Universidade de Aveiro Departamento de Matemática
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Hipermapas Bicontactuais

Bicontactual Hypermaps

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática. Trabalho realizado sob a orientação científica do Doutor António João de Castilho Breda d'Azevedo, Professor Associado do Departamento de Matemática da Universidade de Aveiro.

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Aos meus queridos filhos / To my dear kids,

Bernardo e Daniel

o júri

presidente

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palavras-chave

Hipermapa, mapa, bicontactual, orientável, não-orientável, orientado, regular, quiral, reflexivo.

resumo

Esta tese dedica-se ao estudo de hipermapas regulares bicontactuais, hipermapas com a propriedade que cada hiperface contacta só com outras duas hiperfaces. *Nos anos 70*, S. Wilson classificou os mapas bicontactuais e, em 2003, Wilson e Breda d'Azevedo classificaram os hipermapas bicontactuais no caso não-orientável. Quando esta propriedade é transferida para hipermapas origina três tipos de bicontactualidade, atendendo ao modo como as duas hiperfaces aparecem à volta de uma hiperface fixa: *edge-twin*, *vertex-twin* and *alternate* (dois deles são o dual um do outro).

Um *hipermapa topológico* é um mergulho celular de um grafo conexo trivalente numa superfície compacta e conexa tal que as células são 3-coloridas. Ou de maneira mais simples, um hipermapa pode ser visto como um mapa bipartido.
Um *hipermapa orientado regular* é um triplo ordenado consistindo num conjunto finito e dois geradores, que são permutações (involuções) do conjunto tal que o grupo gerado por eles, chamado o *grupo de monodromia*, actua regularmente no conjunto.

Nesta tese, damos uma classificação de todos os hipermapas orientados regulares bicontactuais e, para completar, reclassificamos, usando o nosso método algébrico, os hipermapas não-orientáveis bicontactuais.

keywords

Hypermap, map, bicontactual, orientable, non-orientable, oriented, regular, chiral, reflexible.

abstract

This thesis is devoted to the study of bicontactual regular hypermaps, hypermaps with the property that each hyperface meets only two others. In the seventies, S. Wilson classified the bicontactual maps and, in 2003, Wilson and Breda d'Azevedo classified the bicontactual non-orientable hypermaps. When this property is transferred for hypermaps it gives rise to three types of bicontactuality, according as the two hyperfaces appear around a fixed hyperface: edge-twin, vertex-twin and alternate (two of which are dual of each other).

A *topological hypermap* is a cellular embedding of a connected trivalent graph into a compact and connected surface such that the cells are 3-colored. Or simply, a hypermap can be seen as a bipartite map.
A *regular oriented-hypermap* is an ordered triple, consisting of a finite set and two generators, which are permutations of the set, such that the group generate by them, called *monodromy group*, acts regularly on the set.

In this thesis, we give a classification of all bicontactual regular oriented-hypermaps and, for completion, we reclassify, using our algebraic method, the bicontactual non-orientable hypermaps.

Contents

Notation

$\mathrm{v}_{\mathcal{V}}$	valency of the hypervertices
$\mathrm{v}_{\mathcal{E}}$	valency of the hyperedges
$\mathrm{v}_{\mathcal{F}}$	valency of the hyperfaces
$\mathrm{n}_{\mathcal{V}}$	number of hypervertices
$\mathrm{n}_{\mathcal{E}}$	number of hyperedges
$\mathrm{n}_{\mathcal{F}}$	number of hyperfaces
$\langle X \rangle$	subgroup generate by X
$<$	subgroup of
$\lhd$	normal subgroup of
$\overline{R}$	normal closure of R
$\langle X \rangle^G$	normal closure of X in G
F/N	factor group
$\mathfrak{R}(a_1, a_2, \ldots, a_m)$	set of word relations, in function of a_1, a_2, ..., a_m
$G = \langle X \vert \mathcal{R} \rangle$	group presentation of the group G
$Mon(\mathcal{H})$	monodromy group of the hypermap $\mathcal{H}$
$\mathcal{X}(\mathcal{H})$, $\mathcal{X}$	Euler characteristic of the hypermap $\mathcal{H}$
$\mathbb{X}_{\mathcal{H}}$	chirality group of a hypermap $\mathcal{H}$

$\kappa = \kappa(\mathcal{H})$	chirality index of a hypermap $\mathcal{H}$
C_n	cyclic group
D_n	dihedral group
S_n	symmetric group
1	empty word (identity element of the group)
x^y	conjugate of x by y ($y^{-1}xy$)
$\widehat{k}$	representative element of the equivalence class
(k, c)	greatest common divisor between k and c
$d \vert x$	d divides x or x is multiple of d
$a \equiv b\ (mod\ c)$	c divides $a - b$
$\leftrightharpoons$	commutes
$AltB$	family of alternate bicontactual regular oriented-hypermaps
$FETB$	family of fundamental edge-twin bicontactual regular oriented-hypermaps
ETB	family of edge-twin bicontactual regular oriented-hypermaps

Introduction

The theory of hypermaps, a subject of Combinatorial and Topological Geometry, is closely related to the classical theory of maps on surfaces. A *map* is a cellular embedding of a connected graph in a compact and connected surface (with or without boundary). It is often viewed as a cellular division of a surface into simply-connected open regions by *arcs*, called the *edges* of the map; the regions are called the *faces* of the map and the endpoints of each edge are the *vertices* of the map. In this definition, if we replace "graph" by "hypergraph" we have a *hypermap*. By [25] a hypergraph is seen as a generalisation of a graph and, from this point of view, a *hypermap* is a generalisation of a map, by allowing edges to connect more than two vertices. The orientability of a hypermap is determinated by the orientability of the underlying surface. An *oriented-hypermap* is obtained by fixing an orientation in the underlaying surface. A *bicontactual map* had the property that each face meets exactly two others.

Bicontactual maps were introduced and classified in the seventies by Wilson in his PhD thesis for maps [26]. Later in [30] Wilson and Breda d'Azevedo showed that bicontactual non-orientable regular hypermaps are "essentialy" bicontactual maps with only one exception: $D_{02}(GW_k)$ (adopting their notation). This is the Walsh of a bicontactual reflexible map, and it turns out to be the only "pure" bicontactual non-orientable hypermap. In this thesis we complete the classification of the bicontactual hypermaps started in [30].

In spite of the resemblance, the notion of bicontactuality when transferred to hypermaps gives rise to three types of bicontactuality (two of which are dual of each other): the edge-twin, the vertex-twin and the alternate bicontactuality. In a hypermap that each face meets only two others, if we fix a hyperface, for example F, the two hyperfaces around it, labelled A and B, appear in the pattern $AABBAABBAA...$, with repetitions occurring at the hyperedges (*edge-twin*), or in the same pattern but the repetitions occurring at the hypervertices (*vertex-twin*), or in the pattern $ABABABABA...$ (*alternate*). It is easy to see that bicontactual maps are included in the edge-twin bicontactual hypermaps.

Classifying maps/hypermaps subject to specific conditions, has been one a favorite topic since as early in the 20^{th} century with Brahana proposing an approach to the classification of regular maps on a given surface. For example,

- Classification *by genera*: non-orientable regular maps and hypermaps with size a power of 2 is presented in [30]; in [13], the classification of all regular maps on non-orientable surfaces with a negative odd prime Euler characteristic; classification of regular maps with Euler characteristic $-p^2$ is presented in [17]; by [13, 14, 18] we have a classification of all regular maps of Euler characteristic $-p$ and $-2p$ with p a prime;

- Classification *by number of hyperfaces*: the classifications of the non-orientable regular maps and hypermaps with 1, 2, 3, 5 hyperfaces and of the non-orientable regular maps with a prime number of faces are presented in [30]; regular oriented-hypermaps with a prime number of hyperface in [6];

- Classification *by embedding graphs*: the paper [22] classifies the regular imbeddings of the complete graphs K_n in orientable surfaces;

- Classification *by group*: the classification of some particular hypermaps is described in [1]; in [17], the regular maps with almost Sylow-cyclic automorphism groups;

- Classification *by properties*: Wilson, in [26], classified the bicontactual maps; chiral maps in the orientable case in [18];

just to name but a few. It is in the last topic that my thesis is included. Looking at the list of 14647 pure (or proper) hypermaps given by Conder [15], we found 776 edge-twin biconctactual, 519 vertex-twin bicontactual and 753 alternate bicontactual. In conclusion, only 14 percent of pure hypermaps up to genus 101 are bicontactual.

Although at this stage we avoid details, we would like to note that hypermaps can be described either in a topological or algebraic way. Most of the existing literature considers hypermaps as algebraic objects. Our algebraic classification of regular bicontactual hypermaps consists in classifying (up to an isomorphism and duality) the edge-twin oriented-hypermaps and the alternate oriented-hypermaps.

For a deeper study on hypermaps we recommend reading [3, 8, 9, 12, 19, 25], where the theory of hypermaps is well-developed and thoroughly explained.

We conclude the introduction with a brief outline of the chapters in this thesis.

The first two chapters overview the main relevant results on hypermaps, bicontactual maps and bicontactual hypermaps. Besides presenting well-known material in this area and explaining some of the necessary theoretical background, we introduce new concepts and new results that are relevant for our purposes. For this we establish adequate notation.

In Chapters 3 and 4 we start by developing properties of bicontactual oriented-hypermaps and then we study their classification. The chirality group as well as the respective chirality index of hypermaps have already been investigated in [7, 10, 12]. After the classification, we are particularly interested in identifying the chiral bicontactual hypermaps.

Chapter 5 is devoted to the non-orientable bicontactual hypermaps classification, done by Wilson and Breda d'Azevedo. Here, we rewrite the proof of the classification using our algebraic approach.

The software, for computational discrete algebra, GAP, [21], was used in this work to help us to get confidence on the number of word relations and on the amount of conditions to be imposed to the hypermaps and also to explore the existing data-bases.

This thesis does not include the proofs of non-original results, apart from chapter 5. All the other results and their proofs are original, though some of them being inspired by Wilson's results.

We finish this thesis with some general considerations, such as the classification of the bicontactual hypermaps with boundary and future work. In fact, we have already started studies in the direction of a classification of the bicontactual hypermaps in the pseudo-orientable case.

Chapter 1

Background

In this chapter, some basic definitions and some elementary concepts concerning hypermaps, oriented-hypermaps, presentations of groups, chirality, metacyclic groups and bicontactuality are presented.

In the first two sections an introduction to the theory of hypermaps is given. Section 2 deals with oriented-hypermaps.

In section 4, we introduce the chirality notion for regular oriented-hypermaps while the algebraic definition of bicontactuality comes in the last section.

The contents of this chapter have been adapted from various sources including [**?**, 8, 11, 12, 23, 25, 28, 29, 31].

1.1 Hypermaps

Hypermaps can be defined either in a topological or algebraic way. We will present a topological definition as well as an algebraic one.

From a topological point of view, a *hypermap* $\mathcal{H}$ (without boundary) is a cellular embedding of a connected trivalent graph $\mathcal{G}$ into a compact and connected surface such that the cells are 3-colored (say by black, grey and white colours) with adjacent cells having different colours. The *flags* are the vertices of $\mathcal{G}$, while the faces labeled 0 (dark grey), 1 (grey) and 2 (white), are called, respectively, the *hypervertices*, the *hyperedges* and the *hyperfaces* of $\mathcal{H}$. Labeling the edges of $\mathcal{G}$ with the missing adjacent cell number, the hypergraph formed in this way induces three permutations r_0, r_1 and r_2 on the set of flags Ω; each r_i switches the pairs of vertices connected by $i-$edges (edges labeled i).

Example 1 Figure 1.1, [9, pg 52], shows a hypermap on the double torus with 12 flags, 1 hypervertex, 1 hyperedge and 2 hyperfaces and where ω is an arbitrary flag of the hypermap:

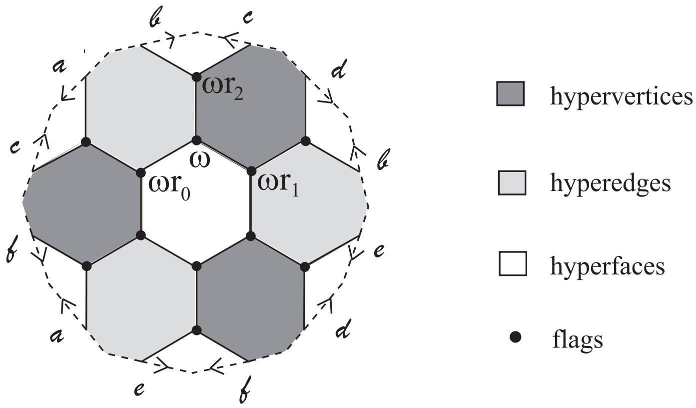


Figure 1.1: A topological hypermap on the torus

These permutations r_i are involutions and the group G generated by r_0, r_1 and r_2 will be called the *monodromy group*, $Mon(\mathcal{H})$, of the hypermap $\mathcal{H}$.

Since the graph $\mathcal{G}$ is connected, the monodromy group acts *transitively*[1] on Ω. This implies $|\Omega| \leq |Mon(\mathcal{H})|$. If the monodromy group $Mon(\mathcal{H})$ acts *semi-regularly*[2] on Ω, the monodromy group $Mon(\mathcal{H})$ acts regularly on Ω. In this case the hypermap is called *regular*, and we have $|Mon(\mathcal{H})| = |\Omega|$.

The orbits of $\langle r_1, r_2\rangle$, $\langle r_0, r_2\rangle$ and $\langle r_0, r_1\rangle$ on Ω determine the hypervertices, the hyperedges and the hyperfaces, respectively. If all $\langle r_i, r_j\rangle$ orbits on Ω have the same cardinality (for each $i, j \in \{0, 1, 2\}$ with $i < j$), then the hypermap $\mathcal{H}$ is *uniform*. For an uniform hypermap, the length $\mathrm{v}_{\mathcal{V}}$, $\mathrm{v}_{\mathcal{E}}$, $\mathrm{v}_{\mathcal{F}}$ of the cycles r_1r_2, r_2r_0 and r_1r_0, respectively, are called the *valency of the hypervertices*, the *valency of the hyperedges* and the *valency of the hyperfaces*, respectively, and they determine the *type* $(\mathrm{v}_{\mathcal{V}}, \mathrm{v}_{\mathcal{E}}, \mathrm{v}_{\mathcal{F}})$ of the hypermap. Let us denote by $\mathrm{n}_{\mathcal{V}}$ the size of $\langle r_1, r_2\rangle$, by $\mathrm{n}_{\mathcal{E}}$ the size of $\langle r_0, r_2\rangle$ and by $\mathrm{n}_{\mathcal{F}}$ the size of $\langle r_0, r_1\rangle$.

From a hypermap $\mathcal{H}$ we can produce six hypermaps on the same surface by permuting the three colours 0, 1, 2 of their cells. Each permutation $\sigma \in S_3$ determines a new hypermap with

[1]that is, $\forall \omega_1, \omega_2 \in \Omega\ \exists g \in Mon(\mathcal{H})$ such that $\omega_2 = \omega_1 g$

[2]that is, $\forall g \in Mon(\mathcal{H})$, $(\exists \omega \in \Omega$ such that $\omega = \omega g) \Rightarrow g = 1$

the same underlying trivalent graph $\mathcal{G}$ and whose hypervertices, hyperedges and hyperfaces are the cells colored 0σ, 1σ, 2σ, respectively. We call it the *σ-dual* of $\mathcal{H}$ and will denote by $D_\sigma(\mathcal{H})$. For example, D_{01} is the hypermap that results from switching labels 0 and 1 in the coloured graph of $\mathcal{H}$; the classic dual, usually denoted by D, corresponds to D_{02}.

The *orientability*, the *Euler characteristic* and the *genus* of a hypermap $\mathcal{H}$ are the orientability, the Euler characteristic and the genus of the respective underlying surface. The *Euler characteristic* is given by the formula

$$\mathcal{X}(\mathcal{H}) = \mathrm{n}_{\mathcal{V}} + \mathrm{n}_{\mathcal{E}} + \mathrm{n}_{\mathcal{F}} - \frac{|\Omega|}{2}$$

or equivalently,

$$\mathcal{X}(\mathcal{H}) = \frac{|\Omega|}{2}\left(\frac{1}{\mathrm{v}_{\mathcal{V}}} + \frac{1}{\mathrm{v}_{\mathcal{E}}} + \frac{1}{\mathrm{v}_{\mathcal{F}}} - 1\right).$$

From an algebraic point of view, a *hypermap* $\mathcal{H}$ is a 4−tuple $\mathcal{H} = (\Omega;\, r_0,\, r_1,\, r_2)$, where Ω is a non-empty finite set and r_0, r_1 and r_2 are three permutations of Ω satisfying $r_0^2 = r_1^2 = r_2^2 = 1$, such that the permutation group generated by r_0, r_1 and r_2 acts transitively on Ω (where 1 is the identity in G).If $(r_0r_2)^2 = 1$ then $\mathcal{H}$ is a map. Here, the *hypervertices* are also the orbits of $\langle r_1, r_2\rangle$, as well as the *hyperedges* are the orbits of $\langle r_0, r_2\rangle$ and the *hyperfaces* are the orbits of $\langle r_0, r_1\rangle$.

There is a one-to-one correspondence (see [25]) between hypermaps and *bipartite*[3] maps. The faces of the bipartite map represent the *hyperfaces* of $\mathcal{H}$ and the two bipartition of vertices (nodes) of the bipartite map (colored white and black) represent the *hyperedges* (the white nodes) and *hypervertices* (the black nodes) of the hypermap. The elements of Ω are called *flags* and are described in the following geometric construction: For each face in the bipartite map $\mathcal{M}$, choose a point in its interior (let us call it the *face-center*). We obtain a subdivision of the bipartite map by joining each face-center to each white node and to each black node. Figure 1.2 illustrates the subdivision and shows the neighborhood of one flag ω.

[3]In a bipartite map its vertices can be colored with two colors, so that no two vertices joined by an edge have the same color.

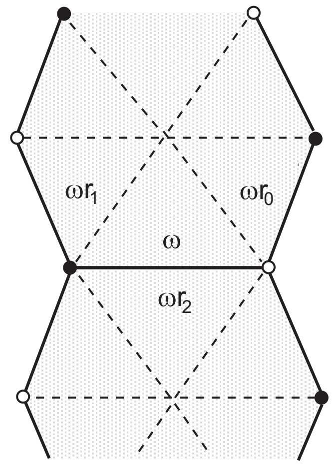


Figure 1.2: Geometric notion of a flag

A *flag* in a hypermap is a mutual incidence of a hypervertex, a hyperedge and a hyperface (for example, in Figure 1.3 the flag $\omega \in \Omega$ corresponds to the triple $(v,\, e,\, F_1)$). From [8], the hypermap $\mathcal{H}$ is orientable if and only if its flags can be 2−coloured with each r_i $(i = 0,\, 1,\, 2)$ transposing the colours. Thus, the flags can be represented by white and black triangles in alternating way. Figure 1.3 also shows our geometric convention for a flag, instead of the previous triangular subregion.

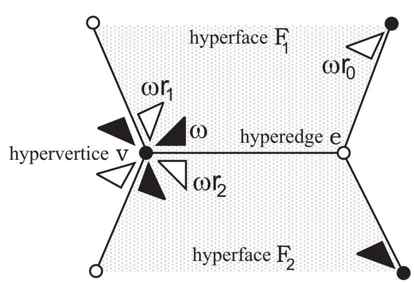


Figure 1.3: Algebraic hypermap

Interchanging the roles of white and black nodes we have the duality D_{01} of hypermaps, that we will refer in the chapter 4.

It may happen that the subgroup of $Mon(\mathcal{H})$ generated by r_1r_2 and r_2r_0 may act on Ω with one or two orbits. When it acts with two orbits, $\mathcal{H}$ is *orientable*. An *oriented-hypermap* is a hypermap with a fixed orientation of the underlying surface. Oriented-hypermaps are described by triples $(D;\, R,\, L)$ where D is a non-empty finite set, which elements are called *darts*, and two permutations R and L of D such that the group $\langle R, L\rangle$ acts transitively on D. If $\mathcal{H}$ is orientable then the set Ω is the union of the two disjoint sets Ω^+ and Ω^- (corresponding the set Ω^+ to the black flags and the set Ω^- to the white flags). In this case, we can derive two associated oriented-hypermaps:

$$\mathcal{H}^+ = (\Omega^+;\, r_1r_{2_{|\Omega^+}},\, r_2r_{0_{|\Omega^+}}) \quad ; \quad \mathcal{H}^- = (\Omega^-;\, r_1r_{2_{|\Omega^-}},\, r_2r_{0_{|\Omega^-}})\,.$$

The hypermaps that satisfy $(r_0r_2)^2 = 1$ are called *maps*. Otherwise, the hypermaps are called *pure hypermaps*.

A *covering* from $\mathcal{H}_1 = (\Omega_1;\, r_0,\, r_1,\, r_2)$ to $\mathcal{H}_2 = (\Omega_2;\, s_0,\, s_1,\, s_2)$ is a function $\phi : \Omega_1 \to \Omega_2$ such that

$$\forall \omega \in \Omega_1\,,\ \forall i \in \{0, 1, 2\}\,,\ (\omega r_i)\phi = \omega\phi s_i\,.$$

Note that a covering is necessary surjective, because of the transitivity of the actions. If there is a *covering*, we say that $\mathcal{H}_1$ *covers* $\mathcal{H}_2$ and we denote this by $\mathcal{H}_1 \to \mathcal{H}_2$. When $\mathcal{H}_1$ covers $\mathcal{H}_2$, it follows that the assignment $r_i \mapsto s_i$, for $i = 0, 1, 2$, extends to a *canonical epimorphism* between the monodromy groups.

An injective covering is an *isomorphism*. We denote an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$ by $\mathcal{H}_1 \cong \mathcal{H}_2$ and we say that $\mathcal{H}_1$ is isomorphic to $\mathcal{H}_2$. The hypermaps $\mathcal{H}^+$ and $\mathcal{H}^-$ associated with a given orientable hypermap $\mathcal{H}$ may or may not be isomorphic.

An *automorphism* of $\mathcal{H} = (\Omega;\, r_0,\, r_1,\, r_2)$ is an isomorphism from $\mathcal{H}$ to $\mathcal{H}$, that is, a permutation of Ω commuting with r_0, r_1 and r_2 (consequently, commuting with all elements of $Mon(\mathcal{H})$). The group of all automorphisms of $\mathcal{H}$, denoted by $Aut(\mathcal{H})$, acts semi-regularly on Ω. So $|Aut(\mathcal{H})| \leq |\Omega|$, and hence we have

$$|Aut(\mathcal{H})| \leq |\Omega| \leq |Mon(\mathcal{H})|\,. \tag{1.1}$$

If $Aut(\mathcal{H})$ acts transitively on Ω, that is, if the action of $Aut(\mathcal{H})$ on Ω is regular, then we said that $\mathcal{H}$ is regular and we have $|Aut(\mathcal{H})| = |\Omega|$. In equation (1.1), an equality on one side implies an equality on the other side.

In an orientable hypermap $\mathcal{H}$, an automorphism $\phi \in Aut(\mathcal{H})$ *preserves the orientation* if

$$\forall \omega \in \Omega^\epsilon\,, \text{ where } \epsilon \in \{+, -\}\,,\ \omega\phi \in \Omega^\epsilon\,.$$

Otherwise, ϕ *reverses the orientation* and it is called a *mirror symmetry*, or *reflection*, or *inversion*. If the hypermap has a mirror symmetry then it is called *mirror symmetric*, else $\mathcal{H}$ is *mirror asymmetric*.

The hypermap $\mathcal{H}$ is said to be *orientably-regular*, or *rotary*, if the subgroup $Aut^+(\mathcal{H})$ of the automorphisms preserving orientation acts transitively on Ω^+. In this case, $|Mon(\mathcal{H})| = \frac{|\Omega|}{2}$. An orientably-regular hypermap is *reflexible* if and only if it has an automorphism that reverses the orientation. A *chiral* hypermap is an orientably-regular hypermap which is not reflexible. The number of flags of a reflexible hypermap must be even.

