ and ψ) weakly defined by

$$A_\sigma f := \int_X \langle f, \pi(\sigma(x))\psi\rangle_{\mathcal{H}}\, \pi(\sigma(x))\psi\, d\mu_\sigma(x), \tag{1.19}$$

i.e.

$$\langle A_\sigma f, g\rangle_{\mathcal{H}} = \int_X \langle f, \pi(\sigma(x))\psi\rangle_{\mathcal{H}}\, \overline{\langle \pi(\sigma(x))\psi, g\rangle_{\mathcal{H}}}\, d\mu_\sigma(x) \quad \text{for all } f \in \mathcal{H} \tag{1.20}$$

is bounded and has a bounded inverse A_σ^{-1}. Then we have that

$$\langle A_\sigma f, f\rangle_{\mathcal{H}} = \int_X |\langle f, \pi(\sigma(x))\psi\rangle_{\mathcal{H}}|^2 d\mu_\sigma(x) < \infty \quad \text{for all } f \in \mathcal{H}. \tag{1.21}$$

The function ψ is called therefore *admissible* for (π, σ) and the section σ is admissible for (π, ψ). If A_σ is a multiple of the identity then ψ is called strictly admissible.

We define a set of coherent states based on X, $S_\sigma := \{\pi(\sigma(x))\psi : x \in X\}$. However, to obtain genuine coherent states in a covariant way we have to define the family of vectors

$S_\sigma = \{\psi_{\sigma(x)} = \sqrt{\lambda(\sigma(x), x)}\ \pi(\sigma(x))\psi, x \in X\}$. Thus, these new states are obtained by transporting a fixed vector ψ over X under the action of G, in a covariant way.

A wavelet, or voice transform, is defined by

$$V_\psi f(x) := \langle f, \pi(\sigma(x))\psi \rangle_{\mathcal{H}}, \quad x \in X. \tag{1.22}$$

The set S_σ is total in $\mathcal{H}$ and $V_\psi \in L^2(X)$ by (1.21). If π is strictly square integrable mod (H, σ) for $\psi \in \mathcal{H}$, then it is well known that V_ψ is an isometry from $\mathcal{H}$ onto the reproducing kernel Hilbert space

$$\mathcal{M}_2 = \{F \in L^2(X) : \langle F, R(x, \cdot) \rangle = F(x)\},$$

with Hermitian reproducing kernel

$$\begin{aligned} R(x,y) = R_{\psi,\sigma}(x,y) &:= V_\psi(\pi(\sigma(x))\psi)(y) \\ &= \langle \pi(\sigma(x))\psi, \pi(\sigma(y))\psi \rangle_{\mathcal{H}} \\ &= \left\langle \psi, \pi(\sigma(x)^{-1}\sigma(y))\psi \right\rangle_{\mathcal{H}}. \end{aligned}$$

By the Schwarz's inequality ess $\sup\limits_{x,y \in X} |R(x,y)| \leq ||\psi||^2_{\mathcal{H}}$. Thus, V_ψ can be inverted on its range $\mathcal{M}_2$ by its adjoint V_ψ^* given by

$$V_\psi^* F(s) := \int_X F(x)\pi(\sigma(x))\psi(s)\, d\mu_\sigma(x)$$

This gives us a reconstruction formula for $f \in \mathcal{H}$ given by

$$f = V_\psi^* V_\psi f = \int_X \langle f, \pi(\sigma(x))\psi \rangle_{\mathcal{H}}\, \pi(\sigma(x))\psi\, d\mu_\sigma(x)$$

to be read in a weak-sense.

In the general case, when $A_\sigma \neq \lambda I$ we define a second transform

$$W_\psi f(x) := V_\psi(A_\sigma^{-1} f)(x) = \left\langle A_\sigma^{-1} f, \pi(\sigma(x))\psi \right\rangle_{\mathcal{H}} = \left\langle f, A_\sigma^{-1}\pi(\sigma(x))\psi \right\rangle_{\mathcal{H}}, \quad x \in X.$$

On the one hand substituting f by $A_\sigma^{-1} f$ in (1.20) we obtain that

$$\langle f, g \rangle_{\mathcal{H}} = \langle W_\psi f, V_\psi g \rangle_X \quad f, g \in \mathcal{H}. \tag{1.23}$$

On the other hand substituting g by $A_\sigma^{-1} g$ in (1.20) we obtain that

$$\langle f, g \rangle_{\mathcal{H}} = \langle V_\psi f, W_\psi g \rangle_X, \quad f, g \in \mathcal{H}. \tag{1.24}$$

Combining (1.23) and (1.24) we obtain that

$$\langle f, g \rangle_{\mathcal{H}} = \langle W_\psi f, V_\psi g \rangle_X = \langle V_\psi f, W_\psi g \rangle_X, \quad f, g \in \mathcal{H}, \tag{1.25}$$

By (1.25), we see that

$$\begin{aligned} V_\psi f(x) &= \langle f, \pi(\sigma(x))\psi\rangle_{\mathcal{H}} = \langle V_\psi f, W_\psi(\pi(\sigma(x))\psi)\rangle_X = \langle V_\psi f, R(x,\cdot)\rangle_X \\ W_\psi f(x) &= \left\langle f, A_\sigma^{-1}\pi(\sigma(x))\psi\right\rangle_{\mathcal{H}} = \langle W_\psi f, W_\psi(\pi(\sigma(x))\psi)\rangle_X = \langle W_\psi f, R(x,\cdot)\rangle_X, \end{aligned}$$

where

$$R(x,y) = R_\psi(x,y) := W_\psi(\pi(\sigma(x))\psi)(y) = \left\langle A_\sigma^{-1}\pi(\sigma(x))\psi, \pi(\sigma(y))\psi\right\rangle_{\mathcal{H}}. \tag{1.26}$$

Clearly, $R(y,x) = \overline{R(x,y)}$ for all $x, y \in X$. Moreover, we have for all $x, y \in X$ that

$$|R(x,y)| \leq ||A_\sigma^{-1}||\,||\psi||^2_{\mathcal{H}}.$$

The following facts about square-integrable representations π modulo (H, σ) and admissible functions are well-known (see [2], Theorem 7.3.1).

- The set S_σ is total in $\mathcal{H}$, i.e. $S_\sigma^\perp = \{0\}$. Since A_σ is continuously invertible, we see that also the set $A_\sigma^{-1}(S_\sigma) = \{A_\sigma^{-1}\pi(\sigma(x))\psi : x \in X\}$ is total in $\mathcal{H}$.
- The mappings V_ψ and W_ψ are bijective mappings of $\mathcal{H}$ onto the reproducing kernel Hilbert space
$$\mathcal{M}_2 := \{F \in L^2(X) : \langle F, R(x,\cdot)\rangle_X = F(x) \text{ a.e.}\},$$
- The mappings $\tilde{V}_\psi : L^2(X) \to \mathcal{H}$ and $\tilde{W}_\psi : L^2(X) \to \mathcal{H}$, weakly defined by
$$\begin{aligned} \tilde{V}_\psi F &:= \int_X F(x) A_\sigma^{-1}\pi(\sigma(x))\psi \, d\mu_\sigma(x), \\ \tilde{W}_\psi F &:= \int_X F(x)\pi(\sigma(x))\psi \, d\mu_\sigma(x), \end{aligned}$$
i.e.
$$\left\langle \tilde{V}_\psi F, g\right\rangle_{\mathcal{H}} = \langle F, W_\psi g\rangle_X, \quad \left\langle \tilde{W}_\psi F, g\right\rangle_{\mathcal{H}} = \langle F, V_\psi g\rangle_X.$$
satisfy, by (1.25) the following relations
$$V_\psi \tilde{V}_\psi F(x) = \langle F, R(x,\cdot)\rangle_X, \quad W_\psi \tilde{W}_\psi F(x) = \langle F, R(x,\cdot)\rangle_X, \quad \text{for all } F \in \mathcal{M}_2.$$
Moreover, as the adjoint mappings of $\tilde{V}_\psi$ and $\tilde{W}_\psi$ are W_ψ and V_ψ, respectively, i.e. $\tilde{V}_\psi^* = W_\psi$ and $\tilde{W}_\psi^* = V_\psi$ we obtain the reconstruction formulas
$$f = \tilde{V}_\psi V_\psi f, \quad f = \tilde{W}_\psi W_\psi f, \quad \text{for all } f \in \mathcal{H},$$
to be read in a weak sense.

Remark 1.4.1 *It may also happen that the integral (1.21) with the translated measure* $d\mu_\sigma(x) = \lambda(\sigma(x), x)d\mu(x)$ *actually diverges, but that the corresponding one with another quasi-invariant measure* ν, *equivalent to* μ_σ, *converges. In that case, one may still define a useful family of vectors, called quasi-coherent states, as* $\{\widetilde{\psi}_{\sigma(x)}, x \in X\}$, *which enjoy all the nice properties of the coherent states (overcompleteness, resolution of a positive operator* A_σ, *reproducing kernel), but not covariance. In this case, there are no "true" coherent states associated with the section* σ *and the given representation* π.

All this formalism can be used for the construction of wavelets on some manifolds and the reproduction formulae obtained are on the basis of the theory of coorbit spaces (see [25], [26], [24]).

Chapter 2

The Lorentz group $\mathrm{Spin}^+(1, n)$

In physics, the Lorentz group is the classical setting for all (nongravitational) physical phenomena. The mathematical form of the kinematical laws of special relativity, Maxwell's field equations in the theory of electromagnetics and Dirac's equation in the theory of the electron are each invariant under Lorentz transformations. Therefore, the Lorentz group can be said to express a fundamental symmetry of many of the known fundamental laws of nature.

The Lorentz group is a subgroup of the Poincaré group, the group of all isometries of the Minkowski spacetime. The Lorentz transformations are precisely the isometries which leave the origin fixed. Thus, the Lorentz group is an isotropy subgroup of the isometry group of Minkowski spacetime. For this reason, the Lorentz group is sometimes called the *homogeneous Lorentz group* while the Poincaré group is called the *inhomogeneous Lorentz group*.

This chapter is devoted to the study of the proper Lorentz group $\mathrm{Spin}^+(1, n)$, which is the symmetry group of conformal geometry on the unit sphere. First, we will introduce the subject of Clifford Algebras. Of special importance are the Clifford group $\Gamma(p, q)$, the $\mathrm{Pin}(p, q)$ and $\mathrm{Spin}(p, q)$ groups. In Section 2.2 we will describe Möbius transformations in $\mathbb{R}^{p,q}$ through 2×2 Clifford valued matrices. This is the starting point for the derivation of the conformal group of the unit ball in $\mathbb{R}^n$. We will study some decompositions of the group $\mathrm{Spin}^+(1, n)$, namely its decomposition in terms of rotations of the form $R_s(x) := sx\bar{s}, s \in \mathrm{Spin}(n)$ and of Möbius transformations of the form $\varphi_a(x) := (x - a)(1 + ax)^{-1}$, $a \in B^n$, together with its Cartan decomposition. The introduction of the structure of gyrogroup in Section 2.6 allow us to understand the algebraic structure behind this decomposition. From the definition of the gyro-subgroups $(D_\omega^{n-1}, \oplus)$ and $(L_\omega, \oplus)$ we factorize the gyrogroup $(B^n, \oplus)$ and we define global and local sections for the unit ball. These sections are then extended to the proper Lorentz group. In Section 2.11 we compare the algebraic structure of $\mathrm{Spin}^+(1,n)$ with the algebraic structure of the group of automorphisms of the unit disc in $\mathbb{C}$. In Section 2.12 we will consider the particular case of $n=3$ and we establish the relationship between $(B^3,\oplus)$ and $SL(2,\mathbb{C})$, which is then generalized (in a non-trivial way) in Section 2.13. Finally, we will end this chapter with some integration's formulae regarding the Lorentz group $\mathrm{Spin}^+(1,n)$.

2.1 Real Clifford algebras

2.1.1 Definitions and basic properties

Let $\mathbb{R}^{p,q}$ be the real $n-$dimensional space, where $n=p+q$, endowed with the nondegenerate bilinear symmetric form $\mathcal{B}(\cdot,\cdot)$ of signature (p,q). We assume that an orthonormal basis $e_1,\ldots,e_n$ is given, such that

$$\begin{array}{rcll} \mathcal{B}(e_i,e_i) & = & -1, & i=1,\ldots,p \\ \mathcal{B}(e_i,e_i) & = & 1, & i=p+1,\ldots,n \\ \mathcal{B}(e_i,e_j) & = & 0, & i\neq j. \end{array}$$

We noticed that with this notation the Euclidean space with standard inner product is written as $\mathbb{R}^{0,n}$ (which sometimes will be abbreviated to $\mathbb{R}^n$).

The real Clifford algebra over $\mathbb{R}^{p,q}$ will be denoted by $\mathbb{R}_{p,q}$. It is the free algebra generated by the scalar identity 1 and the e_i's modulo the relations

$$e_ie_j+e_je_i=-2\mathcal{B}(e_i,e_j), \tag{2.1}$$

which means that we have as multiplication rules

$$\begin{array}{rl} e_i^2=+1, & i=1,\ldots,p \\ e_i^2=-1, & i=p+1,\ldots,n \\ e_ie_j=-e_je_i, & i\neq j. \end{array}$$

The last relation shows the noncommutative character of this algebra. Elements of $\mathbb{R}_{p,q}$ are called Clifford numbers. Take now a product of basic vectors $e_{i_1}\ldots e_{i_s}$. As e_i and e_j anticommute if $i\neq j$ it is possible to rearrange the factors in this product to obtain a product of the form $\pm e_{j_1},\ldots e_{j_s}$ where $j_1\leq\ldots\leq j_s$. On the other hand $e_i^2=\pm1$, and so the product can be reduced (possibly up to the sign) to $e_{t_1}\ldots e_{t_k}$, where $t_1<\ldots<t_k$.

The set of all $e_A=e_{t_1}\ldots e_{t_k}$, where $A=\{t_1,\ldots,t_k\}\subset N=\{1,\ldots,n\}$ for $1\leq t_1<\ldots<t_k\leq n$ and $e_\emptyset=1$ forms a basis for the real Clifford algebra $\mathbb{R}_{p,q}$. The algebra is universal if its dimension over $\mathbb{R}$ is equal to 2^n.

If A has k elements, then e_A is called a $k-$vector. Any linear combination of $k-$vectors is called a $k-$vector, and the vector space of $k-$vectors is written as $\mathbb{R}^k_{p,q}$. Then $\mathbb{R}_{p,q} = \bigoplus_{k\leq n} \mathbb{R}^k_{p,q}$ and the projection of a Clifford number a on $\mathbb{R}^k_{p,q}$ will be denoted by $[a]_k$. Instead of $1-$vectors the term vectors is used. Also the term bivectors is used for $2-$vectors. The $n-$vector $e_1 \dots e_n$ is called the pseudoscalar.

The product of two vectors is sometimes called the geometric product and it is decomposed in symmetric and antisymmetric parts, accordingly to

$$xy = \frac{1}{2}(xy + yx) + \frac{1}{2}(xy - yx). \tag{2.2}$$

Thus, we define a *inner product* ($\langle \cdot, \cdot \rangle$) and a *wedge product* ($\wedge$) by

$$\langle x, y\rangle = \frac{1}{2}(xy + yx) = -\mathcal{B}(x, y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$$

and

$$x \wedge y = \frac{1}{2}(xy - yx) = \sum_{i<j} (x_i y_j - x_j y_i) e_i e_j.$$

The *inner* and *wedge* products can be extended to the whole algebra $\mathbb{R}_{p,q}$. For our purpose we only need these definitions since all the arguments are vector valued.

If $\langle x, y\rangle = 0$ (resp. $x \wedge y = 0$) then we say that the vector x is orthogonal (resp. parallel) to the vector y. Therefore, if the vector x is orthogonal to the vector y then $xy = -yx$, while if the vector x is parallel to the vector y then $xy = yx$. By (2.2), $x^2 = \langle x, x\rangle$ is real for any vector x and the vector x, as an element of the Clifford algebra, is then invertible if and only if $x^2 \neq 0$. In this case $x^{-1} = \frac{-x}{|x|^2}$, while if $x^2 = 0$ then either x is zero or it is a zero divisor and hence not invertible.

There are two linear anti-automorphisms (reversion and conjugation) and a linear automorphism (main involution) defined on the Clifford algebra $\mathbb{R}_{p,q}$:

- the *main involution* is defined by

$$e_i' = -e_i, \quad 1' = 1, \quad (i = 1, \dots, n), \quad (ab)' = a'b', \quad \forall\, a, b \in \mathbb{R}_{p,q};$$

- the *reversion* is defined by

$$e_i^* = e_i, \quad 1^* = 1, \quad (i = 1, \dots, n), \quad (ab)^* = b^* a^*, \quad \forall\, a, b \in \mathbb{R}_{p,q};$$

- the *conjugation* is defined by

$$\overline{e_i} = -e_i, \quad \overline{1} = 1, \quad (i = 1, \dots, n), \quad \overline{ab} = \overline{b}\,\overline{a}, \quad \forall\, a, b \in \mathbb{R}_{p,q}.$$

We remark that $\overline{x} = (x')^* = (x^*)'$, $\forall x \in \mathbb{R}_{p,q}$. From the definition we can derive the action on the basis elements $e_A = e_{t_1} \dots e_{t_k}$, $1 \le t_1 < \dots < t_k \le n$ by the rule:

$$(e_A)' = (-1)^k e_A, \qquad (e_A)^* = (-1)^{\frac{k(k-1)}{2}} e_A, \qquad \overline{e_A} = (-1)^{\frac{k(k+1)}{2}} e_A.$$

In particular, for a vector x we have $\overline{x} = x' = -x$ and $x^* = x$.

2.1.2 Some groups in Clifford algebra

The set of linear mappings $T : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ preserving the bilinear form $\mathcal{B}$, i.e. for which $\mathcal{B}(Tx, Ty) = \mathcal{B}(x, y)$, for all $x, y \in \mathbb{R}^{p,q}$, forms a group under the operation of composition, called the *pseudo-orthogonal group* of $\mathbb{R}^{p,q}$ and denoted by $O(p,q)$. Alternatively, $O(p,q)$ may be considered as the set of those invertible $n \times n$ matrices Q satisfying $Q^T A Q = A$, where $A = (a_{i,j})$ is the matrix of $\mathcal{B}$ relative to the standard orthonormal basis of $\mathbb{R}^{p,q}$.

The subgroup of $O(p,q)$ consisting of those Q having determinant 1 is called the group of *special orthogonal transformations* or *rotations* of $\mathbb{R}^{p,q}$ and is denoted by $SO(p,q)$. The elements $O(p,q) \backslash SO(p,q)$ are called *antirotations*. For any p, q, the groups $O(p,q)$ and $O(q,p)$ are isomorphic, as are the groups $SO(p,q)$ and $SO(q,p)$.

Clifford algebras allow us to construct two-fold covering groups for the orthogonal groups and for the rotation groups in particular.

The even subalgebra $\mathbb{R}^+_{p,q}$ is the set of all Clifford numbers a such that $a = a'$. Alternatively, it can be defined as the subalgebra generated by all finite products of even number of vectors. In Clifford algebra the following groups are of special interest:

- The *Clifford group* $\Gamma(p,q)$, sometimes called the Lipschitz group. It is generated, as a multiplicative group, by all finite products of invertible vectors:

$$\Gamma(p,q) = \left\{ \prod_{i=1}^{k} s_i : s_i \in \mathbb{R}^{p,q}, s_i^2 \neq 0,\, i = 1, \dots, k,\, k \in \mathbb{N} \right\}. \tag{2.3}$$

The *even Clifford group* $\Gamma^+(p,q)$ arises as a subgroup of $\Gamma(p,q)$

$$\Gamma^+(p,q) = \Gamma(p,q) \cap \mathbb{R}^+_{p,q}.$$

- The *Pin group* $\mathrm{Pin}(p,q)$ is the group generated by all finite products of unit vectors, i.e. vectors x such that $x^2 = \pm 1$:

$$\mathrm{Pin}(p,q) = \{ s \in \Gamma(p,q) : s\overline{s} = \pm 1 \}.$$

We define the *Pin plus group* $\mathrm{Pin}^+(p,q)$ as a subgroup of $\mathrm{Pin}(p,q)$:

$$\mathrm{Pin}^+(p,q) = \{s \in \Gamma(p,q) : s\bar{s} = +1\}.$$

For $p = 0$, $\mathrm{Pin}^+(0,n) = \mathrm{Pin}(0,n)$.

- The *Spin group* $\mathrm{Spin}(p,q)$ is the subgroup arising from the intersection of $\mathrm{Pin}(p,q)$ with $\mathbb{R}^+_{p,q}$. Hence, it consists on all finite products of an even numbers of invertible unit vectors:

$$\mathrm{Spin}(p,q) = \{s \in \Gamma^+(p,q) : s\bar{s} = \pm 1\} = \mathrm{Pin}(p,q) \cap \mathbb{R}^+_{p,q}.$$

  Analogously we define the group $\mathrm{Spin}^+(p,q)$ as a subgroup of $\mathrm{Spin}(p,q)$:

$$\mathrm{Spin}^+(p,q) = \{s \in \Gamma^+(p,q) : s\bar{s} = +1\} = \mathrm{Pin}^+(p,q) \cap \mathbb{R}^+_{p,q}.$$

$\mathrm{Spin}^+(p,q)$ is a two fold covering group of $SO^+(p,q)$ (also denoted by $SO_0(p,q)$), which in its turn is the identity component of $SO(p,q)$, called the *proper Lorentz group* of $\mathbb{R}^{p,q}$. It is related with the preservation of the orientation of the time axes.

For example, in the space-time $\mathbb{R}^{1,3}$ the orthogonal group $O(1,3)$ is called the *Lorentz group*, its elements being called *Lorentz transformations*. The proper Lorentz group $SO^+(1,3)$ is also called the *Lorentz rotation group*.

If $p = 0$ we often use the notations $\Gamma(n)$, etc. instead of $\Gamma(0,n)$, etc., in analogy with the short notation $\mathbb{R}^n$ for $\mathbb{R}^{0,n}$. In $\mathbb{R}^n$, any non-zero vector is invertible and hence the groups $\Gamma(n), \mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$ have a simple description. For example

$$\mathrm{Pin}(n) = \left\{ \prod_{i=1}^{k} w_i : w_i \in S^{n-1}, k \in \mathbb{N} \right\}$$

and

$$\mathrm{Spin}(n) = \left\{ \prod_{i=1}^{2k} w_i : w_i \in S^{n-1}, k \in \mathbb{N} \right\},$$

where S^{n-1} denotes the $(n-1)-$sphere embedded in $\mathbb{R}^n$. It should be noticed that for any element of the Clifford group we have the relations

$$a' = a \quad \text{and} \quad \bar{a} = a^* \quad \text{if} \quad a \in \mathbb{R}^+_{p,q},$$

and

$$a' = -a \quad \text{and} \quad \bar{a} = -a^* \quad \text{otherwise}.$$

To see that $\mathrm{Pin}(p,q)$ gives a double covering group of $O(p,q)$ we define for every element s of the Pin group a transformation of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p,q}$ by $\chi(s)x = sxs'^{-1}$. This defines a group homomorphism satisfying $\chi(s_1)\chi(s_2) = \chi(s_1 s_2)$. Hence we restrict ourselves to the action $\chi(s)$, where $s \in \mathbb{R}^{p,q} \cap \Gamma(p,q)$. A vector x can be split into a part λs parallel to the vector s (λ real), and a part $x^\perp$ orthogonal to s and hence anticommunting with s. Since $s' = -s$ we obtain

$$\begin{aligned}\chi(s)x &= sxs'^{-1} = -s(\lambda s + x^\perp)s^{-1}\\ &= -\lambda s(ss^{-1}) + x^\perp(ss^{-1})\\ &= -\lambda s + x^\perp.\end{aligned}$$

Thus, $\chi(s)x$ describes the orthogonal reflection of x with respect to the hyperplane orthogonal to s, and therefore, for any $s \in \mathrm{Pin}(p,q)$, $\chi(s)$ is an orthogonal transformation. As the kernel of this homomorphism is $\{-1,+1\}$, $\mathrm{Pin}(p,q)$ gives a double covering of the orthogonal group $O(p,q)$, while $\mathrm{Spin}(p,q)$ gives a double covering of the special orthogonal group $SO(p,q)$.

Lemma 2.1.1 *If a or b belongs to $\Gamma(p,q)$ then ab^* and a^*b are simultaneously in $\mathbb{R}^{p,q}$.*

Proof: For every $s \in \Gamma(p,q)$ the mapping $\chi(s)x = sxs'^{-1}$, $x \in \mathbb{R}^{p,q}$ is an orthogonal transformation. Hence, sxs^* and $sx\overline{s} \in \mathbb{R}^{p,q}$.

We suppose that $a \in \Gamma(p,q)$ and $b \neq 0$. If $ab^* \in \mathbb{R}^{p,q}$ then $a^*(ab^*)a \in \mathbb{R}^{p,q}$. As $a^*a = \pm|a|^2 \neq 0$ then $b^*a \in \mathbb{R}^{p,q}$ and thus, $a^*b = (b^*a)^* \in \mathbb{R}^{p,q}$.

If $a^*b \in \mathbb{R}^{p,q}$ then $a(a^*b)a^* = (aa^*)(ba^*) \in \mathbb{R}^{p,q}$, hence $ba^* \in \mathbb{R}^{p,q}$ and $ab^* = (ba^*)^* \in \mathbb{R}^{p,q}$.

Analogous reasoning is valid if $b \in \Gamma(p,q)$ and $a \neq 0$.

■

For more details about Clifford Algebras we refer e.g. to [28] and [22].

2.2 Möbius transformations in $\mathbb{R}^{p,q}$

Möbius transformations on a space $\mathbb{R}^{p,q}$, $p+q \geq 3$, are the only conformal mappings of $\mathbb{R}^{p,q}$ which map spheres onto spheres.

The equation of a sphere $\mathbf{s}$ in $\mathbb{R}^{p,q}$ with center m and radius τ (where τ^2 is real, but not necessarily positive) can be written as $\mathcal{B}(y-m, y-m) = \tau^2$ or

$$-y^2 - 2\mathcal{B}(y,m) + (-m^2 - \tau^2) = 0.$$

We call $\mathbf{s}$ a positive or negative sphere according to whether τ^2 is positive or negative. If $\tau^2 = 0$ we have a zero sphere.

The group of Möbius transformations in $\mathbb{R}^{p,q}$ will be denoted by $\mathcal{M}(p,q)$, while the group of sense preserving Möbius transformations is denoted by $\mathcal{M}^+(p,q)$. It is well known that the orthogonal group $O(p+1,q+1)$ gives a double covering of $\mathcal{M}(p,q)$. In Clifford algebra the use of 2×2 Clifford valued matrices to describe Möbius transformations was proposed by Vahlen in 1902, [74]. Ahlfors rediscovered these matrices in 1986 and made an important study from the point of view of differential geometry [1]. A complete geometrical relation between $\mathcal{M}(p,q)$ and $\mathrm{Pin}(p+1,q+1)$ was given by Fillmore and Springer with a characterization of matrices in the $\mathrm{Pin}(p+1,q+1)$ group [35]. The nice feature of this approach is that the projective space over $\mathbb{R}^{p+1,q+1}$ is identified with the set of spheres of $\mathbb{R}^{p,q}$.

The construction is based on an algebra isomorphism between the Clifford algebra $\mathbb{R}_{p+1,q+1}$ and the algebra $(\mathbb{R}_{p,q})^{2\times 2}$ of 2×2 matrices with entries in $\mathbb{R}_{p,q}$ (see [22] for the details of the construction).

Each sphere $\mathbf{s}$ in $\mathbb{R}^{p,q}$ with center m and radius τ is associated with a ray of matrices by the mapping

$$T:\{y\,|\,\mathcal{B}(y-m,y-m)=\tau^2\}\mapsto\lambda\begin{pmatrix} m & -m^2-r^2\\ 1 & -m\end{pmatrix}.$$

A point $x\in\mathbb{R}^n$ is associated with a zero radius sphere with center x and thus it is represented by the matrix $\begin{pmatrix} x & -x^2\\ 1 & x\end{pmatrix}$.

The following theorems characterize the groups $\Gamma(1,n+1)$ and $\Gamma(p+1,q+1)$ (see [21]):

Theorem 2.2.1 *(Vahlen) A matrix* $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ *represents a Möbius transformation in* $\mathbb{R}^n$ *if and only if*

1. $a,b,c,d\in\Gamma(n)\cup\{0\}$;

2. $ab^*,cd^*,c^*a,d^*b\in\mathbb{R}^n$;

3. *the pseudodeterminant of* A, $\lambda=ad^*-bc^*$, *is real and non zero.*

Theorem 2.2.2 *The Clifford group* $\Gamma(p+1,q+1)$ *coincides with the set of matrices* $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ *satisfying*

(i) $a,b,c,d\in T(p,q)$, *where* $T(p,q)$ *is the set of all products of vectors in* $\mathbb{R}^{p,q}$;

(ii) $bd^*,ac^*,a^*b,c^*d\in\mathbb{R}^{p,q}$;

(iii) *the pseudodeterminant of A, $\lambda = ad^* - bc^*$ is real and non-zero.*

Then Möbius transformations in $\mathbb{R}^{p,q}$ corresponds to the orthogonal rotations in the projective space $P\mathbb{R}^{p+1,q+1}$ by the mapping

$$g : \mathbf{s} \mapsto A\mathbf{s}A'^{-1}, \tag{2.4}$$

$$\begin{pmatrix} m & -m^2-r^2 \\ 1 & -m \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} m & -m^2-r^2 \\ 1 & -m \end{pmatrix}\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}.$$

We can write a Möbius transformation as a fractional linear transformation:

$$\varphi(x) = \frac{ax+b}{cx+d} := (ax+b)(cx+d)^{-1}.$$

Remark 2.2.3 *In a non-commutative algebra setting the ambiguous notation $\frac{a}{b}$ always means ab^{-1}. Consequently $\frac{ac}{bc} = \frac{a}{b}$ but $\frac{ca}{cb} \neq \frac{a}{b}$ in general.*

2.3 Conformal group of the unit ball

Theorem 2.3.1 *The group $\mathcal{M}(B^n)$ of all conformal mappings of the unit ball B^n onto itself admits the matricial representation*

$$\begin{pmatrix} u & v' \\ v & u' \end{pmatrix}, \quad u, v \in \Gamma(n) \cup \{0\}, \quad uv^* \in \mathbb{R}^n, \quad |u|^2 - |v|^2 = 1. \tag{2.5}$$

Its inverse is $\begin{pmatrix} u & v' \\ v & u' \end{pmatrix}^{-1} = \begin{pmatrix} \overline{u} & -\overline{v} \\ -v^* & u^* \end{pmatrix}.$

Proof: As the unit sphere S^{n-1} corresponds to the matrix $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we have to verify that matrices (2.5) preserve s by the action described in (2.4). Indeed,

$$\begin{pmatrix} u & v' \\ v & u' \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u^* & v^* \\ \overline{v} & \overline{u} \end{pmatrix} = \begin{pmatrix} v' & -u \\ u' & -v \end{pmatrix}\begin{pmatrix} u^* & v^* \\ \overline{v} & \overline{u} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and thus the corresponding Möbius transformations preserve the unit sphere. Finally, to see that they preserve the unit ball we observe that we can associate to the matrix (2.5) the transformation $\phi(x) = (ux+v')(vx+u')^{-1}$ and therefore, it satisfies $|\varphi(0)| = |\frac{v'}{u'}| = \frac{|v|}{|u|} < 1$ by the constraint $|u|^2 - |v|^2 = 1$.

■

We remark that $\mathcal{M}(B^n)$ can be identified with the group $\mathrm{Pin}^+(1,n)$ via the matrix representation (2.5)(see [22]). The constraint $|u|^2-|v|^2=1$ makes the representation (2.5) of $\mathcal{M}(B^n)$ somewhat difficult to use. However, we can identify the unit ball with the right coset $\mathrm{Pin}(n)\backslash\mathcal{M}(B^n)$ and therefore, $\mathcal{M}(B^n)\sim \mathrm{Pin}(n)\times B^n$. We have the following decomposition:

$$\begin{aligned}\begin{pmatrix} u & v' \\ v & u' \end{pmatrix} &= |u|\begin{pmatrix} \frac{u}{|u|} & 0 \\ 0 & \frac{u'}{|u|} \end{pmatrix}\begin{pmatrix} 1 & \frac{\overline{u}}{|u|^2}v' \\ \frac{u^*}{|u|^2}v & 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{1-|a|^2}}\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}\begin{pmatrix} 1 & a' \\ a & 1 \end{pmatrix},\end{aligned} \tag{2.6}$$

where

$$w=\frac{u}{|u|},\quad a=\frac{u^*}{|u|^2}v,\quad \sqrt{1-|a|^2}=|u|^{-1}. \tag{2.7}$$

As $uv^*\in\mathbb{R}^n$ we conclude that $u^*v\in\mathbb{R}^n$ by Lemma 2.1.1 and therefore $a\in\mathbb{R}^n$. Moreover, $|a|^2=\frac{|v|^2}{|u|^2}<1$, since $|u|^2>|v|^2$.

For $a\in B^n$ and $w\in\mathrm{Pin}(n)$, we shall denote by $\varphi_{(a,\omega)}(x)$ the transformations associated with the matrix representation

$$\frac{1}{\sqrt{1-|a|^2}}\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}\begin{pmatrix} 1 & a' \\ a & 1 \end{pmatrix}=\frac{1}{\sqrt{1-|a|^2}}\begin{pmatrix} w & wa' \\ w'a & w' \end{pmatrix}. \tag{2.8}$$

These transformations constitute the group $\mathcal{M}(B^n)$.

Lemma 2.3.2 *The following commutation relation holds*

$$\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}\begin{pmatrix} 1 & a' \\ a & 1 \end{pmatrix}=\begin{pmatrix} 1 & wa'w^* \\ w'a\overline{w} & 1 \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}.$$

Moreover, $w'a\overline{w}=waw^*$, for all $a\in B^n$ and $w\in\mathrm{Pin}(n)$.

Proof: As $w\in\mathrm{Pin}(n)$ then $w\overline{w}=1$, and $w^*w'=(\overline{w}w)'=1$. Hence,

$$\begin{pmatrix} 1 & wa'w^* \\ w'a\overline{w} & 1 \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}=\begin{pmatrix} w & wa'w^*w' \\ w'a\overline{w}w & w' \end{pmatrix}=\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}\begin{pmatrix} 1 & a' \\ a & 1 \end{pmatrix}.$$

If $w\in\mathrm{Spin}(n)=\mathrm{Pin}(n)\cap\mathbb{R}^+_{0,n}$, where $\mathbb{R}^+_{0,n}$ denotes the even subalgebra of $\mathbb{R}_{0,n}$, then $w'=w$ and hence $w^*=\overline{w'}=\overline{w}$. If $w\in\mathrm{Pin}(n)\backslash\mathrm{Spin}(n)$ then $w'=-w$ and hence $w^*=\overline{w'}=-\overline{w}$. In both cases the relation $w'a\overline{w}=waw^*$ holds.

■

Theorem 2.3.3 *The product of two elements of* $\mathrm{Pin}^+(1,n)$ *yields again an element of* $\mathrm{Pin}^+(1,n)$:

$$\frac{1}{\sqrt{1-|a|^2}}\frac{1}{\sqrt{1-|b|^2}}\begin{pmatrix} w_1 & 0 \\ 0 & w_1' \end{pmatrix}\begin{pmatrix} 1 & a' \\ a & 1 \end{pmatrix}\begin{pmatrix} w_2 & 0 \\ 0 & w_2' \end{pmatrix}\begin{pmatrix} 1 & b' \\ b & 1 \end{pmatrix}$$
$$= \frac{1}{\sqrt{1-|y|^2}}\begin{pmatrix} w_1w_2z & 0 \\ 0 & (w_1w_2z)' \end{pmatrix}\begin{pmatrix} 1 & y' \\ y & 1 \end{pmatrix},$$

where $y = (1-w_2^*aw_2b)^{-1}(w_2^*aw_2+b)$ *and* $z = \frac{1-w_2^*aw_2b}{|1-w_2^*aw_2b|}$.

Proof: By Lemma 2.3.2 we have

$$\begin{pmatrix} w_1 & 0 \\ 0 & w_1' \end{pmatrix}\begin{pmatrix} 1 & a' \\ a & 1 \end{pmatrix}\begin{pmatrix} w_2 & 0 \\ 0 & w_2' \end{pmatrix}\begin{pmatrix} 1 & b' \\ b & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} w_1 & 0 \\ 0 & w_1' \end{pmatrix}\begin{pmatrix} 1 & a' \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & w_2b'w_2^* \\ w_2'b\overline{w_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & b' \\ b & 1 \end{pmatrix}$$
$$= \begin{pmatrix} w_1 & 0 \\ 0 & w_1' \end{pmatrix} t \begin{pmatrix} 1 & t^{-1}(w_2b'w_2^*+a') \\ t^{-1}(a+w_2'b\overline{w_2}) & t^{-1}(aw_2b'w_2^*+1) \end{pmatrix}\begin{pmatrix} w_2 & 0 \\ 0 & w_2' \end{pmatrix},$$

where $t = 1 + a'w_2'b\overline{w_2}$.

As a and b are vectors, $a' = \overline{a} = -a^* = -a$ and $b' = \overline{b} = -b^* = -b$. Using also the relation $w_2'b\overline{w_2} = w_2bw_2^*$, we can make the following simplifications

$$(1+a'w_2'b\overline{w_2})^{-1}(aw_2b'w_2^*+1) = (1-aw_2bw_2^*)^{-1}(-aw_2bw_2^*+1) = 1,$$

$$\begin{aligned}(1+a'w_2'b\overline{w_2})^{-1}(w_2b'w_2^*+a') &= w_2(w_2-aw_2'b)^{-1}(-w_2b-aw_2')w_2^* \\ &= w_2(1-\overline{w_2}aw_2'b)^{-1}w_2^{-1}w_2(-b-\overline{w_2}aw_2')w_2^* \\ &= w_2xw_2^*,\end{aligned}$$

$$\begin{aligned}(1+a'w_2'b\overline{w_2})^{-1}(a+w_2'b\overline{w_2}) &= (1-aw_2bw_2^*)^{-1}(a+w_2'b\overline{w_2}) \\ &= w_2'(w_2'-aw_2b)^{-1}(aw_2+w_2'b)\overline{w_2} \\ &= w_2'(1-w_2^*aw_2b)^{-1}w_2'^{-1}w_2'(w_2^*aw_2+b)\overline{w_2} \\ &= w_2yw_2^*,\end{aligned}$$

with $x = (1-\overline{w_2}aw_2'b)^{-1}(-b-\overline{w_2}aw_2')$ and $y = (1-w_2^*aw_2b)^{-1}(w_2^*aw_2+b)$. It is easy to see that $x = y'$.

Therefore, the product of the four initial matrices is equal to

$$\begin{pmatrix} w_1 & 0 \\ 0 & w_1' \end{pmatrix}(1+a'w_2'b\overline{w_2})\begin{pmatrix} 1 & w_2y'w_2^* \\ w_2yw_2^* & 1 \end{pmatrix}\begin{pmatrix} w_2 & 0 \\ 0 & w_2' \end{pmatrix}$$

$$= \begin{pmatrix} w_1 & 0 \\ 0 & w_1' \end{pmatrix}(1+a'w_2'b\overline{w_2})\begin{pmatrix} w_2 & 0 \\ 0 & w_2' \end{pmatrix}\begin{pmatrix} 1 & y' \\ y & 1 \end{pmatrix}$$

$$= \begin{pmatrix} w_1(1+a'w_2'b\overline{w_2})w_2 & 0 \\ 0 & w_1'(1+a'w_2'b\overline{w_2})w_2' \end{pmatrix}\begin{pmatrix} 1 & y' \\ y & 1 \end{pmatrix}$$

$$= \begin{pmatrix} w_1w_2(1-w_2^*aw_2b) & 0 \\ 0 & w_1'w_2'(1-\overline{w_2}aw_2'b) \end{pmatrix}\begin{pmatrix} 1 & y' \\ y & 1 \end{pmatrix}.$$

We note that $w_2' = \epsilon w_2$, where $\epsilon = +1$ if $w_2 \in \mathrm{Spin}(n)$ and $\epsilon = -1$ if $w_2 \in \mathrm{Pin}(n)\backslash\mathrm{Spin}(n)$. Hence, $w_1'(1+a'w_2'b\overline{w_2})w_2' = w_1'w_2(1+\overline{w_2}a'w_2'b)\epsilon = w_1'w_2'(1-\overline{w_2}aw_2'b)$.

To finish the proof we need to calculate the product of the factors $(1-|a|^2)^{-1/2}$ and $(1-|b|^2)^{-1/2}$ in terms of the variable y. We remark that $y = (1-w_2^*aw_2b)^{-1}(w_2^*aw_2+b) = \varphi_{-b}(w_2^*aw_2)$ (see Lemma 2.5.3). By the relation

$$\frac{1-|\varphi_a(x)|^2}{1-|x|^2} = \frac{1-|a|^2}{|1+ax|^2}$$

we obtain

$$1-|y|^2 = 1-|\varphi_{-b}(w_2^*aw_2)|^2 = \frac{(1-|w_2^*aw_2|^2)(1-|b|^2)}{|1-bw_2^*aw_2|^2}$$

or

$$(1-|a|^2)^{-1/2}(1-|b|^2)^{-1/2} = (1-|y|^2)^{-1/2}|1-bw_2^*aw_2|^{-1}.$$

Combining all the results yields

$$\frac{1}{\sqrt{1-|a|^2}}\frac{1}{\sqrt{1-|b|^2}}\begin{pmatrix} w_1 & 0 \\ 0 & w_1' \end{pmatrix}\begin{pmatrix} 1 & a' \\ a & 1 \end{pmatrix}\begin{pmatrix} w_2 & 0 \\ 0 & w_2' \end{pmatrix}\begin{pmatrix} 1 & b' \\ b & 1 \end{pmatrix} =$$

$$= \frac{1}{\sqrt{1-|y|^2}}\begin{pmatrix} w_1w_2z & 0 \\ 0 & (w_1w_2z)' \end{pmatrix}\begin{pmatrix} 1 & y' \\ y & 1 \end{pmatrix},$$

where $y = (1-w_2^*aw_2b)^{-1}(w_2^*aw_2+b)$ and $z = \frac{1-w_2^*aw_2b}{|1-w_2^*aw_2b|}$.

∎

Corollary 2.3.4 *The following relations hold:*

1. $\varphi_{(w_1,0)}\varphi_{(w_2,0)} = \varphi_{(w_1w_2,0)}$;
2. $\varphi_{(1,a)}\varphi_{(1,b)} = \varphi_{(w,c)}$, *where* $w = \frac{1-ab}{|1-ab|}$ *and* $c = (1-ab^{-1})(a+b)$;
3. $\varphi_{(1,a)}\varphi_{(1,-a)} = I$, *thus* $\varphi^{-1}_{(1,a)} = \varphi_{(1,-a)}$;
4. $\varphi^{-1}_{(1,a)}(0) = a$ *and* $\varphi^{-1}_{(1,a)}(-a) = 0$.

Definition 2.3.5 *For $w_1, w_2 \in \mathrm{Pin}(n)$ and $a, b \in B^n$ we define the product*

$$(w_1, a)(w_2, b) = (w_1w_2\frac{1-w_2^*aw_2b}{|1-w_2^*aw_2b|}, (1-w_2^*aw_2b)^{-1}(w_2^*aw_2+b)). \tag{2.9}$$

Remark 2.3.6 *Similarly, if we consider $\mathcal{M}(B^n) \sim B^n \times \mathrm{Pin}(n)$, i.e. we identify the open unit ball B^n with the left cosets $\mathcal{M}(B^n)/\mathrm{Pin}(n)$, this product can be written as*

$$(a, w_1)(b, w_2) = ((w_1bw_1^*+a)(1-aw_1bw_1^*)^{-1}, \frac{1-aw_1bw_1^*}{|1-aw_1bw_1^*|}w_1w_2). \tag{2.10}$$

As we can see, the unit ball B^n is a homogeneous space. In Section 2.6 we will see that it has also an algebraic structure.

2.4 Cartan or KAK decomposition

The decomposition (2.6) is the polar decomposition of a unimodular matrix onto the product of its Hermitian and orthogonal parts. It is a special case of the global Cartan decomposition of a Lie group associated with a Riemannian symmetric space of noncompact type (see [47]). From now on we will restrict ourselves to conformal mappings which are space and time preserving, i.e. proper orthochronous Lorentz transformations. This means that we will restrict to the subgroup $\mathrm{Spin}^+(1,n)$, which is a double covering group of the proper orthochronous Lorentz group $\mathrm{SO}^+(1,n)$. We will derive the KAK decomposition for the proper Lorentz group $\mathrm{Spin}^+(1,n)$, where K is the maximal compact subgroup $\mathrm{Spin}(n)$, and $A = \mathrm{Spin}(1,1)$, which is a double covering of $\mathrm{SO}(1,1)$.

Lemma 2.4.1 *Each $a \in B^n$ can be described as*

$$a = sre_n\overline{s} \tag{2.11}$$

where $r \in [0,1[$ and $s = s_1 \dots s_{n-1} \in \mathrm{Spin}(n)$, with

$$s_i = \cos\frac{\theta_i}{2} + e_{i+1}e_i \sin\frac{\theta_i}{2}, \quad i = 1,\dots,n-1,$$

where $0 \le \theta_1 < 2\pi$ $\;0 \le \theta_i < \pi$, $i = 2,\dots,n-1$, and $\theta_{n-1} := \phi$, This follows from the description of $a \in B^n$ in spherical coordinates

$$a : \begin{cases} a_1 & = & r\sin\theta_1\, \sin\theta_2 \cdots \sin\theta_{n-2}\sin\phi \\ a_2 & = & r\cos\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2}\sin\phi \\ a_3 & = & r\cos\theta_2 \sin\theta_3 \cdots \sin\theta_{n-2}\sin\phi \\ \vdots & & \\ a_{n-1} & = & r\cos\theta_{n-2}\sin\phi \\ a_n & = & r\cos\phi \end{cases} \tag{2.12}$$

using the rotors $\mathrm{e}^{e_{i+1}e_i \frac{\theta_i}{2}} = \cos\left(\frac{\theta_i}{2}\right) + e_{i+1}e_i \sin\left(\frac{\theta_i}{2}\right)$, $i = 1, \ldots, n-1$. For $s = \cos\left(\frac{\theta}{2}\right) + e_ie_j \sin\left(\frac{\theta}{2}\right)$, $i \neq j$ we have

$$s x \overline{s} = (\cos\theta\, x_i - \sin\theta\, x_j)e_i + (\cos\theta\, x_j + \sin\theta\, x_i)e_j + \sum_{\substack{k=1 \\ k\neq i,j}}^{n} x_k e_k,$$

which represents a rotation of angle θ in the $e_{i+1}e_i-$plane. In general we have $s_is_j \neq s_js_i$, $i \neq j$. The order of the rotors is important since different choices leads to different systems of coordinates. Due to the relevance of the rotor s_{n-1} we shall denote $\theta_{n-1} := \phi$.

Remark 2.4.2 *From now on we will denote by s_* the product $s_1 \cdots s_{n-2}$. It is an element of Spin$(n-1)$ and it leaves the x_n-axis invariant. This is due to the decomposition Spin(n)/ Spin$(n-1) \cong S^{n-1}$, where Spin$(n-1)$ is the stabilizer of the vector $e_n = (0,\ldots,0,1)$. The symbol "s" usually will denote an element of Spin(n).*

Since $a \in B^n, a' = \overline{a} = -a$. By Lemma 2.4.1 we have the following decomposition

$$\begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & -sre_n\overline{s} \\ sre_n\overline{s} & 1 \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 1 & -re_n \\ re_n & 1 \end{pmatrix}\begin{pmatrix} \overline{s} & 0 \\ 0 & \overline{s} \end{pmatrix} \tag{2.13}$$

and hence, the global Cartan decomposition or KAK decomposition for an arbitrary element of $\mathrm{Spin}^+(1,n)$ is

$$\begin{pmatrix} u & v' \\ v & u' \end{pmatrix} = \frac{1}{\sqrt{1-r^2}}\begin{pmatrix} ws & 0 \\ 0 & w's \end{pmatrix}\begin{pmatrix} 1 & -re_n \\ re_n & 1 \end{pmatrix}\begin{pmatrix} \overline{s} & 0 \\ 0 & \overline{s} \end{pmatrix}. \tag{2.14}$$

It remains to show that the second matrix on the left hand side of (2.14) corresponds to the subgroup $\mathrm{Spin}(1,1)$ of hyperbolic rotations. For that purpose we will use the hyperbolic model of space-time in Clifford Analysis (see [20], [22]).

