Universidade de Aveiro Departamento de Matemática
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MÉTODOS VARIACIONAIS E TOPOLÓGICOS

NONLINEAR ELLIPTIC EQUATIONS WITH NONSMOOTH POTENTIAL

VARIATIONAL AND TOPOLOGICAL METHODS

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica de Vasile Staicu, Professor Catedrático do Departamento de Matemática da Universidade de Aveiro e de Nikolaos S. Papageorgiou, Professor Catedrático do Departamento de Matemática da National Technical University, em Atenas, Grécia

Dissertation presented to University of Aveiro to fullfil the necessary requirements to the acquirement of the Ph.D. degree in Mathematics, performed under the scientific supervising of Professor Vasile Staicu, Full Professor of the Department of Mathematics of University of Aveiro, and Professor Nikolaos S. Papageorgiou, Full Professor of Department of Mathematics of National Technical University of Athens, Greece.

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Às três pessoas mais importantes na minha vida:
a minha mãe Helena, Mário e à memória do meu pai José

To the three most important people in my life:
my mother Helena, Mário and in memory of my father José

o júri

Presidente

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Professor Auxiliar do Departamento de Matemática da Universidade de Aveiro

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palavras-chave

Solução de sinal constante, potencial não suave, operador do tipo $(S)_+$, teoria de grau, subdiferencial generalizado, p-Laplaciano, operador diferencial assimptótico, função localmente lipschitziana, ressonância dupla, *local linking*, condição de Palais-Smale, *linking set*, condição de Ambrosetti-Rabinowitz, soluções múltiplas.

resumo

Nesta tese de doutoramento, estudamos a existência e a multiplicidade de soluções para algumas classes de equações elípticas não lineares com potencial não suave. Os resultados originais foram obtidos, utilizando métodos variacionais e da teoria de grau.

A nossa abordagem variacional é baseada em descobertas recentes na teoria não suave (nonsmooth) dos pontos críticos. A teoria de grau é aplicada a determinadas perturbações multívocas de operadores de tipo monótono (operadores do tipo $(S)_+$).

O primeiro problema que consideramos é um problema de valor próprio semi-linear com potencial não suave (ver Capítulo 3). O resultado de existência obtido estende para uma versão não suave, e sob hipóteses de crescimento mais fracas, um resultado obtido por Rabinowitz para potenciais suaves. Mais, sob condições no potencial que permitem ressonância, quer em zero, quer no infinito, provamos um resultado de multiplicidade.

Para um problema elíptico não linear derivado do p-Laplaciano e com um potencial não suave (ver Capítulo 4), estabelecemos a existência de, pelo menos, três soluções suaves, não triviais e distintas, sendo duas delas de sinal constante (uma positiva e uma negativa).

Problemas semi-lineares de Neumann, que são duplamente ressonantes na origem, relativamente a qualquer intervalo espectral $[\lambda_k,\lambda_{k+1}]$, são estudados no Capítulo 5. O resultado de multiplicidade obtido para um potencial não suave estende resultados existentes para o caso do potencial suave, nos quais a ressonância é completa relativamente a λ_k, mas incompleta relativamente a λ_{k+1}.

Respondemos afirmativamente à questão aberta em relação à validade do resultado de multiplicidade, quando ocorre, também, ressonância completa relativamente a λ_{k+1} (situação de dupla ressonância).

A última parte da tese (Capítulo 6) é dedicada ao estudo de uma classe de problemas de Neumann, em que o operador diferencial não é homogéneo, nem variacional. Portanto, os métodos mini-max da teoria dos pontos críticos (suave e não-suave) não podem ser utilizados. Usando o espectro do operador diferencial assimptótico, juntamente com métodos da teoria de grau, estabelecemos a existência de soluções suaves não triviais.

keywords

Constant sign solution, nonsmooth potential, (S)+ operator, degree theory, generalized subdifferential, p-Laplacian, asymptotic differential operator, locally Lipschitz function, double resonance, local linking, Palais-Smale condition, linking set, Ambrosetti-Rabinowitz condition, multiple solutions.

abstract

In this Ph.D. thesis, we study the existence and the multiplicity of solutions to some classes of nonlinear elliptic equations with a nonsmooth potential. Our new results were obtained by using variational and degree theoretic methods.

The variational approach we used is based on recent developments in nonsmooth critical point theory. The degree theory we used concerns certain multivalued perturbations of a class of monotone type operators (the $(S)_+$ type operators).

The first problem we consider is a semilinear eigenvalue problem with a nonsmooth potential (see Chapter 3). The existence result we obtained extends to nonsmooth setting and under weaker growth assumptions, a result obtained by Rabinowitz for smooth potentials. Moreover, under conditions on the potential which allow resonance both at zero and at infinity, we prove a multiplicity result.

For a nonlinear elliptic problem driven by the p-Laplacian and with a nonsmooth potential (see Chapter 4), we establish the existence of at least three distinct nontrivial smooth solutions, two of them with constant sign (one positive and one negative).

Semilinear Neumann problems which are doubly resonant at the origin with respect to any spectral interval $[\lambda_k,\lambda_{k+1}]$ were studied in Chapter 5.

The multiplicity result we obtained for nonsmooth potential, extend results known for the case of smooth potential, where the resonance is complete with respect to λ_k, but incomplete (nonuniform nonresonance) with respect to λ_{k+1}. We give a positive answer to an open question asking whether the multiplicity result also holds when complete resonance occurs also with respect to λ_{k+1} (double resonance situation).

The last part of the thesis (Chapter 6) is devoted to the study of a class of Neumann problems where the differential operator driving the problem is neither homogeneous, nor variational. So the minimax methods of critical point theory (smooth and nonsmooth alike) fail. Using the spectrum of the asymptotic differential operator together with degree theoretic methods, we establish the existence of nontrivial smooth solutions.

Contents

Chapter 1

Introduction

The goal of this thesis is to use variational methods and degree theoretical tecniques to study the existence and multiplicity of solutions to some nonlinear elliptic problems with nonsmooth potential (hemivariational inequalities).

While the foundation of variational inequalities, mainly concerned with convex energy functionals, is from Fichera, Lions and Stampacchia, and it dates back to the 1960's, hemivaritational inequalities were introduced by Panagiotopoulos about two decades ago and are closely related with the development of the new concept of Clarke's generalized gradient.

This new type of inequalities arise in mechanics and engineering, if one wants to consider more realistic laws of set-valued and nonmonotone nature, which correspond to nonsmooth and nonconvex energy functionals and their study requires tools and techniques from nonsmooth and multivalued analysis.

The variational method we use, in this thesis, is based on the nonsmooth critical point theory, which uses the subdifferential theory for locally Lipschitz functions. In Chapter 2 we present some basic definitions and facts from nonsmooth analysis (the subdifferential of Clarke) and from nonsmooth critical point theory. We also introduce degree theory for maps of monotone type and the spectrum of the Dirichlet and Neumann Laplacian and p-Laplacian, which will be used in the sequel.

In Chapter 3, we consider a semilinear eigenvalue problem with a nonsmooth potential. Eigenvalue problems for hemivariational inequalities have attracted the interest of many authors. We mention the works of Barletta-Marano [6], Cirstea-Radulescu [12], Goeleven-Motreanu-Panagiotopoulos [24], Marano-Molica Bisci-Motreanu [37], Motreanu-Panagiotopoulos [41], [42] (for semilinear problems) and by Gasinski-Papageorgiou [19],

[20], Motreanu-Motreanu-Papageorgiou [40], Motreanu-Radulescu [45] and Papageorgiou-Papageorgiou [47] (for quasilinear problems). The existence result presented in this chapter (proved in [48]) extends to nonsmooth setting the result obtained for smooth potential by Rabinowitz ([54], p.30). For an extension to hemivariational inequalities of Rabinowitz's result, we refer to Barletta-Marano [6]. The so-called Ambrosetti-Rabinowitz condition, assumed in both [54] and [6], dictates a superquadratic behavior for the potential function. They use a local version of this condition and employ an additional one. To obtain our result, we require only a local Ambrosetti-Rabinowitz and this makes our existence theorem (see Theorem 3.8), a more genuine nonsmooth generalization of Theorem 5.16, p.30, of Rabinowitz [54]. In fact, even when restricted to the smooth case, our result improves and refines the afforementioned theorem of Rabinowitz [54]. Moreover, under conditions on the potential function, which permit resonance both at zero and at infinity, we prove a multiplicity theorem (see Theorem 3.22). Our approach is variational, using minimax methods from nonsmooth critical point theory.

Following [50], in Chapter 4 we establish the existence of at least three distinct nontrivial smooth solutions for a nonlinear elliptic problem driven by the p-Laplacian and with a nonsmooth potential. Although, recently, three solutions theorems for the p-Laplacian equation with a smooth potential were proved by Liu [36] and [35], our conditions imply a different behavior of the generalized subdifferential near the origin. While the approach of the two quoted papers which is variational, based on critical point theory, we use degree theory for certain multivalued perturbations of nonlinear $(S)_+$-operators, due to Hu-Papageorgiou [25] (see also Hu-Papageorgiou [26], Section 4.4). Two of the three solutions obtained have a constant sign (one is positive and one is negative). We should mention the works of Alves-Ding [2], Garcia Azorero-Manfredi-Peral Alonso [18], where the authors examine eigenvalue problems driven by the partial p-Laplacian. In both works, they treat problems with concave-convex nonlinearities and exclude asymptotically (at $\pm\infty$) p-linear problems, which is the class of problems considered here.

A semilinear second order elliptic problem with Neumann boundary conditions and a nonsmooth potential is studied in Chapter 5.

Such a problem, but with a smooth potential, was recently studied by Tang-Wu in [59], where they proved a multiplicity result for problems which are resonant at zero between two successive eigenvalues λ_k, λ_{k+1}. The resonance is complete with respect to λ_k, but incomplete (nonuniform nonresonance) with respect to λ_{k+1}.

It was left as an open problem, whether their multiplicity result is actually valid when complete resonance occurs also with respect to λ_{k+1} (double resonance situation; see Remark 4 of Tang-Wu [59]).

The results presented in this chapter were obtained in [49]. We answer to the open problem of Tang-Wu [59] and prove a multiplicity result for semilinear Neumann problems which are doubly resonant at the origin with respect to any spectral interval $[\lambda_k, \lambda_{k+1}]$. We also relax the hypotheses of Tang-Wu [59] and we allow the potential function to be nonsmooth. Our approach is variational based on the nonsmooth critical point theory.

Existence theorems for semilinear resonant Neumann problems were proved by Iannacci-Nkashama [29], [30], Kuo [33], Mawhin-Ward-Willem [39] and Rabinowitz [55]. In the first three papers the approach is degree theoretic. Mawhin-Ward-Willem [39] use the monotonicity condition, while Rabinowitz [55] uses the periodicity condition. In both papers the approach is variational based on critical point theory.

In all the aforementioned works, with the exception of Iannacci-Nkashama [30], the resonance is with respect to the principal eigenvalue $\lambda_0 = 0$. None of these works deals with the doubly resonant situation and also they do not address the question of existence of multiple solutions.

The last problem, presented in Chapter 6, is a nonlinear Neumann problem with a non-smooth potential, we studied in [51]. In the last decade, nonlinear elliptic problems driven by the p-Laplacian differential operator have attracted a lot of interest. Most of the works focused on the Dirichlet with a smooth potential. The study of the corresponding Neumann problem is lagging behind. In this direction we mention the works of Anello-Cordaro [4], Arcoya-Orsina [5], Binding-Drabek-Huang [7], Faraci [16], Godoy-Gossez-Paczka [23], Huang [28] (problems with a smooth potential) and Filippakis-Gasinski-Papageorgiou [17], Hu-Papageorgiou [27], Marano-Motreanu [38], Motreanu-Papageorgiou [43], Papalini [52], [53] (problems with a nonsmooth potential). In all the aforementioned works the p-Laplacian differential operator, which is $(p-1)$-homogeneous, is used and so the Lusternik-Schnirelmann theory can be used to determine its spectral properties. Moreover, the operator is variational and so the methods of critical point theory can be used to obtain solutions of the boundary value problems. For this reason, in all the above works the approach is variational.

In contrast, the differential operator used in the problem we consider, is neither homogeneous nor variational. So the minimax methods of critical point theory (smooth and nonsmooth alike) fail and we need to device new techniques in order to deal with our problem. Using the spectrum of the asymptotic differential operator together with degree theoretic methods, based on the degree map for multivalued perturbations of $(S)_+$-operators, due to Hu-Papageorgiou [25] (see also Hu-Papageorgiou [26]) we are able to establish the existence of nontrivial smooth solutions.

Chapter 2

Preliminaries

Let X be a Banach space. As usual, $2^X \setminus \{\varnothing\}$ stands for the family of all nonempty subsets of X and $\overline{\Omega}$ denotes the closure of a set $\Omega \in 2^X \setminus \{\varnothing\}$. By X^* we denote its topological dual and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X). The norms in X or X^* will be denoted by $\|\cdot\|$.

Definition 2.1 *A function $\varphi : X \to \mathbb{R}$ is said to be* locally Lipschitz, *if for every $x \in X$ we can find an open set U containing x and a constant $k_U > 0$ (depending on U) such that*

$$|\varphi(y) - \varphi(z)| \leq k_U \|y - z\| \text{ for all } y,\, z \in U.$$

Definition 2.2 *We say that $\varphi : X \to \mathbb{R}$ is* coercive, *if $\varphi(x) \to +\infty$, as $\|x\| \to \infty$.*

The nonsmooth critical point theory which we employ in the variational arguments, is based mainly on the subdifferential theory for the locally Lipschitz functions. So, we recall some basic notions and facts from this theory. For further details we refer to the book of Clarke [13].

Definition 2.3 *The function $\varphi^0 : X \times X \to \mathbb{R}$ defined by*

$$\varphi^0(x; h) = \limsup_{x' \to x,\ \lambda \downarrow 0} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}$$

is called the generalized directional derivative *of φ. It is easy to check that $\varphi^0(x; \cdot)$ is continuous, sublinear and, so, it is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq X^*$, defined by*

$$\partial\varphi(x) = \left\{ x^* \in X^* : \langle x^*, h \rangle \leq \varphi^0(x; h) \text{ for all } h \in X \right\}, \text{ for all } x \in X.$$

The multifunction (or set-valued map) $x \to \partial\varphi(x)$ is known as the generalized subdifferential *of φ.*

If $\varphi \in C^1(X)$, then φ is locally Lipschitz and $\partial\varphi(x) = \{\varphi'(x)\}$, for all $x \in X$.

If φ is continuous and convex, then φ is locally Lipschitz and the generalized subdifferential coincides with the subdifferential in the sense of convex analysis, i.e.

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) \text{ for all } y \in X\}.$$

If $\varphi, \psi : X \to \mathbb{R}$ are two locally Lipschitz functions, then

$$\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x) \text{ for all } x \in X.$$

Definition 2.4 *We say that $x \in X$ is a* critical point *of a locally Lipschitz function $\varphi : X \to \mathbb{R}$ if*

$$0 \in \partial\varphi(x).$$

If $x \in X$ is a critical point then $c = \varphi(x)$ is called a critical value *of φ.*

It can be easily checked that, any local extremum of φ (i.e. a local maximum or a local minimum) is a critical point of φ.

The nonsmooth version of the *Palais-Smale condition* (*nonsmooth PS-condition* for short) takes the following form:

Definition 2.5 *A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the* nonsmooth PS-condition, *if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that*

$$\varphi(x_n) \to c \text{ (for some } c \in \mathbb{R}) \text{ and } m(x_n) := \inf\{\|x^*\| : x^* \in \partial\varphi(x_n)\} \to 0 \text{ as } n \to \infty,$$

has a strongly convergent subsequence.

Sometimes, it is convenient to use a weaker condition, known as the nonsmooth *Cerami condition* (*nonsmooth C-condition* for short), which has the following form:

Definition 2.6 *A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the* nonsmooth C-condition *if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that*

$$\varphi(x_n) \to c \text{ (for some } c \in \mathbb{R}) \text{ and } (1 + \|x_n\|)\, m(x_n) \to 0 \text{ as } n \to \infty,$$

has a strongly convergent subsequence.

If φ is bounded below, then the two notions are equivalent (see Gasisnki-Papageorgiou [21], p.127)

The topological notion of linking sets is crucial in the minimax characterization of the critical values of a locally Lipschitz function.

Definition 2.7 *Let X be a Hausdorff topological space, E_0, E and D are nonempty closed subsets of X, with $E_0 \subseteq E$. We say that the pair $\{E, E_0\}$* is linking with *D in X, if*

(a) $E_0 \cap D = \varnothing$;

(b) for any $\gamma \in C(E, X)$, with $\gamma|_{E_0} = id|_{E_0}$,

we have $\gamma(E) \cap D \neq \varnothing$.

Using this notion, we have the following general minimax principle for the critical values of a locally Lipschitz function (see Kourogenis-Papageorgiou [32]).

Theorem 2.8 *If X is a reflexive Banach space, $\varphi : X \to \mathbb{R}$ is locally Lipschitz which satisfies the nonsmooth C-condition, $\{E, E_0\}$ is linking with D in X, $\sup\limits_{E_0}\varphi \le \inf\limits_{D}\varphi$, and if*

$$c = \inf_{\gamma\in\Gamma}\sup_{x\in E}\varphi(\gamma(x)), \qquad \textit{with } \Gamma = \{\gamma \in C(E, X) : \gamma|_{E_0} = id|_{E_0}\},$$

then $c \ge \inf\limits_{D}\varphi$ and c is a critical value of φ, i.e. there exists a critical point $x_0 \in X$ of φ, such that $\varphi(x_0) = c$.

With suitable choices of linking sets, can be obtained nonsmooth versions of well-known minimax theorems (see [21], p.138). We only mention the *nonsmooth mountain pass theorem*, which we shall need in the sequel.

Theorem 2.9 *If x_0, $x_1 \in X$ with $\|x_1 - x_0\| > r > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} \le \inf\{\varphi(x) : \|x - x_0\| = r\}$$

and φ satisfies the nonsmooth PS-condition, with

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}\varphi(\gamma(t)), \textit{ where } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\},$$

then $c \ge \inf\{\varphi(x) : \|x - x_0\| = r\}$ and c is a critical value of φ.

Moreover, if $c = \inf\{\varphi(x) : \|x - x_0\| = r\}$, then there exists a critical point x_0 of φ, with $\varphi(x_0) = c$ and $\|x - x_0\| = r$.

More about the nonsmooth critical point theory can be found in the books of Carl-Le-Motreanu [10], Gasinski-Papageorgiou [20] and Motreanu-Radulescu [44].

Recently, Kandilakis-Kourogenis-Papageorgiou [31] (see also Gasinski-Papageorgiou [20], p.178), obtained the following nonsmooth version of the local linking theorem of Brezis-Nirenberg [8]:

Theorem 2.10 *If X is a Banach space, $X = Y \oplus V$, with $\dim Y < +\infty$, $\varphi : X \to \mathbb{R}$ is Lipschitz continuous on bounded sets, φ satisfies the nonsmooth PS-condition, $\varphi(0) = 0$, φ is bounded below, $\inf_X \varphi < 0$ and there exists $r > 0$, such that*

$$\begin{cases} \varphi(y) \leq 0 & \text{if } y \in Y,\ \|y\| \leq r \\ \varphi(v) \geq 0 & \text{if } v \in V,\ \|v\| \leq r \end{cases},$$

then φ has at least two nontrivial critical points.

In this dissertation we, also, employ degree theoretic techniques based on a degree map defined on a certain multivalued perturbations of nonlinear $(S)_+$-operators.

Degree theory is a basic tool of nonlinear analysis and produces powerful existence and multiplicity results for nonlinear boundary value problems. Such theory was introduced by Brouwer in 1912.

Let consider an operator equation of the form $\varphi(x) = y_0$, where φ is a map (often continuous) of $\overline{U}$, the closure of an open set U of the domain space X, into the range space Y and $y_0 \in Y$ satisfies $y_0 \notin \varphi(\partial U)$. Then the Brouwer degree of φ at y_0 relative to U, written $d_B(\varphi, U, y_0)$, is an algebraic count of the number of solutions of the equation $\varphi(x) = y_0$.

In particular the equation $\varphi(x) = y_0$ will have solutions in U, whenever $d_B(\varphi, U, y_0) \neq 0$.

In the next theorem we summarize the basic properties of Brouwer's degree.

Theorem 2.11 *If $U \subseteq \mathbb{R}^N$ is a bounded open set, $\varphi \in C(\overline{U}, \mathbb{R}^N)$ and $y \notin \varphi(\partial U)$ then:*

(i) (Normalization:) *$d_B(Id, U, y) = 1$ for all $y \in U$;*

(ii) (Aditivity with respect to domain:) *if U_1, U_2 are disjoint open subsets of U and $y \notin \varphi\left(\overline{U} \setminus (U_1 \cup U_2)\right)$, then*

$$d_B(\varphi, U, y) = d_B(\varphi, U_1, y) + d_B(\varphi, U_2, y);$$

(iii) (Homotopy invariance:) *if $h\colon [0,1]\times\overline{U} \longrightarrow \mathbb{R}^N$ is a continuous map and $y \notin h(t, \partial U)$ for all $t \in [0,1]$, then $d_B(h(t,\cdot), U, y)$ is independent of $t \in [0,1]$;*

(iv) (Dependence on the boundary values:) *if* $\widehat{\varphi} \in C(\overline{U}, \mathbb{R}^N)$ *and* $\varphi|_{\partial U} = \widehat{\varphi}|_{\partial U}$*, then*

$$d_B(\varphi, U, y) = d_B(\widehat{\varphi}, U, y);$$

(v) (Excision property:) *if* $K \subseteq \overline{U}$ *is closed and* $y \notin \varphi(K)$*, then*

$$d_B(\varphi, U, y) = d_B(\varphi, U \setminus K, y);$$

(vi) (Continuity with respect to φ:) *if* $\widehat{\varphi} \in C(\overline{U})$ *and* $\|\varphi - \widehat{\varphi}\|_\infty < d\,(y, \varphi(\partial U))$*, then* $d_B(\widehat{\varphi}, U, y)$ *is defined and equals* $d_B(\varphi, U, y)$*;*

(vii) (Existence property:) *if* $d_B(\varphi, U, y) \neq 0$ *then* $\varphi^{-1}(y) \neq \varnothing$*.*

Suppose now that X is a reflexive Banach space. Then by the Troyanski renorming theorem (see Gasinski-Papageorgiou [22], p.911), we can equivalently renorm X so that both X and X^* are locally uniformly convex and with Fréchet differentiable norms. So, in what follows, we assume that both X and X^* are locally uniformly convex. Hence, if $\mathcal{F} : X \to X^*$ is the duality map defined by

$$\mathcal{F}\,(x) = \left\{x^* \in X^* : \|x\|^2 = \|x^*\|^2\right\},$$

we have that $\mathcal{F}$ is a homeomorphism.

Definition 2.12 *An operator* $A : X \to X^*$*, which is single-valued and everywhere defined, is said to be* of type $(S)_+$*, if for every sequence* $\{x_n\}_{n\geq 1} \subseteq X$ *such that*

$$x_n \overset{w}{\to} x \text{ in } X \text{ and } \limsup_{n\to\infty} \langle A\,(x_n)\,, x_n - x\rangle \leq 0, \text{ one has } x_n \to x \text{ in } X.$$

Definition 2.13 *An operator* $A : X \to X^*$ *is* demicontinuous *if and only if*

$$x_n \to x \text{ in } X \text{ implies that } A\,(x_n) \overset{w^*}{\to} A\,(x) \text{ in } X^*.$$

Let U be a bounded open set in X and let $A : \overline{U} \to X^*$ be a demicontinuous operator of type $(S)_+$. Let $\{X_\alpha\}_{\alpha\in J}$ be the family of all finite dimensional subspaces of X and let A_α be the Galerkin approximation of A with respect to X_α, that is,

$$\langle A_\alpha\,(x)\,, y\rangle_{X_\alpha} = \langle A\,(x)\,, y\rangle \text{ for all } x \in \overline{U} \cap X_\alpha \text{ and all } y \in X_\alpha.$$

Definition 2.14 *For* $x^* \notin A\,(\partial U)$*, the degree map* $d_{(S)_+}\,(A, U, x^*)$ *is defined by*

$$d_{(S)_+}\,(A, U, x^*) = d_B\,(A_\alpha, U \cap X_\alpha, x^*)$$

for X_α large enough (in the sense of inclusion), where d_B stands for the classical Brouwer degree map (see [9]). If X is separable and A is bounded (maps bounded sets to bounded ones), then we can use only a countable subfamily $\{X_n\}_{n\geq 1}$ of $\{X_\alpha\}_{\alpha\in J}$ such that

$$\overline{\underset{n\geq 1}{\cup} X_n} = X.$$

For further details on the degree map $d_{(S)_+}$ we refer to Browder [9] and Skrypnik [57].

Definition 2.15 *A multifunction $G : X \to 2^{X^*}\setminus\{\varnothing\}$ is said to be* upper semicontinuous *(*usc *for short) if, for every set $C \subseteq X^*$, we have that*

$$G^-(C) = \{x \in X : G(x) \cap C \neq \varnothing\}$$

is closed in X.

Notice that the generalized subdifferential multifunction $x \to \partial\varphi(x)$ is usc from X with the norm topology into X^* furnished with the w^*-topology.

Definition 2.16 *We say that a multifunction $G : X \to 2^{X^*}\setminus\{\varnothing\}$* belongs in class (P), *if it is usc, for every $x \in X$, $G(x)$ is closed, convex and for every $A \subseteq X$ bounded, we have*

$$G(A) := \underset{x\in A}{\cup} G(x)$$

is relatively compact in X^*.

From Cellina [11] (see also Hu-Papageorgiou [26], p.106), we know that

Theorem 2.17 *If $D \subseteq X$ is an open set and if $G : D \to 2^{X^*}\setminus\{\varnothing\}$ is an usc multifunction with closed and convex values, then given $\varepsilon > 0$, we can find a continuous map $g_\varepsilon : D \to X^*$ (called approximate selection of G) such that*

$$g_\varepsilon(x) \in G((x + B_\varepsilon) \cap D) + B^*_\varepsilon, \text{ for all } x \in D$$

and $g_\varepsilon(D) \subseteq \overline{co}(G(D))$, with

$$B_\varepsilon = \{x \in X : \|x\| < \varepsilon\} \quad \text{and} \quad B^*_\varepsilon = \{x^* \in X^* : \|x^*\| < \varepsilon\}.$$

Note that, if the multifunction G belongs in class (P), then the continuous approximate selection g_ε is compact.

Definition 2.18 *If $G : X \to 2^{X^*} \setminus \{\varnothing\}$ is a multifunction belonging in class (P) ,then for every $x^* \notin (A+G)(\partial U)$, $\widehat{d}(A+G,U,x^*)$ is defined by*

$$\widehat{d}(A+G,U,x^*) = d_{(S)_+}(A+g_\varepsilon,U,x^*)$$

for $\varepsilon > 0$ small, where g_ε is the continuous approximate selector of G given by the previous Theorem.

Note that since G belongs in class (P), $g_\varepsilon : \overline{U} \to X^*$ is compact and so $x \longmapsto A(x)+g_\varepsilon(x)$ is still of type $(S)_+$.

More about the degree map $\widehat{d}$, can be found in Hu-Papageorgiou [25], [26].

In order to formulate the homotopy invariance property for the degree map $\widehat{d}$, we need to define the admissible homotopies for the operator A and the multifunction G.

Definition 2.19 *(a) A one-parameter family $\{A_t\}_{t\in[0,1]}$ of maps from $\overline{U}$ into X^* is said to be a* homotopy of class $(S)_+$, *if for any $\{x_n\}_{n\geq 1} \subseteq \overline{U}$ such that $x_n \xrightarrow{w} x$ in X and for any $\{t_n\}_{n\geq 1} \subseteq [0,1]$ with $t_n \to t$ for which*

$$\limsup_{n\to\infty} \langle A_{t_n}(x_n), x_n - x\rangle \leq 0,$$

one has $x_n \to x$ in X and $A_{t_n}(x_n) \xrightarrow{w} A_t(x_n)$ in X^* as $n \to \infty$.