The $\mathcal{H}-$*sequence* of a hypermap $\mathcal{H}$ is the sequence

$$[\,\mathrm{N}\,;\,\mathrm{v}_{\mathcal{V}},\,\mathrm{v}_{\mathcal{E}},\,\mathrm{v}_{\mathcal{F}}\,;\,\mathrm{n}_{\mathcal{V}},\,\mathrm{n}_{\mathcal{E}},\,\mathrm{n}_{\mathcal{F}}\,;\,|Mon(\mathcal{H})|\,]\,,$$

where N is the negative of the Euler characteristic of the hypermap, which is constant for all hypermaps in a fixed surface S.

Each hypermap $\mathcal{H}$ determines a transitive permutation representation $\pi\,:\,\Delta \to Mon(\mathcal{H})$ of the *triangular group* $\Delta := \langle R_0, R_1, R_2 \,|\, R_0^2 = R_1^2 = R_2^2 = 1\rangle \cong C_2 * C_2 * C_2$ (a free product of three cyclic groups of order 2), given by $R_i \mapsto r_i$. We call to the stabiliser H in Δ of a flag ω, that is $H = \Delta_\omega = \{g \in Mon(\mathcal{H}) \,|\, \omega g = \omega\}$, *hypermap subgroup* of the hypermap $\mathcal{H}$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypermaps with hypermap subgroups H_1 and H_2, respectively. Then $\mathcal{H}_1$ *covers* $\mathcal{H}_2$ if and only if $\exists d \in \Delta$ such that $H_1^d < H_2$. The covering is given by

$$\Delta\diagup H_1 \to \Delta\diagup H_2\,,\;H_1 d \mapsto H_2 d\,.$$

It is important, for our work, to refer three of the seven epimorphisms $\Delta \to C_2$, which have kernels $\Delta^0 = \langle R_0,\, R_1R_2\rangle^\Delta$, $\Delta^{\hat{0}} = \langle R_1,\, R_2\rangle^\Delta$ and $\Delta^+ = \langle R_1R_2,\, R_2R_0\rangle^\Delta$, where $\langle ...\rangle^\Delta$ denote the normal closure in Δ. The last subgroup Δ^+ is the *even subgroup* of Δ, which is generated by a pair of free generators R_1R_2 and R_2R_0. These generators projects to the rotations R and L around hypervertices and hyperedges, the generators of the monodromy group of oriented-hypermaps (see the following section). The inclusions between the preceding subgroups of Δ are those shown below:

$$\begin{tikzcd} & \Delta \arrow[dl] \arrow[d] \arrow[dr] & \\ \Delta^+ & \Delta^0 & \Delta^{\hat{0}} \end{tikzcd}$$

Being $H < \Delta$ the hypermap subgroup of $\mathcal{H}$, the hypermap $\mathcal{H}$ is

1. *regular* or Δ-*regular* if $H \lhd \Delta$;

2. *oriented* if $H < \Delta^+$;

3. *regular oriented* or Δ^+-*regular* if $H \lhd \Delta^+$;

4. *chiral* if $H \lhd \Delta^+$ and $H \ntriangleleft \Delta$;

5. *pseudo-oriented* if $H < \Delta^0$;

6. *regular pseudo-oriented* or Δ^0-*regular* if $H \lhd \Delta^0$;

7. *bipartite* if $H < \Delta^{\hat{0}}$.

Walsh showed in [25] that there is a bijection between hypermaps and bipartite maps on the same surface. As a hypermap can be regarded as a bipartite map, a bipartite map can also be regarded as a hypermap (see an example in figure below).

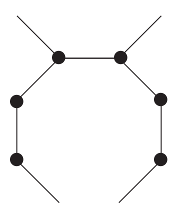


Figure 1.4: Bipartite map

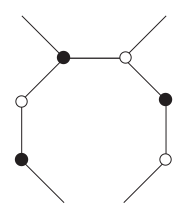


Figure 1.5: Hypermap

The edges of the bipartite map are the *arcs* of the hypermap. So, the notions of *edge* in maps and *arc* in hypermaps are similar, as is illustrate in the following figure:

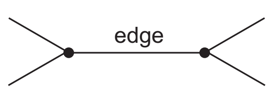


Figure 1.6: (map)

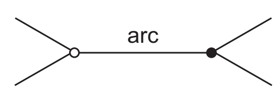


Figure 1.7: (hypermap)

The *Walsh map* of a hypermap $\mathcal{H}$, $W(\mathcal{H})$, is a bipartite map, the set of vertices corresponding to the hypervertices and hyperedges of $\mathcal{H}$, while the faces and edges of $W(\mathcal{H})$ correspond to the hyperfaces and arcs of $\mathcal{H}$, respectively. Conversely, any bipartite map $\mathcal{M}$, with two monochrome sets of vertices (usually coloured black and white), corresponds to a hypermap

$W^{-1}(\mathcal{M})$ where one of two monochromatic sets of vertices represents the hypervertices and the other the hyperedges of the hypermap, the faces and edges of $\mathcal{M}$ correspond to the hyperfaces and arcs of $W^{-1}(\mathcal{M})$, respectively.

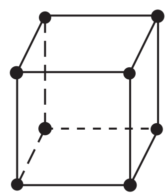


Figure 1.8: Cube $\mathcal{C}$

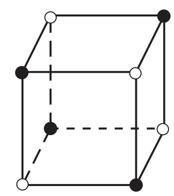


Figure 1.9: $W^{-1}(\mathcal{C})$

Remark 1 If a bipartite uniform map $\mathcal{M}$ has type $(\mathrm{v}_{\mathcal{V}},\, 2,\, \mathrm{v}_{\mathcal{F}})$, then the hypermap $W^{-1}(\mathcal{M})$ has type $\left(\mathrm{v}_{\mathcal{V}},\, \mathrm{v}_{\mathcal{V}},\, \frac{\mathrm{v}_{\mathcal{F}}}{2}\right)$.

1.2 Oriented-hypermaps

As we said before, an *oriented-hypermap* is a triple $\mathcal{Q} = (D;\, R,\, L)$ consisting of a non-empty finite set D, which elements are called *darts*, and two permutations R and L of D such that the group $\langle R, L\rangle$ acts transitively on D. This group will be called the *monodromy group* of the hypermap $\mathcal{Q}$, denoted by $Mon(\mathcal{Q})$ or G. We will assume that the set D corresponds to the set Ω^+.

The orbits of R, L and RL on D will be called the *hypervertices*, *hyperedges* and *hyperfaces*, respectively. The least common multiples $\mathrm{v}_{\mathcal{V}}$, $\mathrm{v}_{\mathcal{E}}$, $\mathrm{v}_{\mathcal{F}}$ of the length of the cycles R, L and RL on D, respectively, determine the *type* $(\mathrm{v}_{\mathcal{V}},\, \mathrm{v}_{\mathcal{E}},\, \mathrm{v}_{\mathcal{F}})$ of the oriented-hypermap $\mathcal{Q}$.

Since an oriented-hypermap is an orientable hypermap with a fixed orientation, we choose, for instance, the counter-clockwise orientation[4]. We will consider R the one step counter-clockwise rotation of the darts around hypervertices ($R = r_1r_2$) and L the one step counter-clockwise rotation of the darts around hyperedges ($L = r_2r_0$). In this thesis we will work only with oriented-hypermaps which are regular, thus each dart corresponds to an element of $Mon(\mathcal{Q})$. So, we will use G instead of D.

[4]We will abreviate *clockwise* and *counterclockwise* by *CW* and *CCW*, respectively.

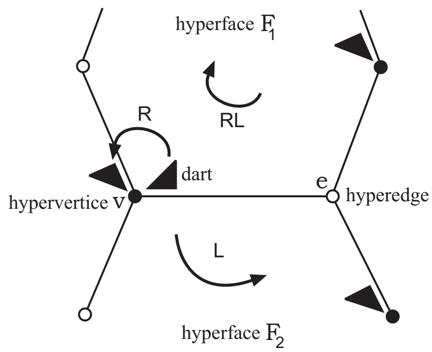


Figure 1.10: Regular Oriented-hypermap

If L is an involution, then $\mathcal{Q}$ is an *oriented map*.

Given two oriented hypermaps $\mathcal{Q}_1 = (D_1; R_1, L_1)$ and $\mathcal{Q}_2 = (D_2; R_2, L_2)$, a *covering* $\mathcal{Q}_1 \rightarrow \mathcal{Q}_2$ is a function $\phi : D_1 \rightarrow D_2$ such that $R_1\phi = \phi R_2$ and $L_1\phi = \phi L_2$. By the connectivity of $\mathcal{G}$ any covering is necessarily onto. If ϕ is injective the covering is an *isomorphism* of hypermaps.

An *automorphism* (or *symmetry*) of a hypermap $\mathcal{Q} = (D; R, L)$ is an isomorphism of $\mathcal{Q}$ into itself; in other words, a permutation of D that commutes with R and L. The automorphism group of $\mathcal{Q}$ acts semi-regularly on D while the monodromy group acts transitively. Hence

$$|Aut(\mathcal{Q})| \leq |D| \leq |Mon(\mathcal{Q})|$$

and if one of the equalities holds then the other equality holds as well, and $\mathcal{Q}$ is said *regular*. If, in addition, $\mathcal{Q}$ has an orientation inverting automorphism (also called *mirror symmetry*), that is, a permutation ψ of D such that $R\psi = \psi R^{-1}$ and $L\psi = \psi L^{-1}$, then $\mathcal{Q}$ is said *reflexible*. If $\mathcal{Q}$ is regular but not reflexible then $\mathcal{Q}$ is *chiral*.

The *Euler characteristic* of a regular oriented-hypermap $\mathcal{Q}$ is given by the formula

$$\mathcal{X}(\mathcal{Q}) = \mathrm{n}_{\mathcal{V}} + \mathrm{n}_{\mathcal{E}} + \mathrm{n}_{\mathcal{F}} - |G| = |G| \left(\frac{1}{\mathrm{v}_{\mathcal{V}}} + \frac{1}{\mathrm{v}_{\mathcal{E}}} + \frac{1}{\mathrm{v}_{\mathcal{F}}} - 1 \right)$$

and the *genus* of $\mathcal{Q}$ is

$$g(\mathcal{Q}) = \frac{2 - \mathcal{X}(\mathcal{Q})}{2}.$$

1.3 Presentations of groups

All the contents of this section were extracted from the Johnson's book [23], though we had established, sometimes, our own notation.

1.3.1 Free presentations of groups

In the theory of group presentations, the notion of free group is fundamental. A group F is called *free* if it has a subset S with the property that every element of F can be written uniquely is a product of elements of S and their inverses.

Definition 1 *Let X be a set, $F = F(X)$ a free group on X, $\mathcal{R}$ a subset of F, $N = \overline{\mathcal{R}}$ the normal closure of $\mathcal{R}$ in F, and G the factor group F/N.*
We write $G = \langle X|\mathcal{R}(x_1, \ldots, x_m)\rangle$, or just $G = \langle X|\mathcal{R}\rangle$, and call this a free presentation, or simply a presentation of G. The elements of X are called generators and the elements of $\mathcal{R}$ are called word relations.

From now, we will assume the notation presents in the preceding definition and each element of $\mathcal{R}$ is denoted by $\omega_X = \omega_{x_1,\ldots,x_n}$, a word in X (or simply, ω).

A group G is called *finitely presented* if it has a presentation with both X and $\mathcal{R}$ finite sets.

Remark 2 Let ω_1 and ω_2 be word relations. A word relation of the form $\omega_1 = \omega_2$ corresponds to the *defining relator* $\omega_1\omega_2^{-1}$.

Proposition 1 *Every group has a presentation, and every finite group is finitely presented. [23, pg 42]*

Proposition 2 ***(von Dyck)*** *If $G = \langle X|\mathcal{R}\rangle$ and $H = \langle X|\mathcal{S}\rangle$, where $\mathcal{R} \subseteq \mathcal{S} \subseteq F(X)$, then there is an epimorphism $\phi : G \to H$ fixing every $x \in X$ and such that*

$$Ker(\phi) = \overline{S \setminus R}.$$

Conversely, every factor group of $G = \langle X|\mathcal{R}\rangle$ has a presentation $\langle X|\mathcal{S}\rangle$ with $\mathcal{S} \supseteq \mathcal{R}$. [23, pg 43]

We shall be concerned with the problem of constructing presentations of groups, specifically monodromy groups of bicontactual hypermaps. The presentation of a group give rises to the problem of deducing properties of this group.

For the problem of finding a presentation P for a group G, we need

1. to determine a set of generators for G, X;
2. to write, in terms of X, the word relations of $\mathcal{R}$ that are valid in G and that are enough to define G;
3. letting $P = \langle X|\mathcal{R}\rangle$, the last step is to find $|P|$. If $|P| = |G|$ then the procedure stops, otherwise we will go back to step 2.

In [23], chapter 5, there are several examples which illustrate the various techniques involved.

Given two presentations P_1 and P_2 we write $P_1 \sim P_2$ if they are presentations of the same group.

1.3.2 Presentations of group extensions

We start with two basic definitions that we need in course of this work.

Definition 2 *Let S be subgroup of a group G. The cosets of S are a partition of G and, by choosing one element from each, we obtain a transversal for S in G.*

Definition 3 *Let U be a transversal for a subgroup of a free group $F(t_1, t_2, \ldots, t_k)$. It is a Schreier transversal if U has the Schreier property, that is, for all $t_i \in \{t_1, t_2, \ldots, t_k\}$ if $t_1 t_2 \cdots t_n \in U$ then $t_1 t_2 \cdots t_{n-1} \in U$.*

Consequently, any Schreier transversal contains the empty word 1.

Definition 4 *An extension of a group G by a group A is a group $\widetilde{G}$ having a normal subgroup N such that*

$$A \cong N \ , \ \ \widetilde{G}/N \cong G \ . \tag{1.2}$$

Let $\widetilde{G}$ be an extension of G by $K = Ker(\upsilon)$

$$1 \to K \xrightarrow{\iota} \widetilde{G} \xrightarrow{\upsilon} G \to 1$$

with presentations $G = \langle X \mid \mathcal{R} \rangle$ and $K = \langle Y \mid \mathcal{S} \rangle$. If we have a transversal T for K in $\widetilde{G}$, we can set up a presentation for $\widetilde{G}$. This is done in the following proposition.

Proposition 3 *[23, pg 139] Let $\widetilde{Y} = \{\widetilde{y} = y\iota \mid y \in Y\}$ and let $\widetilde{S} = \{\widetilde{s} \mid s \in S\}$ be the set of words in $\widetilde{Y}$ obtained from S by replacing each y by $\widetilde{y}$ wherever it appears. For each element $x \in X$ we choose $\widetilde{x} \in T$ such that $\tilde{x}\upsilon = x$. In this way, we obtain the generators $\widetilde{X} = \{\widetilde{x} \mid x \in X\}$. For each $r \in R$, let $\tilde{r}$ be the word in $\widetilde{X}$ obtained from r by replacing each x by $\tilde{x}$. Also each $r \in R$ represents the identity of the group G, then $\tilde{r} \in Ker(\upsilon) = Im\ \iota$. So each $\tilde{r}$ can be written as a word in the $\tilde{y}$ and them create the set $\widetilde{R}$. Since $K \triangleleft G$, the conjugates $\tilde{y}^{\tilde{x}} \in K$, with $\tilde{x} \in \widetilde{X}$ and $\tilde{y} \in \widetilde{Y}$, and so are words in the $\tilde{y}$. They form the set $\widetilde{T}$. Then group $\widetilde{G}$ has a presentation*

$$\langle \widetilde{X}, \widetilde{Y} \mid \widetilde{R}, \widetilde{S}, \widetilde{T} \rangle \ . \tag{1.3}$$

Corollary 1 *Let $\widetilde{G}$ be an extension of G by A. If G and A are finitely presented, then so is $\widetilde{G}$. [23, pg 140]*

Corollary 2 *Let H be a subgroup of finite index in a group G. If H is finitely presented, then so is G. [23, pg 141]*

1.4 Chirality

Chirality can be seen as a property of a geometrical figure which occurs when its "image in a mirror, ideally realized, cannot be brought into coincidence with itself", [10].

Chirality of hypermaps can be studied qualitatively by the chirality group and quantitatively by the chirality index (being this the size of the chirality group).

In this section, $\mathcal{Q} = (G; R, L)$ denotes a regular oriented-hypermap with hypermap subgroup Q and $\langle R, L \mid \mathfrak{R}(R, L) \rangle$ be a presentation of G, $G \cong \Delta^+ \diagup Q$.

Automorphisms of regular oriented-hypermaps give rise to orientation-preserving self-homeomorphisms of the underlying surface. There are however external symmetries (the mirror symmetries) that comes from orientation-reversing self-homeomorphisms of the surface. Recall that a regular oriented-hypermap is *chiral* if it admits no mirror symmetry.

Considering $\Delta^+ = \langle \rho, \lambda \rangle$ with $\rho = R_1R_2$ and $\lambda = R_2R_0$ (see section 1.1), each regular oriented-hypermap $\mathcal{Q}$ corresponds to a transitive permutation representation

$$\mu : \Delta^+ \to Mon(\mathcal{Q})$$

given by $\rho \mapsto R$, $\lambda \mapsto L$. Observe that the conjugation by R_2 induces an automorphism of Δ^+ inverting ρ and λ. Since $Q \lhd \Delta^+$, its conjugates in Δ are Q and $Q^{R_0} = Q^{R_1} = Q^{R_2}$. We will denote by Q^r the common conjugate Q^{R_i}.

Definition 5 *The mirror image of $\mathcal{Q}$ is $Q^r = (G; R^{-1}, L^{-1})$.*

If $\mathcal{Q} \cong Q^r$ (that is, if the assignment $R \to R^{-1}$ and $L \to L^{-1}$ extends to a group automorphism of G) then $\mathcal{Q}$ is reflexible, otherwise $\mathcal{Q}$ is chiral.

The largest normal subgroup of Δ contained in Q is the group $Q_{\Delta} = Q \cap Q^r$, and the smallest normal subgroup of Δ containing Q is the group $Q^{\Delta} = QQ^r$.

From [10] and by the third isomorphism theorem, we have

Proposition 4 *The four groups Q^{Δ}/Q, Q/Q_{Δ}, Q^{Δ}/Q^r and Q^r/Q_{Δ} are all isomorphic to each other.*

Definition 6 *The chirality group of $\mathcal{Q}$ is the factor group Q/Q_{Δ} and its order is the chirality index.*

We will denote[5] by $\mathbb{X}_{\mathcal{H}}$ and by $\kappa = \kappa(\mathcal{H})$ the chirality group and the chirality index, respectively, of a hypermap $\mathcal{H}$. From Proposition 4,

$$\mathbb{X}_{\mathcal{H}} \cong Q^{\Delta}/Q < \Delta^+/Q \, ,$$

that is, the chirality group $\mathbb{X}_{\mathcal{H}}$ is a subgroup of the monodromy group of $\mathcal{H}$. As a consequence, the chirality index divides the number of darts.

[5] Notation adopted in [7].

The most extreme type of chirality arises when the chirality group coincides with the monodromy group. Such hypermaps are called *totally chiral*.

We can obtain the chirality group of a hypermap from the presentation of the respective monodromy group, as follows:

Proposition 5 *[5] From the presentation of $\mathcal{Q}$, the chirality group $\mathbb{X}_{\mathcal{Q}}$ is the normal closure of $\langle \mathfrak{R}(R^{-1}, L^{-1})\rangle$ in G.*

As a summarise we have

Lemma 1 *A regular oriented-hypermap $\mathcal{Q}$ is*

1. *chiral if and only if $\kappa(\mathcal{Q}) > 1$;*

2. *reflexible if and only if $\kappa(\mathcal{Q}) = 1$.*

Regularity and chirality are duality invariants:

Proposition 6 *[12] Let $\mathcal{Q}$ be a hypermap and σ be any permutation of the symmetric group S_3. Then*

1. *$\mathcal{Q}$ is regular if and only if $D_\sigma(\mathcal{Q})$ is regular;*

2. *$\mathcal{Q}$ is chiral if and only if $D_\sigma(\mathcal{Q})$ is chiral.*

where

$$\begin{array}{rrcl} & \mathcal{Q} & = & (G; R, L) \\ \sigma = (1,2)\,, & D_\sigma(\mathcal{Q}) & = & (G; R^{-1}, RL) \\ \sigma = (0,1)\,, & D_\sigma(\mathcal{Q}) & = & (G; R^{-1}, L^{-1}) \\ \sigma = (0,1,2)\,, & D_\sigma(\mathcal{Q}) & = & (G; (RL)^{-1}, L) \\ \sigma = (0,2)\,, & D_\sigma(\mathcal{Q}) & = & (G; RL, L^{-1}) \\ \sigma = (0,2,1)\,, & D_\sigma(\mathcal{Q}) & = & (G; R, (RL)^{-1}) \end{array}$$

1.5 Metacyclic groups

A group G is a metacyclic group if it has a normal subgroup H such that both H and G/H are cyclic.

Proposition 7 *Consider the group* $G = \langle x, y \mid x^m = 1,\, x^y = x^r,\, y^n = x^s \rangle$, *where* m, n, r, s *are positive integers such that* $r, s \leq m$ *satisfying the conditions* $r^n \equiv 1 \pmod m$ *and* $rs \equiv s \pmod m$. *Then* $N = \langle x \rangle$ *is a normal subgroup of* G *such that* $N \cong \mathbb{Z}_m$ *and* $G/N \cong \mathbb{Z}_n$. *Thus,* G *is a finite metacyclic group.*
Conversely, if G *is a finite metacyclic group, then* G *has a presentation of the form*

$$\langle x, y \mid x^m = 1,\, x^y = x^r,\, y^n = x^s \rangle\,, \tag{1.4}$$

where m, n, r, s *are integers such that* $r, s \leq m$ *and satisfying the conditions* $r^n \equiv 1 \pmod m$ *and* $rs \equiv s \pmod m$. *[23, pg 60]*

Consequently, $|G| = |G : N|\,|N| = nm$.

We will denote the group with presentation (1.4) by $M(m, n, s, r)$.

1.6 Bicontactuality

Two arbitrary hyperfaces, for instance F_1 and F_2, of a hypermap are *arc-adjacent*, or F_1 meets F_2, if they have in common an arc. We denote by F_1F_2-*arc* all the arcs where F_1 and F_2 are arc-adjacent. For example, the figure

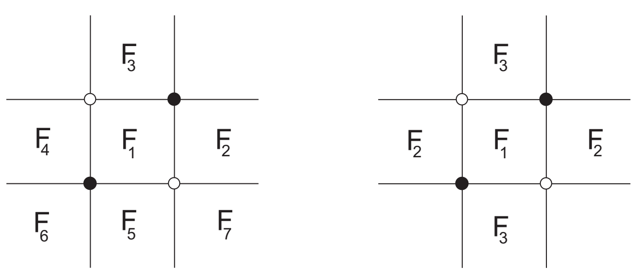


Figure 1.11: Hypermaps $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively

shows that in the hypermap $\mathcal{H}_1$ (left) the hyperfaces F_1 and F_i (with $2 \leq i \leq 5$) are arc-adjacent, but the hyperfaces F_1 and F_j (with $6 \leq j \leq 7$) are not arc-adjacent. In the hypermap $\mathcal{H}_2$ (right), the hyperface F_1 *meets* the hyperface F_2, or F_3, along two arcs.

Definition 7 *Let $\mathcal{H}$ be a hypermap. The hypermap $\mathcal{H}$ is called bicontactual if each hyperface is arc-adjacent to just 2 hyperfaces. So around a fixed hyperface there are just two different hyperfaces of $\mathcal{H}$.*

In the figure 1.11, the hypermap $\mathcal{H}_2$ is bicontactual.

As a consequence of what we have just said, the proof of the following lemma is straightforward.