We consider the Minkowski space $\mathbb{R}^{1,n}$ and its associated Clifford algebra $\mathbb{R}_{1,n}$, together with the special identification $\epsilon := e_{n+1}$, the vector that spans the time axis.

A pure boost in the direction $\omega \in S^{n-1}$ (or hyperbolic rotation) is viewed as a transformation $B(\omega)$ which belongs to the Lie algebra generated by the bivectors of the form $\epsilon\omega$. It has the general form

$$s_\omega = \cosh\frac{\alpha}{2} + \epsilon\omega\sinh\frac{\alpha}{2},\ \alpha \in \mathbb{R},\ \omega \in S^{n-1} \tag{2.15}$$

and it acts on space-time vectors $X = x + t\epsilon$, with $x \in \mathbb{R}^n, t \in \mathbb{R}$, according to the transformations rules $X \mapsto Y = s_\omega X\overline{s_\omega}$, and on functions via the (Spin-invariant) L or $H-$representations

$$\begin{aligned} F(X) &\mapsto L(s_\omega)F(X) = s_\omega F(\overline{s_\omega}Xs_\omega) \\ F(X) &\mapsto H(s_\omega)F(X) = s_\omega F(\overline{s_\omega}Xs_\omega)\overline{s_\omega}. \end{aligned}$$

Proposition 2.4.3 *Let $x = \sum_{i=1}^n x_i e_i \in S^{n-1}$. The boost $s_\omega x\overline{s_\omega}$ yields the point $\zeta \in S^{n-1}$:*

$$\zeta = \frac{x + ((\cosh\alpha - 1)\langle\omega, x\rangle - \sinh\alpha)\omega}{\cosh\alpha - \sinh\alpha\langle x, \omega\rangle}. \tag{2.16}$$

Proof: We extend the point x to the Minkowski space $\mathbb{R}^{1,n}$ by considering the point $X = x + \epsilon$ in the intersection of the Null Cone with the hyperplane $T = 1$. Since $\overline{\epsilon\omega} = -\epsilon\omega$, $\epsilon x = -x\epsilon$ and $\epsilon^2 = +1$, we obtain

$$\begin{aligned} s_\omega X\overline{s_\omega} &= \left(\cosh\frac{\alpha}{2} + \epsilon\omega\sinh\frac{\alpha}{2}\right)(x+\epsilon)\left(\cosh\frac{\alpha}{2} - \epsilon\omega\sinh\frac{\alpha}{2}\right) \\ &= \cosh^2\left(\frac{\alpha}{2}\right)x + \cosh\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\alpha}{2}\right)(x\omega + \omega x)\epsilon + \left(\cosh^2\frac{\alpha}{2} + \sinh^2\frac{\alpha}{2}\right)\epsilon \\ &\quad -2\sinh\left(\frac{\alpha}{2}\right)\cosh\left(\frac{\alpha}{2}\right)\omega - \sinh^2\left(\frac{\alpha}{2}\right)\omega x\omega. \end{aligned}$$

As

$$\omega x\omega = (-\langle\omega, x\rangle + \frac{1}{2}(\omega x - x\omega))\omega = -\langle\omega, x\rangle\omega + \frac{1}{2}\omega x\omega + \frac{1}{2}|w|^2 x$$

we conclude that $\omega x\omega = -2\langle\omega, x\rangle\omega + x$. Moreover, $x\omega + \omega x = -2\langle\omega, x\rangle$. Therefore,

$$\begin{aligned} Y = s_\omega X\overline{s_\omega} &= x + \left(2\sinh^2\left(\frac{\alpha}{2}\right)\langle\omega, x\rangle - \sinh\alpha\right)\omega + (\cosh\alpha - \sinh\alpha\langle\omega, x\rangle)\epsilon \\ &= x + ((\cosh\alpha - 1)\langle\omega, x\rangle - \sinh\alpha)\omega + (\cosh\alpha - \sinh\alpha\langle\omega, x\rangle)\epsilon. \end{aligned}$$

By homogeneity, i.e. by restricting this point to the hyperplane $T = 1$ we obtain the desired result:

$$\xi = \frac{x + ((\cosh\alpha - 1)\langle\omega, x\rangle - \sinh\alpha)\omega}{\cosh\alpha - \sinh\alpha\langle\omega, x\rangle}.$$

By some algebraic manipulations we can easily prove that $\xi \in S^{n-1}$.

■

Corollary 2.4.4 *The fixed points of the transformation (2.16) are ω and $-\omega$.*

Remark 2.4.5 *A pure boost $B(\omega)$ can always be described as the composition $R(e_n,\omega)\circ B(e_n)\circ R(\omega,e_n)$, where $R(\omega,x)$ stands for the rotation which maps $\omega\in S^{n-1}$ into $x\in S^{n-1}$. Indeed, by Lemma 2.4.1 we can write $\omega = se_n\overline{s}$. Since $\epsilon s = s\epsilon$ then we have $s_\omega = ss_{e_n}\overline{s}$ (here s_{e_n} is the boost in the e_n-direction). Hence $s_\omega X\overline{s_\omega} = ss_{e_n}\overline{s}Xs\overline{s_{e_n}}\,\overline{s}$. This is the equivalent description of the KAK decomposition of the group $SO_0(1,n)$ by matrices, presented in [67], considering $\alpha\in\mathbb{R}^+$.*

We will consider the subgroup $\mathrm{Spin}(1,1)$ as the subgroup of Lorentz boosts in the e_n-direction. In this case Formula (2.16) takes a simpler expression:

$$s_{e_n}x\overline{s_{e_n}} = \sum_{i=1}^{n-1}\frac{x_i}{\cosh\alpha - \sinh\alpha\, x_n}e_i + \frac{\cosh\alpha\, x_n - \sinh\alpha}{\cosh\alpha - \sinh\alpha\, x_n}e_n. \tag{2.17}$$

Instead of working with matrices of the form (2.6) we will consider the transformations:

- $R_s(x) := sx\overline{s},\ s\in\mathrm{Spin}(n)$ denotes a rotation in $\mathbb{R}^n$;
- $\varphi_a(x) := (x-a)(1+ax)^{-1}, a\in B^n$ is a Möbius transformation.

As an immediate consequence of the KAK decomposition of $\mathrm{Spin}^+(1,n)$ we obtain the polar decomposition of φ_a.

Lemma 2.4.6 *For $a = sre_n\overline{s}$, (c.f. Lemma 2.4.1) we have*

$$\varphi_a(x) = \varphi_{sre_n\overline{s}}(x) = s\varphi_{re_n}(\overline{s}xs)\overline{s}. \tag{2.18}$$

Thus, a Möbius transformation φ_a can be described by a Möbius transformation φ_{re_n}, where re_n is a point belonging to the intersection of the unit ball with the positive x_n-axis and a convenient rotation induced by s.

This decomposition is not unique (and, therefore, neither it is the Cartan decomposition). The centralizer C of $A = \mathrm{Spin}(1,1)$ in $K = \mathrm{Spin}(n)$, i.e.

$$C = \{s\in\mathrm{Spin}(n) : \overline{s}\varphi_{re_n}(x)s = \varphi_{re_n}(\overline{s}xs)\},$$

corresponds to the subgroup $\mathrm{Spin}(n-1)$ of rotations around the x_n-axis. Thus, if $s_1\in C$ and $a = sre_n\overline{s}$ then

$$ss_1\varphi_{re_n}(\overline{ss_1}xss_1)\overline{ss_1} = \varphi_{ss_1re_n\overline{s_1}\,\overline{s}}(x) = \varphi_{sre_n\overline{s}}(x) = \varphi_a(x)$$

which shows that

$$\varphi_a(x) = s\varphi_{re_n}(\overline{s}xs)\overline{s} = ss_1\varphi_{re_n}(\overline{s_1}\,\overline{s}xss_1)\overline{s_1}\,\overline{s}.$$

Therefore, only the rotation $s_{n-1} = \cos\frac{\phi}{2} + e_n e_{n-1}\sin\frac{\phi}{2}$ affects the operator φ_{re_n}. This is very important for the study of a local dilation on the unit sphere S^{n-1}. In Chapter 4 we study how the parameters r and ϕ do influence local dilations around the North Pole of S^{n-1}.

Now we show that there is an isomorphism between the subgroup of Lorentz boosts in a fixed direction $\omega \in S^{n-1}$ and the subgroup of Möbius transformations φ_a, for $a = t\omega$, with $t \in]-1, 1[$.

For the following we need to know the component functions of φ_a.

Proposition 2.4.7 *If $x \in B^n$ then*

$$\varphi_a(x) = \frac{(1-|a|^2)x - (1+|x|^2 - 2\langle a, x\rangle)a}{1 - 2\langle a, x\rangle + |a|^2|x|^2}. \tag{2.19}$$

Moreover, if $x \in S^{n-1}$ we obtain

$$\varphi_a(x) = \frac{(1-|a|^2)x - 2(1-\langle a, x\rangle)a}{1 - 2\langle a, x\rangle + |a|^2}. \tag{2.20}$$

Proof: Consider $x \in B^n$. Then

$$\varphi_a(x) = (x-a)(1+ax)^{-1} = \frac{(x-a)(1+xa)}{|1+ax|^2} = \frac{x - |x|^2 a - a - axa}{1 - 2\langle a, x\rangle + |a|^2|x|^2}$$

As

$$axa = (-\langle a, x\rangle + \frac{1}{2}(ax - xa))a = -\langle a, x\rangle a + \frac{1}{2}axa + \frac{1}{2}|a|^2 x$$

we conclude that $axa = -2\langle a, x\rangle a + |a|^2 x$. Thus, we obtain

$$\varphi_a(x) = \frac{(1-|a|^2)x - (1+|x|^2 - 2\langle a, x\rangle)a}{1 - 2\langle a, x\rangle + |a|^2|x|^2}.$$

If $x \in S^{n-1}$, then $|x|^2 = 1$ and we obtain the expression (2.20).

∎

Now we can relate transformations (2.16) and (2.20).

Proposition 2.4.8 *([22]) We assume, in (2.20), $a = t\omega$, with $-1 < t < 1$ and $\omega \in S^{n-1}$. Then transformations (2.16) and (2.20) are related by*

$$\cosh\alpha = \frac{1+t^2}{1-t^2} \quad \textit{and} \quad \sinh\alpha = \frac{2t}{1-t^2} \tag{2.21}$$

$$\alpha = \ln\left(\frac{1+t}{1-t}\right) \quad \textit{and} \quad t = \frac{e^\alpha - 1}{e^\alpha + 1} = \tanh\left(\frac{\alpha}{2}\right). \tag{2.22}$$

Proof: If $x \in S^{n-1}$ and $a = t\omega$ then

$$\varphi_{t\omega}(x) = \frac{(1-t^2)x - 2(1-t\langle\omega, x\rangle)t\omega}{1-2t\langle\omega, x\rangle + t^2} = \frac{x + \left(\frac{2t^2}{1-t^2}\langle\omega, x\rangle - \frac{2t}{1-t^2}\right)\omega}{\frac{1+t^2}{1-t^2} - \frac{2t}{1-t^2}\langle\omega, x\rangle}.$$

The result follows by comparing this expression with (2.16).

■

By this isomorphism it is easy to see that φ_{te_n}, with $t \in]-1, 1[$ corresponds to the Spin$(1, 1)$ group of hyperbolic rotations in the e_n-direction.

It is interesting to see the action of Spin$(1, 1)$ on the tangent plane of S^{n-1}, at the North Pole $e_n = (0, \ldots, 0, 1)$.

Lemma 2.4.9 *In spherical coordinates the action of an element of Spin$(1, 1)$ in the form (2.15) $(\omega = e_n)$ on a given point $x = x(\theta_1, \ldots, \theta_{n-2}, \phi) \in S^{n-1}$ is fully determined by*

$$x \mapsto x_\alpha = \{(\theta_j)_\alpha, \phi_\alpha\}_{j=1}^{n-2},$$

where

$$(\theta_j)_\alpha = \theta_j,\ j = 1\ldots, n-2, \quad \textit{and} \quad \tan\frac{\phi_\alpha}{2} = e^\alpha \tan\frac{\phi}{2}. \tag{2.23}$$

Proof: If we consider $x = (x_1, \ldots, x_n) \in S^{n-1}$ written in spherical coordinates as in (2.12) then the relation between spherical and cartesian coordinates is given by

$$\cos\theta_k = \frac{x_{k+1}}{r_{k+1}}, \quad \sin\theta_k = \frac{r_k}{r_{k+1}}, \quad r_k^2 = x_1^2 + \ldots + x_k^2,\ \ k = 1, \ldots, n-1,$$

with the identification $\theta_{n-1} := \phi$. By (2.17) we obtain

$$(\theta_j)_\alpha = \theta_j,\ j = 1, \ldots, n-2, \quad \text{and}$$

$$\tan\frac{\phi_\alpha}{2} = \sqrt{\frac{1-\cos\phi_\alpha}{1+\cos\phi_\alpha}} = \sqrt{\frac{1 - \frac{\cosh\alpha\, x_n - \sinh\alpha}{\cosh\alpha - \sinh\alpha\, x_n}}{1 + \frac{\cosh\alpha\, x_n - \sinh\alpha}{\cosh\alpha - \sinh\alpha\, x_n}}} = \sqrt{\frac{\cosh\alpha + \sinh\alpha}{\cosh\alpha - \sinh\alpha}}\sqrt{\frac{1-x_n}{1+x_n}}.$$

Using the identities $\cosh\alpha = \frac{e^\alpha + e^{-\alpha}}{2}$ and $\sinh\alpha = \frac{e^\alpha - e^{-\alpha}}{2}$ it follows that

$$\tan\frac{\phi_\alpha}{2} = \mathrm{e}^\alpha \tan\frac{\phi}{2}.$$

■

Sometimes it is useful to consider the stereographic projection of S^{n-1} onto its tangent plane.

Definition 2.4.10 *The stereographic projection of S^{n-1} onto its tangent plane at the point $e_n = (0,\ldots,0,1)$ is given by the mapping*

$$\Phi_1 : S^{n-1}\backslash\{-e_n\} \to \mathbb{R}^{n-1}$$

$$(x_1,\ldots,x_n) \mapsto \left(\frac{2x_1}{1+x_n},\ldots,\frac{2x_{n-1}}{1+x_n}\right). \tag{2.24}$$

The inverse mappig is given by

$$\Phi_1^{-1} : \mathbb{R}^{n-1} \to S^{n-1}\backslash\{-e_n\}$$

$$(y_1,\ldots,y_{n-1}) \mapsto \left(\frac{4y_1}{4+r^2},\ldots,\frac{4y_{n-1}}{4+r^2},\frac{4-r^2}{4+r^2}\right), \tag{2.25}$$

with $r^2 = y_1^2 + \ldots + y_{n-1}^2$.

Remark 2.4.11 *We will consider the stereographic projection mapping (2.24) since it is commonly used in the literature about this subject (see [7] and [8]) and it gives a nice description between spherical wavelet theory and Euclidean wavelet theory. Considering different projection mappings will alter some of the algebraic details and formulae but not change the gist of the reasoning. In this thesis we will sometimes use the stereographic projection mapping from the North Pole e_n to the hyperplane at the origin, given by*

$$\Phi_2 : S^{n-1}\backslash\{e_n\} \to \mathbb{R}^{n-1}$$

$$(x_1,\ldots,x_n) \mapsto \left(\frac{x_1}{1-x_n},\ldots,\frac{x_{n-1}}{1-x_n}\right). \tag{2.26}$$

Proposition 2.4.12 *We have the intertwining relation*

$$\Phi_1(\varphi_{te_n}(x)) = \frac{1+t}{1-t}\Phi_1(x), \tag{2.27}$$

with $t \in]-1,1[$ and $x \in S^{n-1}$.

Proof: By (2.24) we have

$$r = \sqrt{\frac{4(x_1^2+\ldots+x_{n-1}^2)}{(1+x_n)^2}} = 2\sqrt{\frac{1-x_n}{1+x_n}} = 2\tan\frac{\phi}{2},$$

where $r = ||\Phi_1(x)||$. Thus, by (2.23) it follows $r_\alpha = \mathrm{e}^\alpha r$. Finally by the isomorphism (2.22) we have the following expansion factor $r_t = \frac{1+t}{1-t}r$.

■

2.5 Properties of the automorphisms of the unit ball

In the previous section we considered the following transformations:

$$R_s(x) := sx\bar{s},\ \ s \in \mathrm{Spin}(n) \qquad \text{and} \qquad \varphi_a(x) := (x-a)(1+ax)^{-1},\ a \in B^n.$$

We will proceed by deducing the main properties of Möbius transformations φ_a and its relations with rotations.

Proposition 2.5.1 *The composition of two Möbius transformations of type φ_a is again a Möbius transformation of the same type, up to a rotation:*

$$(\varphi_a \circ \varphi_b)(x) = q\,\varphi_c(x)\,\overline{q}, \quad \textit{with} \quad c = (1-ab)^{-1}(a+b) \quad \textit{and} \quad q = \frac{1-ab}{|1-ab|}. \tag{2.28}$$

Proof: If $a = 0$ then $\varphi_a = Id$. Hence, we assume $a, b \neq 0$. Then

$$\begin{aligned}
(\varphi_a \circ \varphi_b)(x) &= ((x-b)(1+bx)^{-1} - a)(1 + a(x-b)(1+bx)^{-1})^{-1} \\
&= (x - b - a(1+bx))(1+bx)^{-1}(1+bx)(1+bx+a(x-b))^{-1} \\
&= ((1-ab)x - (a+b))(1-ab+(a+b)x)^{-1} \\
&= (1-ab)(x-(1-ab)^{-1}(a+b))(1+(1-ab)^{-1}(a+b)x)^{-1}(1-ab)^{-1} \\
&= \frac{1-ab}{|a-b|}(x-(1-ab)^{-1}(a+b))(1+(1-ab)^{-1}(a+b)x)^{-1}\frac{\overline{1-ab}}{|1-ab|}
\end{aligned}$$

Thus,

$$(\varphi_a \circ \varphi_b)(x) = q\,\varphi_{(1-ab)^{-1}(a+b)}(x)\,\overline{q}\,, \quad \text{with} \quad q = \frac{1-ab}{|1-ab|}. \tag{2.29}$$

As $q = \frac{1-ab}{|1-ab|} = \frac{a}{|a|}\frac{(a^{-1}-b)}{|a^{-1}-b|}$, if $|a| \neq 0$, then q is a product of two unit vectors, and therefore, it is an element of $\mathrm{Spin}(n)$.

■

Definition 2.5.2 *We denote by $b \oplus a := (1-ab)^{-1}(a+b)$ the symbol of the new Möbius transformation and by $gyr[a,b]c := \frac{1-ab}{|1-ab|}c\frac{\overline{1-ab}}{|1-ab|}$ the action of the rotation induced by q.*

We choose the notation $b \oplus a$ because first we apply φ_b and only then we apply φ_a. We will see on Section 2.6 that this structure will give rise to a left gyrogroup. The composition of Möbius transformations is non commutative in general, hence $b \oplus a \neq a \oplus b$ in general. However, there are special cases in which the operation is commutative as we will see later.

Lemma 2.5.3 *The symbol can be expressed as a Möbius transformation:*

$$b \oplus a = (1-ab)^{-1}(a+b) = (a+b)(1-ba)^{-1} = \varphi_{-b}(a). \tag{2.30}$$

Proof: By simple calculations we have that

$$b \oplus a = (1-ab)^{-1}(a+b) = \frac{\overline{(1-ab)}(a+b)}{|1-ab|^2} = \frac{(1-ba)(a+b)}{|1-ab|^2} = \frac{a+b-baa-bab}{|1-ab|^2} =$$

$$= \frac{a+b-aab-bab}{|1-ab|^2} = \frac{(a+b)(1-ab)}{|1-ab|^2} = \frac{(a+b)\overline{(1-ba)}}{|1-ba|^2} = (a+b)(1-ba)^{-1} = \varphi_{-b}(a).$$

■

For every $a, b \in B^n$ we have the following relation:

$$1 - |b \oplus a|^2 = 1 - |\varphi_{-b}(a)|^2 = \frac{(1-|a|^2)(1-|b|^2)}{|1-ba|^2}. \tag{2.31}$$

Now we establish some properties involving Möbius transformations and rotations.

Lemma 2.5.4 *For $s \in \mathrm{Spin}(n)$ and $a \in B^n$ we have:*

$$(i) \quad \varphi_a(sx\overline{s}) = s\varphi_{\overline{s}as}(x)\overline{s}\,; \tag{2.32}$$

$$(ii) \quad \varphi_a(x) = s\varphi_{\overline{s}as}(\overline{s}xs)\overline{s}\,; \tag{2.33}$$

$$(iii) \quad s\varphi_a(x)\overline{s} = \varphi_{sa\overline{s}}(sx\overline{s})\,. \tag{2.34}$$

Proof: By direct calculations we have

$$\varphi_a(sx\overline{s}) = (sx\overline{s}-a)(1-asx\overline{s})^{-1} = s(x-\overline{s}as)\overline{s}s(1-\overline{s}asx)^{-1}\overline{s} = s\varphi_{\overline{s}as}(x)\overline{s},$$

which proves (i). By the change of variables $y = sx\overline{s}$ in (i) we obtain (ii). The equality (iii) is easily deduced from (ii).

■

Corollary 2.5.5 *For $s \in \mathrm{Spin}(n)$ and $a, b \in B^n$ the following equalities hold:*

$$(i) \quad (sa\overline{s}) \oplus b = s(a \oplus (\overline{s}bs)\overline{s}\,; \tag{2.35}$$

$$(ii) \quad a \oplus b = s((\overline{s}as) \oplus sb\overline{s})\overline{s}\,; \tag{2.36}$$

$$(iii) \quad (sa\overline{s}) \oplus (sb\overline{s}) = s(a \oplus b)\overline{s}\,. \tag{2.37}$$

The proof follows from (2.30) and Lemma 2.5.4.

The relation (2.37) defines a homomorphism of $\mathrm{Spin}(n)$ onto the groupoid $(B^n, \oplus)$. Each φ_a can be naturally identified with a point $a \in B^n$ and the composition of two Möbius transformations $\varphi_a \circ \varphi_b$ can be identified with the operation $b \oplus a$.

2.5.1 Non-commutativity of $a \oplus b$ on B^n

Without loss of generality, we assume $a, b \in B^n$ such that $a, b \neq 0$. If we switch the roles of a and b in $b \oplus a = (a+b)(1-ba)^{-1}$, the first term will remain the same, but the second factor will transform to its conjugate and hence will not be the same unless ba is real. But ba is real if and only if a and b are parallel. In that case $ba = -\langle b, a\rangle$. Thus $b \oplus a$ is equal to $a \oplus b$ if and only if the vectors a and b are parallel. In general we have

$$b \oplus a = (a+b)(1-ba)^{-1} = (b+a)(1-ab)^{-1}(1-ab)(1-ba)^{-1} = (a \oplus b)\, R, \tag{2.38}$$

where $R = (1-ab)(1-ba)^{-1}$ is an element of $\mathrm{Spin}(n)$. To see this we note that

$$R = \frac{(1-ab)(1-ab)}{|1-ab|^2} = \frac{a}{|a|}\frac{a^{-1}-b}{|a^{-1}-b|}\frac{a}{|a|}\frac{a^{-1}-b}{|a^{-1}-b|}\,, \tag{2.39}$$

with $a^{-1} = \overline{a}/|a|^2 = -a/|a|^2$. Thus R is an even product of unit vectors which proves that it is an element of $\mathrm{Spin}(n)$. If $a = t_1\omega$ and $b = t_2\omega$ for some $t_1, t_2 \in]-1,1[$ and $\omega \in S^{n-1}$, then $R = 1$.

We know that $b \oplus a$ and $a \oplus b$ are again points on B^n and $|b \oplus a| = |a \oplus b|$. Thus, the multiplication of $a \oplus b$ by the Clifford number R defines a rotation of the vector $a \oplus b$. Indeed, the element $q = \frac{1-ab}{|1-ab|} \in \mathrm{Spin}(n)$ is such that

$$b \oplus a = \overline{q}\ (a \oplus b)\ q. \tag{2.40}$$

In the case $n = 3$ we can easily characterize such rotation. The non-commutativity of $b \oplus a$ is given by an operator of rotation around the axis through the origin in the direction $b \times a$, by an angle β in the plane generated by b and a. Here $b \times a$ denotes the standard cross product between the vectors a and b and the angle β is defined by $\tan(\beta/2) = -\frac{|b||a|\sin\theta}{1+|b||a|\cos\theta}$, where θ is the angle between the vectors a and b. For details see [38].

For higher dimensions this rotation can also be characterized using the language of Clifford algebras by a plane and an angle of rotation (see [73]).

2.5.2 Non-associativity of $a \oplus b$ on B^n

We want to study the associative law for the operation $\oplus$ on B^n.

Proposition 2.5.6 *The operation $\oplus$ satisfies the following relation*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus (qc\overline{q}), \quad \textit{with} \quad q = \frac{1-ab}{|1-ab|}, \tag{2.41}$$

for all $a, b, c \in B^n$.

Proof: By the associativity of the composition of Möbius transformations we have on the one hand

$$\begin{aligned}((\varphi_c \circ \varphi_b) \circ \varphi_a)(x) &= ((q_1 \varphi_{b\oplus c} \overline{q_1}) \circ \varphi_a)(x) \\ &= q_1 \varphi_{b\oplus c}(\varphi_a(x)) \overline{q_1} \\ &= q_1 q_2 \varphi_{a\oplus(b\oplus c)}(x) \overline{q_2}\, \overline{q_1} \\ &= q_1 q_2 \varphi_{a\oplus(b\oplus c)}(x) \overline{q_2}\, \overline{q_1},\end{aligned}$$

with $q_1 = \frac{1-cb}{|1-cb|}$ and $q_2 = \frac{1-(b\oplus c)a}{|1-(b\oplus c)a|}$.

On the other hand we have

$$\begin{aligned}(\varphi_c \circ (\varphi_b \circ \varphi_a))(x) &= (\varphi_c \circ (q_3 \varphi_{a\oplus b} \overline{q_3}))(x) \\ &= \varphi_c(q_3 \varphi_{a\oplus b}(x) \overline{q_3}) \\ &= q_3 \varphi_{\overline{q_3} c q_3}(\varphi_{a\oplus b}(x)) \overline{q_3} \\ &= q_3 q_4 \varphi_{(a\oplus b)\oplus(\overline{q_3} c q_3)}(x) \overline{q_4}\, \overline{q_3},\end{aligned}$$

with $q_3 = \frac{1-ba}{|1-ba|}$ and $q_4 = \frac{1-(\overline{q_3} c q_3)(a\oplus b)}{|1-(\overline{q_3} c q_3)(a\oplus b)|}$.

Thus,

$$q_1 q_2 = q_3 q_4 \quad \text{and} \quad a \oplus (b \oplus c) = (a \oplus b) \oplus (qc\overline{q}), \text{ with } q = \frac{1-ab}{|1-ab|}. \tag{2.42}$$

Indeed, by direct calculations we have that

$$q_1 q_2 = \frac{1-cb-ca-ba}{|1-cb-ca-ba|} = q_3 q_4$$

$$\begin{aligned}a \oplus (b \oplus c) &= (1-(b\oplus c)a)^{-1}(b \oplus c + a) \\ &= (1-(1-cb)^{-1}(c+b)a)^{-1}((1-cb)^{-1}(c+b)+a) \\ &= (1-cb-(c+b)a)^{-1}(1-cb)(1-cb)^{-1}(c+b+(1-cb)a) \\ &= (1-cb-ca-ba)^{-1}(c+b+a-cba);\end{aligned}$$

$$\begin{aligned}(a\oplus b) \oplus (\overline{q_3} c q_3) &= (1-\overline{q_3} c q_3 (a\oplus b))^{-1}(\overline{q_3} c q_3 + a \oplus b) \\ &= (1-(1-ba)^{-1}c(1-ba)(1-ba)^{-1}(b+a))^{-1} \\ &\quad ((1-ba)^{-1}c(1-ba)+(1-ba)^{-1}(b+a)) \\ &= (1-ba-c(b+a))^{-1}(1-ba)(1-ba)^{-1}(c(1-ba)+b+a) \\ &= (1-ba-cb-ca)^{-1}(c-cba+b+a).\end{aligned}$$

■

Corollary 2.5.7 *For all $a, b, c \in B^n$ the operation $\oplus$ satisfies*

$$\begin{aligned} (a \oplus b) \oplus c &= a \oplus (b \oplus \overline{q} c q), && (2.43) \\ a \oplus (b \oplus c) &= q((b \oplus a) \oplus c)\overline{q}, && (2.44) \end{aligned}$$

with $q = \frac{1-ab}{|1-ab|}$.

Proof: By (2.41) we obtain the property (2.43)

$$a \oplus (b \oplus \overline{q} c q) = (a \oplus b) \oplus (q \overline{q} c q \overline{q}) = (a \oplus b) \oplus c.$$

By (2.41), (2.35), and (2.40) we have

$$a \oplus (b \oplus c) = (a \oplus b) \oplus (q c \overline{q}) = q((\overline{q}(a \oplus b) q) \oplus c)\overline{q} = q((b \oplus a) \oplus c)\overline{q},$$

which proves (2.44).

■

Nevertheless, the associativity happens in some special situations.

Lemma 2.5.8 *If $a, b, c \in B^n$ such that $a // b$ or ($a \perp c$ and $b \perp c$) then the operation $\oplus$ is associative, i.e.*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c.$$

Proof: We have to calculate $q c \overline{q}$ (c.f. (2.41)):

$$q c \overline{q} = \frac{1 - ab}{|1 - ab|} c \frac{\overline{1 - ab}}{|1 - ba|} = \frac{c - abc - cba + abcba}{1 + 2\langle a, b \rangle + |a|^2 |b|^2}. \tag{2.45}$$

As

$$\begin{aligned} abc &= -2\langle a, b \rangle c - bac \\ &= -2\langle a, b \rangle c - (-2\langle a, c \rangle b - bca) \\ &= -2\langle a, b \rangle c - (-2\langle a, c \rangle b - (-2\langle b, c \rangle a - cba)) \\ &= -2\langle a, b \rangle c + 2\langle a, c \rangle b - 2\langle b, c \rangle a - cba \end{aligned}$$

and

$$\begin{aligned} abcba &= a(-2\langle b, c \rangle b + |b|^2 c)a \\ &= -2\langle b, c \rangle (-2\langle a, b \rangle a + |a|^2 b) + |b|^2(-2\langle a, c \rangle a + |a|^2 c) \\ &= 4\langle a, b \rangle \langle b, c \rangle a - 2\langle b, c \rangle |a|^2 b - 2\langle a, c \rangle |b|^2 a + |a|^2 |b|^2 c \end{aligned}$$

we will eventually find that

$$
\begin{aligned}
q c \overline{q} \;=\; & \frac{(1+2\langle a,b\rangle + |a|^2|b|^2)c - 2(\langle a,c\rangle + \langle b,c\rangle |a|^2)b}{1+2\langle a,b\rangle + |a|^2|b|^2} + \\
& \frac{2(\langle b,c\rangle (1+2\langle a,b\rangle) - \langle a,c\rangle |b|^2)a}{1+2\langle a,b\rangle + |a|^2|b|^2}.
\end{aligned} \tag{2.46}
$$

Thus $qc\overline{q} = c$ if and only if

$$(\langle a,c\rangle + \langle b,c\rangle |a|^2)b = (\langle b,c\rangle (1+2\langle a,b\rangle) - \langle a,c\rangle |b|^2)a\,.$$

The last equality can only be satisfied when $a//b$, i.e. $a = t_1\omega$ and $b = t_2\omega$ for some $-1 < t_1, t_2 < 1$ and $\omega \in S^{n-1}$ or when $c \perp b$ and $c \perp a$. In the first case we will obtain the identity $(t_1t_2 + t_1^2t_2^2)\langle c,\omega\rangle\,\omega = (t_1t_2 + t_1^2t_2^2)\langle c,\omega\rangle\,\omega$ and in the second case we immediately obtain $0 = 0$.

■

2.6 Gyrogroups

Gyrogroups are grouplike structures which first arose in the study of Einstein's velocity addition in the theory of special relativity.

Gyrogroups are special loops which share remarkable analogies with groups. They have been studied intensively by Abraham Ungar ([69], [70], [71], [72]). The first known gyrogroup is the relativistic gyrogroup $(B^3, \oplus)$, that appeared in 1988 [71], consisting of the unit ball $B^3 = \{x \in \mathbb{R}^3 : ||x|| < 1\}$ of the Euclidean 3−space, endowed with Einstein's velocity addition. Einstein's addition $\oplus$ of relativistically admissible velocities is a binary operation in the unit ball B^3, where the vacuum speed of light is normalized to $c = 1$. Counterintuitive, the Einstein velocity addition $\oplus$ is neither commutative nor associative. The group structure that has been lost in the transition from the group $(\mathbb{R}^3, +)$ to the groupoid $(B^3, \oplus)$ is replaced by a loop structure using a peculiar rotation called the *Thomas precession*. This measures the deviation of the addition of relativistically admissible velocities from being associative. The notion of gyrogroup appears by the extension of the Einstein relativistic groupoid $(B^3, \oplus)$ with its Thomas precession by abstraction, where the abstract Thomas precession is called the *Thomas gyration*. In order to elaborate a precise language for dealing with analytic hyperbolic geometry Ungar has adopted the term "gyro" since 1991. The resulting gyrolanguage rests on the unification of Euclidean and hyperbolic geometry in terms of analogies they share (see [69]).

Definition 2.6.1 *([69]) A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms:*

(G1) *In G there is at least one element 0, called a left identity satisfying $0 \oplus a = a$, for all $a \in G$;*

(G2) *For each $a \in G$ there is an element $\ominus a \in G$, called a left inverse of a, satisfying $(\ominus a) \oplus a = 0$;*

(G3) *For any $a, b, c \in G$ there exists a unique element $gyr[a, b]c \in G$ such that the binary operation obeys the left gyroassociative law*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c;$$

(G4) *The map $gyr[a, b] : G \to G$ given by $c \mapsto gyr[a, b]c$, is an automorphism of $(G, \oplus)$;*

(G5) *The gyroautomorphism $gyr[a, b]$ possesses the left loop property*

$$gyr[a, b] = gyr[a \oplus b, b].$$

Gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 2.6.2 *([69]) A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation satisfies the gyrocommutative law*

$$a \oplus b = gyr[a, b](b \oplus a), \quad \text{for all } a, b \in G.$$

In order to capture useful analogies between gyrogroups and groups and uncovers duality symmetries Ungar defined a second binary operation, called the gyrogroup cooperation.

Definition 2.6.3 *([69]) Let $(G, \oplus)$ be a gyrogroup. The gyrogroup cooperation (or coaddition) is a second binary operation $\boxplus$, in G, given by*

$$a \boxplus b = a \oplus gyr[a, \ominus b]b, \quad \text{for all } a, b, \in G.$$

and the cosubtraction is defined as $a \boxminus b = a \ominus gyr[a, b]b$.

We will list some of the main identities in gyrogroups $(G, \oplus)$ that need not be gyrocommutative [69].

List of identities in gyrogroups $(G, \oplus)$	
1)	$a \oplus (\ominus a \oplus b) = b$ (Left cancellation law)
2)	$(b \ominus a) \boxplus a = b$ (Right cancellation law)
3)	$(b \boxminus a) \oplus a = b$ (Right cancellation law)
4)	$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c$ (Left gyroassociative law)
5)	$(a \oplus b) \oplus c = a \oplus (b \oplus gyr[b, a]c)$ (Right gyroassociative law)
6)	$(a \boxplus b) \oplus c = a \oplus gyr[a, \ominus b](b \oplus c)$
7)	$gyr[a, b]c = \ominus(a \oplus b) \oplus [a \oplus (b \oplus c)]$
8)	$gyr[\ominus a, \ominus b] = gyr[a, b]$ (Even simmetry)
9)	$gyr^{-1}[a, b] = gyr[b, a]$ (Inversive simmetry)
10)	$gyr[a, b \oplus a] = gyr[a, b]$ (Right loop property)
11)	$gyr[a \boxminus b, b] = gyr[a, b]$ (Left coloop property)
12)	$gyr[a, b \boxminus a] = gyr[a, b]$ (Right coloop property)
13)	$\ominus(a \oplus b) = gyr[a, b](\ominus b \ominus a)$
14)	$(a \oplus gyr[a, b]c) \boxminus (b \oplus c) = a \boxminus b$
15)	$(\ominus a \oplus b) \oplus gyr[\ominus a, b](\ominus b \oplus c) = \ominus a \oplus c$
16)	$(a \boxminus b) \oplus (b \boxminus c) = a \ominus gyr[a, b]gyr[b, c]c$
17)	$gyr[b, \ominus gyr[b, a]a] = gyr[a, b]$
18)	$gyr[\ominus gyr[a, b]b, a] = gyr[a, b]$.

This list shows that gyrogroups have new and interesting relations. Some of these properties had been already deduced in our case without knowledge of this more abstract formalism, recently developed. In our case the left and right cancelation laws are very important for our work.

Lemma 2.6.4 *For all $a, b \in B^n$ it holds*

$$(-b) \oplus (b \oplus a) = a \tag{2.47}$$

$$(a \oplus b) \oplus (q(-b)\overline{q}) = a, \tag{2.48}$$

with $q = \frac{1-ab}{|1-ab|}$.

A gyrogroup is a special case of a loop. Loops are a important subcategory of the category of groupoids.

Definition 2.6.5 *A loop is a groupoid $(S, \cdot)$ with an identity element in which each of the two equations $a \cdot x = b$ and $y \cdot a = b$ possesses a unique solution for the unknowns x and y.*

The solution of these basic equations in the theory of gyrogroups is given in the following theorem.

Theorem 2.6.6 *([69]) Let $(G,\oplus)$ be a gyrogroup, and let $a,b \in G$. The unique solution of the equation $a \oplus x = b$ in G, for the unknown x, is given by $x = \ominus a \oplus b$, while the unique solution of the equation $y \oplus a = b$ in G, for the unknown y, is given by $y = b \boxminus a = b \ominus gyr[b,a]a$.*

Definition 2.6.7 *([69]) Let $(G,\oplus)$ be a gyrogroup, and let $Aut(G)$ be the automorphism group of G. A gyroautomorphism group $Aut_0(G)$ is any subgroup of Aut(G) containing all the gyroautomorphisms gyr[a, b] of G, with $a,b \in G$.*

One of the most important results in this theory is the proof of the fact that the gyrosemidirect product of a gyrogroup $(G,\oplus)$ with a gyroautomorphism group $H \subset Aut(G,\oplus)$ is a group.

Theorem 2.6.8 *([69]) Let $(G,\oplus)$ be a gyrogroup, and let $Aut_0(G,\oplus)$ be a gyroautomorphism group of G. Then the gyrosemidirect product $G \times Aut_0(G)$ is a group, with group operation given by the gyrosemidirect product*

$$(x,X)(y,Y) = (x + Xy, gyr[x,Xy]XY). \tag{2.49}$$

As we saw in Section 2.3 we can define two different group operations depending if we work with left or right cosets. To distinguish both gyrosemidirect products we will use the symbols $\times^l$ and $\times^r$ in agreement with the left or right cosets. Thus, $(\mathrm{Spin}(n) \times B^n, \times^r)$ is a group with the right gyrosemidirect product given by

$$(s_1,a) \times^r (s_2,b) = (s_1 s_2 q_1, b \oplus (\overline{s_2} a s_2)) \tag{2.50}$$

while $(B^n \times \mathrm{Spin}(n), \times^l)$ is a group with the left gyrosemidirect product given by

$$(a,s_1) \times^l (b,s_2) = (a \oplus (s_1 b \overline{s_1}), q_2 s_1 s_2), \tag{2.51}$$

with $q_1 = \frac{1-\overline{s_2} a s_2 b}{|1-\overline{s_2} a s_2 b|}$ and $q_2 = \frac{1-a s_1 b \overline{s_1}}{|1-a s_1 b \overline{s_1}|}$.

We will prove this result only for the group operation (2.50), being the proof similar for the group operation (2.51).

Proposition 2.6.9 $(\mathrm{Spin}(n) \times B^n, \times^r)$ *is a group with operation given by the gyrosemidirect product*

$$(s_1,a) \times^r (s_2,b) = (s_1 s_2 q, b \oplus (\overline{s_2} a s_2)) \tag{2.52}$$

with $s_1,s_2 \in \mathrm{Spin}(n)$, $a,b \in B^n$, and $q = \frac{1-\overline{s_2} a s_2 b}{|1-\overline{s_2} a s_2 b|}$.

Proof: We will prove the group axioms by order:

(i) The operation $\times^r$ is well defined since the result $(s_1,a)\times^r(s_2,b)$ is an element of $\mathrm{Spin}(n)\times B^n$.

(ii) Existence of a left identity: the element $(1,0)$, where 0 is the identity element of $(B^n,\oplus)$ and 1 is the identity of $\mathrm{Spin}(n)$, satisfies

$$(1,0)\times^r(s_1,a)=(1s_11,a\oplus(\overline{s_1}0s_1))=(s_1,a\oplus 0)=(s_1,a),$$

for all $(s_1,a)\in\mathrm{Spin}(n)\times B^n$.

(iii) Existence of a left inverse: for all $(s_1,a)\in\mathrm{Spin}(n)\times B^n$ it exists an element $(\overline{s_1},-s_1a\overline{s_1})\in\mathrm{Spin}(n)\times B^n$ such that

$$\begin{aligned}(\overline{s_1},-s_1a\overline{s_1})\times^r(s_1,a) &= \left(|s_1|^2\frac{1+\overline{s_1}s_1a\overline{s_1}s_1a}{|1+\overline{s_1}s_1a\overline{s_1}s_1a|},a\oplus(-\overline{s_1}s_1a\overline{s_1}s_1)\right)\\ &= (1,a\oplus(-a))\\ &= (1,0).\end{aligned}$$

(iv) Associative law: for arbitrary $(s_1,a),(s_2,b),(s_3,c)\in\mathrm{Spin}(n)\times B^n$ we have on the one hand,

$$(s_1,a)\times^r((s_2,b)\times^r(s_3,c))=(s_1,a)\times^r\left(s_2s_3\frac{1-\overline{s_3}bs_3c}{|1-\overline{s_3}bs_3c|},c\oplus(\overline{s_3}bs_3)\right)$$

$$=\left(s_1s_2s_3\frac{1-\overline{s_3}bs_3c}{|1-\overline{s_3}bs_3c|}\frac{1-\overline{d}ad(c\oplus(\overline{s_3}bs_3))}{|1-\overline{d}ad(c\oplus(\overline{s_3}bs_3))|},(c\oplus\overline{s_3}bs_3)\oplus(\overline{d}ad)\right),$$

with $d:=s_2s_3\frac{1-\overline{s_3}bs_3c}{|1-\overline{s_3}bs_3c|}$,

while, on the other hand,

$$((s_1,a)\times^r(s_2,b))\times^r(s_3,c)=\left(s_1s_2\frac{1-\overline{s_2}as_2b}{|1-\overline{s_2}as_2b|},b\oplus(\overline{s_2}as_2)\right)\times^r(s_3,c)$$

$$=\left(s_1s_2\frac{1-\overline{s_2}as_2b}{|1-\overline{s_2}as_2b|}s_3\frac{1-\overline{s_3}(b\oplus(\overline{s_2}as_2))s_3c}{|1-\overline{s_3}(b\oplus(\overline{s_2}as_2))s_3c|},c\oplus(\overline{s_3}(b\oplus(\overline{s_2}as_2))s_3\right)$$

$$=\left(s_1s_2s_3\frac{1-\overline{s_3}\,\overline{s_2}as_2s_3\overline{s_3}bs_3}{|1-\overline{s_3}\,\overline{s_2}as_2s_3\overline{s_3}bs_3|}\frac{1-(\overline{s_3}bs_3)\oplus(\overline{s_3}\,\overline{s_2}as_2s_3))c}{|1-(\overline{s_3}bs_3)\oplus(\overline{s_3}\,\overline{s_2}as_2s_3))c|},c\oplus(\overline{s_3}(b\oplus(\overline{s_2}as_2))s_3)\right),$$

where we have employed the relation (2.37) in the last equality. Using the following notation:

$$\overline{s_3}bs_3:=u\quad\text{and}\quad\overline{s_3}\,\overline{s_2}as_2s_3:=v$$

we have to establish the following identity

$$\left(s_1s_2s_3q\frac{1-\bar{q}vq(c\oplus u)}{|1-\bar{q}vq(c\oplus u)|},(c\oplus u)\oplus(\bar{q}vq)\right)=\left(s_1s_2s_3\frac{1-vu}{|1-vu|}\frac{1-(u\oplus v)c}{|1-(u\oplus v)c|},c\oplus(u\oplus v)\right),$$

with $q = \frac{1-uc}{|1-uc|}$.

The identities

$$(c \oplus u) \oplus (\overline{q}vq) = c \oplus (u \oplus v)$$

and

$$\frac{1-uc}{|1-uc|}\frac{1-\overline{q}vq(c\oplus u)}{1-\overline{q}vq(c\oplus u)} = \frac{1-vu}{|1-vu|}\frac{1-(u\oplus v)c}{|1-(u\oplus v)c|}$$

are true by (2.42).

■

Up to the rotation induced by q, the formula (2.52) resembles the definition of a semidirect product. However we remark that it is impossible to build a semidirect product since $SO(n)$ has no outer automorphisms. Simple calculations shows that $\mathrm{Spin}(n)$ is not normal on $\mathrm{Spin}^+(1,n)$. The conjugation of an element $s \in \mathrm{Spin}(n)$ by $a \in B^n$ is not in general an element of $\mathrm{Spin}(n)$:

$$(\varphi_a \circ R_s \circ \varphi_{-a})(x) = \varphi_a(s\varphi_{-a}(x)\overline{s}) = s\varphi_{\overline{s}as}(\varphi_{-a}(x))\overline{s} = sq\,\varphi_{(\overline{s}as)\oplus(-a)}(x)\,\overline{q}\,\overline{s}$$

with $q = \frac{1+\overline{s}asa}{|1+\overline{s}asa|}$. When s is a rotation that leaves invariant $a \in B^n$ we obtain $(\varphi_a \circ R_s \circ \varphi_{-a})(x) = sx\overline{s}$.

On the other hand, the conjugation of an element of B^n by an element of $\mathrm{Spin}(n)$ gives again an element of B^n

$$(R_s \circ \varphi_a \circ R_{\overline{s}})(x) = s\varphi_a(\overline{s}xs)\overline{s} = s\overline{s}\varphi_{sa\overline{s}}(x)s\overline{s} = \varphi_{\overline{s}as}(x).$$

The multiplication defined in (2.52) is a generalization of the familiar semidirect product of groups (c.f. [52]). We recall that if G is a group with subgroups K and H, where K is normal, $G = KH$, and $K \cap H = \{1\}$, then G is said to be an *internal* semidirect product of K and H. On the other hand, if K and H are groups and $\sigma : H \to Aut(K), h \mapsto \sigma_h$ is a homomorphism, then the *external* semidirect product of K and H given by σ, denoted by $K \rtimes_\sigma H$, is the set $K \times H$ with the multiplication

$$(k_1, h_1)(k_2, h_2) = (k_1 \cdot \sigma_{h_1}(k_2), h_1h_2), \quad k_1, k_2 \in K,\ h_1, h_2 \in H. \tag{2.53}$$

2.7 Gyro-subgroups

Definition 2.7.1 *Let H be a non-empty subset of $(G,\oplus)$. A gyro-subgroup H of $(G,\oplus)$ is a gyrogroup with operation induced from $(G,\oplus)$ and $gyr[a,b]\in Aut(H)$ for all $a,b\in H$.*

Some gyro-subgroups of $(B^n,\oplus)$ arise in a nice geometrical way. The gyro-subgroups D^{n-1}_ω and L_ω that will be defined below are of great importance in this thesis.