(b) A one-parameter family $\{G_t\}_{t\in[0,1]}$ of multifunctions $G_t : \overline{U} \to 2^{X^*} \setminus \{\varnothing\}$ is said to be a homotopy of class (P), *if $(t,x) \longmapsto G_t(x)$ is usc from $[0,1] \times X$ into $2^{X^*} \setminus \{\varnothing\}$, for every $(t,x) \in [0,1] \times \overline{U}$ the set $G_t(x) \subseteq X^*$ is closed, convex and*

$$\overline{\cup \left\{G_t(x) : t \in [0,1],\ x \in \overline{U}\right\}}$$

is compact in X^*.

With these admissible homotopies for A and G, the homotopy invariance property of $\widehat{d}$ can be formulated as follows:

> "If $\{A_t\}_{t\in[0,1]}$ is a homotopy of class $(S)_+$ such that for every $t \in [0,1]$, A_t is bounded, $\{G_t\}_{t\in[0,1]}$ is a homotopy of class (P) and $x^* : [0,1] \to X^*$ is a continuous map such that
>
> $$x_t^* \notin (A_t + G_t)(\partial U) \text{ for all } t \in [0,1],$$
>
> then $\widehat{d}(A_t + G_t, U, x_t^*)$ is independent of $t \in [0,1]$."

Concerning the normalization property of the degree map, we have

$$\widehat{d}(\mathcal{F},U,x^*) = d_{(S)_+}(\mathcal{F},U,x^*) = 1 \text{ for all } x^* \in \mathcal{F}(U).$$

Both degree maps $d_{(S)_+}$ and $\widehat{d}$ exhibit all the usual properties mentioned for Brouwer's degree in Theorem 2.11, such as normalization, homotopy invariance, solution property, additivity with respect to the domain, excision property, etc.

Let now $\mathbb{R}^N$ be the usual N-dimensional Euclidean space, with norm $\|\cdot\|_{\mathbb{R}^N}$. Let $Z \subseteq \mathbb{R}^N$ be an open bounded set, with boundary ∂Z. If $u : Z \to \mathbb{R}$ then the ∇u stands for the gradient of u, Δu denotes the Laplacian of u and for $1 < p < \infty$, $\Delta_p u := div\left(\|\nabla u\|_{\mathbb{R}^N}^{p-2} \nabla u\right)$ is the p-Laplacian of u.

As usual, we denote by $L^p(Z)$, with $1 \leq p < \infty$, the space of measurable functions $u : Z \to \mathbb{R}$ such that $\|u\|_p := \left(\int_Z |u(z)|^p \, dz\right)^{\frac{1}{p}} < \infty$ and by $L^\infty(Z)$, we denote the space of measurable functions u, such that $|u(z)| \leq C$ a.e. in Z, with norm $\|u\|_\infty := \inf\{C \geq 0 : |u(z)| \leq C \text{ a.e. in } Z\}$. By $L^\infty(Z)_+$, we denote the subspace of $L^\infty(Z)$ of functions with strictly positive essential infimum.

The space of infinitely differentiable functions with compact support in Z (resp. $\mathbb{R}^N$) is denoted by $C_0^\infty(Z)$ (resp. $C_0^\infty(\mathbb{R}^N)$), while $C^k(\overline{Z})$ denotes the space of k-times continuously differentiable functions on $\overline{Z}$ (with ∂Z assumed smooth).

We also denote by $W_0^{1,p}(Z)$ (resp. $W^{1,p}(Z)$), the Sobolev space obtained by completion of $C_0^\infty(Z)$ (resp. $C^\infty(\overline{Z})$), with norm $\|u\| := \left(\int_Z [|\nabla u(z)|^p + |u(z)|^p] \, dz\right)^{\frac{1}{p}}$, $1 \leq p < \infty$. When $p = 2$, we write $H_0^1(Z)$ (resp. $H^1(Z)$), instead of $W_0^{1,2}(Z)$ (resp. $W^{1,2}(Z)$)

Finally, let us recall some basic facts about the spectrum of the negative Laplacian with Dirichlet boundary conditions. For details we refer, for example, to Gasinski-Papageorgiou [22]. Let $Z \subseteq \mathbb{R}^N$ be a bounded domain and consider the following weighted linear eigenvalue problem with weight $a \in L^\infty(Z)_+$

$$\begin{cases} -\Delta x(z) = \lambda a(z) x(z), & \text{a.e. on } Z \\ x|_{\partial Z} = 0 \end{cases}, \tag{2.1}$$

By an *eigenvalue* of (2.1), we mean a number $\lambda \in \mathbb{R}$ for which problem (2.1) has a nontrivial solution $u \in H_0^1(Z)$, which is an *eigenfunction* corresponding to $\lambda \in \mathbb{R}$. By $E(\lambda)$, we denote the eigenspace corresponding to the eigenvalue λ, that is, the linear subspace generated by the eigenfunctions corresponding to λ.

It is easy to see, that a necessary condition for $\lambda \in \mathbb{R}$ to be an eigenvalue, is that $\lambda \geq 0$. Note that $\lambda = 0$ is an eigenvalue and the corresponding eigenspace is $\mathbb{R}$ (the space of constant functions).

This problem has a sequence of distinct eigenvalues $\{\lambda_m\}_{m\geq 1}$, which are all positive, $\lambda_m < \lambda_{m+1}$, for all $m \geq 1$, $\lambda_m \to +\infty$ as $m \to \infty$ and $\lambda_1 > 0$ is simple (i.e., the corresponding eigenspace, $E(\lambda_1)$, is one-dimensional). If $\{u_n\}_{n\geq 1} \subseteq H_0^1(Z)$ are the eigenfunctions corresponding to these eigenvalues, then $u_n \in H_0^1(Z) \cap C^\infty(Z)$ and

$$\int_Z (Du_n, Du_k)_{\mathbb{R}^N}\, dz = 0, \int_Z a(z)\, u_n(z)\, u_k(z)\, dz = 0, \text{ for all } n \neq k.$$

Moreover, if ∂Z is a C^k-manifold ($1 \leq k \leq \infty$), then $u_n \in C^k(\overline{Z})$, for all $n \geq 1$. For every integer $m \geq 1$, by $E(\lambda_m)$ we denote the eigenspace corresponding to the eigenvalue λ_m. This space has the *unique continuation property*, namely if $u \in E(\lambda_m)$, then the set $\{z \in Z : u(z) = 0\}$ has an empty interior.

We set

$$Y_m = \overset{m}{\underset{i=1}{\oplus}} E(\lambda_i) \quad \text{and} \quad V_m = \overline{\underset{i\geq m}{\oplus} E(\lambda_i)}, \quad \text{for all } m \geq 1.$$

We have the following variational characterizations of the eigenvalues (using the so-called Rayleigh quotient):

$$\lambda_1 = \min\left[\frac{\|Dx\|_2^2}{\int_Z ax^2 dz} : x \in H_0^1(Z),\, x \neq 0\right] \tag{2.2}$$

and for $m \geq 2$,

$$\begin{aligned} \lambda_m &= \max_{x\in Y_m, x\neq 0} \frac{\|Dx\|_2^2}{\int_Z ax^2 dz} \\ &= \min_{x\in V_m,\, x\neq 0} \frac{\|Dx\|_2^2}{\int_Z ax^2 dz} \\ &= \min\left\{\max_{x\in Y,\, x\neq 0} \frac{\|Dx\|_2^2}{\int_Z ax^2 dz} : Y \subseteq H_0^1(Z),\ \dim Y = m\right\} \end{aligned} \tag{2.3}$$

We shall need the following simple facts about the component spaces Y_m and V_m.

Lemma 2.20 *(a) If $\theta \in L^\infty(Z)_+$, $\theta(z) \leq \lambda_m$ a.e. on Z and $\theta \neq \lambda_m$, then there exists $\xi_0 > 0$, such that*

$$\psi_0(x) = \|Dx\|_2^2 - \int_Z \theta(z)\, a(z)\, x(z)^2\, dz \geq \xi_0 \|Dx\|_2^2, \text{ for all } x \in V_m.$$

(*b*) If $\gamma \in L^{\infty}(Z)_{+}$, $\gamma(z) \geq \lambda_m$ a.e. on Z and $\gamma \neq \lambda_m$, then there exists $\xi_1 > 0$, such that

$$\psi_1(x) = \int_Z \gamma(z)\, a(z)\, x(z)^2\, dz - \|Dx\|_2^2 \geq \xi_1 \|Dx\|_2^2, \text{ for all } x \in Y_m.$$

Proof. (*a*) We proceed by contradiction. So suppose that the result is not true. Exploiting the 2-homogeneity of ψ_0, we can find $\{x_n\}_{n\geq 1} \subseteq V_m$, such that $\|Dx_n\|_2 = 1$, for all $n \geq 1$ and $\psi_0(x_n) \downarrow 0$. We may assume that

$$x_n \xrightarrow{w} x \text{ in } H_0^1(Z) \text{ and } x_n \to x \text{ in } L^2(Z).$$

Hence in the limit as $n \to \infty$, we obtain

$$\|Dx\|_2^2 \leq \int_Z \theta a x^2 dz \leq \lambda_m \int_Z a x^2 dz.$$

If $x = 0$, then $Dx_n \to 0$ in $L^2(Z, \mathbb{R}^N)$, a contradiction, since $\|Dx_n\|_2 = 1$, for all $n \geq 1$. So, $x \neq 0$ and $x \in V_m$. Hence, by virtue of (2.3), we have $\|Dx\|_2^2 = \lambda_m \|x\|_2^2$ and so $x \in E(\lambda_m)$. Because $E(\lambda_m)$ has the unique continuation property, it follows that $x(z) \neq 0$ a.e. on Z. Therefore

$$\|Dx\|_2^2 < \lambda_m \int_Z a x^2 dz,$$

a contradicition to (2.3).

(*b*) The proof of this part is done similarly as (*a*) and so it is ommitted.

■

For the spectrum of the negative Laplacian with Neumann boundary conditions (i.e. of $(-\triangle, H^1(Z))$), we consider the following linear eigenvalue problem

$$\begin{cases} -\triangle x(z) = \lambda x(z) & \text{a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 & \text{on } \partial Z,\ \lambda \in \mathbb{R}. \end{cases} \tag{2.4}$$

Using the spectral theorem for compact self-adjoint operators on a Hilbert space (see Gasisnki-Papageorgiou [22], p.296), we show that (2.4) has a sequence $\{\lambda_k\}_{k\geq 0}$ of distinct eigenvalues, $\lambda_k \to +\infty$ as $k \to +\infty$ and $\lambda_0 = 0$. We can, also, choose a corresponding sequence of eigenfunctions, which form an orthonormal basis for $L^2(Z)$ and an orthogonal basis for $H^1(Z)$. If $E(\lambda_k)$ denotes the eigenspace corresponding to the eigenvalue λ_k, $k \geq 0$, then we have the orthogonal direct sum decomposition

$$H^1(Z) = \underset{k\geq 0}{\oplus} E(\lambda_k).$$

Moreover, we have the following variational characterizations of the eigenvalues

$$\lambda_k = \min\left\{\frac{\|Dx\|_2^2}{\|x\|_2^2} : x \in \overline{\underset{i\geq k}{\oplus} E(\lambda_i)}, x \neq 0\right\} \tag{2.5}$$
$$= \max\left\{\frac{\|Dx\|_2^2}{\|x\|_2^2} : x \in \overset{k}{\underset{i=0}{\oplus}} E(\lambda_i), x \neq 0\right\}.$$

By regularity theory (see, e.g., [21], p. 112), we have that every eigenfunction $u \in H^1(Z)$ belongs in $C^1(\overline{Z})$. Moreover, the eigenfunctions for λ_k, $k \geq 1$, change sign (nodal functions).

Next we turn our attention to the spectrum of the negative partial p-Laplacian with Dirichlet boundary conditions (i.e. of $\left(-\triangle_p, W_0^{1,p}(Z)\right)$,with $\triangle_p$ denoting the p-Laplacian). So we consider the following nonlinear weighted (with weight m) eigenvalue problem, with $m \in L^\infty(Z)_+$, $m \neq 0$

$$\begin{cases} -div\left(\|Du(z)\|^{p-2} Du(z)\right) = \widehat{\lambda} m(z) |u(z)|^{p-2} u(z) \text{ a.e. on } Z \\ u|_{\partial Z} = 0, \ \widehat{\lambda} \in \mathbb{R} \end{cases} \tag{2.6}$$

From nonlinear regularity theory (see for example Gasinski-Papageorgiou [22], p.737-738), we have that every eigenfunction u belongs to $C_0^1(\overline{Z})$.

Problem (2.6) has a smallest eigenvalue denoted by $\widehat{\lambda}_1(m)$, which is positive, isolated and simple. Moreover, $\widehat{\lambda}_1(m) > 0$ admits the following variational characterization

$$\widehat{\lambda}_1(m) = \inf\left\{\frac{\|Du\|_p^p}{\int_Z m|u|^p dz} : u \in W_0^{1,p}(Z), \ u \neq 0\right\}. \tag{2.7}$$

In (2.7) the infimum is actually realized at a corresponding eigenfunction $u_1 \in C_0^1(\overline{Z})$. Note that if u_1 is a solution of the minimization problem (2.7), then so does $|u_1|$ and so we may assume that $u_1(z) \geq 0$ for all $z \in \overline{Z}$. In fact invoking the strict maximum principle of Vasquez [60], we have

$$u_1(z) > 0 \text{ for all } z \in Z \text{ and } \frac{\partial u_1}{\partial n}(z) < 0 \text{ for all } z \in \partial Z. \tag{2.8}$$

If m, $m' \in L^\infty(Z)_+$, $0 \leq m(z) \leq m'(z)$ a.e. on Z with strict inequalities on sets (not necessarily the same) of positive measure, then $\widehat{\lambda}_1(m') < \widehat{\lambda}_1(m)$.

If $m \equiv 1$, then we write $\lambda_1 = \widehat{\lambda}_1(1)$.

Finally if $u \in W_0^{1,p}(Z)$ is an eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(m)$, then $u \in C_0^1(\overline{Z})$ must change sign.

The Banach space $C_0^1\left(\overline{Z}\right)=\left\{x\in C^1\left(\overline{Z}\right):x\mid_{\partial Z}=0\right\}$, is an ordered Banach space with order cone

$$C_0^1\left(\overline{Z}\right)_+=\left\{x\in C_0^1\left(\overline{Z}\right):x\left(z\right)\geq 0\text{ for all }z\in\overline{Z}\right\}.$$

This order cone has a nonempty interior, given by

$$int\ C_0^1\left(\overline{Z}\right)_+=\left\{x\in C_0^1\left(\overline{Z}\right):x\left(z\right)>0\ \forall z\in Z\text{ and }\frac{\partial x}{\partial n}\left(z\right)<0\ \forall z\in\partial Z\right\}.\qquad(2.9)$$

Note that from (2.8) and (2.9), we infer that $u_1\in int\ C_0^1\left(\overline{Z}\right)_+$.

Chapter 3

Eigenvalue Problems for Semilinear Hemivariational Inequalities

3.1 Introduction

Let $Z \subseteq \mathbb{R}^n$ be a bounded domain with a C^2-boundary ∂Z. The goal of this chapter is to study the following semilinear eigenvalue problem with a nonsmooth potential

$$\begin{cases} -\Delta x(z) - \lambda a(z) x(z) \in \partial j(z, x(z)) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0. \end{cases} \tag{3.1}$$

Here $a \in L^{\infty}(Z)_{+}$ is a function with strictly positive essential infimum, $j(z,x)$ is a measurable potential function which is locally lipschitz and, in general, nonsmooth in the $x \in \mathbb{R}$ variable and $\partial j(z,x)$ is the generalized subdifferential of $j(z,\cdot)$ (see Definition 2.3). Such problems are known as *hemivariational inequalities.* For concrete applications, we refer to the book of Naniewicz-Panagiotopoulos [46]. Their study requires tools and techniques from nonsmooth and multivalued analysis.

In his work, Rabinowitz [54], p.30, assumed that $j \in C^1(\overline{Z}, \mathbb{R})$ and that $f(z,x) = \partial j(z,x)$ satisfies the sign condition

$$xf(z,x) \geq 0 \text{ for all } (z,x) \in \overline{Z} \times \mathbb{R}.$$

The work of Rabinowitz was extended to hemivariational inequalities by Barletta-Marano [6]. Both works assume the so-called Ambrosetti-Rabinowitz condition (AR-condition, for short), which dictates a superquadratic behavior for the potential $x \to j(z,x)$. They use a local version of the AR-condition (i.e. it is valid only for $|x|$ large). Barletta-Marano [6], also employ an additional condition (see (j_5) in [6]). Here, we use only a local AR-condition (see Hypothesis $H(j)_1(v)$) and this makes our existence theorem (see Theorem 3.8), a more genuine nonsmooth generalization of Theorem 5.16, p.30, of Rabinowitz [54]. In fact, even when restricted to the smooth case, our result improves and refines the afforementioned theorem of Rabinowitz [54]. Moreover, for the scalar problem (i.e., $N = 1$, hence (3.1) becomes an ordinary differential inclusion), we are able to replace the AR-condition by a weaker one (see hypothesis $H(j)_2(vi)$). Finally, when $\lambda = \lambda_1$, with $\lambda_1 > 0$ being the principal (first) eigenvalue of $(-\Delta, H_0^1(Z), a)$ ($a \in L^\infty(Z)_+$ being a weight function), under conditions on the potential function $j(z,\cdot)$, which permit resonance both at zero and at infinity, we prove a multiplicity theorem (see Theorem 3.22).

Our approach is variational, using minimax methods from nonsmooth critical point theory.

3.2 Existence of Solutions

The hypotheses on the nonsmooth potential $j(z,x)$ are the following:

$H(j)_1:$ $j : Z \times \mathbb{R} \to \mathbb{R}_+$ is a function such that $j(z,0) = 0$ a.e. on Z and

(*i*) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(*ii*) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(*iii*) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have

$$|u| \leq a_0(z) + c_0 |x|^{r-1},$$

with $a_0 \in L^\infty(Z)_+$, $c_0 > 0$ and $2 < r < 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } n > 2 \\ +\infty & \text{if } n = 1,\ 2 \end{cases}$;

(*iv*) $\lim_{x \to 0} \frac{j(z,x)}{x^2} = 0$ uniformly for a.a. $z \in Z$;

(*v*) there exist $\mu > 2$ and $M > 0$, such that for almost all $z \in Z$ and all $|x| \geq M$

$$0 < \widehat{c} \leq \mu j(z,x) \leq -j^0(z,x;-x).$$

Remark 3.1 *Hypothesis $H(j)_1(v)$ is a nonsmooth analog of the local AR-condition. In Lemma 3.3 below, we show that this hypothesis implies the superquadratic growth $x \to j(z,x)$.*

Example 3.2 *The following functions satisfy hypotheses $H(j)_1$. In what follows $a_0 \in L^{\infty}(Z)_+$.*

$$j_1(z,x) = \begin{cases} \frac{a_0(z)}{3}|x|^3 & \text{if } |x| \leq 1 \\ \frac{1}{\mu}|x|^{\mu} - x^2 \ln|x| - \frac{1}{\mu} + \frac{a_0(z)}{3} & \text{if } |x| > 1 \end{cases}, \ \mu > 2,$$

$$j_2(z,x) = \begin{cases} \frac{a_0(z)}{2}x^2 \ln(|x|+1) & \text{if } |x| \leq 1 \\ \frac{1}{\mu}|x|^{\mu} - \frac{1}{\mu} + \frac{a_0(z)}{2}\ln 2 & \text{if } |x| > 1 \end{cases}, \ \mu > 2, \quad \text{and}$$

$$j_3(z,x) = \frac{a_0(z)}{\mu}|x|^{\mu} + \frac{1}{2}x^2 \ln(|x|+1), \ \mu > 2.$$

Note that $j_2(z,x)$ does not satisfy the hypotheses of Barletta-Marano [6], who assume the global AR-condition. Also note that $j_3(z,x)$ is a C^1 function with respect to x.

We start with a simple lemma, which highlights the consequences of the nonsmooth AR-condition (see hypothesis $H(j)_1(v)$).

Lemma 3.3 *If $j_0 : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and we can find $\mu > 2$ and $M > 0$, such that*

$$0 < \mu j_0(x) \leq -(j_0)^0(x; -x), \text{ for all } |x| \geq M,$$

then there exist $c_1, c_2 > 0$, such that

$$c_1 |x|^{\mu} - c_2 \leq j_0(x), \text{ for all } x \in \mathbb{R}.$$

Proof. Let $x \in \mathbb{R}$, with $|x| \geq M$ and consider the function $\xi_0 : [1, \infty) \to \mathbb{R}_+$ defined by

$$\xi_0(r) = j_0(rx), r \geq 1.$$

Evidently ξ_0 is locally Lipschitz. From the nonsmooth chain rule (see Clarke [13], p.32), we have

$$\partial_r(\xi_0(r)) \subseteq x \partial j_0(rx). \tag{3.2}$$

The function $r \to \xi_0(r)$ is differentiable almost everywhere and if $r \in [1, \infty)$ is such a point of differentiability of $\xi_0(\cdot)$, we have $\frac{d}{dr}\xi_0(r) \in \partial \xi_0(r)$ (see Clarke [13], p.32). So, from (3.2) and the hypothesis of the Lemma, we have

$$\mu \xi_0(r) = \mu j_0(rx) \leq -(j_0)^0(rx; -rx) \leq r \frac{d}{dr}\xi_0(r), \text{ for a.a. } r \geq 1,$$

hence

$$\frac{\mu}{r} \leq \frac{\frac{d}{dr}\xi_0(r)}{\xi_0(r)}, \text{ for a.a. r} \geq 1.$$

Integrating this inequality from 1 to $r \geq 1$, we obtain

$$\ln r^{\mu} \leq \ln \frac{\xi_0(r)}{\xi_0(1)},$$

so

$$r^{\mu}\xi_0(1) \leq \xi_0(r),$$

therefore

$$r^{\mu} j_0(x) \leq j_0(rx), \text{ for all } r \geq 1 \text{ and all } |x| \geq M. \tag{3.3}$$

Then, for all $|x| \geq M$, we have

$$\begin{aligned} j_0(x) &= j_0\left(\frac{|x|}{M} M \frac{x}{|x|}\right) \geq \frac{|x|^{\mu}}{M^{\mu}} j_0\left(\frac{Mx}{|x|}\right) \quad (\text{see } (3.3)) \\ &\geq \frac{|x|^{\mu}}{M^{\mu}} \min\{j_0(M), j_0(-M)\} = c_1 |x|^{\mu} \end{aligned} \tag{3.4}$$

for some $c_1 > 0$.

On the other hand, if $|x| < M$, then we can find $\widehat{c}_1 > 0$, such that

$$|j_0(x)| \leq \widehat{c}_1. \tag{3.5}$$

From (3.4) and (3.5) it follows that

$$j_0(x) \geq c_1 |x|^{\mu} - c_2,$$

for all $x \in \mathbb{R}$, with $c_2 = \widehat{c}_1 + c_1 M^{\mu} > 0$. ■

Let $\varphi_\lambda : H_0^1(Z) \to \mathbb{R}$ be the Euler functional for problem (3.1), defined by

$$\varphi_\lambda(x) = \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda}{2}\int_Z a(z)\, x(z)^2\, dz - \int_Z j(z, x(z))\, dz$$

for all $x \in H_0^1(Z)$.

We know that φ_λ is Lipchitz continuous on bounded sets, hence it is locally Lipschitz (see Clarke [13], p.83).

Proposition 3.4 *If hypotheses $H(j)_1$ hold and $\lambda_k \leq \lambda < \lambda_{k+1}$ for some $k \geq 1$, then φ_λ satisfies the nonsmooth PS-condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ be a sequence such that

$$|\varphi_\lambda(x_n)| \leq M_1, \tag{3.6}$$

for some $M_1 > 0$, all $n \geq 1$ and

$$m_\lambda(x_n) = \inf\left[\|x^*\| : x^* \in \partial\varphi_\lambda(x_n)\right] \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Since $\partial\varphi_\lambda(x_n) \subseteq H^{-1}(Z) = H_0^1(Z)^*$ is nonempty and weakly compact and in any Banach space, the norm functional is weakly lower semicontinuous, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi_\lambda(x_n)$ such that $m_\lambda(x_n) = \|x_n^*\|$ for all $n \geq 1$. Let $A \in \mathcal{L}(H_0^1(Z), H^{-1}(Z))$ be defined by

$$\langle A(x), y\rangle = \int_Z (Dx, Dy)_{\mathbb{R}^N}\, dz, \text{ for all } x,\, y \in H_0^1(Z).$$

We have

$$x_n^* = A(x_n) - \lambda a x_n - u_n, \text{ with } u_n \in N(x_n), \tag{3.8}$$

where

$$N(v) = \left\{u \in L^{r'}(Z) : u(z) \in \partial j(z, v(z)) \text{ a.e. on } Z\right\},$$

for all $v \in H_0^1(Z)$ and with $\frac{1}{r} + \frac{1}{r'} = 1$.