Lemma 2 *In a bicontactual regular or orientably-regular (or regular pseudo-oriented) hypermap the number of hyperfaces is at least three.*

Proof :
In a bicontactual hypermap we must have at least 2 distinct hyperfaces about any given hyperface F; if F is adjacent to itself then, by regularity, the hypermap would have only one hyperface. □

Consequently,

1. in a bicontactual regular oriented-hypermap the number of darts is ≥ 6;

2. in a bicontactual regular hypermap the least number of flags is 6, if it has boundary, and 12 otherwise.

Chapter 2

Regular bicontactual hypermaps

This chapter aims at laying the foundations of the theory of bicontactual hypermaps, specially in the second section.

Firstly, we sum up Wilson's thesis [26] and the results in [27], in order to present his classification of rotary bicontactual maps, in the cases orientable and non-orientable, which are all reflexible.

Then, in section 2, we extend some those results, definitions and lemmas necessary for the notion of bicontactuality in regular hypermaps, several properties for bicontactual hypermaps are presented and new results are formulated.

2.1 Regular bicontactual maps

In this section we will elaborate a summary of the results established by Wilson [26, 27] for bicontactual maps. Recalling map and Wilson's notation, $\{p, q\}$ means the family of orientably-regular (or rotary) maps of type $(q, 2, p)$ while $\{p, q\}_r$ means the family of maps in $\{p, q\}$ with petrie-length r.

Let us start with the obvious bicontactual maps (keeping Wilson's notation):

D$\epsilon_{\mathbf{n}}$ - This is also known as the dipole $\mathcal{D}_n$, a regular and reflexible orientable bicontactual map on the sphere.

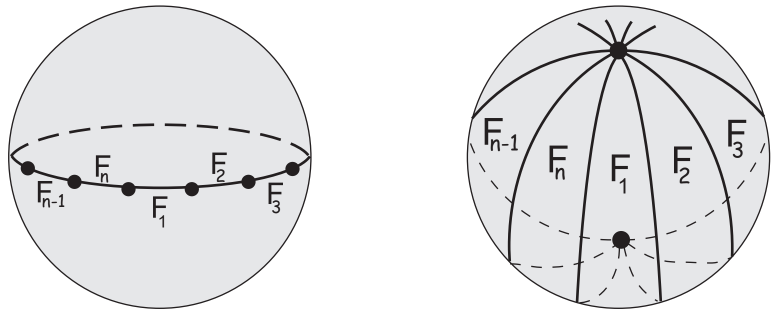


Figure 2.1: ε_n (left) and D_{ε_n} (right); only D_{ε_n} is bicontactual

D$\delta_\mathbf{n}$ - This is the projective version of the dipole, a regular and reflexible bicontactual map on the projective plane.

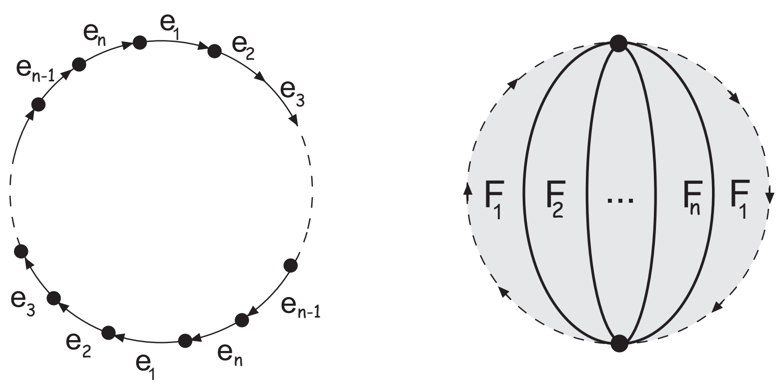


Figure 2.2: δ_n (left) and D_{δ_n} (right); only D_{δ_n} is bicontactual

$\mathbf{M'_{k,i}}$ **-** This 2-face bicontactual map is the dual of M_k, the one-face map obtained from a regular $2k$-gon with its edges crossly identified ([27, pg 439]).

To follow the spirit of Wilson' classification [27] we start to describe the overlapping families of non-obvious bicontactual maps in the same order as they appear in [27]. We must say that, in the paper, Wilson describes bicontactual maps with overlapping families, forgetting to prune them in order to get non-overlapping families. Such prune can however be found, though not so easily, in his Thesis [26].

1. $\mathbf{B(n, 2c)}$. This family, together with the next family $B^*(n, 2c)$, of regular bicontactual maps was first described by means of a quite complicated schema (or diagram). The

maps $B(n, 2c)$ have even number of faces n. Each member of this family is orientable, regular and reflexible, has $\mathcal{H}-$*sequence*

$$[\, nc-(2c,n)-n\,;\ \frac{2nc}{(2c,n)},\, 2,\, 2c\,;\, (2c,n),\, nc,\, n\,;\, 2nc\,]\,,$$

and its monodromy group has presentation

$$\langle R, L \,|\, L^2=(RL)^{2c}=1,\, (LR)^2=(RL)^2,\, (LRL^{-1}R^{-1})^{\frac{n}{2}}=1\rangle\,.$$

2. $\mathbf{B^*(n, 2c)}$. This family divides into two subclasses, when n and c are both even, and when n and c are both odd. It has $\mathcal{H}-$*sequence*

$$[\, nc-(2c,n+c)-n\,;\ \frac{2nc}{(2c,n+c)},\, 2,\, 2c\,;\, (2c,n+c),\, nc,\, n\,;\, 2nc\,]\,,$$

   (a) $\mathbf{n, c}$ **both even**. Each bicontactual map in this family is orientable, regular and reflexible. The monodromy group has presentation

$$\langle R, L \,|\, L^2=(RL)^{2c}=1,\, (LR)^2=(RL)^2,\, (LRL^{-1}R^{-1})^{\frac{n}{2}}=(RL)^c\rangle\,.$$

   (b) $\mathbf{n, c}$ **both odd**. Each bicontactual map in this family is orientable, regular and reflexible. The monodromy group has presentation

$$\langle R, L \,|\, L^2=(RL)^{2c}=1,\, (LR)^2=(RL)^2,\, R^n=(RL)^{n-c}\rangle\,.$$

3. **Opp** $(\mathbf{B(n, 2c)})$. Briefly, the operator *Opp* applied to a map $\mathcal{M}$ corresponds to the permutation given by $r_2 \mapsto r_2r_0$ in the word relations which define $\mathcal{M}$ (for more information, read [28, 26]).

   In this family, where n is even, each member is orientable, regular and reflexible, has $\mathcal{H}-$*sequence*

$$[\,(c-1)(n-1)-1\,;\, n,\, 2,\, 2c\,;\, 2c,\, nc,\, n\,;\, 2nc\,]\,,$$

   and its monodromy group has presentation

$$\langle R, L \,|\, L^2=(RL)^{2c}=1,\, (RLR)^2=1,\, R^n=1\rangle\,.$$

4. **Opp** $(\mathbf{B^*(n, 2c)})$. This family divides into two subclasses, when n and c are both even, and when n and c are both odd.

(a) **n, c both even**. Each bicontactual map in this family is orientable, regular and reflexible, has $\mathcal{H}-$*sequence*

$$[\,(c-1)(n-1)-1\,;\,2n,\,2,\,2c\,;\,c,\,nc,\,n\,;\,2nc\,]\,,$$

and its monodromy group has presentation

$$\langle R, L \,|\, L^2 = (RL)^{2c} = 1,\, (RLR)^2 = 1,\, R^n = (RL)^c \rangle\,.$$

(b) **n, c both odd**. Each bicontactual map in this family is non-orientable, regular and reflexible, has $\mathcal{H}-$*sequence*

$$[\,(c-1)(n-1)-1\,;\,2n,\,2,\,2c\,;\,c,\,nc,\,n\,;\,4nc\,]\,,$$

and its monodromy group has presentation

$$\langle r_0, r_1, r_2 \,|\, r_i^2 = L^2 = (RL)^{2c} = 1,\, (RLR)^2 = 1,\, (r_2 r_0 r_1)^n = (RL)^{-n+c} \rangle\,.$$

5. $\mathbf{\Gamma_{\frac{n}{3}}}$. This family of (rotary and reflexible) bicontactual maps are described as follows:

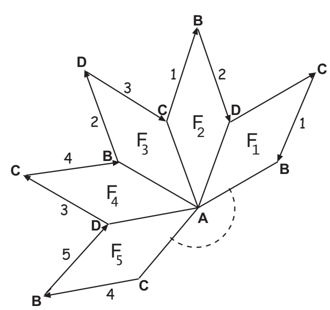


Figure 2.3: Some faces of $\Gamma_{\frac{n}{3}}$

The map $\Gamma_{\frac{n}{3}}$ constructed in this way has 4 vertices of valency n, where n is the number of faces and it is a multiple of 3. For example, the following figure shows the map Γ_2.

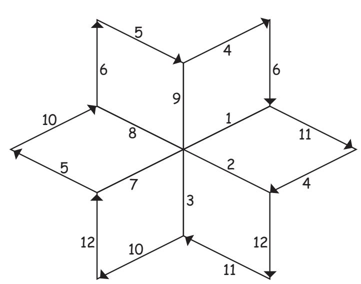


Figure 2.4: Map Γ_2

Γ_1 is the *hemi-cube*, a map on the projective plane derived from the cube by identifying antipodal points.

Each member of this family is non-orientable and it is a fundamental bicontactual map[1]. The $\mathcal{H}-$*sequence* of each bicontactual map is

$$[\, n-4\,;\; n,\, 2,\, 4\,;\; 4,\, 2n,\, n\,;\; 8n\,]\,,$$

and its monodromy group has presentation

$$\langle r_0, r_1, r_2 \,|\, r_i^2 = L^2 = (RL)^4 = 1,\; R^n = 1,\; (RL)^2 = r_0^R\rangle\,.$$

6. $\mathbf{B}(\mathbf{n}, \mathbf{2c}, \rho, \sigma)$. Each member of this family has n (even) faces and it is orientable. Moreover, as shown by Wilson, each bicontactual map $B(n, 2c, \rho, \sigma)$ is rotary and reflexible if the parameters satisfy

$$\rho^2 \equiv\; 1\;(mod\; c)\;,\;\; 2\sigma \equiv\; \frac{n}{2}(\rho+1)\;(mod\; c)\;\; \text{and}\;\; \rho\sigma \equiv\; \sigma\;(mod\; c)\,.$$

Each element of the family has the $\mathcal{H}-$*sequence*[2]

$$[\, n(c-1) - 2(c,\sigma)\,;\; n|\sigma|_c,\, 2,\, 2c\,;\; 2(c,\sigma),\, nc,\, n\,;\; 2nc\,]\,,$$

and its monodromy group has presentation

$$\langle R, L \,|\, L^2 = (RL)^{2c} = 1,\; (LR)^2 = (RL)^{2\rho},\; R^n = (RL)^{2\sigma}\rangle\,.$$

The families described above are not distinct, in fact when n is even $B(n, 2c)$, $B^*(n, 2c)$, D_{ε_n}, $Opp\,(B(n, 2c))$ and $Opp\,(B^*(n, 2c))$ derive from $B(n, 2c, \rho, \sigma)$. That is, these families overlap as follows:

- $B(n, 2c) = B(n, 2c, 1, \frac{n}{2})$.
- $B^*(n, 2c) = B(n, 2c, 1, \frac{n+c}{2})$.
- $D\epsilon_n = \begin{cases} B(n, 2), & \text{if n is even;} \\ B^*(n, 2), & \text{if n is odd.} \end{cases}$
- $OppB(n, 2c, \rho, \sigma) = B(n, 2c, -\rho, \frac{n}{2} - \sigma)$.

[1] That is, the valency of the vertices is n.

[2] $|\sigma|_c = \frac{c}{(c,\sigma)}$

The classification

Now Wilson's classification of bicontactual maps prune to the following non-overlapping families:

A Two families of orientable bicontactual rotary maps (all reflexible):

1. $B(n, 2c, \rho, \sigma)$,

with $n \equiv 0\ (mod\ 2)$, $\rho^2 \equiv 1\ (mod\ c)$, $2\sigma \equiv \frac{n}{2}(\rho+1)\ (mod\ c)$ and $\rho\sigma \equiv \sigma\ (mod\ c)$;

2. $B^*(n, 2c)$, with $n \equiv 1\ (mod\ 2)$ and $c \equiv 1\ (mod\ 2)$.

B Three families of non-orientable bicontactual regular maps:

1. *opp* $B^*(n, 2c)$, with $n \equiv 1\ (mod\ 2)$ and $c \equiv 1\ (mod\ 2)$,

2. $\Gamma_{\frac{n}{3}}$, with $n \equiv 0\ (mod\ 3)$,

3. D_{δ_n}.

2.2 Bicontactual hypermaps - generalisation

All hypermaps considered here are regular. Now we can ask the following: *How many ways can we place two hyperfaces around a hyperface F such that they are arc-adjacent to F?*

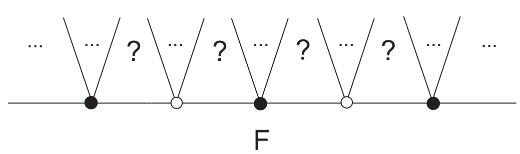


Figure 2.5: How can we distribute hyperfaces around F?

In a bicontactual hypermap there should be just two different hyperfaces around any hyperface F. There are 3 ways we can place 2 hyperfaces around F, giving rise to three types of bicontactuality: the *edge-twin* bicontactual when F_1, F_2 appear around F with pattern $F_1, F_1, F_2, F_2, F_1, F_1, F_2, F_2, \ldots$, in circular order being the repetitions happening at hyperedges (Fig 2.6); the *vertex-twin* bicontactual when $D_{01}(\mathcal{Q})$ is edge-twin bicontactual (Fig 2.7); and the *alternate* bicontactual when F_1, F_2 appear around F with alternate pattern $F_1, F_2, F_1, F_2, \ldots$ (Fig 2.8).

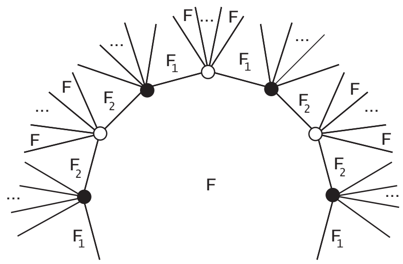


Figure 2.6: Edge-twin

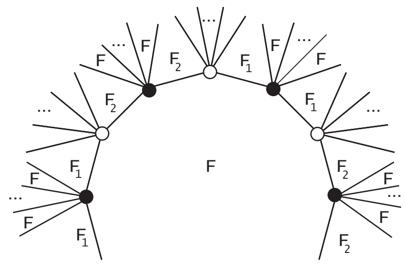


Figure 2.7: Vertex-twin

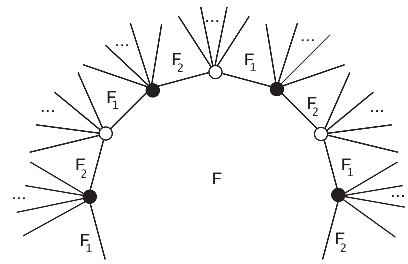


Figure 2.8: Alternate

In short, a regular bicontactual hypermap can be edge-twin, vertex-twin or alternate bicontactual. The first two cases, which correspond to dual cases, $r_0 \leftrightarrow r_1$, form the twin bicontactual hypermaps.

In the particular case of maps, there exists only one type of bicontactuality: the edge-twin bicontactual, termed 'bicontactual' by Wilson in [26]:

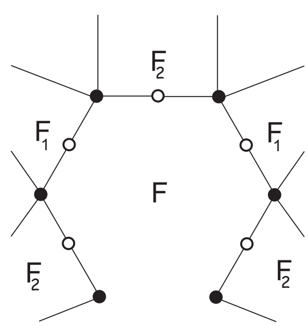


Figure 2.9: Bicontactual Map

The algebraic traduction of the previous definitions is given by the following lemma, without proof:

Lemma 3 *Let $\mathcal{H}$ be a regular hypermap. Then $\mathcal{H}$ is*

1. edge-twin bicontactual iff

$$r_0^{r_2} \in \langle r_0, r_1 \rangle,\ r_1^{r_2} \notin \langle r_0, r_1 \rangle \textit{ and } r_0^{r_1 r_2} \in \langle r_0, r_1 \rangle\ ;$$

2. *alternate bicontactual iff*

$$(r_1 r_0)^{r_2} \in \langle r_0, r_1 \rangle \text{ and } r_1^{r_2} \notin \langle r_0, r_1 \rangle \ .$$

Recall that in any map the word relation $(r_2 r_0)^2 = 1$ holds and that every bicontactual regular map is an edge-twin bicontactual hypermap. So bicontactuality in a regular map is algebraically characterized by

$$r_0^{r_1 r_2} \in \langle r_0, r_1 \rangle \text{ and } r_1^{r_2} \notin \langle r_0, r_1 \rangle \ .$$

The condition $r_1^{r_2} \notin \langle r_0, r_1 \rangle$ can be omitted if the number of hyperfaces is big enough. The proof of the following lemma is straightforward and so omitted.

Lemma 4 *Let $\mathcal{H}$ be a regular hypermap. Then:*

1. *$\mathcal{H}$ has one face if and only if $r_2 \in \langle r_0, r_1 \rangle$.*
2. *$\mathcal{H}$ has two faces if and only if $r_2 \notin \langle r_0, r_1 \rangle$ and $r_0^{r_2}, r_1^{r_2} \in \langle r_0, r_1 \rangle$.*
3. *$\mathcal{H}$ has more than two faces if and only if $r_0^{r_2}, r_1^{r_2} \notin \langle r_0, r_1 \rangle$.*

Lemma 5 *Let $\mathcal{H}$ be a regular hypermap with three or more hyperfaces. Then*

$$r_0^{r_2}, r_0^{r_1 r_2} \in \langle r_0, r_1 \rangle \Rightarrow r_1^{r_2} \notin \langle r_0, r_1 \rangle \ .$$

Remark 3 By Lemma 2 and from the previous Lemma 5, we don't need to refer to the condition $r_1^{r_2} \notin \langle r_0, r_1 \rangle$ to characterise algebraically any regular bicontactual hypermap.

2.2.1 General properties of edge-twin bicontactual hypermaps

In this section $\mathcal{H}_{ET}$ stands for a regular edge-twin bicontactual hypermap with n hyperfaces (without boundary), where $n \geq 3$. Recall that we are assuming that $\mathcal{H}_{ET}$ is orientable.

We will study the basic properties of $\mathcal{H}_{ET}$ which will be important to find the word relations that algebraically define $\mathcal{H}_{ET}$. Most of these properties are just extensions of the Wilson's properties for bicontactual maps (note that bicontactual maps are edge-twin bicontactual hypermaps).

An important term used here is *neighborhood* of a hypervertex, or hyperedge or hyperface.

Definition 8 *The face-neighborhood of a hypervertex (or hyperedge or hyperface) is the set of the hyperfaces that are incident to that hypervertex (or hyperedge or hyperface).*

Let F be an arbitrary hyperface of the hypermap $\mathcal{H}_{ET}$. Around F, there are the hyperfaces F_1, F_1, F_2, F_2, F_1, F_1, *etc* in circular order. So, the proof of the following lemma is obvious.

Lemma 6 *In a regular edge-twin bicontactual hypermap, the valency of hyperfaces must be even.*

Let us analyse the neighborhood of a hyperedge of the hypermap $\mathcal{H}_{ET}$.

Lemma 7 *The neighborhood of an arbitrary hyperedge e contains only two incident hyperfaces, as shown below.*

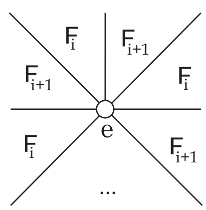


Figure 2.10: neighborhood of a hyperedge

Proof :
By the first sentence of the lemma 3, the condition $r_0^{r_2} \in \langle r_0, r_1 \rangle$ holds in $\mathcal{H}_{ET}$. □

We can say, from figure 2.10, that the face-neighborhood of the hyperedge e is the set $\{F, F_1\}$. Consequently, the next result is straightforward.

Lemma 8 *In the hypermap $\mathcal{H}_{ET}$, the valency of hyperedges is even.*

Definition 9 *Let e be an arbitrary hyperedge of $\mathcal{H}_{ET}$. We define the edge-repetition degree as being the integer number r_e such that the sequence F, F_1 occurs r_e times around e.*

The following proposition proves the existence of repetitions of the hyperedges incidents to each hyperface of a pure hypermap $\mathcal{H}_{ET}$. If $\mathcal{H}_{ET}$ is a regular map then the size of $Mon(\mathcal{H}_{ET})$ is determinated by the number of edges. More precisely,

$$| \, Mon(\mathcal{H}_{ET}) \, |= 2 \, | \, \mathcal{E} \, | \, .$$

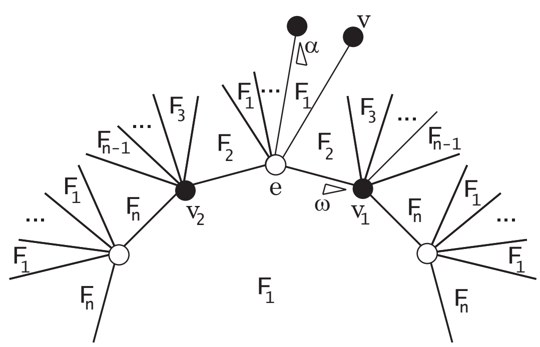
Proposition 8 *If about a hyperface the incident hyperedges do not repeat themselves then $\mathcal{H}_{ET}$ is a map.*

Proof :

Let $\mathcal{H}_{ET}$ be a regular edge-twin bicontactual hypermap, and let F_1 an arbitrary hyperface of $\mathcal{H}_{ET}$.

Let us suppose that about the hyperface F_1 the incident hyperedges do not repeat.

Let e be an hyperedge incident to F_1 and let v_1, v_2 be its neighborhood hypervertices in F_1. Let ω be a flag in F_1 incident to both v_1 and e. Since e does not repeat in F_1, at F_1 it is incident only to v_1 and v_2.



The flag $\alpha = \omega L^2$ is incident to both F_1 and e, therefore, it must be incident to v_1 or v_2. If α is incident to v_1 then $\mathcal{H}_{ET}$ must be a map (since we must have $\alpha = \omega$). Suppose that α is incident to v_2; then $v = v_1$ and we have two consecutive repeating hypervertices. By regularity $\mathcal{H}_{ET}$ has only one hypervertex and consequently the group generated by R and L is cyclic. By observing the figure above we must have $\alpha = \omega r_0$, but this implies that $\mathcal{Q}$ is non-orientable, which is against our supposition. □

Let us examine the neighborhood of a hypervertex of $\mathcal{H}_{ET}$.

Proposition 9 *The neighborhood of an arbitrary hypervertex v of $\mathcal{H}_{ET}$ contains all the hyperfaces of the hypermap.*

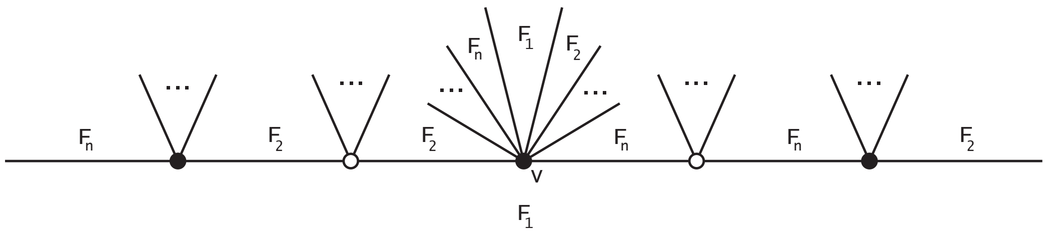
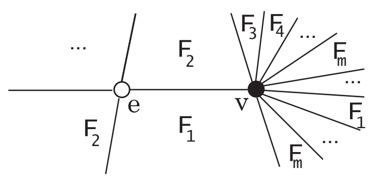


Figure 2.11: neighborhood of a hypervertex

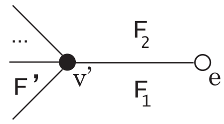
Proof :

Let $\mathcal{H}_{ET}$ be a regular edge-twin bicontactual hypermap, and let $\mathcal{F} = \{F_1, F_2, ..., F_n\}$ be the set of the hyperfaces of $\mathcal{H}_{ET}$. Let v be a hypervertex incident to F_1, and e a hyperedge incident to both v and F_1.