2.7.1 Gyro-subgroups $(D^{n-1}_\omega,\oplus)$

Let $\omega\in S^{n-1}$. We consider the hyperplane $H_\omega=\{x\in\mathbb{R}^n:\langle\omega,x\rangle=0\}$. Hence, $D^{n-1}_\omega=H_\omega\cap B^n$ denotes a hyperdisc of dimension $n-1$.

Proposition 2.7.2 *For each $\omega\in S^{n-1}$, $(D^{n-1}_\omega,\oplus)$ is a gyro-subgroup of $(B^n,\oplus)$.*

Proof: Consider a and b two arbitrary points of D^{n-1}_ω. Then

$$\langle a,\omega\rangle=0 \quad\text{and}\quad \langle b,\omega\rangle=0. \tag{2.54}$$

As

$$b\oplus a=\varphi_{-b}(a)=(a+b)(1-ba)^{-1}=\frac{(1-|b|^2)a+(1+|a|^2+2\langle a,b\rangle)b}{1+2\langle a,b\rangle+|a|^2|b|^2}$$

we can easily see, by (2.54), that $\langle b\oplus a,\omega\rangle=0$. In an analogous way,

$$a\oplus b=\varphi_{-a}(b)=(a+b)(1-ab)^{-1}=\frac{(1-|a|^2)b+(1+|b|^2+2\langle a,b\rangle)a}{1+2\langle a,b\rangle+|a|^2|b|^2}$$

and therefore $\langle a\oplus b,\omega\rangle=0$.

Thus, $a\oplus b\in D^{n-1}_\omega$ and $b\oplus a\in D^{n-1}_\omega$.

Finally, it is easy to see that $-a\in D^{n-1}_\omega$ for each $a\in D^{n-1}_\omega$ and by (2.46) it follows that $gyr[a,b]c\in D^{n-1}_\omega$, for all $a,b,c\in D^{n-1}_\omega$.

■

These gyro-subgroups are not abelian. In fact, $a\oplus b=b\oplus a$ if and only if $ab=ba$, i.e. $a\wedge b=0$, which means that a and b must be parallel vectors.

2.7.2 Gyro-subgroups $(L_\omega,\oplus)$

Consider again $\omega\in S^{n-1}$. Let L_ω be the segment resulting from the intersection of the unit ball with the straight line passing through the origin and spanned by ω.

Proposition 2.7.3 *For each $\omega\in S^{n-1}$, $(L_\omega,\oplus)$ is an abelian subgroup of $(B^n,\oplus)$.*

Proof: Consider a and b two arbitrary points of L_ω. Then $a = t_1\omega$ and $b = t_2\omega$, with $-1 < t_1, t_2 < 1$. As

$$b \oplus a = (a+b)(1-ba)^{-1} = \frac{t_1+t_2}{1+t_1t_2}\omega = a \oplus b$$

and $-1 < \frac{t_1+t_2}{1+t_1t_2} < 1$, we obtain that $a \oplus b \in L_\omega$.

Finally $-a \in L_\omega$ for each $a \in L_\omega$ and $gyr[a,b]c = c$, for all $a, b, c \in L_\omega$.

■

2.8 Factorizations of the gyrogoup of the unit ball

2.8.1 Factorizations of type I

The factorization of the gyrogroup $(B^n, \oplus)$ by a given gyro-subgroup $(D_\omega^{n-1}, \oplus)$ will be called factorization of type I. We will see that the equivalence relation used in the factorization of a group by a given subgroup cannot be applied for the factorization of a gyrogroup by a gyro-subgroup.

The following theorem is the basis of our construction. It gives us an unique decomposition for each point $c \in B^n$ with respect to the operation $\oplus$.

Theorem 2.8.1 *For each $c \in B^n$ there exist unique $a \in D_{e_n}^{n-1}$ and $b \in L_{e_n}$ such that $c = b \oplus a$.*

Proof: Let $c = (c_1, \ldots, c_n) \in B^n$ be an arbitrary point. By Lemma 2.4.1 we can write $c = s_* c_* \overline{s_*}$, where $s_* = s_1 \ldots s_{n-2} \in \mathrm{Spin}(n-1)$, and $c_* = (0, \ldots, 0, c_{n-1}^*, c_n)$, with $c_{n-1}^* = r\sin\phi$ and $c_n = r\cos\phi$, $r = |c| \in [0,1[$ and $\phi = \arccos(c_n) \in [0,\pi]$. If $c_{n-1}^* = 0$, then we take $a = 0$ and $b = c_*$; otherwise, we consider $a = \lambda e_{n-1}$ and $b = te_n$ where

$$\lambda = \frac{|c_*|^2 - 1 + \sqrt{((c_n+1)^2 + (c_{n-1}^*)^2)((c_n-1)^2 + (c_{n-1}^*)^2)}}{2c_{n-1}^*} \quad \text{and} \quad t = \frac{c_n}{c_{n-1}^*\lambda + 1}. \quad (2.55)$$

We can see that $-1 < \lambda, t < 1$. Thus, $a \in D_{e_n}^{n-1}$ and $b \in L_{e_n}$. Taking into account that $a \perp b$, that is $\langle a, b\rangle = 0$, we obtain

$$b \oplus a = \left(0, \ldots, 0, \frac{\lambda(1-t^2)}{1+\lambda^2t^2}, \frac{t(1+\lambda^2)}{1+\lambda^2t^2}\right). \quad (2.56)$$

Substituting λ and t in the coordinates of (2.56) we obtain by straightforward computations that $b \oplus a = c_*$.

Consider now $a_* = s_* a \overline{s_*}$. Obviously, $a_* \in D_{e_n}^{n-1}$ because the rotation induced by s_* leaves the x_n-axis invariant. Then by (2.35) we have

$$b \oplus (s_* a \overline{s_*}) = s_*(\overline{s_*} b s_* \oplus a)\overline{s_*} = s_*(b \oplus a)\overline{s_*} = s_* c_* \overline{s_*} = c,$$

which shows that $c = b \oplus a_*$. Hence the existence of the decomposition is proved.

To prove the uniqueness we suppose that there exist $a, d \in D_{e_n}^{n-1}$ and $b, f \in L_{e_n}$ such that $c = b \oplus a = f \oplus d$. Then $a = (-b) \oplus (f \oplus d)$, by (2.47). As $b \perp d$ and $f \perp d$ we have $a = ((-b) \oplus f) \oplus d$, by Lemma 2.5.8. Since by hypothesis $a, d \in D_{e_n}^{n-1}$ then $(-b) \oplus f$ must be an element of $D_{e_n}^{n-1}$. This is true if and only if $(-b) \oplus f = 0$. This implies $b = f$ and $a = d \oplus 0 = d$, as we wish to prove.

■

Let us consider the gyro-subgroup $D_{e_n}^{n-1}$ and the relation R defined by

$$\forall\, c, d \in B^n, \quad c\, R\, d \;\Leftrightarrow\; \exists\, a \in D_{e_n}^{n-1} : c = d \oplus a.$$

It is a reflexive relation but it is not symmetric nor transitive because the operation $\oplus$ is not commutative nor associative. Therefore R is not an equivalence relation.

However, an equivalence relation on B^n can be built if we construct a partition of B^n.

Proposition 2.8.2 *The family $\{S_b^l : b \in L_{e_n}\}$, where $S_b^l = \{b \oplus a : a \in D_{e_n}^{n-1}\}$, is a disjoint partition of B^n.*

Proof: We first prove that this family is indeed disjoint. Let $b = t_1 e_n$ and $c = t_2 e_n$ with $t_1 \neq t_2$, and assume that $S_b^l \cap S_c^l \neq \emptyset$. Then there exists $f \in B^n$ such that $f = b \oplus a$ and $f = c \oplus d$ for some $a, d \in D_{e_n}^{n-1}$. By (2.47) and (2.41) we have

$$a = (-b) \oplus (c \oplus d) = ((-b) \oplus c) \oplus (q d \overline{q}), \quad \text{with} \quad q = \frac{1 + bc}{|1 + bc|}.$$

As $q = \frac{1+bc}{|1+bc|} = \frac{1-t_1 t_2}{|1-t_1 t_2|} = 1$, then $a = ((-b) \oplus c) \oplus d$. Since $a, d \in D_{e_n}^{n-1}$ then $(-b) \oplus c \in D_{e_n}^{n-1}$. Therefore, $(-b) \oplus c = 0$, i.e. $b = c$. But this contradicts our assumption. Thus, $S_b^l \cap S_c^l = \emptyset$, for $b \neq c$.

Finally, by Theorem 2.8.1 we have that $\cup_{b \in L_{e_n}} S_b^l = B^n$.

■

This partition induces an equivalence relation on B^n. Two points $c, d \in B^n$ are said to be in relation if and only if there exists $b \in L_{e_n}$ and $a, f \in D_{e_n}^{n-1}$ such that $c, d \in S_b$, i.e.

$$\forall\, c, d \in B^n, \quad c \sim_l d \;\Leftrightarrow\; \exists\, b \in L_{e_n},\; \exists\, a, f \in D_{e_n}^{n-1} : \; c = b \oplus a \;\text{ and }\; d = b \oplus f. \tag{2.57}$$

We will use the symbol $\sim_l$ to denote the left action, i.e. we are considering left coset spaces. This relation is equivalent to

$$\forall\, c, d \in B^n, \quad c \sim_l d \;\Leftrightarrow\; \exists\, b \in L_{e_n},\; \exists\, a, f \in D^{n-1}_{e_n} :\; c \oplus (q_1(-a)\overline{q_1}) = d \oplus (q_2(-f)\overline{q_2}),$$

with $q_1 = \frac{1-ab}{|1-ab|}$ and $q_2 = \frac{1-fb}{|1-fb|}$.

Thus, we have proved the following isomorphism:

$$B^n/(D^{n-1}_{e_n}, \sim_l) \cong L_{e_n}.$$

We wish to give a characterization of the surfaces S^l_b, $b \in L_{e_n}$.

Proposition 2.8.3 *For each $b = te_n \in L_{e_n}$ the surface S^l_b is a surface of revolution in turn of the x_n-axis, obtained by the intersection of B^n with the sphere orthogonal to S^{n-1}, with center in the point $C = (0, \ldots, 0, \frac{1+t^2}{2t})$ and radius $\tau = \frac{1-t^2}{2|t|}$.*

Proof: Let $b = te_n \in L_{e_n}$, $c = \lambda e_{n-1} \in D^{n-1}_{e_n}$ and

$$P_\lambda := b \oplus c = \left(0, \ldots, 0, \frac{\lambda(1-t^2)}{1+\lambda^2 t^2}, \frac{t(1+\lambda^2)}{1+\lambda^2 t^2}\right).$$

Let $C_b = \{b \oplus c : c = \lambda e_{n-1}, 0 \le \lambda < 1\}$ denote the arc inside the unit ball in the $x_{n-1}x_n-$plane. We consider now $s_* = s_1 \ldots s_{n-2} \in \mathrm{Spin}(n-1)$ (c.f. Remark 2.4.2). Obviously each $a \in D^{n-1}_{e_n}$ can be described as $a = s_* c \overline{s_*}$. Then, by (2.35)

$$b \oplus (s_* c \overline{s_*}) = s_*((\overline{s_*} b s_*) \oplus c)\overline{s_*} = s_*(b \oplus c)\overline{s_*}.$$

Thus, S^l_b is a surface of revolution obtained by the revolution in turn of the x_n-axis of the arc C_b. The last coordinate of the surface S^l_b is $\frac{t(1+\lambda^2)}{1+\lambda^2 t^2}$ and it gives us information about the orientation of its concavity.

For all $\lambda \in [0, 1[$, we have that $||P_\lambda - C||^2 = \tau^2$, with $C = (0, \ldots, 0, \frac{1+t^2}{2t})$ and $\tau = \frac{1-t^2}{2|t|}$, there is, the arc C_b lies on the sphere centered at C and radius τ. Moreover, as t tends to zero the radius of the sphere tends to infinity thus proving that the surface S^l_0 coincides with the hyperdisc $D^{n-1}_{e_n}$.

Each S^l_b is orthogonal to S^{n-1} because $||C||^2 = 1 + \tau^2$. We recall that two spheres, S_1 and S_2, with centers m_1 and m_2 and radii τ_1 and τ_2, respectively, intersect orthogonally if and only if $\langle m_1 - y, m_2 - y\rangle = 0$, for all $y_1 \in S_1 \cap S_2$, or equivalently, if $||m_1 - m_2||^2 = \tau_1^2 + \tau_2^2$.

■

We can perform their projection into the $x_{n-1}x_n-$plane for an easy visualization.

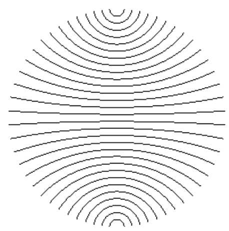


Figure 2.1: Cut of the surfaces S_b^l in the $x_{n-1}x_n-$plane.

We now generalize the Proposition 2.8.1 to arbitrary gyro-subgroups $(D_\omega^{n-1}, \oplus)$ and $(L_\omega, \oplus)$.

Theorem 2.8.4 *For each $d \in B^n$, there exists unique $u \in D_\omega^{n-1}$ and $v \in L_\omega$ such that $d = v \oplus u$.*

Proof: By Lemma 2.4.1 there exists $s = s_1 \dots s_{n-1} \in \text{Spin}(n)$ such that $\omega = se_n\overline{s}$. For each $d \in B^n$ we take $c \in B^n$ such that $d = sc\overline{s}$. By Theorem 2.8.1, there exist unique $a \in D_{e_n}^{n-1}$ and $b \in L_{e_n}$ such that $c = b \oplus a$. Then,

$$d = sc\overline{s} = s(b \oplus a)\overline{s} = (sb\overline{s}) \oplus (sa\overline{s})$$

by the relation (2.37). As $\omega = se_n\overline{s}$, we have that $u = sa\overline{s} \in D_\omega^{n-1}$ and $v = sb\overline{s} \in L_\omega$.

■

The family $\{sS_b^l\overline{s} : b \in L_{e_n}\}$ is obviously a partition of B^n and it induces the following equivalence relation:

$$\forall\, c, d \in B^n, \quad c \sim_l d \;\Leftrightarrow\; \exists\, v \in L_\omega,\; \exists\, u, w \in D_\omega^{n-1} :\; c = v \oplus u \text{ and } d = v \oplus w. \tag{2.58}$$

Corollary 2.8.5 *We have the isomorphism $B^n/(D_\omega^{n-1}, \sim_l) \cong L_\omega$.*

Since $(B^n, \oplus)$ is a non gyrocommutative gyrogroup we can consider right coset spaces arising from the decomposition of B^n by the gyro-subgroups D_ω^{n-1}. We will proceed in an analogous way as for the left action. First we obtain an analogue version of Theorem 2.8.1.

Theorem 2.8.6 *For each $c \in B^n$ there exist unique $a \in D_{e_n}^{n-1}$ and $b \in L_{e_n}$ such that $c = a \oplus b$.*

The proof is analogous to the proof of Theorem 2.8.1. For $c_* = (0, \ldots, 0, c_{n-1}^*, c_n)$, as in the proof of Theorem 2.8.1, we have again two cases: if $c_n = 0$ then we take $b = 0$ and $a = c_*$, otherwise we consider $a = \lambda e_{n-1}$ and $b = t e_n$ such that

$$t = \frac{|c_*|^2 - 1 + \sqrt{((c_{n-1}^* + 1)^2 + c_n^2)((c_{n-1}^* - 1)^2 + c_n^2)}}{2c_n} \quad \text{and} \quad \lambda = \frac{c_{n-1}^*}{c_n t + 1}.$$

Proposition 2.8.7 *The family $\{S_b^r : b \in L_{e_n}\}$, where $S_b^r = \{a \oplus b : a \in D_{e_n}^{n-1}\}$, is a disjoint partition of B^n.*

Proof: Let $b = t_1 e_n$ and $c = t_2 e_n$ with $t_1 \neq t_2$, and assume that $S_b^r \cap S_c^r \neq \emptyset$. Then there exist $d \in B^n$ such that $d = a \oplus b$ and $d = f \oplus c$ for some $a, f \in D_{e_n}^{n-1}$. By (2.41) and (2.47) we have

$$b = (-a) \oplus (f \oplus c) = ((-a) \oplus f) \oplus (qc\overline{q}), \quad \text{with} \quad q = \frac{1 + af}{|1 + af|}.$$

As $a \perp c$ and $f \perp c$ it follows by Lemma 2.5.8 that $b = ((-a) \oplus f) \oplus c$. As $b, c \in L_{e_n}$ we must have $(-a) \oplus f \in L_{e_n}$. But this happens if and only if $(-a) \oplus f = 0$. Thus, $b = 0 \oplus c = c$, which contradicts our assumption. Thus, $S_b^r \cap S_c^r = \emptyset$, for $b \neq c$.

By Theorem 2.8.6 it follows that $\cup_{b \in L_{e_n}} S_b^r = B^n$.

Thus, the family $\{S_b^r : b \in L_{e_n}\}$ is a disjoint partition of B^n.

■

Corollary 2.8.8 *The following isomorphism hold*

$$B^n / (D_{e_n}^{n-1}, \sim_r) \cong L_{e_n}.$$

Proposition 2.8.9 *For each $b = t e_n \in L_{e_n}$ the surface S_b^r is a surface of revolution in turn of the x_n-axis obtained by the intersection of B^n with the sphere with center in the point $C^r = (0, \ldots, 0, \frac{t^2-1}{2t})$ and radius $\tau = \frac{1+t^2}{2|t|}$.*

Proof: Let $b = t e_n \in L_{e_n}$ and $c = \lambda e_{n-1} \in D_{e_n}^{n-1}$ and

$$Q_\lambda := c \oplus b = \left(0, \ldots, 0, \frac{\lambda(1 + t^2)}{1 + \lambda^2 t^2}, \frac{t(1 - \lambda^2)}{1 + \lambda^2 t^2}\right)$$

Let $C_b^r = \{c \oplus b : c = \lambda e_{n-1}, 0 \leq \lambda < 1\}$ denotes the arc inside the unit ball in the $x_{n-1}x_n-$plane. We consider now $s_* = s_1 \ldots s_{n-2} \in \text{Spin}(n-1)$ (c.f. Remark 2.4.2). Obviously each $a \in D_{e_n}^{n-1}$ can be described as $a = s_* c \overline{s_*}$. Then, by (2.36)

$$(s_* c \overline{s_*}) \oplus b = s_*(c \oplus (\overline{s_*} b s_*))\overline{s_*} = s_*(c \oplus b)\overline{s_*}.$$

Thus S_b^r is a surface of revolution obtained by revolution in turn of the x_n-axis of the arc C_b^r. The last coordinate of the surface S_b^r is $\frac{t(1-\lambda^2)}{1+\lambda^2 t^2}$ and it encloses information about its orientation.

For all $\lambda \in [0,1[$, we have that $||Q_\lambda - C^r||^2 = \tau^2$, with $C^r = (0,\dots,0,\frac{t^2-1}{2t})$ and $\tau = \frac{1+t^2}{2|t|}$, there is, the arc C_b^r lies on the sphere centered at C^r and radius τ. Moreover, as t tends to zero the radius of the sphere tends to infinity thus proving that the surface S_0^r coincides with the hyperdisc $D_{e_n}^{n-1}$.

■

We remark that these spheres are not orthogonal to S^{n-1} because they do not satisfy the relation $||C^r||^2 = 1 + \tau^2$.

We can observe the cut of the surfaces S_b^r in the $x_{n-1}x_n-$plane.

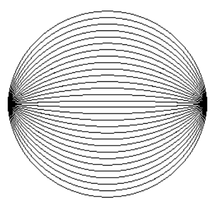


Figure 2.2: Cut of the surfaces S_b^r, in the $x_{n-1}x_n-$plane.

Finally we have the following results for an arbitrary $\omega \in S^{n-1}$.

Proposition 2.8.10 *For each $d \in B^n$, there exists unique $u \in D_\omega^{n-1}$ and $v \in L_\omega$ such that $d = u \oplus v$.*

The proof is analogous to the proof of Theorem 2.8.4 and is based on the decomposition obtained in Theorem 2.8.6.

The family $\{sS_b^r\overline{s} : b \in L_{e_n}\}$ with $s = \in \mathrm{Spin}(n)$ such that $\omega = se_n\overline{s}$ is obviously a partition of B^n and it induces the following equivalence relation:

$$\forall\, c,d \in B^n, \quad c \sim_r d \Leftrightarrow \exists\, v \in L_\omega,\ \exists\, u,w \in D_\omega^{n-1} :\ c = u \oplus v \text{ and } d = w \oplus v. \tag{2.59}$$

Corollary 2.8.11 *The following isomorphism hold*

$$B^n/(D_\omega^{n-1}, \sim_r) \cong L_\omega.$$

2.8.2 Factorizations of type II

Factorizations of type II correspond to factorizations of $(B^n, \oplus)$ by subgroups $(L_\omega, \oplus)$, $\omega \in S^{n-1}$. Due to the duality role between L_ω and D_ω^{n-1}, the results will be analogous to the ones in the D_ω^{n-1} case. Therefore, some proofs will be omitted. We begin by considering the left coset space $B^n/(L_{e_n}, \sim_l)$.

Proposition 2.8.12 *The family $T^l = \{T_a^l : a \in D_{e_n}^{n-1}\}$, with $T_a^l = \{a \oplus b : b \in L_{e_n}\}$ is a disjoint partition of B^n.*

This partition induces the following equivalence relation on B^n :

$$\forall\, c, d \in B^n, \quad c \sim_l d \;\Leftrightarrow\; \exists\, a \in D_{e_n}^{n-1},\ \exists\, b, f \in L_{e_n} :\ c = a \oplus b \ \text{ and } \ d = a \oplus f. \qquad (2.60)$$

Corollary 2.8.13 *We have the isomorphism $B^n/(L_{e_n}, \sim_l) \;\cong\; D_{e_n}^{n-1}$.*

Proposition 2.8.14 *For an arbitrary $a \in D_{e_n}^{n-1}$, T_a^l is the intersection of B^n with the circumference of radius $\tau = \frac{1-|a|^2}{2|a|}$ and center in the point $s_* C_0^r \overline{s_*}$, in the plane $s_* H \overline{s_*}$, where $s_* = s_1 \ldots s_{n-2}$ is such that $\overline{s_*} a s_* = \lambda e_{n-1}$, $\lambda \in]-1, 1[$, $C_0^r = (0, \ldots, 0, \frac{1+\lambda^2}{2\lambda}, 0)$, and $s_* H \overline{s_*}$ is the rotation around the x_n-axis of the $x_{n-1}x_n-$plane or $H-$plane. Moreover, we have that each T_a^l is orthogonal to S^{n-1}.*

Proof: Let $c = \lambda e_{n-1} \in D_{e_n}^{n-1}$, $b = te_n \in L_{e_n}$, with $-1 < \lambda, t < 1$, and

$$P_t := c \oplus b = \left(0, \ldots, 0, \frac{\lambda(1+t^2)}{1+\lambda^2 t^2}, \frac{t(1-\lambda^2)}{1+\lambda^2 t^2}\right).$$

Let $T_c^l = \{c \oplus b : b \in L_{e_n}\}$ be the curve inside the unit ball in the $x_{n-1}x_n-$plane or $H-$plane. For all $t \in]-1, 1[$, we have $||P_t - C_0^l||^2 = \tau^2$, with $C_0^l = (0, \ldots, 0, \frac{1+\lambda^2}{2\lambda}, 0)$ and $\tau = \frac{1-\lambda^2}{2|\lambda|} = \frac{1-|c|^2}{2|c|}$. Thus, the curve T_c^l lies on the circumference with center in C_0^l and radius τ in the $H-$plane. When λ tends to zero, the radius of this circumference tends to infinity, thus proving that the curve T_0^l coincides with the segment L_{e_n}.

For an arbitrary $a \in D_{e_n}^{n-1}$ we have $a = s_* c \overline{s_*}$ (c.f. Remark 2.4.2). By (2.35) we get

$$a \oplus b = (s_* c \overline{s_*}) \oplus b = s_*(c \oplus (\overline{s_*} b s_*))\overline{s_*} = s_*(c \oplus b)\overline{s_*},$$

that is, T_a^l is the rotation under s_* of the curve T_c^l. Therefore T_a^l is obtained from the intersection between B^n and the circumference with center in the point $s_* C_0^l \overline{s_*}$ and radius $\tau = \frac{1+|a|^2}{2|a|}$, in the plane $s_* H \overline{s_*}$. To see that each curve T_a^l is orthogonal to S^{n-1} it suffices to verify that $||s_* C_0^r \overline{s_*}||^2 = 1 + \tau^2$.

∎

We can observe the cut of such arcs in the $x_{n-1}x_n-$plane in Fig. 2.3.

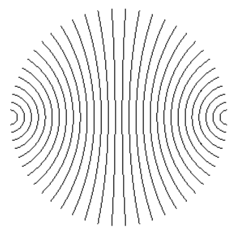


Figure 2.3: Cut of the partition T^l in the $x_{n-1}x_n-$plane.

Now we will consider the right coset space $B^n/(L_{e_n},\sim_r)$.

The family $T^r = \{T_a^r : a \in D_{e_n}^{n-1}\}$, where $T_a^r = \{b \oplus a : b \in L_{e_n}\}$, is again a partition of B^n and it induces the following equivalence relation on B^n :

$$\forall c, d \in B^n, \quad c \sim_r d \;\Leftrightarrow\; \exists\, a \in D_{e_n}^{n-1},\ \exists\, b, f \in L_{e_n} :\ c = b \oplus a \ \text{ and } \ d = f \oplus a. \tag{2.61}$$

Proposition 2.8.15 *For an arbitrary* $a \in D_{e_n}^{n-1}$*, the curve* T_a^r *is obtained from the intersection between* B^n *and the circumference of radius* $\tau = \frac{1+|a|^2}{2|a|}$ *and center in the point* $s_* C_0^r \overline{s_*}$*, in the plane* $s_* H \overline{s_*}$*, where* $s_* = s_1 \ldots s_{n-2}$ *is such that* $\overline{s_*} a s_* = \lambda e_{n-1}, \lambda \in]-1,1[$*,* $C_0^l = (0,\ldots,0,\frac{\lambda^2-1}{2\lambda},0)$*, and* $s_* H \overline{s_*}$ *denotes the rotation of the* $x_{n-1}x_n-$*plane (or* $H-$*plane).*

The proof is analogous to the proof of Proposition 2.8.14. We can observe a projection of such curves in Fig. 2.4.

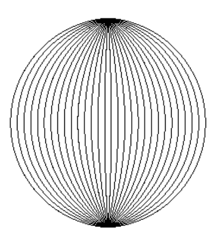


Figure 2.4: Cut of the partition T^r in the $x_{n-1}x_n-$plane.

Thus, we have the isomorphism $B^n/(L_{e_n},\sim_l) \;\cong\; D_{e_n}^{n-1}$ and more generally the isomorphisms $B^n/(L_\omega,\sim_l) \cong D_\omega^{n-1}$ and $B^n/(L_\omega,\sim_r) \cong D_\omega^{n-1}$ hold.

2.9 Sections on the unit ball

Let G be a group, H a closed subgroup of G (not necessarily normal) and $X = G/H$. We consider $\pi : G \to X$, $\pi(g) = gH$, the canonical surjection (or projection map). In the language of fiber bundles, if G is a Lie group and H is a closed subgroup then G is a principal $H-$bundle over the (left) coset space G/H, with projection map π. The fibers, i.e. the pre-images $\pi^{-1}(x)$, $x \in X$, are the left cosets of G/H. A (global) *section* on X is a map $\sigma : X \to G$ such that $\pi(\sigma(x)) = x$, for all $x \in X$. In general, bundles may not have globally defined sections and, therefore, we may only define local sections. In our case, we can define both type of sections.

To be concise in our exposition we will present the definition of a fiber bundle.

Definition 2.9.1 *A fiber bundle is a 4-tuple (E, B, π, F), where $E, B,$ and F are topological spaces and $\pi : E \to B$ is a continuous surjection such that, for any $x \in B$, there is an open neighborhood U of x such that $\pi^{-1}(U)$ is homeomorphic to the product space $U \times F$, in such a way that π carries over to the projection onto the first factor. Thus, the following diagram should commute, where $\mathrm{proj}_1 : U \times F \to U$ is the natural projection and $\varphi : \pi^{-1}(U) \to U \times F$ is a homeomorphism.*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

Figure 2.5: Fiber bundle diagram.

The space E is called the *total space*, B is called the *base space* and F is called the *fiber space*. For any $x \in B$ the pre-image $\pi^{-1}(x)$ is homeomorphic to F and it is called the *fiber* over x. Although we are dealing with gyrogroups and gyro-subgroups we can still define sections since we have a fiber bundle structure. Thus, the concepts defined above for groups can be extended to gyrogroups.

2.9.1 Global sections

For each factorization obtained previously we can define a fiber bundle structure and global sections. We will begin by considering factorizations of type I. Let $X_1 = B^n/(D^{n-1}_{e_n}, \sim_l)$. We define the projection map $\pi : B^n \to X_1$ such that $\pi(c) = [c]$, where $[c]$ is the equivalence class of $c \in B^n$ on X_1. By Proposition 2.8.2 we know that the equivalence class $[c]$ coincides with one of the surfaces S^l_b, $b \in L_{e_n}$ (see also Theorem 2.8.1). Thus, by the isomorphism $B^n/(D^{n-1}_{e_n}, \sim_l) \cong L_{e_n}$, we can define a second projection $\widetilde{\pi} : B^n \to L_{e_n}$ such that $\widetilde{\pi}(c) = b$. The fibers generated by π and $\widetilde{\pi}$ are the same. The action of the gyrogroup $(B^n, \oplus)$ on X_1 is given by the mapping

$$h : B^n \times X_1 \to X_1, \quad h(c, [a]) = [c \oplus a]. \tag{2.62}$$

The 4−tuple (B^n, X_1, π, S^l_b) is a fiber bundle whose fibers are the surfaces S^l_b. Given a neighborhood of $b \in L_{e_n}$, $U_\epsilon = \{x \in L_{e_n} : |x - b| < \epsilon\}$, with $\epsilon < 1 - |b|$, the following diagram commutes

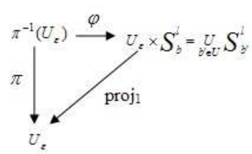


Figure 2.6: Fiber bundle (B^n, X_1, π, S^l_b).

We will consider L_{e_n} as the fundamental section σ^l_0. From Proposition 2.8.3, an entire family of sections $\sigma^l : B^n/(D^{n-1}_{e_n}, \sim_l) \to B^n$ can be obtained from L_{e_n} considering

$$\sigma^l(te_n) = te_n \oplus f(t)e_{n-1} = \left(0, \ldots, 0, \frac{f(t)(1-t^2)}{1+(tf(t))^2}, \frac{t(1+f(t)^2)}{1+(tf(t))^2}\right), \tag{2.63}$$

where $f :]-1,1[\to]-1,1[$. The function f will be called the *generating function* of the section σ^l. Depending on the properties of the function f, we can have sections that are Borel maps and also smooth sections.

The first simple example is given by the generating function $f(t) = \lambda$, for all $t \in]-1,1[$, with $\lambda \in]-1,1[$. Then we obtain the family of sections

$$\sigma^l_\lambda(te_n) = \left(0, \ldots, 0, \frac{\lambda(1-t^2)}{1+(t\lambda)^2}, \frac{t(1+\lambda^2)}{1+(t\lambda)^2}\right).$$

These sections belong to the set of orbits arising in the decomposition $B^n/(L_{e_n}, \sim_r)$.

Another family of sections is $\sigma_c^l = (0, \ldots, 0, c\sin\phi, -\cos\phi)$, with $\phi \in]0, \pi[$ for each $c \in]-1, 1[$ fixed. For $c \in]-1, 1[\backslash\{0\}$ the generating function is given by

$$f(t) = \begin{cases} \sqrt{\frac{t^2-1+\sqrt{(1-t^2)^2+4c^4t^2}}{2t^2c^2}}, & t \in]-1,1[\backslash\{0\} \\ c, & t = 0 \end{cases} .$$

Thus, we obtain

$$\cos\phi = \begin{cases} -\frac{t(1+f(t)^2)}{1+(tf(t))^2} = -\frac{t^2(2c^2+1)-1+\sqrt{(1-t^2)^2+4c^4t^2}}{t(2c^2+t^2-1+\sqrt{(1-t^2)^2+4c^4t^2})}, & t \in]-1,1[\backslash\{0\} \\ 0, & t = 0 \end{cases} .$$

In the next chapters we will make clear the importance of these and other sections in the definition of spherical continuous wavelet transforms on the unit sphere.

For the right coset space $X_2 = B^n/(D_{e_n}^{n-1}, \sim_r)$, we have that L_{e_n} is again the fundamental section. From Proposition 2.8.9 an entire class of sections can be obtained by considering

$$\sigma^r(te_n) = g(t)e_{n-1} \oplus te_n = \left(0, \ldots, 0, \frac{g(t)(1+t^2)}{1+(tg(t))^2}, \frac{t(1-g(t)^2)}{1+(tg(t))^2}\right),$$

with $g :]-1,1[\to]-1,1[$. Choosing the generating function $g(t) = \lambda$, with $\lambda \in]-1,1[$ we obtain the family of sections

$$\sigma_\lambda^r(te_n) = \left(0, \ldots, 0, \frac{\lambda(1+t^2)}{1+(t\lambda)^2}, \frac{t(1-\lambda^2)}{1+(t\lambda)^2}\right),$$

These sections belong to the set of orbits obtained in the decomposition $B^n/(L_{e_n}, \sim_l)$. Again we can consider the family of sections $\sigma_c^r = (0, \ldots, 0, c\sin\phi, -\cos\phi)$, $\phi \in [0, \pi[$, for each $c \in]-1, 1[$ fixed. The generating function is now given by

$$g(t) = \begin{cases} \sqrt{\frac{t^4+2t^2(1-2c^2)+1-(1+t^2)\sqrt{t^4+2t^2(2c^4-4c^2+1)+1}}{2t^2c^2(t^2-1)}}, & t \in]-1,1[\backslash\{0\} \\ c, & t = 0 \end{cases} .$$

Thus, we obtain

$$\cos\phi = \begin{cases} -\frac{t(1-g(t)^2)}{1+(tg(t))^2} = -\frac{t^2(2c^2-1)-1+\sqrt{t^4+2t^2(2c^4-4c^2+1)+1}}{t(t^2-2c^2+1-\sqrt{t^4+2t^2(2c^4-4c^2+1)+1})}, & t \in]-1,1[\backslash\{0\} \\ 0, & t = 0 \end{cases} .$$

We emphasize that there are many other sections that we could obtain. However, we will not describe there here.

The construction of sections for the homogeneous spaces $B^n/(D^{n-1}_\omega, \sim_l)$ and $B^n/(D^{n-1}_\omega, \sim_r)$ can be made considering a convenient rotation of the sections described above.

For our work we only need to consider factorizations of type I as we will see on the next chapter.

With respect to factorizations of type II we can also construct sections for the respective homogeneous spaces. For instance, let us consider the space $X_3 = B^n/(L_{e_n}, \sim_l)$. Given a function $h : D^{n-1}_{e_n} \to]-1,1[$, we define $\sigma : X_3 \to B^n$ by $\sigma([a]) = a \oplus h(a)e_n$, for each $a \in D^{n-1}_{e_n}$. Thus, σ is a global section for the homogeneous space $B^n/(L_{e_n}, \sim_l)$. For example, for each $\lambda \in]-1,1[$, if $h(a) = \lambda$ for all $a \in D^{n-1}_{e_n}$, then the family of sections obtained coincides with the set of orbits obtained in the decomposition $B^n/(D^{n-1}_{e_n}, \sim_r)$. For $\lambda = 0$, we obtain the section $D^{n-1}_{e_n}$, which can be considered the fundamental section in this case.

Analogously, we can construct sections for the spaces $B^n/(L_{e_n}, \sim_r)$, $B^n/(L_\omega, \sim_r)$, and $B^n/(L_\omega, \sim_l)$.

There is a duality relation between factorizations of type I and factorizations of type II that we will describe in the next theorem.

Theorem 2.9.2 *The following duality relations hold:*

1. *The orbits of the decomposition $B^n/(L_{e_n}, \sim_r)$ are global sections for the homogeneous spaces $B^n/(D^{n-1}_{e_n}, \sim_l)$ and $B^n/(D^{n-1}_{e_n}, \sim_r)$, and vice versa.*

2. *The orbits of the decomposition $B^n/(L_{e_n}, \sim_l)$ are global sections for the homogeneous space $B^n/(D^{n-1}_{e_n}, \sim_r)$ and vice versa.*

3. *The orbits of $B^n/(L_{e_n}, \sim_l)$ are local sections for the space $B^n/(D^{n-1}_{e_n}, \sim_l)$ and vice versa.*

The relation between the orbits of $B^n/(L_{e_n}, \sim_l)$ and $B^n/(D^{n-1}_{e_n}, \sim_l)$ leads to the concept of local sections.

2.9.2 Local sections

A *local section* of a fiber bundle (E, B, π, F) is a continuous map $\sigma : U \to E$, where U is an open set in B and $\pi(\sigma(x)) = x$ for all $x \in U$.

In our case we can also construct local sections. We will give some examples. For each $\phi \in]0, \pi/2[\cup]\pi/2, \pi[$ the family of sections $\sigma^l = (0, \dots, 0, r\sin\phi, r\cos\phi)$, with $r \in]-1,1[$, is a family of local sections for the homogeneous space $X_1 = B^n/(D^{n-1}_{e_n}, \sim_l)$. For each $\phi \in]0, \pi/2[$ the generating function is given by

$$f_1 : V_1 = \left]-\frac{\cos\phi}{1+\sin\phi}, \frac{\cos\phi}{1+\sin\phi}\right[\to]-1,1[,$$

$$f_1(t) = \begin{cases} \frac{(1-t^2)\cos\phi - \sqrt{(1+t^2)^2\cos^2\phi - 4t^2}}{2t\sin\phi}, & t \in V_1\backslash\{0\} \\ 0, & t = 0 \end{cases}. \tag{2.64}$$

The parameter r is related with the generating function f_1 by

$$r = \frac{f_1(t)}{t\cos\phi f_1(t) + \sin\phi} = \begin{cases} \frac{(t^2-1)\cos\phi - \sqrt{(1+t^2)^2\cos^2\phi - 4t^2}}{t\left((1+t^2)\cos^2\phi - 2 - \cos\phi\sqrt{(1+t^2)^2\cos^2\phi - 4t^2}\right)}, & t \in V_1\backslash\{0\} \\ 0, & t = 0 \end{cases}.$$

For each $\phi \in]\pi/2, \pi[$ the generating function is given by

$$f_2 : V_2 = \left]\frac{\cos\phi}{1+\sin\phi}, -\frac{\cos\phi}{1+\sin\phi}\right[\to]-1,1[,$$

$$f_2(t) = \begin{cases} \frac{(1-t^2)\cos\phi + \sqrt{(1+t^2)^2\cos^2\phi - 4t^2}}{2t\sin\phi}, & t \in V_2\backslash\{0\} \\ 0, & t = 0 \end{cases}.$$

Therefore, the parameter r is related with the generating function by

$$r = \frac{f_2(t)}{t\cos\phi f_2(t) + \sin\phi} = \begin{cases} \frac{(t^2-1)\cos\phi + \sqrt{(1+t^2)^2\cos^2\phi - 4t^2}}{t\left((1+t^2)\cos^2\phi - 2 + \cos\phi\sqrt{(1+t^2)^2\cos^2\phi - 4t^2}\right)}, & t \in V_2\backslash\{0\} \\ 0, & t = 0 \end{cases}.$$

More generally, given $\omega \in S^{n-1}\backslash\{\pm e_n\}$ such that $< \omega, e_n > \neq 0$, the family of sections $\sigma^l = r\omega$, with $r \in]-1,1[$ is a family of local sections for the homogeneous space X_1.

For the homogeneous space $X_2 = B^n/(D^{n-1}_{e_n}, \sim_r)$ it is also possible to construct local sections. We will construct a simple example. For each $k \in]-1,0[\cup]0,1[$ we consider the family of local sections $\sigma^r = (0, \ldots, 0, p, k)$, with $p \in [0, \sqrt{1-k^2}[$.

For $k \in]0,1[$ the generating function is given by

$$g_1 :]k, 1[\to]-1,1[; \quad t \mapsto g_1(t) = \sqrt{\frac{t-k}{t(1+kt)}}$$

and for $k \in]-1,0[$ the generating function is given by

$$g_2 :]-1, k[\to]-1,1[, \quad t \mapsto g_2(t) = \sqrt{\frac{t-k}{t(1+kt)}}.$$

The parameter r is related with the generating functions by

$$p = g_i(t)(1+kt) = \sqrt{\frac{(t-k)(1+kt)}{t}}, \quad i = 1, 2.$$

For the homogeneous spaces obtained from factorizations of type II it is also possible to construct examples of local sections. For instance, for $k \in]-1,1[$, the family of hyperdiscs $D_k = \{a \in B^n : a_n = k\}$ is a family of local sections for the homogeneous space $X_3 = B^n/(L_{e_n}, \sim_l)$.

We remark that the construction of families of local sections is a quite arbitrary procedure and we have not exhausted all examples.

2.10 Factorizations of the proper Lorentz group

Until now we only have considered factorizations of the gyrogroup of the unit ball. In order to incorporate rotations in our scheme we have to extend the equivalence relations obtained in both factorizations of the unit ball to the whole of the group $\text{Spin}^{+}(1,n)$, according to the group operations (2.50) or (2.51). Is is possible to use either the left or right gyrosemidirect products to make this extension. However, due to the structure of the gyrosemidirect product, the extension becomes different for right and left cosets. We only deal with factorizations of type I, since the extension of the equivalence relations obtained in factorizations of type II is analogous.

Let 1 denote the identity rotation. It is easy to see that the direct products $\{1\} \times D_{e_n}^{n-1}$ and $D_{e_n}^{n-1} \times \{1\}$ are gyrogroups. Our goal is to define an equivalence relation $\sim_l^{*,1}$ on $\text{Spin}(n) \times B^n$, which is an extension of the equivalence relation $\sim_l$ on B^n, such that the resulting homogeneous space $\widetilde{X_1} = (\text{Spin}(n) \times B^n)/(\{1\} \times D_{e_n}^{n-1}, \sim_l^{*,1})$ is isomorphic as a set to $\text{Spin}(n) \times L_{e_n}$. For $(s_1, c), (s_2, d) \in \text{Spin}(n) \times B^n$ we define

$$(s_1, c) \sim_l^{*,1} (s_2, d) \Leftrightarrow \exists\, s_3 \in \text{Spin}(n),\ \exists\, b \in L_{e_n},\ \exists\, a, f \in D_{e_n}^{n-1} :$$

$$(s_1, c) = \left(s_3 \frac{\overline{1 - [(-a) \oplus (b \oplus a)]a}}{|1 - [(-a) \oplus (b \oplus a)]a|}, (-a) \oplus (b \oplus a) \right) \times^r (e, a)$$

and

$$(s_2, d) = \left(s_3 \frac{\overline{1 - [(-f) \oplus (b \oplus f)]f}}{|1 - [(-f) \oplus (b \oplus f)]f|}, (-f) \oplus (b \oplus f) \right) \times^r (e, f). \tag{2.65}$$

The equivalence relation (2.65) reduces to

$$(s_1, c) \sim_l^{*,1} (s_2, d) \Leftrightarrow \exists\, b \in L_{e_n},\ \exists\, a, f \in D_{e_n}^{n-1} : s_1 = s_2 \wedge (c = b \oplus a\ \wedge\ d = b \oplus f).$$

Thus, it is easy to see that the equivalence class associated to (s_1, c) is equal to $\{s_1\} \times [c]$, with $[c] = S_b^l \in B^n/(D_{e_n}^{n-1}, \sim_l)$. Moreover, the quotient space is $\widetilde{X}_1 = \{\{s\} \times S_b^l : s \in \mathrm{Spin}(n), b \in L_{e_n}\} \cong \mathrm{Spin}(n) \times L_{e_n}$.

The group $\mathrm{Spin}^+(1,n)$ acts on $\widetilde{X_1}$ according to (2.50) by the mapping

$$g : \mathrm{Spin}^+(1,n) \times \widetilde{X_1} \to \widetilde{X_1}$$

$$((s_1, a), (s_2, [c])) \mapsto \left(s_1 s_2 \frac{1 - \overline{s_2} a s_2 c}{|1 - \overline{s_2} a s_2 c|}, [c \oplus (\overline{s_2} a s_2)] \right).$$

The equivalence relation $\sim_l^{*,1}$, associated to the group operation (2.50), appears here as the natural extension of the relation $\sim_l$ of the factorization $B^n/(D_{e_n}^{n-1}, \sim_l)$.

It is also possible to use the group operation (2.51) to obtain a similar isomorphism. But, in this case, the extension of $\sim_l$ is now a right equivalence relation. For $(c, s_1), (d, s_2) \in B^n \times \mathrm{Spin}(n)$ it is defined by

$$(c, s_1) \sim_r^{*,1} (d, s_2) \Leftrightarrow \exists\, s_3 \in \mathrm{Spin}(n),\ \exists\, b \in L_{e_n},\ \exists\, a, f \in D_{e_n}^{n-1} :$$

$$(c, s_1) = (a, e) \times^l \left((-a) \oplus (b \oplus a), \frac{\overline{1 - a[(-a) \oplus (b \oplus a)]}}{|1 - a[(-a) \oplus (b \oplus a)]|} s_3 \right)$$

and

$$(d, s_2) = (f, e) \times^l \left((-f) \oplus (b \oplus f), \frac{\overline{1 - f[(-f) \oplus (b \oplus f)]}}{|1 - f[(-f) \oplus (b \oplus f)]|} s_3 \right),$$

that is

$$(c, s_1) \sim_r^{*,2} (d, s_2) \Leftrightarrow \exists\, b \in L_{e_n},\ \exists\, a, f \in D_{e_n}^{n-1} : s_1 = s_2 \wedge (c = a \oplus b \ \wedge\ d = f \oplus b),$$

Therefore, we obtain the homogeneous space $(B^n \times \mathrm{Spin}(n))/(D_{e_n}^{n-1} \times \{1\}, \sim_r^{*,1}) \cong L_{e_n} \times \mathrm{Spin}(n)$.

We can construct global and local sections for $\widetilde{X_1}$ using the knowledge obtained in sub-Sections 2.9.1 and 2.9.2. In fact, if $\sigma : B^n/(D_{e_n}^{n-1}, \sim_l) \to B^n$ is a section for $B^n/(D_{e_n}^{n-1}, \sim_l)$ then, $\sigma^* : \widetilde{X_1} \to \mathrm{Spin}(n) \times B^n$, defined as $\sigma^*(\{s\} \times S_b^l) = (s, \sigma(S_b^l))$ is a section for $\widetilde{X_1}$.