Let $\eta \in (2, \mu)$. We have

$$\frac{\eta}{2}\|Dx_n\|_2^2 - \frac{\lambda\eta}{2}\int_Z a x_n^2 dz - \int_Z \eta j(z, x_n)\, dz \leq \eta M_1 \text{ (see (3.6))} \tag{3.9}$$

and

$$\left|\langle A(x_n), x_n\rangle - \lambda\int_Z a x_n^2 dz - \int_Z u_n x_n dz\right| \leq \varepsilon_n \|x_n\|,$$

with $\varepsilon_n \downarrow 0$ (see (3.7), (3.8)), therefore

$$-\|Dx_n\|_2^2 + \lambda\int_Z a x_n^2 dz + \int_Z u_n x_n dz \leq \varepsilon_n \|x_n\|,$$

so

$$-\|Dx_n\|_2^2 + \lambda\int_Z a x_n^2 dz - \int_Z j^0(z, x_n; -x_n)\, dz \leq \varepsilon_n \|x_n\|. \tag{3.10}$$

Adding (3.9) and (3.10), we obtain

$$\left(\frac{\eta}{2} - 1\right)\|Dx_n\|_2^2 - \lambda\left(\frac{\eta}{2} - 1\right)\int_Z a x_n^2 dz - \int_Z \left[\eta j(z, x_n) + j^0(z, x_n; -x_n)\right] dz$$
$$\leq \varepsilon_n \|x_n\| + \eta M_1, \tag{3.11}$$

hence

$$\left(\frac{\eta}{2}-1\right)\|Dx_n\|_2^2 - c_3\|x_n\|_2^2 - \int_Z \left[\eta j(z,x_n) + j^0(z,x_n;-x_n)\right] dz$$
$$\le \varepsilon_n \|x_n\| + \eta M_1, \text{ with } c_3 = c_3(\lambda) = \lambda\left(\frac{\eta}{2}-1\right)\|a\|_\infty > 0. \tag{3.12}$$

Note that

$$\begin{aligned}
&\int_Z \left[\eta j(z,x_n) + j^0(z,x_n;-x_n)\right] dz \\
&= \int_Z \left[\mu j(z,x_n) + j^0(z,x_n;-x_n)\right] dz - (\mu-\eta)\int_Z j(z,x_n)\, dz \\
&= \int_{\{|x_n|<M\}} \left[\mu j(z,x_n) + j^0(z,x_n;-x_n)\right] dz \\
&+ \int_{\{|x_n|\ge M\}} \left[\mu j(z,x_n) + j^0(z,x_n;-x_n)\right] dz - (\mu-\eta)\int_Z j(z,x_n)\, dz \\
&\le c_4 - (\mu-\eta)\int_Z j(z,x_n)\, dz, \text{ for some } c_4>0
\end{aligned} \tag{3.13}$$

(see hypotheses $H(j)_1(iii)$,(v)). Returning to (3.12) and using (3.13), we have

$$\left(\frac{\eta}{2}-1\right)\|Dx_n\|_2^2 - c_3\|x_n\|_2^2 - c_4 + (\mu-\eta)\int_Z j(z,x_n)\, dz \le \varepsilon_n\|x_n\| + \eta M_1,$$

so

$$\left(\frac{\eta}{2}-1\right)\|Dx_n\|_2^2 - c_3\|x_n\|_2^2 + (\mu-\eta)c_1\|x_n\|_\mu^\mu \le \varepsilon_n\|x_n\| + c_5, \tag{3.14}$$

for some $c_5>0$ (see Lemma 3.3). Since $\mu>2$ and using Young's inequality with $\varepsilon>0$, we obtain

$$\|x_n\|_2^2 \le c_6\|x_n\|_\eta^2 \le \varepsilon\|x_n\|_\eta^\eta + c_7(\varepsilon),$$

for some c_6, $c_7(\varepsilon)>0$. So (3.14) becomes

$$\left(\frac{\eta}{2}-1\right)\|Dx_n\|_2^2 + ((\mu-\eta)c_1-\varepsilon)\|x_n\|_\eta^\eta \le \varepsilon_n\|x_n\| + c_8(\varepsilon), \tag{3.15}$$

with $c_8(\varepsilon) = c_5 + c_7(\varepsilon) > 0$. Choosing $0<\varepsilon\le(\mu-\eta)c_1$, from (3.15) and Poincaré's inequality, we conclude that $\{x_n\}_{n\ge1}\subseteq H_0^1(Z)$ is bounded. Hence by passing to a suitable subsequence, if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } H_0^1(Z)$$

and

$$x_n \to x \text{ in } L^2(Z) \text{ and in } L^r(Z) \text{ (recall } r<2^*).$$

From (3.7), we know that

$$\left|\langle A(x_n), x_n - x\rangle - \lambda \int_Z a x_n (x_n - x)\, dz - \int_Z u_n (x_n - x)\, dz\right| \leq \varepsilon_n \|x_n - x\|. \quad (3.16)$$

Evidently, we have

$$\int_Z a x_n (x_n - x)\, dz \to 0 \text{ and } \int_Z u_n (x_n - x)\, dz \to 0 \text{ as } n \to \infty.$$

Hence, from (3.16), we obtain

$$\langle A(x_n), x_n - x\rangle \to 0 \text{ as } n \to \infty.$$

Note that $A(x_n) \stackrel{w}{\to} A(x)$ in $H^{-1}(Z)$. Hence

$$\|Dx_n\|_2^2 = \langle A(x_n), x_n\rangle \to \langle A(x), x\rangle = \|Dx\|_2^2.$$

But $Dx_n \stackrel{w}{\to} Dx$ in $L^2(Z, \mathbb{R}^N)$. So, from the Kadec-Klee property of Hilbert spaces (see Gasinski-Papageorgiou [21], p.722), it follows that $Dx_n \to Dx$ in $L^2(Z, \mathbb{R}^N)$, therefore $x_n \to x$ in $H_0^1(Z)$. This proves that for $\lambda \in [\lambda_k, \lambda_{k+1})$, the functional φ satisfies the nonsmooth PS-condition. ■

Proposition 3.5 *If hypotheses $H(j)_1$ hold and $\lambda < \lambda_1$, then φ_λ satisfies the nonsmooth PS-condition.*

Proof. Since by hypothesis $\lambda < \lambda_1$, we have that

$$|x|^2 = \|Dx\|_2^2 - \lambda \int_Z a x^2 dz,$$

is an equivalent norm for $H_0^1(Z)$ (see Lemma 2.20 (a)). Let $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ be a PS-sequence (i.e., it satisfies (3.6) and (3.7)). From (3.11) and (3.13) we have

$$\left(\frac{\eta}{2} - 1\right) |x_n|^2 + (\mu - \eta) \int_Z j(z, x_n)\, dz \leq c_9,$$

for some $c_9 > 0$, all $n \geq 1$, therefore

$$\left(\frac{\eta}{2} - 1\right) |x_n|^2 + c_{10} \|x_n\|_\mu^\mu \leq c_{11}, \quad (3.17)$$

for some $c_{10}, c_{11} > 0$, all $n \geq 1$, (see Lemma 3.3).

Since $\eta \in (2, \mu)$, from (3.17) we infer that $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ is bounded and so we finish this proof as that of Proposition 3.4. ■

In the sequel, we set

$$H_- = \overset{k-1}{\underset{i=1}{\oplus}} E(\lambda_i) \text{ and } H_+ = \overline{\underset{i\geq k+1}{\oplus} E(\lambda_i)}.$$

Proposition 3.6 *If hypotheses $H(j)_1$ hold and $\lambda_k \leq \lambda < \lambda_{k+1}$, for some $k \geq 1$, then we can find $\rho > 0$ and $\beta > 0$, such that*

$$\varphi_\lambda(x) \geq \beta > 0$$

for all $x \in H_+$, with $\|x\| = \rho$.

Proof. By virtue of hypothesis $H(j)_1(iv)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$, such that

$$j(z,x) \leq \frac{\varepsilon}{2}x^2 \tag{3.18}$$

for a.a. $z \in Z$ and all $|x| \leq \delta$.

On the other hand, hypothesis $H(j)_1(iii)$ and the mean value theorem for locally Lipschitz functions (see Clarke [13], p.41) imply that

$$j(z,x) \leq c_{12}|x|^\tau \tag{3.19}$$

for a.a. $z \in Z$, all $|x| > \delta$ and some $c_{12} > 0$, $\tau > 2$. From (3.18) and (3.19) it follows that

$$j(z,x) \leq \frac{\varepsilon}{2}|x|^2 + c_{12}|x|^\tau \tag{3.20}$$

for a.a. $z \in Z$, all $x \in \mathbb{R}$.

Let $x \in H_+$. Then

$$\begin{aligned}
\varphi_\lambda(x) &= \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda}{2}\int_Z ax^2dz - \int_Z j(z,x(z))\,dz \\
&\geq \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda}{2}\int_Z ax^2dz - \frac{\varepsilon}{2}\|x\|_2^2 - c_{13}\|x\|^\tau \text{ for some } c_{13} > 0 \\
&\geq \frac{c_{14}}{2}\|Dx\|_2^2 - \frac{\varepsilon c_{15}}{2}\|Dx\|_2^2 - c_{16}\|Dx\|_2^\tau \text{ for some } c_{14}, c_{15}, c_{16} > 0
\end{aligned} \tag{3.21}$$

(see (3.20) and Lemma 2.20 (a)). From (3.21) we see that, if $\varepsilon < \frac{c_{14}}{c_{15}}$, then

$$\varphi_\lambda(x) \geq c_{17}\|Dx\|_2^2 - c_{16}\|Dx\|_2^\tau \tag{3.22}$$

for all $x \in H_+$ and some $c_{17} > 0$. Because $\tau > 2$, from (3.22) and Poincaré's inequality, we see that if $\rho \in (0,1)$ is small, then

$$\varphi_\lambda(x) \geq \beta > 0,$$

for all $x \in H_+$, with $\|x\| = \rho$. ■

We continue to assume that $\lambda_k \leq \lambda < \lambda_{k+1}$, for some $k \geq 1$. Let $e \in E(\lambda_{k+1})$, with $\|De\|_2 = \rho$ and consider the following half-ball

$$E = \{w = v + re : v \in V = H_- \oplus E(\lambda_k),\ \|w\| \leq R,\ r \geq 0\},$$

with $R > \rho$ to be fixed in the process of the proof. We have

$$\partial E = E_0 = E_1 \cup E_2,$$

where

$$E_1 = \{w = v \in V : \|w\| \leq R\} \text{ (the basis of the half-ball)}$$

and

$$E_2 = \{w = v + re : v \in V,\ \|w\| = R,\ r \geq 0\} \text{ (the hemisphere).}$$

Proposition 3.7 *If hypotheses $H(j)_1$ hold and $\lambda_k \leq \lambda < \lambda_{k+1}$ for some $k \geq 1$, then $\varphi|_{E_0} \leq 0$.*

Proof. First we examine what happens on E_1. If $w \in E_1$, then $w = v \in V$ and $\|v\| \leq R$. So

$$\begin{aligned}\varphi(v) &= \frac{1}{2}\|Dv\|_2^2 - \frac{\lambda}{2}\int_Z av^2 dz - \int_Z j(z, v)\, dz \\ &\leq \frac{1}{2}\|Dv\|_2^2 - \frac{\lambda}{2}\int_Z av^2 dz \text{ (recall } j \geq 0) \\ &\leq \frac{1}{2}(\lambda_k - \lambda)\int_Z av^2 dz \text{ (recall (2.3))} \\ &\leq 0 \text{ (since } \lambda_k \leq \lambda). \end{aligned} \tag{3.23}$$

Next we examine what happens on E_2. So, let $w \in E_2$. Exploiting the orthogonality of the component spaces, we have

$$\begin{aligned}\varphi(w) &= \frac{1}{2}\|Dw\|_2^2 - \frac{\lambda}{2}\int_Z aw^2 dz - \int_Z j(z, w)\, dz \\ &\leq \frac{1}{2}\|Dv\|_2^2 - \frac{\lambda}{2}\int_Z av^2 dz + \frac{r^2}{2}\|De\|_2^2 - \frac{\lambda r^2}{2}\int_Z ae^2 dz - c_{18}\|w\|_\mu^\mu + c_{19} \end{aligned} \tag{3.24}$$

for some $c_{18}, c_{19} > 0$ (see Lemma 3.3).

We have

$$\frac{1}{2}\|Dv\|_2^2 - \frac{\lambda}{2}\int_Z av^2 dz \leq 0 \text{ (see Lemma 2.20 } (b)) \tag{3.25}$$

and

$$\frac{r^2}{2}\left(\|De\|_2^2 - \lambda \int_Z ae^2 dz\right) = \frac{r^2}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\|De\|_2^2 \text{ (since } e \in E(\lambda_{k+1})\text{)}. \tag{3.26}$$

Returning to (3.24) and using (3.25) and (3.26), we obtain

$$\varphi(w) \leq \frac{r^2}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\rho^2 - c_{18}\|w\|_\mu^\mu + c_{19} \tag{3.27}$$

for some $c_{18}, c_{19} > 0$. The space $W = V \oplus \mathbb{R}e = H_- \oplus E(\lambda_k) \oplus \mathbb{R}e$ is finite dimensional. So, all norms are equivalent. Hence we can find $c_{20} > 0$, such that $\|w\| \leq c_{20}\|w\|_\mu$, for all $w \in W$. Using this fact in (3.27), we obtain

$$\begin{aligned} \varphi(w) &\leq \frac{r^2}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\rho^2 - c_{21}\|w\|^\mu + c_{19}, \text{ with } c_{21} = \frac{c_{18}}{c_{20}} > 0 \\ &= \frac{r^2}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\rho^2 - c_{21}R^\mu + c_{19} \text{ (since } w \in E_2\text{)}. \end{aligned} \tag{3.28}$$

From (3.28) it is clear that if we choose $R > \rho$ large enough, we shall have

$$\varphi|_{E_2} \leq 0. \tag{3.29}$$

From (3.23) and (3.29), it follows that $\varphi|_{E_0} \leq 0$. ■

Now we are ready for the existence theorem.

Theorem 3.8 *If hypotheses $H(j)_1$ hold, then for every $\lambda \in \mathbb{R}$, problem* (3.1) *has a nontrivial solution $x \in C^1(\overline{Z})$.*

Proof. First we assume that $\lambda_k \leq \lambda < \lambda_{k+1}$, for some $k \geq 1$.

Let E be the half-ball introduced earlier,

$$E_0 = \partial E = E_1 \cup E_2 \text{ and } D = H_+ \cap \partial B_\rho.$$

Claim 1 *$\{E, E_0\}$ is linking with D in $H_0^1(Z)$.*

Clearly $D \cap E_0 = \varnothing$. Also, let $\gamma \in \Gamma = \{\gamma \in C(E, H_0^1(Z)) : \gamma|_{E_0} = id|_{E_0}\}$. We need to show that $\gamma(E) \cap D \neq \varnothing$. To this end, let p_V be the orthogonal projection onto V and let $\widehat{r}_{E_0} : W \backslash \{e\} \to E_0$ be a retraction map. Suppose that $\gamma(E) \cap D = \varnothing$. Then the map

$$x \to \widehat{r}_{E_0}\left(p_V(\gamma(x)) + \frac{1}{\rho}\|(I - p_V)\gamma(x)\| e\right)$$

is a retraction of E onto $\partial E = E_0$. But E is homeomorphic to a finite dimensional ball and as it is well-known, in a finite dimensional space no such retraction is possible (see, for example, Denkowski-Migorski-Papageorgiou [14], p.196). Therefore $\gamma(E) \cap D \neq \varnothing$ and so $\{E, E_0\}$ and D link in $H_0^1(Z)$. Because of Proposition 3.4, 3.6 and 3.7, we can apply Theorem 2.8 and obtain $x \in H_0^1(Z)$ such that

$$\varphi(x) \geq \beta > 0 = \varphi(0) \tag{3.30}$$

and

$$0 \in \partial\varphi(x) \text{ (i.e., } x \text{ is a critical point of } \varphi). \tag{3.31}$$

From (3.30), it is clear that $x \neq 0$, while from (3.31) we have

$$A(x) - \lambda a x = u,$$

with $u \in N(x)$,therefore

$$-\Delta x(z) - \lambda a(z) x(z) = u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z,\ x|_{\partial Z} = 0.$$

Moreover, standard regularity theory implies $x \in C^1(\overline{Z})$. Next, we assume that $\lambda < \lambda_1$. As we already pointed earlier, in this case,

$$|x|^2 = \|Dx\|_2^2 - \lambda \int_Z a x^2 dz,\ x \in H_0^1(Z),$$

is an equivalent norm on the Sobolev space $H_0^1(Z)$. If $y \in C_0^1(\overline{Z})$, $y(z) > 0$ for all $z \in Z$, then

$$\begin{aligned} \varphi(ty) &\leq c_{22} t^2 \|Dy\|_2^2 - \int_Z j(z, ty(z))\, dz, \text{ for some } c_{22} > 0 \text{ and all } t > 0 \\ &\leq c_{22} t^2 \|Dy\|_2^2 - c_{23} t^\mu \|y\|_\mu^\mu + c_{24}, \text{ for some } c_{23}, c_{24} > 0 \text{ (see Lemma 3.3)} \end{aligned} \tag{3.32}$$

Since $\mu > 2$, from (3.32), it follows that

$$\varphi(ty) \to -\infty, \text{ as } t \to +\infty. \tag{3.33}$$

On the other hand, for every $x \in H_0^1(Z)$, we have

$$\begin{aligned} \varphi(x) &= \frac{1}{2}|x|^2 - \int_Z j(z, x(z))\, dz \\ &\geq \frac{1}{2}|x|^2 - \frac{\varepsilon}{2}\|x\|_2^2 - c_{12}\|x\|^\tau \text{ (see (3.20))} \\ &\geq \frac{1}{2}(c_{25} - \varepsilon)\|x\|^2 - c_{12}\|x\|^\tau, \text{ for some } c_{25} > 0. \end{aligned}$$

Choosing $\varepsilon < c_{25}$, we infer that

$$\varphi(x) \geq c_{26} \|x\|^2 - c_{12} \|x\|^\tau, \tag{3.34}$$

for some $c_{26} > 0$ and for all $x \in H_0^1(Z)$. Because $\tau > 2$, from (3.34), we infer that, if we choose $\rho \in (0,1)$ small, then

$$\varphi(x) \geq \widehat{\beta} > 0 = \varphi(0) \geq \varphi(ty), \tag{3.35}$$

for all $x \in H_0^1(Z)$, with $\|x\| = \rho$, and $t > \rho$, such that $\|ty\| > \rho$ (see (3.33)).

Because of (3.35) and Proposition 3.5, we can apply Theorem 2.9 and obtain $x \in H_0^1(Z)$, such that

$$\varphi(x) \geq \widehat{\beta} > 0 = \varphi(0) \tag{3.36}$$

and

$$0 \in \partial\varphi(x). \tag{3.37}$$

As before, from (3.36), we have that $x \neq 0$, while from (3.37) it follows that $x \in C^1(\overline{Z})$ and solves (3.1). ■

In the scalar case (i.e. $N = 1$, ordinary differential inclusion), we can weaken the hypotheses. So, we consider the following scalar Dirichlet problem

$$\begin{cases} -x''(t) - \lambda a(t) x(t) \in \partial j(t, x(t)) & \text{a.e. on } T = [0,b], \\ x(0) = x(b) = 0. \end{cases} \tag{3.38}$$

The hypotheses on the nonsmooth potential are the following:

$H(j)_2$: $j : T \times \mathbb{R} \to \mathbb{R}_+$ is a function, such that $j(t,0) = 0$ a.e. on T and

- (i) for all $x \in \mathbb{R}$, $t \to j(t,x)$ is measurable;
- (ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;
- (iii) for almost all $t \in T$, all $x \in \mathbb{R}$ and all $u \in \partial j(t,x)$

$$|u| \leq a_0(t) + c_0 |x|^r,$$

with $a_0 \in L^1(T)_+$, $c_0 > 0$ and $1 < r$;

(iv) $\lim\limits_{x\to 0}\frac{j(t,x)}{x^2}=0$, uniformly, for a.a. $t\in T$;

(v) $\lim\limits_{|x|\to\infty}\frac{j(t,x)}{x^2}=+\infty$, uniformly, for a.a. $t\in T$;

(vi) there exist $\theta>r-1$ and $M>0$, such that for almost all $t\in T$ and all $|x|\geq M$, we have

$$\widehat{c}\,|x|^{\theta}\leq -j^0\left(t,x;-x\right)-2j\left(t,x\right),\ \text{ with } \widehat{c}>0.$$

Remark 3.9 *We no longer have the AR-condition. Instead, we use hypothesis* $H\left(j\right)_2(vi)$*.*

As the next examples illustrate, there are functions which satisfy $H\left(j\right)_2(vi)$*, but not the AR-condition.*

Example 3.10 *We consider the following functions. For the sake of simplicity, we drop the* t*-dependence.*

$$j_1\left(x\right)=\begin{cases}\frac{1}{5}\left|x\right|^5 & \text{if } |x|\leq 1\\ x^2\ln|x|+c\left|x\right|+\frac{1}{5}-c & \text{if } |x|>1\end{cases},\ \text{ with } c\geq 0$$

and

$$j_2\left(x\right)=x^2\ln\left(|x|+1\right).$$

Note that $j_2\in C^1\left(\mathbb{R}\right)$*, while, if* $c=\frac{1}{5}$*, then* $j_1\in C^1\left(\mathbb{R}\right)$*, too. Both these functions satisfy hypotheses* $H\left(j\right)_2$*, but not the AR-condition.*

The Euler functional $\varphi_\lambda: W_0^{1,2}\left(0,b\right)\to\mathbb{R}$ for problem (3.38) is defined by

$$\varphi_\lambda\left(x\right)=\frac{1}{2}\left\|x'\right\|_2^2-\frac{\lambda}{2}\int_0^b ax^2dt-\int_0^b j\left(t,x\left(t\right)\right)dt,\ \text{ for all } x\in W_0^{1,2}\left(0,b\right).$$

We know that φ_λ is Lipschitz continuous on bounded sets, hence it is locally Lipschitz (see Clarke [13], p.83).

Proposition 3.11 *If hypotheses* $H\left(j\right)_2$ *hold and* $\lambda_k\leq\lambda<\lambda_{k+1}$*, for some* $k\geq 1$*, then* φ_λ *satisfies the nonsmooth C-condition.*

Proof. Let $\left\{x_n\right\}_{n\geq 1}\subseteq W_0^{1,2}\left(0,b\right)$ be a sequence such that

$$\left|\varphi_\lambda\left(x_n\right)\right|\leq M_2,\ \text{for some } M_2>0,\ \text{all } n\geq 1 \tag{3.39}$$

and

$$(1+\|x_n\|)\, m_\lambda(x_n) \to 0, \text{ as } n\to\infty, \tag{3.40}$$

where $m_\lambda(x_n)=\inf\{\|x^*\| : x^*\in\partial\varphi_\lambda(x_n)\}$.

As before, we can find $x_n^*\in\partial\varphi_\lambda(x_n)$, satisfying $m_\lambda(x_n)=\|x_n^*\|$, for all $n\geq 1$. We know that

$$x_n^* = A(x_n) - \lambda a x_n - u_n,\ u_n\in N(x_n),$$

with $A\in\mathcal{L}\left(W_0^{1,2}(0,b), W^{-1,2}(0,b)\right)$ defined by

$$\langle A(x), y\rangle = \int_0^b x'y'dt, \text{ for all } x,y\in W_0^{1,2}(0,b)$$

and, for all $x\in W_0^{1,2}(0,b)$,

$$N(x)=\left\{u\in L^{\frac{r+1}{r}}(o,b) : u(t)\in\partial j(t,x(t)) \text{ a.e. on } (0,b)\right\}.$$

Claim 2 *The sequence $\{x_n\}_{n\geq 1}\subseteq W_0^{1,2}(0,b)$ is bounded.*

We argue indirectly. So, suppose that the Claim 2 is not true. By passing to a subsequence, if necessary, we may assume that $\|x_n'\|_2\to\infty$.

From (3.39) and (3.40), we have

$$|2\varphi_\lambda(x_n)| = \left|\|x_n'\|_2^2 - \lambda\int_0^b a x_n^2 dt - \int_0^b 2j(t,x_n)\,dt\right| \leq 2M_2, \tag{3.41}$$

for all $n\geq 1$ and

$$|\langle x_n^*, x_n\rangle| = \left|\|x_n'\|_2^2 - \lambda\int_0^b a x_n^2 dt - \int_0^b u_n x_n dt\right| \leq \varepsilon_n, \tag{3.42}$$

for all $n\geq 1$, with $\varepsilon_n\downarrow 0$. From (3.41) and (3.42), it follows that

$$\int_0^b (u_n x_n - 2j(t,x_n))\,dt \leq M_3 \text{ for some } M_3>0, \text{ all } n\geq 1,$$

hence

$$\int_0^b \left[-j^0(t,x_n;-x_n) - 2j(t,x_n)\right] dt \leq M_3,$$

therefore

$$\int_{\{|x_n|\geq M\}} \left[-j^0(t,x_n;-x_n) - 2j(t,x_n)\right] dt$$
$$+ \int_{\{|x_n|<M\}} \left[-j^0(t,x_n;-x_n) - 2j(t,x_n)\right] dt$$
$$\leq M_3$$

so

$$\widehat{c}\int_{\{|x_n|\geq M\}} |x_n|^{\theta}\, dt \leq M_4, \text{ for some } M_4 > 0 \text{ and all } n \geq 1,$$

(see $H(j)_2\,(iii)\,,(vi)$), hence

$$\{x_n\}_{n\geq 1} \subseteq L^{\theta}(0,b) \text{ is bounded.} \tag{3.43}$$

We consider the orthogonal direct sum decomposition

$$W_0^{1,2}(0,b) = H_- \oplus E(\lambda_k) \oplus H_+.$$

We can write, in an unique way,

$$x_n = \overline{x}_n + x_n^0 + \widehat{x}_n,$$

with $\overline{x}_n \in H_-$, $x_n^0 \in H_0 = E(\lambda_k)$, $\widehat{x}_n \in H_+$, $n \geq 1$.

From (3.40), we have

$$|\langle x_n^*, u\rangle| \leq \frac{\varepsilon_n}{1+\|x_n\|}\|u\|, \text{ for all } u \in W_0^{1,2}(0,b).$$

Let $u = \widehat{x}_n$. Exploiting the orthogonality of the component spaces, we have

$$|\langle x_n^*, \widehat{x}_n\rangle| = \left| \|\widehat{x}_n'\|_2^2 - \lambda \int_0^b a\widehat{x}_n^2 dt - \int_0^b u_n \widehat{x}_n dt \right| \leq \varepsilon_n. \tag{3.44}$$

Since, by hypothesis, $\lambda \in [\lambda_k, \lambda_{k+1})$, from Lemma 2.20 (a), we have

$$\xi_0 \|\widehat{x}_n'\|_2^2 \leq \|\widehat{x}_n'\|_2^2 - \lambda \int_0^b a\widehat{x}_n^2 dt. \tag{3.45}$$

Also, we have

$$\int_0^b u_n \widehat{x}_n dt \leq \int_0^b |u_n|\,|\widehat{x}_n|\, dt \leq c_{27}\|\widehat{x}_n'\|_2 \int_0^b |u_n|\, dt, \text{ for some } c_{27} > 0$$
$$\leq c_{27}\|\widehat{x}_n'\|_2 \int_0^b \left(a_0(t) + c_0|x_n|^r\right) dt \text{ (see hypothesis } H(j)\,(iii)\text{)}$$
$$\leq c_{28}\|\widehat{x}_n'\|_2 + c_{29}\|\widehat{x}_n'\|_2 \|x_n\|_r^r, \text{ for some } c_{28}, c_{29} > 0. \tag{3.46}$$

Assuming without any loss of generality that $\theta \leq r$ and using the interpolation inequality (see, for example, Gasinski-Papageorgiou [22], p.905), we have

$$\|x_n\|_r \leq \|x_n\|_\theta^{1-t} \|x_n\|_\infty^t \text{, where } t \in (0,1)\text{, } \frac{1-t}{\theta} = \frac{1}{r},$$

so

$$\|x_n\|_r^r \leq c_{30} \|x_n'\|^{tr} \text{, for some } c_{30} > 0 \text{, all } n \geq 1 \text{ (see (3.43))}.$$

Therefore, we have

$$\int_0^b u_n \widehat{x}_n dt \leq c_{28} \|\widehat{x}_n'\|_2 + c_{31} \|\widehat{x}_n'\|_2 \|x_n'\|^{tr} \text{, for some } c_{31} > 0 \text{, all } n \geq 1.$$

Returning to (3.44) and using (3.45) and (3.46) ,we obtain

$$\xi_0 \|\widehat{x}_n'\|_2^2 \leq \varepsilon_n + c_{28} \|\widehat{x}_n'\|_2 + c_{31} \|\widehat{x}_n'\| \|x_n'\|^{tr},$$

hence

$$\xi_0 \frac{\|\widehat{x}_n'\|_2^2}{\|x_n\|^2} \leq \frac{\varepsilon_n}{\|x_n\|^2} + \frac{c_{32}}{\|x_n\|} + c_{33} \frac{1}{\|x_n\|^{1-tr}},$$

for some $c_{32}, c_{33} > 0$, all $n \geq 1$.

Note that $\theta > r - 1$ is equivalent to $tr < 1$. So, passing to the limit as $n \to \infty$, we obtain

$$\frac{\|\widehat{x}_n'\|_2}{\|x_n\|} \to 0 \text{, as } n \to \infty,$$

therefore

$$\frac{\|\widehat{x}_n\|}{\|x_n\|} \to 0 \text{, as } n \to \infty. \tag{3.47}$$

In a similar fashion, using as a test function $u = \overline{x}_n$ and Lemma 2.20 (b), we show that

$$\frac{\|\overline{x}_n\|}{\|x_n\|} \to 0 \text{, as } n \to \infty. \tag{3.48}$$

Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$, for all $n \geq 1$, and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,2}(0,b) \text{ and } y_n \to y \text{ in } C\left(T, \mathbb{R}^N\right).$$

From (3.47) and (3.48), it follows that $y \in E(\lambda_k)$. Also, due to (3.43), we have $y = 0$. Then the finite dimensionality of the space H_0 implies

$$y_n^0 = \frac{x_n^0}{\|x_n\|} \to 0 \text{ in } W_0^{1,2}(0,b). \tag{3.49}$$

Combining (3.47), (3.48) and (3.49), we have

$$1 = \|y_n\| \leq \frac{\|\overline{x}_n\| + \|x_n^0\| + \|\widehat{x}_n\|}{\|x_n\|} \to 0, \text{ as } n \to \infty,$$

a contradiction. This proves the Claim 2.