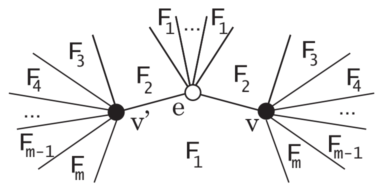
Let m be the number of hyperfaces incident to v. Without loss of generality, assume that F_1, F_2, ..., F_m are the hyperfaces incident to v, in the clockwise way (see figure):



Since the hypermap is bicontactual, F_1 is adjacet to only F_2 and F_m. By edge-twin bicontactuality, the hyperedge e is only incident to two hyperfaces (in this case, hyperfaces F_1 and F_2). Any other hypervertex $v' \neq v$ incident to e is also incident to F_1 and F_2. Let us see the hyperfaces that lie around v' in clockwise order. Let F' be the first after F_1.



We have $F' = F_m$, because F_1 has only two adjacent hyperfaces. Similarly, F_m has also only two adjacent hyperfaces and they must be F_1 and F_{m-1}, therefore the next hyperface incident to v' is F_{m-1}. By repeating this argument, we conclude that F_1, F_m, ..., F_2 are the hyperfaces incident to v' (in this order).



Thus two consecutive hypervertices has the same set of incident hyperfaces. By regularity, the set $\{F_1, F_2, \ldots, F_m\}$ will consists of all the hyperfaces of $\mathcal{H}_{ET}$. Hence $m = n$.

□

Remark 4 We saw that the face-neighborhood of any hypervertex v is all the set $\mathcal{F}$ of the hyperfaces of $\mathcal{H}_{ET}$. From this moment on we may assume, without loss of generality, we are able to number the hyperfaces $1, 2, \ldots, n$, so that the hyperface F_i meets the hyperfaces F_{i-1} and $F_{i+1}\,(mod\,n)$ only.

Since all the hyperfaces are incident to any given hypervertex v, all the hypervertices of $\mathcal{H}_{ET}$ will lie around each hyperface of $\mathcal{H}_{ET}$. This proves the following

Corollary 3 *All the hypervertices of an edge-twin bicontactual hypermap are incident to any given hyperface.*

Consequently, the next result is straightforward.

Lemma 9 *In the hypermap $\mathcal{H}_{ET}$, the valency of hypervertices is a multiple of n.*

From the previous proposition, we can conclude that if around a hypervertex v of $\mathcal{H}_{ET}$ the hyperfaces of the hypermap appear in order CW then in any hypervertex adjacent to v the hyperfaces appear in order CCW and conversely.

Definition 10 *Let v be an arbitrary hypervertex of a regular edge-twin bicontactual hypermap $\mathcal{H}_{ET}$. We define vertex-repetition degree as being the integer number r_v such that the sequence $F_1,\ F_2,\ \ldots,\ F_n$ occurs r_v times around v.*

We know, from Proposition 9, that all the hyperfaces of a regular edge-twin bicontactual hypermap must lie around any given hypervertex. However this can be achieved with repetitions. So, we need to introduce the notion of fundamental in regular edge-twin bicontactual hypermaps.

Definition 11 *If $\mathcal{H}_{ET}$ is a regular edge-twin bicontactual hypermap with n hyperfaces such that $v_{\mathcal{V}} = n$, then $\mathcal{H}_{ET}$ is called fundamental.*

Therefore, in a regular fundamental edge-twin bicontactual hypermap with n hyperfaces, the vertex-repetition degree of an arbitrary hypervertex is equal to 1 (that is, $r_v = 1$). Moreover,

Lemma 10 *In a regular fundamental edge-twin bicontactual hypermap with n hyperfaces of valency $2c$, the number of hypervertices is $2c$.*

Proof :
By the corollary 3 and with $r_v = 1$, we obtain $\mathrm{n}_{\mathcal{V}} = \mathrm{v}_{\mathcal{F}}$. □

2.2.2 General properties of alternate bicontactual hypermaps

In this section $\mathcal{H}_{Alt}$ stands for a regular alternate bicontactual orientable hypermap with n hyperfaces (without boundary), where $n \geq 3$.

We will study the basic properties of the hypermap $\mathcal{H}_{Alt}$ which will be important to find the word relations that define it algebraically.

Lemma 11 *If $\mathcal{H}$ is a regular alternate bicontactual hypermap then the dual $D_{01}(\mathcal{H})$ of $\mathcal{H}$ interchanging hypervertices by hyperedges is also a regular alternate bicontactual hypermap.*

Proof :
The algebraic description $(r_1 r_0)^{r_2} \in \langle r_0, r_1 \rangle$ and $r_1^{r_2} \notin \langle r_0, r_1 \rangle$ (Lemma 3) is invariant if we swap r_0 with r_1. □

Let F be an arbitrary hyperface of the hypermap $\mathcal{H}_{Alt}$. Around F, there are the hyperfaces F_1, F_2, F_1, F_2, F_1, F_2, *etc* in circular order. So, the proof of the following lemma is obvious.

Lemma 12 *In a regular alternate bicontactual hypermap, the valency of hyperfaces can be any positive integer greater or equal to 2.*

In a regular edge-twin bicontactual hypermap, only the neighborhood of a hypervertex has all the hyperfaces of the hypermap incident. Next proposition and corollary show that in an alternate bicontactual hypermap, both the neighborhoods of a hypervertex and of a hyperedge have all the hyperfaces of the hypermap incident.

Proposition 10 *The neighborhood of an arbitrary hypervertex v of the hypermap $\mathcal{H}_{Alt}$ contains all the hyperfaces of the hypermap.*

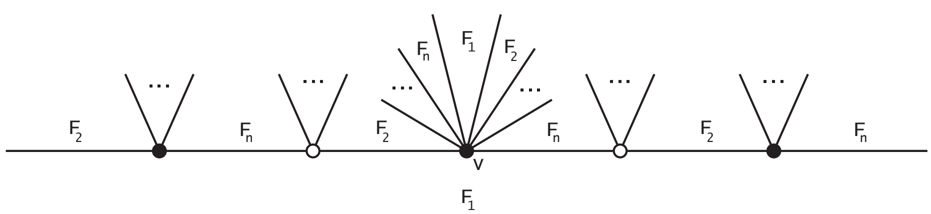
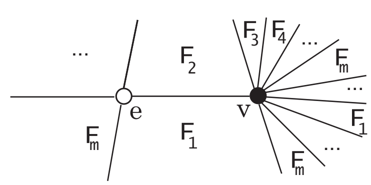


Figure 2.12: neighborhood of a hypervertex

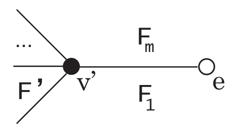
Proof :

Let $\mathcal{H}_{Alt}$ be a regular alternate bicontactual hypermap, and let $\mathcal{F} = \{\mathrm{F}_1, \mathrm{F}_{2,\dots}, \mathrm{F}_n\}$ be the set of the hyperfaces of $\mathcal{H}_{Alt}$. Let v be a hypervertex incident to F_1, and e an incident hyperedge to both v and F_1.

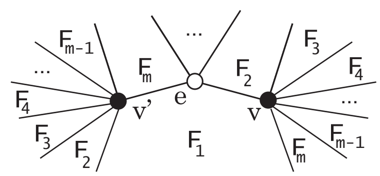
Let m be the number of hyperfaces incident to v. Without losing generality, we assume that F_1, F_2, ..., F_m are the hyperfaces incident to v, in the clockwise way (see figure):



By bicontactuality, F_1 is adjacent to only F_2 and F_m. Then, as $\mathcal{H}_{Alt}$ is alternate, the hyperedge e is incident, at least, to the hyperfaces F_2 and F_m. Take another hypervertex $v' \neq v$ incident to e and F_1. Let us analyse the hyperfaces that lie around v' in clockwise order. Let F' be the first after F_1:



Since F_1 has only two adjacent hyperfaces, $F' = F_2$. Similarly, F_2 has only two adjacent hyperfaces and they must be F_1 and F_3, therefore the next hyperface incident to v' is F_3. By repeating this argument, we conclude that F_1, F_2, ..., F_m are the hyperfaces incident to v' (in this order).



So we have two consecutive hypervertices having the same set $\{F_1, F_2, \ldots, F_m\}$ of incident hyperfaces then, by regularity, the set $\{F_1, F_2, \ldots, F_m\}$ contains all the hyperfaces of $\mathcal{H}_{Alt}$. Hence $m = n$.

$\square$

Also for this class of hypermaps, we will assume that the hyperface F_i meets the hyperfaces F_{i-1} and $F_{i+1}\,(mod\,n)$ only.

By duality (Lemma 11), the preceding proposition is also valid for hyperedges, as follows

Corollary 4 *In a regular alternate bicontactual hypermap, all the hyperfaces are incident to any given hyperedge.*

Summing up, the set $\mathcal{F}$ of the hyperfaces of $\mathcal{H}$ is both the face-neighborhood of any hyperedge as well as the face-neighborhood of any hypervertex. Consequently, all the hypervertices and all the hyperedges of $\mathcal{H}_{Alt}$ will lie around each hyperface of $\mathcal{H}_{Alt}$. This proves,

Corollary 5 *All the hypervertices and, also all the hyperedges, of hypermap $\mathcal{H}_{Alt}$ actually are incident to any given hyperface.*

Consequently, the next result is straightforward.

Lemma 13 *In the hypermap $\mathcal{H}_{Alt}$, the valency of the hypervertices, as well as the valency of the hyperedges, is a multiple of n.*

In conclusion, from proposition 4 and from of the proof of the proposition 10, we obtain (see figure below):

- around two consecutive hypervertices (or hyperedges) the hyperfaces appear in the same order;
- the hyperfaces around a hypervertex (or hyperedge) appear in inverse order to the order in which they appear around incident hyperedges (or hypervertices).

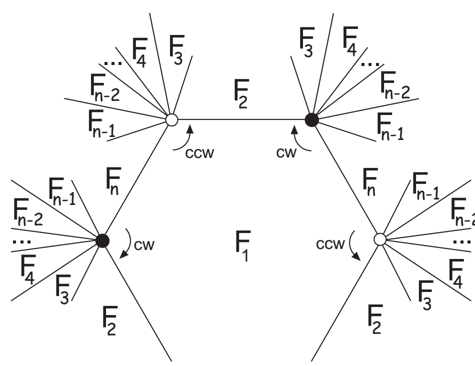


Figure 2.13: orientations in hypervertices and hyperedges of $\mathcal{H}_{Alt}$

Definition 12 *Let v be an arbitrary hypervertex of a regular alternate bicontactual hypermap $\mathcal{H}_{Alt}$ and let e be an arbitrary hyperedge of $\mathcal{H}_{Alt}$. We define:*

- *vertex repetition-degree as being the integer $r_v^{'}$ such that the sequence F_1, F_2, ..., F_n occur $r_v^{'}$ times around v;*
- *edge repetition-degree as being the integer $r_e^{'}$ such that the sequence F_1, F_2, ..., F_n occur $r_e^{'}$ times around e.*

We know, from Proposition 10 and from Proposition 4, that all the hyperfaces of a regular alternate hypermap must lie around any given hypervertex and around any given hyperedge. However this can be achieved with repetitions. Although we will introduce the notion of fundamental regular alternate hypermaps, this notion, here, will not have a role as important as it will have for twin bicontactual hypermaps.

Definition 13 *The hypermap $\mathcal{H}_{Alt}$ is called* fundamental *if the hyperfaces do not repeat neither around hypervertices nor around hyperedges (that is, $r_e^{'} = 1$ and $r_v^{'} = 1$).*

Moreover,

Lemma 14 *In a regular fundamental alternate bicontactual hypermap with n hyperfaces of valency c, the number of hypervertices, and also de number of hyperedges, is c.*

Proof :
By the corollary 5 and with $r_v^{'} = 1 = r_e^{'}$, we conclude that $\mathrm{n}_{\mathcal{V}} = \mathrm{v}_{\mathcal{F}} = \mathrm{n}_{\mathcal{E}}$. $\square$

We will see later that any regular alternate bicontactual oriented-hypermap will cover a fundamental one (Theorem 2).

Chapter 3

Alternate bicontactual oriented-hypermaps

In this chapter we present the classification of the regular alternate bicontactual oriented-hypermaps.

In the following section, we rewrite some properties, obtained in the section 2.2.2, for alternate bicontactual hypermaps (in function of the generators R and L).

Section 3.2 presents the classification of alternate bicontactual oriented-hypermaps, concluding that the monodromy group of an alternate bicontactual oriented-hypermap is a metacyclic group.

In the last section of this chapter, we will show that some alternate bicontactual oriented-hypermaps are chiral while others are not. Using the results obtained by Breda and R. Nedela in [12] for metacyclic groups, we establish the conditions for an alternate bicontactual oriented-hypermap to be chiral.

In what follows, let $\mathcal{Q}_{Alt} = (G; R, L)$ be a regular alternate bicontactual oriented-hypermap with n hyperfaces of valency c.

3.1 Some general properties

The *algebraic definition* of regular alternate bicontactual hypermaps, without boundary, was presented in lemma 3. We consider alternate bicontactual regular oriented-hypermaps. Towards that, we start by rewriting the properties of section 2.2 in terms of R and L.

Lemma 15 *Let $\mathcal{H} = (G; R, L)$ be a regular oriented-hypermap. Then $\mathcal{H}$ is alternate bicontactual if and only if*

$$LR \in \langle RL \rangle \tag{3.1}$$

Proposition 11 *The neighborhood of an arbitrary hyperedge of the hypermap $\mathcal{Q}_{Alt}$ contains all the hyperfaces of the hypermap. Therefore, $L^n \in \langle RL \rangle$ and $L^{nr'_e} = 1$.*

Proposition 12 *The neighborhood of an arbitrary hypervertex of the hypermap $\mathcal{Q}_{Alt}$ contains all the hyperfaces of the hypermap. Therefore, $R^n \in \langle RL \rangle$ and $R^{nr'_v} = 1$.*

Remark 5 If $\mathcal{Q}_{Alt}$ is a fundamental hypermap, then $L^n = 1$ and $R^n = 1$.

3.2 Classification

The objective of this section is the classification of the family of the regular alternate bicontactual oriented-hypermaps.

The following word relations, for the hypermap $\mathcal{Q}_{Alt}$, are easily obtained from the basic properties of section 3.1.

- $(RL)^c = 1$, for some positive integer c

By Lemma 15 and the following propositions, we have

- $LR = (RL)^d$, for some integer $d \in \{0, \cdots, c-1\}$
- $R^n = (RL)^{t_v}$, for some $t_v \in \{0, \cdots, c-1\}$
- $L^n = (RL)^{t_e}$, for some $t_e \in \{0, \cdots, c-1\}$

Lemma 16 *Assume the first two relations. Then*

$$R^n = (RL)^{t_v} \;\Rightarrow\; L^n = (RL)^{t_e}, \textit{ for some } t_e\,.$$

Conversely,

$$L^n = (RL)^{t_e} \;\Rightarrow\; R^n = (RL)^{t_v}, \textit{ for some } t_v\,.$$

Proof :

To see this it is necessary to prove first, by induction, that

$$L^i R^i = (RL)^{d^i+d^{i-1}+\ldots+d^2+d}\,, \tag{3.2}$$

for all integer i. Since we have $LR = (RL)^d$, for some integer $d \in \{0, \cdots, c-1\}$, the equation (3.2) holds for $i = 1$. We assume that, for all j in $\{1, \ldots, i\}$, the equation $L^j R^j = (RL)^{d^j+d^{j-1}+\ldots+d^2+d}$ holds. For $i+1$ we obtain

$$\begin{aligned} L^{i+1}R^{i+1} &= LL^iR^iR \\ &= L(RL)^{d^i+d^{i-1}+\ldots+d^2+d}R \\ &= (LR)^{d^i+d^{i-1}+\ldots+d^2+d+1} \\ &= ((RL)^d)^{d^i+d^{i-1}+\ldots+d^2+d+1} \\ &= (RL)^{d(d^i+d^{i-1}+\ldots+d^2+d+1)} \\ &= (RL)^{d^{i+1}+d^i+\ldots+d^2+d} \end{aligned}$$

So, we conclude that for $i+1$ the equation (3.2) is valid. Putting $i = n$ in the equation (3.2), on the one hand, we get

$$L^n = (RL)^{d^n+d^{n-1}+\ldots+d^2+d-t_v}\,. \tag{3.3}$$

Then, we can assume $t_e = d^n + d^{n-1} + \ldots + d^2 + d - t_v$.

On the other hand, we get

$$R^n = (RL)^{d^n+d^{n-1}+\ldots+d^2+d-t_e}\,. \tag{3.4}$$

Then, we can assume $t_v = d^n + d^{n-1} + \ldots + d^2 + d - t_e$. □

From now on, we assume $(RL)^c = 1$, $LR = (RL)^d$, *and* $R^n = (RL)^{t_v}$, where c is a positive integer and d, t_v integers in $\{0, \cdots, c-1\}$.

As we shall see later, the four parameters involved in preceding word relations are sufficient to classify all the regular alternate bicontactual oriented-hypermaps.

Being n the smallest positive integer such that $R^n \in \langle RL\rangle$ by Lemma 16, n is also the smallest positive integer such that $L^n \in \langle RL\rangle$. Then the parameters r_e' and r_v' are determinated as follows:

$$\begin{array}{lllll} L^n = (RL)^{t_e} & \Rightarrow & \mid L^n \mid = \mid (RL)^{t_e} \mid & \Leftrightarrow & r_e' = \frac{c}{(c,t_e)} \\ R^n = (RL)^{t_v} & \Rightarrow & \mid R^n \mid = \mid (RL)^{t_v} \mid & \Leftrightarrow & r_v' = \frac{c}{(c,t_v)} \end{array}$$

Theorem 1 *If $\mathcal{Q}_{Alt}$ is an alternate bicontactual oriented-hypermap with n hyperfaces, then*

$$Mon(\mathcal{Q}_{Alt}) = \langle R, L \mid (RL)^c = 1,\ LR = (RL)^d,\ R^n = (RL)^{t_v} \rangle$$

for some integer $c \geq 2$ and integers $d, t_v \in \{0, \ldots, c-1\}$ such that

$$(i) \quad dt_v \equiv t_v \ (mod\ c)\,;$$

$$(ii) \quad d^n \equiv 1 \ (mod\ c)\,.$$

Reciprocally, any hypermap $\mathcal{Q} = (G; R, L)$ defined above is an alternate bicontactual oriented-hypermap with n hyperfaces.

Proof :

Let $\mathcal{Q}_{Alt} = (G; R, L)$ be an alternate bicontactual oriented-hypermap with n hyperfaces. By lemma 15, G is a factor group of

$$\langle R, L \mid (RL)^c \underset{(1)}{=} 1,\ LR \underset{(2)}{=} (RL)^d,\ R^n \underset{(3)}{=} (RL)^{t_v} \rangle \tag{3.5}$$

for some integers $c \geq 1$ and $d, t_v \in \{0, \cdots, c-1\}$. Without lost of generality we assume that c is the order of RL.

Let us show that the conditions (i) and (ii) hold. From 3^{rd} relation, we have $R^n = (LR)^{t_v}$. Hence, $(RL)^{t_v} = (LR)^{t_v}$. Conjugating the 2^{nd} relation by L we obtain $RL = L^{-1}(RL)^d L$. Thus, we have $(RL)^{t_v} = L^{-1}(RL)^{dt_v} L$. Conjugating by L^{-1} we get $(LR)^{t_v} = (RL)^{dt_v}$. So $(RL)^{t_v} = (RL)^{dt_v}$. That is, $dt_v \equiv t_v \ (mod\ c)$. For convenience, let us change generators to $R = R$ and $S = RL$. The new generators R and S of G satisfy the new word relations:

$$\langle R, S \mid S^c \underset{(1)}{=} 1,\ S^R \underset{(2)}{=} S^d,\ R^n \underset{(3)}{=} S^{t_v} \rangle \tag{3.6}$$

To prove condition (ii), we first show, by induction, that

$$S^{d^i} = S^{R^i}\,, \tag{3.7}$$

for all integer i. The equation (3.7) holds for $i = 1$ by the 2^{nd} relation.

Assuming that the equation $S^{d^j} = S^{R^j}$ holds, for all j in $\{1, \dots, i\}$, then

$$\begin{aligned} S^{d^{i+1}} &= (S^{d^i})^d \\ &= (R^{-i}SR^i)^d \\ &= R^{-i}S^dR^i \\ &= R^{-i}S^RR^i \\ &= R^{-i-1}SR^{i+1} \end{aligned}$$

So we conclude that for $i+1$ the equation (3.7) is valid. Thus, putting $i = n$ in the equation (3.7) we get by the 3^{rd} relation

$$S^{d^n} = R^{-n}SR^n = S^{-t_v}SS^{t_v} \Leftrightarrow S^{d^n-1} = 1\,.$$

That is, $d^n \equiv 1\ (mod\ c)$.

It remains to show that the group with presentation (3.5) satisfying the conditions (i) and (ii), has size nc. In fact, since $dt_v \equiv t_v\ (mod\ c)$ and $d^n \equiv 1\ (mod\ c)$, the presentation (3.6) is of a metacyclic group $M(c, n, t_v, d)$ of order nc (see Proposition 7).

The converse statement is obvious. □

Theorem 1 determines a family of (regular) bicontactual oriented-hypermaps with n hyperfaces which are alternate. Such family will be denote by $AltB(n, c, d, t_v)$.

The *H-sequence* associated to this family is

$$\left[nc - (c,t_e) - (c,t_v) - n;\ \frac{nc}{(c,t_v)},\ \frac{nc}{(c,t_e)},\ c;\ (c,t_v),\ (c,t_e), n;\ nc\right],$$

where $t_e = d^n + d^{n-1} + \ldots + d^2 + d - t_v$.

Next result makes reference to fundamental alternate oriented-hypermaps.

Theorem 2 *If $\mathcal{Q}_{Alt}$ is an alternate bicontactual oriented-hypermap with n hyperfaces, then $\mathcal{Q}_{Alt}$ covers a fundamental alternate bicontactual hypermap with n hyperfaces.*

Proof :

Let $\mathcal{Q}_{Alt} = (D; R, L)$ be an alternate bicontactual oriented-hypermap with n hyperfaces.

Since $\mathcal{Q}_{Alt}$ is a alternate bicontactual hypermap with n hyperfaces, $LR = (RL)^d$ and $R^n = (RL)^{t_v}$ for some d, t_v in $\{0, \dots, c-1\}$. Consequently, we have $L^n = (RL)^{t_e}$ with $t_e = d^n + d^{n-1} + \ldots + d^2 + d - t_v$.

Let $N_v = \langle R^n \rangle$ be a subgroup of G. We have

$$(R^n)^{L^{-1}} = ((RL)^{t_v})^{L^{-1}} = (LR)^{t_v} = (R^{-1}RLR)^{t_v} = ((RL)^{t_v})^R = (R^n)^R = R^n \,,$$

and, consequently, $N_v \lhd G$. Let $N_e = \langle L^n \rangle$ be a subgroup of G. Also we have

$$(L^n)^R = ((RL)^{t_e})^R = (LR)^{t_e} = (LRLL^{-1})^{t_e} = ((RL)^{t_e})^{L^{-1}} = (L^n)^{L^{-1}} = L^n \,,$$

and, consequently, $N_e \lhd G$.

Therefore, the subgroup $N = \langle R^n, L^n \rangle$ is normal in G. So $\mathcal{Q}_{Alt}/N$ is fundamental and

$$\mathcal{Q}_{Alt} \longrightarrow \mathcal{Q}_{Alt}/N \,.$$

$\square$

Remark 6 From the previous proof and knowing that c is the valency of the hyperfaces in $\mathcal{Q}_{Alt}$, we have that $1 = R^n = (RL)^{t_v}$ and $1 = L^n = (RL)^{t_e}$ hold in the hypermap $\mathcal{Q}_{Alt}/N$. Therefore, $(RL)^c = 1 = (RL)^{t_v} = (RL)^{t_e}$. Consequently, we obtain

$$(RL)^{\mu} = 1$$

in the fundamental hypermap $\mathcal{Q}_{ET}/N$, where $\mu = (c, t_v, t_e)$.

