Universidade de Aveiro Departamento de Matemática
2008

GIORGI BOGVERADZE

TEORIA DE FREDHOLM PARA OPERADORES DE WIENER-HOPF MAIS HANKEL

FREDHOLM THEORY FOR WIENER-HOPF PLUS HANKEL OPERATORS

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tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Doutor Luís Filipe Pinheiro de Castro, Professor Associado com Agregação do Departamento de Matemática da Universidade de Aveiro

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o júri

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palavras-chave

operador de Wiener-Hopf, operador de Hankel, operador de Toeplitz, propriedade de Fredholm, invertibilidade, índice de Fredholm, factorização, função quase periódica, função semi-quase periódica, função quase periódica por troços, função sectorial, discontinuidade quase periódica, cálculo simbólico.

resumo

Na presente tese consideramos combinações algébricas de operadores de Wiener-Hopf e de Hankel com diferentes classes de símbolos de Fourier. Nomeadamente, foram considerados símbolos matriciais na classe de elementos quase periódicos, semi-quase periódicos, quase periódicos por troços e certas funções matriciais sectoriais. Adicionalmente, foi dedicada atenção também aos operadores de Toeplitz mais Hankel com símbolos quase periódicos por troços e com símbolos escalares possuindo n pontos de discontinuidades quase periódicas usuais.

Em toda a tese, um objectivo principal teve a ver com a obtenção de descrições para propriedades de Fredholm para estas classes de operadores.

De forma a deduzir a invertibilidade lateral ou bi-lateral para operadores de Wiener-Hopf mais Hankel com símbolos matriciais AP foi introduzida a noção de factorização assimétrica AP. Neste âmbito, foram dadas condições suficientes para a invertibilidade lateral e bi-lateral de operadores de Wiener-Hopf mais Hankel com símbolos matriciais AP. Para tais operadores, foram ainda exibidos inversos generalizados para todos os casos possíveis.

Para os operadores de Wiener-Hopf-Hankel com símbolos matriciais SAP e PAP foi deduzida a propriedade de Fredholm e uma fórmula para a soma dos índices de Fredholm destes operadores de Wiener-Hopf mais Hankel e operadores de Wiener-Hopf menos Hankel. Uma versão mais forte destes resultados foi obtida usando a factorização generalizada AP à direita.

Foram analisados os operadores de Wiener-Hopf-Hankel com símbolos que apresentam determinadas propriedades pares e também com símbolos de Fourier que contêm matrizes sectoriais. Em adição, para operadores de Wiener-Hopf-Hankel, foi obtido um resultado correspondente ao teorema clássico de Douglas e Sarason conhecido para operadores de Toeplitz com símbolos sectoriais e unitários.

Condições necessárias e suficientes foram também deduzidas para que os operadores de Wiener-Hopf mais Hankel com símbolos L^∞ sejam de Fredholm (ou invertíveis). Para se obter tal resultado, trabalhou-se com certas factorizações ímpares dos símbolos de Fourier.

Os operadores de Toeplitz mais Hankel gerados por símbolos que possuem n pontos de discontinuidades quase periódicas usuais foram também considerados. Foram obtidas condições sob as quais estes operadores são invertíveis à direita e com dimensão de núcleo infinita, invertíveis à esquerda e com dimensão de co-núcleo infinita ou não normalmente solúveis.

A nossa atenção foi também colocada em operadores de Toeplitz mais Hankel com símbolos matriciais contínuos por troços. Para tais operadores, condições necessárias e suficientes foram obtidas para se ter a propriedade de Fredholm. Tal foi realizado usando a abordagem do cálculo simbólico, determinados operadores auxiliares emparelhados com símbolos semi-quase periódicos e várias relações de equivalência após extensão entre operadores.

keywords

Wiener-Hopf operator, Hankel operator, Toeplitz operator, Fredholm property, invertibility, Fredholm index, factorization, almost periodic function, semi-almost periodic function, piecewise almost periodic function, sectorial function, almost periodic discontinuity, symbol calculus.

abstract

In this thesis we considered algebraic combinations of Wiener-Hopf and Hankel operators with different classes of Fourier symbols. Namely, matrix symbols from the almost periodic, semi-almost periodic, piecewise almost periodic and certain sectorial matrix functions were considered. In addition, attention was also paid to Toeplitz plus Hankel operators with piecewise almost periodic symbols and with scalar symbols having n points of standard almost periodic discontinuities.

In the entire thesis a main goal is to obtain Fredholm properties description of those classes of operators.

To deduce the lateral or both sided invertibility theory for Wiener-Hopf plus Hankel operators with AP matrix symbols was introduced the notion of an AP asymmetric factorization. In this framework were given sufficient conditions for the lateral and both sided invertibility of the Wiener-Hopf plus Hankel operators with matrix AP symbols. For such kind of operators were also exhibited generalized inverses for all the possible cases.

For the Wiener-Hopf-Hankel operators with matrix SAP and PAP symbols the Fredholm property and a formula for the sum of the Fredholm indices of these Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators were derived. A stronger version of these results was obtained by using the generalized right AP factorization.

It was analyzed the Wiener-Hopf-Hankel operators with symbols presenting some even properties, and also with Fourier symbols which contain sectorial matrices. In addition, for Wiener-Hopf-Hankel operators, it was obtained a corresponding result to the classical theorem by Douglas and Sarason known for Toeplitz operators with sectorial and unitary valued symbols.

Necessary and sufficient condition for the Wiener-Hopf plus Hankel operators with L^{∞} symbols to be Fredholm (or invertible) were also derived. To obtain such a result we dealt with certain odd asymmetric factorization of the Fourier symbols.

The Toeplitz plus Hankel operators generated by symbols which have n points of standard almost periodic discontinuities were also considered. Conditions were obtained under which these operators are right-invertible and with infinite kernel dimension, left-invertible and with infinite cokernel dimension or simply not normally solvable.

We also focused our attention to Toeplitz plus Hankel operators with piecewise almost periodic matrix symbols. For such operators necessary and sufficient conditions were obtained to have the Fredholm property. This was done using a symbol calculus approach, certain auxiliary paired operators with semi-almost periodic symbols, and several equivalence after extension operator relations.

Contents

List of Symbols

$\mathfrak{A}$	$= \mathrm{alg}(C(\bar{\mathbb{R}}), W^0(PC))$ the C^*-algebra generated by the operators aW_b^0 with $a \in C(\bar{\mathbb{R}})$ and $b \in PC$, 143
A	index set, 29
$\mathfrak{A}^\pi$	$= \mathfrak{A}/\mathcal{K}$ quotient space of $\mathfrak{A}$ over the set of all compact operators $\mathcal{K}$, 143
A^π	coset $A + \mathcal{K}$, 143
AP	class of almost periodic functions, 28
AP_-	smallest closed subalgebra of $L^\infty(\mathbb{R})$ which contains all the functions e_λ, $\lambda \leq 0$, 29
AP_+	smallest closed subalgebra of $L^\infty(\mathbb{R})$ which contains all the functions e_λ, $\lambda \geq 0$, 29
AP^0	almost periodic polynomials, 37
AP_{Γ_0}	almost periodic functions on the unit circle, 34
APW	Wiener subalgebra of AP, 29
$A(t, x, \mu)$	symbol of the operator A, 143
$\arg z$	argument of z, 28
$\mathfrak{B}$	$= \mathrm{alg}(S_\mathbb{R}, SAP^{N\times N})$, 145
B	isometrical isomorphism of $L^2(\Gamma_0)$ onto $L^2(\mathbb{R})$, 22
$\mathfrak{B}^0$	set of all operators of the form $B = \sum A_\lambda e_\lambda I$, where $A_\lambda \in \mathfrak{A}$ and λ ranges over arbitrary finite subsets of $\mathbb{R}$, 145
B^{-1}	isometrical isomorphism of $L^2(\mathbb{R})$ onto $L^2(\Gamma_0)$, inverse of B, 22
B^2	the Besicovitch space, 37
B_0	isometrical isomorphism of $L^\infty(\mathbb{R})$ onto $L^\infty(\Gamma_0)$, 22
B_0^{-1}	isometrical isomorphism of $L^\infty(\Gamma_0)$ onto $L^\infty(\mathbb{R})$, inverse of B_0, 22
B^N	vectors with N components, 3
$B^{N\times M}$	matrices with N rows and M columns, 3

$\widetilde{B}(t)$	$=\Pi_t(B)$ for $t\in\mathbb{R}$, 146
$\widetilde{B}(\infty,\mu)$	$=\Pi_{\infty,\mu}(B)$ for $\mu\in[0,1]$, 146
$b_\lambda(x)$	$=b(x+\lambda)$, 145
$BC(\mathcal{M},\mathbb{C}^{2\times 2})$	bounded and continuous functions from $\mathcal{M}$ into $\mathbb{C}^{2\times 2}$, 143
$\mathfrak{C}$	$=\mathrm{alg}(S_{\mathbb{R}},AP^{N\times N})$, 145
$\mathbb{C}_+$	$=\{z\in\mathbb{C}:\Im m\ z>0\}$, 2
$\mathbb{C}_-$	$=\{z\in\mathbb{C}:\Im m\ z<0\}$, 2
$C(\bar{\mathbb{R}})$	continuous functions with possible jump at infinity, 30
$C(\dot{\mathbb{R}})$	(bounded and) continuous functions on $\dot{\mathbb{R}}$, 30
$C_0(\dot{\mathbb{R}})$	functions from $C(\dot{\mathbb{R}})$ with zero limits at infinity, 30
χ_+	characteristic function for $\mathbb{R}_+$, 62
χ_-	characteristic function for $\mathbb{R}_-$, 148
Coker$T$	cokernel of the operator T, 4
const	constant quantity, 128
$\mathfrak{D}$	$=\mathrm{alg}(PC,W^0(PC))$, 143
$\mathbb{D}^+$	interior of the unit circle on the complex plane, 2
$\mathbb{D}^-$	exterior of the unit circle on the complex plane, 2
$\mathcal{D}_\Gamma^+$	interior of the closed rectifiable Jordan curve Γ, 2
$\mathcal{D}_\Gamma^-$	exterior of the closed rectifiable Jordan curve Γ, 2
$\mathfrak{D}_\Phi$	$=\mathrm{diag}[WH_\Phi,W_\Phi-H_\Phi]$, 53
$\mathbf{d}(\Phi)$	geometric mean value of Φ, 54
$d\mu$	the normalized Haar measure on $\mathbb{R}_B$, 37
$d(T)$	$=\dim\mathrm{Coker}T$, 4
det	determinant, 32
$\mathrm{diag}[a_1,\dots,a_n]$	diagonal matrix with diagonal entries equal to $a_1,\dots,a_n$, 40
$\dim X$	dimension of the space X, 4
$\mathrm{dist}(X,Y)$	distance from the linear space X to the linear space Y, 33
$\mathscr{E}$	$=L(\phi_o^{-1})(I+J_{\Gamma_0})P_{\Gamma_0}L(\phi_-^{-1})$, 105
e_λ	$=e^{i\lambda x}$, $\lambda\in\mathbb{R}$ almost periodic function on $\mathbb{R}$, 29
ess sup	essential supremum, 2
$\mathbb{F}$	class of functions admitting generalized factorization, 120
$\mathcal{F}$	Fourier transform, 9
$\mathcal{F}^{-1}$	inverse of the Fourier transform, 9
$\mathbb{F}_\infty$	class of functions admitting generalized factorization with infinite index, 123
F_B	Bohr-Fourier transformation, 38
$\widetilde{f}(x)$	$=f(-x)$ in case f is defined on $\mathbb{R}$, 11

$\widetilde{f}(t)$	$= f(t^{-1})$ in case f is defined on Γ_0, 3
Γ	closed rectifiable Jordan curve in the complex plane, 2
Γ_0	unit circle of the complex plane, 2
$\mathcal{G}B$	group of the invertible elements from the Banach algebra B, 3
H_Φ	Hankel operator with symbol Φ, 11
$\mathcal{H}_\mathbb{R}$	closed two-sided ideal of $\mathfrak{S}$ generated by all the commutators $cS_\mathbb{R} - S_\mathbb{R}cI$ where $u \in PC$ and with $c(+\infty) = c(-\infty)$, 144
$\mathcal{H}_\infty$	closed two-sided ideal of $\mathfrak{S}$ generated by all the commutators $uS_\mathbb{R} - S_\mathbb{R}uI$ with $u \in C(\bar{\mathbb{R}})$, 144
$H^2_+(\mathbb{R})$	"plus" Hardy space on $\mathbb{R}$, 3
$H^2_-(\mathbb{R})$	"minus" Hardy space on $\mathbb{R}$, 3
$H^\infty(\mathbb{C}_-)$	set of all bounded analytic functions in $\mathbb{C}_-$, 2
$H^\infty(\mathbb{C}_+)$	set of all bounded analytic functions in $\mathbb{C}_+$, 2
$H^\infty_-(\mathbb{R})$	set of all elements in $L^\infty(\mathbb{R})$ that are non-tangential limits of functions in $H^\infty(\mathbb{C}_-)$, 2
$H^\infty_+(\mathbb{R})$	set of all elements in $L^\infty(\mathbb{R})$ that are non-tangential limits of functions in $H^\infty(\mathbb{C}_+)$, 2
I	identity operator, 13
I_X	identity operator on the Banach space X, 4
$I_{N\times N}$	$N \times N$ identity matrix, 19
$\{I_\alpha\}_{\alpha\in A}$	$= \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ family of intervals $I_\alpha \subset \mathbb{R}$ such that $\lvert I_\alpha\rvert = y_\alpha - x_\alpha \to \infty$ as $\alpha \to \infty$, 29
$\Im m\ z$	imaginary part of the complex number z, 2
$\mathrm{Im}T$	image (range) of the operator T, 4
$\mathrm{Ind}T$	$= n(T) - d(T)$ Fredholm index of the operator T, 4
$\mathrm{Ind}_\mathbb{R}\mathcal{T}^\#$	$= \frac{1}{2\pi}\lim_{t\to+\infty}\frac{1}{t}\int_0^t(\psi_\mu(x) - \psi_\mu(-x))dx$, 155
inf	infimum, 28
J	reflection operator on $\mathbb{R}$, 11
J_{Γ_0}	reflection operator on Γ_0, 14
$k(\varphi)$	Bohr mean motion of the scalar almost periodic function φ, 29
$k(\Phi)$	vector containing the right AP indices, 54
$\mathcal{K}(X)$	set of all compact operators acting on Banach space X, 5
$\mathcal{K}(X,Y)$	set of all compact operators acting between Banach spaces X and Y, 5
$\mathrm{Ker}T$	kernel of the operator T, 4
$L(\Phi)$	multiplication operator by Φ, 22
$\mathcal{L}(X)$	Banach space of all linear bounded operators acting from X into X, 3
$\mathcal{L}(X,Y)$	Banach space of all linear bounded operators acting from X into Y, 3

$\mathcal{LF}(X)$	Banach space of all linear finite rank operators acting from X into X, 5
ℓ_0	zero extension operator from $L^2(\mathbb{R}_+)$ onto $L^2_+(\mathbb{R})$, 15
ℓ^e	even extension operator from $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R})$, 12
ℓ^o	odd extension operator from $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R})$, 12
$\ell^2(X)$	collection of all functions $f : X \to \mathbb{C}$ such that the set $\{\lambda \in X : f(\lambda) \neq 0\}$ is at most countable and $\|\|f\|\|^2_{\ell^2(X)} := \sum \|f(\lambda)\|^2 < \infty$, 38
$\ell^\infty(X)$	set of all functions $f : X \to \mathbb{C}$ such that $\|\|f\|\|_{\ell^\infty(X)} := \sup_{\lambda\in X} \|f(\lambda)\| < \infty$, 38
$L^\infty(\mathbb{R})$	set of all essentially bounded functions on $\mathbb{R}$, 2
$L^p(\Gamma_0)$	Lebesgue space of p-th power integrable functions on the unit circle, 2
$L^p(\mathbb{R})$	Lebesgue space of p-th power integrable functions on the real line, 1
$L^2_{\rm even}(X)$	class of even functions from the space $L^2(X)$ ($X = \mathbb{R}$ or $X = \Gamma_0$), 121
$L^2_{\rm odd}(X)$	class of odd functions from the space $L^2(X)$ ($X = \mathbb{R}$ or $X = \Gamma_0$), 100
$L^p_+(\mathbb{R})$	subspace of $L^p(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_+$, 2
$L^p_-(\mathbb{R})$	subspace of $L^p(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_-$, 2
$\ell^2(\mathbb{R}, \mathbb{C}^2)$	collection of all functions $f : \mathbb{R} \to \mathbb{C}^2$ for which the set $\{\lambda \in \mathbb{R} : f(\lambda) \neq 0\}$ is at most countable and $\|\|f\|\|^2_{\ell^2(\mathbb{R},\mathbb{C}^2)} := \sum_\lambda \|\|f(\lambda)\|\|^2 < \infty$, 145
$\ell^2(\mathbb{Z}_+, \mathbb{C}^N)$	$\mathbb{C}^N$-valued ℓ^2 space over $\mathbb{Z}_+$, 13
$\mathcal{M}$	$= (\mathbb{R} \times \{\infty\} \times [0, 1]) \cup (\{\infty\} \times \mathbb{R} \times [0, 1]) \cup ((\infty, \infty) \times \{0, 1\})$, 143
M	$= (\dot{\mathbb{R}} \times \dot{\mathbb{R}}) \setminus (\mathbb{R} \times \mathbb{R}) = (\mathbb{R} \times \{\infty\}) \cup (\{\infty\} \times \mathbb{R}) \cup \{(\infty, \infty)\}$ maximal ideal space of Z^π, 143
$M(\Phi)$	Bohr mean value of the almost periodic function Φ, 29
$\mathbf{M}(Z)$	maximal ideal space of Z, 142
$\mathcal{M}^0$	$= (\mathbb{R} \times \{\infty\} \times [0, 1]) \cup (\{\infty\} \times \mathbb{R} \times [0, 1])$, 144
$n(T)$	$= \dim \operatorname{Ker} T$, 4
$(Nf)(x)$	$= \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\tau)}{\tau+x} d\tau\,, \quad x \in \mathbb{R}$, 151
$N_{\mathbb{R}_+}$	singular integral operator over $\mathbb{R}_+ : (N_{\mathbb{R}_+}\Phi)(x) = \frac{1}{\pi i} \int_{\mathbb{R}_+} \frac{\Phi(\tau)}{\tau+x} d\tau$, 152
$\nu(\mu)$	$= \sqrt{\mu(1 - \mu)}$, 143
$\mathcal{O}(t_i)$	neighborhood of the point t_i, 131
$\Omega(\psi)$	$= \{\lambda \in \mathbb{R} : M(\psi e_{-\lambda}) \neq 0\}$ the Bohr-Fourier spectrum of $\psi \in AP$, 30
$\mathcal{P}$	all trigonometrical polynomials $\sum_{k=-n}^{n} f_k t^k,\ t \in \Gamma_0$, 25
$\widetilde{P}$	$= F_B \chi_+ F_B^{-1}$, 62
$\mathcal{P}_+$	canonical projection acting from the space $L^2(\mathbb{R})$ onto the space $L^2_+(\mathbb{R})$, 15
$\mathcal{P}_-$	canonical projection acting from the space $L^2(\mathbb{R})$ onto the space $L^2_-(\mathbb{R})$, 15

$P_{\mathbb{R}}$ Riesz projection acting from $L^2(\mathbb{R})$ onto $H^2_+(\mathbb{R})$, 13

$P_{\mathbb{R}_+}$ $= (I + S_{\mathbb{R}_+})/2$, 152

Π_t mapping from $\mathfrak{B}$ to $\mathcal{L}(\mathbb{C}^2)$ acting by the rule $\Pi_t(\sum_\lambda A_\lambda e_\lambda I) = \sum_\lambda A_\lambda(t, \infty, 1)e^{i\lambda t}I$, 146

$\Pi_{\infty,\mu}$ mapping from $\mathfrak{B}$ to $\mathcal{L}(\ell^2(\mathbb{R}, \mathbb{C}^2))$ acting by the rule $[\Pi_{\infty,\mu}(\sum_\lambda A_\lambda e_\lambda I)f](x) = \sum_\lambda A_\lambda(\infty, x, \mu)f(x + \lambda)$, 146

$P_{J_{\Gamma_0}}$ $= \frac{I+J_{\Gamma_0}}{2}$, 104

Φ_ℓ left local representative of a function Φ, 31

Φ_ϕ $= P_{\Gamma_0}L(\phi)Q_{J_{\Gamma_0}}$, 108

Ψ_ψ $= Q_{J_{\Gamma_0}}L(\psi)P_{\Gamma_0}$, 108

Φ_r right local representative of a function Φ, 31

$\psi(a)$ $= F_B a F_B^{-1}$, 38

$\psi_\mu(x)$ $= \arg \mathcal{T}(x, \mu)$, 155

PAP class of piecewise almost periodic functions, 31

PC bounded and piecewise continuous functions on $\dot{\mathbb{R}}$, 30

PC_0 functions from PC with zero limits at infinity, 30

$Q_{J_{\Gamma_0}}$ $= I - P_{J_{\Gamma_0}}$, 104

$Q_{\mathbb{R}}$ $= I - P_{\mathbb{R}}$ complementary projection for the Riesz projection, 13

$Q_{\mathbb{R}_+}$ $= I - P_{\mathbb{R}_+}$, 152

$\mathfrak{R}$ linear space of all trigonometric polynomials, 104

$\mathbb{R}$ set of real numbers, 1

$\dot{\mathbb{R}}$ $= \mathbb{R} \cup \{\infty\}$ one point compactification of the real line $\mathbb{R}$, 30

$\widetilde{\mathbb{R}}$ $= \mathbb{R} \cup (\{\infty\} \times [0, 1])$, 145

r_+ restriction operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}_+)$, 10

$\mathbb{R}_+$ positive half-line, 2

$\mathbb{R}_-$ negative half-line, 2

$\mathbb{R}_B$ Bohr compactification of $\mathbb{R}$, 37

$\mathcal{R}(f)$ essential range of $f \in L^\infty(\mathbb{R})$, 7

$r_B(f)$ $= \sup\{|\lambda| : \lambda \in \mathrm{sp}_B[f]\}$ spectral radius of f, 6

$\rho_B(f)$ $= \mathbb{C} \setminus \mathrm{sp}_B[f]$, 6

$\Re e\ z$ real part of the complex number z, 33

$\mathfrak{S}$ $= \mathrm{alg}(S_{\mathbb{R}}, [C(\bar{\mathbb{R}})]^{N\times N})$, 144

$\mathscr{S}$ $= W^0_{\varphi_o^{-1}}(I - J)\ell_0 W_{\varphi_-^{-1}} : L^2(\mathbb{R}) \to L^2_{\mathrm{even}}(\mathbb{R})$, 102

$\mathcal{S}$ set of all sectorial functions on the real line, 33

S^- reflexive generalized inverse of the operator S, 45

$S_{\mathbb{R}}$ Cauchy singular integral operator on the real line, 12

$S_{\mathbb{R}_+}$ singular integral operator over $\mathbb{R}_+ : (S_{\mathbb{R}_+}\Phi)(x) = \frac{1}{\pi i}\int_{\mathbb{R}_+} \frac{\Phi(\tau)}{\tau - x} d\tau$, 152

SAP	class of semi-almost periodic functions, 30
$SAPD$	standard almost periodic discontinuity, 34
sup	supremum, 2
Sym	symbol map, 144
$\mathrm{sp}[T]$	$= \{\lambda \in \mathbb{C} : T - \lambda I_X \notin \mathcal{GL}(X)\}$, 7
$\mathrm{sp}_B[f]$	$= \{\lambda \in \mathbb{C} : f - \lambda \notin \mathcal{G}B\}$ spectrum of f with respect to the Banach algebra B, 6
$\mathrm{sp}_{\mathrm{ess}}[T]$	$= \{\lambda \in \mathbb{C} : T - \lambda I_X$ is not a Fredholm operator$\}$, 7
$\sigma_{t_0}(\varphi)$	real functional introduced by Dybin and Grudsky, 36
T_Δ	auxiliary operator in the Δ-relation after extension, 8
T_Φ	Toeplitz operator with symbol Φ, 13
T^*	adjoint operator for the operator T, 5
T_ℓ^-	left-inverse of the operator T, 4
T_r^-	right-inverse of the operator T, 4
TH_Φ	Toeplitz plus Hankel operator with symbol Φ, 15
$T \overset{*}{\Delta} S$	T being Δ-related after extension with S, 8
$\mathcal{U}$	class of model functions such that $h \in L_+^\infty(\Gamma_0)$ and $h^{-1} \in L_-^\infty(\Gamma_0)$, 35
U^n	isometrical isomorphism of $L^2(\Gamma_0)$ into $L^2(\Gamma_0)$, $U^n f(t) = t^n f(t)$, 25
$u_\mathbb{R}$	$: \mathbb{R} \to \mathcal{L}(L^2(\mathbb{R}))$ unitary representation of the discrete group $\mathbb{R}$ given by $u_\mathbb{R} : \lambda \to e_\lambda I$, 145
V	$= \chi_+ N \chi_+ I + \chi_- N \chi_- I$, 151
V_{Γ_0}	$: L^2(\Gamma_0) \to L^2(\Gamma_0)$, $(V_{\Gamma_0} f)(t) = f(-t)$, 109
$\mathbb{V}_\psi$	$= \{x \in \mathbb{R} : \psi(x) = \psi(-x) = 0\}$, 115
$\mathcal{W}$	$: \mathbb{W} \to \mathcal{L}(L_+^2(\mathbb{R}), L^2(\mathbb{R}_+)), \quad \phi \to W_\phi$, 10
$\mathbb{W}$	Wiener algebra, 10
W_Φ	Wiener-Hopf operator with symbol Φ, 10
W_a^0	convolution operator with symbol a, 10
WH_Φ	Wiener-Hopf plus Hankel operator with symbol Φ, 12
$X \cong Y$	X being isometrically isomorphic to Y, 116
$\mathbb{Z}$	set of integer numbers, 87
Z	$= \mathrm{alg}(C(\dot{\mathbb{R}}), W^0(C(\dot{\mathbb{R}})))$, 143
$\mathbb{Z}_+$	$= \{0, 1, 2, \ldots\}$, 13
$\lvert\lvert \cdot \rvert\rvert_X$	norm on a linear space X, 1
$\{\cdot\}$	fractional part of a real number, 59
$\lvert \cdot \rvert$	absolute value of the quantity, 1
Δ	Δ-relation after extension, 8
$\oplus$	direct sum, 8

Introduction

This thesis, belonging to the area of Operator Theory and Functional Analysis, deals with special kinds of singular integral and convolution type operators, which are constituted by Wiener-Hopf, Hankel, and Toeplitz operators, and certain algebraic combinations of them. The aim of the thesis is to present some recent results in view of a Fredholm theory of Wiener-Hopf-Hankel and Toeplitz-Hankel operators. We start by describing shortly the historical foundations of the operators under consideration.

The PhD thesis of Hankel [44] initiated the study of Hankel operators. In his thesis Hankel considered finite matrices with entries depending only on the sum of the coordinates. More precisely, Hankel studied determinants of finite complex matrices with entries defined by $c_{jk} = c_{j+k}$ $(j, k \geq 0)$, where $\{c_j\}_{j\geq 0}$ is a sequence of complex numbers. Such type of matrices are called Hankel matrices and therefore have the form

$$\begin{bmatrix} c_0 & c_1 & c_2 & \dots \\ c_1 & c_2 & c_3 & \dots \\ c_2 & c_3 & c_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This was the starting point for the theory of Hankel operators. The study of Wiener-Hopf operators started after the joint work of Wiener and Hopf [73]. In this paper they introduced a new method of solving Riemann-Hilbert problems, based on the decomposition of the Fourier transform of the convolution kernel. This method of decomposition was afterwards named as "Wiener-Hopf factorization" in honor to the authors of that paper.