For the natural extension of the right equivalence relation $\sim_r$ of $B^n/(D_{e_n}^{n-1}, \sim_r)$ we will use the operation (2.50). For $(c, s_1), (d, s_2) \in B^n \times \mathrm{Spin}(n)$ we define

$$(c, s_1) \sim_r^{*,2} (d, s_2) \Leftrightarrow \exists\, s_3 \in \mathrm{Spin}(n),\ \exists\, b \in L_{e_n},\ \exists\, a, f \in D_{e_n}^{n-1} :$$

$$(c, s_1) = (a, e) \times^l \left(b, \frac{\overline{1 - ab}}{|1 - ab|} s_3 \right) \quad \text{and} \quad (d, s_2) = (f, e) \times^l \left(b, \frac{\overline{1 - fb}}{|1 - fb|} s_3 \right). \tag{2.66}$$

Hence, the equivalence relation (2.66) reduces to

$$(c, s_1) \sim_r^{*,1} (d, s_2) \Leftrightarrow \exists\, b \in L_{e_n},\ \exists\, a, f \in D_{e_n}^{n-1} : s_1 = s_2 \wedge (c = b \oplus a \ \wedge\ d = b \oplus f).$$

and we obtain the homogeneous space $\widetilde{X}_2 = (B^n \times \mathrm{Spin}(n))/(D_{e_n}^{n-1} \times \{1\}, \sim_r^{*,2}) = \{S_b^r \times \{s\}, b \in L_{e_n}, s \in \mathrm{Spin}(n)\} \cong L_{e_n} \times \mathrm{Spin}(n)$.

Using the operation (2.50) it is possible to define the following left equivalence relation:

$$(s_1, c) \sim_l^{*,2} (s_2, d) \Leftrightarrow \exists\, s_3 \in \mathrm{Spin}(n),\ \exists\, b \in L_{e_n},\ \exists\, a, f \in D_{e_n}^{n-1} :$$

$$(s_1, c) = \left(s_3 \frac{\overline{1 - ba}}{|1 - ba|}, b \right) \times^r (e, a) \quad \text{and} \quad (s_2, d) = \left(s_3 \frac{\overline{1 - bf}}{|1 - bf|}, b \right) \times^r (e, f),$$

that is

$$(s_1, c) \sim_l^{*,2} (s_2, d) \Leftrightarrow \exists\, b \in L_{e_n},\ \exists\, a, f \in D_{e_n}^{n-1} : s_1 = s_2 \wedge (c = a \oplus b \ \wedge\ d = f \oplus b).$$

Thus, we obtain the homogeneous space $(\mathrm{Spin}(n) \times B^n)/(\{1\} \times D_{e_n}^{n-1}, \sim_l^{*,2}) \cong \mathrm{Spin}(n) \times L_{e_n}$.

For the factorizations of type II we can proceed in a similar way and extend the equivalence relations obtained in 2.8.2 to the whole of the group $\mathrm{Spin}^+(1,n)$ using the group operations (2.50) or (2.51).

The spaces $\widetilde{X_1}$ and $\widetilde{X_2}$ can be used for the construction of spherical continuous wavelet transforms as we will see on the next chapter.

2.11 Connections with the group of automorphisms of the unit disc in $\mathbb{C}$

Since the algebraic structure of the Clifford algebra $\mathbb{R}_{0,n}$ is a generalization of the algebraic structure of the complex plane there should exist a relationship between the Lorentz group $\mathrm{Spin}^+(1,n)$ and the group of conformal automorphisms of the unit disc D in $\mathbb{C}$. The main difference is that the set of complex numbers is a field with a commutative multiplication whereas in higher dimensions we have only an algebra with a non-commutative multiplication. Nevertheless, we will show that there are strong analogies namely the gyrogroup structure is already presented in the group of the automorphisms of the unit disc.

The group of automorphisms of the unit disc is the set $\{e^{i\theta}\widetilde{\varphi}_a(z), \theta \in [0, 2\pi[, a \in D\}$, i.e. it consists of rotations given by $e^{i\theta}(\theta \in [0, 2\pi[)$ and Möbius transformations of the form $\widetilde{\varphi}_a(z) = \frac{z-a}{1-\bar{a}z}$. We will show that the set of Möbius transformations $\widetilde{\varphi}_a(z)$ in $\mathbb{C}$ share the same properties of $\varphi_a(x)$ in $\mathbb{R}^n$.

With respect to the composition of two Möbius transformations we have

$$(\widetilde{\varphi}_a \circ \widetilde{\varphi}_b)(z) = \frac{1+a\bar{b}}{1+\overline{a}b}\,\widetilde{\varphi}_{b\oplus a}(z), \quad \forall\, a,b \in D,\ \forall z \in D,$$

with $b \oplus a = \frac{a+b}{1+\bar{b}a} = \widetilde{\varphi}_{-b}(a)$. Clearly the element $\lambda = \frac{1+a\bar{b}}{1+\overline{a}b}$ defines a rotation since $|\lambda| = 1$. It is easy to see that $(D, \oplus)$ is a gyrocommutative gyrogroup with the left gyroassociative law given by $a \oplus (b \oplus c) = (a \oplus b) \oplus (\lambda c)$, and the gyrocommutative law given by $a \oplus b = \lambda(b \oplus a)$.

For all $a, b, z \in D$ and $\theta \in [0, 2\pi[$ the following properties hold:

1) $\widetilde{\varphi}_a(e^{i\theta})(z) = e^{i\theta}\widetilde{\varphi}_{e^{-i\theta}a}(z)$; 2) $e^{i\theta}\widetilde{\varphi}_a(z) = \widetilde{\varphi}_{e^{i\theta}a}(e^{i\theta}z)$;

3) $(e^{i\theta}a) \oplus b = e^{i\theta}(a \oplus (e^{-i\theta}b))$; 4) $e^{i\theta}(a \oplus b) = (e^{i\theta}a) \oplus (e^{i\theta}b)$.

The main difference between $(D, \oplus)$ and $(B^n, \oplus), n \geq 3$, consists in the existence of subgroups and gyro-subgroups. In the case of $(D, \oplus)$ there are only subgroups of the type $(L_\omega, \oplus)$, with $L_\omega = \{t\omega, t \in]-1, 1[\}$, for each $\omega \in S^1$.

The Unique Decomposition Theorems (2.8.1 and 2.8.6) are valid in this case, but now the decomposition is done in terms of the subgroups L_i and L_1. In higher dimensions the subgroup L_i is replaced by the subgroup L_{e_n} and the subgroup L_1 is replaced by the gyro-subgroup $D_{e_n}^{n-1}$.

Theorem 2.11.1 *For each $c \in D$ there exist unique $a, d \in L_i$ and $b, f \in L_1$ such that $c = a \oplus b$ and $c = f \oplus d$.*

This theorem allow us to construct partitions of D and homogeneous spaces for D as we made in Section 2.8. The orbits obtained have the same structure because they express essentially the hyperbolic geometry, independent of the dimension. Therefore, the theorem about duality relations holds also in $\mathbb{C}$ and the theory of sections on homogeneous spaces can also be developed. As we can see, there exists a strong analogy between the algebraic structure of the Lorentz group $\mathrm{Spin}^+(1, n)$ in $\mathbb{R}_{0,n}, n \geq 3$, and the algebraic structure of the group of conformal automorphisms of the unit disc D in $\mathbb{C}$.

2.12 Relation between $(B^3, \oplus)$ and $SL(2, \mathbb{C})$

In this section we restrict ourselves to the case $n = 3$ and we establish a relation between a Möbius transformation φ_a acting on the unit sphere S^2 and an element of the group $SL(2, \mathbb{C})$.

The relationship between $SL(2, \mathbb{C})$ and $SO_0(1, 3)$ is well known. The group $SL(2, \mathbb{C})$ is isomorphic to the group $\mathrm{Spin}(1, 3)$, which is a (simply connected) double-cover of $SO_0(1, 3)$ (see [13]). Thus, there exists the isomorphism

$$SL(2, \mathbb{C})/\{I, -I\} \cong SO_0(1, 3).$$

In order to construct such isomorphism, we consider the approach presented in [39]. To each hermitian matrix of second order

$$X = \begin{pmatrix} x_0 - x_3 & -x_1 - ix_2 \\ -x_1 + ix_2 & x_0 + x_3 \end{pmatrix} \tag{2.67}$$

we associate a vector $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, where x_0 is the component in the time axis. This correspondence is one-one and linear. Therefore, any linear transformation in the space of the matrices X may be considered as a linear transformation in $\mathbb{R}^4$. Any matrix $A \in SL(2, \mathbb{C})$ specifies an unique linear transformation $X \mapsto Y = AXA^*$ (* denotes the conjugate transpose), which in turn, induces a proper Lorentz transformation in $\mathbb{R}^4$.

It is also known that $SL(2, \mathbb{C}) = PSL(2, \mathbb{C}) \cong \text{Möb}^+$, where $PSL(2, \mathbb{C})$ denotes the group of projective transformations of the complex projective space $\mathbb{CP}^1$, and Möb^+ is the group of Möbius transformations

$$z \mapsto \frac{az + b}{cz + d}, \quad z \in \mathbb{C} \tag{2.68}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. They act on the Riemann sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$. The compactified complex plane is usually denoted by $\overline{\mathbb{C}}$, i.e. $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. It is easy to see that $\text{Möb}^+ \cong SL(2, \mathbb{C})/\{I, -I\}$ and, therefore, $\text{Möb}^+ \cong SO_0(1, 3)$.

We have already defined the stereographic projection map on (2.24). In the 2-dimensional case the stereographic projection map $\Phi_1 : S^2 \to \mathbb{C} \cup \{\infty\}$ is a bijection given by

$$\Phi_1(x_1, x_2, x_3) = \frac{2x_1 + 2x_2 i}{1 + x_3}, \quad \text{for} \quad (x_1, x_2, x_3) \in S^2 \backslash \{(0, 0, -1)\},$$

and $\Phi_1(0, 0, -1) = \infty$.

Definition 2.12.1 *The cross ratio of four complex numbers z_0, z_1, z_2 and z_3 is defined as*

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3} \frac{z_1 - z_3}{z_1 - z_2}. \tag{2.69}$$

Here we make the conventions that

$$\frac{a \pm \infty}{b \pm \infty} = 1, (a, b \in \mathbb{C}), \quad \frac{a}{0} = \infty, (a \neq 0), \quad \frac{a}{\infty} = 0, (a \neq \infty),$$

so for example $(z, 1, 0, \infty) = z$.

When z_1, z_2, and z_3 are distinct, and z_0 is replaced by a variable z, this cross ratio can be used to define a Möbius transformation

$$T(z) = (z, z_1, z_2, z_3) = \frac{z - z_2}{z - z_3} \frac{z_1 - z_3}{z_1 - z_2}$$

mapping $T(z_1) = 1$, $T(z_2) = 0$, and $T(z_3) = \infty$. The following theorem is sometimes called Fundamental Theorem of Möbius Geometry.

Theorem 2.12.2 *If z_1, z_2, and z_3 are three distinct complex numbers, and w_1, w_2 w_3 are also three distinct complex numbers then there is a unique Möbius transformation that maps each z_i to w_i.*

One of the implications of this theorem says that the Möbius transformation is solely determined by three complex numbers z_1, z_2 and z_3 sent to $1, 0$, and ∞, respectively.

Theorem 2.12.3 *The cross ratio is invariant under Möbius geometry, i.e.*

$$(z_0, z_1, z_2, z_3) = (T(z_0), T(z_1), T(z_2), T(z_3)).$$

Applying this theorem we deduce the following result.

Theorem 2.12.4 *For each $a = (a_1, a_2, a_3) \in B^3$, the Möbius transformation φ_a acting on S^2 can be described as an element of the group $SL(2, \mathbb{C})$ by the matrix*

$$\frac{1}{\sqrt{1 - |a|^2}} \begin{pmatrix} 1 + a_3 & -2(a_1 + a_2 i) \\ \frac{-a_1 + a_2 i}{2} & 1 - a_3 \end{pmatrix}. \tag{2.70}$$

Proof: We know that Möbius transformations map circumferences into circumferences. As the stereographic projection preserves circumferences, the Möbius transformation φ_a on the unit sphere S^2 can be identified with a Möbius transformation on the compactified complex plane $\overline{\mathbb{C}}$, i.e. an element of $SL(2, \mathbb{C})$. To perform such identification it is only needed to know the image of three points on $\overline{\mathbb{C}}$ and then to solve a cross ratio relation.

For the computations we need to divide the proof in two cases:

Case 1 - $a_1 \neq 0$ and $a_2 \neq 0$:

We consider the points $(1,0,0),(0,0,-1)$ and $(0,0,1)$ on S^2 that corresponds respectively to the points $2,0$ and ∞ on $\overline{\mathbb{C}}$, via the stereographic projection map Φ_1.

Let T denotes the Möbius transformation on $\overline{\mathbb{C}}$ that corresponds to φ_a on S^2. Then $T(2)=\Phi_1(\varphi_a(1,0,0))$, $T(0)=\Phi_1(\varphi_a(0,0,-1))$ and $T(\infty)=\Phi_1(\varphi_a(0,0,1))$. Now, as T is invariant under the cross ratio, we must have

$$(T(z),T(\infty),T(2),T(0))=(z,\infty,2,0).$$

The left hand-side is equal to $(z-2)/z$. Solving this cross ratio we obtain, by straightforward computations,

$$T(z)=\frac{(2+2a_3)z-4a_1-4a_2i}{(-a_1+a_2i)z+2-2a_3}.$$

After normalization we obtain the matrix (2.70).

Case 2 - $a_1=0$ and $a_2=0$:

For this case we consider $T(2)=\Phi_1(\varphi_a(1,0,0))$, $T(0)=\Phi_1(\varphi_a(0,0,-1))$ and $T(-2)=\Phi_1(\varphi_a(-1,0,0))$. The following equality holds

$$(T(z),T(-2),T(2),T(0))=(z,-2,2,0).$$

The right hand-side is equal to $(z-2)/(2z)$. Again, solving this cross ratio we get

$$T(z)=\frac{(1+a_3)z}{1-a_3}.$$

After normalization we obtain the following matrix of $SL(2,\mathbb{C})$

$$\begin{pmatrix} \sqrt{\frac{1+a_3}{1-a3}} & 0 \\ 0 & \sqrt{\frac{1-a_3}{1+a_3}} \end{pmatrix},$$

which is a particular case of (2.70).

■

Remark 2.12.5 *Considering the stereographic projection mapping (2.26) for $n=3$ we obtain the matrix*

$$\frac{1}{\sqrt{1-|a|^2}}\begin{pmatrix} 1-a_3 & -(a_1+a_2i) \\ -a_1+a_2i & 1+a_3 \end{pmatrix}. \tag{2.71}$$

2.13 Stereographic projection in $\mathbb{R}^n$ of $\varphi_a(x)$

In this section we will generalize the relation between $(B^3, \oplus)$ and $SL(2,\mathbb{C})$ previously obtained to higher dimensions, i.e. we will find the Möbius transformations $\widetilde{\varphi_1}$ and $\widetilde{\varphi_2}$ on $\mathbb{R}^{n-1}$ such that $\Phi_1(\varphi_a(x)) = \widetilde{\varphi_1}(\Phi_1(x))$ and $\Phi_2(\varphi_a(x)) = \widetilde{\varphi_2}(\Phi_2(x))$, respectively.

For that purpose we will use the Cayley transformation, which maps the unit ball B^n into the upper half space $H_n^+ = \{x \in \mathbb{R}^n : x_n > 0\}$.

Definition 2.13.1 *The Cayley transformation in $\mathbb{R}^n$ is defined by*

$$y = (x + e_n)(1 + e_n x)^{-1}. \tag{2.72}$$

First we observe that the stereographic projection is the limit case of the Cayley transformation, i.e., if we consider $S^{n-1} = \partial B^n$ then it is mapped by the Cayley transformation to $\mathbb{R}^{n-1} = \partial H_n^+$.

Proposition 2.13.2 *The stereographic projection mapping $\Phi_2 : S^{n-1} \to \mathbb{R}^{n-1}$ can be defined via the Cayley transformation*

$$\Phi_2(x) = (x + e_n)(1 + e_n x)^{-1} = \frac{x - x_n e_n}{1 - x_n} = \left(\frac{x_1}{1 - x_n}, \ldots, \frac{x_{n-1}}{1 - x_n}\right). \tag{2.73}$$

Proof: By some computations we find that

$$(x + e_n)(1 + e_n x)^{-1} = \frac{(x + e_n)\overline{(1 + e_n x)}}{|1 + e_n x|^2} = \frac{2x + (1 - |x|^2 - 2x_n)e_n}{1 - 2x_n + |x|^2}.$$

Thus, considering $x \in S^{n-1}$, i.e. $|x|^2 = 1$, we obtain the stereographic projection mapping (2.73) (projection from the North Pole $e_n = (0, \ldots, 0, 1)$ to the plane $\mathbb{R}^{n-1}$ at the origin).

■

To find the Möbius transformation that intertwines with the stereographic projection we only have to work with the respective Vahlen matrices (at the level of the symbols - projective geometry).

Theorem 2.13.3 *We have the following intertwining relation $\Phi_2(\varphi_a(x)) = \widetilde{\varphi_2}(\Phi_2(x))$, where $\widetilde{\varphi_2}(x)$ is the Möbius transformation in $\mathbb{R}^{n-1}$ defined by the Vahlen matrix*

$$\frac{1}{\sqrt{1 - |a|^2}} \begin{pmatrix} 1 - a_n & -a + a_n e_n \\ a - a_n e_n & 1 + a_n \end{pmatrix}. \tag{2.74}$$

Proof: Solving the system of equations in order to $u, v, w, z \in \Gamma(n)$ we obtain:

$$\begin{pmatrix} 1 & e_n \\ e_n & 1 \end{pmatrix}\begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} = \begin{pmatrix} u & v \\ w & z \end{pmatrix}\begin{pmatrix} 1 & e_n \\ e_n & 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} u + ve_n = 1 + e_n a \\ ue_n + v = -a + e_n \\ w + ze_n = e_n + a \\ we_n + z = 1 - e_n a \end{cases} \Leftrightarrow \begin{cases} u = 1 + e_n a - ve_n \\ v = \frac{-a - e_n a e_n}{2} \\ w = e_n + a - ze_n \\ z = \frac{2 - e_n a - a e_n}{2} \end{cases} \Leftrightarrow \begin{cases} u = 1 - a_n \\ v = -a + a_n e_n \\ w = a - a_n e_n \\ z = 1 + a_n \end{cases}.$$

In this resolution we used the identities $e_n a e_n = -2a_n e_n + a$ and $e_n a + a e_n = -2a_n$.

After normalization we obtain the matrix (2.74), which is in the Clifford group $\Gamma(1, n+1)$, by Theorem 2.2.1, and thus, it represents a Möbius transformation in $\mathbb{R}^{n-1}$.

∎

Remark 2.13.4 *It is also possible to rewrite the stereographic projection map (2.24) by means of the Cayley transformation. The inverse Cayley transformation maps the upper half space H_n^+ onto the unit ball B^n. However, as a mapping from B^n to $\mathbb{R}^n$, it maps S^{n-1} to $\mathbb{R}^{n-1}$ and coincides with the stereographic projection Φ_1 (up to a factor). Indeed, considering the inverse of the Cayley transformation multiplied by the factor 2 we obtain*

$$\Phi_1(x) = 2(x - e_n)(1 - e_n x)^{-1} = \left(\frac{2x_1}{1 + x_n}, \ldots, \frac{2x_{n-1}}{1 + x_n}\right), \quad x \in S^{n-1}.$$

In this case we obtain the intertwining relation $\Phi_1(\varphi_a(x)) = \widetilde{\varphi_1}(\Phi_1(x))$, where $\widetilde{\varphi_1}(x)$ is the Möbius transformation in $\mathbb{R}^{n-1}$ defined by the Vahlen matrix

$$\frac{1}{\sqrt{1 - |a|^2}} \begin{pmatrix} 1 + a_n & 2(-a + a_n e_n) \\ \frac{a - a_n e_n}{2} & 1 - a_n \end{pmatrix}. \tag{2.75}$$

Finally, we want to remark that the intertwining relations obtained are also valid as intertwining relations between the unit ball and the upper half space (by means of the Cayley transformation).

2.14 Integration formulae

We denote by G the group $\mathrm{Spin}^+(1,n)$. For the polar decomposition of G (see (2.6)) we have the following integration formula:

Lemma 2.14.1 *The invariant measure on* $Spin^+(1,n)$ *is given by*

$$\int_G f(g)dg = \int_{\mathrm{Spin}(n)} \int_{B^n} f(s,a) \frac{dB_a}{(1-|a|^2)^n} d\mu(s), \tag{2.76}$$

where $d\mu(s)$ *is the Haar measure on the compact group* $Spin(n)$ *and* dB_a *is the normalized Lebesgue measure of* B^n.

Proof: Since the group $SO_0(1,n)$ is unimodular, also its double covering group $\mathrm{Spin}^+(1,n)$ is unimodular. Hence, it is sufficient to prove the right invariance. By (2.52), we have

$$(s_1, a) \times^r (s_2, b) = \left(s_1 s_2 \frac{1 - \overline{s_2} a s_2 b}{|1 - \overline{s_2} a s_2 b|}, b \oplus (\overline{s_2} a s_2) \right) = (s_3, c).$$

Thus,

$$s_1 = s_3 \frac{1 - b\overline{s_2} a s_2}{|1 - b\overline{s_2} a s_2|} \overline{s_2} \quad \text{and} \quad a = s_2((-b) \oplus c)\overline{s_2} = s_2 \varphi_b(c) \overline{s_2}.$$

As $d\mu(s)$ is the Haar measure on $\mathrm{Spin}(n)$ we obtain $d\mu(s_1) = d\mu(s_3)$. For the change of variable $a = s_2\varphi_b(c)\overline{s_2}$ we obtain the invariant relation

$$\frac{dB_a}{(1-|a|^2)^n} = \frac{1}{(1-|\varphi_b(c)|^2)^n} \left(\frac{1-|b|^2}{|1+bc|^2} \right)^n dB_c = \frac{dB_c}{(1-|c|^2)^n}.$$

Thus, we derive the desired right invariance

$$\begin{aligned} \int_{\mathrm{Spin}(n)} \int_{B^n} f((s_1,a) \times^r (s_2,b)) \frac{dB_a}{(1-|a|^2)^n} d\mu(s_1) &= \int_{\mathrm{Spin}(n)} \int_{B^n} f(s_3, c) \frac{dB_c}{(1-|c|^2)^n} d\mu(s_3) \\ &= \int_{\mathrm{Spin}(n)} \int_{B^n} f(s_1, a) \frac{dB_a}{(1-|a|^2)^n} d\mu(s_1). \end{aligned}$$

■

Corollary 2.14.2 *For the* KAK *decomposition of* $Spin^+(1,n)$ *it is valid the integration formula:*

$$\int_G f(g)dg = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_{\mathrm{Spin}(n)} \int_{S^{n-1}} \int_0^1 f(s_1 r e_n \xi) \frac{r^{n-1}}{(1-r^2)^n} dr d\xi d\mu(s_1). \tag{2.77}$$

The proof follows from considering, in the previous lemma polar coordinates.

Now we establish the link between this integration formula and the integration formula presented in [67].

Corollary 2.14.3 *Using the KAK decomposition* $g = s_1 s_{e_n}(\alpha) s_2$, *where* $s_1, s_2 \in \mathrm{Spin}(n)$ *and* $s_{e_n}(\alpha) = \cosh\frac{\alpha}{2} + \epsilon e_n \sinh\frac{\alpha}{2}$, $\alpha \in \mathbb{R}^+$, *we have the following integration formula:*

$$\int_G f(g)dg = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_{\mathrm{Spin}(n)} \int_{\mathrm{Spin}(n)} \int_0^\infty f(s_1 s_{e_n}(\alpha) s_2) \sinh^{n-1}\alpha \, d\alpha \, d\mu(s_2) \, d\mu(s_1). \quad (2.78)$$

Proof: There is an isomorphism between the groups $(L_{e_n}, \oplus)$ and $(\mathbb{R}^+, \times)$ given by

$$r = \frac{u-1}{u+1}, \quad \text{with } u \geq 1.$$

Making this change of variables we obtain

$$\frac{dB_a}{(1-|a|^2)^n} = \frac{r^{n-1}}{(1-r^2)^n} dr\, d\xi = \left(\frac{(u+1)^2}{4u}\right)^n \left(\frac{u-1}{u+1}\right)^{n-1} \frac{2}{(u+1)^2} du\, d\xi$$

$$= \left(\frac{u^2-1}{2u}\right)^{n-1} \frac{1}{u} du\, d\xi = \left(\frac{e^\alpha - e^{-\alpha}}{2}\right)^{n-1} dt\, d\xi = \sinh^{n-1}\alpha \, d\alpha \, d\xi,$$

where $u = e^\alpha$, with $\alpha \geq 0$. This proves the assertion.

■

Finally, we observe that left invariant measures on homogeneous spaces turn out to be compatible with the left gyro-structure (see [50]). Thus, $d\mu(a) = \frac{dB_a}{(1-|a|^2)^n}$ is the left invariant measure on $(B^n, \oplus)$. However, nothing is known about the existence of right invariant measures and, in general, it is not possible to find a right invariant measure due to the structure of right gyro-translations. Therefore, we cannot expect to extend the concept of modular function for gyrogroups.

Chapter 3

Spherical continuous wavelet transforms

In this chapter we will construct various spherical continuous wavelet transforms (SCWT) arising from sections of the proper Lorentz group, generalizing the approach due to Antoine and Vandergheynst ([7, 8]). For the construction of a transform that is to be used for analyzing a signal locally we are faced with an important decision about what properties we want our transform to have. We need to clarify which transformations we are interested in using to analyze signals on the sphere. Furthermore, since we would like to develop a wavelet-like transform, we would need to describe three transformations which are traditionally used in wavelet analysis: translations, rotations and dilations. Translations are easy to identify, since they correspond to rotations of the homogeneous space $SO(n)/SO(n-1)$. Rotations can be realized as rotations around a certain axis on the $n-$sphere. Thus, both translations and rotations can be associated with the action of $SO(n)$ on S^{n-1}.

Dilation, or scaling, is more difficult to define because of the spherical structure of S^{n-1}. An idea proposed in [7], for the case of the sphere S^2, is to dilate functions on the tangent plane of the sphere at the North Pole by means of stereographic projection. To dilate a function around some point $\omega \in S^2$ three steps are needed. First, ω is rotated to the North Pole by a rotation R. Next, the function is stereographically projected into the tangent plane and, there dilated, in the same way that dilations are performed for the regular $2D$ CWT. Finally, the result is lifted back to the sphere using the inverse stereographic projection and the inverse rotation R^{-1}.

It is possible to introduce local dilations in a quite natural way on the sphere if we use the conformal group, that is, the proper Lorentz group $SO_0(1,n)$. In [7], the authors used the Iwasawa decomposition of $SO_0(1,3)$ (or $KAN-$decomposition, where K is the maximal compact subgroup, $A = SO(1,1) \cong \mathbb{R} \cong \mathbb{R}_*^+$ is the subgroup of Lorentz boosts in the $z-$direction and $N \cong \mathbb{C}$ is a two dimensional abelian subgroup) to construct the parameter space $X = SO_0(1,3)/N \cong SO(3) \cdot \mathbb{R}_*^+$, the product of $SO(3)$ (for motions) by $\mathbb{R}_*^+$ (for dilations). A generalization of this approach for the $(n-1)-$sphere is presented in [8]. There is a correspondence principle between spherical and Euclidean wavelets in the sense that the inverse stereographic projection of a wavelet on the plane gives a wavelet on the sphere (see [77]). Recently, in [48], the authors extend the case of isotropic dilations on the 2-sphere to the case of anisotropic dilations defined on the 2-sphere in two orthogonal directions.

One of the limitations of the SCWT of Antoine and Vandergheynst is that it does not take into account relativistic movements on the sphere. But, in many applications such movements are required, e.g. an observer which moves at relativistically velocity with respect to the Earth would see the appearance of the night sky (as modeled by points on the celestial sphere) transformed by means of a Möbius transformation. With the present approach we are able to connect the geometry of conformal transformations on the sphere with wavelet theory, while incorporating general relativistic boosts. Another motivation for this work comes from the case of the plane where a wide variety of wavelets, such as ridglets, curvelets or shearlets, exists (see e.g. [15], [29], [56]). For future consideration of such transformations on the sphere, it seems to be necessary to incorporate general conformal transformations first. From the physical point of view, we will obtain relativistic coherent states which we will call *spherical conformlets*.

The group $\mathrm{Spin}^+(1,n)$, together with its Cartan decomposition, constitutes a very rich and powerful model for the description of the spherical continuous wavelet transform with a nice geometric description.

Having described the transformations needed to the construction of our transforms, we will apply the group-theoretical framework presented in Chapter 1, by finding a unitary representation of the group that will give rise to the desired transforms. First, we make an equivalence between our model and the work of Antoine and Vandergheynst for the isotropic case. Then we consider the anisotropic case and we extend the theory to general global sections of our homogeneous space. The invertibility of our transforms is also shown. Finally, we discuss the covariance and anisotropy properties of the generalized SCWT.

3.1 Fourier analysis on the unit sphere

We will consider the space $L^2(S^{n-1}, dS)$ where dS is the surface element on the sphere given by $dS = \sin^{n-2}\phi \sin^{n-3}\theta_{n-2} \ldots \sin\theta_2\, d\phi\, d\theta_{n-2} \ldots d\theta_1$, using the spherical coordinates (2.12). In the case of $n = 3$, the surface element reduces to $dS = \sin\phi\, d\phi\, d\theta$. For simplicity, we will write only $L^2(S^{n-1})$ instead of $L^2(S^{n-1}, dS)$.

Fourier analysis on the unit sphere S^{n-1} is connected with representation theory of the compact group $SO(n)$ (see [75]). A natural unitary representation of $SO(n)$ is the quasi-regular representation

$$(U_{qr}(\gamma)f)(\omega) = f(\gamma^{-1}\omega), \quad \gamma \in SO(n), \quad f \in L^2(S^{n-1}).$$

U_{qr} is infinite dimensional and decomposes into the direct sum of all irreducible unitary representations $U_l, l = 0, 1, 2, \ldots$, which are of finite dimension d_l. An orthonormal basis in the d_l-dimensional representation space of U_l is that of hyperspherical harmonics Y_l^M, where $M = (m_1, m_2, \ldots, m_{n-2})$ runs over the set I_l^n of all integers m_j such that $l \geq m_{n-2} \geq \ldots \geq m_2 \geq |m_1| \geq 0$. The dimension of U_l is

$$d_l = \frac{(2l + n - 2)(l + n - 3)!}{(n-2)!\, l!}$$

which equals $2l + 1$ for $n = 3$. For $n = 3, Y_l^M(\omega)$ reduces to the usual spherical harmonic $Y_l^{m_1}(\theta_2, \theta_1)$, while for $n \geq 4$, $Y_l^M(\theta_1, \ldots, \theta_{n-1})$ is the product of $Y_{m_2}^{m_1}(\theta_2, \theta_1)$ and $(n - 3)$ factors $(\sin\theta_j)^{n-j} C_{q_j}^{p_j}(\cos\theta_j)$, for $j = 3, \ldots, n-1$, where C_q^p denotes a Greenbrier polynomial (see [75]). The expansion of an arbitrary function of $L^2(S^{n-1})$ in this hyperspherical basis is given by

$$f(\omega) = \sum_{l=0}^{\infty} \sum_{M \in I_l^n} \widehat{f}(l, M) Y_l^M(\omega).$$

Parseval's theorem asserts that

$$||f||^2 = \sum_{l=0}^{\infty} \sum_{M \in I_l^M} |\widehat{f}(l, M)|^2,$$

where $\widehat{f}(l, M) = \langle Y_l^M, f \rangle$ denotes the Fourier coefficient of f. The Wigner $D-$functions $D_{MM'}^l$ appear by the transformation formula for spherical harmonics:

$$[U_{qr}(\gamma)Y_l^M](\omega) = Y_l^M(\gamma^{-1}\omega) = \sum_{M' \in I_l^n} D_{MM'}^l(\gamma) Y_l^{M'}(\omega), \quad \gamma \in SO(n).$$

Moreover, the Wigner functions satisfy the following orthogonality relations

$$\int_{SO(n)} \overline{D_{MQ}^l(\gamma)} D_{M'Q'}^{l'}(\gamma) d\mu(\gamma) = \frac{1}{d_l} \delta_{ll'} \delta_{MM'} \delta_{QQ'}. \tag{3.1}$$

In the case of $n = 3$, Fourier analysis on the sphere S^2 is based on the spherical harmonics $Y_l^m(\omega)$, with $l \in \mathbb{N}$, $m \in \mathbb{Z}$, and $|m| \leq l$. These spherical harmonics provide an orthonormal basis on $L^2(S^2)$. In coordinates $(\theta, \phi) \in [0, 2\pi] \times [0, \pi[$, the spherical harmonic Y_l^m has the factorization

$$Y_l^m(\theta, \phi) = k_{l,m} P_l^m(\cos\phi) e^{im\phi}, \tag{3.2}$$

where P_l^m is the associated Legendre function of degree l and order m, and $k_{l,m} = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^2$ is a normalization constant. The orthonormality and completeness relations are respectively,

$$\int_0^{2\pi}\int_0^{\pi} \overline{Y_l^m(\theta,\phi)} Y_{l'}^{m'}(\theta,\phi) \sin\phi \, d\phi d\theta = \delta_{ll'}\delta_{mm'},$$

and

$$\sum_{l\in\mathbb{N}}\sum_{|m|\leq l} \overline{Y_l^m(\theta',\phi')} Y_l^m(\theta,\phi) = \delta(\cos\phi' - \cos\phi)\delta(\theta' - \theta).$$

The Wigner $D-$functions $D_{mn}^l(\gamma)$, [12], with $l \in \mathbb{N}$, $m, n \in \mathbb{Z}$, and $|m|, |n| \leq l$, are the matrix elements of the irreducible unitary representations of weight l of the rotation group in the space of square-integrable functions $L^2(SO(3), d\mu)$ on $SO(3)$, and they satisfy the relation

$$[R(\gamma)Y_l^m](\omega) = Y_l^m(\gamma^{-1}\omega) = \sum_{|n|\leq l} D_{mn}^l(\gamma) Y_l^n(\omega), \quad \gamma \in SO(3)$$

and the following orthogonality relation

$$\int_{SO(3)} \overline{D_{mn}^l(\gamma)} D_{m'n'}^{l'}(\gamma) d\mu(\gamma) = \frac{8\pi^2}{2l+1}\delta_{ll'}\delta_{mm'}\delta_{nn'}, \tag{3.3}$$

by the Peter-Weyl Theorem on compact groups.

Remark 3.1.1 *Wigner $D-$functions are defined on the group $SO(n)$. However, since the group Spin(n) is a double covering group of $SO(n)$ they can be extended to the group Spin(n). Thus, we will use $D_{mn}^l(s), s \in \mathrm{Spin}(n)$ by the identification with $D_{mn}^l(\gamma), \gamma \in SO(n)$, via the homomorphism between $SO(n)$ and Spin(n).*

3.2 Representations of the Lorentz group

The representations of the Lorentz group are studied in detail in the mathematical literature (see, e.g. [67], [75]). Among these are the so-called class I representations, for which there exists a vector in the carrier Hilbert space which is invariant under the maximal compact subgroup $SO(n)$. Class I representations are induced by unitary irreducible representations of the minimal parabolic subgroup. They are given by the operators

$$[U^\rho(g)f](\omega) = \lambda(g,\omega)^{\rho/2}\chi(a)f(g^{-1}\omega), \quad g \in SO_0(1,n),\ \rho \in \mathbb{C},\ f \in L^2(S^{n-1}),$$

where $g = \gamma a n$ is the Iwasawa decomposition of the element g, χ is a character of $A = SO(1,1)$, and $\lambda(g,\omega)$ is a multiplier that satisfies the cocycle property

$$\lambda(g_1 g_2,\omega) = \lambda(g_1,\omega)\lambda(g_2, g_1^{-1}\omega).$$

We will choose the trivial character $\chi(a) \equiv 1$. The main properties of these representations are studied in [67], [75] and summarized in the following theorem.

Theorem 3.2.1 *The representation*

$$[U^\rho(g)f](\omega) = \lambda(g,\omega)^{\rho/2} f(g^{-1}\omega),$$

is a strongly continuous representation of $SO_0(1,n)$ in $L^2(S^{n-1})$. It is reducible if $\rho = 0,-1,-2,\ldots$ and cyclic otherwise. It is unitary and irreducible if and only if $\rho = (n-1)/2 + i\tau, \tau \in \mathbb{R}$.

We are interested in the action of dilations and motions on S^{n-1}. We will consider the following unitary operators on $L^2(S^{n-1})$:

- the Spin rotation operator

$$R_s f(x) = f(\overline{s}xs), \quad s \in \mathrm{Spin}(n), \tag{3.4}$$

- the dilation operator

$$D_c f(x) = \left(\frac{1-|c|^2}{|1-cx|^2}\right)^{\frac{n-1}{2}} f(\varphi_{-c}(x)), \quad c \in B^n. \tag{3.5}$$

Since the Jacobian of the Möbius transformation $\varphi_{-c}(x)$ acting on S^{n-1} is the Radon-Nikodym derivative with respect to the rotational invariant Lebesgue measure on S^{n-1}, i.e. $d\mu(\varphi_{-c}(x)) = \lambda(c,x)d\mu(x)$, with $\lambda(c,x) = \left(\frac{1-|c|^2}{|1-cx|^2}\right)^{n-1}$, it satisfies the 1-cocycle property (see [2], p. 49):

$$\lambda(a\oplus b,x) = \lambda(a,x)\lambda(b,\varphi_{-a}(x)), \quad \forall a,b \in B^n, \forall x \in S^{n-1}$$

that is,

$$\left(\frac{1-|a\oplus b|^2}{|1-(a\oplus b)x|^2}\right)^{n-1} = \left(\frac{1-|a|^2}{|1-ax|^2}\right)^{n-1}\left(\frac{1-|b|^2}{|1-b\varphi_{-a}(x)|^2}\right)^{n-1}, \forall a,b \in B^n, \forall x \in S^{n-1}. \tag{3.6}$$

Indeed, by simple computations we have

$$\begin{aligned}
\left(\frac{1-|a\oplus b|^2}{|1-(a\oplus b)x|}\right)^{n-1} &= \left(\frac{1-|\varphi_{-a}(b)|^2}{|1-\varphi_{-a}(b)x|^2}\right)^{n-1}, \text{ by (2.30)}\\
&= \left(\frac{(1-|a|^2)(1-|b|^2)|1-ab|^{-2}}{|1-(1-ba)^{-1}(a+b)x|}\right)^{n-1}, \text{ by (2.31) and (2.30)}\\
&= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ba-(a+b)x|^2}\right)^{n-1}
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{1-|a|^2}{|1-ax|^2}\right)^{n-1}\left(\frac{1-|b|^2}{|1-b\varphi_{-a}(x)|^2}\right)^{n-1} &= \left(\frac{1-|a|^2}{|1-ax|^2}\frac{1-|b|^2}{|1-b(x+a)(1-ax)^{-1}|^2}\right)^{n-1}\\
&= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ba-(a+b)x|^2}\right)^{n-1}. \qquad (3.7)
\end{aligned}$$

Combining the result (3.6) and the Unique Decomposition Theorem (Theorem 2.8.1) we can factorize the action induced by the dilation operator D_c.

Proposition 3.2.2 *Let $c\in B^n$ such that $c=b\oplus a$ with $a\in D^{n-1}_{e_n}$ and $b\in L_{e_n}$. Then*

$$D_cf(x)=D_bD_aR_qf(x), \quad \textit{with} \quad q=\frac{1-ab}{|1-ab|}. \qquad (3.8)$$

Proof: On the one hand we have,

$$\begin{aligned}
D_cf(x)=D_{b\oplus a}f(x) &= \left(\frac{1-|b\oplus a|^2}{|1-(b\oplus a)x|^2}\right)^{\frac{n-1}{2}} f(\varphi_{-b\oplus a}(x))\\
&= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ab-(a+b)x|^2}\right)^{\frac{n-1}{2}} f(\varphi_{-b\oplus a}(x)).
\end{aligned}$$

As $-\varphi_a(x)=\varphi_{-a}(-x)$ then $-b\oplus a=-\varphi_{-b}(a)=\varphi_b(-a)=(-b)\oplus(-a)$.

Moreover, by (2.28) we have that $\varphi_{-a}\circ\varphi_{-b}(x)=q\varphi_{(-b)\oplus(-a)}(x)\overline{q}$, with $q=\frac{1-ab}{|1-ab|}$, and thus

$$\varphi_{-b\oplus a}(x)=\overline{q}(\varphi_{-a}(\varphi_{-b}(x)))q. \qquad (3.9)$$

Therefore, we have

$$\begin{aligned}
D_bD_aR_qf(x) &= \left(\frac{1-|b|^2}{|1-bx|^2}\frac{1-|a|^2}{|1-a\varphi_{-b}(x)|^2}\right)^{\frac{n-1}{2}} f(\overline{q}\varphi_{-a}(\varphi_{-b}(x))q)\\
&= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ab-(a+b)x|^2}\right)^{\frac{n-1}{2}} f(\varphi_{-b\oplus a}(x)), \textit{ by } (3.9) \text{ and } (3.7).
\end{aligned}$$

■

Definition 3.2.3 *The operator $D_a R_q$ ($a \in D^{n-1}_{e_n}, q = \frac{1-ab}{|1-ab|}, b \in L_{e_n}$), is called the* ***anisotropy operator.***

If $a = 0$, i.e. if we consider the restriction to the Spin$(1,1)$ group then the anisotropy operator reduces to the identity operator.

Analogously, by the Unique Decomposition Theorem (Theorem 2.8.6) we can factorize the action induced by the dilation operator D_c, through the right orbits.

Proposition 3.2.4 *Let $c \in B^n$ such that $c = a \oplus b$ with $a \in D^{n-1}_{e_n}$ and $b \in L_{e_n}$. Then*

$$D_c f(x) = D_a D_b R_{\overline{q}} f(x), \quad \text{with} \quad q = \frac{1-ab}{|1-ab|}. \tag{3.10}$$

Proposition 3.2.5 *The following relations hold:*

$$D_a R_s = R_s D_{\overline{s}as} \tag{3.11}$$

$$D_b D_a = R_{\overline{q}} D_a D_b \tag{3.12}$$

for all $s \in$ Spin(n), and $a, b \in B^n$, with $q = \frac{1-ab}{|1-ab|}$.

Proof: We prove first the relation (3.11). On the one hand we have

$$D_a R_s f(x) = \left(\frac{1-|a|^2}{|1-ax|^2}\right)^{\frac{n-1}{2}} f(\overline{s}\varphi_{-a}(x)s).$$

On the other hand we have

$$\begin{aligned} R_s D_{\overline{s}as} f(x) &= \left(\frac{1-|a|^2}{|1-\overline{s}as\overline{s}xs|^2}\right)^{\frac{n-1}{2}} f(\varphi_{-\overline{s}as}(\overline{s}xs)) \\ &= \left(\frac{1-|a|^2}{|1-ax|^2}\right)^{\frac{n-1}{2}} f(\overline{s}\varphi_{-a}(x)s), \text{ by (2.32).} \end{aligned}$$

Now we prove relation (3.12). On the one hand,

$$\begin{aligned} D_b D_a f(x) &= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ab-(a+b)x|^2}\right)^{\frac{n-1}{2}} f(\varphi_{-a}(\varphi_{-b}(x))) \\ &= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ab-(a+b)x|^2}\right)^{\frac{n-1}{2}} f(q\varphi_{-(b\oplus a)}(x)\overline{q}), \text{ by (3.9).} \end{aligned}$$

On the other hand,

$$
\begin{aligned}
R_{\overline{q}} D_a D_b f(x) &= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ba-(a+b)qx\overline{q}|^2}\right)^{\frac{n-1}{2}} f(\varphi_{-b}(\varphi_{-a}(qx\overline{q}))) \\
&= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ba|^2|1-(a+b)(1-ba)^{-1}x|^2}\right)^{\frac{n-1}{2}} f(\varphi_{-(a\oplus b)}(qx\overline{q})), \text{ by (3.9)} \\
&= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ba|^2|1-(1-ab)^{-1}(a+b)x|^2}\right)^{\frac{n-1}{2}} f(q\varphi_{-\overline{q}(a\oplus b)q}(x)), \text{ by (2.5.3), (2.32)} \\
&= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ab-(a+b)x|^2}\right)^{\frac{n-1}{2}} f(q\varphi_{-(b\oplus a)}(x)\overline{q}), \text{ by (2.40).}
\end{aligned}
$$

■

Using the operators (3.4) and (3.5) we define the representation

$$
\pi_1(s,a)\psi(x) := R_s D_a \psi(x) = \left(\frac{1-|a|^2}{|1-a\overline{s}xs|^2}\right)^{\frac{n-1}{2}} \psi(\varphi_{-a}(\overline{s}xs)). \tag{3.13}
$$

Proposition 3.2.6 *π_1 is a unitary representation of the group* $\mathrm{Spin}^+(1,n)$ *in* $L^2(S^{n-1})$.

Proof: We are going to prove that π_1 is a homomorphism of the group $\mathrm{Spin}^+(1,n)$ onto the space of linear maps from the Hilbert space $L^2(S^{n-1})$ onto itself, with respect to the group operation (2.50).

On the one hand we have

$$
\begin{aligned}
\pi_1(s_1,a)(\pi_1(s_2,b)\psi(x)) &= \left(\frac{1-|a|^2}{|1-a\overline{s_1}xs_1|^2}\frac{1-|b|^2}{|1-b\overline{s_2}\varphi_{-a}(\overline{s_1}xs_1)s_2|^2}\right)^{\frac{n-1}{2}} \psi(\varphi_{-b}(\overline{s_2}\varphi_{-a}(\overline{s_1}xs_1)s_2)) \\
&= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-s_2b\overline{s_2}a-(a+s_2b\overline{s_2})\overline{s_1}xs_1|^2}\right)^{\frac{n-1}{2}} \psi(\varphi_{-(b\oplus(\overline{s_2}as_2))}(\overline{s_3}xs_3)),
\end{aligned}
$$

with $s_3 = s_1 s_2 \frac{1-\overline{s_2}as_2b}{|1-\overline{s_2}as_2b|}$.

We observe that

$$
\begin{aligned}
|1-b\overline{s_2}\varphi_{-a}(\overline{s_1}xs_1)s_2|^2 &= |1-s_2b\overline{s_2}(\overline{s_1}xs_1+a)(1-a\overline{s_1}xs_1)^{-1}|^2 \\
&= |1-a\overline{s_1}xs_1|^{-2}|1-a\overline{s_1}xs_1-s_2b\overline{s_2}(\overline{s_1}xs_1+a)|^2 \\
&= |1-a\overline{s_1}xs_1|^{-2}|1-s_2b\overline{s_2}a-(a+s_2b\overline{s_2})\overline{s_1}xs_1|^2.
\end{aligned}
$$

By (2.34), (2.28) and (2.40) it follows

$$
\begin{aligned}
\varphi_{-b}(\overline{s_2}\varphi_{-a}(\overline{s_1}xs_1))s_2) &= \varphi_{-b}(\varphi_{-\overline{s_2}as_2})(\overline{s_2}\,\overline{s_1}xs_1s_2) \\
&= q_1\varphi_{-(b+\overline{s_2}as_2)(1-\overline{s_2}as_2b)^{-1}}(\overline{s_1 s_2}xs_1s_2)\overline{q_1} \\
&= \varphi_{-q_1((\overline{s_2}as_2)\oplus b)\overline{q_1}}(q_1\overline{s_1 s_2}xs_1s_2\overline{q_1}) \\
&= \varphi_{-(b\oplus(\overline{s_2}as_2))}(\overline{s_3}xs_3),
\end{aligned}
$$

where $q_1 = \frac{1-b\overline{s_2}as_2}{|1-b\overline{s_2}as_2|}$.