Because of the Claim 2 and by passing to a suitable subsequence, if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } W_0^{1,2}(0,b) \text{ and } x_n \to x \text{ in } C(T). \tag{3.50}$$

From (3.40), we have

$$\left| \langle A(x_n), x_n - x \rangle - \lambda \int_0^b a x_n (x_n - x)\, dt - \int_0^b u_n (x_n - x)\, dt \right| \leq \varepsilon_n. \tag{3.51}$$

Because of (3.50), we have

$$\int_0^b a x_n (x_n - x)\, dt \to 0 \text{ and } \int_0^b u_n (x_n - x)\, dt \to 0, \text{ as } n \to \infty.$$

So, from (3.51), we have

$$\langle A(x_n), x_n - x \rangle \to 0, \text{ as } n \to \infty.$$

From this, as in the proof of Proposition 3.4, we conclude that $x_n \to x$ in $W_0^{1,2}(0,b)$ and, so, φ_λ satisfies the nonsmooth C-condition. ■

We, also, check the case when $\lambda < \lambda_1$.

Proposition 3.12 *If hypotheses $H(j)_2$ hold and $\lambda < \lambda_1$, then φ_λ satisfies the nonsmooth C-condition.*

Proof. We know that, in this case,

$$|x|^2 = \|x'\|_2^2 - \lambda \int_0^b a x^2 dt$$

is an equivalent norm for the Sobolev space $W_0^{1,2}(0,b)$. Let $\{x_n\}_{n\geq 1} \subseteq W_0^{1,2}(0,b)$ be a C-sequence. Arguing as in proof of Proposition 3.11 and using hypothesis $H(j)_2(vi)$, we show that $\{x_n\}_{n\geq 1} \subseteq L^\theta(Z)$ is bounded. Also, we have

$$|\langle x_n^*, u \rangle| \leq \frac{\varepsilon_n}{1 + \|x_n\|} \|u\|, \text{ for all } u \in W_0^{1,2}(0,b), \text{ with } \varepsilon_n \downarrow 0.$$

Choose $u = x_n$. Then

$$\left| |x_n|^2 - \int_0^b u_n x_n dt \right| \leq \varepsilon_n, \text{ for all } n \geq 1. \tag{3.52}$$

From (3.46) (with $\widehat{x}_n$ replaced by x_n), we have

$$\int_0^b u_n x_n dt \leq c_{34} |x_n| + c_{35} |x_n|^{tr+1}, \text{ for some } c_{34}, c_{35} > 0, \text{ all } n \geq 1. \tag{3.53}$$

We use (3.53) in (3.52) and obtain

$$|x_n|^2 \leq \varepsilon_n + c_{34} |x_n| + c_{35} |x_n|^{tr+1}. \tag{3.54}$$

Recall that $tr + 1 < 2$. Hence, from (3.54), it follows that $\{x_n\}_{n \geq 1} \subseteq W_0^{1,2}(0,b)$ is bounded. Continuing as in the proof of Proposition 3.11, we conclude that φ_λ satisfies the nonsmooth C-condition. ■

Using hypothesis $H(j)_2(iv)$ and arguing as in the proof of Proposition 3.6, we show that:

Proposition 3.13 *If hypotheses $H(j)_2$ hold and $\lambda_k \leq \lambda < \lambda_{k+1}$, for some $k \geq 1$, then we can find $\rho > 0$ and $\beta > 0$, such that*

$$\varphi(x) \geq \beta > 0, \text{ for all } x \in H_+, \text{ with } \|x\| = \rho.$$

We choose $e \in E(\lambda_{k+1})$, with $\|e\| = 1$ and set $Y = V \oplus \mathbb{R}e$, with $V = H_- \oplus E(\lambda_k)$. We consider the following cylinder:

$$E = \{y = v + re : v \in V, \ \|v\| \leq R, \ 0 \leq r \leq R\},$$

with $R > \rho$ (see Proposition 3.6) to be fixed in the process of the proof. We set

$$E_0 = \partial E = E_1 \cup E_2 \cup E_3,$$

with

$$E_1 = \{v \in V = H_- \oplus E(\lambda_k) : \|v\| \leq R\} \text{ (the lower base of the cylinder)},$$

$$E_2 = \{y = v + Re : \|v\| \leq R\} \text{ (the upper base of the cylinder)},$$

$$E_3 = \{y = v + re : 0 < r < R, \ \|v\| = R\} \text{ (the lateral surface of the cylinder)}.$$

We want to estimate the values of φ_λ on E_0. To this end, we shall need the following lemma. We state the result for the more general Sobolev space $W_0^{1,p}(Z)$.

Lemma 3.14 *If $V \subseteq W_0^{1,p}(Z)$ is a nontrivial finite dimensional subspace, then there exists $\eta > 0$, such that*

$$|\{z \in Z : |v(z)| \geq \eta \|v\|\}|_N \geq \eta, \text{ for all } x \in V,\ x \neq 0.$$

Proof. We argue by contradiction. So, suppose that the lemma is not true. Then we can find $\{v_n\}_{n\geq 1} \subseteq V$, $v_n \neq 0$, such that

$$\left| \left\{ z \in Z : |v_n(z)| \geq \frac{1}{n} \|v_n\| \right\} \right|_N < \frac{1}{n}, \text{ for all } n \geq 1.$$

We set $w_n = \frac{v_n}{\|v_n\|}$, for all $n \geq 1$. Then

$$\left| \left\{ z \in Z : |w_n(z)| \geq \frac{1}{n} \right\} \right|_N < \frac{1}{n},$$

hence

$$w_n \to 0 \text{ in the Lebesgue measure.}$$

So, passing to a suitable subsequence, if necessary, we may assume that $w_n(z) \to 0$ a.e. on Z. Note that $\|w_n\| = 1$ and $\{w_n\}_{n\geq 1} \subseteq V$, with V finite dimensional. So, it follows that $w_n \to 0$ in $W_0^{1,p}(Z)$, a contradiction to the fact that $\|w_n\| = 1$, for all $n \geq 1$. ∎

With the help of this lemma, we can now estimate $\varphi_\lambda|_{E_0}$.

Proposition 3.15 *If hypotheses $H(j)_2$ hold and $\lambda_k \leq \lambda < \lambda_{k+1}$, for some $k \geq 1$, then $\varphi_\lambda|_{E_0} < 0$.*

Proof. By virtue of hypothesis $H(j)_2(v)$, given $\xi > 0$, we can find $M_5 = M_5(\xi) > 0$, such that

$$j(t,x) \geq \xi x^2 \text{ for a.a. } t \in T \text{ and all } |x| \geq M_5.$$

For any function $y \in Y = V \oplus \mathbb{R}e$, we have $y = v + re$, with $v \in V$, $r \in \mathbb{R}$ and so

$$\begin{aligned} \varphi_\lambda(y) &= \frac{1}{2}\|v'\|_2^2 + \frac{r^2}{2}\|e'\|_2^2 - \frac{\lambda}{2}\int_0^b av^2 dt - \frac{\lambda r^2}{2}\int_0^b ae^2 dt - \int_0^b j(t,y)\,dt \\ &\leq \frac{r^2}{2}\|e'\|_2^2 - \frac{\lambda r^2}{2}\int_0^b ae^2 dt - \int_0^b j(t,y)\,dt \text{ (see (2.3)).} \end{aligned} \tag{3.55}$$

Lemma 3.14 implies that there exists $\eta > 0$, such that

$$|D_\eta|_1 = |\{t \in (0,b) : |y(t)| \geq \eta \|y\|\}|_1 \geq \eta \tag{3.56}$$

for all $y \in Y = V \oplus \mathbb{R}e$, $y \neq 0$.

Recall that $\xi > 0$ was arbitrary. So, we choose $\xi \geq \frac{1}{\eta^3}$. Then we have

$$j(t, y(t)) \geq \xi \eta^2 \|y\|^2 \tag{3.57}$$

for a.a. $t \in D_\eta$ and all $y \in Y$, with $\|y\| \geq \frac{M_5}{\eta}$. Choose $R \geq \frac{M_5}{\eta}$. Then for $y \in Y = V \oplus \mathbb{R}e$ with $\|y\| = R$, we have

$$\begin{aligned}
\varphi_\lambda(y) &\leq \frac{r^2}{2}\|e'\|_2^2 - \frac{\lambda r^2}{2}\int_0^b ae^2 dt - \int_0^b j(t, y(t))\, dt \text{ (see (3.55))} \\
&= \frac{r^2}{2}\|e'\|_2^2 - \frac{\lambda r^2}{2}\int_0^b ae^2 dt - \int_{D_\eta} j(t, y(t))\, dt - \int_{D_\eta^c} j(t, y(t))\, dt \\
&\leq \frac{r^2}{2}\|e'\|_2^2 - \frac{\lambda r^2}{2}\int_0^b ae^2 dt - \xi\eta^2 \|y\|^2 |D_\eta|_1 \text{ (see (3.57) and recall } j \geq 0)) \\
&\leq \frac{r^2}{2}\|e'\|_2^2 - \frac{\lambda r^2}{2}\int_0^b ae^2 dt - \xi\eta^3 \|y\|^2 \text{ (since } |D_\eta|_1 \geq \eta \text{, see (3.56))}
\end{aligned} \tag{3.58}$$

Now, let $y \in E_1$. Then $r = 0$ and so (3.58) becomes

$$\varphi_\lambda(y) \leq 0,$$

hence

$$\varphi_\lambda(y)|_{E_1} \leq 0. \tag{3.59}$$

Next let $y \in E_2$. Then $y = v + R\,e$, with $\|v\| \leq R$ and so, from (3.58), we have

$$\begin{aligned}
\varphi_\lambda(y) &\leq \frac{R^2}{2}\|e'\|_2^2 - \frac{\lambda R^2}{2}\int_0^b ae^2 dt - \xi\eta^3\|v\|^2 - \xi\eta^3 R^2 \|e\|^2 \\
&\leq \frac{R^2}{2}\|e'\|_2^2 - \frac{\lambda R^2}{2}\int_0^b ae^2 dt - \|v\|^2 - R^2 \text{ (since } \xi \geq \frac{1}{\eta^3} \text{ and } \|e\| = 1) \\
&\leq \frac{R^2}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\|e'\|_2^2 - R^2 \text{ (see (2.3))} \\
&< \frac{R^2}{2} - R^2 \text{ (since } \lambda_k \leq \lambda < \lambda_{k+1} \text{ and } \|e\| = 1) \\
&< 0.
\end{aligned}$$

So

$$\varphi_\lambda(y)|_{E_2} < 0. \tag{3.60}$$

Finally, let $y \in E_3$. We have that $y = v + te$, with $\|v\| = R$, $0 < t < R$. From (3.58), we have

$$\begin{aligned}\varphi_\lambda(y) &\le \frac{t^2}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\|e'\|_2^2 - \|v\|^2 - t^2 \text{ (see (2.3) and} \\ &\quad \text{recall } \xi \ge \frac{1}{\eta^3},\ \|e\| = 1) \\ &\le \frac{t^2}{2} - t^2 \text{ (since } \lambda_k \le \lambda < \lambda_{k+1},\ \|e\| = 1) \\ &< 0,\end{aligned}$$

therefore

$$\varphi_\lambda(y)\,|_{E_3} < 0. \tag{3.61}$$

Combining (3.59), (3.60) and (3.61), we conclude that $\varphi_\lambda(y)\,|_{E_0} \le 0$. ■

Now, we are ready for the existence result, concerning problem (3.38).

Theorem 3.16 *If hypotheses $H(j)_2$ hold, then, for every $\lambda \in \mathbb{R}$, problem* (3.38) *has a nontrivial solution $x \in C_0^1[0,b]$.*

Proof. First, we assume that $\lambda_k \le \lambda < \lambda_{k+1}$, for some $k \ge 1$.

We can check that the sets $\{E, E_0\}$ and $D = H_+ \cap \partial B_\rho$ link in $W_0^{1,2}(0,b)$ (see Gasinski-Papageorgiou [22], p.643). So, because of Propositions 3.11, 3.13 and 3.15, we can apply Theorem 2.8 and obtain $x \in W_0^{1,2}(0,b)$, such that

$$\varphi_\lambda(x) \ge \beta > 0 = \varphi_\lambda(0) \tag{3.62}$$

and

$$0 \in \partial\varphi_\lambda(x) \text{ (i.e. } x \text{ is a critical point of } \varphi_\lambda). \tag{3.63}$$

From (3.62), we see that $x \ne 0$, while from (3.63) we infer that $x \in C_0^1[0,b]$ is a solution of (3.38). Next, let $\lambda < \lambda_1$. We know that, in this case,

$$|x|^2 = \|x'\|_2^2 - \lambda \int_0^b ax^2 dt$$

is an equivalent norm for the Sobolev space $W_0^{1,2}(0,b)$.

Let $u \in E(\lambda_1)$, $u \neq 0$ and $\mu > 0$. Then

$$\varphi_\lambda(\mu u) = \mu^2 |u|^2 - \int_0^b j(t, \mu u)\, dt. \tag{3.64}$$

By virtue of $H(j)_2(iii)$ and (v), given $\xi > 0$, we can find $\gamma_\xi \in L^1(T)_+$, such that

$$\xi |x|^2 - \gamma_\xi(t) \leq j(t, x) \text{ for a.a. } t \in T \text{ and all } x \in \mathbb{R}.$$

Using this in (3.64), we obtain

$$\begin{aligned} \varphi_\lambda(\mu u) &\leq \mu^2 |u|^2 - \xi\mu^2 \|u\|_2^2 + c_{36}, \text{ for some } c_{36} > 0 \\ &\leq \mu^2 (1 - \xi c_{37}) |u|^2 + c_{36}, \text{ for some } c_{37} > 0 \end{aligned} \tag{3.65}$$

(since all norms are equivalent on $E(\lambda_1)$). Since $\xi > 0$ was arbitrary, we choose $\xi > \frac{1}{c_{37}}$ and so, from (3.65), we infer that

$$\varphi_\lambda(\mu u) \to -\infty \text{ as } \mu \to +\infty. \tag{3.66}$$

Note that hypotheses $H(j)_2(iii)$ and (iv), imply that given $\varepsilon > 0$, we can find $c_{38} = c_{38}(\varepsilon) > 0$, such that

$$j(t, x) \leq \varepsilon x^2 + c_{38} |x|^\tau, \text{ with } \tau > 2.$$

Then, choosing $\varepsilon > 0$ small, we see that, for all $x \in W_0^{1,2}(0, b)$, we have

$$\varphi_\lambda(x) \geq c_{39} \|x\|^2 - c_{40} \|x\|^\tau, \tag{3.67}$$

for some $c_{39}, c_{40} > 0$, all $x \in W_0^{1,2}(0, b)$. Because $\tau > 2$, from (3.67) it follows that, if $\rho \in (0, 1)$ is small, then

$$\varphi_\lambda|_{\partial B_\rho} \geq \beta_0 > 0 = \varphi_\lambda(0). \tag{3.68}$$

Then (3.66), (3.67) and Proposition 3.12 permit the application of Theorem 2.9, which gives $x \in W_0^{1,2}(0, b)$ satisfying (3.62), (3.63). Then $x \in C_0^1[0, b]$, $x \neq 0$ and solves problem (3.38). ■

3.3 Multiple Solutions for Resonant Problems

In this section, we prove a multiplicity result for problem (3.1), when $\lambda = \lambda_1$. So, the problem under consideration is:

$$\begin{cases} -\Delta x(z) - \lambda_1 a(z) x(z) \in \partial j(z, x(z)) \quad \text{a.e. on } Z, \\ x|_{\partial Z} = 0. \end{cases} \tag{3.69}$$

The hypotheses on the nonsmooth potential are the following:

$H(j)_3:$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function, such that $j(z, 0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to j(z, x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z, x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z, x)$

$$|u| \le a_0(z) + c_0 |x|, \text{ with } a_0 \in L^\infty(Z)_+, c_0 > 0;$$

(iv) $\lim\limits_{|x|\to\infty} \frac{j(z,x)}{x^2} = 0$ and $\lim\limits_{|x|\to\infty} j(z, x) = -\infty$, uniformly, for a.a. $z \in Z$;

(v) there exist $\delta > 0$ and $\theta \in L^\infty(Z)_+$, $\theta(z) \le \lambda_2 - \lambda_1$ a.e. on Z, $\theta \ne \lambda_2 - \lambda_1$, such that

$$0 \le j(z, x) \le \frac{\theta(z)}{2} a(z) x^2 \text{ for a.a. } z \in Z \text{ and all } |x| \le \delta.$$

Remark 3.17 *These hypotheses imply that we have resonance at zero and nonuniform non resonance at infinity.*

Example 3.18 *The following function satisfy hypotheses $H(j)_3$. For the sake of simplicity, we drop the z-dependence.*

$$j(x) = \begin{cases} \frac{\theta}{2} x^2 & \text{if } |x| \le 1 \\ \xi \ln|x| - c|x| + \frac{\theta + 2c}{2} & \text{if } |x| > 1 \end{cases}, \text{ with } 0 < \theta < \lambda_2 - \lambda_1,\ \xi, c > 0.$$

Note that, if $\xi \ge \lambda_1$ and $c = \xi - \theta > 0$, then $j \in C^1(\mathbb{R})$.

We consider the Euler functional $\varphi : H_0^1(Z) \to \mathbb{R}$ for problem (3.69), defined by

$$\varphi(x) = \frac{1}{2} \|Dx\|_2^2 - \frac{\lambda_1}{2} \int_Z a x^2 dz - \int_Z j(z, x(z))\, dz \text{ for all } x \in H_0^1(Z).$$

We know that φ is Lipschitz continuous on bounded sets, hence it is locally Lipschitz.

Proposition 3.19 *If hypotheses $H(j)_3$ hold, then φ is coercive.*

Proof. We argue indirectly. So, suppose that φ is not coercive. Then we can find $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$, such that

$$\|x_n\| \to \infty \text{ and } \varphi(x_n) \leq M_5, \text{ for some } M_5 > 0, \text{ all } n \geq 1. \tag{3.70}$$

Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$, for all $n \geq 1$, and, so, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H_0^1(Z), \; y_n \to y \text{ in } L^2(Z), \; y_n(z) \to y(z) \text{ a.e. on } Z$$

and

$$|y_n(z)| \leq k(z), \text{ for a.a. } z \in Z, \text{ all } n \geq 1, \text{ with } k \in L^2(Z)_+. \tag{3.71}$$

By virtue of hypotheses $H(j)_3(iii)$ and (iv), we can find $c_{36} > 0$, such that

$$j(z,x) \leq c_{36}, \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R}. \tag{3.72}$$

Then, from (3.70) and (3.72), we have

$$\frac{M_5}{\|x_n\|^2} \geq \frac{1}{2}\|Dy_n\|_2^2 - \frac{\lambda_1}{2}\int_Z a y_n^2 dz - \frac{c_{37}}{\|x_n\|^2}, \text{with } c_{37} = c_{36}|Z|_N, \tag{3.73}$$

so

$$\lambda_1 \int_Z a y^2 dz \geq \|Dy\|_2^2 \text{ (see (3.71))}.$$

From (2.2), it follows that

$$\|Dy\|_2^2 = \lambda_1 \int_Z a y^2 dz,$$

therefore

$$y \in E(\lambda_1). \tag{3.74}$$

If $y = 0$, then from (3.73), it is clear that $\|Dy_n\|_2 \to 0$ and so $y_n \to 0$ in $H_0^1(Z)$, a contradiction to the fact $\|y_n\| = 1$, for all $n \geq 1$. Hence $y \neq 0$. Because $y \in E(\lambda_1)$ (see (3.74)), we have $|y(z)| > 0$, for all $z \in Z$ and so $|x_n(z)| \to +\infty$, for all $z \in Z$, as $n \to \infty$, hence $j(z, x_n(z)) \to -\infty$, for a.a. $z \in Z$, as $n \to \infty$. Then, by Fatou's Lemma (see (3.72)), we have

$$\lim_{n\to\infty} \int_Z j(z, x_n(z))\, dz = -\infty,$$

which contradicts (3.70). This proves that φ is coercive. ■

Corollary 3.20 *If hypotheses $H(j)_3$ hold, then φ is bounded below and satisfies the nonsmooth PS-condition.*

Proof. Since φ is coercive (see Proposition 3.19), it is bounded below.

Also, let $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ be such that

$$|\varphi(x_n)| \leq M_6 \text{ for some } M_6 > 0, \text{ all } n \geq 1 \tag{3.75}$$

and

$$m(x_n) \to 0 \text{ as } n \to \infty. \tag{3.76}$$

Because of (3.75) and the coercivity of φ (see Proposition 3.19), we have that

$$\{x_n\}_{n\geq 1} \subseteq H_0^1(Z) \text{ is bounded.}$$

So, we may assume that

$$x_n \xrightarrow{w} x \text{ in } H_0^1(Z) \text{ and } x_n \to x \text{ in } L^2(Z).$$

From (3.76), we have

$$\left| \langle A(x_n), x_n - x\rangle - \lambda_1 \int_Z a x_n (x_n - x)\, dz - \int_Z u_n (x_n - x)\, dz \right| \leq \varepsilon_n \|x_n - x\|, \tag{3.77}$$

with $\varepsilon_n \downarrow 0$. Evidently, we have

$$\int_Z a x_n (x_n - x)\, dz \to 0 \text{ and } \int_Z u_n (x_n - x)\, dz \to 0, \text{ as } n \to \infty.$$

So, from (3.77), it follows that

$$\lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0,$$

from which, as before, it follows that $x_n \to x$ in $H_0^1(Z)$. Therefore φ satisfies the nonsmooth PS-condition. ■

We consider the orthogonal direct sum decomposition

$$H_0^1(Z) = E(\lambda_1) \oplus V, \text{ with } V = E(\lambda_1)^{\perp}.$$

Proposition 3.21 *If hypotheses $H(j)_3$ hold, then we can find $r > 0$, such that*

$$\begin{cases} \varphi(x) \leq 0 & \text{if } x \in E(\lambda_1),\ \|x\| \leq r \\ \varphi(x) \geq 0 & \text{if } x \in V,\ \|x\| \leq r \end{cases}. \tag{3.78}$$

Proof. Let $x \in E(\lambda_1)$. Since $E(\lambda_1) \subseteq C^1(\overline{Z})$ and all norms on $E(\lambda_1)$ are equivalent (since $\dim E(\lambda_1) = 1$), we can find $r_1 > 0$, such that $x \in E(\lambda_1)$, with $\|x\| \leq r_1$, we have $|x(z)| \leq \delta$, for all $z \in \overline{Z}$. Hence, by virtue of hypothesis $H(j)_3(v)$, we have

$$0 \leq j(z, x(z)) \text{ a.e. on } Z. \tag{3.79}$$

Therefore, if $x \in E(\lambda_1)$, with $\|x\| \leq r_1$, then

$$\varphi(x) \leq \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda_1}{2}\int_Z ax^2 dz \text{ (see (3.79))},$$

so

$$\varphi(x) \leq 0, \text{ for all } x \in E(\lambda_1), \text{ with } \|x\| \leq r_1. \tag{3.80}$$

On the other hand, by virtue of hypotheses $H(j)_3(v)$ and (iii), we have

$$j(z, x) \leq \frac{\theta(z)}{2} a(z) x^2 + c_{38}|x|^{\tau}, \tag{3.81}$$

for a.a. $z \in Z$, all $x \in \mathbb{R}$, with $c_{38} > 0$, $2 < \tau$. Therefore, if $x \in V$, then

$$\begin{aligned} \varphi(x) &= \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda_1}{2}\int_Z ax^2 dz - \int_Z j(z, x)\, dz \\ &\geq \frac{1}{2}\|Dx\|_2^2 - \frac{1}{2}\int_Z (\lambda_1 + \theta(z))\, ax(z)^2\, dz - c_{39}\|Dx\|_2^{\tau}, \text{ for some } c_{39} > 0, \\ &\geq c_{40}\|Dx\|_2^2 - c_{39}\|Dx\|_2^{\tau}, \text{ for some } c_{40} > 0 \text{ (see Lemma 2.20 } (a)\text{)}. \end{aligned} \tag{3.82}$$

Since $\tau > 2$, from (3.82), we see that we can find $r_2 > 0$ such that

$$\varphi(x) \geq 0, \text{ for all } x \in V, \text{ with } \|x\| \leq r_2.$$

Finally, if we set $r = \min\{r_1, r_2\}$, then we see that (3.78) holds. ■

Theorem 3.22 *If hypotheses $H(j)_3$ hold, then problem* (3.69) *has, at least, two nontrivial solutions $x, y \in C_0^1(\overline{Z})$.*

Proof. Note that $\inf \varphi \leq 0$. If $\inf \varphi = 0$, then, by virtue of (3.78), all $x \in E(\lambda_1)$ with $0 < \|x\| \leq r$ are nontrivial critical points of φ, hence solutions of (3.69), which also belong in $C_0^1(\overline{Z})$ (see, for example, regularity results in [21]). If $\inf \varphi < 0$, then, by virtue of Corollary 3.20 and Proposition 3.21, we can apply Theorem 2.10 and produce $x, y \in H_0^1(Z)$, two nontrivial critical points of φ. Then x, y are nontrivial solutions of (3.69) and, also, $x, y \in C_0^1(\overline{Z})$. ■

Chapter 4

Problems with the Dirichlet p-Laplacian

4.1 Introduction

Let $Z \subseteq \mathbb{R}^n$ be a bounded domain with a C^2 boundary ∂Z. In this chapter we study the following nonlinear Dirichlet problem with nonsmooth potential (hemivariational inequality):

$$\begin{cases} -div\left(\|Dx(z)\|^{p-2} Dx(z)\right) \in \partial j(z, x(z)) \text{ a.e. on } Z \\ x \mid_{\partial Z} = 0, \ 1 < p < \infty. \end{cases} \tag{4.1}$$

Here $j : Z \times \mathbb{R} \to \mathbb{R}$ is a jointly measurable potential function, which is locally Lipschitz and in general nonsmooth in the $x \in \mathbb{R}$ variable, and $\partial j(z, x)$ denotes its generalized subdifferential. Our goal is to establish the existence of at least three nontrivial solutions for problem (4.1).

Recently three solutions theorems for the p-Laplacian equation with a smooth potential (i.e. $j(z, .) \in C^1(\mathbb{R})$ for a.a. $z \in Z$) were proved by Liu [36] and Li-Liu [35], but under conditions that implied a symmetric behavior near the origin. In contrast, our hypotheses imply a different behavior of $\partial j(z, .)$ as we approach 0^- and 0^+. In Liu [36], it is required that, if $f(z, x) = \partial j(z, x)$, then the "slope" $\frac{f(z,x)}{|x|^{p-2}x}$ stays strictly above $\lambda_2 > 0$ as $|x| \to 0$. We require that as $x \to 0^+$, the slopes $\left\{\frac{u}{x^{p-1}} : u \in \partial j(z, x)\right\}$ stay above $\lambda_1 > 0$ allowing partial interaction with the eigenvalue (nonuniform nonresonance). On the other hand as $x \to 0^-$ the "slopes" $\left\{\frac{u}{|x|^{p-2}x} : u \in \partial j(z, x)\right\}$ approach zero, which implies a p-sublinear behavior of $\partial j(z, .)$ and this distinguishes our work from that of Li-Liu [35], where $x \to f(z, x) =$

$\partial j(z,x)$ exhibits a p-sublinear behavior as $|x| \to 0$ (symmetric condition). Note that in the above condition $\lambda_2 > \lambda_1 > 0$ correspond to the first two eigenvalues of $\left(-\triangle_p, W_0^{1,p}(Z)\right)$ (see Chapter 2). The approach in both the works of Liu [36] and Li-Liu [35], is variational based on critical point theory. In this thesis, we use degree theory based on the degree map for certain multivalued perturbations of nonlinear $(S)_+$-operators, due to Hu-Papageorgiou [25] (see also Hu-Papageorgiou [26], Section 4.4). It appears that this work, is the first one on the existence of three nontrivial solutions for the partial p-Laplacian, which employs a degree theoretic approach.