The next result gives another covering between alternate bicontactual oriented-hypermaps.

Theorem 3 *Any fundamental alternate bicontactual oriented-hypermap with n hyperfaces covers the fundamental alternate bicontactual oriented-hypermap $W^{-1}D\varepsilon_n$ with the same number of hyperfaces.*

Proof :

Let $\mathcal{P}_{Alt} = (D; R, L)$ be a fundamental alternate bicontactual oriented-hypermap.

Let N be the subgroup generated by RL of order c. By relation $LR \in \langle RL \rangle$, in Lemma 15, the subgroup $N = \langle RL \rangle$ of D is normal. So, we obtain a projection

$$\mathcal{P}_{Alt} \longrightarrow \mathcal{P}_{Alt}/N = (D/N, x, y)$$

where $x = RN$ and $y = LN$.

In the hypermap $\mathcal{P}_{Alt}$ we have $LR = (RL)^d$, for some $d \in \{0, \ldots, c-1\}$. Then we get $yx = LRN \in \langle RL \rangle N$. So, $yx \in \langle xy \rangle$.

Consequently, $\mathcal{M} = \mathcal{P}_{Alt}/N$ is alternate bicontactual.

But $xy = 1$, by $RL \in N$, therefore $x = y$. So, $\mathcal{P}_{Alt}/N = W^{-1}(D\varepsilon_n)$. □

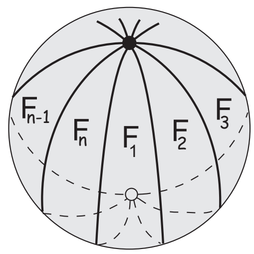


Figure 3.1: Hypermap $W^{-1}(D\varepsilon_n)$

3.3 Chirality

The presentation (3.5), of the monodromy group of an alternate bicontactual oriented-hypermap, coincide with the presentation (3.6) of a metacyclic group.

The chiral hypermaps with metacyclic monodromy groups were identified by Breda d'Azevedo and Nedela in [12]. They give the necessary and sufficient conditions for a regular oriented-hypermap, with metacyclic monodromy group, to be chiral (see section 1.4). Based on this, the following theorem computes the chirality group and the chirality index of an alternate bicontactual regular oriented-hypermap.

Theorem 4 *Let $\mathcal{H}$ be a hypermap of the family $AltB(n, c, d, t_v)$.*

1. *$\mathcal{H}$ is chiral if and only if $d^2 \neq 1 \pmod c$;*

2. *If $d^2 \neq 1 \pmod c$ then $\mathcal{H}$ is chiral with chirality group $\mathbb{X}_{\mathcal{H}} = \langle S^{d^2-1} \rangle$ and chirality index $\kappa = \frac{c}{(c, d^2-1)}$.*

Chapter 4

Edge-twin bicontactual oriented-hypermaps

In this chapter we present the classification of the regular edge-twin bicontactual oriented-hypermaps.

In the first section, we rewrite some properties, obtained in the section 2.2.1, for regular edge-twin bicontactual hypermaps (in function of the generators R and L).

Section 4.2 presents the classification of fundamental edge-twin bicontactual oriented-hypermaps first (subsection 4.2.1) and with this we classify later (subsection 4.2.2) the family of the edge-twin bicontactual oriented-hypermaps.

In the last section of this chapter, we study the chirality of the edge-twin bicontactual oriented-hypermaps. We will show that all them are, in fact, reflexible.

In what follows, let $\mathcal{Q}_{ET} = (G; R, L)$ be a regular edge-twin bicontactual oriented-hypermap with n hyperfaces of valency $2z$.

4.1 Some general properties

The *algebraic definition* of regular edge-twin bicontactual hypermaps, without boundary, was presented in Lemma 3. We consider edge-twin bicontactual regular oriented-hypermaps. Towards that, we start by rewriting the properties of section 2.2 in terms of R and L. These result in the following Lemma and Propositions.

Lemma 17 *Let $\mathcal{H} = (G; R, L)$ be a regular oriented-hypermap. Then $\mathcal{H}$ is edge-twin bicontactual if and only if*

$$L^2 \in \langle RL \rangle \ \ \text{and} \ \ R^L R \in \langle RL \rangle \tag{4.1}$$

Note that the bicontactuality in maps is algebraically characterized only by

$$R^L R \in \langle RL \rangle \, .$$

Lemma 18 *The neighborhood of an arbitrary hyperedge of $\mathcal{Q}_{ET}$ contains only two incident hyperfaces. Consequently, $L^{2r_e} = 1$, where r_e is the edge repetition-degree.*

Proposition 13 *The neighborhood of an arbitrary hypervertex of the hypermap $\mathcal{Q}_{ET}$ contains all the hyperfaces of the hypermap. Therefore, $R^n \in \langle RL \rangle$ and $R^{nr_v} = 1$, where r_v is the vertex repetition-degree.*

Remark 7 If $\mathcal{Q}_{ET}$ is a fundamental hypermap, then $R^n = 1$.

4.2 The classification

The family of the regular fundamental edge-twin bicontactual oriented-hypermaps is essential in our approach since any regular edge-twin bicontactual oriented-hypermap will cover a fundamental one (Theorem 7). The objective of this section is the classification of the above-mentioned family. The classification actually include, by (0,1)-duality, the classification of the regular vertex-twin bicontactual oriented-hypermaps.

Based on what we have previously dealt with, as far as basic properties are concerned, we easily obtain the following word relations for the hypermap $\mathcal{Q}_{ET}$:

- $(RL)^{2z} = 1$ for, some integer z

By Lemma 17 and analysing the respective figures we have

- $L^2 = (RL)^u$, for some **even** integer u in $\{0, \cdots, 2z-1\}$

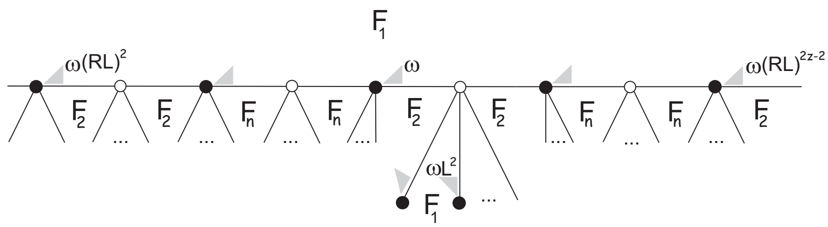


Figure 4.1: ωL^2 belongs to an F_1F_2-arc

- $R^LR = (RL)^s$, for some **even** integer s in $\{0, \cdots, 2z-1\}$

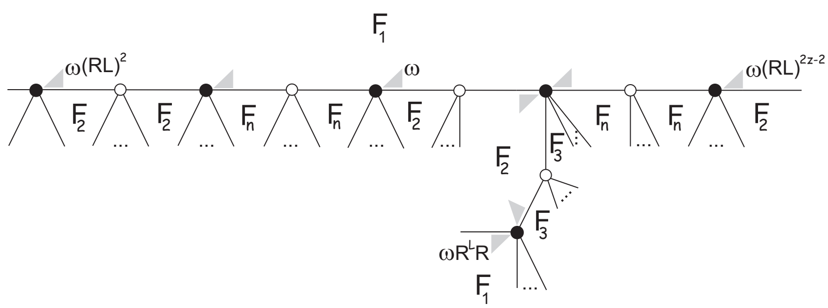


Figure 4.2: ωR^LR belongs to an F_1F_2-arc

- $R^n = (RL)^t$, for some **even** integer t in $\{0, \cdots, 2z-1\}$

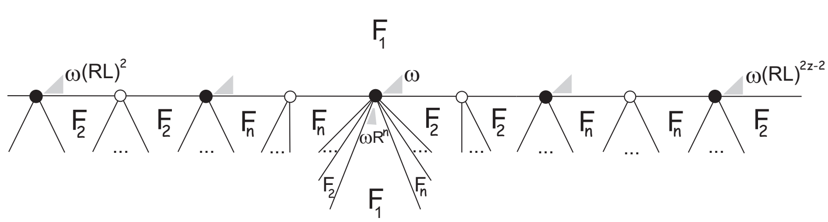


Figure 4.3: ωR^n belongs to an F_1F_2-arc

Remark 8 When $u = 0$ or $z = 1$, the hypermap $\mathcal{Q}_{ET}$ is a map.

For any edge-twin bicontactual oriented-hypermap, we assume

$$(RL)^{2z} = 1,\ L^2 = (RL)^{2a},\ R^LR = (RL)^{2b} \text{ and } R^n = (RL)^{2q},$$

where z is a positive integer and a, b, q are integers in $\{0, \cdots, z-1\}$.

As we shall see later, the five parameters n, z, a, b, q involved in preceding word relations are sufficient to classify all the regular edge-twin bicontactual oriented-hypermaps. The edge and vertex repetition-degree parameters r_e and r_v are determinate by the previous five parameters as follows:

$$\begin{array}{lcl} L^2 = (RL)^{2a} & \Rightarrow \quad \mid L^2 \mid=\mid (RL)^{2a} \mid & \Leftrightarrow \quad r_e = \frac{2z}{(2z,2a)} = \frac{z}{(z,a)} \\ R^n = (RL)^{2q} & \Rightarrow \quad \mid R^n \mid=\mid (RL)^{2q} \mid & \Leftrightarrow \quad r_v = \frac{2z}{(2z,2q)} = \frac{z}{(z,q)} \end{array}$$

from which we obtain the type $\left(\frac{nz}{(z,q)},\, \frac{2z}{(z,a)},\, 2z\right)$ of an edge-twin bicontactual oriented-hypermap.

4.2.1 The fundamental case

From now on let us assume that the hypermap $\mathcal{Q}_{ET}$ is fundamental with n hyperfaces of valency $2c$. So,

$$(RL)^{2c} = 1,\ L^2 = (RL)^{2a},\ R^L R = (RL)^{2b} \text{ and } R^n = 1\,,$$

where c is a positive integer and a, b are integers in $\{0, \cdots, c-1\}$.

Before we calculate the necessary and sufficient conditions for the parameters n, c, a, b that give rise to a fundamental oriented-hypermap with n hyperfaces (Theorem 5), we will need to establish some preliminary results.

Lemma 19 *Let g and h be two arbitrary elements of the group $Mon(\mathcal{Q}_{ET})$. If they satisfy the equations $hgh = g^{2b+1}$ and $hg^2 = g^{2k}h$ (for some positive integer k), then the equation*

$$h^i g h^i = g^{2b(k^{i-1}+k^{i-2}+\cdots+k^2+k+1)+1} \tag{4.2}$$

holds for all $i \in \mathbb{N}$.

Proof :

We will prove the claimed equation by induction.

It holds for $i = 1$, because we have the relation $hgh = g^{2b+1}$.

Assume that for all $j \in \{1, \ldots, i\}$ the equation $h^j g h^j = g^{2b(k^{j-1}+k^{j-2}+\cdots+k^2+k+1)+1}$ holds.

For $i+1$ we obtain

$$\begin{aligned}
h^{i+1}gh^{i+1} &= h(h^i g h^i)h \\
&= hg^{2b(k^{i-1}+k^{i-2}+\cdots+k^2+k+1)+1}h \\
&= hg^{2b(k^{i-1}+k^{i-2}+\cdots+k^2+k+1)}gh \\
&= h\ \underbrace{g^2g^2\ldots\ldots\ldots\ldots\ldots g^2}_{b(k^{i-1}+k^{i-2}+\cdots+k^2+k+1)\text{ times}}\ gh \\
&= g^{(2k)b(k^{i-1}+k^{i-2}+\cdots+k^2+k+1)}hgh \\
&= g^{2b(k^i+k^{i-1}+\cdots+k^2+k)}hgh \\
&= g^{2b(k^i+k^{i-1}+\cdots+k^2+k)}g^{2b+1} \\
&= g^{2b(k^i+k^{i-1}+\cdots+k^2+k+1)+1}
\end{aligned}$$

So, we can conclude that for $i+1$ the equation (4.2) is valid. □

Lemma 20 *Let g and h be two arbitrary elements of the group $Mon(\mathcal{Q}_{ET})$. If g^2 and h^2 commute then*

$$h^i g^t h^{-i} = \begin{cases} g^t, & i \text{ even;} \\ hg^t h^{-1}, & i \text{ odd.} \end{cases} \tag{4.3}$$

holds for all $i \in \mathbb{N}$ and for any even integer t.

Proof :

Let t an even integer and let i an arbitrary integer.

By the condition $g^2 \leftrightarrows h^2$, we get $h^i g^t h^{-i} = h^i h^{-i} g^t = g^t$, for i even.

When i is odd, we have $h^i g^t h^{-i} = hh^{i-1}g^t h^{-i+1}h^{-1}$. Since $i-1$ is even, then from the equation 4.3 (even power), we get $h^i g^t h^{-i} = hg^t h^{-1}$. □

Theorem 5 *If $\mathcal{Q}_{ET}$ is a fundamental edge-twin bicontactual oriented-hypermap with n hyperfaces of valency $2c$, then*

$$Mon(\mathcal{Q}_{ET}) = \langle R, L \mid (RL)^{2c} = R^n = 1,\ L^2 = (RL)^{2a},\ R^L R = (RL)^{2b}\rangle$$

for some integers $c \geq 1$ and $a, b \in \{0, \cdots, c-1\}$ such that $k = a+b$ satisfies

(i) $\quad k^2 \equiv 1\ (mod\ c)\,;$

(ii) $\quad ak \equiv a\ (mod\ c)\,;$

(iii) $\quad b(k^{n-1} + \cdots + k + 1) \equiv 0\ (mod\ c)\,.$

Conversely, any hypermap $\mathcal{H} = (G; R, L)$ defined above is a fundamental edge-twin bicontactual oriented-hypermap with n hyperfaces.

Proof :

Let $\mathcal{Q}_{ET} = (G; R, L)$ be a fundamental edge-twin bicontactual oriented-hypermap with n hyperfaces. By lemma 17 and by remark 7, G is a factor group of

$$\langle R, L \mid (RL)^{2c} \underset{(1)}{=} 1,\ R^n \underset{(2)}{=} 1,\ L^2 \underset{(3)}{=} (RL)^{2a},\ R^L R \underset{(4)}{=} (RL)^{2b} \rangle \tag{4.4}$$

for some integers $c \geq 1$ and $a, b \in \{0, \cdots, c-1\}$. Without lost of generality we assume that $2c$ is the order of RL and n is the order of R.

Let us show that the conditions (i), (ii) and (iii) hold. Conjugating the 3^{rd} relation by L^{-1} we obtain $L^2 = (LR)^{2a}$. Since we have $L^2R = (RL)^{2a}R$, then the relation $L^2R = RL^2$ holds. That is, L^2 belongs to the centre of G. From the 4^{th} relation we get $(LR)^2 = L^2(RL)^{2b}$ and, by the 3^{rd} relation, $(LR)^2 = (RL)^{2k}$ with $k = a+b$. Thus, we have $(LR)^{2a} = (RL)^{2ka} \Leftrightarrow (RL)^{2a(k-1)} = 1$. That is, $ak \equiv a\ (mod\ c)$. For convenience, let us change generators R, L to $X = RL$ and $Y = R^L$. Thus, the new generators X and Y of G satisfy the new word relation:

$$\langle X, Y \mid X^{2c} \underset{(1)}{=} 1,\ Y^n \underset{(2)}{=} 1,\ Y^{-1}XY^{-1} \underset{(3)}{=} X^{2a-1},\ YXY \underset{(4)}{=} X^{2b+1} \rangle \tag{4.5}$$

Now, from 3^{rd} and 4^{th} relations, we obtain $YX^2Y^{-1} = X^{2k}$ and $Y^{-1}X^2Y = X^{2k}$. Then $X^2 \rightleftharpoons Y^2$ and also we have $YX^2Y^{-1} = X^{2k} \Leftrightarrow X^2 = (Y^{-1}X^2Y)^k = (X^{2k})^k \Leftrightarrow X^{2k^2-2} = 1$. So, $k^2 \equiv 1\ (mod\ c)$.

Putting $i = n$ in equation (4.2) for the elements X and Y, and having in account the 2^{nd} relation, we get the following outcome: $X^{2b(k^{n-1}+k^{n-2}+\cdots+k^2+k+1)} = 1$. Consequently,

$$b(k^{n-1} + \cdots + k + 1) \equiv 0\ (mod\ c)\ . \tag{4.6}$$

As result of $k^2 \equiv 1\ (mod\ c)$, we have $k^i + k^{i-1} + \cdots + k + 1 \equiv \frac{i}{2}k + \frac{i+2}{2}\ (mod\ c)$ or $\frac{i+1}{2} + k\frac{i+1}{2}\ (mod\ c)$ according i is even or odd, respectively. So, the congruence (4.6) is equivalent to the congruences

$$\begin{cases} \frac{n}{2}b(k+1) \equiv 0\ (mod\ c)\ , & n \text{ even}\ ; \\ b\left(\frac{n-1}{2}(k+1)+1\right) \equiv 0\ (mod\ c)\ , & n \text{ odd}\ . \end{cases} \tag{4.7}$$

We will show that, when n is odd, combining the congruence $ak \equiv a\ (mod\ c)$ with the second of (4.7) we conclude

$$k \equiv 1\ (mod\ c)\ . \tag{4.8}$$

In fact, (4.7) implies that $b(k-1)(\frac{n-1}{2}(k+1)+1)$ is also congruent to 0 module c. But

$$b(k-1)(\frac{n-1}{2}(k+1)+1) = b\frac{n-1}{2}(k^2-1)+b(k-1)\ .$$

By $k^2 \equiv 1\ (mod\ c)$, we obtain that $bk \equiv b\ (mod\ c)$. This congruence and the congruence $ak \equiv a\ (mod\ c)$ give rise to the congruence $k^2 \equiv k\ (mod\ c)$. By transitivity, we conclude (4.8).

It remains to show that the group with presentation (4.4) satisfying the conditions (i), (ii) and (iii) has size $2nc$. This is done by next lemma.

Lemma 21 *The relations $X^{2c} = Y^n = 1$, $Y^{-1}XY^{-1} = X^{2a-1}$ and $YXY = X^{2b+1}$, where $a, b \in \{0, \cdots, c-1\}$ satisfy the conditions $ak \equiv a\ (mod\ c)$, $k^2 \equiv 1\ (mod\ c)$, and $b(k^{n-1} + \cdots + k + 1) \equiv 0\ (mod\ c)$, with $k = a+b$, determine a group of order $2nc$.*

Proof :

Let K be the group with presentation

$$\langle X, Y \mid X^{2c} = Y^n = 1, Y^{-1}XY^{-1} = X^{2a-1}, YXY = X^{2b+1}\rangle$$

where $a, b \in \{0, \cdots, c-1\}$.

Let N be the subgroup generated by X; its order divides $2c$. One can see that N divides K into (no more than) n cosets N, NY, NY^2, $\cdots$, NY^{n-2} and NY^{n-1}. Let i be in $\mathbb{N}$ and t an even integer. Since $X^2 \leftrightharpoons Y^2$, equation (4.3) holds. Using $YX^2Y^{-1} = X^{2k}$, the equation (4.3) can be rewriten as

$$Y^iX^tY^{-i} = \begin{cases} X^t, & i \text{ even;} \\ X^{kt}, & i \text{ odd.} \end{cases}$$

So

$$NY^iX^t = NY^i\ . \tag{4.9}$$

Now, let t be an odd integer. Then we have

$$Y^iX^{t-1}Y^{-i} = \begin{cases} X^{t-1}, & i \text{ even;} \\ X^{k(t-1)}, & i \text{ odd.} \end{cases}$$

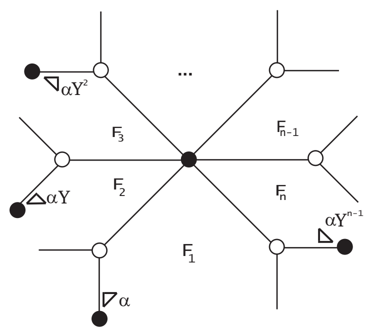
As $Y^iX^tY^i = Y^iX^{t-1}Y^{-i}Y^iXY^i$, equation (4.2) gives

$$Y^iX^tY^i = \begin{cases} X^{t-1}X^{2b(k^{i-1}+k^{i-2}+\cdots+k^2+k+1)+1}, & i \text{ even;} \\ X^{k(t-1)}X^{2b(k^{i-1}+k^{i-2}+\cdots+k^2+k+1)+1}, & i \text{ odd.} \end{cases}$$

Therefore

$$NY^iX^t = NY^{n-i}\,. \tag{4.10}$$

The relations (4.9) and (4.10) are enough to reduce any word $N\omega$, where ω is a word in X and Y, to one of the cosets N, NY, NY^2, ..., NY^{n-2} and NY^{n-1}. The hyperfaces of $\mathcal{Q}_{ET}$ are described by the cosets N, NY, NY^2, ..., NY^{n-2} and NY^{n-1} (as the figure shows).



So $|K| = n|N| \leq 2nc$. That is, the size of K is finite.

The normal subgroup $H = \langle X^2, Y\rangle$ factors K into a cyclic group $C_2 = \langle Y \mid Y^2 = 1\rangle$. Thus $|K : H| = 2$. We are going to use the Reidemeister-Schreier's Rewriting Process to obtain a presentation for the group H. Consulting [23], the method involves four steps. For the first step, we need a Schreier transversal T for H in K. As the quotient group K/H is equal to $\{kH|k \in K\} = \{XH, H\}$ we have $T = \{1, X\}$. The next step is to find the generators for H. This is done in terms of the Schreier transversal T of H, the free generators X and Y of K, and the function $K \to C_2 = K/H$ such that $k \mapsto \widehat{k}$ defined by $kH \cap T = \{\widehat{k}\}$. The elements of the set

$$B^* = \{tu\widehat{tu}^{-1} \mid t \in T,\ u \in \{X, Y\}\}$$

generate a free group. Here, through the properties

$$\widehat{h} = 1,\ \forall h \in H \ \text{ and } \ \widehat{t} = t \Leftrightarrow t \in T\,,$$

we have

$t \diagdown u$	X	Y
1	$\widehat{1X} = \widehat{X} = X$	$\widehat{1Y} = \widehat{Y} = 1$
X	$\widehat{XX} = \widehat{X^2} = 1$	$\widehat{XY} = X$

So $B^* = \{X^2, Y, XYX^{-1}\}$. But, for our convenience, we replace XYX^{-1} by $X^{-1}YX = (XYX^{-1})^{X^2}$. Let $A = X^2$, $B = Y$ and $C = X^{-1}YX$.

For the 3^{rd} step we conjugate the relators of K by T:

$$\begin{aligned}\check{R} &= \{r^t \mid t \in T,\ r \in \{X^{2c},\ Y^n,\ Y^{-1}XY^{-1}X^{-(2a-1)},\ YXYX^{-(2b+1)}\}\} \\ &= \{X^{2c},\ Y^n,\ (Y^{-1}X)^2X^{-2a},\ (YX)^2X^{-2b-2},\ X^{-1}Y^nX, \\ &\quad\ X^{-1}Y^{-1}XY^{-1}X^{2-2a},\ X^{-1}YXYX^{-2b}\}\end{aligned}$$

The last step consists in rewriting each element of $\check{R}$ in terms of A, B and C. The rewritten words will be the relators for a presentation of H. This will be done by inspection, as follows:

$$\begin{aligned}X^{2c} &= A^c \\ Y^n &= B^n \\ Y^{-1}XY^{-1}X^{1-2a} &= Y^{-1}X^2X^{-1}Y^{-1}XX^{-2a} = B^{-1}AC^{-1}A^{-a} \\ YXYX^{-2b-1} &= YX^2X^{-1}YXX^{-2b-2} = BACA^{-(b+1)} \\ X^{-1}Y^nX &= (X^{-1}YX)^n = C^n \\ X^{-1}Y^{-1}XY^{-1}X^{2-2a} &= C^{-1}B^{-1}A^{1-a} \\ X^{-1}YXYX^{-2b} &= CBA^{-b}\end{aligned}$$

Putting together B^* and the rewritten $\check{R}$ we get a presentation for H:

$$\langle A, B, C \mid A^c = 1,\ B^n = 1,\ B^{-1}AC^{-1} = A^a,\ BAC = A^{b+1},\ C^n = 1,\ C^{-1}B^{-1} = A^{a-1},\ CB = A^b\rangle\,.$$

This presentation can be improved. Having in account the last relation we can ignore the generator C, we have $C = A^bB^{-1}$. Thus,

$$\langle A, B \mid A^c \underset{(1)}{=} 1,\ B^n \underset{(2)}{=} 1,\ B^{-1}AB \underset{(3)}{=} A^k,\ BA^{b+1} \underset{(4)}{=} A^{b+1}B,\ (A^bB^{-1})^n \underset{(5)}{=} 1,\ BA^bB^{-1} \underset{(6)}{=} A^{1-a}\rangle\,,$$

where $k = a + b$.