The Toeplitz operators are also known as the discrete analogues of the Wiener-Hopf operators. These operators were first considered by Toeplitz [72]. As in the case of Hankel operators, also in here they have a very interesting description, in terms of the matrices which are generating them. In fact, in certain frameworks, Toeplitz operators are exactly those which can be given by infinite matrices with constant entries on the main diagonals, i.e., matrices of the form:

$$\begin{bmatrix} c_0 & c_{-1} & c_{-2} & \cdots \\ c_1 & c_0 & c_{-1} & \cdots \\ c_2 & c_1 & c_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The theory of Wiener-Hopf and Toeplitz operators was developed in a parallel way, until the works by Rosenblum [67] and Devinatz [30] where they discovered that these two type of operators were unitarily equivalent.

After these brief historical notes we would like to mention that algebraic combinations of Wiener-Hopf and Hankel operators play an important role in various branches of mathematics, such as Analysis, Mathematical Physics, Probability Theory, Control Theory, etc. Therefore, to obtain eventual descriptions of Fredholm properties of classes of operators generated by Wiener-Hopf and Hankel operators is an important goal in present investigations. In this way, we would like to point out that for some particular classes of Fourier symbols the Fredholm theory of Wiener-Hopf plus Hankel operators is already well developed. As an example, we can refer to the class of such operators with continuous Fourier symbols or –in a more general setting– to the corresponding class with piecewise continuous Fourier symbols.

Firstly, let us mention that the Fredholm theory for scalar Wiener-Hopf-Hankel operators with almost periodic, semi-almost periodic and with piecewise almost periodic functions is well documented in the PhD thesis of Nolasco [58]. On the other hand, the Fredholm theory of matrix Wiener-Hopf-Hankel operators with almost periodic, semi-almost periodic or with piecewise almost periodic functions was not yet completely described and developed. Therefore, in this thesis we focus our attention to the Wiener-Hopf-Hankel operators with the just mentioned symbols and moreover with symbols associated in a certain way with unitary and sectorial matrix functions. Furthermore, we treat Toeplitz plus Hankel operators with piecewise almost periodic matrix functions, and also Toeplitz plus Hankel operators with symbols having n points of standard almost periodic discontinuities ($SAPD$). We will study these operators in the framework of L^2 Lebesgue and H^2 Hardy spaces.

This thesis is organized as follows. In the first chapter we give the basic notation and definitions of the main objects under study: Wiener-Hopf, Hankel and Toeplitz operators. The main (here used) concepts from Operator Theory (such as kernel and cokernel, Fredholm index, adjoint operator etc.) are also stated in the first chapter. Moreover, certain equivalence relations between bounded linear operators are defined and some key relations between Wiener-Hopf-Hankel and Toeplitz-Hankel operators are exhibited. The main operator relations which we will use throughout this thesis will be the so-called Δ-relation after extension (introduced by Castro and Speck [20]) and the equivalence relation after extension. Furthermore, in the end of the first chapter we give a necessary condition for Wiener-Hopf (Toeplitz) plus/minus Hankel operators with essentially bounded symbols to be a Fredholm operator.

Chapter 2 concerns the Fourier symbols of the operators under study. This chapter has an introductory nature and starts with the consideration of the algebra of Bohr almost periodic functions (AP). Next, some basic properties and the main characteristics of such functions are given. Then, we will pass to the algebras of semi-almost periodic functions (SAP) and piecewise almost periodic functions (PAP). We will present a formula (Sarason [68]) which allows us to decompose in a more convenient way the functions from the algebra SAP or PAP. Further, unitary and sectorial symbols will be defined and also the classical results involving such functions will be stated. Moreover, the notion of standard almost periodic discontinuity will be given for functions defined on the unit circle of the complex plane. Associated with such type of functions we provide descriptions of the so-called model function, of the class $\mathcal{U}$ and a certain real functional σ_{t_0} [37] will be presented as well. At the end of this chapter we define the Besicovitch space, and present the basic properties of this space.

In Chapter 3 we study Wiener-Hopf plus Hankel operators with symbols from the algebra of matrix almost periodic functions. To deduce one-sided or two-sided invertibility theory for Wiener-Hopf plus Hankel operators with AP matrix symbols we start by considering the Wiener subclass APW of AP. For matrices in APW we introduce the notion of APW asymmetric factorization by following the known scalar case for AP functions (cf. [59] and [60]). Depending on that factorization we give sufficient conditions for one-sided and two-sided invertibility of the Wiener-Hopf plus Hankel operators with matrix APW symbols. Moreover, for such kind of operators we will exhibit the generalized inverses for all the possible cases. The case of Wiener-Hopf plus Hankel operators with matrix symbols in the AP class can be treated in an analogous way as the case of the APW class. The difference occurs only in the uniqueness of the asymmetric factorizations with AP functions. The corresponding theorem for the uniqueness of the AP asymmetric factorization is given at the end of Chapter 3. We point out that the theory of asymmetric factorization in the Banach algebras with factorization property in connection with Toeplitz-Hankel operators was considered in a detailed way in the Habilitation thesis of Ehrhardt [38].

In Chapter 4 we treat the Wiener-Hopf-Hankel operators with matrix SAP symbols. Conditions for the Fredholm property of such kind of operators are developed. Under such conditions, a formula for the sum of the Fredholm indices of the Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators is derived. To achieve such results we will need the right AP factorization of matrix functions and the Δ-relation after extension. In fact, here the Δ-relation after extension is a key tool to obtain the above mentioned results. To achieve a stronger version of the Fredholm property we will make use of the generalized right AP factorization. Obviously, the stronger version of the index formula will be given upon assuming the corresponding Fredholm property. In this situation we follow a standard strategy. Namely, in a first instance we give only an index formula for Wiener-Hopf plus/minus Hankel operators with $SAPW$ symbols. Then, employing an argument based on the passage to the limit and using the fact that $SAPW$ is dense in SAP, we reach to the final goal of the chapter.

Historical notes will be in order. The Toeplitz operators with symbols from the algebra generated by continuous and almost periodic functions (later named as semi-almost periodic functions class) were firstly considered by Sarason [68] (following a suggestion by Gohberg). Sarason worked out the corresponding Fredholm theory for scalar Toeplitz operators acting between L^2 Lebesgue spaces with semi-almost periodic symbols. Later on, for L^p Lebesgue spaces these results were generalized by Duduchava and Saginashvili [35] for scalar Wiener-Hopf operators. The theory of matrix Toeplitz operators with SAP symbols required completely different technics and methods apart to the scalar case and this theory was mostly developed by three authors: Böttcher, Karlovich and Spitkovsky [16]. Scalar Wiener-Hopf plus Hankel operators with SAP symbols were first considered by Nolasco and Castro [61], and in this thesis we give the results for the matrix Wiener-Hopf-Hankel operators with SAP symbols [9].

In Chapter 5 it is discussed the Wiener-Hopf-Hankel operators with matrix PAP symbols. Thus, this chapter generalizes the results obtained in Chapter 4. To reach to the Fredholm property for Wiener-Hopf-Hankel operators with matrix PAP symbols we will need to recall some results [14] and also to generalize for the matrix case the results which were only known for the scalar case. After all this being at our disposal, we will be able to obtain conditions for the Fredholm property and an index formula for Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators. Here, as in Chapter 4, the key ingredient is the Δ-relation after extension. At the final part of this chapter we derive the stronger results, and this will be done by using the same arguments as in Chapter 4. Namely, relying on the generalized right AP factorization and the argument of passage to the limits.

In Chapter 6 we consider the class of Wiener-Hopf-Hankel operators with symbols presenting some even properties (which in particular include unitary matrix functions), and also with Fourier symbols which contain sectorial matrices. This chapter generalizes the classical result known as Douglas-Sarason Theorem [32] for Toeplitz operators with sectorial and unitary symbols. The main result will be obtained by using this classical theorem together with the Δ-relation after extension.

In Chapter 7 we study scalar Wiener-Hopf plus Hankel operators with symbols from the L^∞ functions class. This is motivated by the results obtained by Basor and Ehrhardt [3] for the Toeplitz plus Hankel operators with L^∞ symbols. The main results of this chapter concern necessary and sufficient conditions for the Wiener-Hopf plus Hankel operators with L^∞ symbols to be Fredholm, or invertible. For the Toeplitz plus Hankel operators Basor and Ehrhardt proposed an *even* asymmetric factorization with certain weights, but apart of it, when considering Toeplitz minus Hankel operators, we need to deal with an *odd* asymmetric factorization. Then using the equivalence between the Toeplitz minus Hankel operators and the Wiener-Hopf plus Hankel operators, we will pass from the unit circle to the real line and deduce the main results. It is worth to note that here we encounter "unusual" weights in the form of a so-called weak *odd* asymmetric factorization.

In Chapter 8 we will consider scalar Toeplitz plus Hankel operators generated by symbols which have n points of standard almost periodic discontinuities, and acting between L^2 Lebesgue spaces. Conditions are obtained under which these operators are right-invertible and with infinite dimensional kernel, left-invertible and with infinite dimensional cokernel or simply not normally solvable. This will be done by employing a certain real functional and looking to the resulting signs on the points of the standard almost periodic discontinuities. First of all, we deduce those results for symbols having three points of standard almost periodic discontinuities and then we will be able to generalize the results for functions having n points of standard almost periodic discontinuities. To obtain that conditions we will need to recall the notion of generalized factorization with infinite index, introduced by Dybin and Grudsky [37]. To prove some parts of the main results we will relay on the Δ-relation after extension, but the other parts of the theorems also require different reasonings. Therefore, we will employ an even asymmetric factorization and certain arguments from Complex Analysis to reach the final goal of this chapter.

In Chapter 9 we consider Toeplitz plus Hankel operators with piecewise almost periodic matrix symbols. For such operators, a Fredholm criterion is presented. This is obtained by using a symbol calculus approach ([4], [15]), certain auxiliary paired operators with semi-almost periodic symbols, and several equivalence after extension operator relations. The importance of this theoretical result relays on the fact that we are able to derive necessary and sufficient conditions for the Toeplitz plus Hankel operators with piecewise almost periodic symbols to be Fredholm operators. Moreover, it can also be considered as an enhancement of the results obtained in Chapter 5. At the end of this chapter we obtain a Fredholm index formula for the matrix Toeplitz plus Hankel operators with piecewise almost periodic functions (obviously under the assumption that the operators are Fredholm). This will be done by means of approximation, which in our case means that in a first step we will deal with symbols from $PAPW$, giving the corresponding known index formula for the operators with such type of symbols. After this, employing certain stability properties and passing to the limit, we are able to obtain a Fredholm index formula for matrix Toeplitz plus Hankel operators with PAP symbols.

The new results presented in this thesis are mainly based on the author's published or accepted for publication papers [9], [10], [11], [12], [13], and also on the submitted for publication papers [6], [7] and [8].

Chapter 1

Notation and introductory results

In this chapter we follow several goals. We give the basic notation, introduce the main objects of the study, Wiener-Hopf-Hankel operators, and also the operators closely related to them, Toeplitz-Hankel operators. The connection between the Wiener-Hopf-Hankel operators and the Toeplitz-Hankel operators will be also given. The basic formulas from the theory of Wiener-Hopf-Hankel (Toeplitz-Hankel) operators will be recorded, and in the end of this chapter we state the necessary condition for the semi-Fredholm property of such operators.

Due to the introductory nature of the present chapter, and consequent presentation of known results, we choose only to present proofs of those results which are directly connected with the operators under study and give an insight view of such operators. All the other results are cited or are so general that can be found in any general book on Functional Analysis and Operator Theory.

1.1 Lebesgue and Hardy spaces

For $p = 1$, and $p = 2$, $L^p(\mathbb{R})$ will denote the Banach space of all Lebesgue measurable complex-valued functions on $\mathbb{R}$, for which

$$||f||_{L^p(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(t)|^p dt \right)^{\frac{1}{p}} < \infty \ .$$

By $L^\infty(\mathbb{R})$ will be denoted the Banach space of all essentially bounded and Lebesgue measurable complex-valued functions on the real line, equipped with the norm

$$||f||_{L^\infty(\mathbb{R})} := \operatorname{ess\,sup}\{|f(t)| : t \in \mathbb{R}\} ,$$

where ess sup stands for the essential supremum.

Let Γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$. The interior of this curve will be denoted by $\mathcal{D}_\Gamma^+$, and the exterior by $\mathcal{D}_\Gamma^-$. Further, Γ_0 will stand for the unit circle in the complex plane and in this case we will simply write $\mathbb{D}^\pm$ in the place of $\mathcal{D}_{\Gamma_0}^\pm$.

The Lebesgue spaces $L^p(\Gamma_0)$ $(p = 1, 2, \infty)$ and also $L^p(X)$ (X being an open subset of $\mathbb{R}$, Γ_0, or being an arbitrary rectifiable Jordan curve on the complex plane) are defined analogously to the spaces $L^p(\mathbb{R})$. Further $L^p_\pm(\mathbb{R})$ will denote the subspace of $L^p(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$, respectively.

Let us now introduce the spaces which were first studied by Hardy. He considered the spaces of functions which are analytic and bounded inside the closed unit disk of the complex plain. However our approach will be different that of Hardy.

Let $\mathbb{C}_- := \{z \in \mathbb{C} : \Im m\, z < 0\}$ and $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im m\, z > 0\}$. As usual, let us denote by $H^\infty(\mathbb{C}_\pm)$ the set of all bounded and analytic functions in $\mathbb{C}_\pm$. Fatou's Theorem ensures that functions in $H^\infty(\mathbb{C}_\pm)$ have non-tangential limits on $\mathbb{R}$ almost everywhere. Thus, let $H^\infty_\pm(\mathbb{R})$ be the set of all functions in $L^\infty(\mathbb{R})$ that are non-tangential limits of elements in $H^\infty(\mathbb{C}_\pm)$. Moreover, it is well-known that $H^\infty_\pm(\mathbb{R})$ are closed subalgebras of $L^\infty(\mathbb{R})$.

Let $p = 1$ or $p = 2$. The set of all functions f which are analytic in $\mathbb{C}_\pm$ and satisfy

$$\sup_{\pm y > 0} \int_{\mathbb{R}} |f(x + iy)|^p dx < \infty$$

is denoted by $H^p(\mathbb{C}_\pm)$. Employing again a theorem by Fatou we can ensure that functions in $H^p(\mathbb{C}_\pm)$ have non-tangential limits almost everywhere on $\mathbb{R}$. For the set of corresponding boundary functions we use the notation $H^p_\pm(\mathbb{R})$.

The formulas obtained by Paley and Wiener tells us that the spaces $H^2_\pm(\mathbb{R})$ and $L^2_\pm(\mathbb{R})$ are isometrically isomorphic. Indeed:

$$H^2_+(\mathbb{R}) = \mathcal{F}L^2_+(\mathbb{R}), \quad H^2_-(\mathbb{R}) = \mathcal{F}L^2_-(\mathbb{R}) \ , \tag{1.1.1}$$

where $\mathcal{F}$ stands for the Fourier transform (see the definition below), and hence we have the desired isometrical identification.

We also need the analogues of the above introduced spaces for the unit circle. Let us denote by $H^\infty(\mathbb{D}^\pm)$ the space of all bounded and analytic functions in $\mathbb{D}^\pm$. In case $p = 1$ or $p = 2$ we let $H^p(\mathbb{D}^+)$ denote the set of all functions ϕ which are analytic in $\mathbb{D}^+$ and satisfy

$$\sup_{r\in(0,1)} \int_0^{2\pi} |\phi(re^{i\theta})|^p d\theta < \infty \ ,$$

and $H^p(\mathbb{D}^-)$ will stand for the functions $\phi(z)$ $(z \in \mathbb{D}^-)$ for which $\phi(z^{-1})$ is a function in $H^p(\mathbb{D}^+)$. Again by a theorem of Fatou, functions in $H^p(\mathbb{D}^\pm)$ $(p = 1, 2, \infty)$ have non-tangential limits almost everywhere on Γ_0. We set $H^p_\pm(\Gamma_0)$ for the functions which can be represented as non-tangential limits of functions from $H^p(\mathbb{D}^\pm)$.

Consider a function f given on *the unit circle*: $f : \Gamma_0 \to \mathbb{C}$. By the notation $\widetilde{f}$ we mean the following new function: $\widetilde{f}(t) := f(t^{-1})$, $t \in \Gamma_0$. As usual, on the unit circle Γ_0, we say that a function f is even if $\widetilde{f} = f$ and f is said to be an odd function if $\widetilde{f} = -f$.

For a Banach algebra B, B^N will stand for the vectors with N components, and $B^{N\times M}$ will be the matrices with N rows and M columns. Moreover in our notation $\mathcal{G}B$ will stand for the group of all invertible elements from the Banach algebra B.

1.2 Fredholm, semi-Fredholm and compact operators

Let X and Y be Banach spaces. By $\mathcal{L}(X, Y)$ we denote the Banach space of all linear bounded operators acting from X into Y. In the case $X = Y$ we simply write $\mathcal{L}(X)$.

In this thesis we are using the name of *regularity properties* of a linear operator acting between the Banach spaces for those properties which arise from a direct influence of the kernel and the image of those operators. In more detail, let $T \in \mathcal{L}(X,Y)$, where X and Y are the Banach spaces, and consider the following set:

$$\mathrm{Ker}T := \{x \in X : Tx = 0\} \ .$$

We will refer to it as a kernel of the operator T. The following set is called the image (range) of the operator T and it is defined as follows:

$$\mathrm{Im}T := \{Tx : x \in X\} \ .$$

In case that $\mathrm{Im}T$ is closed we call operator T to be *normally solvable*. Let us observe that $\mathrm{Ker}T$ and $\mathrm{Im}T$ are linear subspaces of X, and that $\mathrm{Ker}T$ is always closed.

Assume that $\mathrm{Im}\,T$ is closed (i.e. T is normally solvable), and let us consider the cokernel of T to be defined by the quotient $\mathrm{Coker}\,T := Y/\mathrm{Im}\,T$. We will recall the numbers, referred as the defect numbers (infinite case is not excluded) of the operator, which are defined by the following formulas:

$$n(T) := \dim \mathrm{Ker}T \ ,$$

and

$$d(T) := \dim \mathrm{Coker}T \ .$$

A normally solvable operator T is called *Fredholm* if both $n(T)$ and $d(T)$ are finite. In this case the Fredholm index of the operator T is defined to be the finite number:

$$\mathrm{Ind}T := n(T) - d(T) \ .$$

A normally solvable operator T is said to be (*properly*) *n-normal* if $n(T) < \infty$ (and $d(T) = \infty$) and (*properly*) *d-normal* if $d(T) < \infty$ (and $n(T) = \infty$). The operators which belong to the set of operators which are Fredholm, n- or d-normal we will call them semi-Fredholm operators, and the operators which belong to the set of operators which are properly n- or d-normal we call them properly semi-Fredholm operators.

Additionally, we say that T is *left-invertible* or *right-invertible* if there exist $T_l^- : Y \to X$ or $T_r^- : Y \to X$ such that $T_l^- T = I_X$ or $TT_r^- = I_Y$, respectively. As usual, in the case when both T_l^- and T_r^- exist the operator T is said to be *two-sided invertible* (or *invertible*). Alternatively, it can be shown that T is left-invertible if and only if T is injective and normally solvable. In the same way, T is right-invertible if and only if T is normally solvable and surjective.

DEFINITION 1.2.1. *Let X and Y be Hilbert spaces and let $T \in \mathcal{L}(X,Y)$. Then T is said to be a finite rank operator if the range of T (i.e., $\mathrm{Im}T$) has a finite dimension.*

It is known (cf., e.g., [31]) that the set of all finite rank operators acting on a Hilbert space X, which we will denote by $\mathcal{LF}(X)$, is a minimal two-sided ideal of the space $\mathcal{L}(X)$.

DEFINITION 1.2.2. *Let X and Y be Hilbert spaces and let $T \in \mathcal{L}(X,Y)$. Then T is said to be a compact operator if the image of any bounded subset of X is relatively compact in Y.*

The set of all compact operators acting between Hilbert spaces X and Y will be denoted by $\mathcal{K}(X,Y)$. It is also a known fact that the norm closure of all finite rank operators acting between the Hilbert spaces coincide with the class of all compact operators (cf., e.g., [31]). Let us observe that $\mathcal{K}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$, where $\mathcal{K}(X)$ stands for the compact operators acting between the Hilbert space X (c.f., e.g., [31]).

The next proposition allows us to give a definition of an adjoint operator.

PROPOSITION 1.2.3. *(cf., e.g., [31]) Let $T \in \mathcal{L}(X)$, where X is a Hilbert space. Then there exist a unique operator $S \in \mathcal{L}(X)$, such that*

$$(Tf,g) = (f,Sg) \ ,$$

where f and g belong to X, and $(\cdot,\cdot)$ denotes the inner product in X.

DEFINITION 1.2.4. *Let $T \in \mathcal{L}(X)$, where X is a Hilbert space. The adjoint operator of T, denoted T^*, is the unique operator on X satisfying $(Tf,g) = (f,T^*g)$ for all f and g in X.*

THEOREM 1.2.5. *(cf., e.g., [14]) Let T^* be the adjoint operator of $T \in \mathcal{L}(X)$, where X is a Hilbert space. Then, T is normally solvable if and only if T^* is normally solvable. Moreover, T is a Fredholm operator if and only if T^* is a Fredholm operator. If T is n-normal, then T^* is d-normal, and if T is d-normal, then T^* is n-normal. Furthermore, the following equalities hold:*

$$n(T^*) = d(T), \quad d(T^*) = n(T), \quad \mathrm{Ind}T^* = -\mathrm{Ind}T \, .$$

THEOREM 1.2.6. *(cf., e.g., [14]) Let $T \in \mathcal{L}(X)$, where X is a Banach space. If T is a Fredholm (resp. n-normal, d-normal) operator and K is a compact operator, then $T + K$ is Fredholm (resp. n-normal, d-normal) operator and* $\mathrm{Ind}(T + K) = \mathrm{Ind}T$.

THEOREM 1.2.7. *(Atkinson) If T and S are Fredholm or n-normal (resp. d-normal) operators acting between the Banach spaces, then TS is Fredholm or n-normal (resp. d-normal) operator and* $\mathrm{Ind}TS = \mathrm{Ind}T + \mathrm{Ind}S$.

We would like to give the notion of the spectrum and the essential spectrum of the operators, and to describe some elementary properties of them.

DEFINITION 1.2.8. *Let B be a Banach algebra with identity I and $f \in B$. We define spectrum of f with respect to B to be the set*

$$\mathrm{sp}_B[f] := \{\lambda \in \mathbb{C} : f - \lambda I \notin \mathcal{G}B\} \, ,$$

and the resolvent set of f to be the set

$$\rho_B(f) := \mathbb{C} \setminus \mathrm{sp}_B[f] \, .$$

The spectral radius is the following real number

$$r_B(f) := \sup\{|\lambda| : \lambda \in \mathrm{sp}_B[f]\} \, .$$

In some cases we will use the brief notation $\mathrm{sp}[f]$ instead of $\mathrm{sp}_B[f]$. In the same way we will simplify the notation for the spectral radius and for the resolvent set. The fact that for a given Banach algebra B (over the complex field) the set $\mathrm{sp}[f]$, with $f \in B$, is nonempty and compact is well-known (cf., e.g., [31]). Moreover we have that $r_B(f) \leq ||f||_B$. In case $B = L^\infty(\mathbb{R})$ the spectrum of $f \in L^\infty(\mathbb{R})$ is called the essential range of f and is denoted by $\mathcal{R}(f)$. Hence, by the definition we have:

$$\mathcal{R}(f) := \mathrm{sp}_{L^\infty(\mathbb{R})}[f] = \{\lambda \in \mathbb{C} : f - \lambda \notin \mathcal{G}L^\infty(\mathbb{R})\} .$$

Consider $T \in \mathcal{L}(X)$, where X is a Banach space. The spectrum of a bounded linear operator T is defined analogously as above:

$$\mathrm{sp}[T] := \{\lambda \in \mathbb{C} : T - \lambda I_X \notin \mathcal{G}\mathcal{L}(X)\} .$$

In addition, the essential spectrum of a linear and bounded operator T is defined in the following way:

$$\mathrm{sp}_{\mathrm{ess}}[T] := \{\lambda \in \mathbb{C} : T - \lambda I_X \text{ is not a Fredholm operator}\} .$$

It is readily seen that $\mathrm{sp}_{\mathrm{ess}}[T] \subset \mathrm{sp}[T]$.

1.3 Relations between operators

To study certain linear bounded operators, very frequently we need to transfer properties from one operator to another somehow equivalent operator. We will recall several kinds of notions of operator equivalence. Let $T \in \mathcal{L}(X_1, Y_1)$ and $S \in \mathcal{L}(X_2, Y_2)$ where $X_{1,2}$ and $Y_{1,2}$ are Banach spaces. We say that T and S are equivalent operators if there exist invertible operators $E \in \mathcal{L}(Y_2, Y_1)$ and $F \in \mathcal{L}(X_1, X_2)$, such that the following equality holds:

$$T = ESF .$$

In addition, if we have an equality $T = ESE^{-1}$ we say that T and S are unitarily equivalent operators. It is clear that if T and S are equivalent operators, then they enjoy the same regularity properties. More precisely, if one of these operators is two-sided invertible, one-sided invertible, Fredholm, (properly) n-normal, (properly) d-normal or normally solvable, then the other one also has exactly the same property.

Another kind of the equivalence relation between linear bounded operators is the notion of equivalence after extension. We say that T and S are equivalent operators after extension, if there exist Banach spaces Z_1 and Z_2, such that $T \oplus I_{Z_1}$ and $S \oplus I_{Z_2}$ are equivalent operators. Here $\oplus$ denotes the direct sum and I_{Z_1} and I_{Z_2} are the identity operators on Z_1 and Z_2 spaces, respectively. In this case we also have that T and S enjoy the same regularity properties.

Further, we will use the notion of Δ-relation after extension introduced by Castro and Speck in [20] for bounded linear operators acting between Banach spaces, e.g. $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$. We say that T *is Δ-related after extension to* S (and use the abbreviation $T \overset{*}{\Delta} S$) if there is an auxiliary bounded linear operator acting between Banach spaces $T_\Delta : X_{1\Delta} \to X_{2\Delta}$, and bounded invertible operators E and F such that

$$\begin{bmatrix} T & 0 \\ 0 & T_\Delta \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & I_Z \end{bmatrix} F \,, \tag{1.3.1}$$

where Z is an additional Banach space and I_Z represents the identity operator in Z. In the particular case where $T_\Delta = I_{X_{1\Delta}} : X_{1\Delta} \to X_{2\Delta} = X_{1\Delta}$ is the identity operator, we arrive at the above introduced notion of equivalent after extension operators.

The Δ-relation after extension allows us to transfer regularity properties between the bounded linear operators T and S only in one direction, namely from the operator S to the operator T. This means that if T is Δ-related after extension with S, then T belongs to the same regularity class as S does, but not in the other direction from the operator T to the operator S. The reason for this is based on the operator T_Δ appearing in the equality (1.3.1). There are many counterexamples, which show that e.g. it is possible that T is an invertible operator, but T_Δ is not normally solvable, and in this case we have that S is not a normally solvable operator. Hence we have that T is an invertible and S is not a normally solvable operator, thus T and S do not enjoy the same regularity properties.

We would like to observe that Δ-relations after extension are transitive [20, Example 1.6]:

$$\left.\begin{array}{l} T_1 \overset{*}{\Delta} S \\ S \overset{*}{\Delta} T_2 \end{array}\right\} \Longrightarrow T_1 \overset{*}{\Delta} T_2 \,.$$

This property is important due to the reason that the equivalence after extension is a particular case of the Δ-relation after extension. This means that if we have $T_1 \overset{*}{\Delta} S$ and S is equivalent after extension with T_2, then $T_1 \overset{*}{\Delta} T_2$. We will use these facts to obtain important equivalence relations between the Wiener-Hopf-Hankel and pure Wiener-Hopf operators (cf. subsequent sections).