In the works of Alves-Ding [2] and Garcia Azorero-Manfredi-Peral Alonso [18], where the authors examine eigenvalue problems driven by the partial p-Laplacian, the right hand side nonlinearity has the form

$$f(x) = \lambda |x|^{q-2} x + |x|^{r-2} x$$

with $1 < q < p < r \leqslant p^*$ (p^* being the critical Sobolev exponent) and $\lambda > 0$. So they treat problems with concave-convex nonlinearities and exclude asymptotically (at $\pm\infty$) p-linear problems, which is the class of problems considered in this work. In both works the approach is variational and they prove that there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, the nonlinear eigenvalue problem has two positive solutions.

In §4.2 we make the assumptions on the nonsmooth potential $j(z,x)$ and we establish the existence of at least two nontrivial solutions of constant sign. The proof is variational based on notions and techniques from nonsmooth analysis. In §4.3 we generate a third nontrivial solution. This is done using degree theory.

4.2 Two Solutions of Constant Sign

The hypotheses on the nonsmooth potential function $j(z,x)$ are the following:

$H(j)$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z,0) = 0$ a.e. on Z and

- (i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;
- (ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;
- (iii) for every $r > 0$, there exists $a_r \in L^\infty(Z)_+$ such that for almost all $z \in Z$, all $|x| \leq r$ and all $u \in \partial j(z,x)$, we have

$$|u| \leq a_r(z);$$

(*iv*) there exists $\theta \in L^{\infty}(Z)_{+}$ such that $\theta(z) \leq \lambda_1$ a.e. on Z with strict inequality on a set of positive measure and

$$\limsup_{x\to+\infty} \frac{u}{|x|^{p-2} x} \leq \theta(z)$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$;

(*v*) there exist functions $\eta_1,\, \eta_2 \in L^{\infty}(Z)_{+}$ with $\lambda_1 \leq \eta_1(z) \leq \eta_2(z)$ a.e. on Z, the first inequality is strict on a set of positive measure,

$$\eta_1(z) \leq \liminf_{x\to 0^+} \frac{u}{x^{p-1}} \leq \limsup_{x\to 0^+} \frac{u}{x^{p-1}} \leq \eta_2(z)$$

and

$$\lim_{x\to 0^-} \frac{u}{|x|^{p-2} x} = 0,$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$;

(*vi*) for a.a. $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$ we have

$$ux \geq 0. \qquad \text{(sign condition)}$$

Remark 4.1 *Note that at 0^+ and $\pm\infty$, we allow partial interaction with the principal eigenvalue $\lambda_1 > 0$ (nonuniform nonresonance). When $p = 2$ (semilinear problems), hypotheses $H(j)$ incorporate in our framework of analysis, the so-called asymptotically linear problems, which attracted considerable interest since the appearance of the pioneering work of Amann-Zehnder [3]. We point out, that the hypotheses are asymmetric with respect to 0^+ and 0^-. Moreover, it is worth mentioning that as we move from 0^+ to $+\infty$, we cross the principal eigenvalue $\lambda_1 > 0$. The following simple nonsmooth, locally Lipschitz function satisfies all hypotheses $H(j)$. For simplicity we drop the z dependence in its definition:*

$$j(x) = \begin{cases} \frac{c}{p}|x|^p + \frac{1}{r} - \frac{c}{p} & \text{if } x < -1 \\ \frac{1}{r}|x|^r & \text{if } x \in [-1,0] \\ \frac{\eta}{p}x^p & \text{if } x \in [0,1] \\ \frac{\theta}{p}x^p + \frac{1}{p}\ln x^p + \frac{\eta-\theta}{p} & \text{if } x > 1 \end{cases},$$

with $1 < p < r < \infty$ and $c,\, \theta < \lambda_1 < \eta$.

In this section, employing a variational approach based on nonsmooth analysis, we establish the existence of two smooth constant sign solutions for problem (4.1). To this end, we need to truncate the potential function and consider the energy functional corresponding to the truncated potential. So let $\tau_{\pm} : \mathbb{R} \to \mathbb{R}_{+}$ be the Lipschitz continuous truncation maps defined by

$$\tau_{+}(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases} \qquad \text{and } \tau_{-}(x) = \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases} .$$

We set

$$j_{\pm}(z, x) = j(z, \tau_{\pm}(x)).$$

Clearly for every $x \in \mathbb{R}$, $z \to j_{\pm}(z, x)$ are measurable and for almost all $z \in Z$, $x \to j_{\pm}(z, x)$ are locally Lipschitz. Note that, since by hypothesis $j(z, 0) = 0$ a.e. on Z, for almost all $z \in Z$ and all $x \leq 0$ (resp.$x \geq 0$) we have $j_{+}(z, x) = 0$ (resp.$j_{-}(z, x) = 0$). Moreover, from the nonsmooth chain rule (see Clarke [13], p.42), we have

$$\partial j_{+}(z, x) \subseteq \begin{cases} \{0\} & \text{if } x < 0 \\ \{r \partial j(z, 0) : r \in [0, 1]\} & \text{if } x = 0 \\ \partial j(z, x) & \text{if } x > 0 \end{cases} \tag{4.2}$$

and

$$\partial j_{-}(z, x) \subseteq \begin{cases} \partial j(z, x) & \text{if } x < 0 \\ \{r \partial j(z, 0) : r \in [0, 1]\} & \text{if } x = 0 \\ \{0\} & \text{if } x > 0 \end{cases} . \tag{4.3}$$

We consider the functionals $\varphi_{\pm} : W_0^{1,p}(Z) \to \mathbb{R}_{+}$ defined by

$$\varphi_{\pm}(x) = \frac{1}{p} \|Dx\|_p^p - \int_Z j_{\pm}(z, x(z))\, dz \text{ for all } x \in W_0^{1,p}(Z).$$

Then $\varphi_{\pm}$ are Lipschitz continuous on bounded sets, hence locally Lipschitz (see Clarke [13], p.83). In what follows, for notational simplicity we set

$$W_{+} = W_0^{1,p}(Z)_{+} = \left\{x \in W_0^{1,p}(Z) : x(z) \geq 0 \text{ a.e. on } Z\right\}$$

and

$$C_{+} = C_0^1\left(\overline{Z}\right)_{+} = \left\{x \in C_0^1\left(\overline{Z}\right) : x(z) \geq 0 \text{ for all } z \in Z\right\}$$

As we already mentioned in Chapter 2, *int* $C_{+} \neq \varnothing$ and *int* C_{+} is given by (2.9).

We consider the nonlinear operator $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|^{p-2} (Dx, Dy)_{\mathbb{R}^N} \, dz, \text{ for all } x, y \in W_0^{1,p}(Z), \tag{4.4}$$

where $W^{-1,p'}(Z) = W_0^{1,p}(Z)^*$,with $\frac{1}{p} + \frac{1}{p'} = 1$.

It is straightforward to check that A is bounded, continuous, monotone, hence it is maximal monotone. Also let $N_\pm : L^p(Z) \to 2^{L^{p'}(Z)}$ be the multifunctions defined by

$$N_\pm(x) = \left\{u \in L^{p'}(Z) : u(z) \in \partial j_\pm(z, x(z)) \text{ a.e. on } Z\right\} \tag{4.5}$$

for all $x \in W_0^{1,p}(Z)$. These are the multivalued Nemytskii operators corresponding to the subdifferentials $x \to .\partial j_\pm(z, x)$. We have

$$\partial\varphi_\pm(x) = A(x) - N_\pm(x) \text{ for all } x \in W_0^{1,p}(Z). \tag{4.6}$$

The next proposition is crucial in obtaining the constant sign solutions of problem (4.1). It underlines the significance of the nonuniform nonresonance condition at $\pm\infty$ (hypothesis $H(j)$ (iv)) and implies that the functionals $\varphi_\pm$ are coercive. This fact makes possible the use of variational techniques.

Proposition 4.2 *If $\theta \in L^\infty(Z)_+$ satisfies $\theta(z) \le \lambda_1$ a.e. on Z with strict inequality on a set of positive measure, then there exists $\xi_0 > 0$ such that*

$$\psi(x) = \|Dx\|_p^p - \int_Z \theta(z) |x(z)|^p \, dz \ge \xi_0 \|Dx\|_p^p \text{ for all } x \in W_0^{1,p}(Z). \tag{4.7}$$

Proof. From the variational characterization of λ_1 (see (2.7)) we have that $\psi \ge 0$. Suppose that (4.7) is not true. Since ψ is p-positively homogeneous, we can find $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(Z)$ such that

$$\|x_n\| = 1 \text{ for all } n \ge 1 \text{ and } \psi(x_n) \downarrow 0 \text{ as } n \to \infty.$$

We may assume that

$$x_n \xrightarrow{w} x \text{ in } W_0^{1,p}(Z), \; x_n \to x \text{ in } L^p(Z), \; x_n(z) \to x(z) \text{ a.e. on } Z$$

and

$$|x_n(z)| \le k(z) \text{ a.e. on } Z, \text{ for all } n \ge 1 \text{ with } k \in L^p(Z)_+.$$

We have $Dx_n \xrightarrow{w} Dx$ in $L^p\left(Z, \mathbb{R}^N\right)$ and so

$$\|Dx\|_p^p \le \liminf_{n\to\infty} \|Dx_n\|_p^p. \tag{4.8}$$

Also from the dominated convergence theorem, we have

$$\int_Z \theta(z)\,|x_n(z)|^p\,dz \to \int_Z \theta(z)\,|x(z)|^p\,dz \text{ as } n \to \infty. \tag{4.9}$$

From (4.8) and (4.9), we have

$$\psi(x) \le \lim_{n\to\infty} \psi(x_n) = 0,$$

hence

$$\|Dx\|_p^p \le \int_Z \theta(z)\,|x(z)|^p\,dz \le \lambda_1 \|x\|_p^p. \tag{4.10}$$

Because of (2.7) and since the infimum in that expression is attained at u_1, from (4.10) it follows that $x = 0$ or $x = \pm u_1$.

If $x = 0$, then $\|Dx_n\|_p \to 0$ and so by Poincaré's inequality $x_n \to 0$ in $W_0^{1,p}(Z)$, a contradiction to the fact that $\|x_n\| = 1$ for all $n \ge 1$.

If $x = \pm u_1$, then from the first inequality in (4.10) and since $|x(z)| = u_1(z) > 0$ for all $z \in Z$, due to the hypothesis on θ, we obtain

$$\|Dx\|_p^p < \lambda_1 \|x\|_p^p,$$

a contradiction to (2.7). This proves (4.7). ■

Using this proposition and a variational argument based on notions from nonsmooth analysis, we can produce the first two solutions of constant sign for problem (4.1).

Theorem 4.3 *If hypotheses $H(j)$ hold, then problem* (4.1) *has two solutions $x_0 \in int\ C_+$ and $v_0 \in -int\ C_+$.*

Proof. By virtue of hypothesis $H(j)$ (iv), given $\varepsilon > 0$, we can find $M = M(\varepsilon) > 0$ such that for almost all $z \in Z$, all $x \ge M$ and all $u \in \partial j(z,x) = \partial j_+(z,x)$, we have

$$u \le (\theta(z) + \varepsilon)\, x^{p-1}. \tag{4.11}$$

On the other hand, hypothesis $H(j)$ (iii) and (4.2) imply that there exists $a_\varepsilon \in L^\infty(Z)_+$ such that for almost all $z \in Z$, all $0 \le x < M$ and all $u \in \partial j_+(z,x)$, we have

$$u \le a_\varepsilon(z). \tag{4.12}$$

Finally note that for almost all $z \in Z$, all $x < 0$ and all $u \in \partial j_+(z,x)$, we have $u = 0$ (see (4.2)) From this together with (4.11) and (4.12), we deduce that for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j_+(z,x)$, one has

$$u \leq (\theta(z) + \varepsilon)|x|^{p-1} + a_\varepsilon(z) \tag{4.13}$$

By hypothesis $H(j)$ (ii), for all $z \in Z \backslash D$ with $|D|_N = 0$, the function $x \to j(z,x)$ is locally Lipschitz and so, by Rademacher's theorem, it is almost everywhere differentiable. Moreover, at any such point $r \in \mathbb{R}$ of differentiability, we have

$$\frac{d}{dr} j_+(z,r) \in \partial j_+(z,r)$$

(see Clarke [13], p.32), hence

$$\frac{d}{dr} j_+(z,r) \leq (\theta(z) + \varepsilon) r^{p-1} + a_\varepsilon(z) \text{ for a.a. } z \in Z$$

(see (4.13)). Integrating this inequality on $[0,x]$, $x > 0$, we obtain

$$j_+(z,x) \leq \frac{1}{p}(\theta(z) + \varepsilon) x^p + a_\varepsilon(z) x \text{ for a.a. } z \in Z, \text{ all } x \geq 0 \tag{4.14}$$

(recall that $j_+(z,0) = 0$ a.e. on Z). So, if $x \in W_+$, we have

$$\begin{aligned}
\varphi_+(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z j_+(z,x(z))\,dz \\
&\geq \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \theta|x|^p\,dz - \frac{\varepsilon}{p}\|x\|_p^p - \|a_\varepsilon\|_\infty \|x\|_1 \\
&\geq \frac{\xi_0}{p}\|Dx\|_p^p - \frac{\varepsilon}{p\lambda_1}\|Dx\|_p^p - c_1\|Dx\|_p \\
&= \frac{1}{p}\left(\xi_0 - \frac{\varepsilon}{\lambda_1}\right)\|Dx\|_p^p - c_1\|Dx\|_p,
\end{aligned} \tag{4.15}$$

for some $c_1 > 0$ (see Proposition 4.2 and (2.7)). If we choose $0 < \varepsilon < \lambda_1 \xi_0$, from (4.15) and Poincaré's inequality, we infer that $\varphi_+|_{W_+}$ is coercive. Moreover, due to the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we verify easily that φ_+ is weakly lower semicontinuous. So by the Weierstrass theorem, we can find $x_0 \in W_+$ such that

$$-\infty < m_+ = \inf_{x \in W_+} \varphi_+(x) = \varphi_+(x_0).$$

Hypothesis $H(j)(v)$ implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that for almost all $z \in Z$, all $0 < x \leq \delta$ and all $u \in \partial j_+(z,x)$, we have

$$(\eta_1(z) - \varepsilon) x^{p-1} \leq u \tag{4.16}$$

From (4.16) as above, we obtain

$$\frac{1}{p}\left(\eta_1(z)-\varepsilon\right)x^p \leq j_+(z,x) \text{ for a.a. } z\in Z, \text{ all } 0<x\leq\delta. \tag{4.17}$$

If $u_1 \in int\ C_+$ is the L^p-normalized principal eigenfunction (see Chapter 2), we can find $\beta > 0$ small, such that

$$\beta u_1(z) \in (0,\delta] \text{ for all } z\in Z. \tag{4.18}$$

Then by (4.17) and (4.18) we get

$$\begin{aligned}\varphi_+(\beta u_1) &= \frac{\beta^p}{p}\|Du_1\|_p^p - \int_Z j_+(z,\beta u_1(z))\,dz \\ &\leq \frac{\beta^p}{p}\|Du_1\|_p^p - \frac{\beta^p}{p}\int_Z (\eta_1(z)-\varepsilon)|u_1(z)|^p\,dz \\ &= \frac{\beta^p}{p}\left[\int_Z (\lambda_1-\eta_1(z))\,u_1(z)^p\,dz+\varepsilon\right]\end{aligned} \tag{4.19}$$

(since we assumed $\|u_1\|_p = 1$). Since $u_1(z) > 0$ for all $z \in Z$, we see that

$$\widehat{\xi} = \int_Z (\lambda_1-\eta_1(z))\,u_1(z)^p\,dz < 0.$$

Thus, if we choose $\varepsilon < -\widehat{\xi}$, from (4.19) we see that

$$\varphi_+(\beta u_1) < 0,$$

hence

$$m_+ = \varphi_+(x_0) < 0 = \varphi_+(0)\,, \text{ i.e. } x_0 \neq 0$$

From Clarke [13], p.32, we have

$$0 \in \partial\varphi_+(x_0) + N_{W_+}(x_0)\,, \tag{4.20}$$

with $N_{W_+}(x_0)$ being the normal cone to W_+ at x_0, defined by

$$N_{W_+}(x_0) = \left\{u^* \in W^{-1,p'}(Z) : \langle u^*, y-x_0\rangle \leq 0 \text{ for all } y\in W_+\right\} \tag{4.21}$$

(see Gasinski-Papageorgiou [22], p.526). From (4.20), we can find $x^* \in \partial\varphi_+(x_0)$ such that

$$-x^* \in N_{W_+}(x_0)\,. \tag{4.22}$$

From (4.6), we know that

$$x^* = A(x_0) - u_0 \text{ with } u_0 \in N_+(x_0)\,.$$

So from (4.21) and (4.22), we have

$$0 \leq \langle A(x_0) - u_0, y - x_0 \rangle \text{ for all } y \in W_+. \tag{4.23}$$

Let $\varepsilon > 0$ and $h \in W_0^{1,p}(Z)$ be given and set

$$y = (x_0 + \varepsilon h)^+ = x_0 + \varepsilon h + (x_0 + \varepsilon h)^- \in W_+.$$

We use this as a test function in (4.23) and we obtain

$$0 \leq \varepsilon \langle x^*, h \rangle + \left\langle x^*, (x_0 + \varepsilon h)^- \right\rangle$$

hence

$$-\left\langle x^*, (x_0 + \varepsilon h)^- \right\rangle \leq \varepsilon \langle x^*, h \rangle. \tag{4.24}$$

We let $Z_\varepsilon^- = \{z \in Z : (x_0 + \varepsilon h)(z) < 0\}$. We know that

$$D\left[(x_0 + \varepsilon h)^-\right](z) = \begin{cases} -D(x_0 + \varepsilon h)(z) & \text{if } z \in Z_\varepsilon^- \\ 0 & \text{otherwise} \end{cases}. \tag{4.25}$$

Then

$$\begin{aligned} & -\left\langle x^*, (x_0 + \varepsilon h)^- \right\rangle \\ & = -\left\langle A(x_0), (x_0 + \varepsilon h)^- \right\rangle + \int_Z u_0 (x_0 + \varepsilon h)^- dz \\ & = -\int_Z \|Dx_0\|^{p-2} \left(Dx_0, D(x_0 + \varepsilon h)^-\right)_{\mathbb{R}^N} dz + \int_Z u_0 (x_0 + \varepsilon h)^- dz. \end{aligned} \tag{4.26}$$

We estimate both integrals in the right hand side of (4.26). So by using (4.25) we have

$$\begin{aligned} & -\int_Z \|Dx_0\|^{p-2} \left(Dx_0, D(x_0 + \varepsilon h)^-\right)_{\mathbb{R}^N} dz \\ & = \int_{Z_\varepsilon^-} \|Dx_0\|^{p-2} (Dx_0, D(x_0 + \varepsilon h))_{\mathbb{R}^N} dz \\ & \geq \varepsilon \int_{Z_\varepsilon^-} \|Dx_0\|^{p-2} (Dx_0, Dh)_{\mathbb{R}^N} dz \\ & = \varepsilon \int_{Z_\varepsilon^- \cap \{x_0 > 0\}} \|Dx_0\|^{p-2} (Dx_0, Dh)_{\mathbb{R}^N} dz \end{aligned} \tag{4.27}$$

since $x_0 \in W_+$ and by Stampacchia's theorem, we have

$$Dx_0(z) = 0 \text{ a.e. on } \{x_0 = 0\}$$

(see Gasinski-Papageorgiou [22], p.195). Also we have

$$\int_Z u_0(x_0+\varepsilon h)^- \, dz = -\int_{Z_\varepsilon^-} u_0(x_0+\varepsilon h)\, dz \geq 0 \tag{4.28}$$

(see hypothesis $H(j)(vi)$).

We return to (4.26) and use (4.27) and (4.28). Then

$$-\left\langle x^*, (x_0+\varepsilon h)^- \right\rangle \geq \varepsilon \int_{Z_\varepsilon^- \cap \{x_0>0\}} \|Dx_0\|^{p-2} (Dx_0, Dh)_{\mathbb{R}^N}\, dz,$$

hence

$$\langle x^*, h\rangle \geq \int_{Z_\varepsilon^- \cap \{x_0>0\}} \|Dx_0\|^{p-2} (Dx_0, Dh)_{\mathbb{R}^N}\, dz \tag{4.29}$$

(see (4.24)). Since $|Z_\varepsilon^- \cap \{x_0>0\}|_N \to 0$ as $\varepsilon \downarrow 0$, if we pass to the limit as $\varepsilon \downarrow 0$ in (4.29), we obtain

$$0 \leq \langle x^*, h\rangle \text{ for all } h \in W_0^{1,p}(Z),$$

hence

$$A(x_0) = u_0. \tag{4.30}$$

From (4.30) we infer that

$$\begin{cases} -div\left(\|Dx_0(z)\|^{p-2} Dx_0(z)\right) = u_0(z) \text{ a.e. on } Z \\ x_0|_{\partial Z} = 0. \end{cases} \tag{4.31}$$

From.(4.31) and nonlinear regularity theory, we have that $x_0 \in C_+$, $x_0 \neq 0$. Then (4.31) and hypothesis $H(j)(vi)$ imply

$$div\left(\|Dx_0(z)\|^{p-2} Dx_0(z)\right) \leq 0 \text{ a.e. on } Z. \tag{4.32}$$

From (4.32) and the nonlinear strict maximum principle of Vasquez [60], we obtain $x_0 \in int\ C_+$. So from (4.2) we conclude that $x_0 \in int\ C_+$ is a solution of problem (4.1).

Similarly, working with the truncated locally Lipschitz functional φ_- and adapting the hypotheses $H(j)$ in a symmetric way, we obtain a second solution $v_0 \in -int\ C_+$. ∎

4.3 Three Nontrivial Smooth Solutions

Let $\varphi : W_0^{1,p}(Z) \to \mathbb{R}$ be the Euler functional for problem (4.1) defined by

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j(z, x(z))\, dz \text{ for all } x \in W_0^{1,p}(Z).$$

Evidently, φ is Lipschitz continuous on bounded sets, hence locally Lipschitz. Moreover, we have

$$\partial\varphi(x) = A(x) - N(x) \text{ for all } x \in W_0^{1,p}(Z), \tag{4.33}$$

where $N : L^p(Z) \to 2^{L^{p'}(Z)}$ is the multifunction defined by

$$N(x) = \left\{u \in L^{p'}(Z) : u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z\right\}, \text{ for all } x \in L^p(Z).$$

As we already mentioned, to produce the third nontrivial smooth solutions, we will use a degree theoretic argument, based on the degree map $\widehat{d}$. So we need to check that the items in (4.33) fit in the framework of that theory.

Proposition 4.4 *The nonlinear operator* $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ *defined by* (4.4) *is a bounded, continuous* $(S)_+$*-operator.*

Proof. From §4.2, we already know that A is bounded, continuous and maximal monotone. Suppose that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and assume that

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \leq 0. \tag{4.34}$$

Because A is maximal monotone, it is generalized pseudomonotone (see Gasinski-Papageorgiou [22], p.330). So from (4.34) it follows that

$$\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle,$$

hence

$$\|Dx_n\|_p \to \|Dx\|_p.$$

Note that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$. Because $L^p(Z, \mathbb{R}^N)$ is uniformly convex, it has the Kadec-Klee property (see Gasinski-Papageorgiou [21], p.722) and so $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$.

Hence by Poincaré's inequality, we have $x_n \to x$ in $W_0^{1,p}(Z)$, which shows that A is an $(S)_+$-operator. ■

Also for the multivalued Nemytskii operator $N : L^p(Z) \to 2^{L^{p'}(Z)}$ from Aizicovici-Papageorgiou-Staicu [1], we have:

Proposition 4.5 *If hypotheses $H(j)$ hold, then the multivalued operator $N : L^p(Z) \to 2^{L^{p'}(Z)}$ has nonempty, w-compact, convex values and it is usc from $L^p(Z)$ with the norm topology into $L^{p'}(Z)$ with the weak topology.*

From the Sobolev embedding theorem (see Gasinski-Papageorgiou [21], p.6), we know that $W_0^{1,p}(Z)$ is embedded compactly and densely in $L^p(Z)$. It follows that $L^{p'}(Z) = L^p(Z)^*$ is embedded compactly and densely in $W^{-1,p'}(Z) = W_0^{1,p}(Z)^*$. So from Proposition 4.5, we deduce the following:

Corollary 4.6 *If hypotheses $H(j)$ hold and*

$$N = N|_{W_0^{1,p}(Z)} : W_0^{1,p}(Z) \to 2^{W^{-1,p'}(Z)} \backslash \{\varnothing\},$$

then N is a multifunction of class (P).

From Theorem 4.3 and its proof, we know that $x_0 \in int\ C_+$ (resp. $v_0 \in -int\ C_+$) is a minimizer of φ_+(resp. φ_-). Since $\varphi_+ |_{C_+} = \varphi$ (resp. $\varphi_- |_{C_-} = \varphi$) we infer that x_0, v_0 are both local $C_0^1(\overline{Z})$ – minimizers of φ. But then from Gasinski-Papageorgiou [21], p.655-656 (see also Kyritsi-Papageorgiou [34]), we have that x_0 and v_0 are local $W_0^{1,p}(Z)$ - minimizers of φ. Therefore from Aizicovici-Papageorgiou-Staicu [1], we have

Proposition 4.7 *If hypotheses $H(j)$ hold and $x_0 \in int\ C_+$, $v_0 \in -int\ C_+$ are the solutions obtained in Theorem 4.3, then there exists $r > 0$ small such that*

$$\widehat{d}(\partial\varphi, B_r(x_0), 0) = \widehat{d}(\partial\varphi, B_r(v_0), 0) = 1.$$

Next we calculate the $\widehat{d}$ degree of $\partial\varphi$ for small balls.

Proposition 4.8 *If hypotheses $H(j)$ hold, then there exists $\rho_0 > 0$ such that*

$$\widehat{d}(\partial\varphi, B_\rho, 0) = \widehat{d}(A - N, B_\rho, 0) = 0 \text{ for all } 0 < \rho \leq \rho_0.$$

Proof. Let $K_+ : W_0^{1,p}(Z) \to W_0^{-1,p'}(Z)$ be the mapping defined by

$$K_+(x)(\cdot) = \left(x^+(\cdot)\right)^{p-1} \text{ for all } x \in W_0^{1,p}(Z).$$

Evidently, this is a completely continuous map (recall that $L^{p'}(Z)$ is embedded compactly into $W_0^{-1,p'}(Z) = W_0^{1,p}(Z)^*$). So, if we consider the map $h_1 : [0,1] \times W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ defined by

$$h_1(t,x) = A(x) - (1-t)\eta_1 K_+(x) - tN(x),$$

for all $(t,x) \in [0,1] \times W_0^{1,p}(Z)$, then $h_1(.,.)$ is an admissible homotopy.