The 4^{th} relation shows that $A^{b+1} \leftrightharpoons B$.

From $ak \equiv a\ (mod\ c)$ we obtain

$$k(b+1) - (b+1) = ab + b^2 + (\underbrace{a+b}_{k})a - 1 = k^2 - 1\,.$$

Then $k(b+1) \equiv (b+1)\ (mod\ c)$ because we have $k^2 \equiv 1\ (mod\ c)$. The 3^{rd} relation gives rise to $B^{-1}A^{b+1}B = A^{k(b+1)}$, which implies that $B^{-1}A^{b+1}B = A^{b+1}$ (that is, the 4^{th} relation). So, this relation is redundant.

We have $B^{-1}A^aB = (B^{-1}AB)^a$ and, by the 3^{rd} relation, we get $B^{-1}A^aB = (A^k)^a$. From $ak \equiv a\ (mod\ c)$ we obtain $B^{-1}A^aB = A^a$. So, A^a and B commute.

From the 6^{th} relation, we obtain that $A^bB^{-1} = B^{-1}A^{1-a}$. Replacing in the 5^{th} relation, we conclude, by $A^a \leftrightharpoons B$, that

$$(B^{-1}A^{1-a})^n = 1 \Leftrightarrow (B^{-1}A)^n = A^{na} .$$

The 3^{rd} relation can be rewrite as $A = BA^aA^bB^{-1}$. Knowing that $A^a \leftrightharpoons B$, we conclude that $A^{1-a} = BA^bB^{-1}$. Therefore, we can eliminate the 6^{th} relation.

From relation $A^{1-a} = BA^bB^{-1}$, we have

$$A^{1-a} = BA^{b+1}A^{-1}B^{-1} \Leftrightarrow A^{1-a-b-1} = BA^{-1}B^{-1} \Leftrightarrow A^{-k} = BA^{-1}B^{-1} .$$

Let us show that the relation $B^n = 1$ is redundant. To do that we need to prove, by induction, that

$$B^i(A^bB^{-1})^i = A^{-k^i}A^{b(k^{i-1}+\cdots+k+1)+1} \tag{4.11}$$

for any positive integer i. We start showing that it holds when $i = 1$:

$$B(A^bB^{-1}) = BA^{-1}A^{b+1}B^{-1} = BA^{-1}B^{-1}A^{b+1} = A^{-k}A^{b+1} .$$

Assume that for all $j \in \{1, \ldots, i\}$ the equation $B^j(A^bB^{-1})^j = A^{-k^j}A^{b(k^{j-1}+\cdots+k+1)+1}$ holds. For $i + 1$ we obtain

$$\begin{aligned}
B^{i+1}(A^bB^{-1})^{i+1} &= BB^i(A^bB^{-1})^iA^bB^{-1} \\
&= BA^{-k^i}A^{b(k^{i-1}+\cdots+k+1)+1}A^bB^{-1} \\
&= BA^{-k^i}A^{b(k^{i-1}+\cdots+k+1)}B^{-1}A^{b+1} \\
&= (BAB^{-1})^{-k^i+b(k^{i-1}+\cdots+k+1)}A^{b+1} \\
&= A^{k(-k^i+b(k^{i-1}+\cdots+k+1))}A^{b+1} \\
&= A^{-k^{i+1}+b(k^i+\cdots+k^2+k)}A^{b+1}
\end{aligned}$$

So for $i + 1$ the equation (4.11) is valid and, consequently, (4.11) is valid. Now $B^n = 1$ is redundant. In fact,

$$B^n = B^n1 = B^n(A^bB^{-1})^n = A^{-k^n}A^{b(k^{n-1}+\cdots+k+1)}A .$$

We have $k^2 \equiv 1\ (mod\ c)$ and if n is odd, $k \equiv 1\ (mod\ c)$[1], so in both cases $k^n \equiv 1\ (mod\ c)$. Hence, by congruence (4.6),

$$B^n = A^{-1}A^0A = 1 .$$

[1]congruence (4.8)

Finally, the presentation of H is as follows

$$H = \langle A, B \mid A^c = 1,\ A^B = A^k,\ (B^{-1}A)^n = A^{na} \rangle \ ,$$

where A and B corresponds to X^2 and Y in K.

Changing the generators $\{A,\ B\}$ for the generators $\{x = A^{-1},\ y = A^{-1}B\}$, we get

$$H = \langle x, y \mid x^c = 1,\ x^y = x^k,\ y^n = x^{na} \rangle \ .$$

As $k(na) = n(ak) \equiv na \ (mod\ c)$ then H is a metacyclic group $M(c, n, na, k)$ and hence $|H| = nc$. So

$$|K| = |K : H||H| = 2nc\,.$$

◇

□

Theorem 5 determines a family of (regular) bicontactual oriented-hypermaps with n hyperfaces which are fundamental edge-twin.

Let P_1 be the presentation in equation (4.4) and P_2 be the presentation in equation (4.5). For such family, we have $P_1 \sim P_2$. We will denote by $FETB(n, c, a, b)$ the group presented by either presentations.

From Lemma 18 and by the value of the edge repetition-degree r_e, we obtain $\mathrm{v}_\varepsilon = \frac{2c}{(c,a)}$. The *H-sequence* associated to this family is

$$\left[2nc - n(c,a) - 2c - n;\ n,\ \frac{2c}{(c,a)},\ 2c;\ 2c,\ n(c,a), n;\ 2nc\right] \ .$$

Next result together with Theorem 7 says that any edge-twin bicontactual regular oriented-hypermap covers a bicontactual regular oriented map. This result, that was one of ours first, had led us to endure a classification of edge-twin bicontactual hypermaps by lifting Wilson's classification. It may still be possible to achieve that this way, however we found this approach too hard to keep pursuing.

Theorem 6 *Any fundamental edge-twin bicontactual oriented-hypermap covers a bicontactual oriented map $\mathcal{M}$.*

Proof :

Consider a hypermap $\mathcal{Q}_{ET} = FETB(n,c,a,b) = (G;R,L)$.

We have $L^2 \in \langle RL \rangle$. As $\mathcal{Q}_{ET}$ is oriented then $L^2 = (RL)^u$ for some even integer u.

Let $N = \langle L^2 \rangle = C_b$, which is normal in G. So, we have a projection

$$\mathcal{Q}_{ET} \longrightarrow \mathcal{Q}_{ET}/N = (G/N, x, y)$$

where $x = RN,\ y = LN$. Clearly that $\mathcal{M} = \mathcal{Q}_{ET}/N$ is a map since $y^2 = 1$.

From the relation $R^L R = (RL)^{2b}$, we get $x^y x = R^L RN \in \langle RL \rangle N$. So, $x^y x \in \langle xy \rangle$.

Consequently, $\mathcal{M} = \mathcal{Q}_{ET}/N$ is bicontactual. □

4.2.2 The general case

In this subsection the notation $\mathcal{Q}_{ET}$ stands now for an edge-twin bicontactual regular oriented-hypermap with n hyperfaces of valency $2z$. The vertex repetition-degree r_v is no longer necessarily 1.

From Proposition 13 and by Figure 4.3 the relation

$$R^n = (RL)^{2q}\,, \tag{4.12}$$

for some integer $q \in \{0, \cdots, c-1\}$, holds in the hypermap $\mathcal{Q}_{ET}$.

Lemma 22 *The generators R and L of the group $Mon(\mathcal{Q}_{ET})$ satisfy the relation*

$$(RL)^{2q} = (LR)^{2q}\,. \tag{4.13}$$

Proof :

Already we have seen that the relation $R^n = (RL)^{2q}$, for some integer $q \in \{0, \cdots, c-1\}$, is valid in the hypermap $\mathcal{Q}_{ET}$. Then we get $R^n = R^{-1}(RL)^{2q}R = (LR)^{2q}$ conjugating by R. So $(RL)^{2q} = (LR)^{2q}$. □

The next result will allow us to classify the edge-twin bicontactual oriented-hypermaps from fundamental edge-twin oriented-hypermaps.

Theorem 7 *If $\mathcal{Q}_{ET}$ is an edge-twin bicontactual oriented hypermap with n hyperfaces of valency $2z$, then $\mathcal{Q}_{ET}$ covers a fundamental edge-twin bicontactual hypermap with the same n hyperfaces.*

Proof :

Let G be the monodromy group of $\mathcal{Q}_{ET}$.

By Lemma 22, the subgroup $N = \langle R^n \rangle$ is normal in G.

Thus, the hypermap $\mathcal{Q}_{ET}/N = (G/N; NR, NL)$ is a fundamental edge-twin bicontactual oriented-hypermap, also with n hyperfaces, and

$$\mathcal{Q}_{ET} \longrightarrow \mathcal{Q}_{ET}/N \, .$$

Equation (4.12) with $R^n = 1$, in the hypermap $\mathcal{Q}_{ET}/N$, says that $(RL)^{2c} = 1$, where $c = (z, q)$, holds in $\mathcal{Q}_{ET}/N$. Consequently, $\mathrm{v}_{\mathcal{F}} = 2c$ in the fundamental edge-twin bicontactual oriented-hypermap $\mathcal{Q}_{ET}/N$. □

The presentation of the group $Mon(\mathcal{Q}_{ET})$ will follow from the known presentation of the monodromy group of the fundamental hypermap $\mathcal{Q}_{ET}/N$ (Theorem 5). For this we need to know some results of the extension group theory, described in the first chapter, section 3.

Proposition 14 *The monodromy group of a regular edge-twin bicontactual oriented-hypermap is an extension of the monodromy group of a regular fundamental edge-twin bicontactual oriented-hypermap.*

Proof :

Having in account the epimorphism present in the proof of Theorem 7 and using the definition 4, the proof is straightforward. □

Theorem 8 *If $\mathcal{Q}_{ET}$ is an edge-twin bicontactual oriented-hypermap with n hyperfaces of valency $2z$, then*

$$Mon(\mathcal{Q}_{ET}) = \langle R, L \mid (RL)^{2z} = 1,\ L^2 = (RL)^{2a},\ R^L R = (RL)^{2b},\ R^n = (RL)^{2q} \rangle$$

for some integers $z \geq 1$ and $a, b, q \in \{0, \cdots, z-1\}$ such that $k = a + b$ satisfies

$$\begin{array}{ll} (i) & k^2 \equiv 1 \ (mod\ z)\,; \\ (ii) & ak \equiv a \ (mod\ z)\,; \\ (iii) & qk \equiv q \ (mod\ z)\,; \\ (iv) & 2q - b(k^{n-1} + \cdots + k + 1) \equiv 0 \ (mod\ z)\,. \end{array}$$

Conversely, any hypermap $\mathcal{H} = (G; R, L)$ defined above is an edge-twin bicontactual oriented-hypermap with n hyperfaces.

Proof :

Let $\mathcal{Q}_{ET} = (G; R, L)$ be an edge-twin bicontactual oriented-hypermap with n hyperfaces of valency $2z$. We will denote by $\mathcal{P} = (P; r, l)$ the fundamental edge-twin bicontactual oriented-hypermap covered by $\mathcal{Q}_{ET}$ (such hypermap exists by Theorem 7).

Recall that $N = \langle R^n \rangle \lhd G$. By equation (4.12), we can assume that $R^n = (RL)^{2q}$, where $q \in \{0, \cdots, z-1\}$.

From the cover $\upsilon : \mathcal{Q}_{ET} \longrightarrow \mathcal{Q}_{ET}/N$, we know that $G/N \cong P$. The epimorphism υ is such that $R \mapsto r$ and $L \mapsto l$.

Our aim it to get a presentation of G from presentations of P and N. We can write $P = FETB(n, c, a, b)$, that is,

$$P = \langle r, l \mid (rl)^{2c} = 1,\, l^2 = (rl)^{2a},\, r^l r = (rl)^{2b},\, r^n = 1 \rangle \,,$$

for some integers $c \geq 1$ and $a, b \in \{0, \cdots, c-1\}$ such that the parameters satisfying the following conditions

$$\begin{array}{ll} (I) & k^2 \equiv 1 \ (mod\ c)\,; \\ (II) & ak \equiv a \ (mod\ c)\,; \\ (III) & b(k^{n-1} + \cdots + k + 1) \equiv 0 \ (mod\ c)\,. \end{array}$$

with $k = a + b$ and $c = (z, q)$.

The presentation of N is

$$N = \langle y = (RL)^{2q} \mid y^{\frac{z}{c}} = 1 \rangle \,.$$

Let X and Y be set of generators of P and N, respectively, and let $\mathcal{R}$ and $\mathcal{S}$ be set of defining relators of P and N, respectively.

Following the notation described in the section 1.3.2 (Proposition 1.3.3) we have $\widetilde{Y} \equiv Y$ and, consequently, $\widetilde{S} \equiv S$. The elements R and L, being transversal elements and generating G, form the set $\widetilde{X}$.

Writing each element of $\mathcal{R}$ in terms of $\widetilde{X}$ and then as a word in $\widetilde{Y}$, we get

$$\widetilde{R} = \{(RL)^{2c} = y,\, L^2(RL)^{-2a} = 1,\, R^L R(RL)^{-2b} = 1,\, R^n = y\}\,.$$

Conjugating y by the elements of $\widetilde{X}$, we obtain

$$\widetilde{T} = \{y^R = y,\, y^L = y\}\,.$$

By equation (1.3), the group G has a presentation given by generators R, L, $y = (RL)^{2q}$ and word relations

$$y^{\frac{z}{c}} = 1,\, (RL)^{2c} = y,\, L^2(RL)^{-2a} = 1,\, R^L R(RL)^{-2b} = 1,\, R^n = y,\, y^R = y \text{ and } y^L = y\,.$$

This is equivalent to a presentation with generators R, L and word relations

$$\begin{aligned} &((RL)^{2c})^{\frac{z}{c}} = 1,\, L^2 = (RL)^{2a},\, R^L R = (RL)^{2b},\, R^n = (RL)^{2q},\\ &((RL)^{2q})^R = (RL)^{2q},\, ((RL)^{2q})^L = (RL)^{2q}\,. \end{aligned}$$

which have the final form, given by

$$G \;=\; \langle R, L \mid (RL)^{2z} \underset{(1)}{=} 1,\, L^2 \underset{(2)}{=} (RL)^{2a},\, R^L R \underset{(3)}{=} (RL)^{2b},\, R^n \underset{(4)}{=} (RL)^{2q}\rangle\,. \tag{4.14}$$

The size of G is

$$|G| = |P||N| = 2nc\frac{z}{c} = 2nz\,.$$

Recovering some relations and results from the proof of Theorem 5, we obtain the conditions (i), (ii), (iii) and (iv). The first two are straightforward. The last two need a little extra work. We had the relation $(LR)^2 = (RL)^{2k}$, obtained from the similar conditions (2) and (3). Thus $2ak$ is congruent to $2a$ and $2qk$ is congruent to $2q$, both module the valency of the hyperfaces, $2z$.

In the same manner, we also change the generators to X and Y (recall that, $X = RL$ and $Y = R^L$). The presentation of G becomes this way

$$\langle X, Y \mid X^{2z} \underset{(1)}{=} 1,\, Y^{-1}XY^{-1} \underset{(2)}{=} X^{2a-1},\, YXY \underset{(3)}{=} X^{2b+1},\, Y^n \underset{(4)}{=} X^{2q}\rangle\,. \tag{4.15}$$

For $i = n$ in equation (4.2), outcome the following equation jointly with the relation (4):

$$X^{4q-2b(k^{n-1}+k^{n-2}+\cdots+k^2+k+1)} = 1\,.$$

Consequently, $2q - b(k^{n-1} + \cdots + k + 1) \equiv 0\ (mod\ z)$.

Conversely, suppose we have an oriented-hypermap $\mathcal{H}$ with n hyperfaces of valency $2z$ such that the monodromy group has the following presentation

$$Mon(\mathcal{H}) = \langle R, L \mid (RL)^{2z} = 1,\, L^2 = (RL)^{2a},\, R^L R = (RL)^{2b},\, R^n = (RL)^{2q}\rangle$$

for some integers $z \geq 1$ and $a, b, q \in \{0, \cdots, z-1\}$, which parameters satisfy

$$\begin{array}{ll} (i) & k^2 \equiv 1 \ (mod\ z)\,; \\ (ii) & ak \equiv a \ (mod\ z)\,; \\ (iii) & qk \equiv q \ (mod\ z)\,; \\ (iv) & 2q - b(k^{n-1} + \cdots + k + 1) \equiv 0 \ (mod\ z)\,. \end{array}$$

where $k = a + b$.

By Lemma 17, the oriented-hypermap $\mathcal{H}$ defined above is edge-twin bicontactual. We only need to show that the group has size $2nz$.

For convenience, let us change generators R, L to $X = RL$ and $Y = R^L$. Thus, in function of the new generators X and Y, the presentation of $Mon(\mathcal{H})$ takes the form (4.15).

The subgroup $H = \langle X^2, Y\rangle$ of $Mon(\mathcal{H})$ is normal and it factors $Mon(\mathcal{H})$ into a cyclic group $C_2 = \langle Y \mid Y^2 = 1\rangle$. Thus $|Mon(\mathcal{H}) : H| = 2$.

We are going to use the Reidemeister-Schreier's Rewriting Process, in the same way to what we made previously, to obtain a presentation for the group H. It is necessary a Schreier transversal T for H in $Mon(\mathcal{H})$. As the quotient group $Mon(\mathcal{H})/H$ is equal to $\{mH | m \in Mon(\mathcal{H})\} = \{XH, H\}$ we have $T = \{1, X\}$. Next, we will find the free generators that will define, as a quotient, H. This is done in terms of the Schreier transversal T of H, the free generators X and Y and the function $Mon(\mathcal{H}) \to C_2 = Mon(\mathcal{H})/H$ such that $m \mapsto \widehat{m}$ defined by $mH \cap T = \{\widehat{m}\}$. The elements of the set

$$B^* = \{tu\widehat{tu}^{-1} \mid t \in T,\ u \in \{X, Y\}\}$$

generate a free group. Here, through the properties

$$\widehat{h} = 1,\ \forall h \in H \ \text{ and } \ \widehat{t} = t \Leftrightarrow t \in T\,,$$

we have

$t \diagdown u$	X	Y
1	$\widehat{1X} = \widehat{X} = X$	$\widehat{1Y} = \widehat{Y} = 1$
X	$\widehat{XX} = \widehat{X^2} = 1$	$\widehat{XY} = X$

So $B^* = \{X^2, Y, XYX^{-1}\}$. But, for our convenience, we replace XYX^{-1} by $X^{-1}YX = (XYX^{-1})^{X^2}$. Let $A = X^2$, $B = Y$ and $C = X^{-1}YX$.

Conjugating the relators of $Mon(\mathcal{H})$ by T, we obtain

$$\begin{aligned}\check{R} &= \{r^t \mid t \in T,\ r \in \{X^{2z},\ Y^{-1}XY^{-1}X^{-(2a-1)},\ YXYX^{-(2b+1)},\ Y^nX^{-2q}\}\} \\ &= \{X^{2z},\ (Y^{-1}X)^2X^{-2a},\ (YX)^2X^{-2b-2},\ Y^nX^{-2q},\ X^{-1}Y^nX^{-2q}X, \\ &\quad X^{-1}Y^{-1}XY^{-1}X^{2-2a},\ X^{-1}YXYX^{-2b}\}\end{aligned}$$

Finally, we will rewrite each element of $\check{R}$ in terms of A, B and C, where rewritten words will be the relators for a presentation of H. This will be done by inspection, as follows:

$$\begin{aligned} X^{2z} &= A^z \\ Y^nX^{-2q} &= B^nA^{-q} \\ (Y^{-1}X)^2X^{-2a} &= B^{-1}AC^{-1}A^{-a} \\ (YX)^2X^{-2b-2} &= BACA^{-(b+1)} \\ X^{-1}Y^nX^{-2q}X &= (X^{-1}YX)^nX^{-2q} = C^nA^{-q} \\ X^{-1}Y^{-1}XY^{-1}X^{2-2a} &= C^{-1}B^{-1}A^{1-a} \\ X^{-1}YXYX^{-2b} &= CBA^{-b} \end{aligned}$$

Putting together B^* and the rewritten $\check{R}$ we get a presentation for H:

$$\langle A, B, C \mid A^z = 1,\ B^n = A^q,\ B^{-1}AC^{-1} = A^a,\ BAC = A^{b+1},\ C^n = A^q,\ C^{-1}B^{-1} = A^{a-1},\ CB = A^b\rangle\ .$$

This presentation can be improved. Having in account the last relation we can ignore the generator C, we have $C = A^bB^{-1}$. Thus,

$$\langle A, B \mid A^z \underset{(1)}{=} 1,\ B^n \underset{(2)}{=} A^q,\ B^{-1}AB \underset{(3)}{=} A^k,\ BA^{b+1} \underset{(4)}{=} A^{b+1}B,\ (A^bB^{-1})^n \underset{(5)}{=} A^q,\ BA^bB^{-1} \underset{(6)}{=} A^{1-a}\rangle\ ,$$

where $k = a + b$.

The 4^{th} relation shows that A^{b+1} and B commute.

From congruence (ii) we obtain

$$k(b+1) - (b+1) = (\underbrace{a+b}_{k})b + (\underbrace{a+b}_{k})a - 1 = k^2 - 1\,.$$

Then, by congruence (i), we have $k(b+1) \equiv (b+1)\ (mod\ c)$. The 3^{rd} relation gives rise to $B^{-1}A^{b+1}B = A^{k(b+1)}$, which implies that $B^{-1}A^{b+1}B = A^{b+1}$. So, we get the 4^{th} relation and we conclude that this relation is redundant.

By the 3^{rd} relation, we also obtain $B^{-1}A^aB = (B^{-1}AB)^a = (A^k)^a$. Consequently, with congruence (ii), we have $B^{-1}A^aB = A^a \Leftrightarrow A^aB = BA^a$. That is, $A^a \leftrightharpoons B$.

From the 6^{th} relation, we obtain that $A^bB^{-1} = B^{-1}A^{1-a}$. Replacing in the 5^{th} relation we obtain

$$(B^{-1}A^{1-a})^n = A^q \Leftrightarrow (B^{-1}A)^n = A^{q+na} ,$$

because A^{-a} commutes with $B^{-1}A$.

By the 3^{rd} relation we have $A = BA^aA^bB^{-1}$ from which we get $A^{1-a} = BA^bB^{-1}$. Together with $A^a \leftrightharpoons B$ we can eliminate the 6^{th} relation. Since $A^{1-a} = BA^bB^{-1}$, we have

$$A^{1-a} = BA^{b+1}A^{-1}B^{-1} \Leftrightarrow A^{1-a-b-1} = BA^{-1}B^{-1} \Leftrightarrow A^{-k} = BA^{-1}B^{-1} .$$

To show that the 2^{nd} relation is redundant, we need to recall equation (4.11) which holds for all positive integer i. In fact,

$$B^n = B^n1 = B^n(A^bB^{-1})^nA^{-q} = A^{-k^n}A^{2q}AA^{-q} .$$

We have $k^2 \equiv 1$ $(mod\ c)$ and if n is odd, $k \equiv 1$ $(mod\ c)$, so in both cases $k^n \equiv 1$ $(mod\ c)$. Hence, by congruence (4.6),

$$B^n = A^{-1}A^{q+1} = A^q .$$

Finally, the presentation of H is as follows

$$H = \langle A, B \mid A^z = 1,\ A^B = A^k,\ (B^{-1}A)^n = A^{q+na}\rangle ,$$

where A and B corresponds to X^2 and Y in K.