On the other hand we have

$$\begin{aligned}
\pi_1(s_3, b\oplus(\overline{s_2}as_2))f(x) &= \left(\frac{1-|b\oplus(\overline{s_2}as_2)|^2}{|1-(b\oplus(\overline{s_2}as_2))\overline{s_3}xs_3|^2}\right)^{\frac{n-1}{2}} \psi(\varphi_{-(b\oplus(\overline{s_2}as_2))}(\overline{s_3}xs_3)) \\
&= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-s_2b\overline{s_2}a-(a+s_2b\overline{s_2})\overline{s_1}xs_1|^2}\right)^{\frac{n-1}{2}} \psi(\varphi_{-(b\oplus(\overline{s_2}as_2))}(\overline{s_3}xs_3)).
\end{aligned}$$

We note that

$$1-|b\oplus(\overline{s_2}as_2)|^2 = 1-|\varphi_{-b}(\overline{s_2}as_2)|^2 = \frac{(1-|a|^2)(1-|b|^2)}{|1-b\overline{s_2}as_2|^2}$$

and (by (2.40))

$$\begin{aligned}
|1-(b\oplus(\overline{s_2}as_2))\overline{s_3}xs_3|^2 &= |1-q_2(b\oplus(\overline{s_2}as_2))\overline{q_2}\,\overline{s_2}\,\overline{s_1}xs_1s_2|^2 \\
&= |1-(\overline{s_2}as_2\oplus b)\,\overline{s_2}\,\overline{s_1}xs_1s_2|^2 \\
&= |1-b\overline{s_2}as_2|^{-2}|1-b\overline{s_2}as_2-(\overline{s_2}as_2+b)\overline{s_2}\,\overline{s_1}xs_1s_2|^2 \\
&= |1-b\overline{s_2}as_2|^{-2}|\overline{s_2}(1-s_2b\overline{s_2}a)-\overline{s_2}(a+s_2b\overline{s_2})\,\overline{s_1}xs_1|^2|s_2|^2 \\
&= |1-b\overline{s_2}as_2|^{-2}|1-s_2b\overline{s_2}a-(a+s_2b\overline{s_2})\,\overline{s_1}xs_1|^2,
\end{aligned}$$

with $q_2 = \frac{1-\overline{s_2}as_2b}{|1-\overline{s_2}as_2b|}$.

Thus, $\pi_1(s_1,a)(\pi_1(s_2,b)\psi(x)) = \pi_1(s_3, b\oplus(\overline{s_2}as_2))\psi(x)$, i.e. π_1 is a representation of $\mathrm{Spin}^+(1,n)$ on $L^2(S^{n-1})$.

Simple changes of variables show that π_1 is unitary, i.e. $||\pi_1(s,a)\psi||_{L^2(S^{n-1})} = ||\psi||_{L^2(S^{n-1})}$.

$\blacksquare$

Remark 3.2.7 *It is also possible to define the representation*

$$\pi_2(a,s)\psi(x) = D_aR_s\psi(x) = \left(\frac{1-|a|^2}{|1-ax|^2}\right)^{\frac{n-1}{2}} \psi(\overline{s}\varphi_{-a}(x)s). \tag{3.14}$$

associated with the group operation (2.51) on $\mathcal{M}(B^n)$, which arises from the decomposition $\mathcal{M}(B^n) \sim B^n \times \mathrm{Spin}(n)$.

We cannot commute, in general, rotations with dilations, therefore, the representations π_1 and π_2 are different. However, both are related in the following way:

$$\pi_1(s,a)\psi(x) = R_sD_a\psi(x) = D_{sa\overline{s}}R_s\psi(x) = \pi_2(sa\overline{s}, s)\psi(x). \tag{3.15}$$

Moreover, if the rotation s leaves invariant a, then rotations will commute with dilations and both representations are equal.

We remark that the representation π_1 can be also obtained by the method of induced representations of Mackey (see [2] - section 4.2).

Since there exists a convolution on the unit sphere we will construct our spherical wavelet theory based on the representation π_1, in order to interpret the spherical wavelet transform as a spherical convolution.

As an immediate consequence of the $KAK-$decomposition (2.14) of the group $\mathrm{Spin}^+(1,n)$ we can decompose the representation π_1 using relation (2.18), with $a = s_1 r e_n \overline{s_1}$, $s_1 \in \mathrm{Spin}(n)/\mathrm{Spin}(n-1) \cong S^{n-1}$ as described in Lemma 2.4.1:

$$\pi_1(s,a)\psi(x) = \pi_1(s_2, r, s_1)\psi(x) = \left(\frac{1-r^2}{|1 - r e_n \overline{s_2} x s_2|^2}\right)^{\frac{n-1}{2}} \psi(s_1 \varphi_{-re_n}(\overline{s_2} x s_2)\overline{s_1}),$$

where $s_2 = s s_1 \in \mathrm{Spin}(n)$.

Thus, we can study the representations associated to the subgroups $K = \mathrm{Spin}(n)$ and $A = \mathrm{Spin}(1,1)$ separately, namely, the representations:

$$T_s\psi(x) = \psi(\overline{s} x s), \quad s \in \mathrm{Spin}(n)$$

and

$$T_r\psi(x) = \left(\frac{1-r^2}{|1 - r e_n x|^2}\right)^{\frac{n-1}{2}} \psi(\varphi_{-re_n}(x)).$$

By the change of variables $r = \tanh(-\alpha/2)$, with $\alpha \in \mathbb{R}$ (c.f. Proposition 2.4.8 and Formula (2.17)) we obtain the following representation

$$\begin{aligned} T_\alpha\psi(x_1,\ldots,x_n) &= (\cosh\alpha - \sinh\alpha\, x_n)^{-\frac{n-1}{2}} \\ &\quad \psi\left(\frac{x_1}{\cosh\alpha - x_n\sinh\alpha}, \ldots, \frac{x_{n-1}}{\cosh\alpha - x_n\sinh\alpha}, \frac{x_n\cosh\alpha - \sinh\alpha}{\cosh\alpha - x_n\sinh\alpha}\right). \end{aligned}$$

We are dealing here with one specific representation of the Lorentz group realized on the Hilbert space $L^2(S^{n-1})$. We can extend the representation π_1 to the null cone C^n_+ by considering the space $\mathcal{B}^{n\rho}$ of smooth functions on C^n_+ and homogeneous of degree $\rho \in \mathbb{C}$, i.e, $f(\lambda\xi) = \lambda^\rho f(\xi)$, $\lambda > 0$, $\xi \in C^n_+$. This is the approach made in [75] for the study of representations of the proper Lorentz group $SO_0(1,n)$. The complete classification of these representations (principal, complementary and discrete series) is done in [67] and [75].

The representation π_1 belongs to the principal series and it is irreducible on the space $L^2(S^{n-1})$. Unfortunately it is not square integrable because the matrix elements $t^{n\rho}_{KM}(g) = \left\langle T^{n\rho}(g) Y^K_l, Y^M_l \right\rangle_{L^2(S^{n-1})}$, with $K = (k_1, k_2, \ldots, k_{n-2}) \in \mathbb{Z}^{n-2}$, $l \geq k_{n-2} \geq \ldots \geq k_2 \geq |k_1| \geq 0$, $M = (m_1, m_2, \ldots, m_{n-2}) \in \mathbb{Z}^{n-2}$, $l \geq m_{n-2} \geq \ldots \geq m_2 \geq |m_1| \geq 0$, and $g \in SO_0(1,n)$, associated to the orthonormal basis of hyperspherical harmonics Y^M_l are not square integrable with respect to the Haar measure of the group. A detailed discussion of this problem is given in [67] and [75].

In order to continue with our construction we need now a suitable homogeneous space of the group $\mathrm{Spin}^+(1,n)$.

3.3 Spherical continuous wavelet transforms

Our model offers many possibilities for the construction of spherical continuous wavelet transforms. Since there exists already a spherical continuous wavelet transform (SCWT) defined by Antoine and Vandergheynst (see [7] and [8]), we will make first the equivalence between our model and its SCWT.

The SCWT defined in [7] and [8] is based on the section $\sigma_I = (\gamma, u)$, with $\gamma \in SO(n)$ and $u \in \mathbb{R}^+$, in the principal fiber bundle defined by the Iwasawa decomposition $\sigma : X = KAN/N \to KAN$. This section corresponds to the fundamental section $\sigma^* = (s, L_{e_n})$ of the homogeneous spaces $\widetilde{X_1} = (\mathrm{Spin}(n) \times B^n)/(\{1\} \times D^{n-1}_{e_n}, \sim^{*,1}_l)$ and $\widetilde{X}_2 = (B^n \times \mathrm{Spin}(n))/(D^{n-1}_{e_n} \times \{1\}, \sim^{*,2}_r)$. Due to the structure of the orbits we will develop our wavelet theory based on the space $\widetilde{X_1}$, since this homogeneous space allow us to distinguish global and local sections in a clear way.

The equivalence of both models is based on the equivalence of both dilation operators. The dilation operator defined in [8] is

$$D^u\psi(\omega) = \lambda^{1/2}(u,\phi)\psi(\omega_{1/u}),$$

with $\lambda(u,\phi) = \left(\frac{4u^2}{[(u^2-1)\cos\phi + (u^2+1)]^2}\right)^{\frac{n-1}{2}}$ and $\omega_{1/u} = ((\theta_1)_u, \ldots, (\theta_{n-2})_u, \phi_u)$, with $(\theta_j)_u = \theta_j$, $j = 1\ldots, n-2$, and $\tan\frac{\phi_u}{2} = u\tan\frac{\phi}{2}$. We have changed the notation of [8], using $u \in \mathbb{R}^+$ and $\phi \in [0, \pi[$.

Proposition 3.3.1 *The dilation operators D^u and D_{te_n} are equivalent.*

Proof: By Lemma 2.4.9, we know that the Möbius transformation φ_{te_n} corresponds to the $\mathrm{Spin}(1,1)$ action, which is the usual Euclidean dilation lifted on S^{n-1} by inverse stereographic projection. It remains to show that the weights are equal. In fact, by the bijection between $]-1,1[$ and $]0,\infty[$ given by $t = \frac{u-1}{u+1}$, $t \in]-1,1[$, $u > 0$ we have

$$\left(\frac{1-t^2}{|1-te_nx|^2}\right)^{\frac{n-1}{2}} = \left(\frac{1-t^2}{1+2\langle te_n, x\rangle + t^2}\right)^{\frac{n-1}{2}} = \left(\frac{1-\left(\frac{u-1}{u+1}\right)^2}{1+2\frac{u-1}{u+1}x_n + \left(\frac{u-1}{u+1}\right)^2}\right)^{\frac{n-1}{2}} =$$

$$= \left(\frac{4u}{2(u^2+1)+2(u^2-1)x_n} \right)^{\frac{n-1}{2}} = \left(\frac{4u^2}{[(u^2-1)\cos\phi+(u^2+1)]^2} \right)^{\frac{n-1}{4}}.$$

■

Thus, the restriction of the representation π_1 to the section $\sigma^* = (s, L_{e_n})$

$$\pi(\sigma^*)\psi(x) := \pi_1(s,b)\psi(x) = R_s D_b \psi(x) = \left(\frac{1-|b|^2}{|1-b\overline{s}xs|^2} \right)^{\frac{n-1}{2}} \psi(\varphi_{-b}(\overline{s}xs)),\ b \in L_{e_n} \quad (3.16)$$

is equivalent to the representation used in [7] and [8]. Therefore, we will translate the results already obtained to the case of our fundamental section, namely the square-integrability of the fundamental section, the admissibility condition and the inversion formula of the SCWT.

The theory of square-integrable representations modulo a subgroup presented in Section 1.4 depends only on the factorization of the group, the choice of a section and a quasi-invariant measure on the respective homogeneous space, and on the representation of the group. Thus, the general approach also works in our case since we can obtain these terms. We will still use the term square-integrability modulo (H, σ), where H denotes only a gyro-subgroup because the homogeneous space $\widetilde{X_1}$ results from the factorization of a group by a gyro-subgroup. It remains to discuss the problem of the quasi-invariant measure for our homogeneous space and for our sections.

For the fundamental section L_{e_n} we consider the measure $d\mu(b) = \frac{2(1-t)^{n-2}}{(1+t)^n}dt$, $b = te_n$. For all $a \in B^n$ and $b \in L_{e_n}$ such that $a = s_* a_* \overline{s_*}$ we have that $-a \oplus b = (s_*(-a_*)\overline{s_*}) \oplus b = s_*(-a_* \oplus (\overline{s_*} b s_*))\overline{s_*} = s_*(-a_* \oplus b)\overline{s_*}$. Thus, the equivalence classes $[-a \oplus b]$ and $[-a_* \oplus b]$ are equal. This means that we only need to consider the action of $a = (0, \ldots, 0, a_{n-1}, a_n)$ on $b = te_n \in L_{e_n}$. Let $d = -a \oplus b = \varphi_a(b) = \left(0, \ldots, 0, \frac{-(1+t^2-2a_nt)a_{n-1}}{1-2a_nt+|a|^2t^2}, \frac{(1-|a|^2)t-(1+t^2-2a_nt)a_n}{1-2a_nt+|a|^2t^2}\right)$. Applying the projection formulas (2.55) we obtain the new equivalence class, given by

$$d_n = g_a(t) = \frac{-2a_n(1+t^2) + 2(1-a_{n-1}^2+a_n^2)t}{\sqrt{C_1(t)C_2(t)} + (1+|a|^2)(1+t^2) - 4ta_n},$$

with

$$C_1(t) := (1-t)^2 a_{n-1}^2 + (1+t)^2(1-a_n)^2, \quad C_2(t) := (1+t)^2 a_{n-1}^2 + (1-t)^2(1+a_n)^2.$$

Differentiating with respect to t we obtain

$$g_a'(t) = \frac{2(1-t^2)(1+a_{n-1}^2-a_n^2)(1-|a|^2)}{C_1(t)C_2(t) + ((1+|a|^2)(1+t^2)-4a_nt)\sqrt{C_1(t)C_2(t)}}.$$

Therefore, the Radon-Nikodym derivative of $d\mu([-a \oplus b])$ with respect to $d\mu(b)$ is given by:

$$\chi(a,b) = \frac{d\mu([-a \oplus b])}{d\mu(b)} = \frac{(1-g_a(t))^{n-2}(1+t)^n}{(1+g_a(t))^n(1-t)^{n-2}} g_a'(t).$$

Since for each a, $g'_a(t) > 0$, for all $t \in]-1,1[$, we conclude that $\chi(a,b) \in \mathbb{R}^+$, for all $a \in B^n$ and $b \in L_{e_n}$, thus proving that the measure $d\mu(b)$ is quasi-invariant. Moreover this measure is equivalent to the measure $d\mu(u) = \frac{du}{u^n}$, by means of the bijection given by $t = \frac{u-1}{u+1}$ ($u \in \mathbb{R}^+$ and $t \in]-1,1[$). For $f \in L^1(L_{e_n})$ we have

$$\int_{L_{e_n}} f([a \oplus b])d\mu(b) = \int_{L_{e_n}} f(b)d\mu([-a \oplus b]) = \int_{L_{e_n}} f(b)\chi(a,b)d\mu(b).$$

For the special case of $a = (0,\dots,0,a_n) \in L_{e_n}$ we obtain $\chi(a,b) = \left(\frac{1+a_n}{1-a_n}\right)^{n-1}$. The behavior of $\chi(a,b)$ depends on the dimension. For $n = 3$ the function $\chi(a,b)$ is bounded in the variable b, i.e. for each $a \in B^n$ there exists a constant $M(a) = \frac{(1-|a|^2)(1+a_{n-1}^2-a_n^2)}{(1-a_n)^4} \in \mathrm{R}^+$ such that $\chi(a,b) \le M(a)$, for all $b \in L_{e_n}$, since $\chi(a,b)$ is an increasing function in the variable b, for each $a \neq 0$. Thus, $d\mu([-a \oplus b]) \le M(a)\, d\mu(b)$. For $n > 3$ this estimate is not valid since the function $\chi(a,b)$ is not bounded in the variable b.

3.3.1 The case of the sphere S^2

We will restrict now to the case of the sphere S^2. The following theorems about the square-integrability of the fundamental section, the admissibility condition and the inversion formula of the SCWT can be found in [7] and [5] (up to a change of variables).

Theorem 3.3.2 ([7]) *For $s \in \mathrm{Spin}(3)$ and $b \in L_{e_3}$ the representation*

$$\pi_1(s,b)\psi(x) := R_s D_b \psi(x) = \frac{1-|b|^2}{|1-b\bar{s}xs|^2}\psi(\varphi_{-b}(\bar{s}xs)) \tag{3.17}$$

is square-integrable modulo the gyro-subgroup $\{1\} \times D^2_{e_3}$ and the section (s, L_{e_3}) that is, the representation space $L^2(S^2)$ contains a nonzero vector ψ admissible modulo $(\{1\} \times D^2_{e_3}, (s, L_{e_3}))$. Thus, there exists a constant $C > 0$, independent of l, such that

$$C_\psi(l) = \frac{8\pi^2}{2l+1} \sum_{|n|\le l} \int_{-1}^{1} |\widehat{\psi_t}(l,n)|^2 d\mu(te_3) < C, \tag{3.18}$$

where $\widehat{\psi}(l,n) = \langle Y_l^n, \psi\rangle$ stands for the Fourier coefficient of ψ, $\psi_t(x) = D_{te_3}\psi(x)$ and $d\mu(te_3) = \frac{2(1-t)}{(1+t)^3}dt$.

The proof consists in an explicit calculation, using the properties of the Fourier analysis on the sphere (see [7]).

From Theorem 3.3.2 we have

$$\int_{-1}^{1} \int_{\mathrm{Spin}(3)} |\, \langle \pi_1(s,b)\psi, f\rangle \,|^2 = \langle f, Af\rangle < \infty, \tag{3.19}$$

where A_ψ is called the resolution operator (or frame operator). The operator A_ψ is diagonal in Fourier space (i.e. it is a Fourier multiplier)

$$\widehat{A_\psi h}(l,m) = C_\psi(l)\widehat{h}(l,m), \quad \forall h \in L^2(S^2),$$

or equivalently,

$$A_\psi h(\omega) = \sum_{l\in\mathbb{N}} \sum_{|m|\leq l} C_\psi(l)\widehat{h}(l,m)Y_l^m(\omega), \quad \forall h \in L^2(S^2).$$

Any admissible function ψ generates a continuous family $\{\psi_{(s,b)} := R_s D_b \psi(x), s \in \mathrm{Spin}(3), b \in L_{e_3}\}$ of spherical wavelets. It is also a continuous frame, which can be seen in the following proposition.

Proposition 3.3.3 ([7]) *For any admissible vector ψ such that $\int_0^{2\pi}\psi(\theta,\phi)d\theta \neq 0$, the family $\{\psi_{(s,b)}, s \in \mathrm{Spin}(3), b \in L_{e_3}\}$ is a continuous frame, that is, there exist constants $A > 0$ and $B < \infty$ such that*

$$A||f||^2 \leq \int_{\mathrm{Spin}(3)} \int_{-1}^{1} |\langle f, R_s D_{te_3}\psi\rangle|^2 d\mu(te_3)\, d\mu(s) \leq B||f||^2, \quad \forall f \in L^2(S^2). \tag{3.20}$$

Thus, for any admissible function ψ, we get a continuous frame, but not necessarily a tight one, i.e. $A \neq B$ in general. The condition (3.18) is necessary and sufficient for the admissibility of ψ, but it is somewhat complicated to use in practice, since it requires the evaluation of nontrivial Fourier coefficients. Instead, there is a simpler condition, although only a necessary condition.

Proposition 3.3.4 ([7]) *A function $\psi \in L^2(S^2)$ is admissible only if it satisfies the condition*

$$\int_0^{2\pi}\int_0^{\pi} \frac{\psi(\theta,\phi)}{1+\cos\phi}\sin\phi\, d\phi\, d\theta = 0. \tag{3.21}$$

This condition is only necessary in general, but it becomes sufficient under mild regularity conditions on ψ. This is the equivalent of the usual necessary condition for wavelets in the Euclidean plane, $\int_{\mathbb{R}^2}\psi(x)d^2x = 0$. It is a zero mean condition, as in the flat case. Thus, it ensures that the CWT on S^2 act as a local filter, in the sense that it selects the components of a signal that are similar to ψ, which is assumed to be well localized.

Theorem 3.3.2 yields the basic ingredient for writing the CWT on S^2. Given an admissible wavelet $\psi \in L^2(S^2)$, the wavelets on the sphere are the functions $\psi_{(s,b)} = R_s D_b \psi$, and the SCWT is defined by

$$W_\psi[f](s,b) = \langle \psi_{(s,b)}, f\rangle = \int_{S^2} \overline{R_s D_b \psi(x)} f(x) dS_x. \tag{3.22}$$

This last expression can be written as a spherical correlation (see [5]). Moreover, the following reconstruction formula can be derived.

Proposition 3.3.5 ([77]) *Let* $f \in L^2(S^2)$. *If* ψ *is an admissible wavelet such that* $\int_0^{2\pi} \psi(\theta,\phi)d\theta \neq 0$, *then*

$$f(\omega) = \int_{-1}^{1} \int_{\mathrm{Spin}(3)} W_\psi[f](s,t)[R_s A_\psi^{-1} D_{te_3}\psi](\omega)\, d\mu(s)\, d\mu(te_3). \tag{3.23}$$

Proof: Let $\psi_{te_3} := D_{te_3}\psi$. Then we have

$$[R_s A_\psi^{-1}\psi_{te_3}](\omega) = \sum_{l\in\mathbb{N}} \sum_{|m|\le l} \sum_{|n|\le l} \frac{1}{C_\psi(l)} D^l_{mn}(s)\widehat{\psi_{te_3}}(l,m) Y_l^n(\omega). \tag{3.24}$$

The wavelet coefficients $W_f(s,b)$ defined in (3.22) may be written as

$$W_\psi[f](s,b) = \sum_{l\in\mathbb{N}} \sum_{|m|\le l} \sum_{|n|\le l} \overline{D^l_{mn}(s)}\,\overline{\widehat{\psi_{te_3}}(l,m)}\widehat{f}(l,n). \tag{3.25}$$

Inserting the expressions (3.24) and (3.25) in (3.23) and using the orthogonality relation for Wigner $D-$functions, we obtain

$$\begin{aligned}
&\int_{-1}^{1} \int_{\mathrm{Spin}(3)} W_\psi[f](s,t)[R_s A_\psi^{-1} D_{te_3}\psi](\omega)\, d\mu(s)\, d\mu(te_3) = \\
&= \int_{-1}^{1} \int_{\mathrm{Spin}(3)} \sum_{l\in\mathbb{N}} \sum_{|m|\le l} \sum_{|n|\le l} \overline{D^l_{m,n}(s)}\,\overline{\widehat{\psi}_{te_3}(l,m)}\widehat{f}(l,n) \\
&\quad \sum_{l'\in\mathbb{N}} \sum_{|m'|\le l'} \sum_{|n'|\le l'} \frac{1}{C_\psi(l')} D^{l'}_{m'n'}(s)\widehat{\psi}_{te_3}(l',m') Y_{l'}^{n'}(\omega) d\mu(s) d\mu(te_3) \\
&= \sum_{l,l'} \sum_{m,m'} \sum_{n,n'} \int_{-1}^{1} \overline{\widehat{\psi}_{te_3}}(l,m)\widehat{\psi}_{te_3}(l',m') d\mu(te_3)\ \widehat{f}(l,n) Y_l^{n'}(\omega) \\
&\quad \frac{1}{C_\psi(l')} \int_{\mathrm{Spin}(3)} \overline{D^l_{mn}(s)} D^{l'}_{m'n'}(s) d\mu(s) \\
&= \sum_{l\in\mathbb{N}} \frac{1}{C_\psi(l)} \left[\frac{8\pi^2}{2l+1} \sum_{|m|\le l} \int_{-1}^{1} |\widehat{\psi}_{te_3}(l,m)|^2 d\mu(te_3) \right] \sum_{|n|\le l} \widehat{f}(l,n) Y_l^n(\omega) \quad \text{(by (3.3))} \\
&= \sum_{l\in\mathbb{N}} \sum_{|n|\le l} \widehat{f}(l,n) Y_l^n(\omega) \\
&= f(\omega).
\end{aligned}$$

■

Thus, we conclude that the reconstruction formula (3.23) holds if and only if the coefficients $C_\psi(l)$ defined in (3.18) are finite and non-zero for any $l \in \mathbb{N}$.

Corollary 3.3.6 ([7]) *Under the conditions of Proposition 3.3.5, the following Plancherel relation is satisfied*

$$||f||^2 = \int_{-1}^{1}\int_{\mathrm{Spin}(3)} \overline{\widetilde{W}_\psi[f](s,t)} W_\psi[f](s,t)\, d\mu(s)\, d\mu(te_3) \tag{3.26}$$

with

$$\widetilde{W}_\psi[f](s,t) = \left\langle \widetilde{\psi}_{(s,t)}, f \right\rangle = \langle R_s A_\psi \psi_{te_3}, f\rangle . \tag{3.27}$$

One of the most interesting results concerning the SCWT is the correspondence principle between Euclidean and spherical wavelets, which states that the inverse stereographic projection of a wavelet on the plane leads to the definition of a wavelet on the sphere (see [77]).

These results hold for the fundamental section $\sigma^* = (s, L_{e_3})$ and they can be generalized to higher dimensions. Now, we consider general global sections $\sigma^l(te_3) = te_3 \oplus f(t)e_2$ on the space $X_1 = B^3/(D^2_{e_2}, \sim_l)$. Then the pair $\sigma^* = (s, \sigma^l)$ with $s \in \mathrm{Spin}(3)$ corresponds to a section on $\widetilde{X_1}$. We will study now when these sections give rise to spherical continuous wavelet transforms.

Definition 3.3.7 *If $f \in C^k(]-1,1[)$ then the left global section $\sigma^l(te_3) = te_3 \oplus f(t)e_2$ is called a section of class C^k or C^k-section. It is called a smooth section if $f \in C^\infty(]-1,1[)$.*

We will develop our generalized spherical continuous wavelet transforms for global left C^0-sections. For an arbitrary global left C^0-section $\sigma^l(te_3) = te_3 \oplus f(t)e_3$ the measure $d\mu(\sigma^l(te_3)) = \chi(\sigma^l(te_3), t)d\mu(te_3)$ is the standard quasi-invariant measure for the section σ^l (see [2]).

We know from Chapter 1 that if μ_1 and μ_2 are quasi-invariant measures on X then there exists a Borel function $f : X \to \mathbb{R}^+$, $f(x) > 0$, for all $x \in X$, such that $d\mu_1(x) = f(x)d\mu_2(x)$, for all $x \in X$, which means that the quasi-invariant measure on a section is unique (up to equivalence). Thus, we will use the measure $d\mu(te_3)$ for an arbitrary global left C^0-section.

In Section 2.12 we establish a relationship between $(B^3, \oplus)$ and $SL(2,\mathbb{C})$ by the Theorem 2.12.4, which corresponds to the stereographic projection of the action of $\varphi_a(x)$ onto its tangent plane. Now we consider the respective unitary operators and we try to construct an intertwining relation for the unitary stereographic projection.

From now on we will use $f \in L^2(S^2)$ and $F \in L^2(\mathbb{R}^2)$ to distinguish functions from different spaces. We will use the variables $y \in \mathbb{R}^2$ and $z \in \mathbb{C}$ and the identification of $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

Lemma 3.3.8 *The map $\Theta : L^2(S^2, dS) \to L^2(\mathbb{R}^2, rdr\,d\theta)$ defined by*

$$f(\theta,\phi) \mapsto F(\theta,r) = \frac{4}{4+r^2} f(\theta, 2\arctan(r/2)) \tag{3.28}$$

is a unitary map. In cartesian coordinates the map Θ reads as

$$f(x) \mapsto F(y) = \frac{4}{4+|y|^2} f(\Phi_1^{-1}(y)). \tag{3.29}$$

Proof: We have

$$||\Theta f||^2_{L^2(\mathbb{R}^2)} = \int_0^{2\pi}\int_0^{\infty} \left|\frac{4}{4+r^2} f(\theta, 2\arctan(r/2))\right|^2 rdr\,d\theta. \tag{3.30}$$

Let us consider the change of variables $\phi = 2\arctan(r/2)$, which means that $r = 2\tan(\phi/2) = 2\sqrt{\frac{1-\cos\phi}{1+\cos\phi}}$. Then $\cos\phi = \frac{4-r^2}{4+r^2}$ and $\sin\phi = \frac{4r}{4+r^2}$. Thus, the integral (3.30) becomes

$$||\Theta f||^2_{L^2(\mathbb{R}^2)} = \int_0^{2\pi}\int_0^{\pi} |f(\theta,\phi)|^2 \sin\phi\, d\phi\, d\theta = ||f||^2_{L^2(S^2)}.$$

■

The following theorem is a consequence of Theorem 2.12.4.

Theorem 3.3.9 *Let $z = z_1 + z_2 i \in \mathbb{C}$, and $\widetilde{\varphi}(z) := \frac{c_1 z + c_2}{c_3 z + c_4}$ be the Möbius transformation obtained from the matrix (2.70), with $c_1 = \frac{1+a_3}{\sqrt{1-|a|^2}}$, $c_2 = \frac{-2(a_1+a_2 i)}{\sqrt{1-|a|^2}}$, $c_3 = \frac{-a_1+a_2 i}{2\sqrt{1-|a|^2}}$, $c_4 = \frac{1-a_3}{\sqrt{1-|a|^2}}$. Then we have the intertwining relation*

$$\Theta D_a \psi = M \Theta \psi, \tag{3.31}$$

where $MF(z) = \frac{4(1-|a|^2)}{|(a_1-a_2 i)z+2(1+a_3)|^2} F(\widetilde{\varphi}^{-1}(z))$ is the unitary operator associated with $\widetilde{\varphi}^{-1}(z) = \frac{c_4 z - c_2}{-c_3 z + c_1}$.

Proof: By the definition of our operators we have

$$\Theta D_a \psi(z) = \frac{4}{4+|z|^2}\frac{1-|a|^2}{|1-a\Phi^{-1}(z)|^2}\psi\left(\varphi_{-a}(\Phi_1^{-1}(z))\right) \tag{3.32}$$

and

$$M\Theta\psi(z) = \frac{4(1-|a|^2)}{|(a_1-a_2 i)z+2(1+a_3)|^2}\frac{4}{4+|\widetilde{\varphi}^{-1}(z)|^2}\psi\left(\Phi_1^{-1}(\widetilde{\varphi}^{-1}(z))\right). \tag{3.33}$$

First we observe that

$$\varphi_{-a}(\Phi_1^{-1}(z)) = \Phi_1^{-1}(\widetilde{\varphi}^{-1}(z)), \quad \forall z \in \overline{\mathbb{C}}. \tag{3.34}$$

If $\varphi_{-a}(\Phi_1^{-1}(z)) = x \in S^2$ then $z = \Phi_1(\varphi_a(x))$. Moreover, if $\Phi_1^{-1}(\widetilde{\varphi}^{-1}(z)) = x$, then $z = \widetilde{\varphi}(\Phi_1(x))$. Since the relation $\Phi_1(\varphi_a(x)) = \widetilde{\varphi}(\Phi_1(x))$ holds by Theorem 2.12.4 we conclude that the relation (3.34) is true. It remains to prove that the weights in (3.32) and (3.33) are equal. On the one hand,

$$\frac{4}{4+|z|^2}\,\frac{1-|a|^2}{|1-a\Phi_1^{-1}(z)|^2} = \frac{4(1-|a|^2)}{(1+|a|^2)(4+|z|^2)+2a_3(4-|z|^2)+8(a_1z_1+a_2z_2)}.$$

On the other hand, since

$$|\widetilde{\varphi}^{-1}(z)|^2 = \frac{|c_4z-c_2|^2}{|-c_3z+c_1|^2} = 4\frac{(1-a_3)^2|z|^2+4(1-a_3)(a_1z_1+a_2z_2)+4(a_1^2+a_2^2)}{(a_1^2+a_2^2)|z|^2+4(1+a_3)(a_1z_1+a_2z_2)+4(1+a_3)^2}$$

and

$$|(a_1-a_2i)z+2(1+a_3)|^2 = (a_1^2+a_2^2)|z|^2+4(1+a_3)(a_1z_1+a_2z_2)+4(1+a_3)^2$$

we obtain, after some computations

$$\frac{4(1-|a|^2)}{|(a_1-a_2i)z+2(1+a_3)|^2}\,\frac{4}{4+|\widetilde{\varphi}^{-1}(z)|^2} = \frac{4(1-|a|^2)}{(1+|a|^2)(4+|z|^2)+2a_3(4-|z|^2)+8(a_1z_1+a_2z_2)}.$$

Thus,

$$\frac{4}{4+|z|^2}\,\frac{1-|a|^2}{|1-a\Phi_1^{-1}(z)|^2} = \frac{4(1-|a|^2)}{|(a_1-a_2i)z+2(1+a_3)|^2}\,\frac{4}{4+|\widetilde{\varphi}^{-1}(z)|^2}.$$

■

The Iwasawa decomposition of the group $SL(2,\mathbb{C})$ yields the decomposition $SL(2,\mathbb{C}) = KAN$, where $K = SU(2)$ is the maximal compact subgroup, A is abelian and N is nilpotent. The Iwasawa decomposition of a generic element of $SL(2,\mathbb{C})$ reads

$$\begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \delta^{-1/2} & 0 \\ 0 & \delta^{1/2} \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \tag{3.35}$$

where $\alpha, \beta, \xi \in \mathbb{C}$ and $\delta \in \mathbb{R}^+$, $uz - vw = 1$ and

$$\delta = (|u|^2+|w|^2)^{-1}, \qquad \alpha = u\delta^{1/2}, \qquad \beta = -\overline{w}\delta^{1/2},$$

$$\xi = u^{-1}(v+\overline{w}\delta),\ \text{if } u \neq 0 \qquad \text{or} \qquad \xi = w^{-1}(z-\overline{u}\delta),\ \text{if } w \neq 0.$$

The Iwasawa decomposition of the matrix (2.70)

$$\frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} 1+a_3 & -2(a_1+a_2i) \\ \frac{-a_1+a_2i}{2} & 1-a_3 \end{pmatrix} \tag{3.36}$$

yields the parameters

$$\alpha = \frac{2(1+a_3)}{\sqrt{4(1+a_3)^2+a_1^2+a_2^2}}, \quad \beta = \frac{a_1+a_2i}{\sqrt{4(1+a_3)^2+a_1^2+a_2^2}}, \quad \delta = \frac{4(1-|a|^2)}{4(1+a_3)^2+a_1^2+a_2^2}, \quad \xi = \frac{-2(a_1+a_2i)(5+3a_3)}{4(1+a_3)^2+a_1^2+a_2^2}. \tag{3.37}$$

Remark 3.3.10 *The Iwasawa decomposition of the matrix (2.71) yields simpler parameters*

$$\alpha = \frac{1-a_3}{\sqrt{1+|a|^2-2a_3}}, \quad \beta = \frac{a_1+a_2 i}{\sqrt{1+|a|^2-2a_3}}, \quad \delta = \frac{1-|a|^2}{1+|a|^2-2a_3}, \quad \xi = \frac{-2(a_1+a_2 i)}{1+|a|^2-2a_3},$$

however, the gist of the reasoning is not changed and it is equivalent to work with the stereographic projection mappings Φ_1 *or* Φ_2.

From the decomposition (3.35) and the parameters (3.37) we can define the following unitary operators on $L^2(\mathbb{C})$ ($R^{\alpha,\beta}$−complex rotation operator, D^δ−dilation operator and T^ξ−translation operator):

$$R^{\alpha,\beta}F(z) = \frac{1}{|-\beta\overline{z}+\alpha|^2}F\left(\frac{\alpha z+\beta}{-\overline{\beta}z+\overline{\alpha}}\right); \quad D^\delta F(z) = \frac{1}{\delta}F\left(\frac{z}{\delta}\right); \quad T^\xi F(z) = F(z+\xi).$$

As these mappings are isometries whose ranges include the whole space $L^2(\mathbb{C})$ the adjoint operators and inverses are identical.

Lemma 3.3.11 *The adjoint operators to* $R^{\alpha,\beta}$, D^δ *and* T^ξ *are*

$$(R^{\alpha,\beta})^* = R^{\overline{\alpha},-\beta}, \qquad (D^\delta)^* = D^{\frac{1}{\delta}} \qquad \text{and} \qquad (T^\xi)^* = T^{-\xi}. \tag{3.38}$$

Proposition 3.3.12 *The operator* M *admits the following factorization*

$$MF = R^{\alpha,-\beta}D^{\frac{1}{\delta}}T^{-\xi}F. \tag{3.39}$$

Proof: By definition we have

$$MF(z) = \frac{4(1-|a|^2)}{|(a_1-a_2 i)z+2(1+a_3)|^2}F(\widetilde{\varphi}^{-1}(z))$$

and

$$R^{\alpha,-\beta}D^{\frac{1}{\delta}}T^{-\xi}\,F(z) = \delta\frac{1}{|\beta\overline{z}+\alpha|^2}F\left(\delta\,\frac{\alpha z-\beta}{\overline{\beta}z+\alpha}-\xi\right).$$

First we prove that $\widetilde{\varphi}^{-1}(z) = \delta\,\frac{\alpha z-\beta}{\overline{\beta}z+\alpha}-\xi$. On the one hand,

$$\widetilde{\varphi}^{-1}(z) = \frac{c_4 z-c_2}{-c_3 z+c_1} = \frac{2(1-a_3)z+4(a_1+a_2 i)}{(a_1-a_2 i)z+2(1+a_3)}.$$

On the other hand,

$$\delta\,\frac{\alpha z-\beta}{\overline{\beta}z+\alpha}-\xi = \frac{(\delta\alpha-\overline{\beta}\xi)z-(\delta\beta+\alpha\xi)}{\overline{\beta}z+\alpha} = \frac{2(1-a_3)z+4(a_1+a_2 i)}{(a_1-a_2 i)z+2(1+a_3)}.$$

Finally, it is easy to see that $\delta\frac{1}{|\beta\overline{z}+\alpha|^2} = \frac{4(1-|a|^2)}{|(a_1-a_2 i)z+2(1+a_3)|^2}$.

■

Corollary 3.3.13 *The intertwining relation (3.31) can be written as*

$$\Theta D_a \psi = R^{\alpha,-\beta} D^{\frac{1}{\delta}} T^{-\xi} \Theta \psi. \tag{3.40}$$

This intertwining relation generalizes Lemma 3.5 presented in [7], as we can easily see in the next corollary.

Corollary 3.3.14 *For $a = te_3 \in L_{e_3}$ we obtain the intertwining relation*

$$\Theta D_{te_3} \psi = D^{\frac{1+t}{1-t}} \Theta \psi. \tag{3.41}$$

For an arbitrary global left section

$$\sigma^l(te_3) = te_3 \oplus f(t)e_2 = \left(0, \frac{f(t)(1-t^2)}{1+(tf(t))^2}, \frac{t(1+f(t)^2)}{1+(tf(t))^2}\right)$$

the parameters (3.37) become

$$\begin{aligned}
\alpha_t &= \frac{2(1+tf(t)^2)}{\sqrt{(1+t^2+6t)f(t)^2+4(1+t^2f(t)^4)}}, \\
\beta_t &= \frac{(1-t)f(t)}{\sqrt{(1+t^2+6t)f(t)^2+4(1+t^2f(t)^4)}} i, \\
\delta_t &= \frac{4(1-t)(1-f(t)^2)(1+t^2f(t)^2)}{(t+1)((1+t^2+6t)f(t)^2+4(1+t^2f(t)^4))}, \\
\xi_t &= \frac{2(t-1)f(t)((5t^2+3t)f(t)^2+3t+5)}{(t+1)((1+t^2+6t)f(t)^2+4(1+t^2f(t)^4))} i\,.
\end{aligned} \tag{3.42}$$

When $f(t) \equiv 0$, (restriction to the fundamental section - Spin$(1,1)$ case) we obtain $\alpha = 1$, $\beta = 0$, $\xi = 0$ and $\delta = \frac{1-t}{1+t}$, which reflects again the fact that we obtain a pure dilation.

Lemma 3.3.15 *The parameter δ_t can be written as $\delta_t = \frac{1-t}{1+t}\delta_t^*$, with $\delta_t^* = \frac{4(1-f(t)^2)(1+t^2f(t)^2)}{(1+t^2+6t)f(t)^2+4(1+t^2f(t)^4)}$, and δ_t^* satisfies the estimate*

$$0 < \delta_t^* < \frac{2(3-2\sqrt{3})}{3(-2+\sqrt{3})}, \quad \forall t \in]-1,1[.$$

Proof: As $f(t) \in]-1,1[$, for every $t \in]-1,1[$, the study of the behavior of the parameter δ_t^* is equivalent to the study of the behavior of the function of two variables $g(t,\lambda) = \frac{4(1-\lambda^2)(1+t^2\lambda^2)}{(1+t^2+6t)\lambda^2+4(1+t^2\lambda^4)}$, with $t, \lambda \in]-1,1[$. Since for each $\lambda \in]-1,1[$

$$\frac{1-\lambda^2}{1+\lambda^2} < g(t,\lambda) < \frac{1-\lambda^4}{1+\lambda^4-\lambda^2}, \quad \forall\, t \in]-1,1[$$

we conclude that

$$0 < g(t,\lambda) < \frac{2(3-2\sqrt{3})}{3(-2+\sqrt{3})}, \quad \forall\, t,\lambda \in]-1,1[.$$

Therefore,

$$0 < \delta_t^* < \frac{2(3-2\sqrt{3})}{3(-2+\sqrt{3})}, \quad \forall\, t \in]-1,1[.$$

■

Now, we will prove the main theorem of this thesis concerning the square-integrability of the representation π_1 over the sections $\sigma^* = (s, \sigma^l)$.

Theorem 3.3.16 *Let $\psi \in L^2(S^2)$ such that the family $\{\psi_{(s,b)}, s \in \mathrm{Spin}(3), b \in L_{e_3}\}$ is a continuous frame, that is, there exist constants $A > 0$ and $B < \infty$ such that*

$$A||g||^2 \leq \int_{\mathrm{Spin}(3)} \int_{-1}^{1} |\left\langle g, R_s D_{te_3}\psi \right\rangle|^2 d\mu(te_3)\, d\mu(s) \leq B||g||^2, \quad \forall g \in L^2(S^2). \tag{3.43}$$

Then ψ is an admissible function for any global left C^0-section $\sigma^l(te_3)$ and the system $\{\psi_{(s,\sigma^l(te_3))}, s \in \mathrm{Spin}(3), t \in]-1,1[\,\}$ forms a continuous frame, i.e.

$$A||g||^2 \leq \int_{\mathrm{Spin}(3)} \int_{-1}^{1} |\left\langle g, R_s D_{\sigma^l(te_3)}\psi \right\rangle|^2 d\mu(te_3)\, d\mu(s) \leq B||g||^2, \quad \forall g \in L^2(S^2). \tag{3.44}$$

Proof: For every $a \in B^3$ and $g \in L^2(S^2)$ arbitrary we have

$$\begin{aligned}
&\int_{\mathrm{Spin}(3)} \int_{-1}^{1} \left| \langle g, R_s D_a \psi \rangle_{L^2(S^2)} \right|^2 d\mu(te_3) d\mu(s) \\
&= \int_{\mathrm{Spin}(3)} \int_{-1}^{1} \left| \langle R_{\overline{s}}\, g, D_a \psi \rangle_{L^2(S^2)} \right|^2 d\mu(te_3) d\mu(s) \\
&= \int_{\mathrm{Spin}(3)} \int_{-1}^{1} \left| \langle \Theta R_{\overline{s}}\, g, \Theta D_a \psi \rangle_{L^2(\mathbb{R}^2)} \right|^2 d\mu(te_3) d\mu(s) \quad \text{(by Lemma 3.3.8)} \\
&= \int_{\mathrm{Spin}(3)} \int_{-1}^{1} \left| \left\langle \Theta R_{\overline{s}}\, g, R^{\alpha,-\beta} D^{\frac{1}{\delta}} T^{-\xi} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 d\mu(te_3) d\mu(s) \quad \text{(by (3.40))} \\
&= \int_{\mathrm{Spin}(3)} \int_{-1}^{1} \left| \left\langle \Theta R_{\overline{s}}\, g, R^{\alpha,-\beta} T^{-\frac{\xi}{\delta}} D^{\frac{1}{\delta}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 d\mu(te_3) d\mu(s) \\
&= \int_{\mathrm{Spin}(3)} \int_{-1}^{1} \left| \left\langle T^{\frac{\xi}{\delta}} R^{\alpha,\beta} \Theta R_{\overline{s}}\, g, D^{\frac{1}{\delta}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 d\mu(te_3) d\mu(s) \quad \text{(by (3.38))}.
\end{aligned} \tag{3.45}$$

Now we consider an arbitrary global left C^0-section $\sigma^l(te_3)$ and the parameters (3.42). By Lemma 3.3.15 the parameter δ_t can be factorized as $\delta_t = \frac{1-t}{1+t}\delta_t^*$, with $\delta_t^* = \frac{4(1-f(t)^2)(1+t^2 f(t)^2)}{(1+t^2+6t)f(t)^2+4(1+t^2 f(t)^4)}$, Therefore, the integral (3.45) becomes

$$\int_{\mathrm{Spin}(3)} \int_{-1}^{1} \left| \left\langle D^{\delta_t^*} T^{\frac{\xi_t}{\delta_t}} R^{\alpha_t,\beta_t} \Theta R_{\overline{s}}\, g, D^{\frac{1+t}{1-t}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 d\mu(te_3) d\mu(s). \tag{3.46}$$

For each $s \in \mathrm{Spin}(3)$ and $t \in]-1,1[$ we have

$$\left| \left\langle D^{\delta_t^*} T^{\frac{\xi_t}{\delta_t}} R^{\alpha_t,\beta_t} \Theta R_{\overline{s}}\, g, D^{\frac{1+t}{1-t}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 \leq \underbrace{\sup_{t' \in]-1,1[} \left| \left\langle D^{\delta_{t'}^*} T^{\frac{\xi_{t'}}{\delta_{t'}}} R^{\alpha_{t'},\beta_{t'}} \Theta R_{\overline{s}}\, g, D^{\frac{1+t}{1-t}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2}_{I}$$

and

$$\underbrace{\inf_{t' \in]-1,1[} \left| \left\langle D^{\delta_{t'}^*} T^{\frac{\xi_{t'}}{\delta_{t'}}} R^{\alpha_{t'},\beta_{t'}} \Theta R_{\overline{s}}\, g, D^{\frac{1+t}{1-t}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2}_{II} \leq \left| \left\langle D^{\delta_t^*} T^{\frac{\xi_t}{\delta_t}} R^{\alpha_t,\beta_t} \Theta R_{\overline{s}}\, g, D^{\frac{1+t}{1-t}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2.$$

Let $t_1, t_2 \in [-1,1]$ such that

$$I := \left| \left\langle D^{\delta_{t_2}^*} T^{\frac{\xi_{t_2}}{\delta_{t_2}}} R^{\alpha_{t_2},\beta_{t_2}} \Theta R_{\overline{s}}\, g, D^{\frac{1+t}{1-t}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2$$

and

$$II := \left| \left\langle D^{\delta_{t_1}^*} T^{\frac{\xi_{t_1}}{\delta_{t_1}}} R^{\alpha_{t_1},\beta_{t_1}} \Theta R_{\overline{s}}\, g, D^{\frac{1+t}{1-t}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2.$$

Thus, we obtain

$$\int_{\mathrm{Spin}(3)} \int_{-1}^{1} \left| \left\langle D^{\delta_{t_1}^*} T^{\frac{\xi_{t_1}}{\delta_{t_1}}} R^{\alpha_{t_1},\beta_{t_1}} \Theta R_{\overline{s}}\, g, D^{\frac{1+t}{1-t}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 d\mu(te_3) d\mu(s) \leq$$
$$\int_{\mathrm{Spin}(3)} \int_{-1}^{1} \left| \left\langle D^{\delta_t^*} T^{\frac{\xi_t}{\delta_t}} R^{\alpha_t,\beta_t} \Theta R_{\overline{s}}\, g, D^{\frac{1+t}{1-t}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 d\mu(te_3) d\mu(s) \leq$$
$$\int_{\mathrm{Spin}(3)} \int_{-1}^{1} \left| \left\langle D^{\delta_{t_2}^*} T^{\frac{\xi_{t_1}}{\delta_{t_1}}} R^{\alpha_{t_1},\beta_{t_1}} \Theta R_{\overline{s}}\, g, D^{\frac{1+t}{1-t}} \Theta\psi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 d\mu(te_3) d\mu(s).$$

Since for $t \in [-1+\epsilon, 1-\epsilon]$, $\epsilon > 0$, the operators $D^{\delta_t^*}$, $T^{\frac{\xi_t}{\delta_t}}$ and R^{α_t,β_t} are unitary and bijective mappings we only need to study the case of $t \to \pm 1$. In this case, the parameter $\frac{\xi_t}{\delta_t}$ associated to the operator $T^{\frac{\xi_t}{\delta_t}}$ can become infinity. But, it is easy to see that the composition of the operators $D^{\delta_t^*}$ and $T^{\frac{\xi_t}{\delta_t}}$ is well behaved. Thus, for each $g \in L^2(S^2)$ we can find $g_1 \in L^2(S^2)$ and $g_2 \in L^2(S^2)$ such that $D^{\delta_{t_1}^*} T^{\frac{\xi_{t_1}}{\delta_{t_1}}} R^{\alpha_{t_1},\beta_{t_1}} \Theta R_{\overline{s}}\, g = \Theta R_{\overline{s}}\, g_1$ and $D^{\delta_{t_2}^*} T^{\frac{\xi_{t_2}}{\delta_{t_2}}} R^{\alpha_{t_2},\beta_{t_2}} \Theta R_{\overline{s}}\, g = \Theta R_{\overline{s}}\, g_2$ with $||g_1|| = ||g_2|| = ||g||$.