Claim 3 *There exists $\rho_0 > 0$ such that $0 \notin h_1(t,x)$ for all $t \in [0,1]$, all $\|x\| = \rho$ and all $0 < \rho \leq \rho_0$.*

We argue indirectly. So suppose that Claim 3 is not true. Then we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ such that

$$t_n \to t \in [0,1], \ \|x_n\| \to 0 \text{ and } 0 \in h_1(t_n, x_n) \text{ for all } n \geq 1. \tag{4.35}$$

From the inclusion in (4.35), we have

$$A(x_n) = (1-t_n)\eta_1 K_+(x_n) + t_n u_n \text{ with } u_n \in N(x_n) \text{ for all } n \geq 1. \tag{4.36}$$

Setting

$$y_n = \frac{x_n}{\|x_n\|}$$

we have

$$A(y_n) = (1-t_n)\eta_1 K_+(y_n) + t_n \frac{u_n}{\|x_n\|^{p-1}} \text{ for all } n \geq 1.$$

Moreover, by passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(Z),\ y_n \to y \text{ in } L^p(Z),\ y_n(z) \to y(z) \text{ a.e. on } Z$$

and

$$|y_n(z)| \leq k(z) \text{ a.e. on } Z, \text{ for all } n \geq 1 \text{ and some } k \in L^{p'}(Z)_+.$$

By virtue of hypotheses $H(j)$ (iii), (iv) and (v), we have that

$$|u| \leq c_1 |x|^{p-1} \text{ for a.a. } z \in Z, \text{all } x \in \mathbb{R} \text{ and all } u \in \partial j(z,x), \tag{4.37}$$

with $c_1 > 0$. Relation (4.37) implies that

$$\left\{\frac{u_n}{\|x_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'}(Z) \text{ is bounded.}$$

So we may assume that

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h_0 \text{ in } L^{p'}(Z) \text{ for some } h_0 \in L^{p'}(Z). \tag{4.38}$$

For every $\varepsilon > 0$ and $n \geq 1$, we introduce the sets

$$C_{\varepsilon,n}^+ = \left\{z \in Z : x_n(z) > 0,\ \eta_1(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)^{p-1}} \leq \eta_2(z) + \varepsilon\right\}$$

and

$$C_{\varepsilon,n}^{-}=\left\{z\in Z:x_{n}\left(z\right)<0,\ -\varepsilon\leq\frac{u_{n}\left(z\right)}{\left|x_{n}\left(z\right)\right|^{p-2}x_{n}\left(z\right)}\leq\varepsilon\right\}.$$

Note that $x_n(z)\to 0^+$ a.e. on $\{y>0\}$ and $x_n(z)\to 0^-$ a.e. on $\{y<0\}$. So, by virtue of hypothesis $H(j)(v)$, we have

$$\chi_{C_{\varepsilon,n}^{+}}\left(z\right)\to 1 \text{ a.e. on } \{y>0\} \text{ and } \chi_{C_{\varepsilon,n}^{-}}\left(z\right)\to 1 \text{ a.e. on } \{y<0\}.$$

Using (4.38), we obtain

$$\chi_{C_{\varepsilon,n}^{+}}\frac{u_{n}}{\left\|x_{n}\right\|^{p-1}}\xrightarrow{w}h_{0}\text{ in }L^{p'}\left(\{y>0\}\right)$$

and

$$\chi_{C_{\varepsilon,n}^{-}}\frac{u_{n}}{\left\|x_{n}\right\|^{p-1}}\xrightarrow{w}h_{0}\text{ in }L^{p'}\left(\{y<0\}\right).$$

From the definitions of the sets $C_{\varepsilon,n}^{+}$ and $C_{\varepsilon,n}^{-}$, we have

$$\begin{aligned}\chi_{C_{\varepsilon,n}^{+}}\left(z\right)\left(\eta_{1}\left(z\right)-\varepsilon\right)y_{n}\left(z\right)^{p-1}&\leq\chi_{C_{\varepsilon,n}^{+}}\left(z\right)\frac{u_{n}\left(z\right)}{\left\|x_{n}\right\|^{p-1}}\\&\leq\chi_{C_{\varepsilon,n}^{+}}\left(z\right)\left(\eta_{2}\left(z\right)+\varepsilon\right)y_{n}\left(z\right)^{p-1}\text{ a.e. on }\{y>0\}\end{aligned}$$

and

$$\begin{aligned}-\chi_{C_{\varepsilon,n}^{-}}\left(z\right)\varepsilon\left|y_{n}\left(z\right)\right|^{p-1}&\leq\chi_{C_{\varepsilon,n}^{-}}\left(z\right)\frac{u_{n}\left(z\right)}{\left\|x_{n}\right\|^{p-1}}\\&\leq\chi_{C_{\varepsilon,n}^{-}}\left(z\right)\varepsilon\left|y_{n}\left(z\right)\right|^{p-1}\text{ a.e. on }\{y<0\}.\end{aligned}$$

Taking weak limit in $L^{p'}(\{y>0\})$ and $L^{p'}(\{y<0\})$ respectively and using Mazur's lemma we obtain

$$\left(\eta_{1}\left(z\right)-\varepsilon\right)y\left(z\right)^{p-1}\leq h_{0}\left(z\right)\leq\left(\eta_{2}\left(z\right)+\varepsilon\right)y\left(z\right)^{p-1}\text{ a.e. on }\{y>0\}$$

and

$$-\varepsilon\left|y\left(z\right)\right|^{p-1}\leq h_{0}\left(z\right)\leq\varepsilon\left|y\left(z\right)\right|^{p-1}\text{ a.e. on }\{y<0\}.$$

Passing to the limit as $\varepsilon\downarrow 0$, yields

$$\eta_{1}\left(z\right)y\left(z\right)^{p-1}\leq h_{0}\left(z\right)\leq\eta_{2}\left(z\right)y\left(z\right)^{p-1}\text{ a.e. on }\{y>0\}\tag{4.39}$$

and

$$h_0(z) = 0 \text{ a.e. on } \{y < 0\}. \tag{4.40}$$

Moreover, it is clear from (4.37) that

$$h_0(z) = 0 \text{ a.e. on } \{y = 0\}. \tag{4.41}$$

From (4.39), (4.40) and (4.41) it follows that

$$h_0(z) = g_0(z)\, y^+(z)^{p-1} \text{ a.e. on } Z \tag{4.42}$$

with

$$g_0 \in L^\infty(Z)_+, \eta_1(z) \le g_0(z) \le \eta_2(z) \text{ a.e.on } Z.$$

Note that

$$\begin{aligned} &\langle A(y_n), y_n - y\rangle \\ &= \int_Z \left((1-t_n)\,\eta_1 \left(y_n^+\right)^{p-1} + t_n \frac{u_n}{\|x_n\|^{p-1}}\right)(y_n - y)\, dz \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.43}$$

But A is of type $(S)_+$ (see Proposition 4.4). So from (4.43) it follows that

$$y_n \to y \text{ in } W_0^{1,p}(Z), \text{ hence } \|y\| = 1, \text{ i.e. } y \ne 0.$$

Moreover, in the limit as $n \to \infty$, we have

$$A(y) = \xi K_+(y)$$

where $\xi \in L^\infty(Z)_+$, $\xi = (1-t)\,\eta_1 + tg_0$. Hence acting with the test function $-y^- \in W_0^{1,p}(Z)$, we see that $y \ge 0$, $y \ne 0$. Therefore

$$\begin{cases} -div\left(\|Dy(z)\|^{p-2} Dy(z)\right) = \xi(z)\,|y(z)|^{p-2}\, y(z) \text{ a.e. on } Z, \\ y|_{\partial Z} = 0,\ y \ne 0. \end{cases} \tag{4.44}$$

Note that $\lambda_1 \le \xi(z)$ a.e. on Z, $\lambda_1 \ne \xi$. Therefore

$$\widehat{\lambda}_1(\xi) < \widehat{\lambda}_1(\lambda_1) = 1 \tag{4.45}$$

Combining (4.44) and (4.45), we infer that $y \in C^1\left(\overline{Z}\right)$ must change sign, a contradiction to the fact that $y \ge 0$. So Claim 3 is true.

Then the homotopy invariance property, implies

$$\widehat{d}\,(A-N,B_\rho,0)=\widehat{d}\,(A-\eta_1K_+,B_\rho,0)\ \text{ for all } 0<\rho\le\rho_0. \tag{4.46}$$

To compute $\widehat{d}\,(A-\eta_1K_+,B_\rho,0)$, we introduce the homotopy $h_2:[0,1]\times W_0^{1,p}(Z)\to W^{-1,p'}(Z)$ defined by

$$h_2(t,x)=A(x)-\eta_1K_+(x)-t\beta,$$

for all $(t,x)\in[0,1]\times W_0^{1,p}(Z)$, with $\beta\in L^\infty(Z)_+$. Suppose that $h_2(t,x)=0$ for all $t\in[0,1]$ and all $\|x\|=\rho$. Then

$$A(x)=\eta_1K_+(x)+t\beta.$$

Acting with the test function $-x^-\in W_0^{1,p}(Z)$, we obtain $x\ge0$. Hence

$$\begin{cases} -div\left(\|Dx(z)\|^{p-2}Dx(z)\right)=\eta_1(z)\,|x(z)|^{p-2}x(z)+t\beta(z) \text{ a.e. on } Z,\\ x|_{\partial Z}=0,\ x\neq0\end{cases}$$

and this by the antimaximum principle of Godoy-Gossez-Paczka [23] implies that $x\in -int\ C_+$, a contradiction to the fact that $x\ge0$. So

$$\widehat{d}\,(A-\eta_1K_+,B_\rho,0)=\widehat{d}\,(A-\eta_1K_+-\beta,B_\rho,0)=0 \text{ for all } 0<\rho\le\rho_0,$$

hence

$$\widehat{d}\,(A-N,B_\rho,0)=0 \text{ for all } 0<\rho\le\rho_0$$

(see (4.46)). ■

Next we conduct a similar computation for large balls. In this case we have:

Proposition 4.9 *If hypotheses $H(j)$ hold, then there exists $R_0>0$ such that*

$$\widehat{d}\,(\partial\varphi,B_R,0)=\widehat{d}\,(A-N,B_R,0)=1 \textit{ for all } R\ge R_0.$$

Proof. We consider the admissible homotopy $h_3:[0,1]\times W_0^{1,p}(Z)\to 2^{W^{-1,p'}(Z)}$ defined by

$$h_3(t,x)=A(x)-tN(x)-(1-t)\,\theta K(x)$$

where $K(x)=|x|^{p-2}x$.

Claim 4 *There exists $R_0 > 0$ such that $0 \notin h_3(t,x)$ for all $t \in [0,1]$ and all $x \in W_0^{1,p}(Z)$ with $\|x\| = R$ and all $R \geq R_0$.*

As in the previous proof, we argue by contradiction. So suppose we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ such that

$$t_n \to t \in [0,1], \ \|x_n\| \to +\infty$$

and

$$A(x_n) = t_n u_n + (1-t_n)\,\theta K(x_n) \ \text{ with } u_n \in N(x_n), \ n \geq 1.$$

If

$$y_n = \frac{x_n}{\|x_n\|}, n \geq 1,$$

then

$$A(y_n) = t_n \frac{u_n}{\|x_n\|^{p-1}} + (1-t_n)\,\theta K(y_n). \tag{4.47}$$

Arguing as in the previous proof, using this time hypothesis $H(j)(iv)$, we obtain

$$y_n \to y \text{ in } W_0^{1,p}(Z), \ \text{ hence } \ \|y\| = 1,$$

and

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h = g\,|y|^{p-2}\,y \text{ with } g \in L^\infty(Z)_+, \ g(z) \leq \theta(z) \ \text{ a.e. on } Z.$$

So if we pass to the limit as $n \to \infty$ in (4.47), we obtain

$$A(y) = \widehat{\xi} K(y) \ \text{ with } \widehat{\xi} \in L^\infty(Z), \ \widehat{\xi}(z) = tg + (1-t)\,\theta \leq \theta(z) \ \text{ a.e. on } Z,$$

hence

$$\begin{cases} -div\left(\|Dy(z)\|^{p-2} Dy(z)\right) = \widehat{\xi}(z)\,|y(z)|^{p-2}\,y(z) \text{ a.e. on } Z, \\ y|_{\partial Z} = 0. \end{cases} \tag{4.48}$$

But

$$\widehat{\lambda}_1\left(\widehat{\xi}\right) > \widehat{\lambda}_1(\lambda_1) = 1.$$

So from (4.48) it follows that $y = 0$, a contradiction to the fact that $\|y\| = 1$. So Claim 4 is true.

Then the homotopy invariance property implies that

$$\widehat{d}(A-N,B_R,0)=\widehat{d}(A-\theta K,B_R,0) \text{ for all } R\geq R_0 \tag{4.49}$$

But from Drabek-Kufner-Nicolosi [15], we have

$$\widehat{d}(A-\theta K,B_R,0)=1 \text{ for } R>0. \tag{4.50}$$

So from (4.49) and (4.50), we conclude that

$$\widehat{d}(A-N,B_R,0)=1 \text{ for all } R\geq R_0.$$

■

Now we are ready for the three solutions theorem for problem (4.1).

Theorem 4.10 *If hypotheses $H(j)$ hold, then problem* (4.1) *has at least three nontrivial solutions $x_0\in int\ C_+$, $v_0\in -int\ C_+$ and $y_0\in C_0^1(\overline{Z})$.*

Proof. We already have two solutions $x_0\in int\ C_+$, $v_0\in -int\ C_+$ from Theorem 4.3. Then from the domain additivity and excision properties of the degree we have

$$\begin{aligned}\widehat{d}(A-N,B_R,0)&=\widehat{d}(A-N,B_\rho,0)\\&\quad+\widehat{d}(A-N,B_r(x_0),0)+\widehat{d}(A-N,B_r(v_0),0)\\&\quad+\widehat{d}\left(A-N,B_R\backslash\left(\overline{B_\rho\cup B_r(x_0)\cup B_r(v_0)}\right),0\right)\end{aligned} \tag{4.51}$$

with $R\geq R_0$ large and $r>0$ and $0<\rho\leq\rho_0$ small such that

$$\rho<R,\ B_r(x_0)\cap\overline{B_\rho}=\varnothing \text{ and } \overline{B}_r(x_0),\ \overline{B}_r(v_0)\subseteq B_R.$$

Then, using Propositions 4.7, 4.8 and 4.9, from (4.51), we obtain

$$-1=\widehat{d}\left(A-N,B_R\backslash\left(\overline{B_\rho\cup B_r(x_0)\cup B_r(v_0)}\right),0\right).$$

From the solution property, we obtain $y_0\in B_R\backslash\left(\overline{B_\rho\cup B_r(x_0)\cup B_r(v_0)}\right)$, hence $y_0\neq 0$, $y_0\neq x_0$, $y_0\neq v_0$ such that

$$A(y_0)=\widehat{u}_0 \text{ with } \widehat{u}_0\in N(y_0),$$

hence

$$\begin{cases}-div\left(\|D(y_0)(z)\|^{p-2}D(y_0)(z)\right)=\widehat{u}_0(z) \text{ a.e. on } Z,\\ (y_0)|_{\partial Z}=0,\end{cases}$$

therefore from regularity theory $y_0\in C_0^1(\overline{Z})$ is a nontrivial solution of (4.1), distinct from x_0 and v_0. ■

Chapter 5

Semilinear Resonant Neumann Problems

5.1 Introduction

Let $Z \subseteq \mathbb{R}^n$ be a bounded domain with a C^2-boundary ∂Z. We consider the following semilinear Neumann problem with a nonsmooth potential (hemivariational inequality):

$$\begin{cases} -\Delta x\left(z\right) \in \partial j\left(z, x\left(z\right)\right) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0, \text{ on } \partial Z. \end{cases} \tag{5.1}$$

Here $j\left(z,x\right)$ is a measurable potential function, which is locally Lipschitz and in general nonsmooth in the x-variable, and $\partial j\left(z,x\right)$ is the generalized subdifferential of $x \to j\left(z,x\right)$. Also, $n\left(z\right)$ is the outward unit normal at $z \in \partial Z$ and $\frac{\partial x}{\partial n} = \left(Dx, n\right)_{\mathbb{R}^n}$ in the sense of traces.

Tang-Wu [59] studied problem (5.1) with a smooth potential, i. e., $j(z, \cdot) \in C^1\left(\mathbb{R}\right)$.

In this chapter we produce an answer to the open problem of Tang-Wu [59], whether their multiplicity result, for problems which are resonant at zero between two successive eigenvalues λ_k, λ_{k+1}, is actually valid when complete resonance occurs also with respect to λ_{k+1} (double resonance situation; see Remark 4 of Tang-Wu [59]). For this, we relax the hypotheses of Tang-Wu [59] and we also allow the potential function to be nonsmooth.

We also prove a multiplicity result for semilinear Neumann problems which are doubly resonant at the origin with respect to any spectral interval $[\lambda_k, \lambda_{k+1}]$.

We should mention that existence theorems for semilinear resonant Neumann problems were proved by Iannacci-Nkashama [29], [30], Kuo [33], Mawhin-Ward-Willem [39] and Rabinowitz [55]. In Iannacci-Nkashama [29], $N = 1$, i.e., the equation is an ordinary differential equation. In Iannacci-Nkashama [30] and Kuo [33], the authors use variants of the Landesman-Lazer asymptotic conditions. In all three papers the approach is degree theoretic. Mawhin-Ward-Willem [39] use the monotonicity condition, while Rabinowitz [55] uses the periodicity condition. In both papers the approach is variational based on critical point theory.

With the exception of Iannacci-Nkashama [30], in all of the aforementioned works, the resonance is with respect to the principal eigenvalue $\lambda_0 = 0$. None of these works deals with the doubly resonant situation and also they do not address the question of existence of multiple solutions.

Our approach is variational based on the nonsmooth critical point theory.

5.2 Multiplicity Result

The hypotheses on the nonsmooth potential function $j(z,x)$ are the following:

$H(j)$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z,0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have

$$|u| \leq a(z) + c|x|^{r-1},$$

where $a \in L^{\infty}(Z)_{+}$, $\quad c > 0, \quad 1 < r < 2^{*} = \begin{cases} \frac{2N}{N-2} & \text{if } n > 2 \\ +\infty & \text{if } n = 1,\ 2 \end{cases}$;

(iv) $j(z,x) \to -\infty$ for almost all $z \in C \subseteq Z$, with $|C|_N > 0$ as $|x| \to \infty$ and $j(z,x) \leq \eta(z)$ for almost all $z \in Z$, all $x \in \mathbb{R}$ with $\eta \in L^1(Z)_{+}$;

(v) there exist $\delta > 0$ and an integer $m \geq 0$, such that for almost all $z \in Z$, all $0 < |x| \leq \delta$ and all $u \in \partial j(z,x)$

$$\lambda_m \leq \frac{u}{x} \leq \lambda_{m+1}.$$

Remark 5.1 *Note that in hypothesis $H(j)(iv)$ the convergence to $-\infty$ occurs only for $z \in C$ and not for almost all $z \in Z$ as in Tang-Wu [59]. Moreover, the convergence need not to be uniform in $z \in C$, while in Tang-Wu [59] it is uniform for a.a. $z \in Z$ (see*

Theorem 2 in [59]). Hypothesis $H(j)(v)$ is the double resonance condition at $x = 0$ with respect to the spectral interval $[\lambda_m, \lambda_{m+1}]$. Complete resonance is possible at both ends of the interval. In contrast, Tang-Wu [59] allow complete resonance with respect to λ_m and they assume nonuniform nonresonance with respect to λ_{m+1}. If the potential $j(z,x)$ is z-independent, then in the setting of Tang-Wu [59] the quotient $\frac{u}{x}$ stays strictly below λ_{m+1} near zero. Finally in Tang-Wu [59], $j(z,\cdot) \in C^1(\mathbb{R})$ for all $z \in Z$.

Example 5.2 *The following locally Lipschitz function $j(x)$ satisfies hypotheses $H(j)$. For simplicity, we drop the z-dependence.*

$$j(x) = \begin{cases} \frac{\lambda_m}{2} x^2 & \text{if } |x| \leq 1 \\ -|x| + \frac{c}{x^2} + \frac{\lambda_m}{2} + 1 - c & \text{if } |x| > 1 \end{cases},$$

with $m \geq 0$ and $c \in \mathbb{R}$. Note that if $c = -\frac{\lambda_{m+1}}{2}$, then $j \in C^1(\mathbb{R})$.

Example 5.3 *Also, the following function satisfies hypotheses $H(j)$, but not those in Theorem 2 of Tang-Wu [59]:*

$$j(z,x) = \int_0^x f(z,s)\,ds, \text{ with } f(z,x) = \lambda_m x - \chi_C(z)(|x|-1)^+ - \lambda_m (x-1)^+,$$

where $C \subseteq Z$ is measurable with $|C|_N > 0$ and for every $u \in \mathbb{R}$, we denote $u^+ := \max\{u, 0\}$.

Let $\varphi : H^1(Z) \to \mathbb{R}$ be the Euler functional for problem (5.1) defined by

$$\varphi(x) = \frac{1}{2}\|Dx\|_2^2 - \int_Z j(z, x(z))\,dz$$

for all $x \in H_0^1(Z)$. We know that φ is Lipschitz continuous on bounded sets, hence locally Lipschitz (see Clarke [13], p.83).

Proposition 5.4 *If hypotheses $H(j)$ hold, then φ is coercive.*

Proof. We argue indirectly. So suppose that the proposition is not true. We can find $\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ such that $\|x_n\| \to \infty$ and

$$\varphi(x_n) = \frac{1}{2}\|Dx_n\|_2^2 - \int_Z j(z, x_n(z))\,dz \leq M, \tag{5.2}$$

for some $M > 0$, all $n \geq 1$. We consider the orthogonal direct sum decomposition

$$H^1(Z) = E(\lambda_0) \oplus V,$$

with $E(\lambda_0) = \mathbb{R}$, $V = E(\lambda_0)^{\perp}$. For each $n \geq 1$, we write in an unique way

$$x_n = \overline{x}_n + \widehat{x}_n,$$

with $\overline{x}_n \in E(\lambda_0) = \mathbb{R}$ and $\widehat{x}_n \in V$.

Because of hypothesis $H(j)(iv)$ and Lemmata 2 and 3 of Tang-Wu [58], given $\varepsilon > 0$, we can find $D_\varepsilon \subseteq C$ measurable set with $|C \backslash D_\varepsilon|_N < \varepsilon$ and functions $g \in C(\mathbb{R}_+)$, $g \geq 0$, $h \in L^1(C)_+$ such that

$$g \text{ is subadditive (i.e., } g(x+y) \leq g(x) + g(y) \text{ for all } x,\, y \in \mathbb{R}); \tag{5.3}$$

$$g \text{ is coercive (i.e., } g(x) \to +\infty \text{ as } |x| \to \infty); \tag{5.4}$$

$$g(x) \leq 4 + |x| \text{ for all } x \in \mathbb{R} \tag{5.5}$$

and

$$j(z,x) \leq h(z) - g(x) \text{ for a.a. } z \in D_\varepsilon \text{ and all } x \in \mathbb{R}. \tag{5.6}$$

Then, by (5.3),

$$g(\overline{x}_n) = g(x_n(z) - \widehat{x}_n(z)) \leq g(x_n(z)) + g(-\widehat{x}_n(z)),$$

hence

$$g(\overline{x}_n) - g(-\widehat{x}_n(z)) \leq g(x_n(z)) \tag{5.7}$$

for all $z \in Z$ and all $n \geq 1$. Therefore, by (5.6),

$$j(z, x_n(z)) \leq h(z) - g(x_n(z)) \leq h(z) - g(\overline{x}_n) + g(-\widehat{x}_n(z)) \tag{5.8}$$

for all $n \geq 1$, a.a. $z \in D_\varepsilon$ (see (5.7)).

We return to (5.3) and use (5.8) .Then, in view of hypothesis $H(j)(iv)$, for all $n \geq 1$,

$$\begin{aligned}
\varphi(x_n) &= \frac{1}{2}\|D\widehat{x}_n\|_2^2 - \int_{D_\varepsilon} j(z, x_n(z))\,dz - \int_{Z\backslash D_\varepsilon} j(z, x_n(z))\,dz \\
&\geq \frac{1}{2}\|D\widehat{x}_n\|_2^2 + g(\overline{x}_n)\,|D_\varepsilon|_N - \int_{D_\varepsilon} g(-\widehat{x}_n(z))\,dz - \|h\|_1 - \|\eta\|_1 \\
&\geq \frac{1}{2}\|D\widehat{x}_n\|_2^2 + g(\overline{x}_n)\,|D_\varepsilon|_N - \int_{Z} g(-\widehat{x}_n(z))\,dz - c_1, \\
&\geq \frac{1}{2}\|D\widehat{x}_n\|_2^2 + g(\overline{x}_n)\,|D_\varepsilon|_N - c_2\|D\widehat{x}_n\|_2 - c_3,
\end{aligned} \tag{5.9}$$

for $c_1 = \|h\|_1 + \|\eta\|_1$ and some $c_2,\, c_3 > 0$.

In the last inequality we have used (5.5) and the Poincaré-Wirtinger inequality.

Since $\|x_n\| \to \infty$, by the Poincaré-Wirtinger inequality, we have

$$|\overline{x}_n| \to \infty \text{ and/or } \|D\widehat{x}_n\|_2 \to \infty.$$

So from (5.9) and since g is coercive (see (5.4)), we deduce that

$$\varphi(x_n) \to \infty,$$

a contradiction to the fact that

$$\varphi(x_n) \leq M \text{ for all } n \geq 1.$$

This proves the coercivity of all φ. ■

Corollary 5.5 *If hypotheses $H(j)$ hold, then φ is bounded below and satisfies the non-smooth PS-condition.*

Proof. Because φ is coercive (see Proposition 5.4), it is bounded below.

Also, let $\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ be a sequence such that

$$|\varphi(x_n)| \leq \widehat{M} \text{ for some } \widehat{M} > 0, \text{ all } n \geq 1 \text{ and } m(x_n) \to 0 \text{ as } n \to \infty. \tag{5.10}$$

Since $\partial\varphi(x_n) \subseteq H^1(Z)^*$ is w-compact and the norm functional in a Banach space is weakly lower semicontinuous, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$.

Let $A \in \mathcal{L}(H^1(Z), H^1(Z)^*)$ be the operator defined by

$$\langle A(x), y\rangle = \int_Z (Dx, Dy)_{\mathbb{R}^N}\, dz \text{ for all } x, y \in H^1(Z).$$

We know that

$$x_n^* = A(x_n) - u_n,$$

with $u_n \in L^{r'}(Z)$ $(\frac{1}{r} + \frac{1}{r'} = 1)$, $u_n(z) \in \partial j(z, x_n(z))$ a.e. on Z (see Clarke [13] and Gasinski-Papageorgiou [21]).

Because of (5.10) and Proposition 5.4, we deduce that $\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ is bounded. Therefore, by passing to a subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } H^1(Z) \text{ and } x_n \to x \text{ in } L^2(Z) \text{ as } n \to \infty.$$

From (5.10), we have

$$|\langle x_n^*, x_n - x\rangle| \le \varepsilon_n \|x_n - x\| \text{ with } \varepsilon_n \downarrow 0,$$

hence

$$\left|\langle A(x_n), x_n - x\rangle - \int_Z u_n (x_n - x)\, dz\right| \le \varepsilon_n \|x_n - x\|. \tag{5.11}$$

Clearly

$$\int_Z u_n (x_n - x)\, dz \to 0 \text{ as } n \to \infty.$$

So, from (5.11) , it follows that

$$\langle A(x_n), x_n - x\rangle \to 0 \text{ as } n \to \infty. \tag{5.12}$$

Note that $A(x_n) \xrightarrow{w} A(x)$ in $H^1(Z)^*$. So, from (5.12), we have

$$\|Dx_n\|_2^2 = \langle A(x_n), x_n\rangle \to \langle A(x), x\rangle = \|Dx\|_2^2.$$

Since

$$Dx_n \xrightarrow{w} Dx \text{ in } L^2\left(Z, \mathbb{R}^N\right),$$

from the Kadec-Klee property of Hilbert spaces, we infer that

$$Dx_n \to Dx \text{ in } L^2\left(Z, \mathbb{R}^N\right),$$

hence

$$x_n \to x \text{ in } H^1(Z).$$

Therefore φ satisfies the nonsmooth PS-condition. ■

Now, we are ready for the multiplicity result.