Changing the generators $\{A,\ B\}$ for the generators $\{x = A^{-1},\ y = A^{-1}B\}$, we get

$$H = \langle x, y \mid x^z = 1,\ x^y = x^k,\ y^n = x^{q+na}\rangle .$$

As $k(na) = n(ak) \equiv na$ $(mod\ c)$ then H is a metacyclic group $M(z, n, q + na, k)$ and hence $|H| = nz$. So

$$|K| = |K : H||H| = 2nz .$$

□

Theorem 8 determines a family of (regular) bicontactual oriented-hypermaps with n hyperfaces which are edge-twin.

Let Q_1 be the presentation in equation (4.14) and Q_2 be the presentation in equation (4.15). For such family, we have $Q_1 \sim Q_2$. We will denote by $ETB(n, z, a, b, q)$ the group presented either by Q_1 or Q_2.

From Lemma 18, Proposition 13 and by the values of the vertex and edge repetition-degrees r_v and r_e, we obtain the *H-sequence* associated to this family. It is

$$\left[2nz - n(z,a) - 2(z,q) - n\mathbf{;}\ \frac{nz}{(z,q)},\ \frac{2z}{(z,a)},\ 2z\mathbf{;}\ 2(z,q),\ n(z,a), n\mathbf{;}\ 2nz\right].$$

4.2.3 The Wilson's classification of bicontactual oriented maps

In this short subsection we want to show that our classification generalises Wilson's one. For the only two families of bicontactual oriented maps $B(n,2c,\rho,\sigma)$ and $B^*(n,2c)$ it is not difficult to verify that they are related to our ETB hypermaps in the following way:

1. $B(n,2z,\rho,\sigma) = ETB(n,2z,0,\rho,\sigma)$, where n is even

and

2. $B^*(n,2z) = ETB(n,2z,0,1,\frac{n-z}{2}\ (mod\ z))$, where n and z are odd

4.3 Chirality

In this section we show that there are no chiral edge-twin bicontactual oriented-hypermaps.

Theorem 9 *All regular edge-twin bicontactual oriented-hypermap $\mathcal{Q}_{ET}$ are reflexible.*

Proof :
Consider the relations that the generators of $Mon(\mathcal{Q}_{ET})$ satisfy (see Theorem 8). The chirality group $\mathbb{X}(\mathcal{Q}_{ET})$ is the subgroup of $Mon(\mathcal{Q}_{ET})$ given by the normal closure of

$$\langle (R^{-1}L^{-1})^{2z},\ (L^{-1})^{-2}(R^{-1}L^{-1})^{2a},\ (R^{-1})^{L^{-1}}R^{-1}(R^{-1}L^{-1})^{-2b},\ (R^{-1})^{-n}(R^{-1}L^{-1})^{2q}\rangle$$

in $Mon(\mathcal{Q})$. Simplifying, we get

$$\mathbb{X}_{\mathcal{Q}} = \langle (LR)^{-2z},\ L^2(LR)^{-2a},\ LR^{-1}L^{-1}R^{-1}(LR)^{2b},\ R^n(LR)^{-2q}\rangle^{Mon(\mathcal{Q})}.$$

Conjugating the relation $(RL)^{2z} = 1$ by L^{-1} we have $(RL)^{2z} = (LR)^{2z}$. In proof of theorem 5, we obtained $L^2 = (LR)^{2a}$, that is, $(RL)^{2a} = (LR)^{2a}$. Conjugating the relation $R^L R = (RL)^{2b}$ by L^{-1} we conclude that $(LR)^{2b} = RLRL^{-1}$. From Lemma 22, we have $(RL)^{2q} = (LR)^{2q}$. This shows that $\mathbb{X}_{\mathcal{Q}_{ET}} = \langle 1\rangle^{Mon(\mathcal{Q}_{ET})}$. Hence, $\mathbb{X}_{\mathcal{Q}_{ET}} = 1$. So, by Lema 1, we conclude that $\mathcal{Q}_{ET}$ is reflexible. □

Chapter 5

Non-orientable bicontactual regular hypermaps

This chapter presents the classification of non-orientable bicontactual regular hypermaps. The main result in this Chapter (Theorem 10 actually) was obtained by Wilson and Breda d'Azevedo in [30] but for completeness we will do it again and give another proof. It will be rewritten from the word relations defining the monodromy group of an edge-twin bicontactual regular non-orientable hypermap.

For this subject, we recommend reading [31].

Up to a $(0,1)-$duality, there are two types of bicontactuality: edge-twin and alternate. The following Lemma shows that, in the non-orientable case, only edge-twin take place.

Lemma 23 *Any alternate bicontactual regular hypermap is orientable.*

Proof :

Let $\mathcal{H}$ be a regular alternate bicontactual hypermap with n hyperfaces ($n \geq 3$). The *H-sequence* associated to $\mathcal{H}$ is

$$\left[\frac{|Mon(\mathcal{H})|}{2} - \mathrm{n}_{\mathcal{E}} - \mathrm{n}_{\mathcal{V}} - n\mathbf{;}\; n,\ n,\ c\mathbf{;}\; \mathrm{n}_{\mathcal{V}},\ \mathrm{n}_{\mathcal{E}}, n\mathbf{;}\; |Mon(\mathcal{H})|\right].$$

Recalling that, an alternate bicontactual hypermap must satisfy the conditions

$$(r_1r_0)^{r_2} \in \langle r_0, r_1\rangle \;\; and \;\; (r_1r_2)^n \in \langle r_0, r_1\rangle\,. \tag{5.1}$$

Figures says that the preceding word relations do not have odd length.

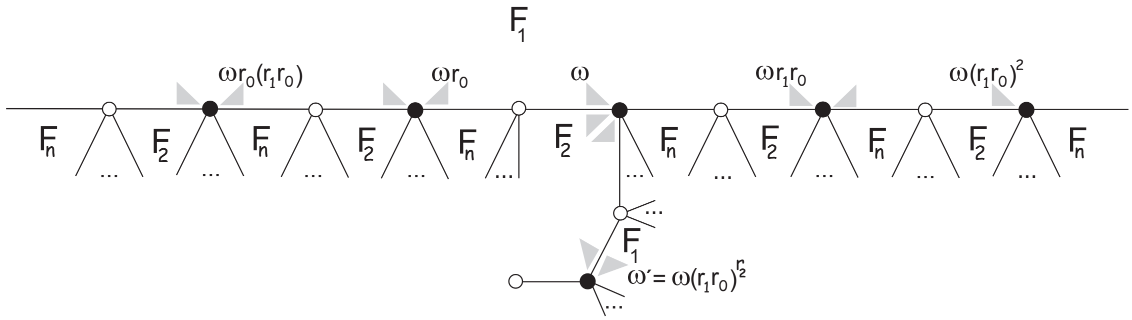


Figure 5.1: The relator of the word relation $(r_1r_0)^{r_2} \in \langle r_0, r_1\rangle$ has even length

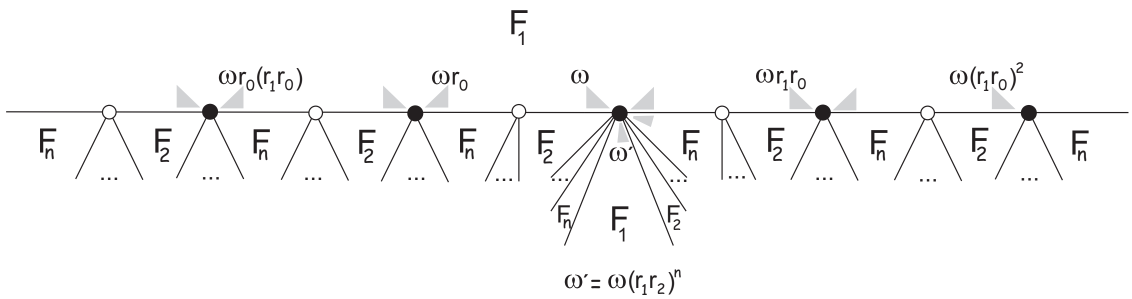


Figure 5.2: The relator of the word relation $(r_1r_2)^n \in \langle r_0, r_1\rangle$ has even length

Then, by the conditions in (5.1), we get $(r_1r_0)^{r_2} = (r_1r_0)^d$ and $(r_1r_2)^n = (r_1r_0)^v$ for some integers d and v. So the monodromy group G of $\mathcal{H}$ is generate by r_0, r_1, r_2 with word relations

$$r_0^2 = 1,\ r_1^2 = 1,\ r_2^2 = 1,\ (r_1r_0)^c = 1,\ (r_1r_0)^{r_2} = (r_0r_1)^d,\ (r_1r_2)^n = (r_0r_1)^t,\ \mathcal{R}(r_0, r_1, r_2)\,,$$

being $\mathcal{R}(r_0, r_1, r_2)$ some more word relations, eventually containing odd length words (if $\mathcal{H}$ is non-orientable).

Is obvious that $N = \langle r_1r_0\rangle$ is a normal subgroup of G. Factorizing G by N we obtain

$$\begin{aligned} G/N &= \langle r_0, r_1, r_2 \mid r_0^2 = 1,\ r_1^2 = 1,\ r_2^2 = 1,\ r_1r_0 = 1,\ (r_1r_2)^n = 1,\ \mathcal{R}'(r_0, r_1, r_2)\rangle \\ &= \langle r_0, r_2 \mid r_0^2 = 1,\ r_2^2 = 1,\ (r_0r_2)^n = 1,\ \mathcal{R}'(r_0, r_1, r_2)\rangle\,, \end{aligned}$$

being $\mathcal{R}'(r_0, r_1, r_2)$ the word relations $\mathcal{R}(r_0, r_1, r_2)$ taking in account that $r_1r_0 = 1$. This factorization give us a hypermap with the same number of hyperfaces and each hyperface with valency 1, that is, the *H-sequence* associated to $\mathcal{H}/N$ has the form

$$[?;\ ?,\ ?,\ 1;\ ?,\ ?, n;\ |G/N|]\ .$$

So the hypermap $\mathcal{H}/N$ is a factorization of $W^{-1}(D\varepsilon_n)$. But $W^{-1}(D\varepsilon_n)$ does not admit non-orientable factorizations since it has only one hypervertex, and one hyperedge, being the hyperedge the antipodal point of the hypervertex. Moreover, since the factorization keeps the number of hyperfaces, we must have $\mathcal{H}/N = W^{-1}(D\varepsilon_n)$. Hence it comes that the surface is orientable. □

Let $\mathcal{H}$ be a non-orientable regular edge-twin bicontactual hypermap with n hyperfaces. We know by Lemma 6 that each hyperface has even valency $2c$. From Lemma 3, the edge-twin bicontactual hypermaps are algebraically characterised by:

$$(r_1r_0)^{2c} = 1 \ , \ r_0^{r_2} \in \langle r_0, r_1\rangle \ , \ r_0^{r_1r_2} \in \langle r_0, r_1\rangle \text{ and } (r_1r_2)^n \in \langle r_0, r_1\rangle \ . \tag{5.2}$$

So $r_0^{r_2} = (r_1r_0)^d$ or $r_0^{r_2} = r_0(r_1r_0)^d$ for some integer d, $r_0^{r_1r_2} = (r_1r_0)^u$ or $r_0^{r_1r_2} = r_0(r_1r_0)^u$ for some integer u, and $(r_1r_2)^n = (r_1r_0)^v$ or $(r_1r_2)^n = r_0(r_1r_0)^v$ for some integer v. All the parameters d, u, v are even. This can be seen by looking at the arc-adjacent of a flag. Applying $r_0^{r_2}$ to some flag ω we get a flag $\omega' = \omega r_0^{r_2}$. Since ω belongs to an F_1F_2-arc so does the flag ω'. This implies that (see Figure 5.3) d must be even. For $r_0^{r_1r_2}$ and $(r_1r_2)^n$ the procedure is similar, the arc-adjacent in Figures 5.4 and 5.5 show that u and v must be even, respectively.

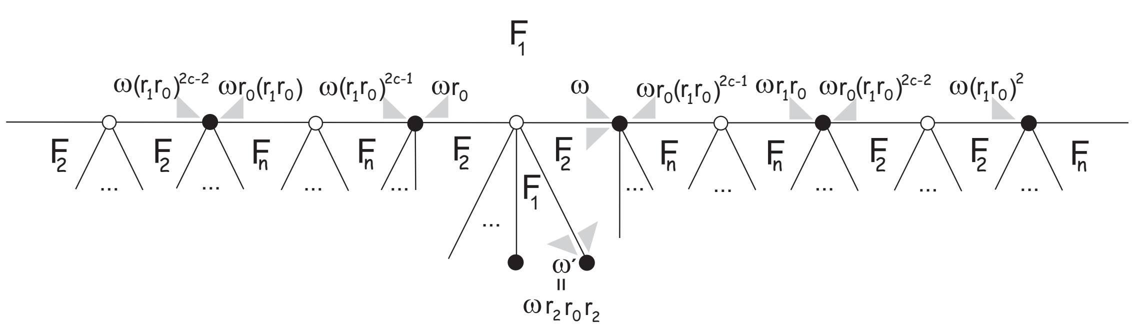


Figure 5.3: The parameter d is even

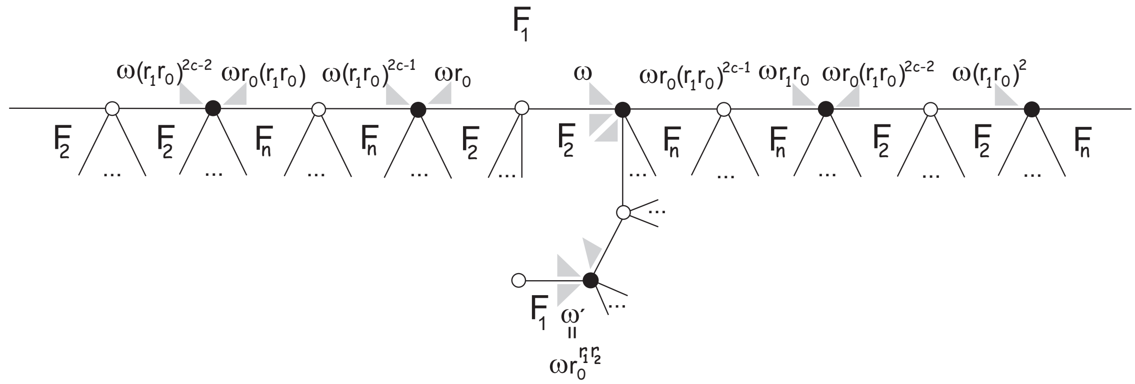


Figure 5.4: The parameter u is even

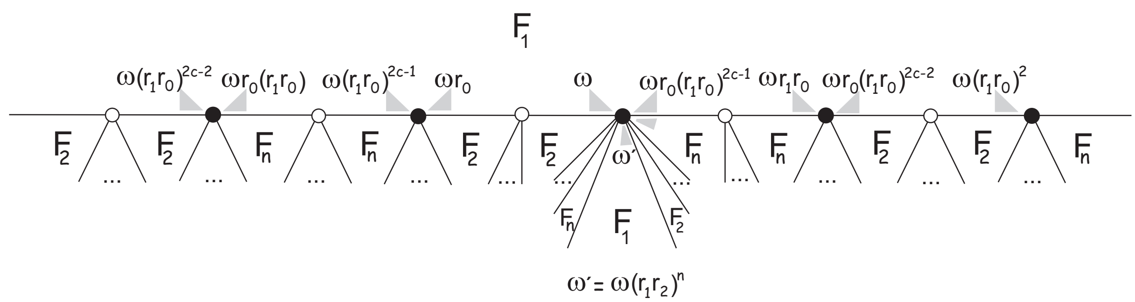


Figure 5.5: The parameter v is even

The main theorem of this chapter states as follows (see [30]).

Theorem 10 *If $\mathcal{H}$ is a non-orientable regular bicontactual hypermap, then $\mathcal{H}$ or $D_{01}(\mathcal{H})$ must be*

1. *the hypermap $D_{02}(GW_k)$,*

2. *the map Γ_k,*

3. *the map $oppB^*(n, 2c)$ for some odd integers n and c, or*

4. *$D\delta_k$ for some k.*

Proof :

Let $\mathcal{H} = (G; r_0, r_1, r_2)$ be a non-orientable edge-twin bicontactual regular hypermap with n hyperfaces of valency $2c$.

Up to a $(0, 1)-$duality the algebraic characterization (5.2), gives rise to 8 possibilities listed below:

Case	word relations in G
I	$r_2r_0r_2 = r_0(r_1r_0)^d$, $r_0^{r_1r_2} = r_0(r_1r_0)^u$, $(r_1r_2)^n = (r_1r_0)^v$
II	$r_2r_0r_2 = r_0(r_1r_0)^d$, $r_0^{r_1r_2} = r_0(r_1r_0)^u$, $(r_1r_2)^n = r_0(r_1r_0)^v$
III	$r_2r_0r_2 = r_0(r_1r_0)^d$, $r_0^{r_1r_2} = (r_1r_0)^u$, $(r_1r_2)^n = r_0(r_1r_0)^v$
IV	$r_2r_0r_2 = r_0(r_1r_0)^d$, $r_0^{r_1r_2} = (r_1r_0)^u$, $(r_1r_2)^n = (r_1r_0)^v$
V	$r_2r_0r_2 = (r_1r_0)^d$, $r_0^{r_1r_2} = r_0(r_1r_0)^u$, $(r_1r_2)^n = r_0(r_1r_0)^v$
VI	$r_2r_0r_2 = (r_1r_0)^d$, $r_0^{r_1r_2} = r_0(r_1r_0)^u$, $(r_1r_2)^n = (r_1r_0)^v$
VII	$r_2r_0r_2 = (r_1r_0)^d$, $r_0^{r_1r_2} = (r_1r_0)^u$, $(r_1r_2)^n = r_0(r_1r_0)^v$
VIII	$r_2r_0r_2 = (r_1r_0)^d$, $r_0^{r_1r_2} = (r_1r_0)^u$, $(r_1r_2)^n = (r_1r_0)^v$

for some positive even integers d, u, v between 0 and $2c$. The **first case** can be discared since it determines orientable regular hypermaps, and they were classified in chapter 4.

Now, we are going to study each of the other cases separately.

Case II: Supposing that $\mathcal{H}$ satisfies the conditions

$$r_2r_0r_2 \underset{(1)}{=} r_0(r_1r_0)^d,\ r_0^{r_1r_2} \underset{(2)}{=} r_0(r_1r_0)^u,\ (r_1r_2)^n \underset{(3)}{=} r_0(r_1r_0)^v$$

with $d, u, v \in \{0, 2, \cdots, 2c-2\}$.

Since $r_0(r_1r_0)^v$ is an involution, we get $(r_1r_2)^{2n} = 1$ and we conclude that the valency of the hypervertices is $2n$. That is, each hypervertex occurs twice in each hyperface. So there are c hypervertices in $\mathcal{H}$.

From the 3^{rd} condition, we get $(r_1r_2)^n r_1 = (r_0r_1)^{v+1}$. Because the first member is an involution then

$$(r_0r_1)^{2(v+1)} = 1 \tag{5.3}$$

So c divides $v+1$. As $v+1 \in \{1, 3, \cdots, 2c-1\}$, we must have $v+1 = c$. We conclude that c is odd. The 3^{rd} condition can then be written in the following way:

$$(r_1r_2)^n = r_0(r_1r_0)^{c-1}\ . \tag{5.4}$$

For convenience, let X and Y be r_1r_0 and $(r_2r_1)^{r_0}$ respectively. Rewriting the word relations, we have:

$$Y^{-1}XY^{-1} \overset{(1)}{=} X^{-d-1},\ YXY \overset{(2)}{=} X^{u+1},\ Y^n \overset{(3)}{=} r_1X^{2-c}\,.$$

From the first two word relations, we obtain $YX^2 = X^{u-d}Y$. So, by Lemma 19, the equation (4.2) is valid here. That is,

$$Y^iXY^i = X^{u(k^{i-1}+k^{i-2}+\ldots+k+1)+1} \tag{5.5}$$

holds for all integer i, where $2k = u-d$ (since $u-d$ is even).

Having in account the 3^{rd} relation and putting $i = n$ in the previous equation, we get $X^{-2} = X^{u(k^{n-1}+k^{n-2}+\cdots+k+1)}$. So, c divides $\frac{u}{2}(k^{n-1}+k^{n-2}+\cdots+k+1)+1$.

We already know that $(r_2r_1)^{2n} = 1$. Thus we have $Y^{2n} = 1$. Putting $i = 2n$ in the equation (5.5) it results the relation $X^{u(k^{2n-1}+k^{2n-2}+\cdots+k+1)} = 1$. So c also divides $\frac{u}{2}(k^{2n-1}+k^{2n-2}+\ldots+k+1)$. Note that

$$\begin{aligned} k^{2n-1}+k^{2n-2}+\ldots+k+1 &= k^{2n-1}+\ldots+k^n+k^{n-1}+\ldots+k+1 \\ &= k^n(k^{n-1}+\ldots+k+1)+k^{n-1}+\ldots+k+1 \\ &= k^n\tfrac{k^n-1}{k-1}+\tfrac{k^n-1}{k-1} \\ &= \tfrac{k^n-1}{k-1}(k^n+1) \end{aligned}$$

Consequently, assuming $t = \frac{u}{2}(k^{n-1}+k^{n-2}+\cdots+k+1)$, we have $c|t+1$ and $c|t(k^n+1)$. Then $c|t+1$ implies that $c|(t+1)k^n$. By the properties of divisibility, we get $c|t-k^n$. From the first two word relations, we obtain $YX^2Y^{-1} = X^{u-d}$ but also $Y^{-1}X^2Y = X^{u-d}$. We have $YX^2Y = (YX^2Y^{-1})Y^2 = X^{u-d}Y^2$ and equivalently $X^2 = Y^{-1}X^{2k}Y = (Y^{-1}X^2Y)^k$. From $Y^{-1}X^2Y = X^{2k}$, we get $X^2 = X^{2k^2}$. So, $k^2 \equiv 1\ (mod\ c)$. For n even, the relation $k^n \equiv 1\ (mod\ c)$ holds and so $c|t-1$. Then, again by the properties of the divisibility, it results that c divides 2. As c is odd, the value of c must be 1. At this moment, the hypermap $\mathcal{H}$ satisfies

$$(r_1r_0)^2 = 1,\ r_2r_0r_2 = r_0,\ r_0^{r_1r_2} = r_0,\ (r_1r_2)^n = r_0\,.$$

We see that $\mathcal{H}$ is a map. Easily we write the H-sequence of the hypermap $\mathcal{H}$ for an even number of hyperfaces n:

$$[-1;\ 2n,\ 2,\ 2;\ 1,\ n,\ n;\ 4n]\,.$$

In conclusion, $\mathcal{H}$ is the map $D\delta_n$ for n even.

To finish the proof of Case I, it is necessary to study the case when the number of hyperfaces is odd. Here, the congruence $k^2 \equiv 1 \ (mod\ c)$ also is valid. Then $k^n \equiv k \ (mod\ c)$ and we obtain $c|t-k$. As we know that $c|t+1$, we get $c|k+1$.

Note that $2k \in \{-2c+2, -2c+4, \ldots, 0, 2, \cdots, 2c-4, 2c-2\}$. Thus we only can have $2k+2 = 2c$, that is, $u = 2c-2+d$. Having in account the values of u and d, the parameter d must be 0 and the parameter u must be $2c-2$. So $\mathcal{H}$ is a map. Among all the possibilities, $\mathcal{H}$ must be $oppB^*(n, 2c)$ with n and c odd.

Case III: Supposing that $\mathcal{H}$ satisfies the conditions

$$r_2r_0r_2 = r_0(r_1r_0)^d,\ r_0^{r_1r_2} = (r_1r_0)^u,\ (r_1r_2)^n = r_0(r_1r_0)^v$$

with $d, u, v \in \{0, 2, \cdots, 2c-2\}$.