1.4 Wiener-Hopf-Hankel operators

In this section we introduce the main objects of our study: Wiener-Hopf-Hankel operators. These operators naturally arise in a variety of Mathematical Physics applications, in Probability Theory, Control Theory, etc. The Hankel operators were first studied in the middle of the 19th century, and the Wiener-Hopf operators came out in the 30s of 20th century. After that, enormous effort was made to describe the regularity properties of such type of operators cf., e.g., the works [3], [14], [16], [18], [20], [22], [26], [27], [30], [32], [34], [35], [37], [38], [40], [41], [42], [47], [54], [56], [64], [66] and the references therein. Therefore, much is known about these kind of operators, and even more remains to be known.

The Fourier transformation $\mathcal{F}$, acting between the Lebesgue spaces $L^2(\mathbb{R})$, is given in the next formula:

$$(\mathcal{F}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{itx} dt, \quad x \in \mathbb{R} \, ,$$

and for the inverse we have:

$$(\mathcal{F}^{-1}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-itx} dx, \quad t \in \mathbb{R} \, .$$

In a formal way the convolution operator W_k^0 (which we will consider in the case when it is acting between $L^2(\mathbb{R})$ Hilbert spaces) is given by the following formula:

$$(W_k^0 f)(x) = \int_{\mathbb{R}} k(x-t) f(t) dt, \quad x \in \mathbb{R} \, .$$

The function k is called the convolution kernel of the operator W_k^0.

Using the Fourier transformation the convolution operator can be formally given by the next formula:

$$W_k^0 = \mathcal{F}^{-1}\hat{k}\mathcal{F},$$

where $\hat{k}$ denotes the Fourier transform of the convolution kernel k.

A relevant class of convolution type operators (both from the theoretical and applied points of views) is constituted by the Wiener-Hopf-Hankel operators. Under the term Wiener-Hopf-Hankel we mean both Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators.

The Wiener-Hopf operators received their name due to the pioneering work of Wiener and Hopf [73] about the study of integral equations of the form

$$c\varphi(x) + \int_0^{+\infty} k(x-y)\varphi(y)\,dy = f(x)\,, \quad x \in \mathbb{R}_+\,,$$

for an unknown φ from $L^2(\mathbb{R}_+)$ where $f \in L^2(\mathbb{R}_+)$ is arbitrarily given, and $c \in \mathbb{C}$ and $k \in L^1(\mathbb{R})$ are fixed and known. Indeed, from these Wiener-Hopf equations arise the (classical) Wiener-Hopf operators defined by

$$W_\phi f(x) = cf(x) + \int_0^{+\infty} k(x-y)f(y)dy \ \ , \quad x \in \mathbb{R}_+, \tag{1.4.1}$$

where ϕ belongs to the Wiener algebra $\mathbb{W} := \{\phi \, : \, \phi = c + \mathcal{F}k,\, c \in \mathbb{C},\, k \in L^1(\mathbb{R})\}$ (which is a Banach algebra when endowed with the norm $\|c + \mathcal{F}k\|_{\mathbb{W}} := |c| + \|k\|_{L^1(\mathbb{R})}$ and the usual multiplication operation). Having in mind the convolution operation, the definition of W_ϕ in (1.4.1) gives rise to an understanding of the Wiener-Hopf operators as convolution type operators. Therefore, they can also be represented as

$$W_\phi = r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F} \ : \ L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+), \tag{1.4.2}$$

where r_+ is the restriction operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}_+)$. Here ϕ is a so-called Fourier symbol of the Wiener-Hopf operator (and from now on we will be briefly refereing to it as a symbol).

Looking now to the structure of the operators in (1.4.2), we recognize that the map

$$\mathcal{W} : \mathbb{W} \to \mathcal{L}(L^2_+(\mathbb{R}), L^2(\mathbb{R}_+)), \quad \phi \to W_\phi,$$

can be extended from $\mathbb{W}$ by continuity to larger algebras, and namely to $L^\infty(\mathbb{R})$.

Within the context of (1.4.1), the Hankel integral operators H have the form

$$Hf(x) = \int_0^{+\infty} k(x+y)f(y)dy \quad , \quad x \in \mathbb{R}_+ \tag{1.4.3}$$

(for some $k \in L^1(\mathbb{R})$). It is well-known that H, as an operator defined between $L^2(\mathbb{R})$ spaces, is a compact operator. However, exactly for the same reasons as above, it is also possible to provide a rigorous meaning to the expression (1.4.3) when the kernel k is a temperate distribution whose Fourier transform belongs to $L^\infty(\mathbb{R})$. Now we can rewrite the Hankel operator in another way using the convolution:

$$H_\phi = r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}J \; : \; L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+), \tag{1.4.4}$$

where J is a so-called reflection operator acting by the rule

$$(Jf)(x) = \widetilde{f}(x) = f(-x), \quad x \in \mathbb{R} \, .$$

Again we will refer to ϕ as the symbol of the Hankel operator.

Although during a long period of time the operators of type (1.4.1) and type (1.4.3) were studied separately, in the last years integral equations governed by algebraic sums of Wiener-Hopf and Hankel operators have been receiving increasing attention (cf. [11], [13], [22], [24], [29], [39], [45], [49], [50], [54], [60], [61], [62], [71]). A great part of the interest is directly originated by concrete mathematical-physics applications where Wiener-Hopf plus Hankel operators appear. This is the case of problems in wave diffraction phenomena which are modeled by boundary-transmission value problems that can be equivalently translated into systems of integral equations characterized by such kind of operators (see, e.g., [23], [25], [26]). Moreover, crack problems considered in the book [34] lead to equations with fixed singularities on finite intervals, which give rise to finite interval Hankel operators.

Considering now Φ to be a matrix function in the formulas (1.4.2) and (1.4.4), we will have that the Wiener-Hopf plus Hankel operator with symbol $\Phi \in [L^\infty(\mathbb{R})]^{N\times N}$ is given by the formula

$$W_\Phi + H_\Phi = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F}(I_{[L^2_+(\mathbb{R})]^N} + J) \; : \; [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N \, .$$

To denote the Wiener-Hopf plus Hankel operators with (matrix) symbol Φ we will use the notation WH_Φ.

As for the Wiener-Hopf minus Hankel operators we have the formula:

$$W_\Phi - H_\Phi = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F}(I_{[L^2_+(\mathbb{R})]^N} - J) \ : \ [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N \ .$$

Let us observe that $I_{[L^2_+(\mathbb{R})]^N} + J$ is an even extension operator acting from $[L^2(\mathbb{R}_+)]^N$ into $[L^2(\mathbb{R})]^N$ and $I_{[L^2_+(\mathbb{R})]^N} - J$ is an odd extension operator acting between the just mentioned spaces. We denote these operators by ℓ^e and ℓ^o, respectively. So, we can rewrite the Wiener-Hopf plus/minus Hankel operators with symbol Φ in the following form:

$$WH_\Phi = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F}\ell^e \ : \ [L^2(\mathbb{R}_+)]^N \to [L^2(\mathbb{R}_+)]^N \ , \tag{1.4.5}$$

and

$$W_\Phi - H_\Phi = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F}\ell^o \ : \ [L^2(\mathbb{R}_+)]^N \to [L^2(\mathbb{R}_+)]^N \ , \tag{1.4.6}$$

respectively. From the formulas (1.4.5) and (1.4.6) we find out that the Wiener-Hopf-Hankel operators are nothing but convolution type operators with symmetry (cf., e.g., [24], [55]).

We will need the formula (1.4.5) in an equivalent form, and it is given by

$$WH_\Phi = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F}\ell^e r_+ \ : \ [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N \ . \tag{1.4.7}$$

1.5 Toeplitz-Hankel operators

Toeplitz operators may be viewed as operators acting between $[H^2_+(\Gamma_0)]^N$ spaces, and which are closely related to the theory of Wiener-Hopf operators acting between Lebesgue spaces on the real line. We start by presenting some additional definitions which will be used to present the Toeplitz operators in a formal way.

Consider Cauchy singular integral operator $S_\mathbb{R}$ acting between $[L^2(\mathbb{R})]^N$ spaces by the formula:

$$(S_\mathbb{R} f)(t) = \frac{1}{\pi i}\int_\mathbb{R} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \mathbb{R} \ ,$$

where the integral is understood in the principal value sense. It is well-known that $S_{\mathbb{R}}^2 = I_{[L^2(\mathbb{R})]^N}$, and therefore it induces two complementary (Riesz) projections, namely:

$$P_{\mathbb{R}} := \frac{I + S_{\mathbb{R}}}{2}, \quad Q_{\mathbb{R}} := \frac{I - S_{\mathbb{R}}}{2} .$$

The Cauchy singular integral operator and the Riesz projections are defined in a similar way for the unit circle, and we will denote them by S_{Γ_0}, P_{Γ_0} and Q_{Γ_0}, respectively.

Consider the following image spaces: $P_{\Gamma_0}([L^2(\Gamma_0)]^N) =: [L^2_+(\Gamma_0)]^N$ and $Q_{\Gamma_0}([L^2(\Gamma_0)]^N) =: [L^2_-(\Gamma_0)]^N$. The Hardy spaces $[H^2_\pm(\Gamma_0)]^N$ can be isometrically identified with the above introduced spaces $[L^2_\pm(\Gamma_0)]^N$. In fact, the "plus" spaces coincide and for the "minus" spaces we have an equality: $[H^2_-(\Gamma_0)]^N = [L^2_-(\Gamma_0)]^N \oplus \mathbb{C}^N$. Here $\oplus$ stands for the direct sum of the spaces.

Let $\Phi \in [L^\infty(\Gamma_0)]^{N\times N}$. The Toeplitz operator acting between $[L^2_+(\Gamma_0)]^N$ spaces is given by

$$T_\Phi := P_{\Gamma_0}\Phi I : [L^2_+(\Gamma_0)]^N \to [L^2_+(\Gamma_0)]^N , \tag{1.5.1}$$

where Φ is called the symbol of the operator and I stands for the identity operator.

Let us also give the discrete analogue of the Toeplitz operators. Let $\Phi \in [L^\infty(\Gamma_0)]^{N\times N}$. For the Fourier coefficients of Φ, we will denote them by $\Phi_k \in \mathbb{C}^{N\times N}$, we have the formula:

$$\Phi_k = \frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{it})e^{-ikt}dt, \quad k \in \mathbb{Z} .$$

The discrete Toeplitz operators are acting between the spaces $\ell^2(\mathbb{Z}_+, \mathbb{C}^N)$, where this space is the $\mathbb{C}^N$-valued ℓ^2 space over $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. We would like to recall that the ℓ^2 space over $\mathbb{Z}_+$ is a space of all infinite sequences $\{z_k\}_{k=0}^\infty$ such that $\sum_{k=0}^\infty |z_k|^2 < \infty$. The operator induced by the matrix

$$\begin{bmatrix} \Phi_0 & \Phi_{-1} & \Phi_{-2} & \ldots \\ \Phi_1 & \Phi_0 & \Phi_{-1} & \ldots \\ \Phi_2 & \Phi_1 & \Phi_0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.5.2}$$

and acting between $\ell^2(\mathbb{Z}_+, \mathbb{C}^N)$ spaces is called the discrete Toeplitz operator with symbol Φ. If we look more carefully to the matrix (1.5.2) we see that the entries in the main diagonal and parallel to it are constant. Therefore we have a one-to-one correspondence between the discrete Toeplitz operators and matrices given by the formula (1.5.2). The origins of such a matrices goes back to the early work of Toeplitz [72] where he investigated finite matrices which are constant on diagonals and the relation of such kind of matrices with the corresponding one- and two-sided infinite limit matrices.

The studies of Toeplitz operators and Wiener-Hopf operators had parallel developments until Rosenblum [67] using certain polynomials proved that those operators are equivalent. A little bit later Devinatz [30] showed that the canonical conformal mapping of the unit disk onto the upper half-plane gives the equivalence between Toeplitz operators and the Fourier transform of Wiener-Hopf operators.

The analogue of the Hankel operator for the unit circle is defined by

$$H_\Phi = P_{\Gamma_0}\Phi J_{\Gamma_0} : [L^2_+(\Gamma_0)]^N \to [L^2_+(\Gamma_0)]^N \ ,$$

where J_{Γ_0} is a Carleman shift operator which acts by the rule:

$$(J_{\Gamma_0} f)(t) = \frac{1}{t} f\left(\frac{1}{t}\right), \quad t \in \Gamma_0.$$

As above we give the discrete analogue of the Hankel operators acting between $\ell^2(\mathbb{Z}_+, \mathbb{C}^N)$ spaces. Let $\Phi \in [L^\infty(\Gamma_0)]^{N\times N}$ and $\Phi_k \in \mathbb{C}^{N\times N}$ be the Fourier coefficients of Φ. Then the operator induced by the matrix

$$\begin{bmatrix} \Phi_0 & \Phi_1 & \Phi_2 & \dots \\ \Phi_1 & \Phi_2 & \Phi_3 & \dots \\ \Phi_2 & \Phi_3 & \Phi_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.5.3}$$

is called the discrete Hankel operator with symbol Φ. Contrary to the Toeplitz matrix here we have that the entries parallel to minor diagonal are constant and that the matrix (1.5.3) is an upside-down Toeplitz matrix. The finite Hankel matrix first appeared in the PhD thesis of Hankel [44].

The Toeplitz plus Hankel operator with symbol Φ will be denoted by TH_Φ, and has the form

$$TH_\Phi = P_{\Gamma_0}\Phi(I + J_{\Gamma_0}) : [L^2_+(\Gamma_0)]^N \to [L^2_+(\Gamma_0)]^N \ .$$

In an analogous way is defined the Toeplitz plus Hankel operator with symbol $\Phi \in [L^\infty(\mathbb{R})]^{N\times N}$ acting between the Hardy spaces $[H^2_+(\mathbb{R})]^N$ (cf., e.g., [14] or [18]). I.e.,

$$TH_\Phi = P_{\mathbb{R}}\Phi(I + J) : [H^2_+(\mathbb{R})]^N \to [H^2_+(\mathbb{R})]^N \ . \tag{1.5.4}$$

1.6 Basic formulas

In this section we give the basic formulas for the Wiener-Hopf-Hankel operators in view of the factorization theory of such an operators. Let ℓ_0 denote the zero extension operator from the space $[L^2(\mathbb{R}_+)]^N$ onto the space $[L^2_+(\mathbb{R})]^N$:

$$\ell_0 : [L^2(\mathbb{R}_+)]^N \to [L^2_+(\mathbb{R})]^N \ .$$

The canonical projection $\mathcal{P}_+$ acting from the space $[L^2(\mathbb{R})]^N$ onto the space $[L^2_+(\mathbb{R})]^N$ and also the complementary projection of it $\mathcal{P}_-$ acting from $[L^2(\mathbb{R})]^N$ onto $[L^2_-(\mathbb{R})]^N$ will be useful to obtain certain results. It is clear that the following equality holds:

$$\mathcal{P}_+ = \ell_0 r_+ : [L^2(\mathbb{R})]^N \to [L^2_+(\mathbb{R})]^N \ .$$

The next proposition gives two basic formulas from the theory of Wiener-Hopf-Hankel operators.

PROPOSITION 1.6.1. *(cf., e.g., [14]) Let* $\Phi, \Psi \in [L^\infty(\mathbb{R})]^{N\times N}$. *Then*

$$\begin{aligned} W_{\Phi\Psi} &= W_\Phi \ell_0 W_\Psi + H_\Phi \ell_0 H_{\widetilde{\Psi}} \ , \qquad (1.6.1) \\ H_{\Phi\Psi} &= W_\Phi \ell_0 H_\Psi + H_\Phi \ell_0 W_{\widetilde{\Psi}} \ . \end{aligned}$$

Proof. Observe that

$$I_{[L^2(\mathbb{R})]^N} = \mathcal{P}_+ + \mathcal{P}_- \ .$$

Moreover, by definition $J^2 = I$ and $\mathcal{P}_\pm^2 = \mathcal{P}_\pm$. Hence we have:

$$I_{[L^2(\mathbb{R})]^N} = \mathcal{P}_+ + \mathcal{P}_- JJ\mathcal{P}_- \ .$$

Taking into account that

$$\mathcal{P}_- J = J\mathcal{P}_+, \quad J\mathcal{P}_- = \mathcal{P}_+ J \ ,$$

we have that

$$I_{[L^2(\mathbb{R})]^N} = \mathcal{P}_+ + J\mathcal{P}_+\mathcal{P}_+ J = \mathcal{P}_+ + J\mathcal{P}_+ J \ .$$

Now the direct computation provides that

$$\begin{aligned}
W_{\Phi\Psi} &= r_+\mathcal{F}^{-1}(\Phi\Psi)\cdot\mathcal{F} = r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}\mathcal{F}^{-1}\Psi\cdot\mathcal{F} \\
&= r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}(\mathcal{P}_+ + J\mathcal{P}_+J)\mathcal{F}^{-1}\Psi\cdot\mathcal{F} \\
&= r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}\mathcal{P}_+\mathcal{F}^{-1}\Psi\cdot\mathcal{F} + r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}J\mathcal{P}_+J\mathcal{F}^{-1}\Psi\cdot\mathcal{F} \\
&= r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}\ell_0 r_+\mathcal{F}^{-1}\Psi\cdot\mathcal{F} + r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}J\ell_0 r_+\mathcal{F}^{-1}\widetilde{\Psi}\cdot\mathcal{F}J \\
&= W_\Phi\ell_0 W_\Psi + H_\Phi\ell_0 H_{\widetilde{\Psi}} \ , \qquad (1.6.2)
\end{aligned}$$

and also

$$\begin{aligned}
H_{\Phi\Psi} &= r_+\mathcal{F}^{-1}(\Phi\Psi)\cdot\mathcal{F}J = r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}\mathcal{F}^{-1}\Psi\cdot\mathcal{F}J \\
&= r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}(\mathcal{P}_+ + J\mathcal{P}_+J)\mathcal{F}^{-1}\Psi\cdot\mathcal{F}J \\
&= r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}\mathcal{P}_+\mathcal{F}^{-1}\Psi\cdot\mathcal{F}J + r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}J\mathcal{P}_+J\mathcal{F}^{-1}\Psi\cdot\mathcal{F}J \\
&= r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}\ell_0 r_+\mathcal{F}^{-1}\Psi\cdot\mathcal{F}J + r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}J\ell_0 r_+\mathcal{F}^{-1}\widetilde{\Psi}\cdot\mathcal{F} \\
&= W_\Phi\ell_0 H_\Psi + H_\Phi\ell_0 W_{\widetilde{\Psi}} \ , \qquad (1.6.3)
\end{aligned}$$

□

Adding the equalities (1.6.2) and (1.6.3) one obtains:

$$WH_{\Phi\Psi} = W_\Phi\ell_0 WH_\Psi + H_\Phi\ell_0 WH_{\widetilde{\Psi}} \ . \qquad (1.6.4)$$

From the last formula it directly follows the following equality:

$$WH_{\Phi\Psi} = WH_{\Phi}\ell_0 WH_{\Psi} + H_{\Phi}\ell_0 WH_{\widetilde{\Psi}-\Psi} \,. \tag{1.6.5}$$

Indeed from (1.6.4) we have that

$$\begin{aligned} WH_{\Phi\Psi} &= W_{\Phi}\ell_0 WH_{\Psi} + H_{\Phi}\ell_0 WH_{\Psi} + H_{\Phi}\ell_0 WH_{\widetilde{\Psi}} - H_{\Phi}\ell_0 WH_{\Psi} \\ &= WH_{\Phi}\ell_0 WH_{\Psi} + H_{\Phi}\ell_0 WH_{\widetilde{\Psi}-\Psi} \,. \end{aligned}$$

We will now deduce two formulas which will be useful for our future goals.

PROPOSITION 1.6.2. *(cf., e.g., [14])*

1. *If* $\Phi \in [H^{\infty}_{-}(\mathbb{R})]^{N\times N}$, *then* $H_{\Phi} = 0$
2. *If* $\Phi \in [H^{\infty}_{+}(\mathbb{R})]^{N\times N}$, *then* $H_{\widetilde{\Phi}} = 0$

Proof. We will prove only the first part of the proposition, since the second part goes analogously to the first part. Let us recall that $H_{\Phi} = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F}J \;:\; [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N$. So, assume that $\Phi \in [H^{\infty}_{-}(\mathbb{R})]^{N\times N}$, and take a function $f \in [L^2_+(\mathbb{R})]^N$. Then $Jf \in [L^2_-(\mathbb{R})]^N$ and using the Paley-Wiener formula (cf. (1.1.1)) we conclude that $\mathcal{F}Jf \in [H^2_-(\mathbb{R})]^N$. The multiplication by $[H^{\infty}_{-}(\mathbb{R})]^{N\times N}$ functions leaves the space $[H^2_-(\mathbb{R})]^N$ invariant, and we get that $\Phi\mathcal{F}Jf \in [H^2_-(\mathbb{R})]^N$. Acting with the inverse of the Fourier transformation and using once again the Paley-Wiener formula one obtains that $\mathcal{F}^{-1}\Phi\mathcal{F}Jf \in [L^2_-(\mathbb{R})]^N$. We are left to observe that the restriction operator gives now the zero function on the space $[L^2(\mathbb{R}_+)]^N$. Hence $H_{\Phi} = 0$. $\square$

The last proposition gives hint how to construct a factorization of matrix functions which allows to factorize corresponding Wiener-Hopf operators.

THEOREM 1.6.3. *(cf., e.g., [14]) Let* $\Phi_{\pm} \in [H^{\infty}_{\pm}(\mathbb{R})]^{N\times N}$ *and* $\Psi \in [L^{\infty}(\mathbb{R})]^{N\times N}$. *We have the following factorization*

$$W_{\Phi_-\Psi\Phi_+} = W_{\Phi_-}\ell_0 W_{\Psi}\ell_0 W_{\Phi_+} \,.$$

Proof. Follows directly from formula (1.6.2) and the previous proposition. □

The importance of the last theorem is that the Wiener-Hopf operator with symbol $\Phi := \Phi_-\Psi\Phi_+$, where $\Phi_\pm \in \mathcal{G}[H^\infty_\pm(\mathbb{R})]^{N\times N}$ is equivalent with the Wiener-Hopf operator with symbol Ψ, due to the reason that $W_{\Phi_\pm}$ are invertible operators, inverse being $\ell_0 W_{\Phi_\pm^{-1}}\ell_0$, since $\ell_0 : [L^2(\mathbb{R}_+)]^N \to [L^2_+(\mathbb{R})]^N$ is also an invertible operator with inverse being $r_+ : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N$. Our aim is to obtain a similar factorization formula for the Wiener-Hopf-Hankel operators. Looking to the formula (1.6.5) we understand that we will have the above mentioned factorization if we have $\Phi = \Phi_-\Phi_e$, where $\Phi_- \in [H^\infty_-(\mathbb{R})]^{N\times N}$ and $\Phi_e \in [L^\infty(\mathbb{R})]^{N\times N}$ is an even function, i.e., $\Phi_e = \widetilde{\Phi_e}$. Consequently, if $\Phi_- \in [H^\infty_-(\mathbb{R})]^{N\times N}$ or $\Phi_e \in [L^\infty(\mathbb{R})]^{N\times N}$ and $\Phi_e = \widetilde{\Phi_e}$ we have a formula:

$$WH_{\Phi_-\Phi_e} = WH_{\Phi_-}\ell_0 WH_{\Phi_e}\ .$$

THEOREM 1.6.4. *(cf., e.g., [59]) Let $\Phi, \Psi, \Theta \in [L^\infty(\mathbb{R})]^{N\times N}$. If $\Phi \in [H^\infty_-(\mathbb{R})]^{N\times N}$ and $\Theta = \widetilde{\Theta}$, then we have the factorization:*

$$WH_{\Phi\Psi\Theta} = W_\Phi \ell_0 WH_\Psi \ell_0 WH_\Theta\ . \tag{1.6.6}$$

Proof. Using formula (1.6.5) and recalling that $H_\Phi = 0$ if $\Phi \in [H^\infty_-(\mathbb{R})]^{N\times N}$ we will have

$$WH_{\Phi\Psi\Theta} = WH_\Phi \ell_0 WH_{\Psi\Theta} = W_\Phi \ell_0 WH_{\Psi\Theta}\ .$$

Now relaying on the fact that Θ is an even function and once again using the formula (1.6.5) we will have that

$$WH_{\Psi\Theta} = WH_\Psi \ell_0 WH_\Theta\ .$$

Combining the last two equalities gives us the assertion:

$$WH_{\Phi\Psi\Theta} = W_\Phi \ell_0 WH_\Psi \ell_0 WH_\Theta\ .$$

□

Altogether we are ready to totally describe the Wiener-Hopf plus Hankel operators with even symbols.

THEOREM 1.6.5. *(cf., e.g., [38]) Let $\Phi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ and Φ_e is even. Then WH_{Φ_e} is invertible with inverse being the following operator:*

$$\ell_0 WH_{\Phi_e^{-1}} \ell_0 : [L^2(\mathbb{R}_+)]^N \to [L^2_+(\mathbb{R})]^N \ .$$

Proof. Clearly we have that

$$WH_{\Phi_e\Phi_e^{-1}}\ell_0 = WH_{I_{N\times N}}\ell_0 = r_+\ell_0 = I_{[L^2_+(\mathbb{R})]^N} \ .$$

Using the fact that inverse of even function is itself even, we obtain from (1.6.5) that

$$WH_{\Phi_e\Phi_e^{-1}} = WH_{\Phi_e}\ell_0 WH_{\Phi_e^{-1}} \ .$$

Multiplying both sides of the last equality by the invertible operator ℓ_0 one obtains:

$$WH_{\Phi_e\Phi_e^{-1}}\ell_0 = WH_{\Phi_e}\ell_0 WH_{\Phi_e^{-1}}\ell_0 = I_{[L^2_+(\mathbb{R})]^N} \ .$$

Similarly we will have that

$$\ell_0 WH_{\Phi_e^{-1}}\ell_0 WH_{\Phi_e} = I_{[L^2_+(\mathbb{R})]^N} \ .$$

Hence we have explicitly shown that WH_{Φ_e} is invertible and the inverse is given by the formula:

$$\ell_0 WH_{\Phi_e^{-1}} \ell_0 : [L^2(\mathbb{R}_+)]^N \to [L^2_+(\mathbb{R})]^N \ .$$

$\square$

Let us emphasize that similar (invertibility) results hold for Wiener-Hopf minus Hankel operators, and also for Toeplitz plus/minus Hankel operators. In the formulas for Toeplitz-Hankel operators (on the unit circle) the function $\widetilde{f}$ must be understood as the function $f(t^{-1})$. For example, the analogue of the formula (1.6.5) will have a form:

$$TH_{\Phi\Psi} = TH_\Phi TH_\Psi + H_\Phi TH_{\widetilde{\Psi}-\Psi} \ ,$$

where $\widetilde{\Psi}(t) = \Psi(t^{-1})$.

1.7 Relation between convolution type operators

In this section we are going to deduce some equivalence relations between Wiener-Hopf-Hankel and pure Wiener-Hopf operators using the Δ-relation after extension, recalled in Section 1.3. Let us start with the Gohberg-Krupnik-Litvinchuk identity, which has the form:

$$\frac{1}{2}\begin{bmatrix} I & I \\ J & -J \end{bmatrix}\begin{bmatrix} T+SJ & 0 \\ 0 & T-SJ \end{bmatrix}\begin{bmatrix} I & J \\ I & -J \end{bmatrix} = \begin{bmatrix} T & S \\ JSJ & JTJ \end{bmatrix},$$

where $T, S \in \mathcal{L}(L^2(\mathbb{R}))$.

Employing the above identity and the methods presented in the paper [20] we can state that the following equality holds:

$$(\mathfrak{D}_\Phi :=)\begin{bmatrix} WH_\Phi & 0 \\ 0 & W_\Phi - H_\Phi \end{bmatrix} = E\begin{bmatrix} W_{\Phi\widetilde{\Phi}^{-1}} & 0 \\ 0 & I_{[L^2(\mathbb{R}_+)]^{N\times N}} \end{bmatrix} F\,, \tag{1.7.1}$$

where $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ and E and F are certain linear bounded invertible operators, which can be computed explicitly (cf., e.g., [20], [58]).