Therefore, we have

$$\int_{\mathrm{Spin}(3)}\int_{-1}^{1}\left|\left\langle \Theta R_{\overline{s}}\, g_1, D^{\frac{1+t}{1-t}}\Theta\psi\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 d\mu(te_3)d\mu(s)\leq$$
$$\int_{\mathrm{Spin}(3)}\int_{-1}^{1}\left|\left\langle D^{\delta_t^*}T^{\frac{\xi_t}{\delta_t}}R^{\alpha_t,\beta_t}\Theta R_{\overline{s}}\, g, D^{\frac{1+t}{1-t}}\Theta\psi\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 d\mu(te_3)d\mu(s)\leq$$
$$\int_{\mathrm{Spin}(3)}\int_{-1}^{1}\left|\left\langle \Theta R_{\overline{s}}\, g_2, D^{\frac{1+t}{1-t}}\Theta\psi\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 d\mu(te_3)d\mu(s). \qquad (3.47)$$

By (3.41), (3.45) and Lemma 3.3.8 condition (3.47) becomes

$$\int_{\mathrm{Spin}(3)}\int_{-1}^{1}\left|\langle g_1, R_sD_{te_3}\psi\rangle_{L^2(S^2)}\right|^2 d\mu(te_3)d\mu(s)\leq$$
$$\int_{\mathrm{Spin}(3)}\int_{-1}^{1}\left|\left\langle g, R_sD_{\sigma^l(te_3)}\psi\right\rangle_{L^2(S^2)}\right|^2 d\mu(te_3)d\mu(s)\leq$$
$$\int_{\mathrm{Spin}(3)}\int_{-1}^{1}\left|\langle g_2, D_{te_3}\Theta\psi\rangle_{L^2(S^2)}\right|^2 d\mu(te_3)d\mu(s). \qquad (3.48)$$

As by hypothesis ψ satisfies condition (3.43) there exist constants $0<A\leq B<\infty$ such that

$$A||g_1||^2\leq\int_{\mathrm{Spin}(3)}\int_{-1}^{1}|\left\langle g, R_sD_{\sigma^l(te_3)}\psi\right\rangle|^2 d\mu(te_3)d\mu(s)\leq B||g_2||^2,\quad \forall g\in L^2(S^2),$$

which means that

$$A||g||^2\leq\int_{\mathrm{Spin}(3)}\int_{-1}^{1}|\left\langle g, R_sD_{\sigma^l(te_3)}\psi\right\rangle|^2 d\mu(te_3)d\mu(s)\leq B||g||^2,\quad \forall g\in L^2(S^2).$$

■

As a consequence of this theorem, every admissible function for the fundamental section (s,te_3) is also an admissible function for any global left C^0-section $(s,\sigma^l(te_3))$. Since the wavelets obtained arise from the action of the conformal group we propose the following definition.

Definition 3.3.17 *For a given global left C^0-section $\sigma^l(te_3)$ and $\psi\in L^2(S^2)$ admissible the wavelets (or system of coherent states) obtained*

$$\psi_{(s,\sigma^l(te_3))}(x)=\pi_1(s,\sigma^l(te_3))\psi(x)$$

are called ***spherical conformlets****.*

For an arbitrary section $(s, \sigma^l(te_3))$ and an admissible function $\psi \in L^2(S^2)$ we define the wavelet transform

$$W_\psi[f](s, \sigma^l(te_3)) = \left\langle \psi_{(s,\sigma^l(te_3))}, f \right\rangle = \int_{S^2} \overline{R_s D_{\sigma^l(te_3)} \psi(x)} f(x) dS_x. \tag{3.49}$$

By Theorem 3.3.16 the generalized wavelet transform (3.49) is a mapping from $L^2(S^2, dS)$ to $L^2(\mathrm{Spin}(3) \times \sigma^l(te_3), d\mu(s) d\mu(te_3))$, from which the following reconstruction formula can be derived.

Proposition 3.3.18 *Let $f \in L^2(S^2)$. If ψ is an admissible wavelet such that $\int_0^{2\pi} \psi(\theta, \phi) d\theta \neq 0$, then*

$$f(x) = \int_{-1}^{1} \int_{\mathrm{Spin}(3)} W_\psi[f](s, \sigma^l(te_3))[R_s (A_\sigma^\psi)^{-1} D_{\sigma^l(te_3)} \psi](x)\, d\mu(s)\, d\mu(t), \tag{3.50}$$

where

$$\widehat{A_\sigma^\psi h}(l, m) = C_\sigma^\psi(l) \widehat{h}(l, m), \quad \forall h \in L^2(S^2),$$

with

$$C_\sigma^\psi(l) := \frac{8\pi^2}{2l+1} \sum_{|n| \le l} \int_{-1}^{1} |\widehat{\psi}_{\sigma^l(te_3)}(l, n)|^2 d\mu(te_3).$$

As we can see most of the properties obtained for the fundamental section (s, te_3) can be generalized for an arbitrary global left C^0-section $(s, \sigma^l(te_3))$.

3.3.2 Generalization to the $(n-1)-$sphere S^{n-1}

In this section we want to generalize propositions and theorems of the previous section in order to construct generalized spherical continuous wavelets transforms for the unit sphere S^{n-1}. Using Clifford algebraic techniques the results are easily extended to higher dimensions. Some proofs will be omitted since they are analogous to the proofs of the previous section.

Let us remark that the surface element dS can be written as $dS = \sin^{n-2} \phi \, d\phi \, dS^{n-2}$, where dS^{n-2} is the surface element of the unit sphere S^{n-2}.

Lemma 3.3.19 *The map $\Theta : L^2(S^{n-1}, dS) \to L^2(\mathbb{R}^{n-1}, r^{n-2} dr\, dS^{n-2})$ defined by*

$$f(\theta_1, \ldots, \theta_{n-2}, \phi) \mapsto F(\theta_1, \ldots, \theta_{n-2}, r) = \left(\frac{4}{4+r^2} \right)^{\frac{n-1}{2}} f(\theta_1, \ldots, \theta_{n-2}, 2\arctan(r/2))$$

is a unitary map. In cartesian coordinates the map Θ reads as

$$f(x) \mapsto F(y) = \left(\frac{4}{4+|y|^2} \right)^{\frac{n-1}{2}} f(\Phi_1^{-1}(y)). \tag{3.51}$$

Proof: We have

$$||\Theta f||^2_{L^2(\mathbb{R}^{n-1})} = \int_{S^{n-2}} \int_0^\infty \left| \left(\frac{4}{4+r^2}\right)^{\frac{n-1}{2}} f(\theta_1, \ldots, \theta_{n-2}, 2\arctan(r/2)) \right|^2 r^{n-2} dr\, dS^{n-2}. \quad (3.52)$$

Let us consider the change of variables $\phi = 2\arctan(r/2)$, which means that $r = 2\tan(\phi/2) = 2\sqrt{\frac{1-\cos\phi}{1+\cos\phi}}$. Then $\cos\phi = \frac{4-r^2}{4+r^2}$ and $\sin\phi = \frac{4r}{4+r^2}$. Thus, the integral (3.52) becomes

$$||\Theta f||^2_{L^2(\mathbb{R}^{n-1})} = \int_{S^{n-2}} \int_0^\pi |f(\theta_1, \ldots, \theta_{n-2}, \phi)|^2 \sin^{n-2}\phi \, d\phi \, dS^{n-2} = ||f||^2_{L^2(S^{n-1})}.$$

■

Theorem 3.3.20 *Let $y \in \mathbb{R}^{n-1}$, and $\widetilde{\varphi}(y) := \frac{c_1 y + c_2}{c_3 y + c_4}$ be the Möbius transformation obtained from the matrix (2.75), with $c_1 = \frac{1+a_n}{\sqrt{1-|a|^2}}$, $c_2 = \frac{2(-a+a_n e_n)}{\sqrt{1-|a|^2}}$, $c_3 = \frac{a-a_n e_n}{2\sqrt{1-|a|^2}}$, $c_4 = \frac{1-a_n}{\sqrt{1-|a|^2}}$. Then we have the intertwining relation*

$$\Theta D_a \psi = M \Theta \psi, \quad (3.53)$$

where $MF(y) = \left(\frac{4(1-|a|^2)}{|-(a-a_n e_n)y + 2(1+a_n)|^2}\right)^{\frac{n-1}{2}} F(\widetilde{\varphi}^{-1}(y))$ is the unitary operator associated with $\widetilde{\varphi}^{-1}(y) = \frac{c_4 y - c_2}{-c_3 y + c_1}$.

The group $SL(2, \Gamma(n) \cup \{0\})$, with entries in the Clifford group $\Gamma(n)$ (see (2.3)) or zero, admits an Iwasawa decomposition similar to the decomposition of the group $SL(2, \mathbb{C})$. The Iwasawa decomposition of a generic element of $SL(2, \Gamma(n) \cup \{0\})$ is

$$\begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \delta^{-1/2} & 0 \\ 0 & \delta^{1/2} \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \quad (3.54)$$

where $\alpha, \beta, \xi \in \Gamma(n) \cup \{0\}$ and $\delta \in \mathbb{R}^+$, $uz^* - vw^* = 1$ and

$$\delta = (|u|^2 + |w|^2)^{-1}, \qquad \alpha = u\delta^{1/2}, \qquad \beta = -\overline{w}\delta^{1/2},$$

$$\xi = u^{-1}(v + \overline{w}\delta), \text{ if } u \neq 0 \qquad \text{or} \qquad \xi = w^{-1}(z - \overline{u}\delta), \text{ if } w \neq 0.$$

The Iwasawa decomposition of the matrix (2.75)

$$\frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} 1 + a_n & 2(-a + a_n e_n) \\ \frac{a - a_n e_n}{2} & 1 - a_n \end{pmatrix}. \quad (3.55)$$

yields the parameters

$$\alpha = \frac{2(1+a_n)}{\sqrt{4(1+a_n)^2 + |a|^2 - a_n^2}}, \qquad \beta = \frac{a - a_n e_n}{\sqrt{4(1+a_n)^2 + |a|^2 - a_n^2}},$$

$$\delta = \frac{4(1-|a|^2)}{4(1+a_n)^2+|a|^2-a_n^2}, \qquad \xi = \frac{2(-a+a_ne_n)(5+3a_n)}{4(1+a_n)^2+|a|^2-a_n^2}. \tag{3.56}$$

From the decomposition (3.54) and the parameters (3.56) we can define the following unitary operators on $L^2(\mathbb{R}^{n-1})$:

$$R^{\alpha,\beta}F(y) = \left(\frac{1}{|-\beta\overline{y}+\alpha|^2}\right)^{\frac{n-1}{2}} F\left(\frac{\alpha y+\beta}{-\overline{\beta}y+\overline{\alpha}}\right);$$

$$D^\delta F(y) = \delta^{-\frac{n-1}{2}} F\left(\frac{y}{\delta}\right); \qquad T^\xi F(y) = F(y+\xi).$$

Lemma 3.3.21 *The adjoint operators to $R^{\alpha,\beta}$, D^δ and T^ξ are*

$$(R^{\alpha,\beta})^* = R^{\overline{\alpha},-\beta}, \qquad (D^\delta)^* = D^{\frac{1}{\delta}} \qquad \text{and} \qquad (T^\xi)^* = T^{-\xi}. \tag{3.57}$$

Proposition 3.3.22 *For $F \in L^2(\mathbb{R}^{n-1})$ and $\psi \in L^2(S^{n-1})$ we have*

$$MF = R^{\alpha,-\beta}D^{\frac{1}{\delta}}T^{-\xi}F, \tag{3.58}$$

$$\Theta D_a\psi = R^{\alpha,-\beta}D^{\frac{1}{\delta}}T^{-\xi}\Theta\psi. \tag{3.59}$$

Corollary 3.3.23 *For $a = te_n \in L_{e_n}$ we obtain the intertwining relation*

$$\Theta D_{te_n}\psi = D^{\frac{1+t}{1-t}}\Theta\psi. \tag{3.60}$$

For an arbitrary global left section

$$\sigma^l(te_n) = te_n \oplus f(t)e_{n-1} = \left(0,\ldots,0,\frac{f(t)(1-t^2)}{1+(tf(t))^2},\frac{t(1+f(t)^2)}{1+(tf(t))^2}\right)$$

the parameters (3.56) become

$$\alpha_t = \frac{2(1+tf(t)^2)}{\sqrt{(1+t^2+6t)f(t)^2+4(1+t^2f(t)^4)}},$$

$$\beta_t = \frac{(1-t)f(t)}{\sqrt{(1+t^2+6t)f(t)^2+4(1+t^2f(t)^4)}}\,e_{n-1},$$

$$\delta_t = \frac{4(1-t)(1-f(t)^2)(1+t^2f(t)^2)}{(t+1)((1+t^2+6t)f(t)^2+4(1+t^2f(t)^4))},$$

$$\xi_t = \frac{2(t-1)f(t)((5t^2+3t)f(t)^2+3t+5)}{(t+1)((1+t^2+6t)f(t)^2+4(1+t^2f(t)^4))}\,e_{n-1}\,. \tag{3.61}$$

Theorem 3.3.24 *Let $\psi \in L^2(S^{n-1})$ such that the family $\{\psi_{(s,b)}, s \in \mathrm{Spin}(n), b \in L_{e_n}\}$ is a continuous frame, that is, there exist constants $A > 0$ and $B < \infty$ such that*

$$A||g||^2 \leq \int_{\mathrm{Spin}(n)} \int_{-1}^{1} |\langle g, R_s D_{te_n}\psi\rangle|^2 d\mu(te_n)\, d\mu(s) \leq B||g||^2, \quad \forall g \in L^2(S^{n-1}).$$

Then ψ is an admissible function for any global left C^0-section $\sigma^l(te_n)$ and the system $\{\psi_{(s,\sigma^l(te_n))}, s \in \mathrm{Spin}(n), t \in]-1,1[\}$ forms a continuous frame, i.e.

$$A||g||^2 \leq \int_{\mathrm{Spin}(n)} \int_{-1}^{1} |\left\langle g, R_s D_{\sigma^l(te_n)}\psi\right\rangle|^2 d\mu(te_n)\, d\mu(s) \leq B||g||^2, \quad \forall g \in L^2(S^{n-1}).$$

As a consequence of this theorem we conclude the result about the admissibility condition (c.f. [8]).

Theorem 3.3.25 *The representation π_1 given in (3.13) is square integrable modulo $(\{1\} \times D_{e_n}^{n-1}, \sim_l^{*,1})$ and the global left C^0-section $(s, \sigma^l(te_n))$, that is, there exists a nonzero admissible vector $\psi \in L^2(S^{n-1})$ satisfying*

$$\frac{1}{d_l} \sum_{M \in I_l^n} \int_{-1}^{1} |\widehat{\psi}_{\sigma^l(te_n)}(l, M)|^2\, d\mu(te_n) < \infty, \tag{3.62}$$

uniformly in l.

For an arbitrary section $(s, \sigma^l(te_n))$ and an admissible function $\psi \in L^2(S^{n-1})$ we define the wavelet transform

$$W_\psi[f](s, \sigma^l(te_n)) = \left\langle \psi_{(s,\sigma^l(te_n))}, f\right\rangle = \int_{S^{n-1}} \overline{R_s D_{\sigma^l(te_n)}\psi(x)} f(x) dS. \tag{3.63}$$

The wavelet transform (3.63) is a mapping from $L^2(S^{n-1}, dS)$ into $L^2(\mathrm{Spin(n)} \times \sigma^l(te_n), d\mu(s)d\mu(te_n))$, from which there is a reconstruction formula and also a Plancherel Theorem.

3.4 Anisotropy and covariance of the generalized SCWT

Having defined a SCWT, depending on the chosen section, it naturally arises the question of its classification according to its properties. One essential question is to understand what kind of dilations are obtained when the Möbius transformation $\varphi_a(x)$ over the fundamental section L_{e_n} is replaced by a Möbius transformation over an arbitrary global left C^0-section

$\sigma^l(te_n)$. This study will be made in the next chapter and we will see that we obtain anisotropic conformal dilations, which in a sense reflects the Möbius deformation of the fundamental section, since $\sigma^l(te_n) = te_n \oplus f(t)e_{n-1}$. Another important question is the characterization of the anisotropy of a given section. By Definition 3.2.3 the operator $D_{f(t)e_{n-1}}R_{q(t)}$, with $q(t) = \frac{1-tf(t)e_{n-1}e_n}{|1-tf(t)e_{n-1}e_n|}$, gives the anisotropy character of an arbitrary global left section. Thus, we propose the following concept of anisotropy.

Definition 3.4.1 *The anisotropy of the section $\sigma^l(te_n) = te_n \oplus f(t)e_{n-1}$ is defined by*

$$\epsilon_f := \int_{-1}^{1} ||D_{f(t)e_{n-1}}R_{q(t)} - I||dt. \tag{3.64}$$

Since the operators $D_{f(t)}$ and $R_{q(t)}$ are unitary we have that $0 \leq \epsilon_f \leq 4$. Thus, the concept of anisotropy is a concept with an important meaning. However, it is very difficult to compute the quantity ϵ_f. Therefore, we will propose an alternative definition using only the generating function of the section $\sigma^l(te_n)$, which can easily computed for every Borel section.

Definition 3.4.2 *For $1 \leq p < \infty$ the $p-$deviation of the global left C^0-section $\sigma^l(te_n)$ is given by*

$$\epsilon^*_{f,p} := \int_{-1}^{1} ||\sigma^l(te_n) - te_n||^p dt$$

and for $p = \infty$ we define the infinity deviation by $\epsilon^*_{f,\infty} := \sup_{t \in]-1,1[} ||\sigma^l(te_n) - te_n||$.

By simple computations we derive the following result:

Proposition 3.4.3 *For $1 \leq p < \infty$ the $p-$deviation of the global left C^0-section $\sigma^l(te_n)$ is given by*

$$\epsilon^*_{f,p} := \int_{-1}^{1} \left(\frac{|f(t)|(1-t^2)}{\sqrt{1+t^2 f(t)^2}} \right)^p dt$$

and the infinity deviation is given by $\epsilon^*_{f,\infty} := \sup_{t \in]-1,1[} \frac{|f(t)|(1-t^2)}{\sqrt{1+t^2 f(t)^2}}$.

For the sections $\sigma^l_\lambda(te_n)$ we obtain $\epsilon^*_{\lambda,p} := |\lambda|^p \int_{-1}^{1} \left(\frac{1-t^2}{\sqrt{1+t^2\lambda^2}} \right)^p dt$, (e.g., $\epsilon^*_{\lambda,1} = \frac{|\lambda|(2\lambda^2 \ln(\lambda+\sqrt{1+\lambda^2}) - \lambda\sqrt{1+\lambda^2} + \ln(\lambda+\sqrt{1+\lambda^2}))}{\lambda^3}$) and $\epsilon^*_{\lambda,\infty} = |\lambda|$. For the sections $\sigma^l_c(te_n)$ we obtain $\epsilon^*_{c,p} := |c|^p \int_0^\pi \sin^p \phi \, d\phi$ (e.g., $\epsilon^*_{c,1} = 2|c|$) and $\epsilon^*_{c,\infty} = |c|$.

The question of covariance of the SCWT (3.63) under rotations on S^{n-1} and dilations is very important. In the flat case, the usual CWT in $\mathbb{R}^n$ is fully covariant with respect to translations, rotations, and dilations, and this property is essential for applications, in particular the covariance under translations. In fact, covariance is a general feature of all coherent state systems directly derived from a square integrable group representation [2]. However, when the representation is only square integrable over a quotient of the group then no general theorem is available. In the case of the sphere the results are the following:

- The SCWT (3.63) is covariant under rotations on S^{n-1} : for any $s_1 \in \mathrm{Spin}(n)$, the transform of the rotated signal $f(\overline{s_1}xs_1)$ is the function $W_\psi[f](\overline{s_1}s, \sigma^l(te_n))$.
- The SCWT (3.63) is not covariant under dilations. The wavelet transform of the dilated signal $D_{\sigma^l(t_1e_n)}f(x)$, is of the form $W_\psi[f](s, (\overline{s}\sigma^l(t_1e_n)s) \oplus (-\sigma^l(te_n)))$ as we can see by direct calculations.

Considering the change of variables $\varphi_{-b}(x) = y \Leftrightarrow x = \varphi_b(y)$, from which $dS_x = \left(\frac{1-|b|^2}{|1+by|^2}\right)^{n-1} dS_y$ we obtain

$$\begin{aligned}
W_\psi[D_bf](s,a) &= \int_{S^{n-1}} \left(\frac{1-|a|^2}{|1-a\overline{s}xs|^2}\right)^{\frac{n-1}{2}} \psi(\varphi_{-a}(\overline{s}xs)) \left(\frac{1-|b|^2}{|1-bx|^2}\right)^{\frac{n-1}{2}} f(\varphi_{-b}(x))dS_x \\
&= \int_{S^{n-1}} \left(\frac{(1-|a|^2)(1-|b|^2)^3}{|1-sa\overline{s}\varphi_b(y)|^2|1-b\varphi_b(y)|^2|1+by|^4}\right)^{\frac{n-1}{2}} \psi(\varphi_{-a}(\overline{s}\varphi_b(y)s))f(y)dS_y \\
&= \int_{S^{n-1}} \left(\frac{(1-|a|^2)(1-|b|^2)}{|1+by-sa\overline{s}(y-b)|^2}\right)^{\frac{n-1}{2}} \psi(\varphi_{(\overline{s}bs)\oplus(-a)}(\overline{s}ys))f(y)dS_y \\
&= W_\psi[f](s,(\overline{s}bs)\oplus(-a)), \quad \forall s \in \mathrm{Spin}(n), \quad \forall a,b \in B^n.
\end{aligned}$$

The SCWT is not covariant under dilations since the parameter space of the SCWT is not a group. This is a general feature of coherent systems based on homogeneous spaces. For applications, of course, it is the covariance under rotations that is essential.

Chapter 4

Local analysis on the unit sphere

Having constructed several spherical continuous wavelet transforms through sections of the proper Lorentz group we now need to understand the role of dilations obtained from the Möbius transformation $\varphi_{\sigma^l(te_n)}(x)$. The easiest way of doing this is by studying the action of $\varphi_a(x)$ on a given spherical cap. Since Möbius transformations map spheres on spheres, a spherical cap will be mapped onto another spherical cap, which is an expansion or a contraction of the initial cap. Following the spirit of the Erlangen program of F. Klein, namely the study of the invariants of a given geometry, we will make a detailed study of the properties of spherical caps under the action of a Möbius transformation $\varphi_a(x)$. As a consequence we arrive at the concept of local dilation around the North Pole and the separation of the unit ball B^n in two regions: the dilation and contraction regions. The construction of the zonal surfaces associated to a given spherical cap allow us to better fully understand the influence of the parameter $a \in B^n$ on each spherical cap.

Since the unit sphere is a compact manifold it is preferable to use localized wavelets in time and frequency than to use spherical harmonics which are global functions. Thus, by defining wavelets with compact support on spherical caps we can perform local analysis on the sphere. The main disadvantage is that there are no good coverings on the unit sphere since there is no equidistribution of points on the unit sphere. Numerical questions and applications will be discussed in Chapter 5.

Another objective of this chapter is to compare every section with the fundamental section and to relate the existence of an attractor point inside the cap with the respective anisotropy of the section. Thus, we arrive at the concept of local anisotropy of a given section.

4.1 Influence of the parameter $a \in B^n$ on spherical caps

For $h \in]-1,1[$ fixed, let $\mathcal{U}_h = \{x \in S^{n-1} : x_n \geq h\}$ be the spherical cap centered at the North Pole, with support in the hyperplane $x_n = h$.

The $(n-2)-$dimensional sphere S in the hyperplane of equation $x_n = h$, defined by

$$S : \begin{cases} x_1^2 + x_2^2 + \ldots + x_{n-1}^2 & = & 1 - h^2 \\ x_n & = & h \end{cases}$$

is said to be the support of $\mathcal{U}_h$. As Möbius transformations map spheres into spheres, it is enough to study the behavior of S under the action of φ_a, with $a \in B^n$. Let S_* denotes the images's support of the new spherical cap, i.e. $S_* = \varphi_a(S)$. The sphere S^* has center in the point

$$P = \begin{bmatrix} \frac{2a_1(a_n-h)((1+|a|^2)h-2a_n)}{k} \\ \vdots \\ \frac{2a_{n-1}(a_n-h)((1+|a|^2)h-2a_n)}{k} \\ \frac{(1-|a|^2+2a_n(a_n-h))((1+|a|^2)h-2a_n)}{k} \end{bmatrix}^T, \tag{4.1}$$

with $k := 4(a_n - h)^2(|a|^2 - a_n^2) + (1 - |a|^2 + 2a_n(a_n - h))^2$, lies in the hyperplane of equation

$$\begin{aligned} 2a_1(a_n - h)x_1 + 2a_2(a_n - h)x_2 + \cdots + 2a_{n-1}(a_n - h)x_{n-1} + \\ (1 - |a|^2 + 2a_n(a_n - h))x_n = (1 + |a|^2)h - 2a_n \end{aligned} \tag{4.2}$$

and its radius is given by

$$\tau_h(a) = \frac{(1 - h^2)^{1/2}(1 - |a|^2)}{k^{1/2}}. \tag{4.3}$$

For $a = s_* a_* \overline{s_*} = s_*(s_{n-1} r e_n \overline{s_{n-1}})\overline{s_*}$, with $s_* \in \mathrm{Spin}(n-1)$ and $s_{n-1} = \mathrm{e}^{e_n e_{n-1} \frac{\phi}{2}}$ the rotor in the $e_n e_{n-1}-$plane (c.f. Lemma 2.4.1 and Remark 2.4.2), we have that

$$\varphi_a(\mathcal{U}_h) = s_* \varphi_{s_{n-1} r e_n \overline{s_{n-1}}}(\overline{s_*} \mathcal{U}_h s_*)\overline{s_*} = s_* \varphi_{s_{n-1} r e_n \overline{s_{n-1}}}(\mathcal{U}_h)\overline{s_*}. \tag{4.4}$$

As a consequence, the rotation induced by $s_* \in \mathrm{Spin}(n-1)$ on $a_* = s_{n-1} r e_n \overline{s_{n-1}}$ acts only as a rotation of the cap $\varphi_{a_*}(\mathcal{U}_h)$. Moreover, only the parameters r and ϕ are intervening in the local dilation of the original cap $\mathcal{U}_h$. Therefore, to prove (4.1) we restrict ourselves to $a_* = (0, \ldots, 0, a_{n-1}, a_n)$. By taking two antipodes points on S (in the $e_{n-1}e_n-$plane), say $P_1 = (0, \ldots, 0, \sqrt{1-h^2}, h)$ and $P_2 = (0, \ldots, 0, -\sqrt{1-h^2}, h)$, then these points are mapped under φ_{a_*} to the following points on $S_* = \varphi_{a_*}(S)$:

$$\widetilde{P_1} = \left(0,\dots,0,\frac{(1-|a|^2)\sqrt{1-h^2}+2(\sqrt{1-h^2}a_{n-1}+ha_n-1)a_{n-1}}{1+|a|^2-2(\sqrt{1-h^2}a_{n-1}+ha_n)},\frac{(1-|a|^2)h+2(\sqrt{1-h^2}a_{n-1}+ha_n-1)a_n}{1+|a|^2-2(\sqrt{1-h^2}a_{n-1}+ha_n)}\right)$$

and

$$\widetilde{P_2} = \left(0,\dots,0,\frac{(|a|^2-1)\sqrt{1-h^2}+2(-\sqrt{1-h^2}a_{n-1}+ha_n-1)a_{n-1}}{1+|a|^2-2(-\sqrt{1-h^2}a_{n-1}+ha_n)},\frac{(1-|a|^2)h+2(-\sqrt{1-h^2}a_{n-1}+ha_n-1)a_n}{1+|a|^2-2(-\sqrt{1-h^2}a_{n-1}+ha_n)}\right).$$

These points are antipodal in S_* since $\langle \overrightarrow{\widetilde{P}\widetilde{P_1}}, \overrightarrow{\widetilde{P},\widetilde{P_2}}\rangle = 0$, for every $\widetilde{P} = \varphi_{a_*}(P)$, with $P = (0,\dots,0,\sqrt{1-h^2}\sin\theta, \sqrt{1-h^2}\cos\theta, h)$, with $\theta \in [0,2\pi]$ and therefore, $\langle \overrightarrow{\widetilde{P}\widetilde{P_1}}, \overrightarrow{\widetilde{P},\widetilde{P_2}}\rangle = 0$, for every $\widetilde{P} \in S_*$. Thus, the center of S_* is the point

$$P_3 = \frac{\widetilde{P_1}+\widetilde{P_2}}{2} = \left(0,\dots,0,\frac{2a_{n-1}(a_n-h)((1+|a|^2)h-2a_n)}{k},\frac{(1-|a|^2+2a_n(a_n-h))((1+|a|^2)h-2a_n)}{k}\right).$$

Hence, for an arbitrary point $a \in B^n$, the center of the sphere S^* is obtained from P_3 by means of the s_*−rotation, thus giving (4.1).

Analyzing expression (4.1) it is easy to see that $k > 0$. Indeed, if $a_n - h = 0$, then $k = (1-|a|^2)^2 > 0$ since $|a| < 1$, while if $|a|^2 - a_n^2 = 0$, then $k = (a_n^2 - 2a_nh + 1)^2 > 0$, due to $|a| < 1$.

Let $\mathcal{U}_{h,a} = \varphi_a(\mathcal{U}_h)$. The center (on the unit sphere) of the cap $\mathcal{U}_{h,a}$ is the point

$$Q = \left(\frac{2a_1(a_n-h)}{k^{1/2}},\dots,\frac{2a_{n-1}(a_n-h)}{k^{1/2}},\frac{1-|a|^2+2a_n(a_n-h)}{k^{1/2}}\right). \tag{4.5}$$

This point is obtained by projecting (4.1) onto the unit sphere and taking into account that Möbius transformations are orientation preserving transformations. We can easily see that the point Q belongs to S^{n-1}, by the definition of k. The distance $d_h(a)$ between points P and Q is given by

$$d_h(a) = 1 + \frac{2a_n - h(1+|a|^2)}{\sqrt{k}}. \tag{4.6}$$

First we observe that we can rewrite k as $k = 4(a_n-h)(a_n-h|a|^2) + (1-|a|^2)^2$. A simple calculation shows that

$$\begin{aligned}
d_h(a)^2 &= \sum_{i=1}^{n-1}\left(\frac{2a_i(a_n-h)(h(1+|a|^2)-2a_n)}{k} - \frac{2a_i(a_n-h)}{k^{1/2}}\right)^2 + \\
&\quad + \left(\frac{(2a_n-h(1+|a|^2))(|a|^2-1-2a_n(a_n-h))}{k} - \frac{1-|a|^2+2a_n(a_n-h)}{k^{1/2}}\right)^2 \\
&= \left(4(a_n-h)^2(|a|^2-a_n^2)+(1-|a|^2+2a_n(a_n-h))^2\right)\frac{((2a_n-h(1+|a|^2))\sqrt{k}+k)^2}{k^3} \\
&= \frac{((2a_n-h(1+|a|^2))\sqrt{k}+k)^2}{k^2}, \quad \text{by definition of k} \\
&= \left(\frac{2a_n-h(1+|a|^2)}{\sqrt{k}}+1\right)^2.
\end{aligned}$$

For each $h \in]-1,1[$ we have that $-1 \leq \frac{2a_n - h(1+|a|^2)}{\sqrt{k}} \leq 1$, for all $a \in B^n$. Therefore, $d_h(a) = 1 + \frac{2a_n - h(1+|a|^2)}{\sqrt{k}}$ and, hence, $0 \leq d_h(a) \leq 2$ for all $a \in B^n$ and all $h \in]-1,1[$.

We consider now the point $a \in B^n$ described in spherical coordinates $a = a(r, \theta_1, \ldots, \theta_{n-1}, \phi)$ (see (2.12)). Thus, expressions (4.3) and (4.6) become

$$\tau_h(r,\phi) = \frac{(1-h^2)^{1/2}(1-r^2)}{\sqrt{k_1}} \qquad \text{and} \qquad d_h(r,\phi) = 1 + \frac{2r\cos\phi - h(1+r^2)}{\sqrt{k_1}}, \tag{4.7}$$

with $k_1 = 4r^2(r\cos\phi - h)^2 \sin^2\phi + (1 - r^2 + 2r\cos\phi(r\cos\phi - h))^2$.

Definition 4.1.1 *The image of the North Pole under the action of φ_a will be called the* ***attractor point*** *and it will be denoted by $\mathcal{A}$. It is given by*

$$\mathcal{A} = \left(\frac{2a_1(a_n - 1)}{1 + |a|^2 - 2a_n}, \cdots, \frac{2a_{n-1}(a_n - 1)}{1 + |a|^2 - 2a_n}, \frac{1 - |a|^2 + 2a_n(a_n - 1)}{1 + |a|^2 - 2a_n} \right). \tag{4.8}$$

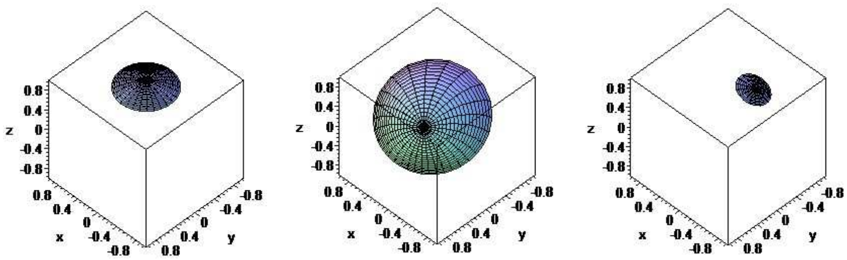
Given a spherical cap $\mathcal{U}_h$, the new spherical cap $\mathcal{U}_{h,a}$ is an expansion or a contraction of $\mathcal{U}_h$. The cap $\mathcal{U}_{h,a}$ is not centered at the North Pole if $a \in B^n \backslash L_{e_n}$ or if $a_n \neq h$. Applying a convenient rotation to the cap $\mathcal{U}_{h,a}$ we can center it at a given arbitrary point of the sphere. In this way we obtain a family of caps $\{\mathcal{U}_{h,r,\phi}^{\theta_1,\cdots,\theta_{n-2}} : r \in [0,1[, \theta_1 \in [0,2\pi[, \theta_2 \ldots \theta_{n-2}, \phi \in [0,\pi[\}$ that will generate our local analysis on a given point of the sphere. For instance, in the case of the sphere S^2, if we consider $s_{h,a} = \cos\beta/2 + \omega \sin\beta/2 \in \mathrm{Spin}(3)$ with

$$\begin{cases} \omega = \left[-\frac{a_2}{\sqrt{a_1^2+a_2^2}}, \frac{a_1}{\sqrt{a_1^2+a_2^2}}, 0 \right] \text{ and } \cos\beta = \frac{1-|a|^2+2a_3(a_3-h)}{k^{1/2}}, & \text{if } a \in B^3 \backslash L_{e_3} \\ \qquad \omega = e_3 \quad \text{and} \quad \beta = 0, & \text{if } a \in L_{e_3} \end{cases}, \tag{4.9}$$

where ω is the axis of the rotation and β is the angle of the rotation, then the set $\{s_{h,a}\mathcal{U}_{h,a}\overline{s_{h,a}} : a \in B^n\}$ stands for a family of caps centered at the North Pole. We remark that, since the point a belongs to the x_3-axis, the North Pole is a fixed point and the cap $\mathcal{U}_{h,a}$ remains centered at the North Pole. Also, if $a_3 = h$ the cap $\mathcal{U}_{h,a}$ remains centered at the North Pole, since in this case $Q = e_n$.

These caps will provide the basis for our local analysis on the sphere. A dilation around an arbitrary point $\omega \in S^{n-1}$ can be obtained by combining the above described dilation around the North Pole with an appropriate rotation.

We illustrate these facts with some concrete examples in $\mathbb{R}^3$.



1 2 3

Figure 4.1: Dilation/contraction of a given cap: **1** - original cap $\mathcal{U}_{\sqrt{3}/2}$, **2** - $\mathcal{U}_{\sqrt{3}/2,(1/8,-\sqrt{3}/8,\sqrt{3}/4)}$, **3** - $\mathcal{U}_{\sqrt{3}/2,(1/4,1/8,-1/4)}$.

The second cap is a dilation of $\mathcal{U}_{\sqrt{3}/2}$, whereas the third cap is a contraction of $\mathcal{U}_{\sqrt{3}/2}$. For each $h \in]-1,1[$ it is possible to split the unit ball into two disjoint regions that will be called ***dilation*** and ***contraction regions***. These regions are separated by a revolution surface $\mathcal{S}_h$ obtained by rotation around the x_n-axis of the arc defined by

$$\overrightarrow{\gamma}(r) = (0, \ldots, 0, r\sqrt{1-(hr)^2}, r^2 h), \quad r \in [0,1[. \tag{4.10}$$

To see that $\overrightarrow{\gamma}(r)$ describes the surface $\mathcal{S}_h$ we substitute $\cos\phi = hr$ and $\sin\phi = \sqrt{1-(hr)^2)}$ in (4.7). We obtain:

$$\begin{aligned}
d_h(r,\phi) &= 1 + \frac{2r\cos\phi - h(1+r^2)}{\sqrt{4r^2(r\cos\phi - h)^2\sin^2\phi + (1-r^2+2r\cos\phi(r\cos\phi - h))^2}} \\
&= 1 + \frac{h(r^2-1)}{1-r^2} \\
&= 1-h, \quad \forall r \in [0,1[,
\end{aligned}$$

which shows that the distance is constant and equal to the distance of the North Pole to the hyperplane which contains the support of the cap $\mathcal{U}_h$.

The surface $\mathcal{S}_h$ can be parameterized by:

$$\mathcal{S}_h : \left\{ \begin{array}{rcl}
s_1 &=& r\,\cos\theta_1'\cos\theta_2'\cdots\cos\theta_{n-2}'(1-(hr)^2)^{1/2} \\
s_2 &=& r\,\sin\theta_1'\cos\theta_2'\cdots\cos\theta_{n-2}(1-(hr)^2)^{1/2} \\
s_3 &=& r\,\sin\theta_2'\cos\theta_3'\cdots\cos\theta_{n-2}'(1-(hr)^2)^{1/2} \\
\vdots && \\
s_{n-1} &=& r\,\sin\theta_{n-2}'(1-(hr)^2)^{1/2} \\
s_n &=& r^2\,h
\end{array} \right. \tag{4.11}$$

with $r \in [0,1[$, $\theta_1' \in [0,2\pi[$, $\theta_2', \ldots, \theta_{n-2}' \in [0,\pi[$.

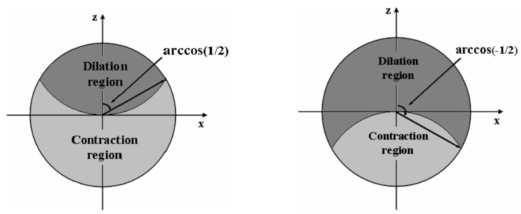


Figure 4.2: Projection of S_h in the $x_{n-1}x_n-$plane: $\mathcal{S}_{1/2}$ (left) and $\mathcal{S}_{-1/2}$ (right).

For example, in Figure 4.2 we can observe a projection of $\mathcal{S}_h$ in the $x_{n-1}x_n-$plane, for $h = 1/2$ and $h = -1/2$.

The dilation (resp. contraction) region is the region in the unit ball above (resp. below) the surface $\mathcal{S}_h$. All the spherical caps obtained from φ_a, $a \in \mathcal{S}_h$, have the same area. However, they differ in the localization of the attractor point as the next proposition states.

Proposition 4.1.2 *Consider $a \in \mathcal{S}_h$, where $h \in]-1,1[$ is fixed. For each $r \in [0,1[$, the corresponding attractor point $\mathcal{A}$ lies in the intersection of the sphere of equation $\mathcal{A}_1^2 + \mathcal{A}_2^2 + \ldots + \mathcal{A}_{n-1}^2 = \frac{4r^2\,(1-(hr)^2)(r^2h-1)^2}{(1+r^2-2r^2h)^2}$ with the hyperplane $\mathcal{A}_n = \frac{1-r^2+2r^2h(r^2h-1)}{1+r^2-2r^2h}$.*

This gives us the advantage of being able to choose a preferable contraction inside the cap by a convenient choice of the position of the *attractor point*. An important information is the arc-length between the attractor point $\mathcal{A}$ and the center Q of the cap $\mathcal{U}_{h,a}$. In cartesian coordinates, it is given by

$$d = \arccos\left(\frac{1 + |a|^4 + 2(2h-1)|a|^2 - 2a_n(1+|a|^2)(1+h) + 4a_n^2}{k(1+|a|^2 - 2a_n)}\right). \tag{4.12}$$

Moreover, distance d provides information on the geometry of the caps (compact support of our future wavelets) under the action of a Möbius transformation and, in particular, of the dilation/contraction effects inside that same cap.

We are interested in the study of the distance (4.7) as a function of these parameters, and this independent of the dimension considered (see Figure 4.3).

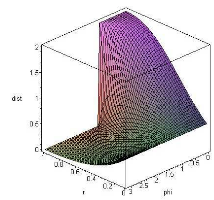


Figure 4.3: Variation of the distance (4.7) for $h = 1/2$.

Since this distance is fundamental for the definition of dilations on the unit sphere we will present on Table 4.1 a detailed study of some of its properties. We will consider its variation for each fixed h and ϕ. Thus, we will denote $d_h(r,\phi)$ by $d(r)$ for simplicity. Moreover, we denote $\phi_{\text{lim}} := \arccos h$.

$h \in]0,1[$	$h = 0$	$h \in]-1,0[$
$\phi \in [0, \phi_{lim}]$ $d(r)$ is strictly increasing	$\phi \in [0, \pi/2[$ $d(r)$ is strictly increasing	$\phi \in [0, \pi/2]$ $d(r)$ is strictly increasing
$\phi \in]\phi_{lim}, \pi/2[$ $d(r)$ has a maximum at $r = \frac{h-\sqrt{h^2-cos^2\phi}}{\cos\phi}$	$\phi = \pi/2$ $d(r) = 1, \forall r \in [0,1[$	$\phi \in]\pi/2, \phi_{lim}[$ $d(r)$ has a minimum at $r = \frac{h+\sqrt{h^2-cos^2\phi}}{\cos\phi}$
$\phi \in [\pi/2, \pi]$ $d(r)$ is strictly decreasing	$\phi \in]\pi/2, \pi]$ $d(r)$ is strictly decreasing	$\phi \in [\phi_{lim}, \pi]$ $d(r)$ is strictly decreasing

Table 4.1: Radial behavior of the distance (4.7).

The angle ϕ_{lim} is related with the separation between the dilation and contraction regions near the boundary of the unit ball (see Fig. 4.2). As we approach the boundary of the unit ball the function $d_h(r,\phi)$ has a discontinuous jump near ϕ_{lim} since

$$\lim_{r\to 1} d_{h,\phi}(r) = \begin{cases} 2, & \text{if } \phi < \arccos h \\ 1, & \text{if } \phi = \arccos h \\ 0, & \text{if } \phi > \arccos h \end{cases}.$$

It is easy to see that maxima and minima presented in Table 4.1 belong to S_h.

Definition 4.1.3 *For fixed $h \in]-1,1[$ and $\rho \in]0,2[$ the surface $S_{h,\rho} = \{a \in B^n : d_h(a) = \rho\}$ will be called a* ***zonal surface****.*

Solving the equation $d_h(r,\phi) = \rho$ in order to ϕ, for fixed $h \in]-1,1[$ and $\rho \in]0,2[$ we obtain a solution $\phi = \phi(r)$ given by

$$\phi = \arccos\left(\frac{(1-\rho)(1-r^2)\sqrt{\rho(2-\rho)(1-h^2)} + h\rho(\rho-2)(1+r^2)}{2r\rho(\rho-2)}\right), \tag{4.13}$$

where the variation of the parameter r is given in Table 4.2.

$h \in]-1,0[$	$h = 0$	$h \in]0,1[$
$0 < \rho \leq 1-h$ $r \in [R_0, 1[$	$0 < \rho < 1$ $r \in [R_0, 1[$	$0 < \rho \leq 1-h \wedge \rho \neq 1+h$ $r \in [R_0, 1[$
$1-h < \rho < 2 \wedge \rho \neq 1+h$ $r \in [-R_0, 1[$	$1 < \rho < 2$ $r \in [-R_0, 1[$	$1-h < \rho < 2$ $r \in [-R_0, 1[$
$\rho = 1+h$ $r \in [h, 1[$	$\rho = 1$ $r \in [0, 1[$	$\rho = 1+h$ $r \in [-h, 1[$

Table 4.2: Variation of the parameter r for the solution (4.13).

The constant R_0 is given by

$$R_0 = \frac{\rho(2-\rho) - \sqrt{\rho(2-\rho)(1-h^2)}}{(\rho-1)\sqrt{\rho(2-\rho)(1-h^2)} + h\rho(\rho-2)}.$$

Considering the solution (4.13) we define the arc $C_{h,\rho}(r) = \{(0,\ldots,0, r\sin\phi(r), r\cos\phi(r)),\ r \in B\}$, where B is one of the sets in Table 4.2, the zonal surface $S_{h,\rho}$ is generated by $C_{h,\rho}(r)$ since $S_{h,\rho} = s_* C_{h,\rho}(r)\overline{s_*}$, for all $s_* \in \mathrm{Spin}(n-1)$.

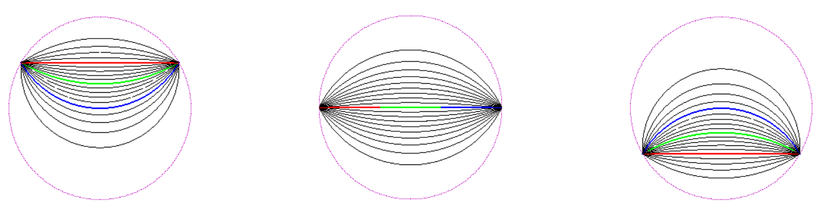


Figure 4.4: Zonal surfaces - cut in the $x_{n-1}x_n-$plane: $S_{1/2,\rho}$(at left); $S_{0,\rho}$(center); $S_{-1/2,\rho}$(at right), with $\rho = i/10,\ i = 1,\ldots,19$. The surfaces $S_{h,1-h}$, $S_{h,1}$ and $S_{h,1+h}$ are represented by the blue, green and red lines, respectively.

The parametrization of the surfaces $S_{h,\rho}$ can be also obtained by solving the equation $d_h(r,\phi) = \rho$ in order to r, for fixed $h \in]-1,1[$ and $\rho \in]0,2[$. In this case, the solutions are real roots of a quartic equation.