Similarity to the preceding case, the valency of the hypervertices is $2n$.

Since $r_0^{r_1r_2}$ is a involution, we have

$$(r_1r_0)^{2u} = 1\ . \tag{5.6}$$

Then $2c$ divides $2u$, that is, $u = c$ with c even. Here the equation (5.4) holds. So

$$r_0^{r_1r_2} = (r_1r_0)^c = r_1r_0(r_1r_0)^{c-1} = r_1(r_1r_2)^n\ .$$

Conjugating the previous relation by r_2r_1, we obtain $r_1r_0 = (r_1r_2)^{n-2}$. Replacing $(r_1r_2)^n = (r_1r_2)^2r_1r_0$ in equation (5.4) it results $(r_1r_2)^2 = r_0(r_1r_0)^{c-2}$. By this involution, we get $(r_1r_2)^4 = 1$. Thus, we must have $n = 2$. But 2 hyperfaces implies non-bicontactual, a contradiction. So, $\mathcal{H}$ does not exist in the conditions of case III.

Case IV: Supposing that $\mathcal{H}$ satisfies the conditions

$$r_2r_0r_2 = r_0(r_1r_0)^d,\ r_0^{r_1r_2} = (r_1r_0)^u,\ (r_1r_2)^n = (r_1r_0)^v$$

with $d, u, v \in \{0, 2, \cdots, 2c-2\}$.

Here the equation (5.6) holds. Then we get $u = c$ and we know that c is even.

By the two first conditions, we obtain $r_2(r_0r_1)^2r_2 = r_0(r_1r_0)^{d+u}$ and, consequently, $(r_0r_1)^4 = 1$. So $2c$ divides 4. Since c is even, we must have $c = 2$. At this moment, we have $c = 2 = u$ and $d, v \in \{0, 2\}$.

If $v = 2$, then $(r_1r_2)^n = r_0^{r_1r_2}$ by conditions 2 and 3. Multiplying on the left by r_1 and after that conjugating by r_1r_2, we get the word relation $r_2(r_1r_2)^{n-1} = r_1r_0$. Since $r_2(r_1r_2)^{n-1}$

is an involution, we have $(r_1r_0)^2 = 1$. So $c = 1$. But this is a contradiction with the fact that c is even.

Now, for $v = 0$, we will analyse two possibilities when $d = 2$ and when $d = 0$. Suppose that we have $v = 0$ and $d = 2$. From the previous relation $r_2(r_0r_1)^2r_2 = r_0(r_1r_0)^{d+u}$, we can write $r_2(r_0r_1)^2r_2 = r_0$ or, equivalently, $r_0r_2(r_0r_1)^2r_2 = 1 = (r_1r_2)^n$. Therefore, we obtain $r_0r_2(r_0r_1)^2 = (r_1r_2)^{n-1}r_1$. Since $(r_1r_2)^{n-1}r_1$ is an involution and $(r_0r_1)^2 = (r_1r_0)^2$, we get $r_0(r_1r_0)^2(r_0r_1)^2 = 1$ by the 2^{nd} condition. Thus $r_0 = 1$. This implies that the hypermap $\mathcal{H}$ is dihedral. So $\mathcal{H}$ is orientable, which is impossible.

The last possibility is $v = 0$ and $d = 0$. Immediately we see that $(r_0r_2)^2 = 1$. So $\mathcal{H}$ is a map. Since $v = 0$, we know that the hypervertices have valency n. Checking all the possibilities, $\mathcal{H}$ must be $\Gamma_{\frac{n}{3}}$ with n a multiple of 3.

Case V: Supposing that $\mathcal{H}$ satisfies the conditions

$$r_2r_0r_2 = (r_1r_0)^d,\ r_0^{r_1r_2} = r_0(r_1r_0)^u,\ (r_1r_2)^n = r_0(r_1r_0)^v$$

with $d, u, v \in \{0, 2, \cdots, 2c-2\}$.

From the 1^{st} condition, we get

$$(r_1r_2)^{2d} = 1\ . \tag{5.7}$$

because $r_2r_0r_2$ is an involution. Then $2c$ divides $2d$, that is, $d = c$ and we know that c is even. By the two first conditions, $r_2(r_0r_1)^2r_2 = (r_1r_0)^{d-u}r_0$ and, consequently, $(r_0r_1)^4 = 1$. So $2c$ divides 4, that is, $c = 1$ or $c = 2$. Since c is even, we must have $c = 2$. As the equation (5.4) holds, the value of v is 1, which is an absurd. We conclude that $\mathcal{H}$ does not exist in the case V.

Case VI: Supposing that $\mathcal{H}$ satisfies the conditions

$$r_2r_0r_2 = (r_1r_0)^d,\ r_0^{r_1r_2} = r_0(r_1r_0)^u,\ (r_1r_2)^n = (r_1r_0)^v$$

with $d, u, v \in \{0, 2, \cdots, 2c-2\}$.

Here the equation (5.7) holds and, observing the previous case, we have $d = c = 2$. Multiplying on the left the word relation $r_2r_0r_2 = (r_1r_0)^d$ by r_0, we obtain that the valency of the hyperedges divides 4 because $r_0(r_1r_0)^d$ is an involution. Thus $\mathcal{H}$ must be a map or a pure hypermap.

At this moment, we have $c = d = 2$ and $u, v \in \{0, 2\}$.

If $u=0$, then the 2^{nd} condition stays $r_0^{r_1r_2}=r_0$. Conjuganting by r_2 and using the 1^{st} condition we get $r_1r_0r_1=(r_1r_0)^2$. Therefore, $r_0=1$. So the hypermap $\mathcal{H}$ is dihedral (that is, orientable), which is impossible.

Now, for $u=2$ suppose that $v=2$. By the relations $(r_1r_2)^n=(r_1r_0)^2$ and $(r_1r_0)^4=1$, we deduce that the valency of the hypervertices is equal to $2n$. Consequently, there are c hypervertices in $\mathcal{H}$ (because each hypervertex occurs twice in each hyperface). From the 2^{nd} condition we have $(r_1r_0)^2=r_0r_0^{r_1r_2}$. Replace in the 3^{rd} condition we obtain $(r_1r_2)^{n-1}=(r_2r_1)^{r_0}$, that is,

$$(r_1r_2)^{-1}(r_1r_0)^2=r_0r_2r_1r_0 \Leftrightarrow r_2r_0=r_0r_2\,.$$

So we conclude that $\mathcal{H}$ is a map. But, searching among all the possibilities, with c even, we find no such map.

We still have one possibility left to analyse $u=2$ and $v=0$. 3^{rd} condition becomes $(r_1r_2)^n=1$. Hence, the valency of the hypervertices is n and the number of hypervertices is equal to the valency of the hyperfaces. From the 1^{st} condition, we get $(r_2r_0)^2=r_1r_0r_1$. Then the valency of the hyperedges is 4 because $r_1r_0r_1$ is an involution. So we conclude that $\mathcal{H}$ is a pure hypermap. We can rewrite de word relation

$$r_2r_0r_2=r_1r_0r_1r_0 \text{ as } r_2r_0r_2r_0r_1r_0r_1=1$$

and this word relation shows that $\mathcal{H}$ is the hypermap $D(GW_{\frac{n}{3}})$, with n a multiple of 3.

Case VII: Supposing that $\mathcal{H}$ satisfies the conditions

$$r_2r_0r_2=(r_1r_0)^d,\ r_0^{r_1r_2}=(r_1r_0)^u,\ (r_1r_2)^n=r_0(r_1r_0)^v$$

with $d,u,v\in\{0,2,\cdots,2c-2\}$.

Here the equations (5.7) and (5.3) hold. Then, from them, we get $d=c$ and $v=c+1$. So, on the one hand, c must be even, but on the other hand c must be odd. It is a contradiction and thus we conclude that $\mathcal{H}$ does not exist in the case VII.

Case VIII: Supposing that $\mathcal{H}$ satisfies the conditions

$$r_2r_0r_2=(r_1r_0)^d,\ r_0^{r_1r_2}=(r_1r_0)^u,\ (r_1r_2)^n=(r_1r_0)^v$$

with $d,u,v\in\{0,2,\cdots,2c-2\}$.

Here the equations (5.7) and (5.6) hold. Then we have $d = c = u$. Using this fact and with the two first conditions, we obtain $(r_0r_1)^2 = 1$. So $c = 1$, which is impossible, because c must be even. $\square$

Final considerations

6.1 Bicontactual hypermaps with boundary

The definition of hypermap *with boundary* follows from section 1.1 with the underlying surface S now having boundary and with the restriction that the boundary does not meet the vertices of the underlying 3-valent graph $\mathcal{G}$. The cells (connected components of $S \setminus \mathcal{G}$) are now homeomorphic to either an open disc (interior of faces) or an half-disc (=intersection of an open disc with the closed upper half-plane $\{(x, y) \mid y \geq 0\}$).

The permutation r_i $(i = 0, 1, 2)$ switches the two flags incident with each $i-$edge not meeting the boundary and fixes the unique flag incident with each $i-$edge meeting the boundary (see figure below).

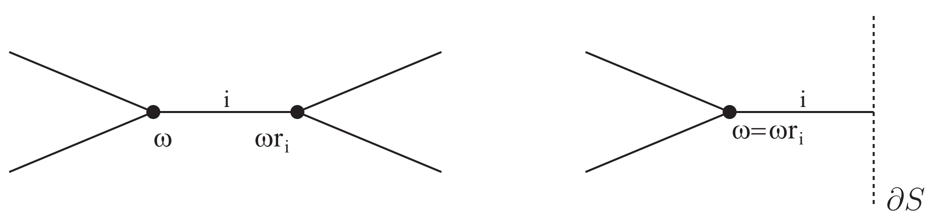


Figure 6.6: Action of r_i's in a hypermap with boundary

This means that $\mathcal{H}$ has boundary if and only if, at least, one of r_i has fixed points. For regular hypermaps we have

Definition 14 *Let $\mathcal{H} = (G; r_0, r_1, r_2)$ be a regular hypermap. Then $\mathcal{H}$ has boundary if and only if, at least, one of r_i is the identity.*

Let $\mathcal{H}_b = (G; r_0, r_1, r_2)$ be a regular hypermap with boundary. From [3] we know that G is cyclic C_2, $V_4 = D_2$ or D_n for some $n \geq 3$. If $|G| \leq 4$ then $\mathcal{H}_b$ has at most 2 hyperfaces and therefore cannot be bicontactual. Hence $\mathcal{H}_b$ to be bicontactual, G must be a dihedral group D_n for some $n \geq 3$. We cannot have $r_2 = 1$ since $\mathcal{H}_b = (D_n; r_0, r_1, 1)$ has only one hyperface.

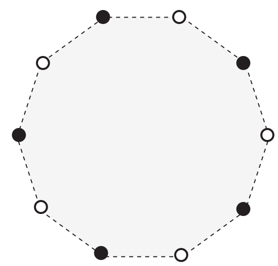


Figure 6.7:

The only possibilities are $r_1 = 1$ or $r_0 = 1$.

Theorem 11 *The bicontactual regular hypermaps with boundary are*

1. $\mathcal{H}_b = (D_n; r_0, 1, r_2)$ *with* $n \geq 3$,

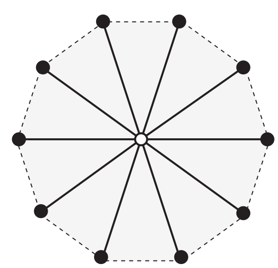


Figure 6.8:

2. $\mathcal{H}_b = (D_n; 1, r_1, r_2)$, *a dual* D_{01} *of the above, with* $n \geq 3$,

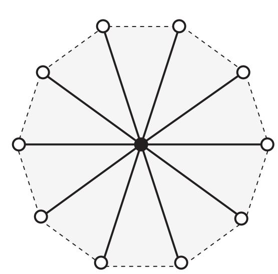


Figure 6.9:

6.2 Future work

A pseudo-oriented map is a map with the property that it is possible to assign an orientation around vertices such that adjacent vertices have opposite orientations, [29].

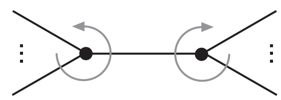


Figure 6.10: A orientation in pseudo-oriented maps

For future work we are interested in the classification of the pseudo-oriented hypermaps. This definition extends to hypermaps in the obvious way (see Figure 6.11).

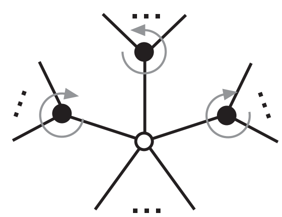


Figure 6.11: A orientation in pseudo-oriented hypermaps

Algebraically, a hypermap $\mathcal{H}$ is pseudo-oriented if and only if its hypermap subgroup H is a subgroup of Δ^0 (see section 1.1), where $\Delta := \langle R_0, R_1, R_2 \mid R_0^2 = R_1^2 = R_2^2 = 1 \rangle \cong C_2 * C_2 * C_2$.

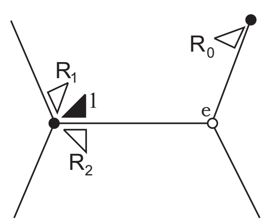


Figure 6.12: Action of each R_i in Δ

If $H \lhd \Delta^0 = \langle R_0, R_1R_2 \rangle^{\Delta}$, then we said that $\mathcal{H}$ is a *regular pseudo-oriented* hypermap.

From [2], any pseudo-oriented hypermap can be described as a 4−tuple $\mathcal{P} = (\Lambda;\, a,\, b,\, z)$, where Λ is the set of "Δ^0-slices" and a, b and z are permutations of Λ satisfying $a^2 = b^2 = 1$ such that the group P generated by a, b and z, the *monodromy group* of $\mathcal{P}$, acts transitively on Λ. If the action of $Mon(\mathcal{P})$ is regular then $\mathcal{P}$ is a *regular* pseudo-oriented hypermap. In general we have $|G| \geq |\Lambda|$, but when $|G| = |\Lambda|$ the pseudo-oriented hypermap $\mathcal{P}$ is *regular*.

The orbits of z, ab^z and ab on Λ are the *hypervertices*, *hyperedges* and *hyperfaces*, respectively, while the positive integers $\mathrm{v}_{\mathcal{V}} = |\langle z\rangle|$, $\mathrm{v}_{\mathcal{E}} = 2|\langle ab^z\rangle|$, $\mathrm{v}_{\mathcal{F}} = 2|\langle ab\rangle|$ determine the *type* $(\mathrm{v}_{\mathcal{V}}, \mathrm{v}_{\mathcal{E}}, \mathrm{v}_{\mathcal{F}})$ of a regular pseudo-oriented hypermap $\mathcal{P}$. Note that a regular pseudo-oriented hypermap is necessarily uniform.

The Δ^0-slices can be seen as the `grey flags` obtained by acting $a = r_0$, $b = r_0^{r_1}$ and $z = r_1 r_2$ on a fixed flag ω (Figure 6.13).

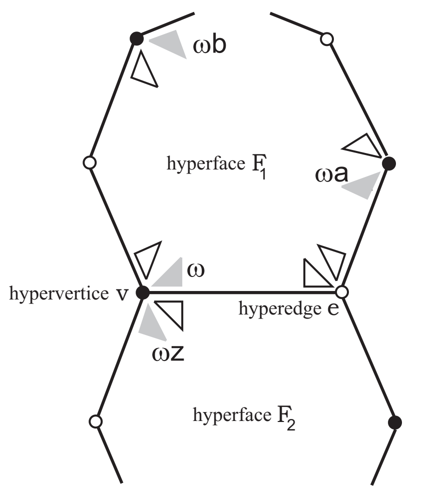


Figure 6.13: Regular pseudo-oriented hypermap

Bicontactual regular pseudo-oriented hypermaps is defined similarly.

Figure 6.14 shows the bicontactual pseudo-oriented hypermap on the torus $(2,4,4)_M$, where $M = \begin{pmatrix} 4 & -1 \\ 4 & 1 \end{pmatrix}$. This is the dual D_{01} of a member of the family of toroidal regular pseudo-oriented hypermaps $(4,2,4)_M$, $M = \begin{pmatrix} k & -m \\ k & m \end{pmatrix}$, ([24]), where the notation follows that given in [20, pg 48].

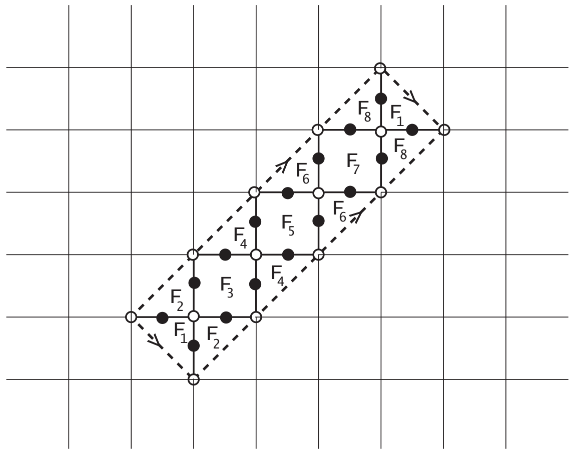


Figure 6.14: The hypermap $(2, 4, 4)_M$

How can 2 hyperfaces be placed around a given hyperface F in a regular pseudo-oriented hypermap?

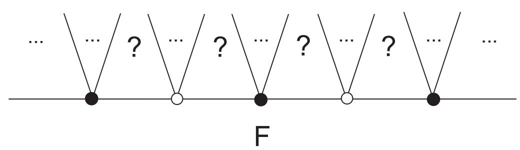


Figure 6.15: How can we distribute hyperfaces around F?

There are 5 ways to do that, namely,

1. **"Alternate" :** where the hyperfaces distribute around F are in alternate way.

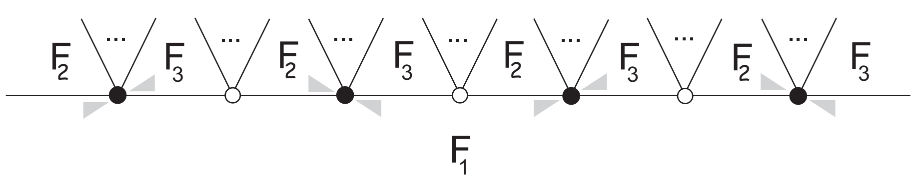


Figure 6.16: Alternate bicontactual

Algebraically, we have

$$zaz \in \langle a, b \rangle,\ zbz \in \langle a, b \rangle \ and\ z^2 \notin \langle a, b \rangle .$$

Example : The bicontactual regular pseudo-oriented hypermap

$$(G;\, a,\, b,\, z) = (D_6;\, x,\, x,\, y) .$$

2. **"Edge-Twin" :** where the hyperfaces distribute around F appear with pattern $F_1, F_1, F_2, F_2,$ $F_1, F_1, F_2, F_2, \ldots,$ in circular order being the repetitions happening at hyperedges.

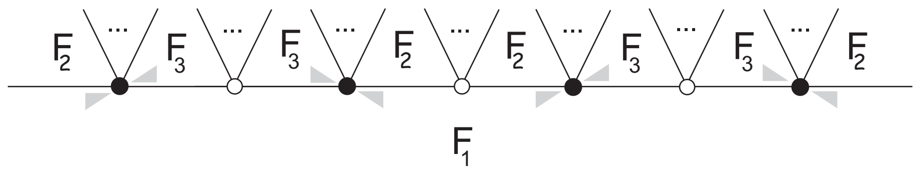


Figure 6.17: Edge-Twin bicontactual

Algebraically, we have

$$a^z \in \langle a, b\rangle,\ a^{z^{-1}} \in \langle a, b\rangle,\ b^z \in \langle a, b\rangle \ and \ z^2 \notin \langle a, b\rangle \, .$$

Example : In [2], the bicontactual regular pseudo-oriented hypermap ($\mathcal{X} = -2$) of type $(8, 2, 4)$ with the extra relation $z^4ab = 1$.

3. **"Vertex-Twin" :** where the hyperfaces distribute around F appear with pattern $F_1, F_1, F_2, F_2,$ $F_1, F_1, F_2, F_2, \ldots,$ in circular order being the repetitions happening at hypervertices.

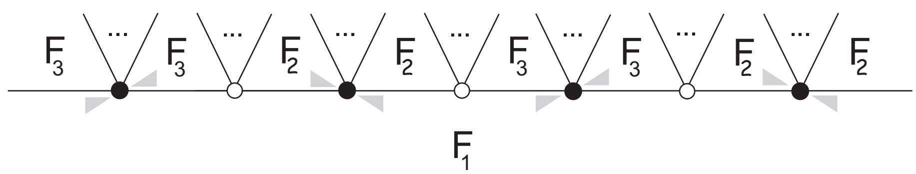


Figure 6.18: Vertex-Twin bicontactual

Algebraically, we have

$$(ab)^z \in \langle a, b\rangle,\ z^2 \in \langle a, b\rangle \ and \ a^z \notin \langle a, b\rangle \, .$$

Example : The bicontactual regular pseudo-oriented hypermap on the torus introduced in Figure 6.14.

4. **"Quadruplet of type I" :** where the hyperfaces distribute around F appear with pattern $F_1, F_1, F_1, F_1,$ $F_2, F_2, F_2, F_2, F_1, F_1, F_1, F_1, \ldots,$ in circular order, as in the following figure.

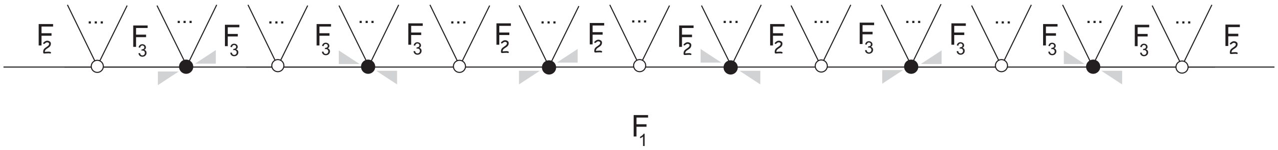


Figure 6.19: Quadruplet of type I bicontactual

Algebraically, we have

$$b^z \in \langle a, b\rangle,\ z^2 \in \langle a, b\rangle,\ (aba)^{z^{-1}} \in \langle a, b\rangle \textit{ and } a^z \notin \langle a, b\rangle\,.$$

Example : The bicontactual regular pseudo-oriented hypermap

$$(G;\, a,\, b,\, z) = (S_4;\, (1,2),\, (1,3)(2,4),\, (1,2,3,4))\,.$$

5. **"Quadruplet of type II" :** where the hyperfaces distribute around F appear with pattern $F_1, F_1, F_1, F_1,\ F_2, F_2, F_2, F_2, F_1, F_1, F_1, F_1, \ldots$, in circular order, as in the following figure.

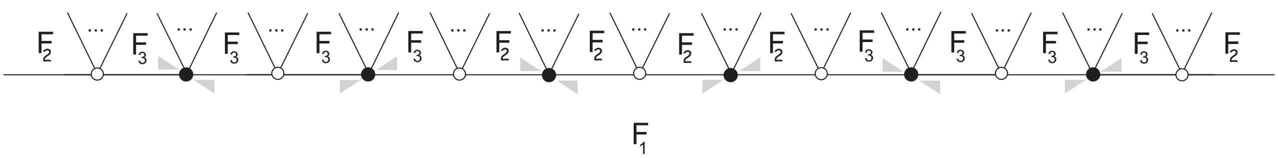


Figure 6.20: Quadruplet of type II bicontactual

Algebraically, we have

$$a^z \in \langle a, b\rangle,\ z^2 \in \langle a, b\rangle,\ (bab)^{z} \in \langle a, b\rangle \textit{ and } b^z \notin \langle a, b\rangle\,.$$

Example : The bicontactual regular pseudo-oriented hypermap

$$(G;\, a,\, b,\, z) = (S_4;\, (1,3)(2,4),\, (3,4),\, (1,2,3,4))\,.$$

As a future work we intend to classify these bicontactual regular pseudo-oriented hypermaps.

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