We list the most important consequences of the last identity:

(i) If $W_{\Phi\widetilde{\Phi}^{-1}}$ is two-sided invertible, one-sided invertible, Fredholm, (properly) n-normal, (properly) d-normal or normally solvable, then the same happens with WH_Φ and $W_\Phi - H_\Phi$.

(ii) $\dim \mathrm{Ker} W_{\Phi\widetilde{\Phi}^{-1}} = \dim \mathrm{Ker} WH_\Phi + \dim \mathrm{Ker}(W_\Phi - H_\Phi)$.

(iii) $\dim \mathrm{Coker} W_{\Phi\widetilde{\Phi}^{-1}} = \dim \mathrm{Coker} WH_\Phi + \dim \mathrm{Coker}(W_\Phi - H_\Phi)$.

(iv) $\mathrm{Ind} W_{\Phi\widetilde{\Phi}^{-1}} = \mathrm{Ind} WH_\Phi + \mathrm{Ind}(W_\Phi - H_\Phi)$.

For the Toeplitz-Hankel operators (on the unit circle) a similar identity as (1.7.1) takes place (with $\widetilde{\Phi}(t) = \Phi(t^{-1})$) and therefore the propositions (i)–(iv) are still in force for these operators.

We stress that it is possible that WH_Φ (TH_Φ) enjoys some regularity property, but $W_{\Phi\widetilde{\Phi}^{-1}}$ ($T_{\Phi\widetilde{\Phi}^{-1}}$) does not have the same property. This happens because of the influence of $W_\Phi - H_\Phi$ ($T_\Phi - H_\Phi$) operator. It is a known fact (cf. [38]) that Wiener-Hopf (Toeplitz) plus Hankel and Wiener-Hopf (Toeplitz) minus Hankel operators with the same symbol may have very different properties.

We would like also to emphasize that from the formula (1.7.1) it directly follows that $\mathfrak{D}_\Phi$ and $W_{\Phi\widetilde{\Phi}^{-1}}$ are equivalent after extension operators.

The next proposition reveals an equivalence between Toeplitz plus Hankel operators and certain singular integral operators (cf., e.g., [1] or [38, Theorem 3.2]).

PROPOSITION 1.7.1. *Let $\Phi \in [L^\infty(\Gamma_0)]^{N\times N}$. The following operators are equivalent after extension:*

(i) TH_Φ acting between $[H^2_+(\Gamma_0)]^N$,

(ii) $\Phi(I + J_{\Gamma_0})P_{\Gamma_0} + Q_{\Gamma_0}$ acting between $[L^2(\Gamma_0)]^N$.

Proof. Recalling the definition of TH_Φ we recognize that TH_Φ acting between $[H^2_+(\Gamma_0)]^N$ spaces is equivalent after extension with $P_{\Gamma_0}\Phi(I+J_{\Gamma_0})P_{\Gamma_0}+Q_{\Gamma_0}$ acting between $[L^2(\Gamma_0)]^N$ spaces. Directly we have that

$$P_{\Gamma_0}\Phi(I + J_{\Gamma_0})P_{\Gamma_0} + Q_{\Gamma_0} = \Big(\Phi(I + J_{\Gamma_0})P_{\Gamma_0} + Q_{\Gamma_0}\Big)\Big(I - Q_{\Gamma_0}\Phi P_{\Gamma_0} - Q_{\Gamma_0}\Phi J_{\Gamma_0}P_{\Gamma_0}\Big). \tag{1.7.2}$$

We are left to observe that $I - Q_{\Gamma_0}\Phi P_{\Gamma_0} - Q_{\Gamma_0}\Phi J_{\Gamma_0}P_{\Gamma_0}$ is an invertible operator with inverse being $I + Q_{\Gamma_0}\Phi P_{\Gamma_0} + Q_{\Gamma_0}\Phi J_{\Gamma_0}P_{\Gamma_0}$. □

REMARK 1.7.2. From the proof presented in the previous proposition it is clear that if we consider TH_Φ acting between the spaces $[H^2_+(\mathbb{R})]^N$, then this operator will be equivalent after extension with $\Phi(I + J)P_\mathbb{R} + Q_\mathbb{R}$ acting between $[L^2(\mathbb{R})]^N$ spaces.

COROLLARY 1.7.3. *TH_Φ acting between $[H^2_+(\Gamma_0)]^N$ (resp. $[H^2_+(\mathbb{R})]^N$) spaces and $\Phi(I + J_{\Gamma_0})P_{\Gamma_0} + Q_{\Gamma_0}$ (resp. $\Phi(I + J)P_\mathbb{R} + Q_\mathbb{R}$) acting between $[L^2(\Gamma_0)]^N$ (resp. $[L^2(\mathbb{R})]^N$) spaces have the same regularity properties.*

Proof. Immediately follows from the previous proposition and the remark cited after this. □

1.8 Certain equivalence relations

In this section we describe the equivalence relation between Wiener-Hopf-Hankel and Toeplitz-Hankel operators. It turns out that Wiener-Hopf plus Hankel operators are equivalent with Toeplitz minus Hankel operators, and Wiener-Hopf minus Hankel operators are equivalent with Toeplitz plus Hankel operators. We will give full proofs of these facts.

First of all we will describe the auxiliary operators which will be needed to obtain that equivalences. Denote by $L(\Phi)$ the multiplication operator by Φ. Hence we can write Toeplitz and Hankel operators acting between $[H^2_+(\Gamma_0)]^N$ spaces in the following way:

$$\begin{aligned} T_\Phi &= P_{\Gamma_0} L(\Phi) P_{\Gamma_0} : [H^2_+(\Gamma_0)]^N \to [H^2_+(\Gamma_0)]^N \ , \\ H_\Phi &= P_{\Gamma_0} L(\Phi) J_{\Gamma_0} P_{\Gamma_0} : [H^2_+(\Gamma_0)]^N \to [H^2_+(\Gamma_0)]^N \ , \end{aligned}$$

and for the Toeplitz plus/minus Hankel operators we have the following:

$$T_\Phi \pm H_\Phi = P_{\Gamma_0} L(\Phi)(I \pm J_{\Gamma_0}) P_{\Gamma_0} : [H^2_+(\Gamma_0)]^N \to [H^2_+(\Gamma_0)]^N \ .$$

Let us consider the useful operator B_0 given by

$$(B_0\Phi)(t) = \Phi\left(i\frac{1+t}{1-t}\right) \ , \quad t \in \Gamma_0 \ . \tag{1.8.1}$$

Obviously $B_0 : [L^\infty(\mathbb{R})]^{N\times N} \to [L^\infty(\Gamma_0)]^{N\times N}$ is an isometrical isomorphism, the inverse of which is given by the following formula:

$$(B_0^{-1}\Psi)(x) = \Psi\left(\frac{x-i}{x+i}\right) \ , \qquad x \in \mathbb{R} \ .$$

In addition, the operator B given by

$$(B\Phi)(x) = \frac{\sqrt{2}}{x+i}\Phi\left(\frac{x-i}{x+i}\right) \ , \qquad x \in \mathbb{R} \ ,$$

is an isometrical isomorphism of $[L^2(\Gamma_0)]^{N\times N}$ onto $[L^2(\mathbb{R})]^{N\times N}$, of $[H^2_+(\Gamma_0)]^{N\times N}$ onto $[H^2_+(\mathbb{R})]^{N\times N}$, and of $t^{-1}[H^2_-(\Gamma_0)]^{N\times N}$ onto $[H^2_-(\mathbb{R})]^{N\times N}$. For the inverse of B we have:

$$(B^{-1}\Psi)(t) = \frac{i\sqrt{2}}{1-t}\Psi\left(i\frac{1+t}{1-t}\right) \ , \qquad t \in \Gamma_0 \ .$$

By using the "convolution" with B operators it is obtained the formula:

$$BL(\Phi)B^{-1} = (B_0^{-1}\Phi)I \ . \tag{1.8.2}$$

A straightforward computation shows that

$$S_{\mathbb{R}} = BS_{\Gamma_0}B^{-1}, \quad P_{\mathbb{R}} = BP_{\Gamma_0}B^{-1}, \quad Q_{\mathbb{R}} = BQ_{\Gamma_0}B^{-1} \ . \tag{1.8.3}$$

The following formula is also of interest:

$$\mathcal{F}^{-1}P_{\mathbb{R}}\mathcal{F} = \ell_0 r_+ = \mathcal{P}_+ : L^2(\mathbb{R}) \to L^2_+(\mathbb{R}) \ . \tag{1.8.4}$$

THEOREM 1.8.1. *Let* $\Phi \in [L^\infty(\mathbb{R})]^{N\times N}$. *Then*

$$W_\Phi \pm H_\Phi : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N$$

is equivalent to

$$T_{(B_0\Phi)} \mp H_{(B_0\Phi)} : [H^2_+(\Gamma_0)]^N \to [H^2_+(\Gamma_0)]^N \ .$$

Proof. Indeed, consider the operator

$$\mathcal{F}^{-1}B[T_{(B_0\Phi)} \mp H_{(B_0\Phi)}]B^{-1}\mathcal{F} : [L^2_+(\mathbb{R})]^N \to [L^2_+(\mathbb{R})]^N \ .$$

then a straightforward computation leads to

$$\begin{aligned}
\mathcal{F}^{-1}B[T_{(B_0\Phi)} \ &\mp \ H_{(B_0\Phi)}]B^{-1}\mathcal{F} = \mathcal{F}^{-1}B[P_{\Gamma_0}L(B_0\Phi)(I \mp J_{\Gamma_0})P_{\Gamma_0}]B^{-1}\mathcal{F} \\
&= \ \underbrace{\mathcal{F}^{-1}BP_{\Gamma_0}B^{-1}\mathcal{F}}_{\ell_0 r_+}\mathcal{F}^{-1}\underbrace{BL(B_0\Phi)B^{-1}}_{L(\Phi)}\mathcal{F}\underbrace{\mathcal{F}^{-1}B(I \mp J_{\Gamma_0})B^{-1}\mathcal{F}}_{I\pm J} \\
&\quad\ \underbrace{\mathcal{F}^{-1}BP_{\Gamma_0}B^{-1}\mathcal{F}}_{\ell_0 r_+} \\
&= \ \ell_0 r_+ \mathcal{F}^{-1}\Phi \cdot \mathcal{F}(I \pm J)\ell_0 r_+ : [L^2_+(\mathbb{R})]^N \to [L^2_+(\mathbb{R})]^N \ .
\end{aligned} \tag{1.8.5}$$

We notice that in the last identities the formulas (1.8.2), (1.8.3) and (1.8.4) were used. In addition, it is clear that we can drop the last $\ell_0 r_+$ operator in (1.8.5) since this is just the identity operator in $[L^2_+(\mathbb{R})]^N$. So ,we have that

$$\mathcal{F}^{-1}B[T_{(B_0\Phi)} \mp H_{(B_0\Phi)}]B^{-1}\mathcal{F} = \ell_0[W_\Phi \pm H_\Phi] \ ,$$

thus we can conclude that $W_\Phi \pm H_\Phi$ is equivalent with $T_{(B_0\Phi)} \mp H_{(B_0\Phi)}$. ∎

From the proof given above it is now clear why the Wiener-Hopf plus Hankel operators are equivalent with the Toeplitz minus Hankel operators and the Wiener-Hopf minus Hankel operators with the Toeplitz plus Hankel operators: the main reason is that in the present situation the operator J is equivalent with $-J_{\Gamma_0}$.

COROLLARY 1.8.2. *Let $\Phi \in [L^\infty(\mathbb{R})]^{N\times N}$. Then $W_\Phi \pm H_\Phi$ and $T_{B_0\Phi} \mp H_{B_0\Phi}$ have the same regularity properties.*

Proof. The proof is a direct consequence of the last theorem. □

1.9 Necessary conditions for the semi-Fredholm property

In this section we show that if the Wiener-Hopf plus Hankel operator is semi-Fredholm then its symbol necessarily needs to be invertible. Although the proof of this result appears in several sources (cf., e.g., [38, Proposition 2.6], [58, Theorem 1.4.1]) for the readers convenience we will reproduce here the full proof.

REMARK 1.9.1. In the proof of the next theorem we will make use of the following known fact about n-normal operators (cf., e.g., [52, Lemma 2.1] or [57, Lemma 2.1]): If (for a Banach space X) $T \in \mathcal{L}(X)$ is an n-normal operator, then there exists a compact operator $K \in \mathcal{K}(X)$ (which can be chosen as a projection onto the space Ker$T$) and a number $\delta > 0$ such that

$$||Tx|| + ||Kx|| \geq \delta ||x||, \quad x \in X.$$

THEOREM 1.9.2. *Let $\Phi \in [L^\infty(\mathbb{R})]^{N\times N}$. If WH_Φ is semi-Fredholm operator, then $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$.*

Proof. Assume that WH_Φ is an n-normal operator. From the equivalence relation (cf. Section 1.8) we obtain that $T_{B_0\Phi} - H_{B_0\Phi}$ is also an n-normal operator. From the previous remark we know that there exist a compact operator K which is a projection onto the $\mathrm{Ker}(T_{B_0\Phi} - H_{B_0\Phi})$ and a positive number $\delta > 0$ such that

$$||(T_{B_0\Phi} - H_{B_0\Phi})f||_{[H^2_+(\Gamma_0)]^N} + ||Kf||_{[H^2_+(\Gamma_0)]^N} \geq \delta||f||_{[H^2_+(\Gamma_0)]^N}$$

for all $f \in [H^2_+(\Gamma_0)]^N$. Making the substitution of $P_{\Gamma_0}f$ instead of f allows us to obtain the following inequality:

$$\begin{aligned}\delta||f||_{[L^2(\Gamma_0)]^N} - \delta||(I_{[L^2(\Gamma_0)]^N} - P_{\Gamma_0})f||_{[L^2(\Gamma_0)]^N} \quad \leq \quad & ||(T_{B_0\Phi} - H_{B_0\Phi})f||_{[L^2(\Gamma_0)]^N} \\ & + ||KP_{\Gamma_0}f||_{[L^2(\Gamma_0)]^N}\end{aligned}$$

for all $f \in [L^2(\Gamma_0)]^N$. Let us now introduce the isometries

$$U^n : [L^2(\Gamma_0)]^N \to [L^2(\Gamma_0)]^N, \quad U^n f(t) = t^n f(t) \; ,$$

where $n \in \mathbb{Z}$. Replacing f by $U^n f$ in the last inequality one obtains:

$$\begin{aligned}||(T_{B_0\Phi} - H_{B_0\Phi})U^n f||_{[L^2(\Gamma_0)]^N} \quad + \quad & ||KP_{\Gamma_0}U^n f||_{[L^2(\Gamma_0)]^N} \\ + \quad & \delta||(I_{[L^2(\Gamma_0)]^N} - P_{\Gamma_0})U^n f||_{[L^2(\Gamma_0)]^N} \geq \delta||U^n f||_{[L^2(\Gamma_0)]^N}\end{aligned}$$

for all $f \in [L^2(\Gamma_0)]^N$. Having in mind that $U^{\pm n}$ are isometries, it follows that

$$\begin{aligned}& ||U^{-n}(T_{B_0\Phi} - H_{B_0\Phi})U^n f||_{[L^2(\Gamma_0)]^N} + ||KP_{\Gamma_0}U^n f||_{[L^2(\Gamma_0)]^N} \\ & + \delta||U^{-n}(I_{[L^2(\Gamma_0)]^N} - P_{\Gamma_0})U^n f||_{[L^2(\Gamma_0)]^N} \geq \delta||f||_{[L^2(\Gamma_0)]^N}\end{aligned} \tag{1.9.1}$$

for all $f \in [L^2(\Gamma_0)]^N$. Since $U^n \to 0$ weakly when $n \to \infty$ on $[L^2(\Gamma_0)]^N$ and using the fact that K is a compact operator, we have

$$KP_{\Gamma_0}U^n \longrightarrow 0 \text{ strongly on } [L^2(\Gamma_0)]^N, \text{when } n \to \infty \; . \tag{1.9.2}$$

Let us consider the dense subset $\mathcal{P}$ of $L^2(\Gamma_0)$ of all trigonometric polynomials

$$\sum_{k=-n}^{n} f_k t^k \quad t \in \Gamma_0 \; .$$

Due to the reason that P_{Γ_0} acts on $\mathcal{P}$ in the following way:

$$P_{\Gamma_0} : \sum_{k=-n}^{n} f_k t^k \to \sum_{k=0}^{n} f_k t^k \; ,$$

we obtain that $U^{-n}P_{\Gamma_0}U^n f$ converges in $L^2(\Gamma_0)$ to f, for all $f \in \mathcal{P}$. Thus by the continuity we can guaranty that

$$U^{-n}P_{\Gamma_0}U^n \to I_{[L^2(\Gamma_0)]^N} \text{ strongly on } [L^2(\Gamma_0)]^N, \text{when } n \to \infty \ . \tag{1.9.3}$$

In a similar way we will have that

$$U^{-n}P_{\Gamma_0}U^{-n} \to 0 \text{ strongly on } [L^2(\Gamma_0)]^N, \text{when } n \to \infty \ . \tag{1.9.4}$$

Recalling the identities $U^n U^{-n} = I_{[L^2(\Gamma_0)]^N}$ and $J_{\Gamma_0}U^n = U^{-n}J_{\Gamma_0}$ provides us to the equalities:

$$\begin{aligned} U^{-n}T_{B_0\Phi}U^n &= (U^{-n}P_{\Gamma_0}U^n)(B_0\Phi)(U^{-n}P_{\Gamma_0}U^n) \ , \\ U^{-n}H_{B_0\Phi}U^n &= (U^{-n}P_{\Gamma_0}U^n)(B_0\Phi)J_{\Gamma_0}(U^{-n}P_{\Gamma_0}U^n) \ . \end{aligned}$$

From (1.9.3) and (1.9.4), it follows that

$$\begin{aligned} U^{-n}T_{B_0\Phi}U^n &\to (B_0\Phi)I_{[L^2(\Gamma_0)]^N} \text{ strongly on } [L^2(\Gamma_0)]^N, \text{ when } n \to \infty, \\ U^{-n}H_{B_0\Phi}U^n &\to 0 \text{ strongly on } [L^2(\Gamma_0)]^N, \text{ when } n \to \infty. \end{aligned}$$

Hence

$$U^{-n}(T_{B_0\Phi} - H_{B_0\Phi})U^n \to (B_0\Phi)I_{[L^2(\Gamma_0)]^N} \text{ strongly on } L^2(\Gamma_0), \text{ when } n \to \infty. \tag{1.9.5}$$

Taking the limit in (1.9.1) when $n \to \infty$ and using (1.9.2), (1.9.3) and (1.9.5), we obtain:

$$||(B_0\Phi)f||_{[L^2(\Gamma_0)]^N} \geq \delta ||f||_{[L^2(\Gamma_0)]^N} \ ,$$

for every $f \in [L^2(\Gamma_0)]^N$. Thus $B_0\Phi \in \mathcal{G}[L^\infty(\Gamma_0)]^{N\times N}$, which means that $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$. The d-normal case follows from the n-normal case by passage to adjoint operators. □

Analogous results hold true for $W_\Phi - H_\Phi$ operators:

THEOREM 1.9.3. *Let $\Phi \in [L^\infty(\mathbb{R})]^{N\times N}$. If $W_\Phi - H_\Phi : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N$ is a semi-Fredholm operator then $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$.*

REMARK 1.9.4. From the previous two theorems it immediately follows that if $\Phi \notin \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$, then $W_\Phi \pm H_\Phi : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N$ are not semi-Fredholm operators.

Chapter 2

The Fourier symbols and the Besicovitch space

In this chapter we give the definitions of certain classes of Fourier symbols of the Wiener-Hopf-Hankel and Toeplitz-Hankel operators, for which we develop the Fredholm theory in the subsequent chapters. We start with complex-valued functions defined on the real line, which is the C^*-algebra of almost periodic functions (AP). Then we introduce semi-almost periodic functions (SAP), and moreover piecewise almost periodic functions (PAP) will be also discussed.

Further, the notion of unitary and sectorial symbols on the real line will be given. After this we turn to the unit circle and give the definition and the important properties of the functions having n points of standard almost periodic discontinuities.

In the end of this chapter we consider the Besicovitch space, which plays an important role in the development of the Fredholm theory for the Wiener-Hopf-Hankel operators with SAP and PAP Fourier symbols.

2.1 Almost periodic functions

The algebra of almost periodic functions was created by Harald Bohr (brother of the famous physician Niels Bohr) in the 1920s. His interest was initially in finite Dirichlet series. He was considering the following elements:

$$n^\sigma e^{(\log n)it} .$$

Taking a finite sum of such terms avoids difficulties of analytic continuation to the region $\sigma < 1$. Further, Mathematical Analysis was applied to discuss the closure of this set of basic functions, in various norms. Bohr himself defined the uniformly almost periodic functions as the closure with respect to the supremum norm in $L^\infty(\mathbb{R})$. He proved that this definition was equivalent to the existence of ε almost periods, for all $\varepsilon > 0$, which means that there are translations $T(\varepsilon) = T$ of the variable t such that

$$|f(t+T) - f(T)| < \varepsilon .$$

The theory was developed using other norms by Besicovitch (to be defined as the Besicovitch space), Stepanov, Weyl, von Neumann, Turing, Bochner and others in the 1920s and 1930s.

Let us also cite the definition by Bochner (1926): A function f is almost periodic if every sequence $\{f(t_n+T)\}$ of translations of f has a subsequence that converges uniformly to f for T in $(-\infty, +\infty)$.

Now, for our purposes we will define the class AP of almost periodic functions in the way that was given by Bohr. A function α of the form

$$\alpha(x) := \sum_{j=1}^{n} c_j \exp(i\lambda_j x) , \qquad x \in \mathbb{R} ,$$

where $\lambda_j \in \mathbb{R}$ and $c_j \in \mathbb{C}$, is called an almost periodic polynomial. If we construct the closure of the set of all almost periodic polynomials by using the supremum norm, we will then obtain the AP class of almost periodic functions.

THEOREM 2.1.1. *(Bohr) Suppose that $\varphi \in AP$ and*

$$\inf_{x\in\mathbb{R}} |\varphi(x)| > 0 . \tag{2.1.1}$$

Then the function $\arg\varphi(x)$ *can be defined so that*

$$\arg\varphi(x) = \lambda x + \psi(x) ,$$

where $\lambda \in \mathbb{R}$ and $\psi \in AP$.

DEFINITION 2.1.2. *(Bohr mean motion) Let $\varphi \in AP$ and let the condition (2.1.1) be satisfied. The Bohr mean motion of the function φ is defined to be the following real number*

$$k(\varphi) := \lim_{\ell\to\infty} \frac{1}{2\ell} \arg \varphi(x)|_{-\ell}^{\ell}\,.$$

The following subclasses of AP are also of interest:

$$AP_+ := \mathrm{alg}_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \geq 0\}, \quad AP_- := \mathrm{alg}_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \leq 0\}\,,$$

where $e_\lambda := e^{i\lambda x}$. In fact, one of the reasons why the last two algebras are very useful is due to the fact that $AP_\pm = AP \cap H^\infty_\pm(\mathbb{R})$ (cf. [14, Corollary 7.7]). The almost periodic functions have a great amount of important and well-known properties. Among them, for our purposes the following ones are the most relevant.

PROPOSITION 2.1.3. *(cf., e.g., [14]) Let $A \subset (0,\infty)$ be an unbounded set and let*

$$\{I_\alpha\}_{\alpha\in A} := \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$$

be a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$ as $\alpha \to \infty$. If $\varphi \in AP$, then the limit

$$M(\varphi) := \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \varphi(x)dx$$

exists, is finite, and is independent of the particular choice of the family $\{I_\alpha\}$.

DEFINITION 2.1.4. *Let $\varphi \in AP$. The number $M(\varphi)$ given by Proposition 2.1.3 is called the* Bohr mean value *or simply the* mean value *of φ.*

In the matrix case the *mean value* is defined entry-wise.

We define the Wiener subalgebra of almost periodic functions in the following way: the elements of APW are those from AP which allow a representation by an absolutely convergent series. In fact, APW is precisely the (proper) subclass of all functions $\varphi \in AP$ which can be written in an absolutely convergent series of the form:

$$\varphi = \sum_j \varphi_j \, e_{\lambda_j} \,, \qquad \lambda_j \in \mathbb{R} \,, \quad \sum_j |\varphi_j| < \infty \,.$$

Let $\Omega(\psi) := \{\lambda \in \mathbb{R} : M(\psi e_{-\lambda}) \neq 0\}$ be the *Bohr-Fourier spectrum* of ψ. Consider APW_- (APW_+) to be the set of all functions $\psi \in APW$ such that $\Omega(\psi) \subset (-\infty, 0]$ ($\Omega(\psi) \subset [0, +\infty)$, respectively). It is therefore clear that $APW_- \subset AP_-$, and $APW_+ \subset AP_+$.

2.2 Semi-almost periodic and piecewise almost periodic functions

Let $C(\dot{\mathbb{R}})$ (with $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$) represent the (bounded and) continuous functions φ on the real line for which the two limits

$$\varphi(-\infty) := \lim_{x \to -\infty} \varphi(x), \quad \varphi(+\infty) := \lim_{x \to +\infty} \varphi(x)$$

exist and coincide. The common value of these two limits will be denoted by $\varphi(\infty)$. Furthermore, $C_0(\dot{\mathbb{R}})$ will stand for the functions $\varphi \in C(\dot{\mathbb{R}})$ for which $\varphi(\infty) = 0$.

We denote by $PC := PC(\dot{\mathbb{R}})$ the C^*-algebra of all bounded piecewise continuous functions on $\dot{\mathbb{R}}$, and we also put $C(\bar{\mathbb{R}}) := C(\mathbb{R}) \cap PC$, where $C(\mathbb{R})$ denote the usual set of continuous functions on the real line. Use will be also made of the C^*-algebra $PC_0 := \{\varphi \in PC : \varphi(\pm\infty) = 0\}$.

The C^*-algebra of semi-almost periodic elements is defined as follows.

DEFINITION 2.2.1. *The C^*-algebra SAP of all* semi-almost periodic functions *on $\mathbb{R}$ is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains AP and $C(\bar{\mathbb{R}})$:*

$$SAP := \mathrm{alg}_{L^\infty(\mathbb{R})}\{AP, C(\bar{\mathbb{R}})\} \,.$$

In [68] Sarason proved the following theorem which reveals in a different way the structure of the SAP algebra.

THEOREM 2.2.2. *Let $u \in C(\bar{\mathbb{R}})$ be any function for which $u(-\infty) = 0$ and $u(+\infty) = 1$. If $\varphi \in SAP$, then there exist $\varphi_\ell, \varphi_r \in AP$ and $\varphi_0 \in C_0(\dot{\mathbb{R}})$ such that*

$$\varphi = (1-u)\varphi_\ell + u\varphi_r + \varphi_0 \ .$$

The functions φ_ℓ, φ_r are uniquely determined by φ, and independent of the particular choice of u. The maps

$$\varphi \mapsto \varphi_\ell, \quad \varphi \mapsto \varphi_r$$

are C^*-algebra homomorphisms of SAP onto AP.

REMARK 2.2.3. The last theorem is also valid in the matrix case.

REMARK 2.2.4. We would like to emphasize that $SAP^{N\times N}$ is an inverse closed subalgebra of $[L^\infty(\mathbb{R})]^{N\times N}$ (i.e., if $\Phi \in SAP^{N\times N}$ is invertible in $[L^\infty(\mathbb{R})]^{N\times N}$, then the inverse Φ^{-1} belongs to $SAP^{N\times N}$).

We set $SAPW$ for all the functions from SAP, for which both the right and the left almost periodic representatives are the functions from the Wiener algebra APW.

Let us consider the closed subalgebra of $L^\infty(\mathbb{R})$ formed by almost periodic and piecewise continuous functions. We will denote it by $PAP := \mathrm{alg}_{L^\infty(\mathbb{R})}\{AP, PC\}$. It is readily seen that $SAP \subset PAP$. In the scalar case it was proved that $PAP = SAP + PC_0$. The same situations is also valid in the matrix case considering the decomposition entrywise. In addition, the next proposition is the matrix version of a known corresponding result for the representation of PAP elements in the scalar case (cf., e.g., [14, Proposition 3.15]).