Theorem 5.6 *If hypotheses $H(j)$ hold, then problem (5.1) has at least two nontrivial solutions x_0, $y_0 \in C^1\left(\overline{Z}\right)$.*

Proof. By virtue of hypothesis $H(j)(v)$, we have

$$\lambda_m x \le u \le \lambda_{m+1} x \text{ for a. a. } z \in Z, \text{ all } x \in (0, \delta] \text{ and all } u \in \partial j(z, x) \tag{5.13}$$

and

$$\lambda_{m+1}x \leq u \leq \lambda_m x \text{ for a. a. } z \in Z, \text{ all } x \in [-\delta, 0) \text{ and all } u \in \partial j(z,x). \tag{5.14}$$

From hypotheses $H(j)(i)$, (ii) and Rademacher's theorem, we know that for all $z \in Z \backslash D$, with $|D|_N = 0$, the function $r \to j(z,r)$ is differentiable a.e. on $\mathbb{R}$ and at a point of differentiability, we have

$$\frac{d}{dr} j(z,r) \in \partial j(z,r)$$

(see Clarke [13]). So, from (5.13) and (5.14) we have

$$\lambda_m r \leq \frac{d}{dr} j(z,r) \leq \lambda_{m+1} r \text{ for a. a. } z \in Z \backslash D \text{ and a. a. } r \in (0, \delta] \tag{5.15}$$

and

$$\lambda_{m+1} r \leq \frac{d}{dr} j(z,r) \leq \lambda_m r \text{ for a. a. } z \in Z \backslash D \text{ and a. a. } r \in [-\delta, 0). \tag{5.16}$$

Integrating (5.15) and (5.16), we obtain

$$\frac{1}{2}\lambda_m x^2 \leq j(z,x) \leq \frac{1}{2}\lambda_{m+1} x^2 \text{ for a. a. } z \in Z \text{ and all } |x| \leq \delta. \tag{5.17}$$

We consider the orthogonal direct sum decomposition

$$H^1(Z) = Y \oplus V,$$

with

$$Y = \bigoplus_{k=0}^{m} E(\lambda_k) \quad \text{and} \quad V = \overline{\bigoplus_{k \geq m+1} E(\lambda_k)} = Y^{\perp}.$$

Since Y is finite dimensional, all norms are equivalent and, because $Y \subseteq C(\overline{Z})$, we can find $c_4 > 0$ such that

$$\|y\|_\infty \leq c_4 \|y\| \text{ for all } y \in Y. \tag{5.18}$$

Therefore, if $y \in Y$ satisfies $\|y\| \leq \frac{\delta}{c_4}$ with $\delta > 0$, as in hypothesis $H(j)(v)$, then from (5.18) we have

$$|y(z)| \leq \delta \text{ for all } z \in \overline{Z}.$$

Hence (5.17) implies

$$\frac{\lambda_m}{2} y(z)^2 \leq j(z, y(z)) \leq \frac{\lambda_{m+1}}{2} y(z)^2 \text{ a.e. on } Z. \tag{5.19}$$

Thus, for $y \in Y$ with $\|y\| \leq \frac{\delta}{c_4}$, we have

$$\varphi(y) = \frac{1}{2}\|Dy\|_2^2 - \int_Z j(z, y(z))\,dz \leq \frac{1}{2}\|Dy\|_2^2 - \frac{\lambda_m}{2}\|y\|_2^2 \leq 0 \tag{5.20}$$

(see (2.5)).

On the other hand, by virtue of hypothesis $H(j)(iii)$, we can find $c_5 > 0$ such that

$$|u| \leq c_5 |x|^{r-1} \text{ for a. a. } z \in Z, \text{ all } |x| > \delta \text{ and all } u \in \partial j(z, x). \tag{5.21}$$

Moreover, without any loss of generality, we can always assume $2 < r < 2^*$. Then, as above, from (5.21) and Rademacher's theorem, after integration we obtain

$$j(z, x) \leq c_6 |x|^r \text{ for a. a. } z \in Z, \text{ all } |x| > \delta \text{ and some } c_6 > 0. \tag{5.22}$$

Let $v \in V$. We have

$$v = u + w,$$

with $u \in E(\lambda_{m+1})$ and $w \in W = \overline{\underset{k \geq m+2}{\oplus} E(\lambda_k)}$.

Let

$$Z_\delta = \{z \in Z : |v(z)| > \delta\}.$$

Then for $z \in Z_\delta$, we have (since $u \in C^1(\overline{Z})$)

$$\begin{aligned} |w(z)| &= |v(z) - u(z)| \geq |v(z)| - |u(z)| \\ &\geq |v(z)| - \|u\|_\infty \geq |v(z)| - c_7\|u\|, \text{ for some } c_7 > 0, \end{aligned} \tag{5.23}$$

since all norms are equivalent on the finite dimensional eigenspace $E(\lambda_{m+1}) \subseteq C^1(\overline{Z})$.

Suppose that $\|v\| \leq \frac{\delta}{2c_7}$. If by p_{m+1} we denote the orthogonal projection operator onto the eigenspace $E(\lambda_{m+1})$, we have

$$\|u\| = \|p_{m+1}(v)\| \leq \|v\| \leq \frac{\delta}{2c_7}. \tag{5.24}$$

From (5.23) and (5.24), we have

$$|w(z)| \geq |v(z)| - \frac{\delta}{2} \geq |v(z)| - \frac{1}{2}|v(z)| = \frac{1}{2}|v(z)|. \tag{5.25}$$

Now, for $v \in V$, with $\|v\| \leq \frac{\delta}{2c_7}$, we have

$$\begin{aligned} \varphi(v) &= \frac{1}{2}\|Dv\|_2^2 - \int_Z j(z, v(z))\,dz \\ &= \frac{1}{2}\|Dv\|_2^2 - \int_{Z_\delta} j(z, v(z))\,dz - \int_{Z \setminus Z_\delta} j(z, v(z))\,dz. \end{aligned} \tag{5.26}$$

Note that

$$\int_{Z\setminus Z_\delta} j\left(z, v\left(z\right)\right) dz = \int_{\{|v(z)|\leq\delta\}} j\left(z, v\left(z\right)\right) dz \leq \frac{\lambda_{m+1}}{2} \left\|v\right\|_2^2 \tag{5.27}$$

(see (5.17)). Also, in view of (5.22) and (5.25) and for $c_8 = 2^r c_6$, we have

$$\begin{aligned}\int_{Z_\delta} j\left(z, v\left(z\right)\right) dz &\leq c_6 \int_{Z_\delta} \left|v\left(z\right)\right|^r dz \leq c_8 \int_{Z_\delta} \left|w\left(z\right)\right|^r dz \\ &\leq c_8 \left\|w\right\|_r^r \leq c_9 \left\|w\right\|^r, \text{ for some } c_9 > 0 \end{aligned} \tag{5.28}$$

(since $H^1\left(Z\right)$ is embedded continuously in $L^r\left(Z\right)$). Using (5.27) and (5.28) in (5.26), we obtain

$$\varphi\left(v\right) \geq \frac{1}{2} \left\|Dv\right\|_2^2 - \frac{\lambda_{m+1}}{2} \left\|v\right\|_2^2 - c_9 \left\|w\right\|^r.$$

Exploiting the orthogonality of the component spaces in the decomposition

$$V = E\left(\lambda_{m+1}\right) \oplus W$$

and since

$$\left\|Du\right\|_2^2 = \lambda_{m+1} \left\|u\right\|_2^2, \text{ for } u \in E\left(\lambda_{m+1}\right),$$

we have

$$\varphi\left(v\right) \geq \frac{1}{2} \left\|Dw\right\|_2^2 - \frac{\lambda_{m+1}}{2} \left\|w\right\|_2^2 - c_9 \left\|w\right\|^r \geq c_{10} \left\|w\right\|^2 - c_9 \left\|w\right\|^r, \tag{5.29}$$

for some $c_{10} > 0$.

Since $2 < r$, from (5.29) and if

$$\left\|w\right\| \leq \left\|v\right\| \leq r \leq \min\left\{\frac{\delta}{2c_7}, \frac{\delta}{c_4}\right\},$$

with $r > 0$ small enough, we have

$$\varphi\left(v\right) \geq 0, \text{ for all } v \in V, \ \left\|v\right\| \leq r. \tag{5.30}$$

If $\inf \varphi = 0 = \varphi\left(0\right)$, then, from (5.20), we see that all $y \in Y$, with $0 < \left\|y\right\| \leq \frac{\delta}{c_4}$, are minimizers of φ, hence critical points of φ. Using Green's identity, we check that the critical points of φ are solutions of (5.1) and regularity theory implies that they belong in $C^1\left(\overline{Z}\right)$.

If $\inf \varphi < 0$, then we can apply the Theorem 2.10 and obtain two nontrivial critical points x_0, $y_0 \in H^1\left(Z\right)$ of φ. Again, using Green's identity (see, for example, Gasinski-Papageorgiou [22], p.209), we verify that both x_0, y_0 are solutions of (5.1) and from regularity theory, we have x_0, $y_0 \in C^1\left(\overline{Z}\right)$. ■

Chapter 6

Nonlinear Nonvariational Neumann Problems

6.1 Introduction

Let $Z \subseteq \mathbb{R}^n$ be a bounded domain with a C^2-boundary ∂Z. We consider the following quasilinear Neumann problem with a nonsmooth potential (hemivariational inequality):

$$\begin{cases} -div\left(A\left(z,x\left(z\right)\right)Dx\left(z\right)\right) \in \partial j\left(z,x\left(z\right)\right) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0, \text{ on } \partial Z. \end{cases} \tag{6.1}$$

Here $A\left(z,x\right)$ is a bounded, $N \times N$-matrix valued Caratheodory function (i.e. it is measurable in $z \in Z$ and continuous in $x \in \mathbb{R}$), $j\left(z,x\right)$ is a measurable potential function which is locally Lipschitz and in general nonsmooth in the $x \in \mathbb{R}$ variable, and $\partial j\left(z,x\right)$ is the generalized subdifferential of $x \to j\left(z,x\right)$.

In this problem, the differential operator $x \to -div\left(A\left(z,x\right)Dx\right)$ is neither homogeneous nor variational. So the minimax methods of critical point theory (smooth and nonsmooth alike) fail and we need to device new techniques in order to deal with problem (6.1). For this reason, we assume that for almost all $z \in Z$, the matrix-valued map $x \to A\left(z,x\right)$ has an asymptotic limit as $|x| \to \infty$. Then, using the spectrum of the corresponding asymptotic linear differential operator, we are able to overcome the lack of homogeneity of the original differential operator and provide conditions for the solvability of problem (6.1). We use the spectrum of the asymptotic differential operator together with degree theoretic methods based on the degree map for multivalued perturbations of $(S)_+$-operators due to Hu-Papageorgiou [25] (see also Hu-Papageorgiou [26]) and we are able to establish the existence of nontrivial smooth solutions.

6.2 Hypotheses

The hypotheses on the matrix-valued function $A(z,x)$ are the following:

$H(A):$ $A : Z \times \mathbb{R} \rightarrow \mathbb{R}^{N\times N}$ is a map such that

(i) for all $x \in \mathbb{R}$, $z \longmapsto A(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \longmapsto A(z,x)$ is continuous;

(iii) there exist constants $0 < c_0 < c_1$ such that

$$c_0 \|\xi\| \leq \|A(z,\xi)\xi\| \leq c_1 \|\xi\| \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R} \text{ and all } \xi \in \mathbb{R}^N;$$

(iv) there exists a constant $c_2 > 0$ such that

$$c_2 \|\xi\|^2 \leq (A(z,\xi)\xi,\xi)_{\mathbb{R}^N} \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R} \text{ and all } \xi \in \mathbb{R}^N;$$

(v) there exists $\widehat{A} \in L^\infty\left(Z,\mathbb{R}^N\right)$ such that

$$A(z,x) \rightarrow \widehat{A}(z) \text{ for almost all } z \in Z, \text{ as } |x| \rightarrow \infty.$$

Using the asymptotic limit function $\widehat{A}(z)$ of hypothesis $H(A)(v)$, we consider the following linear Neumann eigenvalue problem:

$$\begin{cases} -div\left(\widehat{A}(z)\,Dx(z)\right) = \lambda x(z) \text{ a.e.on } Z, \\ \frac{\partial x}{\partial n} = 0, \text{ on } \partial Z \end{cases} \tag{6.2}$$

Let $\widehat{V} \in \mathcal{L}\left(H^1(Z), H^1(Z)^*\right)$ be the continuous linear operator defined by

$$\left\langle \widehat{V}(x), y \right\rangle = \int_Z \left(\widehat{A}(z)\,Dx(z), Dy(z)\right)_{\mathbb{R}^N} dz \text{ for all } x,\, y \in H^1(Z).$$

For every $\varepsilon > 0$ and every $x \in H^1(Z)$, we have:

$$\left\langle \widehat{V}(x), x \right\rangle + \varepsilon \|x\|_2^2 \geq c_2 \|Dx\|_2^2 + \varepsilon \|x\|_2^2 \geq c_3 \|x\|_2^2,$$

with $c_3 = \min\{\varepsilon, c_2\}$.

Then by virtue of Corollary 7D, p.78, of Showalter [56], we know that problem (6.2) has a sequence of eigenvalues $\{\lambda_n\}_{n\geq 0}$, $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \to \infty$, with corresponding eigenfunctions which form an orthonormal basis in $L^2(Z)$ and an orthogonal basis in $H^1(Z)$. Moreover, these eigenvalues admit variational characterization via the corresponding Rayleigh quotients.

Using this spectrum, we can now state the hypotheses on the nonsmooth potential $j(z,x)$:

$H(j)$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z,0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \longmapsto j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \longmapsto j(z,x)$ is locally Lipschitz;

(iii) for every $r > 0$, there exists $a_r \in L^\infty(Z)_+$ such that

$$|u| \leq a_r(z) \text{ for a.a. } z \in Z, \text{ all } |x| \leq r \text{ and all } u \in \partial j(z,x);$$

(iv) there exist an integer $k \geq 0$ and functions $\widehat{\theta}, \theta \in L^\infty(Z)$ such that

$$\lambda_k \leq \widehat{\theta}(z) \leq \theta(z) \leq \lambda_{k+1} \text{ a.e. on } Z,$$

the first and third inequalities are strict on sets (in general different) of positive Lebesgue measure and

$$\widehat{\theta}(z) \leq \liminf_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \leq \limsup_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \leq \theta(z)$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$;

(v) there exist functions $\widehat{\eta}, \eta \in L^\infty(Z)$ such that

$$\eta(z) \leq 0 \text{ a.e. on } Z,$$

the inequality is strict on a set of positive Lebesgue measure and

$$\widehat{\eta}(z) \leq \liminf_{x\to 0} \frac{u}{|x|^{p-2}x} \leq \limsup_{x\to 0} \frac{u}{|x|^{p-2}x} \leq \eta(z)$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$.

Due to the nonsmoothness of the potential function $j(z,x)$, we will use some elements of the subdifferential theory of locally Lipschitz functions (see Chapter 2). Also, due to the nonvariational character of our problem, we will use degree theoretic arguments based on the degree map for multivalued perturbations of $(S)_+$-operators (see Hu-Papageorgiou [25], [26]).

6.3 Existence of Solutions

Let $V : H^1(Z) \to H^1(Z)^*$ be the nonlinear operator defined by

$$\langle V(x), y\rangle = \int_Z (A(z,x)\,Dx, Dy)_{\mathbb{R}^N}\,dz \text{ for all } x,\, y \in H^1(Z).$$

Proposition 6.1 *If hypotheses $H(A)$ hold, then V is an $(S)_+$-operator.*

Proof. Suppose that $x_n \xrightarrow{w} x$ in $H^1(Z)$ and assume that

$$\limsup_{n\to\infty} \langle V(x_n), x_n - x\rangle \le 0. \tag{6.3}$$

By definition

$$\langle V(x_n), x_n - x\rangle = \int_Z (A(z,x_n)\,Dx_n, Dx_n - Dx)_{\mathbb{R}^N}\,dz.$$

We have (see $H(A)$ (iv))

$$\begin{aligned}\langle V(x_n), x_n - x\rangle &= \int_Z (A(z,x_n)\,Dx_n, Dx_n - Dx)_{\mathbb{R}^N}\,dz \\ &= \int_Z (A(z,x_n)\,Dx_n - A(z,x_n)\,Dx, Dx_n - Dx)_{\mathbb{R}^N}\,dz \\ &+ \int_Z (A(z,x_n)\,Dx, Dx_n - Dx)_{\mathbb{R}^N}\,dz \\ &\ge c_2 \|Dx_n - Dx\|_2^2 + \int_Z (A(z,x_n)\,Dx, Dx_n - Dx)_{\mathbb{R}^N}\,dz. \end{aligned} \tag{6.4}$$

Because $x_n \xrightarrow{w} x$ in $H^1(Z)$ and recalling that $H^1(Z)$ is embedded compactly in $L^2(Z)$, we can say that $x_n \to x$ in $L^2(Z)$. By passing to a subsequence, if necessary, we may also assume that

$$x_n(z) \to x(z) \text{ a.e. on } Z$$

and

$$|x_n(z)| \le h(z) \text{ for a. a. } z \in Z, \text{ all } n \ge 1 \text{ and with } h \in L^2(Z)_+.$$

Then

$$A(z, x_n(z))\,Dx_n(z) \to A(z, x(z))\,Dx(z) \text{ a.e. on } Z$$

(see $H(A)$ (ii)). This fact, together with $H(A)$ (iii) and the dominated convergence theorem, imply

$$A(\cdot, x_n(\cdot))\,Dx_n(\cdot) \to A(\cdot, x(\cdot))\,Dx(\cdot) \text{ in } L^2\left(Z, \mathbb{R}^N\right) \text{ as } n \to \infty.$$

Since $Dx_n \xrightarrow{w} Dx$ in $L^2\left(Z, \mathbb{R}^N\right)$, it follows that

$$\int_Z \left(A\left(z, x_n\right) Dx, Dx_n - Dx\right)_{\mathbb{R}^N} dz \to 0 \text{ as } n \to \infty. \tag{6.5}$$

Returning to (6.4), passing to the limit as $n \to \infty$ and using (6.3) and (6.5), we obtain

$$\left\|Dx_n - Dx\right\|_2^2 \to 0 \text{ as } n \to \infty,$$

hence

$$x_n \to x \text{ in } H^1\left(Z\right) \text{ as } n \to \infty.$$

By Urysohn's criterion for convergent sequences, this convergence is true for the original sequence $\{x_n\}_{n\geq 1} \subseteq H^1\left(Z\right)$. ■

Let $N : L^2\left(Z\right) \to 2^{L^2(Z)}$ be the multivalued Nemytskii operator corresponding to the subdifferential multifunction $(z, x) \longmapsto \partial j\left(z, x\right)$, i.e.

$$N\left(x\right) = \left\{u \in L^2\left(Z\right) : u\left(z\right) \in \partial j\left(z, x\left(z\right)\right) \text{ a.e. on } Z\right\}.$$

Proposition 6.2 *If hypotheses $H(j)$ hold, then N has nonempty, weakly compact and convex values in $L^2\left(Z\right)$ and it is usc from $L^2\left(Z\right)$ with the norm topology into $L^2\left(Z\right)$ with the weak topology (denoted by $L^2\left(Z\right)_w$).*

Proof. By virtue of hypotheses $H(j)$ (iii) and (iv), we see that the values of N are $L^2\left(Z\right)$-bounded sets, which are easily seen to be closed and convex. Therefore for every $x \in L^2\left(Z\right)$, the set $N\left(x\right) \subseteq L^2\left(Z\right)$ is weakly compact and convex. We need to show that it is nonempty. For this purpose, let $\{s_n\}_{n\geq 1} \subseteq L^2\left(Z\right)$ be simple functions such that

$$s_n\left(z\right) \to x\left(z\right) \text{ a.e. on } Z \text{ and } \left|s_n\left(z\right)\right| \leq \left|x\left(z\right)\right| \tag{6.6}$$

for a.a. $z \in Z$, all $n \geq 1$.

Because of hypothesis $H(j)$ (i), for every $x \in \mathbb{R}$, the multifunction $z \longmapsto \partial j\left(z, x\right)$ is graph measurable. So, by a straightforward application of the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [26], p.158), we can find a measurable function $f_n : Z \to \mathbb{R}$ such that

$$f_n\left(z\right) \in \partial j\left(z, x\left(z\right)\right)$$

for a.a. $z \in Z$, all $n \geq 1$. Hypotheses $H(j)$ (iii), (iv) imply that

$$\left|f_n\left(z\right)\right| \leq c_3\left(1 + \left|x\left(z\right)\right|\right)$$

for a.a. $z \in Z$, all $n \geq 1$ and some $c_3 > 0$, hence $\{f_n\}_{n\geq 1} \subseteq L^2(Z)$ is bounded. Therefore, we may assume (at least for a subsequence), that

$$f_n \xrightarrow{w} f \text{ in } L^2(Z)$$

Since the subdifferential multifunction has closed and convex values, by Mazur's lemma, we obtain

$$f(z) \in \partial j(z, x(z)) \text{ a.e. on } Z,$$

hence $f \in N(x)$, i.e. $N(x) \neq \varnothing$.

Note that the weak topology on bounded subsets of $L^2(Z)$ is metrizable. Therefore, in order to show the upper semicontinuity of N from $L^2(Z)$ into $L^2(Z)_w$, it suffices to show that its graph

$$GrN = \left\{(x, u) \in L^2(Z) \times L^2(Z) : u \in N(x)\right\}$$

is sequentially closed in $L^2(Z) \times L^2(Z)_w$ (see Hu-Papageorgiou [26], p.38).

So let $\{(x_n, u_n)\}_{n\geq 1} \subseteq Gr\ N$ and assume that $x_n \to x$ in $L^2(Z)$ and $u_n \xrightarrow{w} u$ in $L^2(Z)$ as $n \to \infty$. We have

$$u_n(z) \in \partial j(z, x_n(z)) \text{ for a.a. } z \in Z, \text{ all } n \geq 1.$$

Invoking Proposition 3.9, p.694 of Hu-Papageorgiou [26], in the limit as $n \to \infty$, we obtain

$$u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z,$$

hence

$$u \in N(x).$$

So $Gr\ N$ is sequentially closed in $L^2(Z) \times L^2(Z)_w$ and from this we conclude that N is usc from $L^2(Z)$ into $L^2(Z)_w$. ■

Recalling that $H^1(Z)$ is a separable Hilbert space, which is embedded compactly and densely in $L^2(Z)$, we have that $L^2(Z)^* = L^2(Z)$ is embedded compactly and densely in $H^1(Z)^*$.

Therefore an immediate consequence of Proposition 6.2, is the following corollary:

Corollary 6.3 *If hypotheses $H(j)$ hold, then $N : H^1(Z) \to 2^{H^1(Z)^*} \setminus \{\varnothing\}$ is a multifunction belonging in class (P).*

Proposition 6.1 and Corollary 6.3 permit the definition of the degree $\widehat{d}$ for nonlinear multivalued operator $x \longmapsto V(x) - N(x)$. To compute this degree for various sets we will need the following auxiliary result.

For every integer $k \geq 0$, let $E(\lambda_k)$ denote the eigenspace corresponding to the eigenvalue λ_k. Set

$$\overline{H}_k = \overset{k}{\underset{i=0}{\oplus}} E(\lambda_i) \quad \text{and} \quad \widehat{H}_k = \overline{\underset{i \geq k+1}{\oplus} E(\lambda_i)}.$$

Then we have the orthogonal direct sum decomposition

$$H^1(Z) = \overline{H}_k \oplus \widehat{H}_k.$$

Lemma 6.4 (a) *If $\theta \in L^\infty(Z)_+$ and $\theta(z) \leq \lambda_{k+1}$ a.e. on Z with strict inequality on a set of positive Lebesgue measure, then there exists $\xi_1 > 0$ such that*

$$\psi_1(\widehat{x}) = \int_Z \left(\widehat{A}(z) D\widehat{x}, D\widehat{x}\right)_{\mathbb{R}^N} dz - \int_Z \theta |\widehat{x}|^2 dz \geq \xi_1 \|\widehat{x}\|^2 \text{ for all } \widehat{x} \in \widehat{H}_k.$$

(b) *If $\widehat{\theta} \in L^\infty(Z)_+$ and $\widehat{\theta}(z) \geq \lambda_k$ a.e. on Z with strict inequality on a set of positive Lebesgue measure, then there exists $\xi_2 > 0$ such that*

$$\psi_2(\overline{x}) = \int_Z \widehat{\theta} |\overline{x}|^2 dz - \int_Z \left(\widehat{A}(z) D\overline{x}, D\overline{x}\right)_{\mathbb{R}^N} dz \geq \xi_2 \|\overline{x}\|^2 \text{ for all } \overline{x} \in \overline{H}_k.$$

(c) *If $\eta \in L^\infty(Z)$ and $\eta(z) \leq 0$ a.e. on Z with strict inequality on a set of positive Lebesgue measure, then there exists $\xi_0 > 0$ such that*

$$\psi_0(x) = \int_Z \left(\widehat{A}(z) Dx, Dx\right)_{\mathbb{R}^N} dz - \int_Z \eta |x|^2 dz \geq \xi_0 \|x\|^2 \text{ for all } x \in H^1(Z).$$

Proof. (a) From the variational characterization of λ_{k+1} we have $\psi_1 \geq 0$. Suppose that the result is not true. Exploiting the 2-homogeneity of ψ_1, we can find $\{\widehat{x}_n\}_{n \geq 1} \subseteq \widehat{H}_k$ such that

$$\|\widehat{x}_n\| = 1 \text{ for all } n \geq 1 \text{ and } \psi_1(\widehat{x}_n) \downarrow 0 \text{ as } n \to \infty.$$

We may assume that

$$\widehat{x}_n \xrightarrow{w} \widehat{x} \in \widehat{H}_k \text{ in } H^1(Z) \text{ and } \widehat{x}_n \to \widehat{x} \text{ in } L^2(Z) \text{ as } n \to \infty.$$

Note that

$$\|x\| := \int_Z \left(\widehat{A}(z)\, Dx, Dx\right)_{\mathbb{R}^N} dz,$$

for $x \in H^1(Z)$, defines a norm in $L^2\left(Z, \mathbb{R}^N\right)$ equivalent to the usual norm in $L^2\left(Z, \mathbb{R}^N\right)$ (see hypothesis $H(A)$ (iii)) and recall that a norm in a Banach space is w-lower semicontinuous.

Hence

$$\int_Z \left(\widehat{A}(z)\, D\widehat{x}, D\widehat{x}\right)_{\mathbb{R}^N} dz \leq \int_Z \theta\, |\widehat{x}|^2\, dz \leq \lambda_{k+1} \|\widehat{x}\|_2^2 \tag{6.7}$$

therefore

$$\int_Z \left(\widehat{A}(z)\, D\widehat{x}, D\widehat{x}\right)_{\mathbb{R}^N} dz = \lambda_{k+1} \|\widehat{x}\|_2^2$$

(from the variational characterization of λ_{k+1}).

Hence $\widehat{x} \in E(\lambda_{k+1})$. But the elements of $E(\lambda_{k+1})$ have the unique continuation property (see for example Gasinski-Papageorgiou [22]). So $x(z) \neq 0$ a.e. on Z. Then from (6.7) and using the hypothesis on θ, we have

$$\int_Z \left(\widehat{A}(z)\, Dx, Dx\right)_{\mathbb{R}^N} dz < \lambda_{k+1} \|\widehat{x}\|_2^2,$$

which contradicts the variational characterization of λ_{k+1}.

The proofs of (b) and (c) are similar to those of (a) and so they are ommited.

∎

Using this lemma, we can compute the $\widehat{d}$-degree of $V - N$ for large balls.