Proposition 4.1.4 *For fixed $h \in]-1,1[$ and $\rho \in]0,2[$, $\quad a \equiv a(r, \theta_1, \ldots, \theta_{n-2}, \phi) \in S_{h,\rho}$ if and only if $r = r(\phi)$ is a real root of the fourth degree polynomial*

$$p(r) := \alpha r^4 + \beta r^3 + \gamma r^2 + \beta r + \alpha, \tag{4.14}$$

with $\alpha = h^2 - (\rho-1)^2$, $\beta = 4h\rho(\rho-2)\cos(\phi)$, $\gamma = 4\rho(2-\rho)\cos^2(\phi) + 2(\rho(\rho-2)(1-2h^2) + 1 - h^2)$.

Now we will study the behavior of the orbits constructed in Chapter 2 with respect to the distance (4.6). This study allow us to obtain some special zonal surfaces and to better understand the behavior of the Möbius parameter on the dilation and contraction regions.

We will begin by considering the orbits constructed in Proposition 2.8.3 by means of the factorization $B^n/(D^{n-1}_{e_n}, \sim_l)$. Since the distance (4.6) only depends on the parameters r and ϕ we restrict our study to the $x_{n-1}x_n$−plane.

For each $t \in]-1,1[$ we consider the orbit $O^l_t = \{P_{t,\lambda}, \lambda \in]-1,1[\}$, with $P_{t,\lambda} = \left(0, \ldots, 0, \frac{\lambda(1-t^2)}{1+\lambda^2 t^2}, \frac{t(1+\lambda^2)}{1+\lambda^2 t^2}\right)$. Substituting a given point $P_{t,\lambda}$ of O^l_t in (4.6) we obtain

$$d_{h,t}(\lambda) = 1 + \frac{(1+\lambda^2)(-ht^2 + 2t - h)}{q_{h,t}(\lambda)^{1/2}}, \tag{4.15}$$

with $q_{h,t}(\lambda) = (t^2 - 2ht + 1)^2(1 + \lambda^4) + 2((2h^2 - 1)t^4 - 4ht^3 + 6t^2 - 4ht + 2h^2 - 1)\lambda^2$. Differentiating (4.15) with respect to λ we obtain

$$d'_{h,t}(\lambda) = \frac{4\lambda(1-h^2)(1-\lambda^2)(t^2-1)^2(-ht^2 + 2t - h)}{q_{h,t}(\lambda)^{3/2}}.$$

For $h \in]-1,1[$ fixed, Table 4.3 shows the behavior of distance (4.15) for each $h, t \in]-1,1[$.

$h \in]-1,1[\setminus\{0\}$	$h = 0$
$t < \frac{1-\sqrt{1-h^2}}{h}$ $d_{h,t}(\lambda)$ has a maximum at $\lambda = 0$ $0 < d_{h,t}(\lambda) \le d_{h,t}(0)$, $\forall \lambda \in]-1,1[$	$t < 0$ $d_{0,t}(\lambda)$ has a maximum at $\lambda = 0$ $0 < d_{0,t}(\lambda) \le d_{0,t}(0)$, $\forall \lambda \in]-1,1[$
$t = \frac{1-\sqrt{1-h^2}}{h}$ $d_{h,t}(\lambda) = 1$, $\forall \lambda \in]-1,1[$	$t = 0$ $d_{0,0}(\lambda) = 1$, $\forall \lambda \in]-1,1[$
$t > \frac{1-\sqrt{1-h^2}}{h}$ $d_{h,t}(\lambda)$ has a minimum at $\lambda = 0$ $d_{h,t}(0) \le d_{h,t}(\lambda) < 2$, $\forall \lambda \in]-1,1[$	$t > 0$ $d_{0,t}(\lambda)$ has a minimum at $\lambda = 0$ $d_{0,t}(0) \le d_{0,t}(\lambda) < 2$, $\forall \lambda \in]-1,1[$

Table 4.3: Behavior of distance (4.15)

We illustrate this behavior in Figure 4.5 for $h = 1/2$. The orbit $O_{\frac{1-\sqrt{1-h^2}}{h}} = O_{2-\sqrt{3}}$ is represented there by the black curve. The brown curve represents the separation between dilation and contraction regions. It is readily seen that these two curves touch each other at the critical angle $\phi_{\lim} = \arccos(h)$, on the boundary of the unit ball.

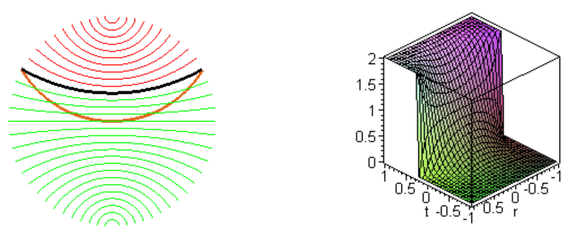


Figure 4.5: At left: Separation of the curves O_t^l for $h = 1/2$ (projection in the $yz-$plane). At right: Variation of the distance (4.6) through the orbits O_t^l.

Now we consider the orbits constructed in Proposition 2.8.9 associated to the factorization $B^n/(D_{e_n}^{n-1}, \sim_r)$.

For each $t \in]-1,1[$ we consider the curve $O_t^r = \{Q_{t,\lambda}, \lambda \in]-1,1[\}$, with $Q_{t,\lambda} = \left(0,\ldots,0,\frac{\lambda(1+t^2)}{1+\lambda^2 t^2},\frac{t(1-\lambda^2)}{1+\lambda^2 t^2}\right)$. Substituting a given point of O_t^r in (4.6) we obtain

$$d_{h,t}(\lambda) = 1 + \frac{2t(1-\lambda^2) - (1+\lambda^2)(1+t^2)h}{n_{h,t}(\lambda)^{1/2}}, \tag{4.16}$$

with $n_{h,t}(\lambda) = (t^2+2ht+1)^2\lambda^4 + 2((2h^2-1)t^4 - 2t^2 - 1 + 2h^2)\lambda^2 + (t^2-2ht+1)^2$. Differentiating (4.16) with respect to λ we obtain

$$d'_{h,t}(\lambda) = \frac{-4(1+t^2)(t^2-1)^2(1-\lambda^2)(1-h^2)h\lambda}{n_{h,t}(\lambda)^{3/2}}.$$

Hence, for $h \in]-1,0[$, $d_{h,t}$ has a maximum at $\lambda = 0$ and $0 < d_{h,t}(\lambda) \leq d_{h,t}(0)$, for all $\lambda \in]-1,1[$; for $h \in]0,1[$, $d_{h,t}$ has a minimum at $\lambda = 0$ and $d_{h,t}(0) \leq d_{h,t}(\lambda) < 2$, for all $\lambda \in]-1,1[$. The most interesting case is $h = 0$. In this particular case we obtain $d_{0,t}(\lambda) = \frac{(1+t)^2}{1+t^2}$. Thus, $d_{0,t}(\lambda)$ is constant for each orbit O_t^r and we have the following proposition:

Proposition 4.1.5 *The zonal surfaces $S_{0,\rho}$ coincide with the equivalence classes of the decomposition $B^n/(D_{e_n}^{n-1}, \sim_r)$.*

We finalize our study with the dual case of the orbits arising from the factorization of the unit ball by the gyro-subgroup L_{e_n}. For the decomposition $B^n/(L_{e_n}, \sim_l)$ we look for the orbits constructed in Proposition 2.8.14. For each $\lambda \in]-1,1[$ let $O^l_\lambda = \{P_{\lambda,t}, t \in]-1,1[\}$, with $P_{\lambda,t} = \left(0,\dots,0,\frac{\lambda(1+t^2)}{1+\lambda^2t^2},\frac{t(1-\lambda^2)}{1+\lambda^2t^2}\right)$. Substituting a given point $P_{\lambda,t}$ of O^l_λ in (4.6) we obtain

$$d_{h,\lambda}(t) = 1 + \frac{2t(1-\lambda^2)-(1+\lambda^2)(1+t^2)h}{n_{h,\lambda}(t)^{1/2}}, \tag{4.17}$$

with $n_{h,\lambda}(t) = n_{h,t}(\lambda)$. Differentiating (4.17) in order to t we obtain

$$d'_{h,\lambda}(t) = \frac{-2(\lambda^2-1)^2(1-h^2)(1-t^2)((\lambda^2-1)t^2+2(1+\lambda^2)ht+\lambda^2-1)}{n_{h,\lambda}(t)^{3/2}}.$$

The behavior of the distance (4.18) is summarized in the following table.

$h \in]-1,0[$	$h=0$	$h \in]0,1[$
$\lambda \in]-1,\frac{\sqrt{1-h^2}}{-1+h}[$ $d_{h,\lambda}(t)$ has a minimum at $t=t_1$	$d_{0,\lambda}(t) = \frac{(1+t)^2}{1+t^2}$	$\lambda \in]-1,-\frac{\sqrt{1-h^2}}{1+h}[$ $d_{h,\lambda}(t)$ has a maximum at $t=t_2$
$\lambda \in [-\frac{\sqrt{1-h^2}}{1-h},\frac{\sqrt{1-h^2}}{1-h}]$ $d_{h,\lambda}(t)$ is strictly increasing	$d'_{0,\lambda}(t) = \frac{2(1-t^2)}{(1+t^2)^2} > 0,$	$\lambda \in [-\frac{\sqrt{1-h^2}}{1+h},\frac{\sqrt{1-h^2}}{1+h}]$ $d_{h,\lambda}(t)$ is strictly increasing
$\lambda \in]\frac{\sqrt{1-h^2}}{1-h},1[$ $d_{h,\lambda}(t)$ has a minimum at $t=t_1$	$d_{0,\lambda}$ is strictly increasing	$\lambda \in]\frac{\sqrt{1-h^2}}{1+h},1[$ $d_{h,\lambda}(t)$ has a maximum at $t=t_2$

Table 4.4: Behavior of distance (4.18) according to the parameter h.

The values t_1 and t_2 are respectively

$$t_1 := \frac{-2h(1+\lambda^2)-2\sqrt{((h+1)\lambda^2+h-1)((h-1)\lambda^2+h+1)}}{2(\lambda^2-1)};$$

$$t_2 := \frac{-2h(1+\lambda^2)+2\sqrt{((h+1)\lambda^2+h-1)((h-1)\lambda^2+h+1)}}{2(\lambda^2-1)}.$$

For the decomposition $B^n/(L_{e_n},\sim_r)$ we look into the orbits constructed in Proposition 2.8.15. For each $\lambda \in]-1,1[$, let $O^r_\lambda = \{Q_{\lambda,t}, t \in]-1,1[\}$, with $Q_{\lambda,t} = \left(0,\dots,0,\frac{\lambda(1-t^2)}{1+\lambda^2t^2},\frac{t(1+\lambda^2)}{1+\lambda^2t^2}\right)$. Substituting a given point of O^r_λ in (4.6) we obtain

$$d_{h,\lambda}(t) = 1 + \frac{(1+\lambda^2)(-ht^2+2t-h)}{q_{h,\lambda}(t)^{1/2}}, \tag{4.18}$$

with $q_{h,\lambda}(t) = q_{h,t}(\lambda)$. Differentiating (4.18) in order to t we obtain

$$d'_{h,\lambda}(t) = \frac{2(1+\lambda^2)(1-\lambda^2)^2(1-h^2)(1-t^2)(t^2-2ht+1)}{q_{h,\lambda}(t)^{3/2}}.$$

Therefore, it is easy to conclude that $d'_{h,\lambda}(t) > 0$, for all $h \in]-1,1[$ and all $\lambda \in]-1,1[$ and, therefore, $d_{h,\lambda}(t)$ is a strictly increasing function of t.

4.2 Conformal group of the hemisphere

Let $n \geq 3$. We consider $S_+^{n-1} = \{x \in S^{n-1} : x_n > 0\}$ as the upper hemisphere embedded in $\mathbb{R}^n$. With the construction made in Section 4.1 we are able to derive the conformal group of S_+^{n-1}. It will consist of Möbius transformations according to Liouville's Theorem (see [9]):

Theorem 4.2.1 *Let U, V be open connected subsets of $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, $n \geq 3$, and let $f : U \mapsto V$ be a conformal map. Then f is a restriction of a Möbius transformation which is uniquely determined by f.*

The group $\mathrm{Spin}(n-1)$ of rotations leaving the x_n-axis invariant belong to the conformal group of the hemisphere. With respect to Möbius transformations φ_a, we need to chose those that leave invariant S_+^{n-1}.

Lemma 4.2.2 *For $a \in D_{e_n}^{n-1}$ the transformations φ_a leave invariant S_+^{n-1}.*

Proof: For $\mathcal{U}_0 = S_+^{n-1} \cup \{x \in S^{n-1} : x_n = 0\}$ we have that

$$d_0(r, \phi) = 1 + \frac{2r\cos\phi}{\sqrt{(1-r^2)^2 + (2r\cos\phi)^2}}.$$

Thus, $d_0(r, \phi) = 1$ if and only if $a \in D_{e_n}^{n-1}$. As for $a \in D_{e_n}^{n-1}$ the center of the cap $\varphi_a(\mathcal{U}_0)$ coincides with the North Pole, the result follows.

■

Proposition 4.2.3 *The conformal group of the hemisphere S_+^{n-1} is $(\mathrm{Spin}(n-1) \times D_{e_n}^{n-1}, \times^r)$, where $\times^r$ is the gyrosemidirect product*

$$(s_1, a) \times^r (s_2, b) = (s_1 s_2 q, b \oplus (\overline{s_2} a s_2)),$$

with $s_1, s_2 \in \mathrm{Spin}(n-1)$, $a, b \in D_{e_n}^{n-1}$, and $q = \frac{1 - \overline{s_2} a s_2 b}{|1 - \overline{s_2} a s_2 b|}$.

Proof: We have only to show that the operation $\times^r$ is well defined. The remaining group axioms are proved in the same way as in Proposition 2.6.9.

Let c be a point on the x_n-axis. As $b \perp c$ and $a \perp c$ then $bc = -cb$ and $ac = -ac$. Moreover, $sc\overline{s} = c$ for $s \in \mathrm{Spin}(n-1)$. Therefore, $qc\overline{q} = cq\overline{q} = c$, which shows that q is again a rotation of $\mathrm{Spin}(n-1)$. Finally, it is easy to see that $b \oplus (\overline{s_2} a s_2) \in D_{e_n}^{n-1}$, for every $a, b \in D_{e_n}^{n-1}$ and $s_2 \in \mathrm{Spin}(n-1)$. Thus, the operation $\times^r$ is well defined.

■

The importance of this result is that the hemisphere, as a manifold, has its own conformal group and, then, it is possible to develop a wavelet theory on it. It is well know that the upper hemisphere (as well as the lower hemisphere) is a model for the *$n-$dimensional hyperbolic space* or *Lobachevskii space* (see [9]). The upper hemisphere is related with the unit ball, another model for the *$n-$dimensional hyperbolic space*, through the Klein model for the unit ball. Thus, a wavelet theory for the unit ball is deeply related to a wavelet theory for the upper (or lower) hemisphere. We will not make this approach since we are mainly interested on the unit sphere S^{n-1}.

More generally, the following result holds.

Proposition 4.2.4 *For every fixed $\omega \in S^{n-1}$, let $Spin_\omega(n-1)$ be the group of rotations leaving the ω-axis invariant and $D_\omega^{n-1} = \{a \in B^n :< a, \omega >= 0\}$. Then $(\mathrm{Spin}_\omega(n-1) \times D_\omega^{n-1}, \times^r)$ is a group for the right gyrosemidirect product*

$$(s_1, a) \times^r (s_2, b) = (s_1 s_2 q, b \oplus (\overline{s_2} a s_2)),$$

with $s_1, s_2 \in \mathrm{Spin}_\omega(n-1)$, $a, b \in D_\omega^{n-1}$, and $q = \frac{1-\overline{s_2}as_2b}{|1-\overline{s_2}as_2b|}$.

4.3 Admissible sections

In Chapter 3 we considered global left C^0-sections for constructing generalized spherical continuous wavelet transforms and we proved that we can construct many systems of wavelets based on the conformal group of the sphere.

In the case of wavelets defined on spherical caps there is a class of sections with the interesting property of intersecting only one time each zonal surface, and thus, establishing a bijection between the domain of the generating function and the interval of scales $]0, 2[$, measured by the distance function (4.6). These sections allow us to define local conformal dilation operators.

We will consider the homogeneous spaces $B^n/(D_{e_n}^{n-1}, \sim_l)$ and $B^n/(D_{e_n}^{n-1}, \sim_r)$ and we will discuss the existence of global and local admissible sections.

Definition 4.3.1 *Let $V =]t_1, t_2[$ such that $-1 \leq t_1 < t_2 \leq 1$. A (global or local) section σ, with generating function $f : V \to]-1, 1[$, is said to be admissible if the function f is of class $C^1(V)$ and $d_h(\sigma) : V \to]0, 2[$, $t \mapsto d_h(\sigma(te_n))$ is bijective.*

4.3.1 Admissible global sections

We saw in Section 2.9 that we can construct an entire class of global sections $\sigma^l : X_1 = B^n/(D^{n-1}_{e_n}, \sim_l) \to B^n$ by considering

$$\sigma^l(te_n) = te_n \oplus f(t)e_{n-1} = \left(0, \ldots, 0, \frac{f(t)(1-t^2)}{1+(tf(t))^2}, \frac{t(1+f(t)^2)}{1+(tf(t))^2}\right), \tag{4.19}$$

where $f :]-1,1[\to]-1,1[$.

In the next proposition we will characterize the admissible global sections defined by (4.19).

Proposition 4.3.2 *Let $h \in]-1,1[$ be fixed. If $f :]-1,1[\to]-1,1[$ is a function of class $C^1(]-1,1[)$ and the function $p_h(t) = 2(t^2-1)(ht^2-2t+h)f(t)f'(t)+(2ht-t^2-1)(f(t)^4-1)$ is strictly positive for all $t \in]-1,1[$, then the section $\sigma^l(te_n)$ is admissible for h.*

Proof: We suppose that f is a function of class $C^1(]-1,1[)$. Replacing a by $\sigma^l(te_n)$ in (4.6) we obtain

$$d_h(\sigma^l(te_n)) = 1 + \frac{(1+f(t)^2)(-ht^2+2t-h)}{q_h(t)^{1/2}}, \tag{4.20}$$

with $q_h(t) = (t^2-2ht+1)^2(1+f(t)^4) + 2((2h^2-1)t^4 - 4ht^3 + 6t^2 - 4ht + 2h^2 - 1)f(t)^2$. Differentiating (4.20) with respect to t we get

$$d'_h(\sigma^l(te_n)) = \frac{2(1-h^2)(1-t^2)(1-f(t)^2)p_h(t)}{q_h(t)^{3/2}},$$

with $p_h(t) = 2(t^2-1)(ht^2-2t+h)f(t)f'(t)+(2ht-t^2-1)(f(t)^4-1)$. Thus, $d'_h(\sigma^l(te_n)) > 0$ if and only if $p_h(t) > 0$ for all $t \in]-1,1[$. Moreover, due to $\lim\limits_{t\to -1^+} d_h(\sigma^l(te_n)) = 0$ and $\lim\limits_{t\to 1^-} d_h(\sigma^l(te_n)) = 2$, we conclude that $d_h(\sigma^l(te_n))$ is a bijection between $]-1,1[$ and $]0,2[$.

■

Since an admissible section generates a curve inside the unit ball we will frequently use the term *section* when referring to its curve. For the space X_1 we will consider curves described from the South Pole $(-e_n)$ to the North Pole $(+e_n)$. In this way, any section will cross first the contraction region, and only then, the dilation one.

Examples 4.3.3 *1. The function $f(t) = \lambda$, with $\lambda \in]-1,1[$, induces the sections $\sigma^l_\lambda(te_n) = \left(0, \ldots, 0, \frac{\lambda(1-t^2)}{1+\lambda^2t^2}, \frac{t(1+\lambda^2)}{1+\lambda^2t^2}\right)$. These sections are admissible for all $h, \lambda \in]-1,1[$ since $p_{h,\lambda}(t) = (t^2+1-2ht)(1-\lambda^4)$ is strictly positive for all $h, t, \lambda \in]-1,1[$. For $\lambda = 0$ we obtain the fundamental section $\sigma^l_0(te_n)$.*

2. *For fixed* $c \in]-1,1[$ *fixed we consider the sections* $\sigma_c^l = (0,\ldots,0,c\sin\phi,-\cos\phi)$, *with* $\phi \in]0,\pi[$. *For* $c \in]-1,1[\backslash\{0\}$, *the generating function is given by* $f(t) = \sqrt{\frac{t^2-1+\sqrt{(1-t^2)^2+4c^4t^2}}{2t^2c^2}}$. *By straightforward computations we get*

$$\begin{aligned} p_{h,c}(t) &= \frac{(1-t^2)(-1+(1-2c^4)t^4+2h(2c^4-c^2-1)t^3+2c^2(2-c^2)t^2+2h(1-c^2)t}{2c^4t^4}+ \\ &(1-t^2)(t^6+2h(c^2(2c^4-1)-1)t^5+(4c^2(-2c^4+c^2+1)-1)t^4+ \\ &+4h(c^4(c^2-2)+1)t^3+ \\ &(4c^2(c^2-1)-1)t^2+2h(c^2-1)t+1)\,/\,(2c^4t^4\sqrt{t^4+2(2c^4-1)t^2+1}). \end{aligned}$$

It is not easy to see that $p_{h,c}(t) > 0$ *for all* $h,t \in]-1,1[$ *and for all* $c \in]-1,1[\backslash\{0\}$. *For this we will use the definition of the section. Substituting* σ_c^l *in (4.6) and differentiating it with respect to* ϕ, *we obtain*

$$\begin{aligned} d'_{h,c}(\phi) &= 2(1-h^2)(c^2-1)^2(2-\sin^2\phi+2h\cos\phi)\sin^3\phi\,/\,[(c^4-2c^2+1)\cos^4\phi+ \\ &4h(c^2-1)\cos^3\phi+2(c^2(2-2h^2-c^2)+2h^2+1)\cos^2\phi-4h(1+c^2)\cos\phi \\ &+c^4+2c^2(2h^2-1)+1]^{3/2}. \end{aligned}$$

Hence, it is now easy to see that $d'_{h,c}(\phi) > 0$ *for all* $\phi \in]0,\pi[$, $h \in]-1,1[$, *and* $c \in]-1,1[\backslash\{0\}$ *and, therefore, also* $p_{h,c}(t) > 0$, *for all* $h,t \in]-1,1[$ *and* $c \in]-1,1[\backslash\{0\}$. *For* $c = 0$, *the generating function is given by* $f(t) = 0$, *and we obtain again the fundamental section* $\sigma_0^l(te_n) = L_{e_n}$, *with* $t = -\cos\phi$.

3. *Another class of admissible sections can be constructed by considering* $f(t) = |t|^n, n \in \mathbb{N}\backslash\{1\}$. *Then, we obtain the sections* $\sigma_n^l(te_n) = \left(0,\ldots,0,\frac{|t|^n(1-t^2)}{1+|t|^{2n}t^2},\frac{t(1+|t|^{2n})}{1+|t|^{2n}t^2}\right)$ *for which*

$$p_{h,n}(t) = 2n|t|^{2n-1}(t^2-1)(t^2h-2t+h)+(2ht-t^2-1)(|t|^{4n}-1) > 0.$$

These sections satisfy $\lim_{n\to+\infty}\sigma_n^l(te_n) = \sigma_0^l(te_n) = L_{e_n}$. *Moreover, they can be extended to an arbitrary parameter* $\alpha \in \mathbb{R}^+$ *by putting* $f(t) = |t|^\alpha$. *However, it is easy to see that the point* $t = 0$ *becomes then a singularity for* $\alpha \in]0,1]$.

4. *Let* $P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n$ *be a Legendre polynomial of degree* n *and* $f_n(t) = (P_n(t)+1)/2$. *Polynomials of degree* 1 *to* 4 *give rise to admissible sections for all* $h \in]-1,1[$. *Polynomials of degree* 5, 6 *and* 7 *give rise to admissible sections only for a subset of* $]-1,1[$ *of values of* h. *For degree higher or equal to 8 the respective sections are non-admissible for all* $h \in]-1,1[$ *because these polynomials are highly oscillating.*

Remark 4.3.4 *If* σ^l *is an admissible section and* $a \in B^n$ *then* $\sigma^l(te_n)\oplus a$ *is not an admissible section in general. The same is true for* $a \oplus \sigma^l(te_n)$.

Now we characterize admissible global sections for $X_2 = B^n/(D^{n-1}_{e_n}, \sim_r)$. We saw in Section 2.9 that we can construct an entire class of global sections $\sigma^r : X_2 \to B^n$ by considering

$$\sigma^r(te_n) = g(t)e_{n-1} \oplus te_n = \left(0, \dots, 0, \frac{g(t)(1+t^2)}{1+(tg(t))^2}, \frac{t(1-g(t)^2)}{1+(tg(t))^2}\right), \tag{4.21}$$

with $g :]-1,1[\to]-1,1[$. Now, we will characterize the admissible right global sections.

Proposition 4.3.5 *Let $h \in]-1,1[$ be fixed. If $g :]-1,1[\to]-1,1[$ is a function of class $C^1(]-1,1[)$, the function $q(t) := 2h(t^4-1)g(t)g'(t) + (t^2+2ht+1)g(t)^4 - 2(t^2+1)g(t)^2 + t^2 - 2ht + 1$ is strictly positive for all $t \in]-1,1[$ and*

$$\begin{cases} \lim\limits_{t\to -1^+} g(t) \in \left]-\dfrac{\sqrt{1-h^2}}{1-h}, \dfrac{\sqrt{1-h^2}}{1-h}\right[, & \text{if } h \in]-1,0] \\ \lim\limits_{t\to 1^-} g(t) \in \left]-\dfrac{\sqrt{1-h^2}}{1+h}, \dfrac{\sqrt{1-h^2}}{1+h}\right[, & \text{if } h \in [0,1[\end{cases} \tag{4.22}$$

then the section $\sigma^r(te_n)$ is admissible for h.

Proof: We suppose that g is a function of class $C^1(]-1,1[)$. Replacing a by $\sigma^r(te_n)$ in (4.6) we obtain

$$d_h(\sigma^r(te_n)) = 1 - \frac{(ht^2+2t+h)g(t)^2 + ht^2 - 2t + h}{q_{h,g}(t)^{1/2}}, \tag{4.23}$$

with $q_{h,g}(t) = (t^2+2ht+1)^2 g(t)^4 + 2((2h^2-1)t^4 - 2t^2 - 1 + 2h^2)g(t)^2 + (t^2-2ht+1)^2$. Differentiating (4.23) with respect to t we will eventually find

$$d_h'(\sigma^r(te_n)) = \frac{2(1-h^2)(1-t^2)(1-g(t)^2)p_{h,g}(t)}{q_{h,g}(t)^{3/2}},$$

with $p_{h,g}(t) = 2h(t^4-1)g(t)g'(t) + (t^2+2ht+1)g(t)^4 - 2(t^2+1)g(t)^2 + t^2 - 2ht + 1$. Thus, $d'_{h,g}(\sigma^r(te_n)) > 0$ if and only if $p_{h,g}(t) > 0$ for all $t \in]-1,1[$. To see that $d_{h,g}$ is a bijective function we study the following two limits

$$\lim_{t\to -1^+} d_{h,g}(t) = 1 + \frac{(1-h)g^+(-1)^2 - (h+1)}{|(1-h)g^+(-1)^2 - (h+1)|}$$

and

$$\lim_{t\to 1^-} d_{h,g}(t) = 1 - \frac{(1+h)g^-(1)^2 + h - 1}{|(1+h)g^-(1)^2 + h - 1|},$$

where $g(-1^+) = \lim\limits_{t\to -1^+} g(t)$ and $g(1^-) = \lim\limits_{t\to 1^-} g(t)$. Since we want to have $\lim\limits_{t\to -1^+} d_{h,g}(t) = 0$ and $\lim\limits_{t\to 1^-} d_{h,g}(t) = 2$ we must impose conditions (4.22).

■

Corollary 4.3.6 *If $h = 0$ then every function $g \in]-1,1[\to]-1,1[$ of class $C^1(]-1,1[)$ generates an admissible global right section σ^r.*

Proof: For $h = 0$, the function $q(t)$ of Proposition 4.3.5 reduces to $q(t) = (t^2+1)(g(t)^2-1)^2$ which is then strictly positive for all $t \in]-1,1[$ and for all functions $g \in C^1(]-1,1[)$. Moreover, conditions (4.22) are trivially satisfied in this case.

■

This is easy to understand since the zonal surfaces $S_{0,\rho}$ coincide with the equivalence classes of the decomposition $B^n/(D^{n-1}_{e_n}, \sim_r)$ (see Proposition 4.1.5) and, therefore, for $h = 0$ each point $\sigma^r(te_n)$ belongs to a different zonal surface $S_{0,\rho}$. Thus, we obtain the desired bijection necessary for obtaining an admissible section.

Examples 4.3.7 1. *Let $h \in]-1,1[$ be fixed and $g(t) = \lambda$, with $\lambda \in]-1,1[$. Then we obtain the sections $\sigma^r_\lambda(te_n) = \left(0,\ldots,0,\frac{\lambda(1+t^2)}{1+\lambda^2t^2},\frac{t(1-\lambda^2)}{1+\lambda^2t^2}\right)$. These sections are admissible if*

$$\begin{cases} \lambda \in \left]\frac{-\sqrt{1-h^2}}{1+h},\frac{\sqrt{1-h^2}}{1+h}\right[, & \text{if } h \in [0,1[\\ \lambda \in \left]\frac{-\sqrt{1-h^2}}{1-h},\frac{\sqrt{1-h^2}}{1-h}\right[, & \text{if } h \in]-1,0[\end{cases}. \tag{4.24}$$

We remark the analogy of the condition (4.24) with the conclusions obtained in the Table 4.4. If we first fix λ then the section σ^r_λ is admissible only for $h \in \left]-\frac{1-\lambda^2}{1+\lambda^2},\frac{1-\lambda^2}{1+\lambda^2}\right[$.

2. *For every $c \in]-1,1[$ the global sections $\sigma^r_c = (0,\ldots,0,c\sin\phi,-\cos\phi)$, with $\phi \in]0,\pi[$ are admissible.*

3. *For each $p \in]-1,1[$ we define the sections $\sigma^r_p = (0,\ldots,0,p,r)$, with $r \in]-\sqrt{1-p^2},\sqrt{1-p^2}[$. The generating function is given by*

$$g(t) = \begin{cases} \frac{1+t^2-\sqrt{(1+t^2)^2-4p^2t^2}}{2pt^2}, & t \in]-1,1[\backslash\{0\} \\ p, & t = 0 \end{cases}.$$

The parameter r is related with the generating function by means of

$$r = \frac{p-g(t)}{tg(t)} = \begin{cases} \frac{t^2(2p^2-1)-1+\sqrt{(1+t^2)^2-4p^2t^2}}{t(1+t^2-\sqrt{(1+t^2)^2-4p^2t^2})}, & t \in]-1,1[\backslash\{0\} \\ 0, & t = 0 \end{cases}.$$

It is difficult to verify the conditions of Proposition 4.3.5 in this case. Again, we use the definition of the section. Substituting σ_p^r in (4.6) and differentiating it with respect to the parameter r, we obtain

$$d'_{h,p}(r) = \frac{2(1-h^2)(r^2+p^2-1)(p^2-1+2hr-r^2)}{(4(r-h)^2p^2+(1-p^2-r^2+2r(r-h))^2)^{3/2}}.$$

Since $r^2+p^2<1$ then $d'_{h,p}(r)>0$ for all $r\in]-\sqrt{1-p^2},\sqrt{1-p^2}[$ if and only if $p^2-1+2hr-r^2<0$, for all $r\in]-\sqrt{1-p^2},\sqrt{1-p^2}[$. This is true if $p\in]-\sqrt{1-h^2},\sqrt{1-h^2}[$. Moreover, as $\lim_{r\to-\sqrt{1-p^2}} d_{h,p}(r)=0$ and $\lim_{r\to\sqrt{1-p^2}} d_{h,p}(r)=2$, we conclude that the sections σ_p^r are admissible for $p\in]-\sqrt{1-h^2},\sqrt{1-h^2}[$.

As we have seen, a global section is not always admissible. The same holds for local sections.

4.3.2 Admissible local sections

When we consider a local section it may happen that it doesn't give rise to a dilation operator on the sphere with complete scale. We will begin by giving a characterization of admissible local left sections.

Proposition 4.3.8 *Let $h\in]-1,1[$ be fixed and $-1<t_1<\frac{1-\sqrt{1-h^2}}{h}<t_2<1$. If $f:]t_1,t_2[\to]-1,1[$ is a function of class $C^1(]t_1,t_2[)$, the function $p(t):=2(t^2-1)(ht^2-2t+h)f(t)f'(t)+(2ht-t^2-1)(f(t)^4-1)$ is strictly positive for all $t\in]t_1,t_2[$ and $\lim_{t\to t_1^+} f(t)=\pm1$, $\lim_{t\to t_2^-} f(t)=\pm1$ then $\sigma^l(te_n)$ is an admissible local left section.*

Proof: Replacing a by $\sigma^l(te_n)$ in (4.6) we obtain

$$d_h(\sigma^l(te_n)) = 1+\frac{(1+f(t)^2)(-ht^2+2t-h)}{q_h(t)^{1/2}}, \tag{4.25}$$

with $q_h(t)=(t^2-2ht+1)^2(1+f(t)^4)+2((2h^2-1)t^4-4ht^3+6t^2-4ht+2h^2-1)f(t)^2$. Differentiating (4.25) with respect to t we will obtain

$$d'_h(\sigma^l(te_n)) = \frac{2(1-h^2)(1-t^2)(1-f(t)^2)p_h(t)}{q_h(t)^{3/2}},$$

with $p_h(t)=2(t^2-1)(ht^2-2t+h)f(t)f'(t)+(2ht-t^2-1)(f(t)^4-1)$. Thus, $d'_h(\sigma^l(te_n))>0$ if and only if $p_h(t)>0$ for all $t\in]t_1,t_2[$. It remains to prove that $\lim_{t\to t_1^+} d_h(\sigma^l(te_n))=0$ and

$\lim_{t \to t_2^-} d_h(\sigma^l(te_n)) = 2$. Since $-1 < t_1 < \frac{1-\sqrt{1-h^2}}{h} < t_2 < 1$, $\lim_{t \to t_1^+} f(t) = \pm 1$ and $\lim_{t \to t_2^-} f(t) = \pm 1$ we have that

$$\lim_{t \to t_1^+} d_h(\sigma^l(te_n)) = 1 + \frac{2(-ht_1^2 + 2t_1 - h)}{\sqrt{4((1+t_1^2)h - 2t_1)^2}} = 1 + \frac{2(-ht_1^2 + 2t_1 - h)}{2((1+t_1^2)h - 2t_1)} = 0.$$

and

$$\lim_{t \to t_2^-} d_h(\sigma^l(te_n)) = 1 + \frac{2(-ht_2^2 + 2t_2 - h)}{\sqrt{4((1+t_2^2)h - 2t_2)^2}} = 1 + \frac{2(-ht_2^2 + 2t_2 - h)}{-2((1+t_2^2)h - 2t_2)} = 2.$$

■

Examples 4.3.9 *1. The sections constructed in Example 4.3.7 are local sections for $B^n/(D^{n-1}_{e_n}, \sim_l)$ except for the case $p = 0$ where we have a global section (the fundamental section). They are local admissible only for $p \in]-\sqrt{1-h^2}, 0[\cup]0, \sqrt{1-h^2}[$.*

2. For each $\phi \in]0, \pi/2[$ let us consider the local sections $\sigma^l = (0, \ldots, 0, r\sin\phi, r\cos\phi)$, with $r \in]-1, 1[$. These local sections are admissible only for $\phi \in]0, \arccos(h)[$ if $h \in [0, 1[$ and for $\phi \in]0, \arccos(-h)[$ if $h \in]-1, 0[$. Indeed, substituting σ^l in (4.6) and differentiating it with respect to r we will find

$$d'_{h,\phi}(r) = \frac{2(1-r^2)(1-h^2)((1+r^2)\cos\phi - 2hr)}{(4r(r(\cos^2\phi + h^2) - h(1+r^2)\cos\phi) + (r^2-1)^2)^{3/2}}.$$

Thus, $d'_{h,\phi}(r) > 0$ if and only if $(1+r^2)\cos\phi - 2hr > 0$. Therefore, the statement holds.

For the case of $X_2 = B^n/(D^{n-1}_{e_n}, \sim_r)$ there are no admissible local right sections due to structure of the orbits of X_2. As we want that the function $d_h(\sigma^r(te_n))$ is a bijection between $]t_1, t_2[$ and $]0, 2[$, the section $\sigma^r(te_n)$ must approach the boundary of the unit ball, but in that case the section will intersect all orbits of X_2 and, therefore, we can only have admissible global sections.

Remark 4.3.10 *1. The global or local character of an admissible section only depends on the homogeneous space considered. Once a given section is admissible for the left case it is also admissible for the right case, but may have a different character. For example, for $h = 0$ there are admissible global right sections that are only admissible local left sections.*

4.4 Local anisotropy vs attractor point

For a given left section $\sigma^l(te_n)$ on the unit ball (global or local) we want to caracterize its anisotropy by measuring how much sections deviate from the fundamental section L_{e_n}, the section associated with pure dilations on the unit sphere.

Although the parameters $\alpha, \beta, \delta, \xi$ arising from the Iwasawa decomposition (see Chapter 3) are adequate for the study of the global behavior of the generalized SCWT, they are not good when we want to perform a local analysis of signals on the sphere. An isolated analysis of the parameter δ could lead to wrong interpretations. For instance, there is a difference between the families of sections σ^l_λ and σ^l_c : geometrically these sections generate similar curves inside the unit ball, but they have a different behavior near the South Pole. In fact, for $\sigma^l_\lambda(t) = \left(0, \ldots, 0, \frac{\lambda(1-t^2)}{1+t^2\lambda^2}, \frac{t(1+\lambda^2)}{1+t^2\lambda^2}\right)$ we obtain $\delta_\lambda(t) = \frac{4(1-t)(1-\lambda^2)(1+t^2\lambda^2)}{(t+1)((1+t^2+6t)\lambda^2+4(1+t^2\lambda^4))}$, and for each $\lambda \in [0,1[$ it holds $0 < \delta^l_\lambda(t) < \infty$, with $\lim\limits_{t\to 1} \delta^l_\lambda(t) = 0$ and $\lim\limits_{t\to -1} \delta^l_\lambda(t) = \infty$, while for $\sigma^l_c(\phi) = (0, \ldots, 0, c\sin\phi, -\cos\phi)$ we obtain $\delta^l_c(\phi) = \frac{4(1-c^2)\sin^2\phi}{4(1-\cos\phi)^2+c^2\sin^2\phi}$, and for each $c \in]0,1[$ it holds $0 < \delta^l_c(\phi) < \frac{4(1-c^2)}{c^2}$, with $\lim\limits_{\phi\to\pi} \delta^l_c(\phi) = 0$ and $\lim\limits_{\phi\to 0} \delta^l_c(\phi) = \dfrac{4(1-c^2)}{c^2}$. Thus, $\delta_\lambda(t)$ and $\delta_c(\phi)$ are dilation parameters on the tangent plane with different behavior at the infinity. Hence, these parameters are not good for an effective comparison between sections.

In Section 4.1 we observed that if $\mathcal{U}_h$ is a spherical cap centered at the North Pole then the cap $\mathcal{U}_{h,a}$ is not centered at the North Pole if $a \in B^n\backslash L_{e_n}$ or $a_n \neq h$. In order to obtain a dilation operator on the tangent plane we have to move the tangent plane to the center of the cap $\mathcal{U}_{h,a}$ and then to perform the stereographic projection. This is equivalent to rotate all caps to the North Pole and then to perform the stereographic projection. This is exactly our local dilation around the North Pole given in Section 4.1.

Let $\tilde{\mathcal{U}}_{h,a}$ be the spherical cap $\mathcal{U}_{h,a}$ rotated to the North Pole, i.e. $\tilde{\mathcal{U}}_{h,a} = s_{h,a}\mathcal{U}_{h,a}\overline{s_{h,a}}$, for some rotation $s_{h,a} \in \mathrm{Spin}(n)$ (c.f. (4.9) for the case $n = 3$). We want to give a description of the geometry of the ball $\Phi_1(\widetilde{\mathcal{U}}_{h,a})$ on the tangent plane. Since we are dealing with conformal mappings it is enough to study the mapping of the boundary and the mapping of the attractor point to get the mapping property of any point in the ball $\Phi_1(\widetilde{\mathcal{U}}_{h,a})$. We remember that a Möbius transformation on an arbitrary ball $B_r(0)$ (centered at the origin and radius r) is given by $r\widetilde{\varphi}_\rho(y/r)$, $y \in B_r(0)$, where $\widetilde{\varphi}_\rho(y)$ is a Möbius transformation on the unit ball defined by $\widetilde{\varphi}_\rho(y) = (y - \rho)(1 + \rho y)^{-1}$, with $\rho \in B_1(0)$. For the sake of simplicity of the notation, the ball centered at the origin and of radius r will be denoted by $B_r(0)$. It will be clear from the context the dimension of the ball.

Lemma 4.4.1 *Let $a=(0,\ldots,0,a_{n-1},a_n)\in B^n$ and $-1<h<1$. The stereographic projection of the action $s_{h,a}\,\varphi_a(\mathcal{U}_h)\overline{s_{h,a}}$ onto the tangent plane of S^{n-1} is given by*

$$\Phi_1(s_{h,a}\,\varphi_a(x)\,\overline{s_{h,a}})=\widetilde{\varphi}_{\rho_h(a)}(\delta_h(a)\Phi_1(x)),\quad x\in\mathcal{U}_h, \tag{4.26}$$

where $\delta_h(a)=\frac{(1-|a|^2)(1+h)}{(1+|a|^2)h-2a_n+\sqrt{k_h(a)}}$, with $k_h(a):=4(a_n-h)^2a_{n-1}^2+(1-|a|^2+2a_n(a_n-h))^2$. Moreover, $\widetilde{\varphi}_{\rho_h(a)}(y)=r_2(y-\rho_h(a)r_2)(r_2+\rho_h(a)y)^{-1}$, is a Möbius transformation on $B_{r_2}(0)$, with $r_2=\frac{2\delta_h(a)\sqrt{1-h^2}}{1+h}$ and $\rho_h(a)=-\frac{2(1-h)a_{n-1}((1+|a|^2)h-2a_n+\sqrt{k_h(a)})}{\sqrt{1-h^2}(\sqrt{k_h(a)}(1+|a|^2-2a_n)+C_h(a))}e_{n-1}$, with $C_h(a):=k_h(a)-2(1-h)((1+|a|^2-2ha_n)a_n-2ha_{n-1}^2)$.

Proof: Let $P_1=(0,\ldots,0,\sqrt{1-h^2},h)$ and $P_2=(0,\ldots,0,\sqrt{1-h_2^2},h_2)$, with $h_2=1-d_h(a)=\frac{-2a_n+(1+|a|^2)h}{\sqrt{k_h(a)}}$, be two points on the spherical support of the caps $\mathcal{U}_h$ and $\tilde{\mathcal{U}}_{h,a}$, respectively. Then $\Phi_1(P_1)=\left(0,\ldots,0,\frac{2\sqrt{1-h^2}}{1+h}\right)$ and $\Phi_1(P_2)=\left(0,\ldots,0,\frac{2(1-|a|^2)\sqrt{1-h^2}}{(1+|a|^2)h-2a_n+\sqrt{k_h(a)}}\right)$. Thus, the radius of the balls $\Phi_1(\mathcal{U}_h)$ and $\Phi_1(\tilde{\mathcal{U}}_{h,a})$ are given by $r_1=\frac{2\sqrt{1-h^2}}{1+h}$ and $r_2=\frac{2(1-|a|^2)\sqrt{1-h^2}}{(1+|a|^2)h-2a_n+\sqrt{k_h(a)}}$, respectively. The ratio between these radius gives the parameter $\delta_h(a)$, i.e. $\delta_h(a)=\frac{r_2}{r_1}=\frac{(1-|a|^2)(1+h)}{(1+|a|^2)h-2a_n+\sqrt{k_h(a)}}$.

Let d be the spherical distance between the center of the cap $\mathcal{U}_{h,a}$ and its attractor point $\mathcal{A}=\varphi_a(e_n)=\left(0,\ldots,0,\frac{2a_{n-1}(a_n-1)}{1+|a|^2-2a_n},\frac{1-|a|^2+2a_n(a_n-1)}{1+|a|^2-2a_n}\right)$, which is given by

$$\cos(d)=\left(\frac{1+|a|^4+2(2h-1)|a|^2-2a_n(1+|a|^2)(1+h)+4a_n^2}{k_h(a)(1+|a|^2-2a_n)}\right).$$

Since this distance is invariant under rotations we can easily obtain the coordinates of the attractor point $\tilde{\mathcal{A}}$ on the cap $\tilde{\mathcal{U}}_{h,a}$, which are given by $\tilde{\mathcal{A}}=(0,\ldots,0,\epsilon\sqrt{1-\cos^2(d)},\cos(d))$ with $\epsilon=-1$ if $a_{n-1}>0$ or $\epsilon=+1$ if $a_{n-1}\leq 0$. Note that if $a_{n-1}>0$ then the last but one component of $\mathcal{A}$ is negative, otherwise it is non-negative. The stereographic projection of $\tilde{\mathcal{A}}$ yields the point $\Phi_1(\tilde{\mathcal{A}})=\left(0,\ldots,0,\epsilon\frac{2\sqrt{1-\cos^2(d)}}{1+\cos(d)}\right)$. Thus, we obtain the parameter $\rho_h(a)=\epsilon\frac{2\sqrt{1-\cos^2(d)}}{r_2(1+\cos(d))}e_{n-1}$, after rescaling $\Phi_1(\tilde{\mathcal{A}})$ by the radius r_2. By straightforward computations we obtain

$$\rho_h(a)\quad=\quad-\frac{2(1-h)a_{n-1}((1+|a|^2)h-2a_n+\sqrt{k_h(a)})}{\sqrt{1-h^2}((1+|a|^2-2a_n)\sqrt{k_h(a)}+C_h(a))}e_{n-1},$$

with $C_h(a):=k_h(a)-2(1-h)((1+|a|^2-2ha_n)a_n-2ha_{n-1}^2)$.

■

Figure 3 provides an example of Lemma 4.4.1 in $\mathbb{R}^3$, in order to show the pointwisely intertwining relation (4.26).

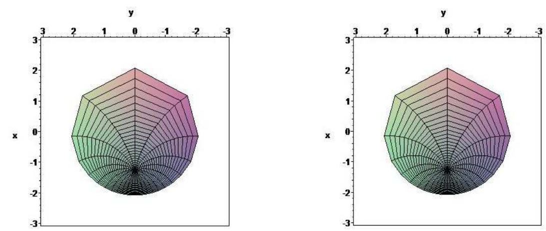


Figure 4.6: R.h.s.: $\Phi_1(s_{h,a}\ \varphi_a(\mathcal{U}_h)\overline{s_{h,a}})$, L.h.s.: $\widetilde{\varphi}_{\rho_h(a)}(\delta_h(a)\Phi_1(\mathcal{U}_h))$, for $h = \sqrt{3}/2$ and $a = (0, 1/2\sin(3\pi/4), \cos(3\pi/4))$.

We can easily generalize relation (4.26) for an arbitrary point $a \in B^n$ since only the position of the attractor point $\rho_h(a)$ is affected by a rotation. Indeed, considering $a = s_* a_* \overline{s_*}$, with $a_* = (0, \ldots, 0, a_{n-1}, a_n)$ and $s_* \in \mathrm{Spin}(n-1)$ we obtain the relation

$$\Phi_1(s_{h,a}\ \varphi_a(\mathcal{U}_h)\ \overline{s_{h,a}}) = \widetilde{\varphi}_{s_* \rho_h(a_*) \overline{s_*}}(\delta_h(a_*)\Phi_1(\mathcal{U}_h)).$$

If $a = te_n \in L_{e_n}$ the parameter $\delta_h(a)$ is independent of h since $\delta(te_n) = \frac{1+t}{1-t}$ and $\rho_h(te_n) = 0$. This shows that the anisotropic effect disappears whenever we consider the fundamental section.