PROPOSITION 2.2.5. *(i) If $\Phi \in PAP^{N\times N}$, then there are uniquely determined matrix-valued functions $\Theta_\ell, \Theta_r \in AP^{N\times N}$ and $\Phi_0 \in PC_0^{N\times N}$ such that*

$$\Phi = (1-u)\Theta_\ell + u\Theta_r + \Phi_0 \ ,$$

where $u \in C(\mathbb{R})$, $0 \le u \le 1$, $u(-\infty) = 0$ and $u(+\infty) = 1$.

(ii) *If* $\Phi \in \mathcal{G}PAP^{N\times N}$, *then there exist matrix-valued functions* $\Theta \in \mathcal{G}SAP^{N\times N}$ *and* $\Xi \in \mathcal{G}PC^{N\times N}$ *such that* $\Xi(-\infty) = \Xi(+\infty) = I_{N\times N}$,

$$\Phi = \Theta\,\Xi\,,$$

(iii) *In addition, the* Θ_ℓ *and* Θ_r *elements used in (i) coincide with the local representatives of* $\Theta \in \mathcal{G}SAP^{N\times N}$ *used in (ii), and their unique existence is ensured by Theorem 2.2.2 and Remark 2.2.3.*

Proof. The proof of the part (i) can be given as the proof of the scalar case (cf. [14, Proposition 3.15]) upon reasoning entrywise, and therefore it is omitted in here.

The proof of part (ii) can also be done in a similar way to the scalar case but contains some additional small differences. Therefore, it will be performed here for the reader convenience. Suppose $\Phi \in \mathcal{G}PAP^{N\times N}$, and put $\Upsilon := (1-u)\Theta_\ell + u\Theta_r$. Then $\Phi = \Upsilon + \Phi_0$. There is an $M \in (0,\infty)$ such that $|\det \Upsilon(x)|$ is bounded away from zero for $|x| > M$, and therefore we can find an element $\Upsilon_0 \in [C_0(\dot{\mathbb{R}})]^{N\times N}$ such that $\Theta := \Upsilon + \Upsilon_0 \in \mathcal{G}SAP^{N\times N}$. This allows us to look to Φ in the form

$$\begin{aligned} \Phi = \Theta + \Phi_0 - \Upsilon_0 \quad &= \quad \Theta[I + \Theta^{-1}(\Phi_0 - \Upsilon_0)] =: \Theta\Xi\,, \\ (= \quad & [I + (\Phi_0 - \Upsilon_0)\Theta^{-1}]\Theta =: \Xi\Theta)\,, \end{aligned}$$

being clear that $\Xi = \Theta^{-1}\Phi \in \mathcal{G}PC^{N\times N}$ and $\Xi(-\infty) = \Xi(+\infty) = I_{N\times N}$.

The part (iii) follows immediately from the construction made for (ii). □

REMARK 2.2.6. Due to the item (iii) of Proposition 2.2.5, we also call Θ_ℓ and Θ_r the local representatives of Φ at $-\infty$ and $+\infty$, respectively.

The class $PAPW$ is defined analogously as the class $SAPW$.

2.3 Unitary and sectorial functions

In this section we are interested in matrix unitary and sectorial functions. We start with the following definition.

DEFINITION 2.3.1. *A matrix function* $\Phi \in [L^\infty(\mathbb{R})]^{N\times N}$ *is called* unitary *if* $\Phi\Phi^* = \Phi^*\Phi = I_{N\times N}$, *where* Φ^* *stands for the conjugate transpose of* Φ.

In other words matrix unitary functions are those whose inverse coincide with its Hermitian transpose matrix.

The sectorial matrix functions are defined as follows.

DEFINITION 2.3.2. *A matrix function* $S \in [L^\infty(\mathbb{R})]^{N\times N}$ *is said to be sectorial if there exist a real number* $\varepsilon > 0$ *and two (constant) matrices* $B, C \in \mathcal{G}\mathbb{C}^{N\times N}$ *such that*

$$\Re e(BS(x)Cz, z) \geq \varepsilon ||z||^2 \,,$$

for almost all $x \in \mathbb{R}$ *and all* $z \in \mathbb{C}^N$.

We will denote by $\mathcal{S}^{N\times N}$ the set of all sectorial matrix functions (in $[L^\infty(\mathbb{R})]^{N\times N}$). In the scalar case, the sectorial functions are exactly those which have an essential range contained in some open half-plane whose boundary passes through the origin (cf., e.g., [14, Section 6.4] or [17, Section 1.7]).

The following result was obtained independently by Pousson and Rabindranathan, and it shows how can we decompose an invertible essentially bounded matrix function through the unitary matrix functions and functions which are invertible in the "plus" Hardy space.

THEOREM 2.3.3 (Pousson [63] and Rabindranathan [65]). *If* $a \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$, *then there exist an unitary matrix function* u *and* $h \in \mathcal{G}[H^\infty_+(\mathbb{R})]^{N\times N}$ *such that* $a = uh$, *almost everywhere on* $\mathbb{R}$.

The next lemma gives a hint how to prove results for Wiener-Hopf operators with sectorial symbols, when the results with unitary symbols are already at our disposal.

LEMMA 2.3.4. *[18, Lemma 2.21] If* E *is a subset of* $[L^\infty(\mathbb{R})]^{N\times N}$ *with the property that* $cg \in E$ *whenever* $c \in \mathbb{C}\setminus\{0\}$ *and* $g \in E$, $\varphi \in [L^\infty(\mathbb{R})]^{N\times N}$ *is unitary, and* $\operatorname{dist}(\varphi, E) < 1$, *then there exist a function* $f \in E$ *and a sectorial function* $s \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ *such that* $\varphi = sf$.

2.4 Functions with n points of standard almost periodic discontinuities

In the present section we will follow some definitions and results which can be found in [37].

Let us transfer to the unit circle Γ_0 the class of almost periodic functions (introduced in Section 2.1 for the real line $\mathbb{R}$), by means of the following operator B_0 (cf. (1.8.1)):

$$(B_0 f)(t) = f\left(i\frac{1+t}{1-t}\right) .$$

To denote the almost periodic functions class in the unit circle, we will use the notation AP_{Γ_0}. Furthermore, almost periodic polynomials on the circle are of the form:

$$a(t) := \sum_{j=1}^{n} c_j \exp\left(\lambda_j \frac{t+1}{t-1}\right), \qquad \lambda_j \in \mathbb{R}.$$

Next, the standard almost periodic discontinuities will be defined for the unit circle.

DEFINITION 2.4.1. *A function $\phi \in L^\infty(\Gamma_0)$ has a standard almost periodic discontinuity ($SAPD$) in the point $t_0 \in \Gamma_0$ if there exists a function $p_0 \in AP_{\Gamma_0}$ and a diffeomorphism $\tau := \omega_0(t)$ of the unit circle Γ_0 onto itself, such that ω_0 preserves the orientation of Γ_0, $\omega_0(t_0) = 1$, the function ω_0 has a second derivative at t_0, and*

$$\lim_{t\to t_0} (\phi(t) - p_0(\omega_0(t))) = 0 , \quad t \in \Gamma_0 . \tag{2.4.1}$$

In such a situation we will say that ϕ has a standard almost periodic discontinuity in the point t_0 with characteristics (p_0, ω_0).

REMARK 2.4.2. Assume that $\phi \in L^\infty(\Gamma_0)$ has a standard almost periodic discontinuity in the point t_0 and let a diffeomorphism ω_0 satisfy the conditions in the definition of a standard almost periodic discontinuity. Then, by means of a simple change of variable, the equality (2.4.1) can be rewritten in the following way:

$$\lim_{\tau\to 1} [\phi(\omega_0^{-1}(\tau)) - p_0(\tau)] = 0, \quad \tau \in \Gamma_0 .$$

2.4.1 Model functions

An invertible function h with properties $h \in L^\infty_+(\Gamma_0)$ and $h^{-1} \in L^\infty(\Gamma_0)$ is called a *model function* on the curve Γ_0. The operator $T_{h^{-1}}$, acting in $L^2_+(\Gamma_0)$, and its kernel $\mathrm{Ker} T_{h^{-1}}$ will be referred to as the *model operator* and the *model subspace* in the space $L^2_+(\Gamma_0)$ generated by the function h, respectively. We say that the model function on the curve Γ_0 belongs to the class $\mathcal{U}$ if $h^{-1} \in L^\infty_-(\Gamma_0)$.

The just described notion of a model function, model operator and model space, can be generalized to the real line, and furthermore for any rectifiable Jordan curve. As an example, take $\exp(i\lambda x)$, with $\lambda > 0$, and we will obtain a model function for the real line $\mathbb{R}$.

In the space $L^2(\Gamma_0)$ let us also consider the pair of complementary projections:

$$P_h := hQ_{\Gamma_0}h^{-1}I, \quad Q_h := hP_{\Gamma_0}h^{-1}I \,,$$

and the subspace

$$\mathcal{M}(h) := P_h(L^2_+(\Gamma_0)).$$

PROPOSITION 2.4.3. *Let $h_j \in \mathcal{U}$, $j = 1, 2, ..., n$. Then $h := \prod_{j=1}^n h_j \in \mathcal{U}$ and*

$$\mathcal{M}(h) = \mathcal{M}(h_1) \oplus h_1\mathcal{M}(h_2) \oplus \ldots \oplus (\prod_{j=1}^{n-1} h_j)\mathcal{M}(h_n) \,.$$

Let $a_k \in \mathbb{C}$, $k = 1, 2, 3, 4$, and assume that $a_1a_4 - a_2a_3 \neq 0$. Consider the following two fractional linear transformations, which are inverses of one another:

$$v(t) = \frac{a_1t + a_2}{a_3t + a_4}, \quad v^{-1}(x) = \frac{a_4x - a_2}{a_1 - a_3x} \,. \tag{2.4.2}$$

If we apply a fractional linear transformation of the form (2.4.2) to the model function $\exp(i\lambda x)$, with $\lambda > 0$, we arrive at the function

$$h_0(t) = \exp(\phi_0(t - t_0)^{-1}), \quad \phi_0 \in \mathbb{C} \setminus \{0\} \,, \tag{2.4.3}$$

which will be considered on the unit circle Γ_0 (and $t_0 \in \Gamma_0$).

PROPOSITION 2.4.4. *The function h_0 given by (2.4.3) is a model function on Γ_0 if and only if* $\arg \phi_0 = \arg t_0$.

The previous proposition is just a particularization of a corresponding result in [37, Proposition 4.2] when passing from the case of simple closed smooth contours to our Γ_0 case.

PROPOSITION 2.4.5. *Suppose that a diffeomorphism $\tau = \omega_0(t)$ of the unit circle Γ_0 onto itself satisfies the conditions in the definition of a standard almost periodic discontinuity at the point $t_0 \in \Gamma_0$. Then the following representation holds on Γ_0 :*

$$\phi(t) = \exp\left(\lambda \frac{\omega_0(t)+1}{\omega_0(t)-1}\right) = h_0(t)c_0(t), \quad \lambda \in \mathbb{R} , \tag{2.4.4}$$

where $c_0 \in \mathcal{G}C(\Gamma_0)$, $h_0 \in L^\infty(\Gamma_0)$ is given by (2.4.3) with $\phi_0 = 2\lambda/\omega_0'(t_0)$, and $C(\Gamma_0)$ is a usual set of continuous functions on Γ_0.

REMARK 2.4.6. Proposition 2.4.4 ensures that whenever on Γ_0 there exists a function ϕ that has a standard almost periodic discontinuity in the point t_0, one of the functions h_0 given by (2.4.3) or h_0^{-1} is a model function on Γ_0. Since here the mapping $\tau = \omega_0(t)$ preserves the orientation of Γ_0, $\arg \phi_0 = \arg t_0$ when $\lambda > 0$ (cf. (2.4.4)) and $\arg \phi_0 = \arg t_0 - \pi$ when $\lambda < 0$.

2.4.2 Functional σ_{t_0}

Let $t_0 \in \Gamma_0$ and let the function $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ be continuous in a neighborhood of t_0, except, possibly, in the point t_0 itself. Let us recall the real functional used by Dybin and Grudsky in [37]:

$$\sigma_{t_0}(\phi) := \lim_{\delta \to 0} \frac{\delta}{4} \left[\arg \phi(t)\right] \Big|_{t=t''}^{t'} = \lim_{\delta \to 0} \frac{\delta}{4} \left(\arg \phi\left(t'\right) - \arg \phi\left(t''\right)\right) \tag{2.4.5}$$

where $t', t'' \in \Gamma_0$, $t' \prec t_0 \prec t''$, $|t' - t_0| = |t'' - t_0| = \delta$.

The notation $t' \prec t_0 \prec t''$, used above, means that when we are tracing the curve in the positive direction we will meet the point t' first, then the point t_0 and then the point t''. The next proposition establishes a connection between the functional $\sigma_{t_0}(\phi)$ and the standard almost periodic discontinuities on Γ_0.

PROPOSITION 2.4.7. *Suppose that the diffeomorphism $\tau = \omega_0(t)$ of the unit circle Γ_0 onto itself satisfies the conditions in the definition of a standard almost periodic discontinuity in the point $t_0 \in \Gamma_0$ and that $p \in \mathcal{G}AP_{\Gamma_0}$. Then $\phi(t) = p(\omega_0(t)) \in \mathcal{G}L^\infty(\Gamma_0)$, $\sigma_{t_0}(\phi)$ exists, and*

$$\sigma_{t_0}(\phi) = k(p)/|\omega_0'(t_0)| \; .$$

2.5 The Besicovitch space

In this section we introduce notations and results about the Besicovitch space. For the corresponding proofs, the reader should consult [14, Chapter 7] and the references therein (cf., e.g., [14, page 130]). Denote by AP^0 the set of all almost periodic polynomials. The Besicovitch space B^2 is defined as the completion of AP^0 with respect to the norm

$$||\varphi||_{B^2} := \left(\sum_\lambda |\varphi_\lambda|^2 \right)^{\frac{1}{2}} \; ,$$

where $\varphi = \sum_\lambda \varphi_\lambda e_\lambda \in AP^0$. Let $\mathbb{R}_B$ denote the Bohr compactification of $\mathbb{R}$ and $d\mu$ the normalized Haar measure on $\mathbb{R}_B$ (see, e.g., [14, Chapter 7]). It is known that AP may be identified with $C(\mathbb{R}_B)$ and that one can identify B^2 with $L^2(\mathbb{R}_B, d\mu)$. Thus, B^2 is a (nonseparable) Hilbert space, and the inner product in $B^2 = L^2(\mathbb{R}_B, d\mu)$ is given by

$$(f, g) := \int_{\mathbb{R}_B} f(\xi)\overline{g(\xi)} d\mu(\xi) \tag{2.5.1}$$

For $f, g \in AP$ we also have the following equality

$$(f, g) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)} dx$$

Since $\mu(\mathbb{R}_B) = 1$ is finite, AP is contained in B^2. Moreover, AP is a dense subset of B^2.

The Cauchy–Schwarz inequality shows that the mean value

$$M(f) := \int_{\mathbb{R}_B} f(\xi) d\mu(\xi)$$

exists and is finite for every $f \in B^2$. For $f \in B^2$, the set

$$\Omega(f) := \{\lambda \in \mathbb{R} : M(fe_{-\lambda}) \neq 0\}$$

is called the Bohr–Fourier spectrum of f and can be shown to be at most countable. Taking into account (2.5.1), one can prove that for every $f \in B^2$,

$$||f||_{B^2}^2 = \sum_{\lambda\in\Omega(f)} |M(fe_{-\lambda})|^2 \; .$$

Let $\ell^2(\mathbb{R})$ denote the collection of all functions $f : \mathbb{R} \to \mathbb{C}$ for which the set $\{\lambda \in \mathbb{R} : f(\lambda) \neq 0\}$ is at most countable and

$$||f||_{\ell^2(\mathbb{R})}^2 := \sum |f(\lambda)|^2 < \infty \; .$$

Further, $\ell^\infty(\mathbb{R})$ is defined as the set of all functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$||f||_{\ell^\infty(\mathbb{R})} := \sup_{\lambda\in\mathbb{R}} |f(\lambda)| < \infty \; .$$

Note that $\ell^2(\mathbb{R})$ is a (nonseparable) Hilbert space with pointwise operations and the inner product

$$(f, g) := \sum_{\lambda\in\mathbb{R}} f(\lambda)\overline{g(\lambda)}$$

and that $\ell^\infty(\mathbb{R})$ is a C^*-algebra with pointwise operations and the norm $||\cdot||_{\ell^\infty(\mathbb{R})}$.

The map $F_B : \ell^2(\mathbb{R}) \to B^2$ which sends a function $f \in \ell^2(\mathbb{R})$ with a finite support to the function

$$(F_B f)(x) = \sum_{\lambda\in\mathbb{R}} f(\lambda)e^{i\lambda x}, \quad x \in \mathbb{R}$$

can be extended by continuity to all of $\ell^2(\mathbb{R})$. It is referred to as the Bohr-Fourier transform. The operator F_B is an isometric isomorphism. The inverse Bohr-Fourier transform acts by the rule

$$F_B^{-1} : B^2 \to \ell^2(\mathbb{R}), \quad (F_B^{-1} f)(\lambda) = M(fe_{-\lambda}), \quad \lambda \in \mathbb{R} \; .$$

If $a \in \ell^\infty(\mathbb{R})$, then the operator $\psi(a) : B^2 \to B^2$ defined by $\psi(a) := F_B a F_B^{-1}$ is bounded.

Chapter 3

Matrix Wiener-Hopf plus Hankel operators with AP Fourier symbols

In this chapter we study matrix Wiener-Hopf plus Hankel operators with AP Fourier symbols. A characterization of the invertibility of such type of matrix operators is obtained based on a factorization of the Fourier symbols, which belong to the class of almost periodic matrix functions.

Therefore, the main aim of the present chapter is to provide an invertibility criterion for the matrix Wiener-Hopf plus Hankel operators with AP Fourier symbols. Thus, we generalize to the matrix case some of the scalar results obtained in [59] and [60].

Note that within this context the representation of the (generalized/one-sided/two-sided) inverses of WH_Φ based on some factorization of the Fourier symbol Φ is an important goal, and will be obtained in Section 3.3. In this way, the main contributions of this chapter are described in Theorem 3.1.3, Theorem 3.2.1, and Theorem 3.3.1.

In addition, we would like to refer here that in the present chapter we will work with APW and AP symbols in view of exhibiting the details in these classes. Anyway, all the results are obtained in the same manner for AP and APW cases except the result about uniqueness of corresponding factorizations which due to this reason we choose to present both versions for the full understanding of those differences.

3.1 Matrix APW asymmetric factorization

Let us start by recalling the definition of a matrix APW asymmetric factorization.

DEFINITION 3.1.1. *We will say that a matrix function $\Phi \in \mathcal{G}APW^{N\times N}$ admits an APW asymmetric factorization if it can be represented in the form*

$$\Phi = \Phi_{-}\mathrm{diag}[e_{\lambda_1}, ..., e_{\lambda_N}]\Phi_e \ , \tag{3.1.1}$$

where $\lambda_k \in \mathbb{R}$, $e_{\lambda_k}(x) = e^{i\lambda_k x}$, $x \in \mathbb{R}$, $\Phi_- \in \mathcal{G}APW_-^{N\times N}$, $\Phi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ and $\widetilde{\Phi_e} = \Phi_e$.

REMARK 3.1.2. We would like to remark that an APW asymmetric factorization, if it exists, is not unique. Anyway, the partial indices of two APW asymmetric factorizations of the same matrix function are unique up to a change in their order (cf. Theorem 3.1.3 below). Consequently the λ_k partial indices can be rearranged in any desired way. Namely, if (3.1.1) is an APW asymmetric factorization of Φ and Π is a permutation constant matrix, then by considering $\Pi^{-1}\mathrm{diag}[e_{\lambda_1}, ..., e_{\lambda_N}]\Pi =: \widehat{\mathrm{diag}}[e_{\lambda_1}, ..., e_{\lambda_N}]$, $\overrightarrow{\Phi_-} := \Phi_-\Pi$, and $\overleftarrow{\Phi_e} := \Pi^{-1}\Phi_e$, we obtain a second asymmetric APW factorization of Φ given by

$$\Phi = \overrightarrow{\Phi_-}\widehat{\mathrm{diag}}[e_{\lambda_1}, ..., e_{\lambda_N}]\overleftarrow{\Phi_e} \, .$$

Besides this last fact, we have the following general result about the uniqueness of these factorizations.

THEOREM 3.1.3. *Let $\Phi \in \mathcal{G}APW^{N\times N}$. Suppose that*

$$\Phi = \Phi_-^{(1)} D^{(1)} \Phi_e^{(1)} \ ,$$

with $D^{(1)} = \mathrm{diag}[e_{\lambda_1}, ..., e_{\lambda_N}]$ and $\lambda_1 \geq \cdots \geq \lambda_N$, is an APW asymmetric factorization of Φ and assume additionally that

$$\Phi = \Phi_-^{(2)} D^{(2)} \Phi_e^{(2)} \ ,$$

with $D^{(2)} = \mathrm{diag}[e_{\mu_1}, ..., e_{\mu_N}]$ *and* $\mu_1 \geq \cdots \geq \mu_N$*, represents any other APW asymmetric factorization of* Φ*. Then*

$$\Phi_-^{(2)} = \Phi_-^{(1)} \Psi^{-1} \ ,$$
$$D^{(1)} = D^{(2)} =: D \ ,$$
$$\Phi_e^{(2)} = D^{-1} \Psi D \Phi_e^{(1)} \ ,$$

where $\Psi(x) = (\psi_{jk}(x))_{j,\, k=1}^N$ *is a matrix function with nonzero and constant determinant, having entries which are entire functions, and*

$$\psi_{jk}(z) = \begin{cases} 0 \ , & \text{if } \ \lambda_j > \lambda_k \ , \\ c_{jk} = \mathrm{const} \neq 0 \ , & \text{if } \ \lambda_j = \lambda_k \ . \end{cases} \tag{3.1.2}$$

Proof. If Φ admits the above mentioned two *APW* asymmetric factorizations, then we can write

$$\Phi = \Phi_-^{(1)} D^{(1)} \Phi_e^{(1)} = \Phi_-^{(2)} D^{(2)} \Phi_e^{(2)} \ , \tag{3.1.3}$$

which leads to

$$(\Phi_-^{(2)})^{-1} \Phi_-^{(1)} D^{(1)} = D^{(2)} \Phi_e^{(2)} (\Phi_e^{(1)})^{-1} \ . \tag{3.1.4}$$

We now define $\Phi_- := (\Phi_-^{(2)})^{-1} \Phi_-^{(1)}$ and $\Phi_e := \Phi_e^{(2)} (\Phi_e^{(1)})^{-1}$. Thus, we have $\Phi_- \in \mathcal{G}APW_-^{N\times N}$, $\Phi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ and $\widetilde{\Phi}_e = \Phi_e$. From (3.1.4), we obtain the following identity for each (j,k) element of that matrix:

$$(\Phi_-)_{jk}(x) e^{i\lambda_k x} = e^{i\mu_j x} (\Phi_e)_{jk}(x) \ ;$$

whence

$$(\Phi_e)_{jk}(x) = (\Phi_-)_{jk}(x) e^{i(\lambda_k - \mu_j)x} \ ,$$

and recall that Φ_e is an even function. Thus

$$(\Phi_-)_{jk}(x) e^{i(\lambda_k - \mu_j)x} = \widetilde{(\Phi_-)}_{jk}(x) e^{i(\mu_j - \lambda_k)x} \ , \tag{3.1.5}$$

and finally we infer from (3.1.5) that

$$(\Phi_-)_{jk}(x) = e^{2i(\mu_j-\lambda_k)x}\widetilde{(\Phi_-)}_{jk}(x)\ . \tag{3.1.6}$$

If $\mu_j \geq \lambda_k$, then the element in the left-hand side of (3.1.6) is in the class APW_-, and the function in the right-hand side belongs to APW_+, which implies that there exist constants c_{jk} such that

$$(\Phi_-)_{jk}(x) = c_{jk} = \widetilde{(\Phi_-)}_{jk}(x)\ e^{2i(\mu_j-\lambda_k)x}\ .$$

Therefore, $c_{jk} = c_{jk}e^{2i(\mu_j-\lambda_k)x}$. Thus, if $\mu_j > \lambda_k$ we obtain $c_{jk} = 0$, and in the case where $\mu_j = \lambda_k$ we conclude that c_{jk} are nonzero constants. Altogether, we have

$$(\Phi_-)_{jk}(x) = \begin{cases} 0, & \text{if} \quad \mu_j > \lambda_k, \\ c_{jk} = \text{const} \neq 0, & \text{if} \quad \mu_j = \lambda_k. \end{cases} \tag{3.1.7}$$

Let us now assume that $\mu_j < \lambda_k$. By the hypothesis, we know that $(\Phi_-)_{jk} \in APW_-$ and so $(\Phi_-)_{jk}$ can be represented in the following form:

$$(\Phi_-)_{jk}(x) = \sum_m (a_m)_{jk}\, e^{i(\nu_m)_{jk}x}\ , \tag{3.1.8}$$

with $\sum_m |(a_m)_{jk}| < \infty$ for all $j, k = \overline{1, N}$. From (3.1.8) we directly have

$$\widetilde{(\Phi_-)}_{jk}(x) = \sum_m (a_m)_{jk}\, e^{-i(\nu_m)_{jk}x}\ . \tag{3.1.9}$$

Combining (3.1.6), (3.1.8) and (3.1.9) we obtain

$$\sum_m (a_m)_{jk}\, e^{i(\nu_m)_{jk}x} = e^{2i(\mu_j-\lambda_k)x} \sum_m (a_m)_{jk}\, e^{-i(\nu_m)_{jk}x}\ ,$$

or equivalently

$$\sum_m (a_m)_{jk}\, e^{i(\nu_m)_{jk}x} = \sum_m (a_m)_{jk}\, e^{i(2(\mu_j-\lambda_k)-(\nu_m)_{jk})x}\ ,$$

and this leads us to the following identity

$$(\nu_m)_{jk} = 2(\mu_j - \lambda_k) - (\nu_m)_{jk}\ .$$

In conclusion, we have in the present case

$$(\nu_m)_{jk} = \mu_j - \lambda_k < 0 \ .$$

So, for any couple (j,k), we will obtain only one real number $(\nu_m)_{jk}$, which is precisely the difference $\mu_j - \lambda_k$ and this means that in the representation of $(\Phi_-)_{jk}$ (cf. (3.1.8)) we need to have $(\Phi_-)_{jk}(x) = c_{jk}e^{i(\nu_m)_{jk}x}$, with some constant $c_{jk} = \text{const}$, for all $j,k = \overline{1,N}$. Thus, we arrive at the conclusion that $(\Phi_-)_{jk}$ are entire functions when $\mu_j < \lambda_k$.

We will now prove that $D^{(1)} = D^{(2)}$, i.e. $\mu_j = \lambda_j$ for all j. Let us first assume that $\mu_j > \lambda_j$, for some j. Then $\mu_l > \lambda_k$ for $l \leq j \leq k$ and from (3.1.7) we infer that $(\Phi_-)_{lk} = 0$ for $l \leq j \leq k$. This and the *Laplace Expansion Theorem* show that $\det \Phi_-(x) = 0$ for all $x \in \mathbb{R}$, which is impossible simply because Φ_- is invertible. If for some j we would assume $\mu_j < \lambda_j$, we can repeat the above reasoning starting from (3.1.3) with $D^{(1)}\Phi_e^{(1)}(\Phi_e^{(2)})^{-1} = (\Phi_-^{(1)})^{-1}\Phi_-^{(2)}D^{(2)}$ instead of (3.1.4) and obtain once again a contradiction. Thus, $\mu_j = \lambda_j$ for all j.