Proposition 6.5 *If hypotheses $H(j)$ hold, then there exists $R_0 > 0$ such that*

$$\widehat{d}(V - N, B_R, 0) = (-1)^{\dim \overline{H}_k} \quad \textit{for all } R \geq R_0.$$

Proof. Let $\widehat{g} \in L^\infty(Z)_+$ be such that

$$\lambda_k \leq \widehat{g}(z) \leq \lambda_{k+1} \text{ a.e. on } Z$$

with strict inequalities on sets (in general different) of positive Lebesgue measure. We consider the admissible homotopy $h_1 : [0,1] \times H^1(Z) \to 2^{H^1(Z)^*} \setminus \{\varnothing\}$ defined by

$$h_1(t, x) = tV(x) + (1-t)\widehat{V}(x) - tN(x) - (1-t)\widehat{g}x.$$

Claim 5 *We can find $R_0 > 0$ such that $0 \notin h_1(t, x)$ for all $t \in [0,1]$ and all $\|x\| = R \geq R_0$.*

We argue indirectly. So suppose Claim 5 is not true. Then we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ such that

$$t_n \to t \in [0,1], \ \|x_n\| \to \infty \text{ and } 0 \in h_1(t_n, x_n) \text{ for all } n \geq 1. \tag{6.8}$$

From the inclusion in (6.8), we know that for every $n \geq 1$, we can find $u_n \in N(x_n)$ such that

$$t_n V(x_n) + (1-t_n)\widehat{V}(x_n) = t_n u_n + (1-t_n)\widehat{g}x_n. \tag{6.9}$$

Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(Z) \text{ and } y_n \to y \text{ in } L^2(Z) \text{ as } n \to \infty.$$

We divide (6.9) by $\|x_n\|$ and we have

$$t_n \frac{V(x_n)}{\|x_n\|} + (1-t_n)\widehat{V}(y_n) = t_n \frac{u_n}{\|x_n\|} + (1-t_n)\widehat{g}y_n \tag{6.10}$$

We take duality brackets with $y_n - y$. Hence

$$t_n \left\langle \frac{V(x_n)}{\|x_n\|}, y_n - y \right\rangle + (1-t_n)\left\langle \widehat{V}(y_n), y_n - y \right\rangle$$
$$= t_n \int_Z \frac{u_n}{\|x_n\|}(y_n - y)\,dz + (1-t_n)\int_Z \widehat{g}y_n(y_n - y)\,dz.$$

From hypotheses $H(j)$ (iii) and (iv), we know that

$$|u| \leq c_3(1+|x|) \tag{6.11}$$

for a.a. $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$. Because of (6.11) and since $y_n \to y$ in $L^2(Z)$, we have

$$\int_Z \frac{u_n}{\|x_n\|}(y_n - y)\,dz \to 0$$

and

$$\int_Z \widehat{g}y_n(y_n - y)\,dz \to 0$$

as $n \to \infty$. Therefore it follows that

$$\lim_{n\to\infty}\left[t_n \left\langle \frac{V(x_n)}{\|x_n\|}, y_n - y \right\rangle + (1-t_n)\left\langle \widehat{V}(y_n), y_n - y \right\rangle\right] = 0. \tag{6.12}$$

We have

$$
\begin{aligned}
&\left\langle \frac{V(x_n)}{\|x_n\|}, y_n - y \right\rangle \\
&= \int_Z (A(z, x_n) Dy_n, Dy_n - Dy)_{\mathbb{R}^N} \, dz \\
&= \int_Z (A(z, x_n)(Dy_n - Dy), Dy_n - Dy)_{\mathbb{R}^N} \, dz \\
&+ \int_Z (A(z, x_n) Dy, Dy_n - Dy)_{\mathbb{R}^N} \, dz \\
&\geq c_2 \|Dy_n - Dy\|_2^2 + \int_Z (A(z, x_n) Dy, Dy_n - Dy)_{\mathbb{R}^N} \, dz
\end{aligned} \tag{6.13}
$$

(see $H(A)$ (iv)). Note that $|x_n(z)| \to +\infty$ a.e. on $\{y \neq 0\}$. Therefore by hypothesis $H(A)$ (v)

$$
A(z, x_n(z)) \to \widehat{A}(z) \text{ a.e. on } \{y \neq 0\} \text{ as } n \to \infty.
$$

Also from Stampacchia's theorem we know that $Dy(z) = 0$ a.e. on $\{y = 0\}$. Therefore, finally, we can say that

$$
A(z, x_n(z)) Dy(z) \to \widehat{A}(z) Dy(z) \text{ a.e. on } Z.
$$

From this convergence, hypothesis $H(A)$ (iii) and the dominated convergence theorem, it follows that

$$
A(\cdot, x_n(\cdot)) Dy(\cdot) \to \widehat{A}(\cdot) Dy(\cdot) \text{ in } L^2\left(Z, \mathbb{R}^N\right),
$$

hence

$$
\int_Z (A(z, x_n(z)) Dy(z), Dy_n(z) - Dy(z))_{\mathbb{R}^N} \, dz \to 0 \text{ as } n \to \infty \tag{6.14}
$$

Moreover,

$$
\begin{aligned}
&\left\langle \widehat{V}(y_n), y_n - y \right\rangle \\
&= \int_Z \left(\widehat{A}(z) Dy_n, Dy_n - Dy\right)_{\mathbb{R}^N} dz \\
&= \int_Z \left(\widehat{A}(z)(Dy_n - Dy), Dy_n - Dy\right)_{\mathbb{R}^N} dz \\
&+ \int_Z \left(\widehat{A}(z) Dy, Dy_n - Dy\right)_{\mathbb{R}^N} dz \\
&\geq c_2 \|Dy_n - Dy\|_2^2 + \int_Z \left(\widehat{A}(z) Dy, Dy_n - Dy\right)_{\mathbb{R}^N} dz
\end{aligned} \tag{6.15}
$$

(see $H(A)$ (iv) and (v)). Because $Dy_n \xrightarrow{w} Dy$ in $L^2(Z,\mathbb{R}^N)$, we have

$$\int_Z \left(\widehat{A}(z)\, Dy, Dy_n - Dy\right)_{\mathbb{R}^N} dz \to 0 \text{ as } n \to \infty. \tag{6.16}$$

Returning to (6.12), using (6.13), (6.14), (6.15) and (6.16) and passing to the limit as $n \to \infty$, we obtain

$$\|Dy_n - Dy\|_2 \to 0 \text{ as } n \to \infty,$$

therefore

$$y_n \to y \text{ in } H^1(Z) \text{ as } n \to \infty.$$

So $\|y\| = 1$, hence $y \neq 0$.

By virtue of (6.11), $\left\{\frac{u_n}{\|x_n\|}\right\}_{n\geq 1} \subseteq L^2(Z)$ is bounded. So we may assume that

$$\frac{u_n}{\|x_n\|} \xrightarrow{w} h \text{ in } L^2(Z) \text{ as } n \to \infty.$$

Given $\varepsilon > 0$ and $n \geq 1$, we introduce the following two sets

$$C^+_{\varepsilon,n} = \left\{z \in Z : x_n(z) > 0,\ \widehat{\theta}(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)} \leq \theta(z) + \varepsilon\right\}$$

and

$$C^-_{\varepsilon,n} = \left\{z \in Z : x_n(z) < 0,\ \widehat{\theta}(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)} \leq \theta(z) + \varepsilon\right\}.$$

Note that

$$x_n(z) \to +\infty \text{ for a.a. } z \in \{y > 0\}$$

and

$$x_n(z) \to -\infty \text{ for a.a. } z \in \{y < 0\}.$$

Then hypothesis $H(j)$ (iv) implies that

$$\chi_{C^+_{\varepsilon,n}}(z) \to 1 \text{ a.e. on } \{y > 0\}$$

and

$$\chi_{C^-_{\varepsilon,n}}(z) \to 1 \text{ a.e. on } \{y < 0\}.$$

Via the dominated convergence theorem, we have

$$\left\| \left(1 - \chi_{C^+_{\varepsilon,n}}\right) \frac{u_n}{\|x_n\|} \right\|_{L^1(\{y>0\})} \to 0$$

and

$$\left\| \left(1 - \chi_{C^-_{\varepsilon,n}}\right) \frac{u_n}{\|x_n\|} \right\|_{L^1(\{y<0\})} \to 0$$

as $n \to \infty$. From the definition of the set $C^+_{\varepsilon,n}$, we have

$$\begin{aligned} \chi_{C^+_{\varepsilon,n}}(z)\left(\widehat{\theta}(z) - \varepsilon\right) y_n(z) &\le \chi_{C^+_{\varepsilon,n}}(z) \frac{u_n(z)}{\|x_n\|} \\ &\le \chi_{C^+_{\varepsilon,n}}(z)\left(\theta(z) + \varepsilon\right) y_n(z) \text{ a.e. on } Z. \end{aligned}$$

Passing to the limit as $n \to \infty$ and using Mazur's lemma, we obtain

$$\left(\widehat{\theta}(z) - \varepsilon\right) y(z) \le h(z) \le \left(\theta(z) + \varepsilon\right) y(z) \text{ a.e. on } \{y > 0\}.$$

Because $\varepsilon > 0$ was arbitrary, we let $\varepsilon \downarrow 0$ and have

$$\widehat{\theta}(z) y(z) \le h(z) \le \theta(z) y(z) \text{ a.e. on } \{y > 0\}. \tag{6.17}$$

Similarly working with the set $C^-_{\varepsilon,n}$, we obtain

$$\theta(z) y(z) \le h(z) \le \widehat{\theta}(z) y(z) \text{ a.e. on } \{y < 0\}. \tag{6.18}$$

Finally, it is clear from (6.11) that

$$h(z) = 0 \text{ a.e. on } \{y = 0\}. \tag{6.19}$$

From (6.17), (6.18) and (6.19), it follows that there exists $g_\infty \in L^\infty(Z)_+$ such that

$$\widehat{\theta}(z) \le g_\infty(z) \le \theta(z) \text{ a.e. on } Z$$

and

$$h(z) = g_\infty(z) y(z) \text{ a.e. on } Z.$$

For every $v \in L^\infty(Z)$, we have

$$\left\langle \frac{V(x_n)}{\|x_n\|}, v \right\rangle = \int_Z \left(A(z, x_n) Dy_n, Dv\right)_{\mathbb{R}^N} dz.$$

Recall that $y_n \to y$ in $H^1(Z)$. So we may assume that

$$Dy_n(z) \to Dy(z) \text{ a. e. on } Z.$$

Since $|x_n(z)| \to +\infty$ a.e. on $\{y \neq 0\}$, hypothesis $H(A)$ (v) implies that

$$A(z, x_n) \to \widehat{A}(z) \text{ a. e. on } \{y \neq 0\}.$$

Also $Dy(z) = 0$ a.e. on $\{y = 0\}$. Hence

$$A(z, x_n) Dy_n(z) \to 0 \text{ a.e. on } \{y = 0\}.$$

Therefore

$$A(z, x_n(z)) Dy_n(z) \to \widehat{A}(z) Dy(z) \text{ a.e. on } Z \text{ as } n \to \infty.$$

From this convergence, hypothesis $H(A)$ (iii) and the dominated convergence theorem, we infer that

$$A(\cdot, x_n(\cdot)) Dy_n(\cdot) \to \widehat{A}(\cdot) Dy(\cdot) \text{ in } L^2(Z, \mathbb{R}^N) \text{ as } n \to \infty.$$

Hence

$$\int_Z (A(z, x_n) Dy_n, Dv)_{\mathbb{R}^N} dz \to \int_Z \left(\widehat{A}(z) Dy, Dv\right)_{\mathbb{R}^N} dz,$$

so

$$\left\langle \frac{V(x_n)}{\|x_n\|}, v \right\rangle \to \left\langle \widehat{V}(y), v \right\rangle \text{ for all } v \in H^1(Z),$$

therefore

$$\frac{V(x_n)}{\|x_n\|} \to \widehat{V}(y) \text{ in } H^1(Z)^*.$$

Returning to (6.10) and passing to the limit as $n \to \infty$, we obtain

$$\widehat{V}(y) = (tg_\infty + (1-t)\widehat{g}) y = gy, \tag{6.20}$$

with $g = tg_\infty + (1-t)\widehat{g} \in L^\infty(Z)_+$, $\widehat{\theta}(z) \leq g(z) \leq \theta(z)$ a.e. on Z. From (6.20) we have

$$\begin{cases} -div\left(\widehat{A}(z) Dy(z)\right) = g(z) y(z) \text{ a.e. on } Z, \\ \frac{\partial y}{\partial n} = 0, \text{ on } \partial Z. \end{cases} \tag{6.21}$$

Exploiting the monotonicity of the eigenvalues on the weight function (see for example Gasinski-Papageorgiou [22]), we have

$$1 = \widehat{\lambda}_k(\lambda_k) > \widehat{\lambda}_k(g)$$

and

$$1 = \widehat{\lambda}_{k+1}(\lambda_{k+1}) < \widehat{\lambda}_{k+1}(g).$$

Using this in (6.21), we infer that $y = 0$, a contradiction to the fact that $\|y\| = 1$. This proves Claim 5.

This claim permits the use of the homotopy invariance property of the degree map $\widehat{d}$. So

$$\widehat{d}(V - N, B_R, 0) = d_{(S)_+}\left(\widehat{V} - \widehat{g}I, B_{R,}0\right) \text{ for all } R \geq R_0. \tag{6.22}$$

We have to compute $d_{(S)_+}\left(\widehat{V} - \widehat{g}I, B_{R,}0\right)$. To this end we consider the orthogonal direct sum decomposition

$$H^1(Z) = \overline{H}_k \oplus \widehat{H}_k.$$

Let $\overline{p}_k$ and $\widehat{p}_k$ be the orthogonal projections on the component spaces $\overline{H}_k$ and $\widehat{H}_k$ respectively. Also let $\mathcal{F} : H^1(Z) \to H^1(Z)^*$ be the duality map for the Sobolev space $H^1(Z)$. We consider the $(S)_+$-homotopy $h_2 : [0,1] \times H^1(Z) \to H^1(Z)^*$ defined by

$$h_2(t,x) = t\left(\overline{p}_k^*\left(\mathcal{F}(\widehat{x})\right) - \overline{x}\right) + (1-t)\left(\widehat{V} - \widehat{g}I\right)(x)$$

where for every $x \in H^1(Z)$, we have $x = \overline{x} + \widehat{x}$ with $\overline{x} = \overline{p}_k(x) \in \overline{H}_k$, $\widehat{x} = \widehat{p}_k(x) \in \widehat{H}_k$.

Next we show that $h_2(t,x) \neq 0$ for all $t \in [0,1]$ and all $x \neq 0$.

Indeed, since on the finite dimensional space $\overline{H}_k$ all norms are equivalent, we have that

$$\begin{aligned}
&\langle h_2(t,x), \widehat{x} - \overline{x}\rangle \\
&\geq t\langle \mathcal{F}(\widehat{x}), \widehat{x}\rangle + tc_4\|\overline{x}\|^2 + (1-t)\left\langle \widehat{V}(x) - \widehat{g}x, \widehat{x} - \overline{x}\right\rangle \\
&\geq tc_4\|x\|^2 + (1-t)\left[\int_Z \left(\widehat{A}(z)D\widehat{x}, D\widehat{x}\right)_{\mathbb{R}^N} dz - \int_Z \widehat{g}\widehat{x}^2 dz + \int_Z \widehat{g}\overline{x}^2 dz - \right. \\
&\left.\int_Z \left(\widehat{A}(z)D\overline{x}, D\overline{x}\right)_{\mathbb{R}^N} dz\right]
\end{aligned}$$

for some $c_4 \in (0,1)$. Here we have used the orthogonality of the component spaces. Using Lemma 6.4 (a), (b), we obtain

$$\langle h_2(t,x), \widehat{x} - \overline{x}\rangle \geq tc_4\|x\|^2 + (1-t)\widehat{\xi}\|x\|^2 \geq c_5\|x\|^2,$$

with $\widehat{\xi} = \min\{\xi_1, \xi_2\}$, for some $c_5 \in (0,1)$, hence

$$h_2(t,x) \neq 0 \text{ for all } t \in [0,1], \text{ all } x \neq 0.$$

Invoking, once again, the homotopy invariance property of the degree map $\widehat{d}$, we have

$$d_{(S)_+}\left(\widehat{V} - \widehat{g}I, B_r, 0\right) = d_{(S)_+}\left(\widehat{p}_k^* \circ \mathcal{F} \circ \widehat{p}_k - \overline{p}_k, B_r, 0\right) \text{ for all } r > 0. \tag{6.23}$$

Set

$$B_{\frac{r}{2}}^{\widehat{H}_k} = \left\{\widehat{x} \in \widehat{H}_k : \|\widehat{x}\| < \frac{r}{2}\right\}$$

and

$$B_{\frac{r}{2}}^{\overline{H}_k} = \left\{\overline{x} \in \overline{H}_k : \|\overline{x}\| < \frac{r}{2}\right\}.$$

Then from the excision and product properties of the degree, we have

$$\begin{aligned} d_{(S)_+}\left(\widehat{p}_k^* \circ \mathcal{F} \circ \widehat{p}_k - \overline{p}_k, B_r, 0\right) &= d_{(S)_+}\left(\mathcal{F}\mid_{\widehat{H}_k}, B_{\frac{r}{2}}^{\widehat{H}_k}, 0\right) \times d_B\left(-I, B_{\frac{r}{2}}^{\overline{H}_k}, 0\right) \\ &= 1 \times (-1)^{\dim \overline{H}_k} \end{aligned} \tag{6.24}$$

From (6.22), (6.23) and (6.24), we conclude that

$$\widehat{d}(V - N, B_R, 0) = (-1)^{\dim \overline{H}_k} \text{ for all } R \geq R_0.$$

■

Next we conduct a similar computation for small balls.

Proposition 6.6 *If hypotheses $H(j)$ hold, then there exists $\rho_0 > 0$ such that*

$$\widehat{d}(V - N, B_\rho, 0) = 1 \text{ for all } 0 < \rho \leq \rho_0.$$

Proof. Let $A_0 \in L^\infty\left(Z, \mathbb{R}^{N\times N}\right)$ be defined by $A_0(z) = A(z,0)$. We introduce the continuous linear operator $V_0 \in \mathcal{L}\left(H^1(Z), H^1(Z)^*\right)$ defined by

$$\langle V_0(x), y\rangle = \int_Z (A_0(z)Dx, Dy)_{\mathbb{R}^N}\, dz \text{ for all } x, y \in H^1(Z).$$

We consider the admissible homotopy $h_3 : [0,1] \times H^1(Z) \to H^1(Z)^*$ defined by

$$h_3(t,x) = tV(x) + (1-t)V_0(x) - tN(x) - (1-t)\widehat{h}x,$$

with $\widehat{h} \in L^\infty(Z)$ satisfying $\widehat{\eta}(z) \leq \widehat{h}(z) \leq \eta(z)$ a.e. on Z.

Claim 6 *There exists* $\rho_0 > 0$ *such that* $0 \notin h_3(t,x)$, *for all* $t \in [0,1]$ *and all* $0 < \|x\| = \rho \leq \rho_0$.

Again we argue by contradiction. So suppose that Claim 6 is not true. Then we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ such that

$$t_n \to t \in [0,1], \ \|x_n\| \to 0 \text{ and } 0 \in h_3(t_n, x_n) \text{ for all } n \geq 1 \tag{6.25}$$

Note that hypothesis $H(j)$ (v) implies that we can find $\delta > 0$ such that

$$|u| \leq c_5 |x|$$

for a.a. $z \in Z$, all $|x| \leq \delta$, all $u \in \partial j(z,x)$ and some $c_5 > 0$.

On the other hand from (6.11), we see that there exists $c_6 = c_6(\delta) > 0$ such that

$$|u| \leq c_6 |x|$$

for a.a. $z \in Z$, all $|x| > \delta$, all $u \in \partial j(z,x)$.

Hence we can say that

$$|u| \leq c_7 |x| \tag{6.26}$$

for a.a. $z \in Z$, all $x \in \mathbb{R}$, all $u \in \partial j(z,x)$, with $c_7 = \max\{c_5, c_6\}$.

From the inclusion in (6.25), we have

$$t_n V(x_n) + (1-t_n) V_0(x_n) = t_n u_n + (1-t_n)\widehat{h} x_n \tag{6.27}$$

with $u_n \in N(x_n)$. We set

$$y_n = \frac{x_n}{\|x_n\|}, n \geq 1.$$

We may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(Z) \text{ and } y_n \to y \text{ in } L^2(Z) \text{ as } n \to \infty.$$

We divide (6.27) with $\|x_n\|$ and obtain

$$t_n \frac{V(x_n)}{\|x_n\|} + (1-t_n) V_0(y_n) = t_n \frac{u_n}{\|x_n\|} + (1-t_n)\widehat{h} y_n. \tag{6.28}$$

Taking duality brackets with $y_n - y$, we have

$$t_n \left\langle \frac{V(x_n)}{\|x_n\|}, y_n - y \right\rangle + (1-t_n)\langle V_0(y_n), y_n - y\rangle$$
$$= t_n \int_Z \frac{u_n}{\|x_n\|}(y_n - y)\,dz + (1-t_n)\int_Z \widehat{h} y_n (y_n - y)\,dz.$$

Note that

$$\int_Z \frac{u_n}{\|x_n\|}(y_n - y)\,dz \to 0$$

(see (6.26)) and

$$\int_Z \widehat{h} y_n (y_n - y)\,dz \to 0 \text{ as } n \to \infty.$$

Since $\|x_n\| \to 0$, we may assume that $x_n(z) \to 0$ a.e. on Z and so

$$A(z, x_n(z)) \to A_0(z) \text{ a.e. on } Z$$

(see $H(A)$ (ii)).

Then arguing as in the proof of Proposition 6.5, we show that

$$y_n \to y \text{ in } H^1(Z)\,, \text{ hence } \|y\| = 1, \text{ i.e. } y \neq 0.$$

In addition, for every $v \in H^1(Z)$, we have

$$\begin{aligned}\left\langle \frac{V(x_n)}{\|x_n\|}, v \right\rangle &= \int_Z (A(z, x_n)\, Dy_n, Dv)_{\mathbb{R}^N}\,dz \to \int_Z (A_0(z)\, Dy, Dv)_{\mathbb{R}^N}\,dz \\ &= \langle V_0(y)\,, v\rangle\,,\end{aligned}$$

hence

$$\frac{V(x_n)}{\|x_n\|} \xrightarrow{w} V_0(y) \text{ in } H^1(Z)^*\,.$$

From (6.26), we see that $\left\{\frac{u_n}{\|x_n\|}\right\}_{n\geq 1} \subseteq L^2(Z)$ is bounded. So we may assume that

$$\frac{u_n}{\|x_n\|} \xrightarrow{w} \beta \text{ in } L^2(Z) \text{ as } n \to \infty.$$

Given $\varepsilon > 0$ and $n \geq 1$, we introduce the sets

$$D^+_{\varepsilon,n} = \left\{ z \in Z : x_n(z) > 0,\ \widehat{\eta}(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)} \leq \eta(z) + \varepsilon \right\}$$

and

$$D^-_{\varepsilon,n} = \left\{ z \in Z : x_n(z) < 0,\ \widehat{\eta}(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)} \leq \eta(z) + \varepsilon \right\}.$$

Since $\|x_n\| \to 0$, we may assume that $x_n(z) \to 0$ a.e. on Z. Hence by virtue of hypothesis $H(j)$ (v), we have

$$\chi_{D^+_{\varepsilon,n}}(z) \to 1 \text{ a.e. on } \{y > 0\}$$

and

$$\chi_{D^-_{\varepsilon,n}}(z) \to 1 \text{ a.e. on } \{y < 0\}.$$

Arguing as in the proof of Proposition 6.5, we infer that

$$\beta = h_0 y,$$

with $h_0 \in L^\infty(Z)$,

$$\widehat{\eta}(z) \le h_0(z) \le \eta(z) \text{ a. e. on } Z.$$

We pass to the limit as $n \to \infty$ in (6.28) and obtain

$$V_0(y) = hy, \tag{6.29}$$

with $h = th_0 + (1-t)\widehat{h} \in L^\infty(Z)$, $\widehat{\eta}(z) \le h(z) \le \eta(z)$ a.e. on Z. We take duality brackets with y. So

$$\int_Z (A_0(z) Dy, Dy)_{\mathbb{R}^N} dz = \int_Z hy^2 dz \le 0$$

therefore

$$\|Dy\|_2 = 0, \text{ i.e. } y = \widehat{c} \in \mathbb{R} \text{ (see } H(A)\ (iv)).$$

Note that $\widehat{c} \neq 0$ (since $\|y\| = 1$). Hence

$$0 = \int_Z (A_0(z) Dy, Dy)_{\mathbb{R}^N} dz = |\widehat{c}|^2 \int_Z h dz < 0,$$

a contradiction. Therefore the Claim 6 is true.

This claim permits the use of the homotopy invariance property and we have

$$\widehat{d}(V - N, B_\rho, 0) = d_{(S)_+}\left(V_0 - \widehat{h}I, B_\rho, 0\right) \text{ for all } 0 < \rho \le \rho_0. \tag{6.30}$$

To compute $d_{(S)_+}\left(V_0 - \widehat{h}I, B_\rho, 0\right)$, we consider the $(S)_+$-homotopy

$$h_4(t,x) = t\left(V_0 - \widehat{h}I\right)(x) + (1-t)\mathcal{F}(x).$$

Then for every $t \in [0,1]$ and $x \neq 0$, we have

$$\begin{aligned} \langle h_4(t,x), x\rangle &= t\left[\int_Z (A_0(z) Dx, Dx)_{\mathbb{R}^N} dz - \int_Z \widehat{h} x^2 dz\right] + (1-t)\|x\|^2 \\ &\ge t\xi_0 \|x\|^2 + (1-t)\|x\|^2 > 0 \end{aligned}$$

(see Lemma 6.4 (c)).

Therefore, once again, the homotopy invariance property implies

$$d_{(S)_+}\left(V_0 - \widehat{h}I, B_\rho, 0\right) = d_{(S)_+}\left(\mathcal{F}, B_\rho, 0\right) = 1 \text{ for all } 0 < \rho,$$

hence

$$d\left(V - N, B_\rho, 0\right) = 1 \text{ for all } 0 < \rho \leq \rho_0$$

(see (6.30)). ∎

Now, we are ready for the existence result concerning problem (6.1).

Theorem 6.7 *If hypotheses $H(A)$ and $H(j)$ hold and* $\dim \overline{H}_k$ *is odd, then problem* (6.1) *has a nontrivial solution $x \in C^1\left(\overline{Z}\right)$.*

Proof. We may assume that $\rho_0 < R_0$ and let $0 < \rho \leq \rho_0$ and $R_0 \leq R$. Then from the additivity and excision properties of the degree map $\widehat{d}$, we have

$$\widehat{d}\left(V - N, B_R, 0\right) = \widehat{d}\left(V - N, B_\rho, 0\right) + \widehat{d}\left(V - N, B_R \backslash \overline{B}_\rho, 0\right),$$

hence

$$(-1)^{\dim \overline{H}_k} = 1 + \widehat{d}\left(V - N, B_R \backslash \overline{B}_\rho, 0\right)$$

(see Propositions 6.5 and 6.6), so

$$\widehat{d}\left(V - N, B_R \backslash \overline{B}_\rho, 0\right) = -2.$$

So from the solution property, we know that we can find $x \in B_R \backslash \overline{B}_\rho$ such that

$$0 \in V(x) - N(x),$$

hence

$$0 = V(x) - u, \text{ with } u \in N(x),$$

so

$$\begin{cases} -div\left(A\left(z, x(z)\right) Dx(z)\right) = u(z) \in \partial j\left(z, x(z)\right) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0, \text{ on } \partial Z. \end{cases}.$$

Moreover, from standard regularity theory we have $x \in C^1\left(\overline{Z}\right)$ (see for example Gasinski-Papageorgiou [22]). So $x \in C^1\left(\overline{Z}\right)$ is a nontrivial solution of (6.1) and note that the Neumann boundary condition is understood pointwise. ∎

Remark 6.8 *If $k = 0$, then $\overline{H}_k = \mathbb{R}$ and so the Theorem applies.*

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