The study of the parameters $\delta_h(a)$ and $\rho_h(a)$ is therefore, very important for our work. Figure 4.7 shows the behavior of $|\rho_h(r,\phi)|$ for $h = 0$, $r \in [0, 1[$ and $\phi \in [0, \pi]$.

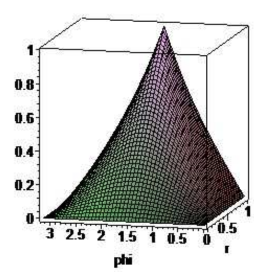


Figure 4.7: Behavior of the parameter $|\rho_h(r,\phi)|$ for $h = 0$.

For an arbitrary (global or local) left section we obtain the parameters

$$\delta_h(\sigma^l(te_n)) = \frac{(1+h)(1-t^2)(1-f(t)^2)}{(1+f(t)^2)(ht^2-2t+h)+\sqrt{K_h(t)}}, \tag{4.27}$$

and

$$\rho_h(\sigma^l(te_n)) = \frac{2(1-h)(1+t)|f(t)|(\sqrt{K_h(t)}+(1+f(t)^2)((1+t^2)h-2t))}{\sqrt{1-h^2}(1-t)((1+f(t)^2)\sqrt{K_h(t)}+C_h(t))}e_{n-1}, \tag{4.28}$$

with

$$\begin{aligned} K_h(t) &= (t^2-2ht+1)^2(1+f(t)^4)+2((2h^2-1)t^4-4ht(1+t^2)+6t^2-1+2h^2)f(t)^2; \\ C_h(t) &= (t^2-2ht+1)(1+f(t)^4)+2((2h-1)(1+t^2)+2(h-2)t)f(t)^2. \end{aligned}$$

For an admissible global section σ^l we have $\lim_{t\to -1^+}\delta_h(\sigma^l(te_n))=0$, $\lim_{t\to -1^+}\rho_h(\sigma^l(te_n))=0$, $\lim_{t\to 1^-}\delta_h(\sigma^l(te_n))=\infty$, and $\lim_{t\to 1^-}\rho_h(\sigma^l(te_n))=0$.

For an admissible local section of domain $V=]t_1,t_2[\subset L_{e_n}$ we have $\lim_{t\to t_1^+}\delta_h(\sigma^l(te_n))=0$ and $\lim_{t\to t_2^-}\delta_h(\sigma^l(te_n))=\infty$, since the section crosses first the contraction region. Regarding the behavior of the parameter $\rho_h(\sigma^l(te_n))$ we only know that such limits exist and belong to the interval $]-1,1[$.

We would like to measure the local anisotropy effect of the cap $\mathcal{U}_{h,a}$ directly on the sphere. The absolute value of the parameter $\rho_h(a)$ gives us information about the local anisotropy of a given section on the tangent plane. By the Unique Decomposition Theorem (Theorem 2.8.1) the subgroup L_{e_n} and the gyro-subgroup $D_{e_n}^{n-1}$ plays an important role as it was observed in Proposition 3.2.2. On the one hand, the subgroup L_{e_n} is associated with pure dilations around the North Pole. On the other hand, by Lemma 4.2.2 we know that Möbius transformations φ_a, with $a\in D_{e_n}^{n-1}$, belong to the conformal group of the hemisphere and thus, they will produce the desired anisotropic effects. As any spherical cap U_h can be mapped onto the hemisphere by an action of the Spin$(1,1)$ group, then any spherical cap $\widetilde{\mathcal{U}}_{h,a}$ can be described by Möbius transformations over L_{e_n} and $D_{e_n}^{n-1}$. The knowledge of the distance function (4.6) and the position of the attractor point $\mathcal{A}$ are therefore, essential in this description.

Proposition 4.4.2 *Let* $a=(0,\dots,0,a_{n-1},a_n)\in B^n$, $-1<h<1$,

$$\widetilde{b_n}=\begin{cases}\frac{1-\sqrt{1-h^2}}{h}, & h\neq 0\\ 0, & h=0\end{cases} \quad \text{and} \quad \widetilde{d_n}=\begin{cases}\frac{1-\sqrt{d_h(a)(2-d_h(a))}}{d_h(a)-1}, & d_h(a)\neq 1\\ 0, & d_h(a)=1\end{cases}.$$

Thus, the following relation holds

$$\widetilde{\mathcal{U}}_{h,a}=s_{h,a}\,\varphi_a(\mathcal{U}_h)\,\overline{s_{h,a}}=\varphi_{\widetilde{d_n}e_n}(\varphi_{-\rho_h(a)}(\varphi_{\widetilde{b_n}e_n}(\mathcal{U}_h))). \tag{4.29}$$

Proof: The proof will be made in three steps.

First step: transformation of the cap $\mathcal{U}_h$ onto the hemisphere $\mathcal{U}_0$.

From the study of the distance function (4.6) through the orbits of $B^n/(D^{n-1}_{e_n},\sim_l)$ it is easy to conclude that $\mathcal{U}_0=\varphi_{\widetilde{b_n}e_n}(\mathcal{U}_h)$.

Second step: transformation of the cap $\mathcal{U}_0$ onto the anisotropic cap $\widetilde{\mathcal{U}}_0$ by the action of an element of $D^{n-1}_{e_n}$.

We need to consider the stereographic projection mapping of S^{n-1} from the South Pole to the hyperplane at the origin which is given by

$$\begin{array}{lcl}\Phi_3: S^{n-1}\backslash\{e_n\}\to\mathbb{R}^{n-1} & \text{and} & \Phi_3^{-1}:\mathbb{R}^{n-1}\to S^{n-1}\backslash\{e_n\}\\ (x_1,\dots,x_n)\mapsto\left(\dfrac{x_1}{1+x_n},\dots,\dfrac{x_{n-1}}{1+x_n}\right) & & (y_1,\dots,y_{n-1})\mapsto\left(\dfrac{2y_1}{1+r^2},\dots,\dfrac{2y_{n-1}}{1+r^2},\dfrac{1-r^2}{1+r^2}\right).\end{array}$$

The mapping Φ_3 establishes a bijection between the unit ball $B_1(0)$ and the hemisphere S^{n-1}_+. On the one hand, for $a=a_{n-1}e_{n-1}$ we obtain the attractor point $\mathcal{A}=\left(0,\dots,0,\frac{-2a_{n-1}}{1+a^2_{n-1}},\frac{1-a^2_{n-1}}{1+a^2_{n-1}}\right)$. On the other hand, $\Phi_3^{-1}(\rho_h(a)e_{n-1})=\left(0,\dots,0,\frac{2\rho_h(a)}{1+\rho_h(a)^2},\frac{1-\rho_h(a)^2}{1+\rho_h(a)^2}\right)$. Therefore, we conclude that $\widetilde{\mathcal{U}}_0=\varphi_{-\rho_h(a)e_{n-1}}(\mathcal{U}_0)$.

Third step: transformation of the cap $\widetilde{\mathcal{U}}_0$ onto the anisotropic cap $\widetilde{\mathcal{U}_{h,a}}$ by an element of $\mathrm{Spin}(1,1)$.

For $a=d_ne_n$ and $h=0$ the distance function (4.6) becomes $d_0(d_ne_n)=\frac{(1+d_n)^2}{1+d_n^2}$. Thus, for $a=(0,\dots,0,a_{n-1},a_n)$, solving the equation $d_h(a)=\frac{(1+d_n)^2}{1+d_n^2}$ in order to d_n we obtain $d_n=\frac{1-\sqrt{d_h(a)(2-d_h(a))}}{d_h(a)-1}$. Therefore, we conclude that $\widetilde{U}_{h,a}=\varphi_{\widetilde{d_n}e_n}(\widetilde{\mathcal{U}}_0)$ and the proof is finished.

■

Our final conclusion is that the absolute value of the parameter $\rho_h(a)$ obtained on the tangent plane can also be used to measure the anisotropy of a given section on the unit sphere. Thus, we can define our concept of local anisotropy of an admissible section acting on a spherical cap $\mathcal{U}_h$.

Definition 4.4.3 *For $1 \leq p < \infty$ the local $p-$anisotropy of a left smooth section (global or local) $\sigma^l(te_n) = te_n \oplus f(t)e_{n-1}$, on a spherical cap $\mathcal{U}_h$ is defined by*

$$\epsilon^p_{f,h} := ||\rho_h(\sigma^l(te_n))||_{L^p([t_1,t_2])},$$

and for $p = \infty$ we define the local infinity anisotropy by $\epsilon^\infty_{f,h} := \sup_{t \in]t_1,t_2[} |\rho_h(\sigma^l(te_n))|$.

We remark that the concept of local anisotropy is not directly related with the definition of global anisotropy given in Chapter 3.

The results obtained in this chapter will help us significantly in the practical applications and implementations of the SCWT.

Chapter 5

Frames on the unit sphere S^2

As in the Euclidean case, the proposed wavelet transform has two aspects: the continuous one and the discrete one. In Chapter 3, we have obtained a reconstruction formula for the generalized SCWT but it did envolve inverting the operator A_ψ and so a simple discretization of (3.50) is not suitable.

Various alternative constructions of discrete spherical wavelets have been proposed. For example, spherical wavelets based on the lifting scheme were introduced by Schröder and Sweldens in [66]. They yield a multiresolution analysis based on a particular parametrization of the sphere. Also, in [37], W. Freeden defines a transformation on S^2 using a special dilation operator defined in the Fourier domain. Polynomial spherical frames have been introduced in [59], where the order of the polynomials plays the role of dilation.

The drawbacks of these methods are that they focus on the frequential aspects of the transformations. In consequence, the spatial localization of these wavelets is either not guaranteed or not precisely controlled.

In [14], T. Büllow did succeed in getting good localization properties by using the evolution of a spherical Gaussian governed by the heat equation on S^2. The set of wavelet filters is obtained by differentiation of the spherical Gaussian. However, this approach is restricted to the Gaussian function and thus it is not as general as the one based on a stereographic dilation applied to an arbitrary admissible wavelet on S^2.

Finally, in [11], I. Bogdanova constructed half-continuous, controlled frames and discrete spherical frames for the SCWT developed in [7].

In this chapter we aim to construct frames for the generalized SCWT and we propose an algorithm for the reconstruction of spherical signals on S^2 based on overlapping domain decomposition of S^2.

5.1 Frames

Let $\mathcal{H}$ be a separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and the associated norm $||f|| = \sqrt{\langle f, f\rangle}$, $f \in \mathcal{H}$.

Definition 5.1.1 *Let $\mathcal{I}$ be an index set. A family $\{\psi_i\}_{i\in\mathcal{I}}$ is a frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that for all $f \in \mathcal{H}$*

$$A||f||^2 \leq \sum_{i\in\mathcal{I}} |\langle\psi_i, f\rangle|^2 \leq B||f||^2, \quad \forall f \in \mathcal{H}. \tag{5.1}$$

The constants A and B are called the lower and upper bounds for the frame, respectively. Those sequences which satisfy only the upper inequality in (5.1) are called Bessel sequences. A frame is tight if $A = B$ and it is called a Parseval frame if $A = B = 1$. Moreover, if a Parseval frame satisfies $||\psi_i|| = 1, \forall i \in \mathcal{I}$, then it is an orthonormal basis. A frame is exact if it ceases to be a frame whenever any single element is deleted from the sequence $\{\psi_i\}_{i\in\mathcal{I}}$.

Given a frame $\{\psi_i\}_{i\in\mathcal{I}}$, we define the *synthesis operator* $T_\psi : l^2(\mathcal{I}) \to \mathcal{H}$ by

$$T_\psi(c) = \sum_{i\in\mathcal{I}} c_i\psi_i, \quad \forall\ \{c_i\}_{i\in\mathcal{I}} \in l^2(\mathcal{I}).$$

The adjoint operator $T_\psi^*(f) : \mathcal{H} \to l^2(\mathcal{I})$, also called *analysis operator*, is given by $T_\psi^*(f) = \{\langle\psi_i, f\rangle\}_{i\in\mathcal{I}}$. The frame operator $S_\psi : \mathcal{H} \to \mathcal{H}$, defined by

$$S_\psi(f) = T_\psi T_\psi^*(f) = \sum_{i\in I} \langle\psi_i, f\rangle\, \psi_i$$

is a bounded, invertible, and positive operator [27]. This provides us with the reconstruction formula

$$f = S_\psi^{-1} S_\psi(f) = \sum_{i\in\mathcal{I}} \langle f, \psi_i\rangle\, \widetilde{\psi}_i = \sum_{i\in\mathcal{I}} \left\langle f, \widetilde{\psi}_i \right\rangle \psi_i, \tag{5.2}$$

where $\widetilde{\psi}_i = S_\psi^{-1}\psi_i$. The family $\{\widetilde{\psi}_i\}_{i\in\mathcal{I}}$ is also a frame for $\mathcal{H}$, called the canonical dual frame of $\{\psi_i\}_{i\in\mathcal{I}}$. Thus, it is possible to reconstruct a function f from its frame coefficients $\langle f, \psi_i\rangle$ and to write f as a superposition of the ψ_i. Since the ψ_i are typically not linearly independent, there exist many different superpositions of the ψ_i, all adding up to f. This reflects the redundancy of a frame.

Given a frame $\{\psi_i\}_{i\in\mathcal{I}}$, the one thing we need to do, in order to apply (5.2), is to compute $\widetilde{\psi}_i = S_\psi^{-1}\psi_i, i \in \mathcal{I}$ From [27] we know that the frame operator satisfies $A\,\mathrm{Id} \leq S_\psi \leq B\,\mathrm{Id}$. If A and B are close to each other then we have the approximate reconstruction formula [27]

$$f \approx \frac{2}{A+B} \sum_{i\in\mathcal{I}} \langle f, \psi_i\rangle\, \psi_i. \tag{5.3}$$

If the frame is tight, i.e. $A = B$, then the reconstruction formula (5.3) is exact.

5.2 Half-continuous frames

In this first discretization we discretize the parameter scale of the generalized SCWT and we let the rotation parameter vary continuously. We choose the half-continuous grid $\Gamma = \{(s, \sigma^l(t_j)) : j \in \mathbb{Z}, t_j > t_{j+1}, s \in \text{Spin}(3)\}$.

In order to get a reconstruction of every function $f \in L^2(S^2)$, a first approach is to impose

$$A||f||^2 \leq \sum_{j\in\mathbb{Z}} \nu_j \int_{\text{Spin}(3)} |W_\psi[f](s, \sigma^l(t_j e_3))|^2 d\mu(s) \leq B||f||^2,$$

with $0 < A \leq B < \infty$ independent of f, and weights $\nu_j > 0$, arising from the chosen quadrature formula. Thus, the family

$$\Psi = \{\psi_{(s,\sigma^l(t_j e_3))} = R_s D_{\sigma^l(te_3)}\psi : s \in \text{Spin}(3), j \in \mathbb{Z}\}$$

constitutes a half-continuous frame for $L^2(S^2)$. A condition in terms of spherical harmonics is given in the following proposition.

Proposition 5.2.1 *Let ψ be an admissible wavelet for the generalized SCWT. If there are two constants $A, B \in \mathbb{R}^+$ such that*

$$A \leq \sum_{j\in\mathbb{Z}} \frac{8\pi^2}{2k+1} \nu_j \sum_{|m|\leq k} |\widehat{\psi}_{\sigma^l(t_j e_3)}(k,m)|^2 \leq B, \quad \textit{for all } k \in \mathbb{N}, \tag{5.4}$$

then

$$A||f||^2 \leq \sum_{j\in\mathbb{Z}} \nu_j \int_{\text{Spin}(3)} |W_\psi[f](s, \sigma^l(t_j e_3))|^2 d\mu(s) \leq B||f||^2. \tag{5.5}$$

Proof: The SCWT of a function $f \in L^2(S^2)$ in the Fourier domain is given by

$$W_\psi[f](s, \sigma^l(t_j e_3)) = \sum_{k\in\mathbb{N}} \sum_{|m|\leq l} \sum_{|n|\leq l} \overline{D^l_{mn}(s)} \overline{\widehat{\psi}_{\sigma^l(t_j e_3)}(l,m)} \widehat{f}(l,n) \tag{5.6}$$

Inserting (5.6) in (5.5) we obtain

$$\sum_{j\in\mathbb{Z}} \nu_j \int_{\text{Spin}(3)} |W_\psi[f](s, \sigma^l(t_j e_3))| d\mu(s) =$$

$$= \sum_{j\in\mathbb{Z}} \nu_j \sum_{k\in\mathbb{N}} \sum_{|m|\leq k} \sum_{|n|\leq k} \overline{\widehat{\psi}_{\sigma^l(t_j e_3)}(k,m)} \widehat{f}(k,n) \sum_{k'\in\mathbb{N}} \sum_{|m'|\leq k'} \sum_{|n'|\leq k'} \overline{\widehat{\psi}_{\sigma^l(t_j e_3)}(k',m') \widehat{f}(k',n')}$$

$$\int_{\text{Spin}(3)} \overline{D^k_{mn}(s)} D^{k'}_{m'n'}(s) d\mu(s)$$

$$= \sum_{j\in\mathbb{Z}} \nu_j \sum_{k\in\mathbb{N}} \sum_{|m|\leq k} \sum_{|n|\leq k} |\widehat{\psi}_{\sigma^l(t_j e_3)}(k,m)|^2 |\widehat{f}(k,n)|^2 \frac{8\pi^2}{2k+1}, \quad \text{by (3.3)}$$

$$= \sum_{k\in\mathbb{Z}} \sum_{|n|\leq k} |\widehat{f}(k,n)|^2 \sum_{j\in\mathbb{N}} \frac{8\pi^2}{2k+1} \nu_j \sum_{|m|\leq k} |\widehat{\psi}_{\sigma^l(t_j e_3)}(k,m)|^2$$

Due to (5.4), condition (5.5) is satisfied.

■

Remark 5.2.2 *Under the restriction to the fundamental section and if ψ is an axisymmetric (or zonal) wavelet (see [11]) then condition (5.4) simplifies to*

$$A \leq \frac{4\pi}{2k+1} \sum_{j \in \mathbb{Z}} \nu_j |\widehat{\psi}_j(k,0)|^2 \leq B, \quad \textit{for all } k \in \mathbb{N},$$

The investigations in [11] have disclosed that the frame obtained is not a tight frame in general. However, using the notion of controlled frames, the frame sequence can converge to a tight frame.

5.3 Discrete spherical frames

The construction of tight frames is in general a hard task and, therefore, another strategy must be found. We want to discretize completely the SCWT using our results from Chapter 4 and Daubechies's ideas in [27]. We will treat only the case of the SCWT based on the fundamental section.

Since we need to use rotations in $\mathbb{R}^3$, we will devote some attention to the use of quaternions to perform rotations.

5.3.1 Spherical geometry of rotations

In this section we will provide the basic properties of quaternions together with a description of the set of unit quaternions isomorphic to the group Spin(3).

It is well-known that a rotation g mapping the unit vector $x \in S^2$ onto the unit vector $y \in S^2$ according to $gx = y$ may be represented by its corresponding (3×3) orthogonal matrix $M(g)$ or by its corresponding real quaternion $q(g)$. The quaternionic representation has some clear advantages which have been discussed by many authors (see e.g. [54]). In our case it will lead us to an elegant description of the discretization of rotations for the spherical wavelet transform in S^2.

The algebra of real quaternions $\mathbb{H}$ is the 4-tuple of $\mathbb{R}^4$ endowed with the operation of quaternion multiplication. The basis elements of $\mathbb{H}$ are $1, i, j, k$, with 1 the unit element and i, j, k the imaginary units. The multiplication rules for the imaginary units are

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j.$$

For $q = q_0 + q_1 i + q_2 j + q_3 k$, $q_i \in \mathbb{R}$, $i = 1 \ldots 4$, the conjugate element $\overline{q}$ is given by $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$ and $q\overline{q} = \overline{q}q = q_0^2 + q_1^2 + q_2^2 + q_3^2 := |q|^2$, where $|q|^2$ denotes the Euclidean norm of q regarded as an element of the vector space $\mathbb{R}^4$. Each non-zero quaternion has a inverse given by $q^{-1} := \frac{\overline{q}}{|q|^2}$.

Let $\mathrm{Sc}\,(q) := q_0$ be the scalar part and $\underline{q} := q_1 i + q_2 j + q_3 k := \vec{(q)}$ be the vector part of the quaternion q. Given two quaternions $p, q \in \mathbb{H}$, their product according to the algebraic rules of multiplication is

$$pq = p_0 q_0 - \langle \underline{p}, \underline{q} \rangle + p_0 \underline{q} + q_0 \underline{p} + \underline{p} \times \underline{q},$$

where $\langle p, q \rangle$ and $p \times q$ represent the standard inner and cross products in $\mathbb{R}^3$. If $\mathrm{Sc}\,(q) = 0$ then q is called a pure quaternion. In this way, $\mathbb{R}^3$ can be embedded in $\mathbb{H}$. A quaternion q such that $|q|^2 = 1$ is called a unit quaternion. It is well-known that $\mathrm{Spin}(3) \simeq S^3$.

Definition 5.3.1 *Two quaternions $p, q \in \mathbb{H}$ are said to be orthogonal if $p\overline{q}$ is a pure quaternion.*

A unit quaternion admits the representation $q = \cos\frac{\alpha}{2} + \omega \sin\frac{\alpha}{2}$, where $\omega \in S^2$ and $\alpha \in [0, 2\pi[$.

Any active rotation $g \in SO(3)$ mapping the unit vector $x \in S^2$ onto the unit vector $y \in S^2$ according to $gx = y$ can be written in terms of its quaternion representation $q = q(g) \in \mathbb{H}$ as $qxq^{-1} = y$. For $q = \cos\frac{\alpha}{2} + \omega \sin\frac{\alpha}{2}$, where ω denotes the axis of rotation and α corresponds to the angle of rotation in turn of ω, the rotation $y = qxq^{-1}$ becomes

$$y = x \cos\alpha + (\omega \times x) \sin\alpha + (1 - \cos\alpha) \langle \omega, x \rangle \omega.$$

Definition 5.3.2 *Let q_1 and q_2 be two orthonormal quaternions. The set of quaternions*

$$q(t) = q_1 \cos t + q_2 \sin t, \ \ t \in [0, 2\pi[\tag{5.7}$$

is called a circle in $\mathbb{H}$ and is denoted by $C(q_1, q_2)$.

The next proposition characterizes the set of unit quaternions mapping a unit vector onto another one.

Proposition 5.3.3 *([58]) Given a pair of unit vectors $(x, y) \in S^2 \times S^2$ with $x \times y \neq 0$, the set of all rotations $gx = y$ may be represented as a circle $C(q_1, q_2)$ of unit quaternions such that $qx\overline{q} = y$, for all $q \in C(q_1, q_2)$, with*

$$q_1 := \frac{1 - yx}{||1 - yx||} = \cos\frac{\theta}{2} + \frac{x \times y}{||x \times y||} \sin\frac{\theta}{2}, \qquad q_2 := \frac{x + y}{||x + y||} \tag{5.8}$$

where θ denotes the angle between x and y, and

$$||1-yx|| = \sqrt{2(1+\cos\theta)} = 2\cos\frac{\theta}{2}, \qquad ||x+y|| = 2\cos\frac{\theta}{2}.$$

The case $y=-x$ must be considered separately. All rotations q with $qx\overline{q} = -x$ are provided by rotations with their axes in the orthogonal complement $h^{\perp}\cap S^2$ of x, which again represents a circle, and their angles are constantly equal to π. The inverse assertion to Proposition 5.3.3 is also true (see [58]).

Hence, the circle $C(q_1,q_2)$ consists of all quaternions $q(t), t\in[0,2\pi[$, with $q(t)x\overline{q(t)} = y$ for all $t\in[0,2\pi[$, and it is uniquely characterized by the pair $(x,y)\in S^2\times S^2$.

For any other $z\in S^2$ with $\langle x,z\rangle = \cos\rho$ the vector $q(t)z\overline{q(t)} =: r(t)\in S^2$ is not a constant unit vector, but it encloses the same angle ρ with y

$$\langle y, r(t)\rangle = \left\langle q(t)x\overline{q(t)}, q(t)z\overline{q(t)}\right\rangle = \langle x,z\rangle = \cos\rho, \quad \forall t\in[0,2\pi[.$$

Let $C(y,\rho)\subset S^2$ denotes the circle with center y and angle ρ given by $C(y,\rho) = \{r\in S^2 : \langle y,r\rangle = \cos\rho\}$.

Proposition 5.3.4 *([58]) The circle $C(q_1,q_2)$ represents rotations mapping the small circle $C(x,\rho)$ onto the small circle $C(y,\rho)$, i.e., for every $s(u)\in C(x,\rho)$ and $q(t)\in C(q_1,q_2)$ it holds*

$$q(t)s(u)\overline{q(t)} = r(u+2t), \ t,u\in[0,2\pi[. \tag{5.9}$$

These results are analogous to the split of $g\in SO(3)$ into $g=(\chi,[w'])$, with $\chi\in SO(2)$ and $w'\in S^2$, which is formally done through a projection $g\mapsto w'(g)$ in the fiber bundle $S^2\simeq SO(3)/SO(2)$, followed by an arbitrary choice of section $w'\mapsto [w']\in SO(3)$.

5.3.2 Discretization of the SCWT

As stated before, we consider only the discretization of the SCWT arising from the fundamental section. For the sake of simplicity and without loss of generality we consider our mother wavelet ψ defined on the hemisphere $\mathcal{U}_0$. For the discretization of the dilation parameter we will consider the sequence $t_n = \frac{3-a_0^n}{1+a_0^n}, n\in\mathbb{N}$, with $a_0>1$ fixed (usually $a_0=2$). It is a strictly decreasing sequence on the interval $]-1,1[$ and converges to $t=-1$. The discretization of the rotation parameter $s\in\mathrm{Spin}(3)$ is obtained in such a way that, for each t, the caps $s_i\mathcal{U}_{0,te_3}\overline{s_i}, i\in\mathcal{I}$ cover the whole sphere in a suitable way. This discretization is analogous to the discretization of the parameters of the CWT on the real line where we choose $a=a_0^m, m\in\mathbb{Z}$ for the discretization of the dilation parameter and we choose $b_0>0$

such that $\psi(x - nb_0)$ cover the whole real line [27]. However, on the real line we obtain a periodic discretization of the translation parameter, while on the sphere it is impossible to obtain an uniform covering with spherical caps due to the geometry of the sphere. We now explain our choice for the construction of a covering of the sphere S^2 with an arbitrary spherical cap $\mathcal{U}_{\cos\phi}$, (ϕ is the "spherical radius" of the cap $\mathcal{U}_{\cos\phi}$). We begin by fixing the cap $\mathcal{U}_{\cos\phi}$ centered at the North Pole. Then we add a first level of caps such that there is an overlapping between neighboring caps at least of spherical size $\phi' = C\phi$, controlled by the constant $0 < C < 2$ (see Figure 5.2). Our aiming is to make this overlapping almost uniform, by adjusting the caps on each horizontal plane. In order to avoid the existence of gaps we begin the third level fixing the first cap on the intersection of two caps of the previous level (see figure 5.1). We proceed in this way until we obtain a quasi-uniform covering of the sphere S^2 (see Figure 5.3).

Taking this into account, we discretize first the coordinate ϕ, thus obtaining the number of levels for the covering of the whole sphere. For each level we discretize the coordinate θ obtaining, in such a way, a quasi uniform covering of the sphere S^2 by means of spherical caps.

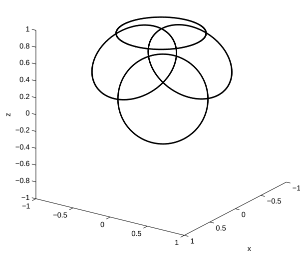


Figure 5.1: Discretization of rotations (with the cap $\mathcal{U}_{\cos(\pi/6)}$).

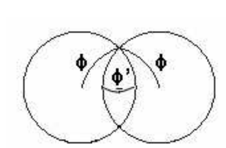


Figure 5.2: Overlapping of two caps $\mathcal{U}_{\cos\phi}$

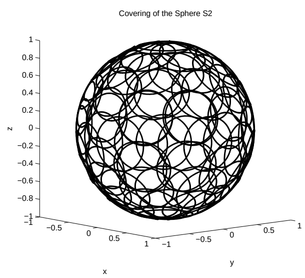


Figure 5.3: Covering of the sphere S^2 with the cap $\mathcal{U}_{\cos(\pi/10)}$.

5.4 Numerical examples

We now construct a numerical algorithm for performing spherical reconstruction of signals based on the following procedure.

Given a frame $\{\psi_{te_3,s}, te_3 \in \mathcal{I}_1, s \in \mathcal{I}_2\}$ we will calculate the coefficients $\left\langle f, \widetilde{\psi}_{t_1e_3,s_1}\right\rangle$ such that

$$\langle f, \psi_{t_2e_3,s_2}\rangle = \sum_{t_1\in\mathcal{I}_1}\sum_{s_1\in\mathcal{I}_2}\left\langle f, \widetilde{\psi}_{t_1e_3,s_1}\right\rangle \langle \psi_{t_1e_3,s_1}, \psi_{t_2e_3,s_2}\rangle\,, \quad \forall t_2 \in \mathcal{I}_1, \forall s_2 \in \mathcal{I}_2. \tag{5.10}$$

In matrix form we have to solve the linear system $AX = b$, where $A = (\langle \psi_{t_1e_3,s_1}, \psi_{t_2e_3,s_2}\rangle)$, $X = \left\langle f, \widetilde{\psi}_{t_1e_3,s_1}\right\rangle$ and $b = \langle f, \psi_{t_2e_3,s_2}\rangle$. The inner products $\langle \psi_{t_1e_3,s_1}, \psi_{t_2e_3,s_2}\rangle$ can be easily computed using the covariance of the SCWT under rotations and an appropriate quadrature rule (e.g. Gauss quadrature). If the Gramm matrix A is sparse and has good properties then the resolution of the system (5.10) can be easily solved by the conjugated gradient method for sparse matrices.

By the results of Subsection 5.3.1 we know how to factorize rotations and how to completely discretize the group Spin(3). If the wavelet ψ is axisymmetric then the discretization of Spin(3) is simpler since it only requires to consider pure quaternions of the form $q = \omega \in S^2$. In this case, the discretization of S^2 is just given by the quasi-uniform covering of S^2. Our numerical algorithm was constructed for axisymmetric wavelets although it can be adapted to non-axisymmetric wavelets. It can be summarized as follows.

1. Choice of the levels (discretization of the dilation parameter);

2. For each dilation level, covering of the unit sphere with the respective spherical cap (discretization of the rotation parameter);

3. Determination of the Gramm matrix $A = (\langle \psi_{t_1e_3,s_1}, \psi_{t_2e_3,s_2}\rangle)$;

4. Determination of the coefficients $b = \langle f, \psi_{t_2 e_3, s_2} \rangle$;

5. Computation of the dual coefficients $X = \left\langle f, \widetilde{\psi}_{t_1 e_3, s_1} \right\rangle$ by solving the linear system $AX = b$;

6. Application of the reconstruction formula

$$f = \sum_{t_1 \in \mathcal{I}_1} \sum_{s_1 \in \mathcal{I}_2} \left\langle f, \widetilde{\psi}_{t_1 e_3, s_1} \right\rangle \psi_{t_1 e_3, s_1}.$$

We now present numerical examples based on the spherical Mexican wavelet on the hemisphere which was obtained from the two dimensional Mexican wavelet $\psi(\vec{x}) = -2(-1 + 8|\vec{x}|^2) \exp(-8|\vec{x}|^2), \vec{x} \in \mathbb{R}^2$ (restricted to the unit disc) by inverse stereographic projection, i.e., $\psi(x, y, z) = \Theta^{-1}\psi(\vec{x}) = -\frac{4}{1+z}(-1 + 8\frac{1-z}{1+z}) \exp(-8\frac{1-z}{1+z}), z \geq 0$. Our signals are defined analytically, and they include a spherical triangle (signal 1), a circle inscribed in a spherical band (signal 2), a C^∞ function with compact support (signal 3), and a C^∞ global signal defined by $f(x, y, z) = \exp(x)z^3 + 10xy^2$ (signal 4).

In our approximation we used 24 levels in the discretization of the dilation parameter and the overlapping constant $C = 17/10$.

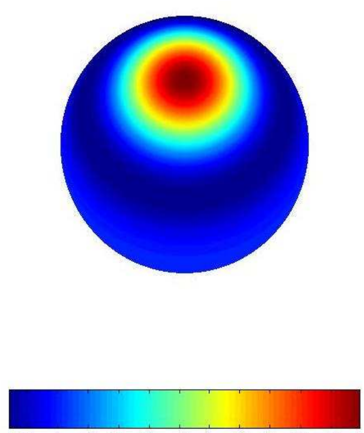


Figure 5.4: Spherical Mexican Hat on the hemisphere

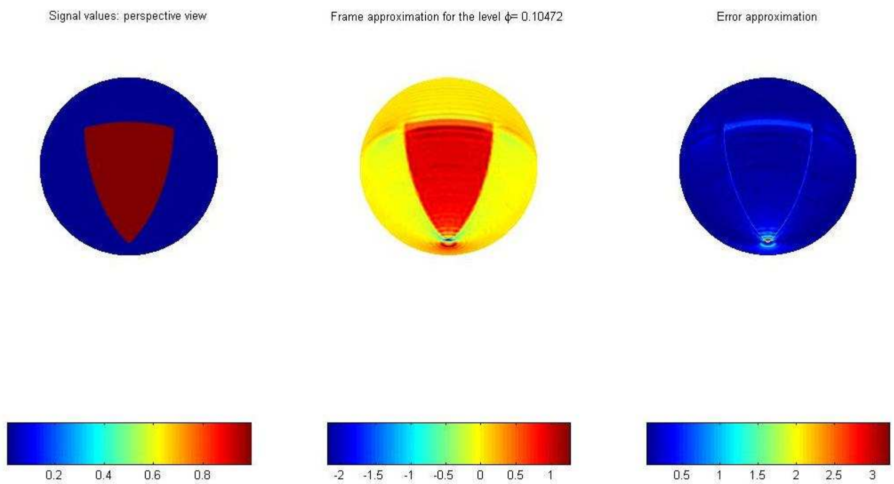


Figure 5.5: Signal 1

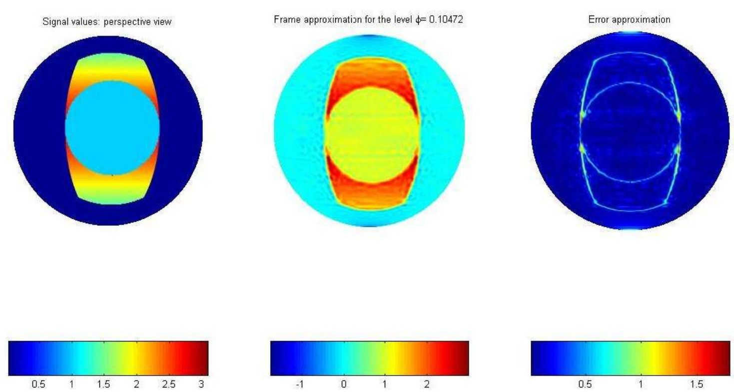


Figure 5.6: Signal 2

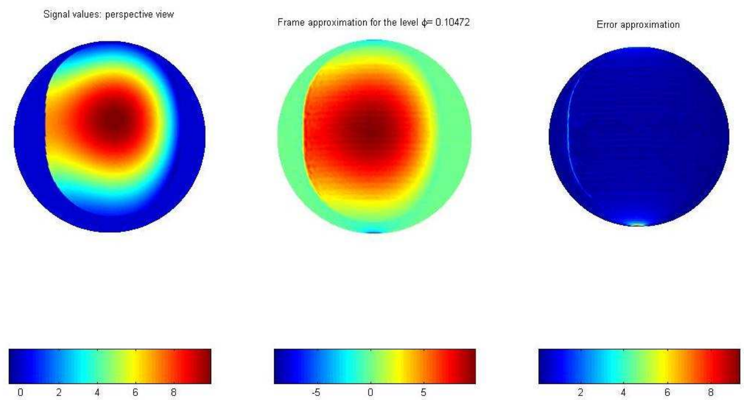


Figure 5.7: Signal 3

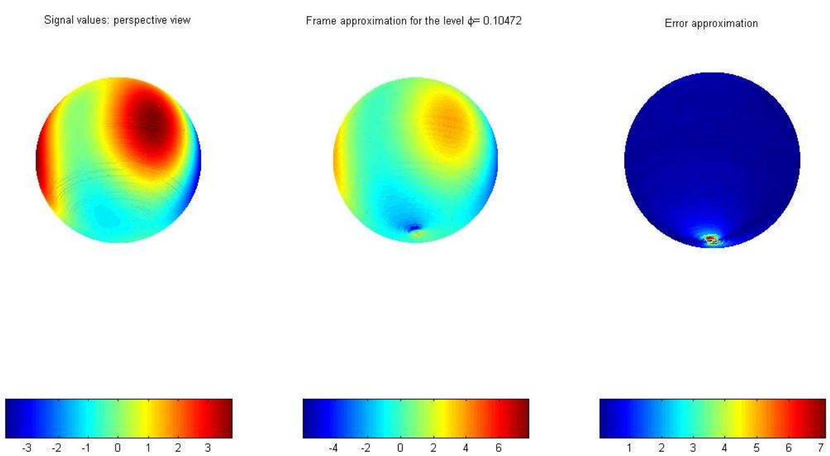


Figure 5.8: Signal 4

The numerical examples show that the proposed algorithm yields in fact the reconstruction of signals on the unit sphere. As a result of the proposed covering of S^2, the concentration of the frame elements near the Poles is different from the other points on the sphere. This yield a bad approximation near the Poles as we can see in our numerical examples. Being the sphere an homogeneous space all points must be equivalent. This suggests that the construction of other coverings of the unit sphere can yield better reconstruction. The overlapping constant depends on the choice of the mother wavelet and it influences also the reconstruction. Figure 5.9 illustrates the sparsity of the Gramm matrix A and Table 5.1 analyzes some of its properties.

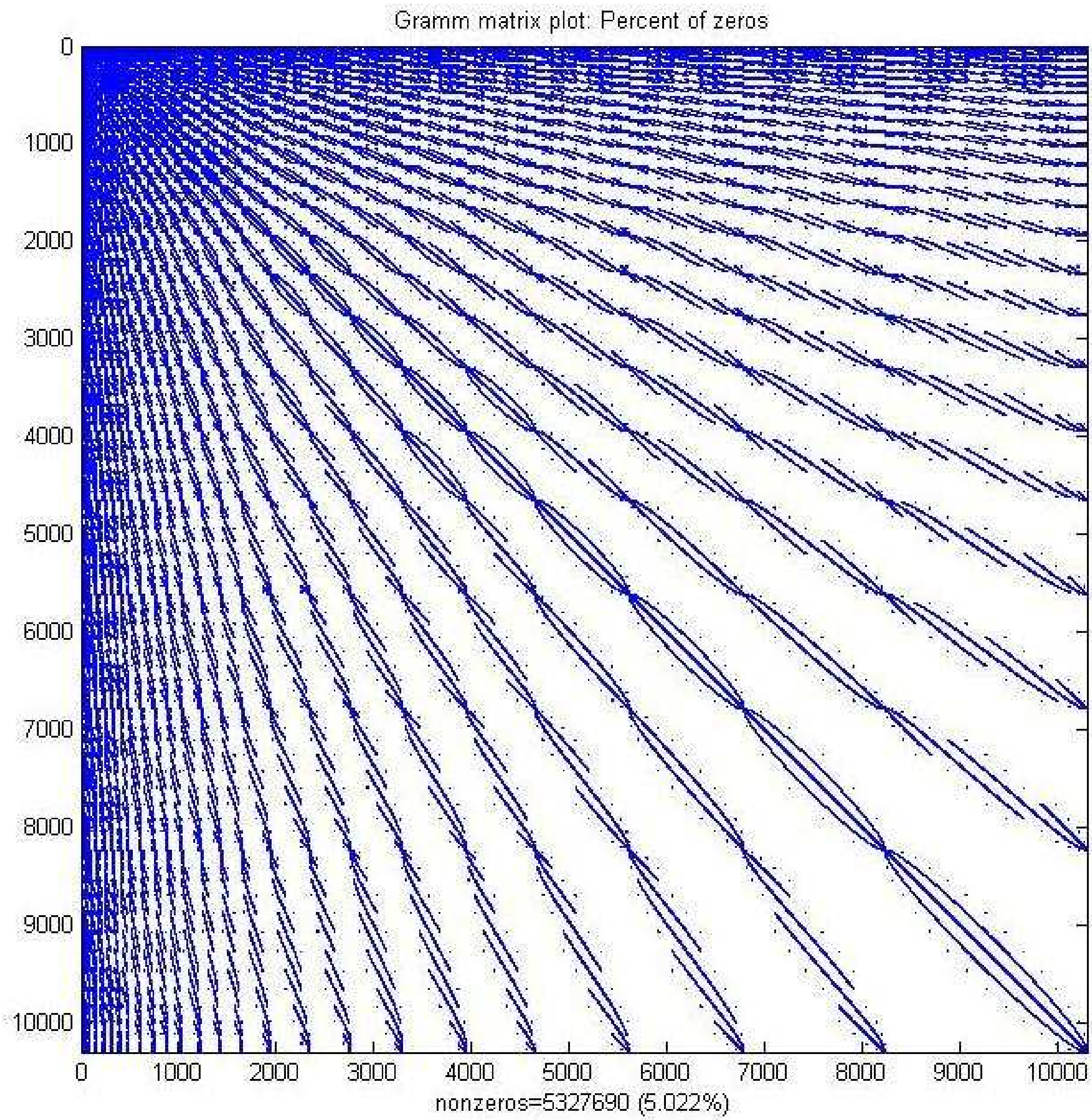


Figure 5.9: Gramm matrix

levels	ansatz	$\lambda_{\max}$	$\lambda_{\min}$	condition number	% non-zeros
1	42	2.5911	1.2062×10^{-1}	2.9599×10^{1}	56.80
2	86	4.8426	1.9443×10^{-2}	4.3508×10^{2}	54.16
3	160	6.3861	6.9223×10^{-4}	2.5378×10^{4}	46.33
4	238	8.0075	1.1343×10^{-5}	1.7880×10^{6}	43.36
5	319	9.2766	3.0747×10^{-6}	1.7247×10^{7}	40.87
6	402	10.5890	1.2607×10^{-6}	4.3155×10^{7}	38.96
7	489	11.8957	1.2883×10^{-6}	5.3389×10^{7}	36.79
8	617	12.6680	5.9112×10^{-7}	1.2917×10^{8}	33.49
9	749	13.8208	5.5416×10^{-7}	1.6902×10^{8}	31.06
10	887	14.9796	1.9118×10^{-8}	4.0688×10^{9}	28.96
11	1029	16.0416	5.7312×10^{-7}	1.6839×10^{8}	27.24
12	1224	17.0978	4.8172×10^{-7}	2.1537×10^{8}	24.97
13	1426	18.0343	4.3317×10^{-7}	2.6876×10^{8}	23.15
14	1644	18.8452	1.5468×10^{-7}	9.7697×10^{8}	21.43
15	1934	19.4115	8.9701×10^{-8}	1.4761×10^{9}	19.38
16	2336	19.9414	4.9531×10^{-8}	2.7267×10^{9}	16.84
17	2763	20.2160	1.2353×10^{-8}	1.0602×10^{10}	14.86
18	3293	20.8766	4.1274×10^{-8}	3.8142×10^{9}	13.05
19	3943	21.2636	1.4457×10^{-8}	1.0618×10^{10}	11.34
20	4647	21.5843	3.1488×10^{-8}	4.3199×10^{9}	9.94
21	5613	21.8381	—	—	8.52
22	6785	22.0304	—	—	7.28
23	8229	22.1681	—	—	6.15
24	10300	22.1730	—	—	5.02

Table 5.1: Study of the Gramm matrix: $\lambda_{\max}$—maximum eigenvalue, $\lambda_{\min}$—minimum eigenvalue.

In Table 5.1 we can see estimates for the upper and lower eigenvalues as well as for the condition numbers of our resulting matrices. Besides an initial grow we can observe that the upper and lower eigenvalue stabilizes around 22 and 10^{-8}, respectively. It shows clearly that we will not get a tight frame. Because of the resulting high condition number we use a preconditioned conjugated gradient (PCG) method for sparse matrices to solve the system. As expected, the sparsity of the matrix decreases with the number of levels.

Conclusion and Outlook

This dissertation connects wavelet analysis on the sphere with Clifford algebra theory. The results obtained allow us to conclude that Clifford algebraic techniques are a powerful tool in the study of this topic. This can be seen in Chapter 2 on the description of the conformal group of the unit sphere, the proper Lorentz group $\mathrm{Spin}^+(1,n)$. The algebraic structure of the unit ball leads us to a gyrogroup, which is a generalization of the notion of group. The representation theory for gyrogroups seems to be an open field to be explored which could give new insights for the continuous wavelet transform on some Euclidean manifolds, like the unit ball or the hemisphere (both having a conformal group).

The connection between gyrogroups and the SCWT is established in this thesis. The factorization of the gyrogroup of the unit ball by its gyro-subgroups allowed us to construct the appropriate homogeneous space for the development of the generalized SCWT, which depends on the choice of the section. Thus, we obtained anisotropic coherent states called spherical conformlets. In the anisotropic case we have a notion of directionality on the unit sphere, produced directly by the action of the group, since the spherical dilation operator $D_{\sigma^l(te_n)}$ is not restricted to pure dilations. Analogous constructions can be made on the plane since the conformal groups of the sphere and the plane are connected by the stereographic projection mapping. The properties of these states should be investigated in more detail. For example, the coorbit space theory developed by Feichtinger and Gröchenig, and recently generalized to quotient spaces need to be investigated in the case of the sphere. The coorbit spaces are defined as the collection of all functions for which the wavelet transform is contained in some weighted L_p-space. A judicious discretization of the representation produces the desired frames for the coorbit spaces. This approach works fine on the whole Euclidean plane and covers, e.g., the classical wavelet and Weil-Heisenberg frames. The generalization of the coorbit theory to homogeneous spaces were developed in a series of papers ([25], [26], [24]). In [25, 26], the authors constructed Gabor frames and modulation spaces on the sphere. The generalized SCWT allow us to obtain new smoothness spaces as it was done in [24]. By varying the specific Borel sections we can obtain some kind of mixed spaces. For example, the family of sections σ_λ^l or σ_c^l can generate a family of smoothness spaces analogous to the $\alpha-$modulation spaces developed in [24].

In Chapter 3 we have studied only left global smooth sections and we showed that they can give rise to SCWT. However, our model has the additional advantage that we can also study SCWT arising from local admissible sections. The introduced concept of anisotropy associated to a section characterizes it geometrically but further applications should be studied, like evolution and non-linear diffusion equations. The combination of sections with different characteristics can generate dictionaries of frames with interesting properties, which are suitable for practical applications of signal detection and spherical reconstruction. This shows the advantage of working with the whole of the conformal group of the sphere.

In Chapter 4 we studied the anisotropic conformal dilations obtained from the Möbius transformation φ_a. This study is very important for numerical applications but also from the theoretical point of view of hyperbolic geometry on the unit sphere.

In Chapter 5 we constructed an algorithm for reconstruction of spherical signals based on our group theoretical approach and we showed its practical feasibility with some academic examples. As we can see from the numerical examples presented, in general, we will not get a tight frame although it remains to be investigated if a different choice of our adaptive parameters or a different covering of the sphere S^2 can significantly lower the condition number of the Gramm matrix. We also would like to make a similar study for the stiffness matrix with applications to partial differential equations on the sphere in mind.

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