Letting $\Psi := \Phi_-$ we immediately have that Ψ is an entire function. Additionally, by virtue of the equality $D^{(1)} = D^{(2)} =: D$ and (3.1.7), Ψ satisfies (3.1.2). The block-triangular structure of Ψ implies that $\det \Psi$ is a constant, and since $\Psi = (\Phi_-^{(2)})^{-1}\Phi_-^{(1)}$ this constant cannot be zero. Finally, identity (3.1.4) gives that $\Phi_e = D^{-1}\Psi D$, and therefore $\Phi_e^{(2)} = D^{-1}\Psi D \Phi_e^{(1)}$. This together with the identity $\Phi_-^{(2)} = \Phi_-^{(1)}\Psi^{-1}$ concludes the proof. $\square$

3.2 Invertibility characterization

For further purposes let us recall that two linear operators T and S are said to be *equivalent* operators if there exist two bounded invertible operators E and F such that $T = ESF$ (recall Chapter 1).

THEOREM 3.2.1. *Let Φ have an APW asymmetric factorization, with partial indices $\lambda_1, ..., \lambda_N$.*

(a) If there exist positive and negative partial indices, then WH_Φ is not semi-Fredholm.

(b) If $\lambda_i \leq 0$, $i = \overline{1, N}$, and if for at least one index i we have $\lambda_i < 0$, then WH_Φ is properly d-normal and right-invertible.

(c) If $\lambda_i \geq 0$, $i = \overline{1, N}$, and if for at least one index i we have $\lambda_i > 0$, then WH_Φ is properly n-normal and left-invertible.

(d) If $\lambda_i = 0$, $i = \overline{1, N}$, then WH_Φ is two-sided invertible.

Proof. Since by hypothesis Φ admits an APW asymmetric factorization, we have

$$\Phi = \Phi_- D \Phi_e \, , \tag{3.2.1}$$

where $\Phi_- \in \mathcal{G}APW_-^{N\times N}$, $D = \mathrm{diag}[e_{\lambda_1}, ..., e_{\lambda_N}]$, and Φ_e is an invertible even element. Without lost of generality (cf. Remark 3.1.2), we will assume that $\lambda_1 \geq \ldots \geq \lambda_N$. As previously observed, from (3.2.1) we therefore obtain the operator factorization (cf. (1.6.6))

$$WH_\Phi = W_{\Phi_-} \ell_0 WH_D \, \ell_0 WH_{\Phi_e} \, . \tag{3.2.2}$$

We know that W_{Φ_-} is invertible because $\Phi_- \in \mathcal{G}APW_-^{N\times N}$ (and its inverse is given by $\ell_0 W_{\Phi_-^{-1}} \ell_0$). Additionally, WH_{Φ_e} is also invertible because Φ_e is an even element (cf. Theorem 1.6.5). Thus, (3.2.2) shows us an operator equivalence relation between WH_Φ and WH_D (recall that $\ell_0 : [L^2(\mathbb{R}_+)]^N \to [L^2_+(\mathbb{R})]^N$ is invertible by $r_+ : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N$). We will therefore analyze the regularity properties of WH_D.

Suppose that at least some of the partial indices are greater than zero, some of them may be equal to zero, and that some of them are less than zero; for instance, $\lambda_1, ..., \lambda_i > 0$, $\lambda_{i+1} = ... = \lambda_j = 0$, and $\lambda_{j+1}, ..., \lambda_N < 0$. This means that

$$\begin{aligned} \ell_0 WH_D &= \mathrm{diag}[\ell_0 WH_{e_{\lambda_1}}, ..., \ell_0 WH_{e_{\lambda_i}}, \ell_0 WH_{e_{\lambda_{i+1}}}, ..., \\ &\qquad \ell_0 WH_{e_{\lambda_j}}, \ell_0 WH_{e_{\lambda_{j+1}}}, ..., \ell_0 WH_{e_{\lambda_N}}] \\ &= \mathrm{diag}[\ell_0 WH_{e_{\lambda_1}}, ..., \ell_0 WH_{e_{\lambda_i}}, I, ..., I, \ell_0 W_{e_{\lambda_{j+1}}}, ..., \ell_0 W_{e_{\lambda_N}}] \, , \end{aligned} \tag{3.2.3}$$

because $WH_{e_{\lambda_k}} = W_{e_{\lambda_k}}$, for $k = \overline{j+1, N}$, due to the condition $\lambda_{j+1} < 0, ..., \lambda_N < 0$ and due to the structure of the Hankel operators (and also because $\ell_0 WH_{e_{\lambda_k}} = I$, $k = \overline{i+1, j}$

due to the condition $\lambda_{i+1} = \ldots = \lambda_j = 0$). The nonzero scalar operators in the diagonal matrix operator (3.2.3) are such that: $WH_{e_{\lambda_1}}, \ldots, WH_{e_{\lambda_i}}$ are properly n-normal and left-invertible (cf. Theorem 6 in [59]); $W_{e_{\lambda_{j+1}}}, \ldots, W_{e_{\lambda_N}}$ are d-normal and right-invertible (cf. the *Gohberg–Feldman–Coburn–Douglas Theorem* [14, Theorem 2.28], [28], [40]). Therefore, WH_D cannot be semi-Fredholm, hence WH_Φ cannot be semi-Fredholm. This proves part (a) of the theorem.

Suppose now that $\lambda_i \leq 0$, $i = \overline{1,N}$. This implies that $D \in AP_-^{N\times N}$. Since $AP_-^{N\times N} = AP^{N\times N} \cap [H_-^\infty(\mathbb{R})]^{N\times N}$, it holds that $D \in [H_-^\infty(\mathbb{R})]^{N\times N}$ and hence $WH_D = W_D$. So, in this case, WH_Φ is equivalent to W_D. If we employ again the *Gohberg–Feldman–Coburn–Douglas Theorem* to the each one of the operators in the main diagonal of the operator W_D, it follows the assertion (b) of the theorem.

Part (c) can be deduced from the assertion (b) by passing to adjoints.

If all partial indices are zero, we have that $\ell_0 WH_D$ is just the identity operator. This, together with the operator equivalence relation (3.2.2) presented in the first part of the proof, leads us to the last assertion (d). □

3.3 Representation of the inverses

We now reach the main goal of this chapter: the representation of generalized/one-sided/two-sided inverses of WH_Φ based on a factorization of the Fourier symbol. This result extends the scalar version obtained in [59, Theorem 7].

Let us first recall that a bounded linear operator $S^- : Y \to X$ (acting between Banach spaces) is called a reflexive generalized inverse of a bounded linear operator $S : X \to Y$ if: (i) S^- is a generalized inverse (or an inner pseudoinverse) of S, i.e., $SS^-S = S$; (ii) S^- is an outer pseudoinverse of S, i.e., $S^-SS^- = S^-$.

THEOREM 3.3.1. *Suppose Φ admits an APW asymmetric factorization and*

$$\begin{aligned} T \;&:=\; \ell_0 r_+ \mathcal{F}^{-1}\Phi_e^{-1} \cdot \mathcal{F}\ell^e r_+ \mathcal{F}^{-1}\mathrm{diag}[e_{-\lambda_1}, \ldots, e_{-\lambda_N}] \cdot \mathcal{F}\ell^e r_+ \mathcal{F}^{-1}\Phi_-^{-1} \cdot \mathcal{F}\ell \\ &\quad : [L^2(\mathbb{R}_+)]^N \to [L^2_+(\mathbb{R})]^N \,, \end{aligned}$$

where Φ_e^{-1} *and* Φ_-^{-1} *are the inverses of the corresponding factors of an* APW *asymmetric factorization of* Φ, $\Phi = \Phi_- D \Phi_e$, *and the operator* $\ell : [L^2(\mathbb{R}_+)]^N \to [L^2_+(\mathbb{R})]^N$ *denotes an arbitrary extension operator (i.e.,* T *is independent of the particular choice of the extension* ℓ*). Then the operator* T *is a reflexive generalized inverse of* WH_Φ *and, in the following special cases, we additionally have that* T *is:*

(a) the right-inverse of WH_Φ, *if* $\lambda_i \leq 0$ *for all* $i = \overline{1, N}$,

(b) the left-inverse of WH_Φ, *if* $\lambda_i \geq 0$ *for all* $i = \overline{1, N}$,

(c) the (two-sided) inverse of WH_Φ, *if* $\lambda_i = 0$ *for all* $i = \overline{1, N}$.

In the case when there exist partial indices with different signs, the operator WH_Φ *is not Fredholm but* T *is still a (reflexive) generalized inverse of* WH_Φ.

Proof. We start with the cases (a), (b) and (c). Since Φ admits an APW asymmetric factorization, we can write

$$\Phi = \Phi_- \mathrm{diag}[e_{\lambda_1}, ..., e_{\lambda_N}] \Phi_e$$

(with the corresponding factor properties). Consequently, from (1.4.7), it follows that

$$WH_\Phi = r_+ A_- E A_e \ell^e r_+ \; ,$$

where $A_- = \mathcal{F}^{-1}\Phi_- \cdot \mathcal{F}$, $E = \mathcal{F}^{-1}\mathrm{diag}[e_{\lambda_1}, ..., e_{\lambda_N}] \cdot \mathcal{F}$ and $A_e = \mathcal{F}^{-1}\Phi_e \cdot \mathcal{F}$.

(a) If $\lambda_i \leq 0$ for all $i = \overline{1, N}$, consider

$$\begin{aligned} WH_\Phi T &= r_+ A_- E A_e \ell^e r_+ \ell_0 r_+ A_e^{-1} \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1} \ell \\ &= r_+ A_- E A_e \ell^e r_+ A_e^{-1} \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1} \ell \; , \end{aligned} \qquad (3.3.1)$$

where the term $\ell_0 r_+$ was omitted due to the fact that $r_+ \ell_0 r_+ = r_+$. Since A_e^{-1} preserves the even property of its symbol, we may also drop the first $\ell^e r_+$ term in (3.3.1), and obtain

$$WH_\Phi T = r_+ A_- E \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1} \ell \; . \qquad (3.3.2)$$

Additionally, due to the definition of E and E^{-1} in the present case ($\lambda_i \leq 0$ for all $i = \overline{1,N}$), we have $\ell_0 r_+ E \ell^e r_+ E^{-1} \ell^e r_+ = \ell_0 r_+$; also because A_- is a *minus type factor* it follows $r_+ A_- = r_+ A_- \ell_0 r_+$. Therefore, from (3.3.2), we have

$$WH_\Phi T = r_+ A_- \ell_0 r_+ A_-^{-1} \ell = r_+ \ell = I_{[L^2(\mathbb{R}_+)]^N} \,. \tag{3.3.3}$$

(b) If $\lambda_i \geq 0$ for all $i = \overline{1,N}$, we will now analyze the composition

$$T\, WH_\Phi = \ell_0 r_+ A_e^{-1} \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1} \ell r_+ A_- E A_e \ell^e r_+ \,. \tag{3.3.4}$$

In the present case, due to the definition of E^{-1}, it follows $\ell^e r_+ E^{-1} \ell^e r_+ = \ell^e r_+ E^{-1}$. The same reasoning applies to the *minus type factor* A_-^{-1}, and therefore the equality (3.3.4) takes the form

$$T\, WH_\Phi = \ell_0 r_+ A_e^{-1} \ell^e r_+ A_e \ell^e r_+ = \ell_0 r_+ \ell^e r_+ = \ell_0 r_+ = I_{[L^2_+(\mathbb{R})]^N} \,, \tag{3.3.5}$$

where we have used the fact that $\ell^e r_+ A_e \ell^e r_+ = A_e \ell^e r_+$.

(c) From the last two cases (a) and (b), it directly follows that in the case of $\lambda_i = 0$ for all $i = \overline{1,N}$, the operator T is the two-sided inverse of WH_Φ (cf. (3.3.3) and (3.3.5)).

Let us now turn to the more general case: assume now that there exist partial indices with different signs.

In this case, we recall that the assertion about the non-Fredholm property was already provided in Theorem 3.2.1, assertion (a).

As about the generalized inverse, we will start by rewriting the operator E in the following new form:

$$\begin{aligned} E &= \operatorname{diag}[\mathcal{F}^{-1} e_{\lambda_{11}} \cdot \mathcal{F}, \ldots, \mathcal{F}^{-1} e_{\lambda_{1N}} \cdot \mathcal{F}]\, \operatorname{diag}[\mathcal{F}^{-1} e_{\lambda_{21}} \cdot \mathcal{F}, \ldots, \mathcal{F}^{-1} e_{\lambda_{2N}} \cdot \mathcal{F}] \\ &=: E_1\, E_2 \,, \end{aligned}$$

where

$$\lambda_{1j} = \begin{cases} \lambda_j & \text{if} \quad \lambda_j \leq 0 \\ 0 & \text{if} \quad \lambda_j \geq 0 \end{cases} \,, \qquad \lambda_{2j} = \begin{cases} \lambda_j & \text{if} \quad \lambda_j \geq 0 \\ 0 & \text{if} \quad \lambda_j \leq 0 \end{cases}$$

for $j = \overline{1, N}$.

We will then directly compute $WH_\Phi\, T\, WH_\Phi$, in the following way:

$$\begin{aligned}
WH_\Phi\, T\, WH_\Phi &= (r_+A_-E_1E_2A_e\ell^e r_+)\left(\ell_0 r_+A_e^{-1}\ell^e r_+E_2^{-1}E_1^{-1}\ell^e r_+A_-^{-1}\ell\right) \\
&\quad (r_+A_-E_1E_2A_e\ell^e r_+) \\
&= r_+A_-E_1E_2A_e\ell^e r_+A_e^{-1}\ell^e r_+E_2^{-1}E_1^{-1}\ell^e r_+A_-^{-1}\ell r_+A_- \qquad (3.3.6) \\
&\quad E_1E_2A_e\ell^e r_+ \\
&= r_+A_-E_1E_2\ell^e r_+E_2^{-1}E_1^{-1}\ell^e r_+E_1E_2A_e\ell^e r_+ \qquad (3.3.7) \\
&= r_+A_-E_1E_2A_e\ell^e r_+ \qquad (3.3.8) \\
&= WH_\Phi
\end{aligned}$$

where in (3.3.7) we omitted the first term $\ell^e r_+$ of (3.3.6) due to the factor (invariance) property of A_e^{-1} that yields $A_e\ell^e r_+A_e^{-1}\ell^e r_+ = \ell^e r_+$. Similarly we dropped the term ℓr_+ in $\ell^e r_+A_-^{-1}\ell r_+A_-$ due to a factor property of A_-^{-1}. Analogous arguments apply to the factors E_1^{-1} and E_2^{-1}. In a more detailed way: (i) if one of the factors E_1 or E_2 equals I, then it is clear that $E_2(\ell^e r_+E_2^{-1}\ell^e r_+)E_2 = E_2(\ell^e r_+E_2^{-1})E_2 = E_2\ell^e r_+$ or $\ell_0 r_+E_1\ell^e r_+E_1^{-1}\ell^e r_+E_1 = \ell_0 r_+E_1$ holds, respectively; (ii) in the general diagonal matrix case, the situation is identical just because in each place of the main diagonal we have the last situation. This justifies the simplification made in obtaining (3.3.8) from (3.3.7).

As about the composition $T\, WH_\Phi\, T$, it follows:

$$\begin{aligned}
T\, WH_\Phi\, T &= \left(\ell_0 r_+A_e^{-1}\ell^e r_+E_2^{-1}E_1^{-1}\ell^e r_+A_-^{-1}\ell\right)(r_+A_-E_1E_2A_e\ell^e r_+) \\
&\quad \left(\ell_0 r_+A_e^{-1}\ell^e r_+E_2^{-1}E_1^{-1}\ell^e r_+A_-^{-1}\ell\right) \\
&= \ell_0 r_+A_e^{-1}\ell^e r_+E_2^{-1}E_1^{-1}\ell^e r_+A_-^{-1}\ell r_+A_-E_1E_2A_e\ell^e r_+A_e^{-1}\ell^e r_+ \qquad (3.3.9) \\
&\quad E_2^{-1}E_1^{-1}\ell^e r_+A_-^{-1}\ell \\
&= \ell_0 r_+A_e^{-1}\ell^e r_+E_2^{-1}E_1^{-1}\ell^e r_+A_-^{-1}\ell \qquad (3.3.10) \\
&= T
\end{aligned}$$

where the third $\ell^e r_+$ is unnecessary in (3.3.9) due to the factor (invariance) property of A_e that yields $A_e\ell^e r_+A_e^{-1}\ell^e r_+ = \ell^e r_+$, and we also can omit the term ℓr_+ in (3.3.9) since

A_-^{-1} is *minus* type. Additionally, a similar reasoning as above was also used for obtaining equality (3.3.10) since due to the definitions of E_1 and E_2 it holds $\ell^e r_+ E_1 \ell^e r_+ = \ell^e r_+ E_1$, and $\ell^e r_+ E_2^{-1} \ell^e r_+ E_2 \ell^e r_+ E_2^{-1} = \ell^e r_+ E_2^{-1}$. $\square$

3.4 Matrix AP asymmetric factorization

We would like now to cite the last paragraph of the paper [10]: *"We end up by mentioning that almost all the above methods also work – without crucial changes – in the case of matrix Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols. However, a corresponding version of Theorem 3.1.3 for invertible $AP^{N\times N}$ elements is an open problem. This has to do with the difficulties in substituting the arguments in the part of the proof of Theorem 3.1.3 where some representations of APW elements are used".*

We have already closed this open problem and the corresponding theorem will be given below after giving the appropriate definition of the AP asymmetric factorization.

DEFINITION 3.4.1. *We will say a matrix function $\Phi \in \mathcal{G}AP^{N\times N}$ admits an AP asymmetric factorization if it can be represented in the form*

$$\Phi = \Phi_- \mathrm{diag}[e_{\lambda_1}, ..., e_{\lambda_N}] \Phi_e \ ,$$

where $\lambda_k \in \mathbb{R}$, $e_{\lambda_k}(x) = e^{i\lambda_k x}$, $x \in \mathbb{R}$, $\Phi_- \in \mathcal{G}AP_-^{N\times N}$, $\Phi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ and $\widetilde{\Phi_e} = \Phi_e$.

The property that we can rearrange the partial indices of such type of factorizations in any desired way is obviously valid in this case too (cf. Remark 3.1.2).

We are ready to give the following general result about the uniqueness of these kind of factorizations.

THEOREM 3.4.2. *Let $\Phi \in \mathcal{G}AP^{N\times N}$. Suppose that*

$$\Phi = \Phi_-^{(1)} D^{(1)} \Phi_e^{(1)} \ ,$$

with $D^{(1)} = \mathrm{diag}[e_{\lambda_1}, ..., e_{\lambda_N}]$ and $\lambda_1 \geq \cdots \geq \lambda_N$, is an AP asymmetric factorization of Φ and assume additionally that

$$\Phi = \Phi_-^{(2)} D^{(2)} \Phi_e^{(2)} \ ,$$

with $D^{(2)} = \mathrm{diag}[e_{\mu_1}, ..., e_{\mu_N}]$ *and* $\mu_1 \geq \cdots \geq \mu_N$*, represents any other AP asymmetric factorization of* Φ*. Then*

$$\Phi_-^{(2)} = \Phi_-^{(1)} \Psi^{-1} ,$$

$$D^{(1)} = D^{(2)} =: D ,$$

$$\Phi_e^{(2)} = D^{-1} \Psi D \Phi_e^{(1)} ,$$

where $\Psi(x) = (\psi_{jk}(x))_{j,\, k=1}^N$ *is a matrix function with nonzero and constant determinant, having entries which are almost periodic functions, such that the Bohr-Fourier spectrum of* $\Omega(\psi_{jk})$ *is contained in* $[2(\mu_j - \lambda_k), 0]$ $(\Omega(\psi_{jk}) \subset [2(\mu_j - \lambda_k), 0]$*, where* $\mu_j < \lambda_k$*), and*

$$\psi_{jk}(z) = \begin{cases} 0 , & \text{if } \lambda_k < \lambda_j , \\ c_{jk} = \text{const} \neq 0 , & \text{if } \lambda_k = \lambda_j . \end{cases} \tag{3.4.1}$$

Proof. Up to equality (3.1.7) the proof of this theorem runs in an analogous way as the proof of Theorem 3.1.3 (with obvious changes in the corresponding necessary different places). Now assume that $\mu_j < \lambda_k$. We will rewrite equality (3.1.6) in the following way:

$$(\Phi_-)_{jk}(x) e^{2i(\lambda_k - \mu_j)x} = \widetilde{(\Phi_-)}_{jk}(x) . \tag{3.4.2}$$

In (3.4.2) the right-hand side belongs to AP_+ class and in the left-hand side we have a product of AP_- and AP_+ functions. Thus to have the equality we must guaranty that the left-hand side is also belongs to AP_+, and therefore $(\Phi_-)_{jk}$ must have Bohr-Fourier spectrum contained in $[2(\mu_j - \lambda_k), 0]$. From (3.4.2) we have that

$$(\Phi_-)_{jk} \in e^{2i(\mu_j - \lambda_k)x} AP_+ \cap AP_- = \{\varphi \in AP : \Omega(\varphi) \subset [2(\mu_j - \lambda_k), 0]\} .$$

To deduce the last formula one needs to note that the multiplication of an almost periodic function by an e_λ element shift the Bohr-Fourier spectra of the first function by the value of λ. Additionally, from (3.4.2) it is readily seen that the almost periodic functions which satisfy that equality have Bohr-Fourier spectrum distributed symmetrically with respect to a "central" point $\mu_j - \lambda_k$. More precisely we have that if $x_0 \in [2(\mu_j - \lambda_k), 0]$, $\mu_j - \lambda_k - x_0 \in \Omega(\varphi)$, and if φ satisfies the equality (3.4.2), then necessarily we have that also $\mu_j - \lambda_k + x_0 \in \Omega(\varphi)$. As a remark, in this case, for APW functions we have that the Bohr-Fourier spectrum of that functions contain only the "central" $\mu_j - \lambda_k$ point.

The proof of the last part of this theorem also runs in a similar way as the proof of Theorem 3.1.3, and therefore is omitted in here (cf. the last two paragraphs in the proof of Theorem 3.1.3). $\square$

Chapter 4

Matrix Wiener-Hopf-Hankel operators with SAP Fourier symbols

In this chapter conditions for the Fredholm property of Wiener-Hopf plus/minus Hankel operators with matrix semi-almost periodic Fourier symbols are exhibited. Under such conditions, a formula for the sum of the Fredholm indices of these Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators is derived. Concrete examples are worked out in view of the computation of the Fredholm indices. Within this context, the main goal of this chapter is to present a formula for the Fredholm index of the matrix operator which has the following diagonal form

$$\mathfrak{D}_\Phi = \begin{bmatrix} W_\Phi + H_\Phi & 0 \\ 0 & W_\Phi - H_\Phi \end{bmatrix} : [L^2_+(\mathbb{R})]^{2N} \to [L^2(\mathbb{R}_+)]^{2N} \tag{4.0.1}$$

in the case where the entries of the matrix Φ are semi-almost periodic functions (and under certain conditions in which we will obtain a Fredholm property characterization of $\mathfrak{D}_\Phi$).

4.1 Matrix AP factorization

Since our results will be obtained through certain factorizations of the involved matrix functions, we will therefore recall the definitions of the so-called right and left AP

factorization.

DEFINITION 4.1.1. *A matrix function* $\Phi \in \mathcal{G}AP^{N\times N}$ *is said to admit a* right AP factorization *if it can be represented in the form*

$$\Phi(x) \;=\; \Phi_-(x)\, D(x)\, \Phi_+(x) \tag{4.1.1}$$

for all $x \in \mathbb{R}$, *with*

$$\Phi_- \in \mathcal{G}AP_-^{N\times N}, \quad \Phi_+ \in \mathcal{G}AP_+^{N\times N},$$

and D *is a diagonal matrix of the form*

$$D(x) = \mathrm{diag}(e^{i\lambda_1 x}, \ldots, e^{i\lambda_N x}), \quad \lambda_j \in \mathbb{R}\,.$$

The numbers λ_j *are called the* right AP indices *of the factorization. A right AP factorization with* $D = I_{N\times N}$ *is referred to be a* canonical right AP factorization.

In another way, it is said that a matrix function $\Phi \in \mathcal{G}AP^{N\times N}$ *admits a left AP factorization if instead of (4.1.1) we have*

$$\Phi(x) \;=\; \Phi_+(x)\, D(x)\, \Phi_-(x)$$

for all $x \in \mathbb{R}$, *and* $\Phi_\pm$ *and* D *having the same properties as above.*

REMARK 4.1.2. It is readily seen from the above definition that if an invertible almost periodic matrix function Φ admits a right AP factorization, then $\widetilde{\Phi}$ admits a left AP factorization, and also Φ^{-1} admits a left AP factorization.

The vector containing the right AP indices will be denoted by $k(\Phi)$, i.e., in the above case $k(\Phi) := (\lambda_1, \ldots, \lambda_N)$. If we consider the case with equal *right AP indices* ($k(\Phi) = (\lambda_1, \lambda_1, \ldots, \lambda_1)$), then the matrix

$$\mathbf{d}(\Phi) := M(\Phi_-)M(\Phi_+)$$

is independent of the particular choice of the right AP factorization (cf., e.g., [14, Proposition 8.4]). In this case the matrix $\mathbf{d}(\Phi)$ is called the *geometric mean of* Φ.

In order to relate operators and to transfer certain operator properties between the related operators, we will also need the known notion of equivalence after extension for bounded linear operators (recall Chapter 1).

4.2 The Fredholm property

We start by recalling a known Fredholm characterization for Wiener-Hopf operators with SAP matrix Fourier symbols having lateral almost periodic representatives admitting right AP factorizations.

THEOREM 4.2.1. *(cf. [14, Theorem 10.11] and [46]) Let $\Phi \in SAP^{N\times N}$ and assume that the almost periodic representatives Φ_ℓ and Φ_r admit a right AP factorization. Then the Wiener-Hopf operator W_Φ is Fredholm if and only if:*

(i) $\Phi \in \mathcal{G}SAP^{N\times N}$;

(ii) The almost periodic representatives Φ_ℓ and Φ_r admit canonical right AP factorizations (therefore with $k(\Phi_\ell) = k(\Phi_r) = (0,\ldots,0)$);

(iii) $\mathrm{sp}[\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)] \cap (-\infty, 0] = \emptyset$, *where* $\mathrm{sp}[\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)]$ *stands for the set of the eigenvalues of the matrix*

$$\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell) := [\mathbf{d}(\Phi_r)]^{-1}\mathbf{d}(\Phi_\ell) .$$

The matrix version of Sarason's Theorem (cf. Theorem 2.2.2) says that if $\Phi \in \mathcal{G}SAP^{N\times N}$ then this matrix admits the following representation

$$\Phi = (1-u)\Phi_\ell + u\Phi_r + \Phi_0 , \tag{4.2.1}$$

where $\Phi_{\ell,r} \in \mathcal{G}AP^{N\times N}$, $u \in C(\bar{\mathbb{R}})$, $u(-\infty) = 0$, $u(+\infty) = 1$, $\Phi_0 \in C_0(\dot{\mathbb{R}})$. From (4.2.1) it follows that

$$\widetilde{\Phi^{-1}} = [(1-\tilde{u})\widetilde{\Phi_\ell} + \tilde{u}\widetilde{\Phi_r} + \widetilde{\Phi_0}]^{-1} ,$$

and

$$\Phi\widetilde{\Phi^{-1}} = [(1-u)\Phi_\ell + u\Phi_r + \Phi_0][(1-\tilde{u})\widetilde{\Phi_\ell} + \tilde{u}\widetilde{\Phi_r} + \widetilde{\Phi_0}]^{-1} . \tag{4.2.2}$$

Using the basic properties of the almost periodic functions (cf., e.g., in Section 2.1 the Bochner definition), from (4.2.2) we obtain that

$$(\Phi\widetilde{\Phi^{-1}})_\ell = \Phi_\ell\widetilde{\Phi_r^{-1}} , \quad (\Phi\widetilde{\Phi^{-1}})_r = \Phi_r\widetilde{\Phi_\ell^{-1}} . \tag{4.2.3}$$

These representations, and the equivalence after extension relation described in Section 1.7 between the operator $\mathfrak{D}_\Phi$ and the pure Wiener-Hopf operator, lead to the following characterization in the case when $\Phi_\ell\widetilde{\Phi_r^{-1}}$ admits a right AP factorization.

THEOREM 4.2.2. *Let $\Phi \in SAP^{N\times N}$, and assume that $\Phi_\ell\widetilde{\Phi_r^{-1}}$ admits a right AP factorization. In this case, the operator $\mathfrak{D}_\Phi$ is Fredholm if and only if the following three conditions are satisfied:*

(j) $\Phi \in \mathcal{G}SAP^{N\times N}$;

(jj) $\Phi_\ell\widetilde{\Phi_r^{-1}}$ *admits a canonical right AP factorization;*

(jjj) $\mathrm{sp}[\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})] \cap i\,\mathbb{R} = \emptyset$

(where, as before, Φ_ℓ and Φ_r are the local representatives at $\mp\infty$ of Φ, cf. (4.2.1)).

Proof. If $\mathfrak{D}_\Phi$ is a Fredholm operator, then by Theorems 1.9.2 and 1.9.3 it follows that $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$, and therefore condition (j) arises (cf. Remark 2.2.4).

Moreover, the Fredholm property of $\mathfrak{D}_\Phi$ also implies that the operator $W_{\Phi\widetilde{\Phi^{-1}}}$ is Fredholm (due to the equivalence after extension relation, cf. (1.7.1)). Employing now Theorem 4.2.1 we will obtain that $\Phi\widetilde{\Phi^{-1}} \in \mathcal{G}SAP^{N\times N}$, $(\Phi\widetilde{\Phi^{-1}})_\ell$ and $(\Phi\widetilde{\Phi^{-1}})_r$ admit a canonical right AP factorizations and

$$\mathrm{sp}[\mathbf{d}^{-1}((\Phi\widetilde{\Phi^{-1}})_r)\mathbf{d}((\Phi\widetilde{\Phi^{-1}})_\ell)] \cap (-\infty, 0] = \emptyset\,. \tag{4.2.4}$$

By virtue of (4.2.3) we conclude that $\Phi_\ell\widetilde{\Phi_r^{-1}}$ admits a canonical right AP factorization. Once again, due to (4.2.3), from (4.2.4) we derive that

$$\mathrm{sp}[\mathbf{d}^{-1}(\Phi_r\widetilde{\Phi_\ell^{-1}})\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})] \cap (-\infty, 0] = \emptyset\,. \tag{4.2.5}$$

In addition, a canonical right AP factorization of $\Phi_\ell\widetilde{\Phi_r^{-1}}$ can be normalized into

$$\Phi_\ell\widetilde{\Phi_r^{-1}} = \Psi_-\Lambda\Psi_+\,, \tag{4.2.6}$$

where $\Psi_\pm$ have the same factorization properties as the original lateral factors of the canonical factorization but with $M(\Psi_\pm) = I$, and where $\Lambda := \mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})$. Thus, (4.2.6) allows

$$\Phi_r \widetilde{\Phi_\ell^{-1}} = \left(\widetilde{\Phi_\ell \widetilde{\Phi_r^{-1}}}\right)^{-1} = \widetilde{\Psi_+^{-1}} \Lambda^{-1} \widetilde{\Psi_-^{-1}}\,,$$

which in particular shows that

$$\mathbf{d}(\Phi_r \widetilde{\Phi_\ell^{-1}}) = \Lambda^{-1}.$$

Thus, (4.2.5) turns out to be equivalent to

$$\mathrm{sp}[\Lambda^2] \cap (-\infty, 0] = \emptyset$$

which directly from the eigenvalue definition leads to

$$\mathrm{sp}[\Lambda] \cap i\mathbb{R} = \emptyset\,.$$

Therefore the proposition (jjj) is satisfied, and the proof of the "only if" part is completed.

Now we will be concerned with the "if" part. From the hypothesis that $\Phi \in \mathcal{G}SAP^{N\times N}$ we can consider $\Phi\widetilde{\Phi^{-1}}$, and therefore this is also invertible in $SAP^{N\times N}$. The left and right representatives of $\Phi\widetilde{\Phi^{-1}}$ are given by the formula (4.2.3). Since $\Phi_\ell \widetilde{\Phi_r^{-1}} = (\Phi\widetilde{\Phi^{-1}})_\ell$ admits a canonical right AP factorization, then

$$(\widetilde{\Phi\widetilde{\Phi^{-1}}})_\ell = \widetilde{\Phi_\ell}\Phi_r^{-1}$$

admits a canonical left AP factorization and

$$[(\widetilde{\Phi\widetilde{\Phi^{-1}}})_\ell]^{-1} = \Phi_r \widetilde{\Phi_\ell^{-1}}$$

admits a canonical right AP factorization. Moreover, using the same reasoning as in the "only if" part, these last two canonical right AP factorizations and condition (jjj) imply that

$$\mathrm{sp}[\mathbf{d}^{-1}((\Phi\widetilde{\Phi^{-1}})_r)\mathbf{d}((\Phi\widetilde{\Phi^{-1}})_\ell)] \cap (-\infty, 0] = \mathrm{sp}[\mathbf{d}^{-1}(\Phi_r \widetilde{\Phi_\ell^{-1}})\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})] \cap (-\infty, 0] = \emptyset.$$

All these facts together with Theorem 4.2.1 give us that $W_{\Phi\widetilde{\Phi^{-1}}}$ is a Fredholm operator. Using now the equivalence after extension relation presented in (1.7.1), we obtain that $\mathfrak{D}_\Phi$ is a Fredholm operator. $\square$

REMARK 4.2.3. In the last Theorem 4.2.2: (i) if the symbol Φ belongs to the Wiener subalgebra of $SAP^{N\times N}$, i.e. $SAPW^{N\times N}$ (recall Chapter 2), then we can drop the initial assumption which states that $\Phi_\ell\widetilde{\Phi_r^{-1}}$ admits a right AP factorization (since this holds always in such a case); (ii) if the lateral representatives of Φ are $N\times N$ constant matrices (having therefore an even more particular situation than in (i)), then the symbol Φ belongs to $[C(\bar{\mathbb{R}})]^{N\times N}$, and the theorem provides an alternative Fredholm property characterization of such operators to the already known characterizations for this particular case (and which were obtained by different methods; cf., e.g., [39] and [54]).

4.3 Index formula for the sum of Wiener-Hopf plus/minus Hankel operators

In the present section we will be concentrated in obtaining a Fredholm index formula for $\mathfrak{D}_\Phi$, i.e., for the sum of Wiener-Hopf plus/minus Hankel operators $W_\Phi \pm H_\Phi$ with Fourier symbols $\Phi \in \mathcal{G}SAP^{N\times N}$ such that $\Phi_\ell\widetilde{\Phi_r^{-1}}$ admits a right AP factorization. Due to this purpose, let us assume from now on that $W_\Phi + H_\Phi$ and $W_\Phi - H_\Phi$ are Fredholm operators.

Let $\mathcal{G}SAP_{0,0}$ denotes the set of all functions $\varphi \in \mathcal{G}SAP$ for which $k(\varphi_\ell) = k(\varphi_r) = 0$. To define the Cauchy index of $\varphi \in \mathcal{G}SAP_{0,0}$ we need the next lemma.

LEMMA 4.3.1. *[14, Lemma 3.12] Let $A \subset (0,\infty)$ be an unbounded set and let*

$$\{I_\alpha\}_{\alpha\in A} := \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$$

be a family of intervals such that $x_\alpha \geq 0$ and $|I_\alpha| = y_\alpha - x_\alpha \to \infty$, as $\alpha \to \infty$. If $\varphi \in \mathcal{G}SAP_{0,0}$ and $\arg\varphi$ is any continuous argument of φ, then the limit

$$\frac{1}{2\pi}\lim_{\alpha\to\infty}\frac{1}{|I_\alpha|}\int_{I_\alpha}((\arg\varphi)(x) - (\arg\varphi)(-x))dx \tag{4.3.1}$$

exists, is finite, and is independent of the particular choices of $\{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ and $\arg\varphi$.

The limit (4.3.1) is denoted by $\operatorname{ind}\varphi$ and is usually called the *Cauchy index* of φ. Moreover, following [58, section 4.3] we can generalize the notion of Cauchy index for SAP functions with $k(\phi_\ell) + k(\phi_r) = 0$ in the way which was introduced in Lemma 4.3.1, i.e.,

$$\operatorname{ind}\varphi = \frac{1}{2\pi} \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} ((\arg\varphi)(x) - (\arg\varphi)(-x))dx \,, \tag{4.3.2}$$

where $\varphi \in \{\phi \in SAP : k(\phi_\ell) + k(\phi_r) = 0\}$.

The following theorem is well-known, and provides a formula for the Fredholm index of matrix Wiener-Hopf operators with SAP Fourier symbols.

THEOREM 4.3.2. *[14, Theorem 10.12] Let $\Phi \in SAP^{N\times N}$. If the almost periodic representatives Φ_ℓ, Φ_r admit right AP factorizations, and if W_Φ is a Fredholm operator, then*

$$\operatorname{Ind} W_\Phi = -\operatorname{ind}\det\Phi - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg\xi_k \right\} \right), \tag{4.3.3}$$

where $\xi_1, \ldots, \xi_N \in \mathbb{C} \setminus (-\infty, 0]$ are the eigenvalues of the matrix $\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)$ and $\{\cdot\}$ stands for the fractional part of a real number. Additionally, when choosing $\arg\xi_k$ in $(-\pi,\pi)$, we have

$$\operatorname{Ind} W_\Phi = -\operatorname{ind}\det\Phi - \frac{1}{2\pi}\sum_{k=1}^{N} \arg\xi_k \,.$$

We will now be concentrated on an index formula for $\mathfrak{D}_\Phi$ (i.e., on a formula for the sum of the Fredholm indices of $W_\Phi + H_\Phi$ and $W_\Phi - H_\Phi$). In fact, it directly follows from the definition of the operator $\mathfrak{D}_\Phi$ (cf. formula (4.0.1)) that

$$\operatorname{Ind}\mathfrak{D}_\Phi = \operatorname{Ind}[W_\Phi + H_\Phi] + \operatorname{Ind}[W_\Phi - H_\Phi] \,.$$

Now, employing the above equivalence after extension relation (cf. (1.7.1)), we deduce that

$$\operatorname{Ind}\mathfrak{D}_\Phi = \operatorname{Ind} W_{\Phi\widetilde{\Phi^{-1}}} \,.$$

Consequently, we have:

$$\mathrm{Ind}[W_\Phi + H_\Phi] + \mathrm{Ind}[W_\Phi - H_\Phi] = \mathrm{Ind} W_{\Phi\widetilde{\Phi^{-1}}} \ .$$

Using (4.3.3) for $W_{\Phi\widetilde{\Phi^{-1}}}$ (which is a Fredholm operator because of the assumption made in the beginning of the present section), we obtain that

$$\mathrm{Ind} W_{\Phi\widetilde{\Phi^{-1}}} = -\mathrm{ind}[\det(\Phi\widetilde{\Phi^{-1}})] - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \eta_k \right\} \right) , \tag{4.3.4}$$

where $\eta_k \in \mathbb{C} \setminus (-\infty, 0]$ are the eigenvalues of the matrix $\mathbf{d}^{-1}((\Phi\widetilde{\Phi^{-1}})_r)\mathbf{d}((\Phi\widetilde{\Phi^{-1}})_\ell) = \mathbf{d}^{-1}(\Phi_r\widetilde{\Phi_\ell^{-1}})\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})$, cf. (4.2.3). However, as pointed out in the proof of Theorem 4.2.2, $\mathbf{d}^{-1}(\Phi_r\widetilde{\Phi_\ell^{-1}})\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}}) = \left[\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})\right]^2$, and therefore (4.3.4) can be rewritten as

$$\mathrm{Ind} W_{\Phi\widetilde{\Phi^{-1}}} = -\mathrm{ind}[\det(\Phi\widetilde{\Phi^{-1}})] - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{\pi} \arg \zeta_k \right\} \right) \tag{4.3.5}$$

where $\zeta_k \in \mathbb{C} \setminus i\mathbb{R}$ are the eigenvalues of the matrix $\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})$.

In addition, in the case when $\arg \zeta_k$ are chosen in $(-\pi/2, \pi/2)$, formula (4.3.5) is reduced to

$$\mathrm{Ind} W_{\Phi\widetilde{\Phi^{-1}}} = -\mathrm{ind}[\det(\Phi\widetilde{\Phi^{-1}})] - \frac{1}{\pi} \sum_{k=1}^{N} \arg \zeta_k \ .$$

Let us now simplify the form of $\mathrm{ind}[\det(\Phi\widetilde{\Phi^{-1}})]$. Recalling that the matrix $\Phi_\ell\widetilde{\Phi_r^{-1}}$ has a canonical right AP factorization (due to the assumption made in the beginning of the present section), it holds $k(\Phi_\ell\widetilde{\Phi_r^{-1}}) = (0, \ldots, 0)$. From here we obtain that $k(\det(\Phi_\ell\widetilde{\Phi_r^{-1}})) = 0$. Consequently, this yields:

$$\begin{aligned}
k((\det\Phi)_\ell) + k((\det\Phi)_r) &= k(\det(\Phi_\ell)) + k(\det(\Phi_r)) \\
&= k(\det(\Phi_\ell)) + k(\widetilde{\det(\Phi_r)}^{-1}) \\
&= k(\det(\Phi_\ell)) + k(\det(\widetilde{\Phi_r^{-1}})) \\
&= k(\det(\Phi_\ell)\det(\widetilde{\Phi_r^{-1}})) \\
&= k(\det(\Phi_\ell\widetilde{\Phi_r^{-1}})) && (4.3.6) \\
&= 0 \ , && (4.3.7)
\end{aligned}$$

because for any $f \in \mathcal{G}AP$ we have $k(f) = k(\widetilde{f^{-1}})$. Additionally, we have used here the fact that $[\det \Phi]_\ell = \det \Phi_\ell$ (which follows from a direct computation). A similar argument also applies to $\widetilde{\Phi^{-1}}$, since from (4.3.6)–(4.3.7) we have:

$$\begin{aligned} k((\det \widetilde{\Phi^{-1}})_\ell) + k((\det \widetilde{\Phi^{-1}})_r) &= k(\det(\widetilde{\Phi_\ell^{-1}})) + k(\det(\widetilde{\Phi_r^{-1}})) \\ &= k(\widetilde{\det(\Phi_\ell)}^{-1}) + k(\det(\widetilde{\Phi_r^{-1}})) \\ &= k(\det(\Phi_\ell)) + k(\det(\widetilde{\Phi_r^{-1}})) \\ &= k(\det(\Phi_\ell \widetilde{\Phi_r^{-1}})) \\ &= 0 \,. \end{aligned} \tag{4.3.8}$$

Therefore, due to (4.3.7), (4.3.8) and (4.3.2), we can perform the following computations:

$$\begin{aligned} \operatorname{ind}[\det(\Phi \widetilde{\Phi^{-1}})] &= \operatorname{ind}[\det \Phi \det \widetilde{\Phi^{-1}}] \\ &= \operatorname{ind} \det \Phi + \operatorname{ind} \det \widetilde{\Phi^{-1}} \\ &= \operatorname{ind} \det \Phi + \operatorname{ind}[\widetilde{\det \Phi^{-1}}] \\ &= \operatorname{ind} \det \Phi - \operatorname{ind}[\det \Phi^{-1}] \\ &= \operatorname{ind} \det \Phi - \operatorname{ind}[\det \Phi]^{-1} \\ &= \operatorname{ind} \det \Phi + \operatorname{ind} \det \Phi \\ &= 2 \operatorname{ind} \det \Phi \,. \end{aligned}$$

Consequently, we have just deduced the following result.

COROLLARY 4.3.3. *Let $\Phi \in \mathcal{G}SAP^{N\times N}$ and assume that $\Phi_\ell \widetilde{\Phi_r^{-1}}$ admits a right AP factorization. If $W_\Phi \pm H_\Phi$ are Fredholm operators, then*

$$\operatorname{Ind}[W_\Phi + H_\Phi] + \operatorname{Ind}[W_\Phi - H_\Phi] = -2 \operatorname{ind} \det \Phi - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{\pi} \arg \zeta_k \right\} \right), \tag{4.3.9}$$

where $\zeta_k \in \mathbb{C} \setminus i\mathbb{R}$ are the eigenvalues of the matrix $\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})$. Moreover, when choosing $\arg \zeta_k$ *in $(-\pi/2, \pi/2)$, the formula (4.3.9) simplifies to the following one:*

$$\operatorname{Ind}[W_\Phi + H_\Phi] + \operatorname{Ind}[W_\Phi - H_\Phi] = -2 \operatorname{ind} \det \Phi - \frac{1}{\pi} \sum_{k=1}^{N} \arg \zeta_k \,.$$

4.4 Generalized AP factorization

In order to obtain stronger versions of the above derived results we need to present the notion of generalized AP factorization. This will be done in the present section.

Let $B^2_\pm$ denote the Hilbert spaces consisting of the functions in B^2 with the Bohr-Fourier spectra in $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x \geq 0\}$ (recall Chapter 2).

DEFINITION 4.4.1. *A generalized right AP factorization of a matrix function* $\Phi \in \mathcal{G}AP^{N\times N}$ *is a representation*

$$\Phi = \Phi_- D \Phi_+ ,$$

where $D = \operatorname{diag}(e_{\lambda_1}, \ldots, e_{\lambda_N})$ *with* $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ *and*

$$\Phi_- \in \mathcal{G}[B^2_-]^{N\times N}, \quad \Phi_+ \in \mathcal{G}[B^2_+]^{N\times N}, \quad \Phi_- \widetilde{P} \Phi_-^{-1} I \in \mathcal{L}(B^2_N) .$$

Here $\widetilde{P}$ *is the projection* $\widetilde{P} := F_B \chi_+ F_B^{-1} \in \mathcal{L}(B^2_N)$ *(with* χ_+ *being the characteristic function of* $\mathbb{R}_+$*).*

The left generalized AP factorization is defined correspondingly (with obvious changes). If we have that Φ admits a right generalized AP factorization, then $\widetilde{\Phi}$ admits a left generalized AP factorization and also Φ^{-1} admits a left generalized AP factorization.

The definition of the geometric mean value is literally the same as in Section 4.1.

4.5 The Fredholm property for matrix SAP symbols

In this section we give a stronger version of the Theorem 4.2.2. To this end we need the stronger version of the Theorem 4.2.1, which is now stated.

THEOREM 4.5.1. *[14, Theorem 18.18] Let* $\Phi \in SAP^{N\times N}$*. The operator* W_Φ *is Fredholm if and only if the following three conditions are satisfied:*

(i) $\Phi \in \mathcal{G}SAP^{N\times N}$,

(ii) W_{Φ_ℓ} *and* W_{Φ_r} *are invertible operators,*

(iii) $\mathrm{sp}[\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)] \cap (-\infty, 0] \neq \emptyset$, *where* $\mathrm{sp}[\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)]$ *stands for the set of the eigenvalues of the matrix*

$$\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell) := [\mathbf{d}(\Phi_r)]^{-1}\mathbf{d}(\Phi_\ell) \, .$$

Equipped with the last theorem we are ready to give a stronger version of Theorem 4.2.2. Since this is obtained by using the same techniques which were exhibited in the proof of Theorem 4.2.2, the proof of the next theorem will be omitted in here.

THEOREM 4.5.2. *Let $\Phi \in SAP^{N\times N}$. Then $\mathfrak{D}_\Phi$ is Fredholm if and only if the following three conditions are satisfied:*

(j) $\Phi \in \mathcal{G}SAP^{N\times N}$,

(jj) $\Phi_\ell \widetilde{\Phi_r^{-1}}$ *admits a canonical generalized right AP factorization,*

(jjj) $\mathrm{sp}[\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})] \cap i\,\mathbb{R} = \emptyset$,

(where, as before, Φ_ℓ and Φ_r are the local representatives at $\mp\infty$ of Φ, cf. (4.2.1)).

4.6 Index formula for the operator $\mathfrak{D}_\Phi$

In this section our aim is to obtain a Fredholm index formula for the operator $\mathfrak{D}_\Phi$ with $\Phi \in SAP^{N\times N}$. Therefore, let us start with a result which is a particular case of Corollary 4.3.3 (considering the Wiener subclass of $SAP^{N\times N}$).

COROLLARY 4.6.1. *If $W_\Phi \pm H_\Phi$ are Fredholm operators for some $\Phi \in \mathcal{G}SAPW^{N\times N}$, then*

$$\mathrm{Ind}[W_\Phi + H_\Phi] + \mathrm{Ind}[W_\Phi - H_\Phi] = -2 \,\mathrm{ind} \det \Phi - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{\pi} \arg \zeta_k \right\} \right) , \quad (4.6.1)$$

where $\zeta_k \in \mathbb{C} \setminus i\mathbb{R}$ are the eigenvalues of the matrix $\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})$. Moreover, when choosing $\arg \zeta_k$ in $(-\pi/2, \pi/2)$, the formula (4.6.1) simplifies to the following one:

$$\mathrm{Ind}[W_\Phi + H_\Phi] + \mathrm{Ind}[W_\Phi - H_\Phi] = -2 \,\mathrm{ind} \det \Phi - \frac{1}{\pi} \sum_{k=1}^{N} \arg \zeta_k \, . \quad (4.6.2)$$

The main result of this section will be now reached by removing the Wiener subclass from the last corollary.

THEOREM 4.6.2. *If $\mathfrak{D}_\Phi$ is a Fredholm operator for some $\Phi \in \mathcal{G}SAP^{N\times N}$, then*

$$\mathrm{Ind}\mathfrak{D}_\Phi = -2 \,\mathrm{ind}\det\Phi - \sum_{k=1}^{N}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{\pi}\arg\zeta_k\right\}\right), \tag{4.6.3}$$

where $\zeta_k \in \mathbb{C}\setminus i\mathbb{R}$ are the eigenvalues of the matrix $\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})$. Moreover, when choosing $\arg\zeta_k$ in $(-\pi/2, \pi/2)$, the formula (4.6.3) simplifies to the following one:

$$\mathrm{Ind}\mathfrak{D}_\Phi = -2 \,\mathrm{ind}\det\Phi - \frac{1}{\pi}\sum_{k=1}^{N}\arg\zeta_k .$$

Proof. Choose functions $\Phi_n \in \mathcal{G}SAPW^{N\times N}$, such that

$$\lim_{n\to\infty}\sup|\Phi_n - \Phi| = 0 .$$

This is always possible because $SAPW$ is a dense subalgebra of SAP. Stability of the Fredholm property under small perturbations implies that $\mathfrak{D}_{\Phi_n}$ are Fredholm operators for sufficiently large n. From here it follows that the index formula for $\mathfrak{D}_{\Phi_n}$ is given by (4.6.1) (the case of the formula (4.6.2) is treated analogously). Hence we can write:

$$\mathrm{Ind}\mathfrak{D}_{\Phi_n} = -2 \,\mathrm{ind}\det\Phi_n - \sum_{k=1}^{N}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{\pi}\arg(\zeta_k)_n\right\}\right). \tag{4.6.4}$$

As far as we have the uniform convergence of Φ_n to Φ and the stability property for the Fredholm index we can pass to the limit in the equality (4.6.4) to deduce the desired formula (4.6.3). □

4.7 Examples

In the present section we provide two concrete examples for the above theory.

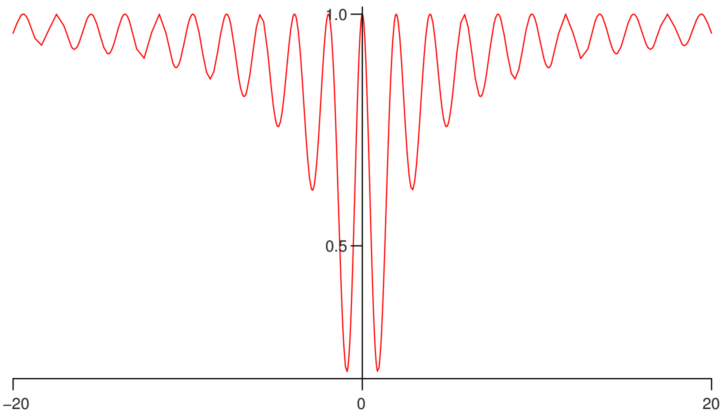

Figure 4.1: The graph of $\det \Phi$.

4.7.1 Example 1

Let us assume that

$$\begin{aligned}\Phi(x) &= (1-u(x))\begin{bmatrix} e^{-i(1+\alpha)x} & 0 \\ e^{-i\alpha x}-1+e^{ix} & e^{i(1+\alpha)x}\end{bmatrix} \\ &\quad +u(x)\begin{bmatrix} e^{i(1+\alpha)x} & 0 \\ e^{i\alpha x}-1+e^{-ix} & e^{-i(1+\alpha)x}\end{bmatrix},\end{aligned}$$

where $\alpha = (\sqrt{5}-1)/2$, and u is the following real-valued function

$$u(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x)\ . \tag{4.7.1}$$

Being clear that $\Phi \in SAP^{2\times 2}$, we will start with showing that $\Phi \in \mathcal{G}SAP^{2\times 2}$. To this end we need first of all to compute the determinant of Φ :

$$\begin{aligned}\det \Phi(x) &= \det \begin{bmatrix} (1-u(x))e^{-i(1+\alpha)x}+u(x)e^{i(1+\alpha)x} & 0 \\ (1-u(x))(e^{-i\alpha x}-1+e^{ix})+u(x)(e^{i\alpha x}-1+e^{-ix}) & (1-u(x))e^{i(1+\alpha)x}+u(x)e^{-i(1+\alpha)x}\end{bmatrix} \\ &= 1-2u(x)(1-u(x))(1-\cos(2(1+\alpha)x))\,.\end{aligned}$$

Recalling that u is a real-valued function given by (4.7.1), we have that

$$2u(x)(1-u(x)) \in \left[0, \frac{1}{2}\right],$$

and where the maximum value 1/2 is obtained only at the point $x = 0$, in view of

$$2u(x)(1-u(x)) = \frac{1}{2} - \frac{2}{\pi^2}\arctan^2(x)\,.$$

Due to the reason that $1-\cos(2(1+\alpha)x) \in [0,2]$ but $(1-\cos(2(1+\alpha)x))\big|_{x=0} = 0$, we have $2u(x)(1-u(x))(1-\cos(2(1+\alpha)x)) \in [0,1)$. Therefore $\det\Phi \in (0,1]$, and thus Φ is invertible.

In this case, although the invertibility of Φ, the Wiener-Hopf operator with symbol Φ is not a Fredholm operator. This happens because the matrix function

$$\begin{bmatrix} e^{-i(1+\alpha)x} & 0 \\ e^{-i\alpha x} - 1 + e^{ix} & e^{i(1+\alpha)x} \end{bmatrix} \in \mathcal{G}AP^{2\times 2}$$

does not have a right AP factorization (cf. [47, pages 284-285] for all the details about this matrix function). However, the Wiener-Hopf plus/minus Hankel operators with the same symbol Φ will have the Fredholm property. Indeed, besides having $\Phi \in \mathcal{G}SAP^{2\times 2}$, let us observe that:

$$\Phi_\ell \widetilde{\Phi_r^{-1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}\,.$$

Consequently, $\Phi_\ell \widetilde{\Phi_r^{-1}}$ obviously admits a canonical right AP factorization and

$$\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}}) = I_{2\times 2}\,.$$

Thus the eigenvalues of this matrix are equal to $1 \notin i\mathbb{R}$. This allows us to conclude that the operator $\mathfrak{D}_\Phi$ is a Fredholm operator (cf. Theorem 4.2.2). This means that the operators $W_\Phi \pm H_\Phi$ have the Fredholm property. Let us now calculate the sum of their Fredholm indices. From the above form of $\det\Phi$, we have already concluded that $\det\Phi \in (0,1]$. Thus, we have that $\det\Phi$ is a real-valued positive function, and so its argument is zero (the graph of $\det\Phi$ is given in the Figure 4.1). Altogether we have:

$$\mathrm{Ind}[W_\Phi + H_\Phi] + \mathrm{Ind}[W_\Phi - H_\Phi] = 0\,,$$

since the eigenvalues of $\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})$ are also real (recall that they are equal to 1), and therefore their arguments are also zero.

4.7.2 Example 2

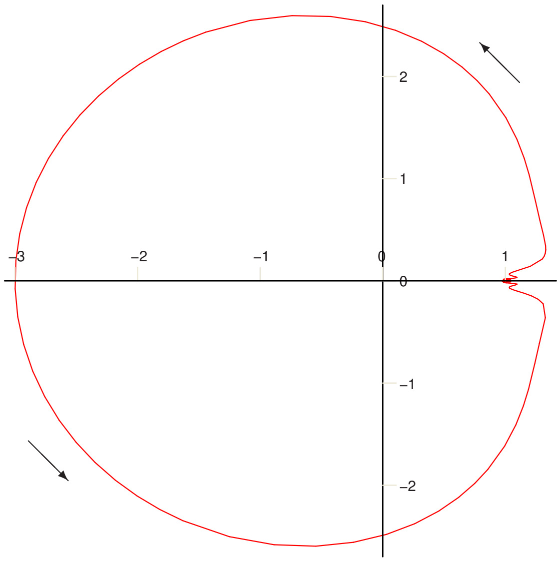

Figure 4.2: The oriented curve representing the range of $\det \Psi$.

Let us now take the following matrix-valued function:

$$\Psi(x) = (1-u(x))\begin{bmatrix} e^{ix} & 0 \\ 0 & e^{-ix}\end{bmatrix} + u(x)\begin{bmatrix} e^{-ix} & 0 \\ 0 & e^{ix}\end{bmatrix} + \begin{bmatrix} 0 & \frac{x-i}{x+i}-1 \\ \frac{x-i}{x+i}-1 & 0\end{bmatrix}, \quad (4.7.2)$$

where u is the same as in the previous example. A direct computation provides that Ψ is invertible. In fact,

$$\det \Psi(x) = (1-u(x))^2 + u^2(x) + 2u(x)(1-u(x))\cos(2x) - \left(\frac{x-i}{x+i}-1\right)^2, \quad (4.7.3)$$

and hence

$$\det \Psi(x) = f(x) + ig(x)\,,$$

where

$$f(x) = 1 - 2u(x)(1-u(x))(1-\cos(2x)) + \frac{4(x^2-1)}{(x^2+1)^2}\,,$$

and

$$g(x) = -\frac{8x}{(x^2+1)^2}\,.$$

From these formulas it follows that f and g do not vanish simultaneously, and consequently $\det \Psi(x) \neq 0$ for all $x \in \dot{\mathbb{R}}$. Hence $\Psi \in \mathcal{G}SAP^{2\times 2}$. Although having this necessary condition for the Fredholm property of the Wiener-Hopf operator with the symbol Ψ, it is easily seen that W_Ψ is not Fredholm. The reason for this is based on the fact that although the matrix-valued functions

$$\Psi_\ell(x) = \begin{bmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{bmatrix}$$

and

$$\Psi_r(x) = \begin{bmatrix} e^{-ix} & 0 \\ 0 & e^{ix} \end{bmatrix}$$

have obvious right AP factorizations (with the identity matrix in the role of the lateral minus and plus factors; cf. (4.1.1)) they do not have a canonical right AP factorization (since $k(\Psi_\ell) = (1,-1)$ and $k(\Psi_r) = (-1,1)$).

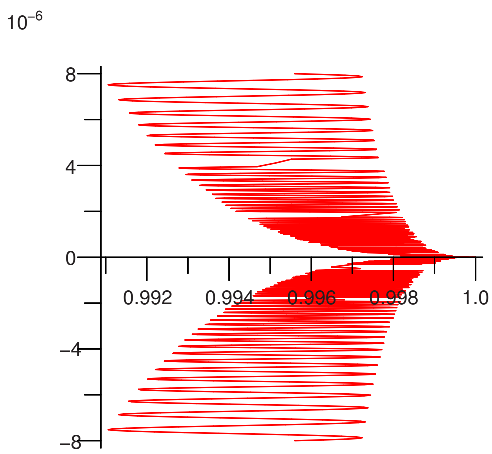

Figure 4.3: The oscillation at infinity of $\det \Psi$.

Despite this situation, the Wiener-Hopf plus/minus Hankel operators with the same symbol Ψ are Fredholm. Indeed, first of all note that we have already deduced that Ψ is invertible in $SAP^{2\times 2}$. Moreover,

$$\Psi_\ell \widetilde{\Psi_r^{-1}} = I_{2\times 2} \tag{4.7.4}$$

and therefore $\Psi_\ell \widetilde{\Psi_r^{-1}}$ has the trivial canonical right AP factorization. From (4.7.4) we also obtain that

$$\mathrm{sp}[\mathbf{d}(\Psi_\ell \widetilde{\Psi_r^{-1}})] = \{1\} \cap i\mathbb{R} = \emptyset \, .$$

These are sufficient conditions for these Wiener-Hopf plus/minus Hankel operators to have the Fredholm property (cf. Theorem 4.2.2).

To calculate the sum of the Fredholm indices of these Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators we need once again to use the above computed determinant of Ψ. From (4.7.3) it follows that $\operatorname{ind}\det\Psi = 1$ (the range of the $\det\Psi$ is given in the Figure 4.2, and it is a closed curve; note also that $\lim_{x\to\pm\infty}\det\Psi(x) = 1$, and in the Figure 4.3 is shown the oscillation of the function $\det\Psi$ at infinity). Therefore, from formula (4.6.2), we obtain that

$$\mathrm{Ind}[W_\Psi + H_\Psi] + \mathrm{Ind}[W_\Psi - H_\Psi] = -2$$

for Ψ in (4.7.2).

Chapter 5

Matrix Wiener-Hopf-Hankel operators with PAP Fourier symbols

In this chapter it is obtained a Fredholm property characterization for matrix Wiener-Hopf plus/minus Hankel operators with piecewise almost periodic Fourier symbols. The conditions that ensure the Fredholm property are mainly based on factorizations of certain almost periodic matrix functions, and spectral properties of some others. In addition, Fredholm index formulas are also obtained based on an extension of the Cauchy index notion which is therefore applied to some new functions derived from the symbols of the operators.

In more detail, the main result in the present chapter (Theorem 5.4.1) provides a Fredholm characterization and an index formula for the following diagonal matrix operator:

$$\mathfrak{D}_\Phi = \begin{bmatrix} W_\Phi + H_\Phi & 0 \\ 0 & W_\Phi - H_\Phi \end{bmatrix} : [L^2_+(\mathbb{R})]^{2N} \to [L^2(\mathbb{R}_+)]^{2N} , \tag{5.0.1}$$

where Φ belongs to the piecewise almost periodic function class (recall Chapter 2). Therefore, the present chapter extends the results of Chapter 4 (cf. also [13]) where the Fredholm property and index of the operator $\mathfrak{D}_\Phi$ were described but only for Fourier symbols in the subclass of semi-almost periodic matrix functions. In addition, it complements some other recent results like the ones of [61].

In the next sections we will prepare the necessary material for the main theorem which will only appear in the Section 5.4. In this sense, in the previous sections to the Section 5.4, we will present some known results and generalize some others to a corresponding matrix version. Moreover, using the generalized AP factorization in Section 5.5 we will be able to give the stronger results, and we will end this chapter with examples illustrating the developed theory.

5.1 Preliminary notation and results

We recall here some of the essential facts from the theory of Wiener-Hopf and Hankel operators. The following equality is well-known (recall (1.6.1)):

$$W_{\Phi\Psi} = W_{\Phi}\ell_0 W_{\Psi} + H_{\Phi}\ell_0 H_{\widetilde{\Psi}} \; , \tag{5.1.1}$$

for $\Phi, \Psi \in [L^\infty(\mathbb{R})]^{N\times N}$. The next proposition is the matrix version of the classical scalar case, which is obviously also valid for the matrix case (one can derive the matrix case result by using the scalar one entrywise).

PROPOSITION 5.1.1. *If $\Theta \in [C(\dot{\mathbb{R}})]^{N\times N}$, then the Hankel operators H_Θ and $H_{\widetilde{\Theta}}$ are compact.*

We can equivalently rewrite (5.1.1) as $W_{\Phi\Psi} - W_{\Phi}\ell_0 W_{\Psi} = H_{\Phi}\ell_0 H_{\widetilde{\Psi}}$, and therefore Proposition 5.1.1 directly yields the following known result.

THEOREM 5.1.2. *If $\Phi, \Psi \in [L^\infty(\mathbb{R})]^{N\times N}$ and at least one of the functions Φ, Ψ belongs to $[C(\dot{\mathbb{R}})]^{N\times N}$, then $W_{\Phi\Psi} - W_{\Phi}\ell_0 W_{\Psi}$ is compact.*

Now, employing a continuous partition of the identity, one can sharpen Theorem 5.1.2 as follows.

THEOREM 5.1.3. *If $\Phi, \Psi \in PC^{N\times N}$ and if at each point $x_0 \in \dot{\mathbb{R}}$ at least one of the functions Φ and Ψ is continuous, then $W_{\Phi\Psi} - W_{\Phi}\ell_0 W_{\Psi}$ is compact.*

Proof. The result can be proved by following the same arguments as in the scalar case [52, Lemma 16.2], with corresponding changes for matrices in the places of functions. Namely, let $x_1, \ldots, x_\ell$ and $x_{\ell+1}, \ldots, x_r$ denote all the points of discontinuity of the matrix functions Φ and Ψ, respectively. Then, let Θ and Ξ be continuous matrix functions on $\dot{\mathbb{R}}$ with the following properties: $\Theta(x_k) = 0_{N\times N},\ k = 1, \ldots, \ell,\ \Xi(x_k) = 0_{N\times N},\ k = \ell+1, \ldots, r$, and $\Theta + \Xi \equiv I_{N\times N}$. This construction of Θ and Ξ turn clear that $\Phi\Theta$ and $\Xi\Psi$ are continuous on $\dot{\mathbb{R}}$. From Theorem 5.1.2 and $\Theta + \Xi = I_{N\times N}$, we have

$$\begin{aligned}
W_{\Phi\Psi} &= W_{\Phi(\Theta+\Xi)\Psi} = W_{\Phi\Theta\Psi} + W_{\Phi\Xi\Psi} = W_{\Phi\Theta}\ell_0 W_\Psi + K_1 + W_\Phi \ell_0 W_{\Xi\Psi} + K_2 \\
&= W_{\Phi\Theta}\ell_0 W_\Psi + W_\Phi \ell_0 W_{\Xi\Psi} + K_3 \\
&= (W_\Phi \ell_0 W_\Theta + K_4)\ell_0 W_\Psi + W_\Phi \ell_0 (W_\Xi W_\Psi + K_5) + K_3 \\
&= W_\Phi \ell_0 W_\Theta \ell_0 W_\Psi + K_6 + W_\Phi \ell_0 W_\Xi \ell_0 W_\Psi + K_7 + K_3 \\
&= W_\Phi \ell_0 (W_\Theta + W_\Xi)\ell_0 W_\Psi + K_8 \\
&= W_\Phi \ell_0 W_\Psi + K_8 \ ,
\end{aligned}$$

where K_i are compact operators ($i = 1, \ldots, 8$). From here we derive that $W_{\Phi\Psi} - W_\Phi \ell_0 W_\Psi$ is compact. □

5.2 Wiener-Hopf operators with PC matrix symbols

For $\Phi \in PC^{N\times N}$, it is well-known the importance for the following auxiliary extension of Φ:

$$\Phi_2(x, \mu) := (1-\mu)\Phi(x-0) + \mu\Phi(x+0), \quad (x, \mu) \in \dot{\mathbb{R}} \times [0, 1] \ ,$$

where $\Phi(x \pm 0)$ denotes the one-sided limits at the point x. This obviously yields $\det \Phi_2$ to have the form

$$\det \Phi_2(x, \mu) = \det[(1-\mu)\Phi(x-0) + \mu\Phi(x+0)], \quad (x, \mu) \in \dot{\mathbb{R}} \times [0, 1] \, ,$$

and maps $\dot{\mathbb{R}} \times [0, 1]$ into $\mathbb{C}$. One of the peculiarities of $\det \Phi_2$ is that it allows the consideration of

$$\mathcal{C} := \{\det \Phi_2(x, \mu) \in \mathbb{C} : x \in \dot{\mathbb{R}} \ , \ \mu \in [0, 1]\} \ ,$$

as a closed continuous curve formed by the union of the curve generated by the image of $\det\Phi$ and the curve that joins $\det\Phi_2(x-0)$ to $\det\Phi_2(x+0)$ through a line segment, at the discontinuity points of $\det\Phi$. In the case of $0 \notin \mathcal{C}$, it is therefore possible to consider the winding number of $\mathcal{C}$, with respect to the origin, as the number of the counter-clockwise circuits around the origin performed by the image of $\det\Phi_2$. In such a case, this winding number will be denoted by $\mathrm{wind}[\det\Phi_2]$.

The next theorem is now considered a classical (Duduchava) result in the Fredholm theory of Wiener-Hopf operators, and there the winding number plays a fundamental role.

THEOREM 5.2.1. *[34, Theorem 4.2] Let $\Phi \in PC^{N\times N}$.*

(a) If $\det\Phi_2(x_0,\mu_0) = 0$ for some $(x_0,\mu_0) \in \dot{\mathbb{R}} \times [0,1]$, then W_Φ is not semi-Fredholm.

(b) If $\det\Phi_2(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$, then W_Φ is Fredholm and its Fredholm index is given by

$$\mathrm{Ind} W_\Phi = -\mathrm{wind}[\det\Phi_2] .$$

Suppose $\det\Phi_2(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$. Then $\Phi(x-0)$ and $\Phi(x+0)$ are invertible for all $x \in \dot{\mathbb{R}}$. Assume in addition that the set $\Delta_\Phi := \{x \in \dot{\mathbb{R}} : \ \Phi(x-0) \neq \Phi(x+0)\}$ is finite. For a connected component ℓ of $\dot{\mathbb{R}} \setminus \Delta_\Phi$, it is denoted by $\mathrm{ind}_\ell[\det\Phi]$ the increment of any continuous argument of $\det\Phi$ on ℓ, times 1 over 2π. Taking into account the possible jump at infinity, the winding number introduced above can be given in the following way (cf., e.g., [14, page 100]):

$$\mathrm{wind}[\det\Phi_2] = \mathrm{ind}[\det\Phi_2] + \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(\infty) \right\} \right) , \tag{5.2.1}$$

where

$$\mathrm{ind}[\det\Phi_2] = \sum_{\ell} \mathrm{ind}_\ell[\det\Phi] + \sum_{x\in\Delta_\Phi} \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(x) \right\} \right) , \tag{5.2.2}$$

and $\xi_1(x), \ldots, \xi_N(x)$ are the eigenvalues of $\Phi^{-1}(x-0)\Phi(x+0)$ for $x \in \Delta_\Phi$, and where $\{c\}$ stands for the fractional part of the number c.

Thus, the last characterization of the Fredholm property can be reformulated in the following way.

THEOREM 5.2.2. *(cf., e.g., [14, Theorem 5.10]) Let $\Phi \in \mathcal{G}PC^{N\times N}$. For W_Φ to be a Fredholm operator it is necessary and sufficient that*

$$\mathrm{sp}[\Phi^{-1}(x-0)\Phi(x+0)] \cap (-\infty, 0] = \emptyset \,,$$

for all $x \in \dot{\mathbb{R}}$. Here $\mathrm{sp}[\Phi^{-1}(x-0)\Phi(x+0)]$ stands for the set of eigenvalues of the matrix $\Phi^{-1}(x-0)\Phi(x+0)$.
If W_Φ is Fredholm and Φ has at most finitely many jumps then

$$\mathrm{Ind} W_\Phi = -\mathrm{wind}[\det \Phi_2] \,,$$

where $\mathrm{wind}[\det \Phi_2]$ is given by (5.2.1)–(5.2.2).

5.3 Wiener-Hopf operators with PAP matrix symbols

The next proposition is the matrix version of a known corresponding result for the scalar case (cf., e.g., [14, Proposition 3.15]).

PROPOSITION 5.3.1. *If $\Phi \in \mathcal{G}PAP^{N\times N}$, then there exist matrix-valued functions $\Theta \in \mathcal{G}SAP^{N\times N}$ and $\Xi \in \mathcal{G}PC^{N\times N}$ such that $\Xi(-\infty) = \Xi(+\infty) = I_{N\times N}$,*

$$\Phi = \Theta\,\Xi \,, \tag{5.3.1}$$

and

$$W_\Phi = W_\Theta \ell_0 W_\Xi + K_1 = W_\Xi \ell_0 W_\Theta + K_2 \tag{5.3.2}$$

with compact operators K_1, K_2.

Proof. The fact that the factorization (5.3.1) is always possible under the conditions of the present theorem was deduced in the proof of Proposition 2.2.5. Hence let us assume that Φ is factorized and is given by the formula (5.3.1). Since Θ is continuous on $\mathbb{R}$ and

Ξ is continuous at ∞, we have that Θ and Ξ do not have common points of discontinuity. Now reasoning in a similar way as in the proof of Theorem 5.1.3 (e.g. considering two continuous matrix functions on $\dot{\mathbb{R}}$, such that the sum of them is the identity matrix, and vanishing at the points of discontinuity of Θ and Ξ) and also taking profit from Theorem 5.1.2 we deduce that (5.3.2) holds for compact operators K_1 and K_2. $\square$

The matrix formulation presented in the next proposition is also an adaptation of the corresponding known scalar case (cf., e.g., [14, Theorem 3.16]).

PROPOSITION 5.3.2. *Let $\Phi \in PAP^{N\times N}$. If $\Phi \notin \mathcal{G}PAP^{N\times N}$, then W_Φ is not semi-Fredholm. Assume now that $\Phi \in \mathcal{G}PAP^{N\times N}$, and Φ_ℓ and Φ_r have a right AP factorization, then W_Φ is Fredholm if and only if*

(i) $k(\Phi_\ell) = k(\Phi_r) = (0, \ldots, 0)$,

(ii) $\mathrm{sp}[\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)] \cap (-\infty, 0] = \emptyset$,

(iii) $\mathrm{sp}[\Phi^{-1}(x-0)\Phi(x+0)] \cap (-\infty, 0] = \emptyset$,

for all $x \in \mathbb{R}$.

In the last case (under conditions (i)–(iii)), the Fredholm index of W_Φ is provided by:

$$\begin{aligned}\mathrm{Ind} W_\Phi = & - \sum_{\ell} \mathrm{ind}_\ell[\det \Xi] - \mathrm{ind}[\det \Theta] - \sum_{x\in\Delta_\Phi} \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(x) \right\} \right) \\ & - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \eta_k \right\} \right) ,\end{aligned} \tag{5.3.3}$$

where $\xi_k(x)$ are the eigenvalues of the matrix function $\Phi^{-1}(x-0)\Phi(x+0)$, and η_k are the eigenvalues of the matrix $\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)$.

Proof. If $\Phi \notin \mathcal{G}PAP^{N\times N}$, then $\Phi \notin \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ and therefore W_Φ is not semi-Fredholm due to the corresponding Simonenko result [69].

Let us now consider $\Phi \in \mathcal{G}PAP^{N\times N}$. Then we can write (cf. formula (5.3.1)) $\Phi = \Theta\Xi$ (with $\Theta \in \mathcal{G}SAP^{N\times N}$, $\Xi \in \mathcal{G}PC^{N\times N}$ and $\Xi(\pm\infty) = I_{N\times N}$) such that

$$W_\Phi = W_\Theta \ell_0 W_\Xi + K , \tag{5.3.4}$$

for a compact operator K. From here we infer that W_Φ is a Fredholm operator if and only if W_Θ and W_Ξ are also Fredholm operators. In the present context, these last two operators are Fredholm if and only if the conditions of the theorem are satisfied. More precisely, since W_Θ is a Wiener-Hopf operator with an invertible semi-almost periodic matrix symbol, and with lateral representatives $\Theta_\ell = \Phi_\ell$ and $\Theta_r = \Phi_r$ (cf. Proposition 2.2.5) which admit right AP factorizations, then W_Θ is Fredholm if and only if (cf. Theorem 4.2.1) $k(\Theta_\ell) = k(\Theta_r) = (0, \dots, 0)$, and $\mathrm{sp}[\mathbf{d}^{-1}(\Theta_r)\mathbf{d}(\Theta_\ell)] \cap (-\infty, 0] = \emptyset$.

We turn now to the operator W_Ξ. This operator has an invertible piecewise continuous matrix symbol. Therefore, applying Theorem 5.2.2, we obtain that W_Ξ is Fredholm if and only if

$$\mathrm{sp}[\Xi^{-1}(x-0)\Xi(x+0)] \cap (-\infty, 0] = \emptyset, \quad x \in \mathbb{R} .$$

Now we simply have to observe that $\Xi^{-1}(x-0)\Xi(x+0) = \Phi^{-1}(x-0)\Phi(x+0)$, to reach the final conclusion (please recall that ℓ_0 is an invertible operator).

To prove the index formula (5.3.3), assume that W_Φ (with $\Phi \in PAP^{N\times N}$) is a Fredholm operator. It is clear that from the equality (5.3.4) we can derive the index formula:

$$\mathrm{Ind} W_\Phi = \mathrm{Ind} W_\Theta + \mathrm{Ind} W_\Xi \, . \tag{5.3.5}$$

Using formulas (4.3.3), (5.2.1) and (5.2.2), from (5.3.5) it follows that

$$\begin{aligned} \mathrm{Ind} W_\Phi = & - \sum_\ell \mathrm{ind}_\ell[\det \Xi] - \mathrm{ind}[\det \Theta] - \sum_{x\in\Delta_\Phi} \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(x) \right\} \right) \\ & - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(\infty) \right\} \right) - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \eta_k \right\} \right) , \end{aligned} \tag{5.3.6}$$

where $\xi_k(x)$ are the eigenvalues of the matrix function $\Phi^{-1}(x-0)\Phi(x+0)$, η_k are the eigenvalues of the matrix $\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)$. Therefore, (5.3.3) follows from (5.3.6) by just taking into account that Ξ does not have a jump at infinity. $\square$

5.4 Wiener-Hopf-Hankel operators with PAP matrix symbols

To give the corresponding result as the Proposition 5.3.2 for the operator $\mathfrak{D}_\Phi$ (cf. (5.0.1)) we employ the notion of equivalence after extension relation (recall Chapter 1).

We are now in a position to present the main theorem of the present chapter.

THEOREM 5.4.1. *Let $\Phi \in \mathcal{G}PAP^{N\times N}$, and assume that $\Phi_\ell\widetilde{\Phi_r^{-1}}$ admits a right AP factorization, then the operator $\mathfrak{D}_\Phi$ is Fredholm if and only if*

(i) $\Phi_\ell\widetilde{\Phi_r^{-1}}$ *admits a canonical right AP factorization, i.e.,* $k(\Phi_\ell\widetilde{\Phi_r^{-1}}) = (0,\ldots,0)$,

(ii) $\mathrm{sp}[\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})] \cap i\mathbb{R} = \emptyset$,

(iii) $\mathrm{sp}[\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)] \cap (-\infty,0] = \emptyset$, $\quad x \in \mathbb{R}$.

In addition, when in the presence of the Fredholm property

$$\begin{aligned}\mathrm{Ind}\mathfrak{D}_\Phi &= -\sum_\ell \mathrm{ind}_\ell[\det\Xi] - \mathrm{ind}[\det\Theta] - \sum_{x\in\Delta_\Phi}\sum_{k=1}^N \left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg\xi_k(x)\right\}\right)\\ &\quad -\sum_{k=1}^N\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{\pi}\arg\eta_k\right\}\right) ,\end{aligned} \tag{5.4.1}$$

where $\Phi\widetilde{\Phi^{-1}} = \Theta\Xi$ is a corresponding factorization in the sense of (5.3.1) for the invertible matrix-valued PAP function $\Phi\widetilde{\Phi^{-1}}$ which appears in the formula (1.7.1), $\xi_k(x)$ are the eigenvalues of the matrix function $\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)$, and η_k are the eigenvalues of the matrix $\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})$.

Proof. The proof of the first part of this theorem follows a similar reasoning as in the proof of Theorem 4.2.2 and Proposition 5.3.2, and therefore it will be omitted in here.

As about the index formula, by using the formula (1.7.1) we obtain that $\mathrm{Ind}\mathfrak{D}_\Phi =$

$\mathrm{Ind} W_{\Phi\widetilde{\Phi^{-1}}}$. Therefore, from (5.3.3), one obtains that

$$\begin{aligned}\mathrm{Ind}\mathfrak{D}_\Phi = & - \sum_\ell \mathrm{ind}_\ell[\det \Xi] - \mathrm{ind}[\det \Theta] - \sum_{x\in\Delta_\Phi}\sum_{k=1}^N \left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg \xi_k(x)\right\}\right) \\ & - \sum_{k=1}^N \left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg \tau_k\right\}\right) ,\end{aligned} \tag{5.4.2}$$

where $\Phi\widetilde{\Phi^{-1}} = \Theta\Xi$ is a corresponding factorization in the sense of (5.3.1) for the invertible PAP function $\Phi\widetilde{\Phi^{-1}}$ which appears in the formula (1.7.1), and which is always possible due to Proposition 5.3.1, $\xi_k(x)$ are the eigenvalues of the matrix function $\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)$, and τ_k are the eigenvalues of the matrix $\mathbf{d}^{-1}(\Phi_r\widetilde{\Phi_\ell^{-1}})\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})$. As we already know that $\mathbf{d}^{-1}(\Phi_r\widetilde{\Phi_\ell^{-1}})\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}}) = \Lambda^2$, then the formula (5.4.2) simplifies to the following one:

$$\begin{aligned}\mathrm{Ind}\mathfrak{D}_\Phi = & - \sum_\ell \mathrm{ind}_\ell[\det \Xi] - \mathrm{ind}[\det \Theta] - \sum_{x\in\Delta_\Phi}\sum_{k=1}^N \left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg \xi_k(x)\right\}\right) \\ & - \sum_{k=1}^N \left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{\pi}\arg \eta_k\right\}\right) ,\end{aligned}$$

where $\Phi\widetilde{\Phi^{-1}} = \Theta\Xi$, $\xi_k(x)$ are as above and η_k are the eigenvalues of the matrix Λ. □

5.5 Generalized factorization and the operator $\mathfrak{D}_\Phi$

To obtain stronger versions of the above results we need the following proposition which can be viewed as a stronger version of Proposition 5.3.2. We would like to emphasize that in this section we will make use of the generalized right AP factorization, which was recalled in Chapter 4.

PROPOSITION 5.5.1. *Let $\Phi \in \mathcal{G}PAP^{N\times N}$. Then W_Φ is Fredholm if and only if*

(i) *Φ_ℓ and Φ_r admit a canonical generalized right AP factorization,*

(ii) $\mathrm{sp}[\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)] \cap (-\infty, 0] = \emptyset$,

(iii) $\mathrm{sp}[\Phi^{-1}(x-0)\Phi(x+0)] \cap (-\infty, 0] = \emptyset$,

for all $x \in \mathbb{R}$.

Proof. The proof runs in an analogous way as the proof of Proposition 5.3.2 (with obvious changes in the corresponding different places). □

We are ready to generalize this last theorem for the Wiener-Hopf-Hankel operators with piecewise almost periodic symbols.

THEOREM 5.5.2. *Let* $\Phi \in \mathcal{G}PAP^{N\times N}$. *Then the operator* $\mathfrak{D}_\Phi$ *is Fredholm if and only if*

(i) $\Phi_\ell \widetilde{\Phi_r^{-1}}$ *admits a canonical generalized right AP factorization,*

(ii) $\mathrm{sp}[\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})] \cap i\mathbb{R} = \emptyset$,

(iii) $\mathrm{sp}[\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)] \cap (-\infty, 0] = \emptyset \,, \quad x \in \mathbb{R}.$

Proof. The proof is the compilation of the techniques examined in Theorem 4.2.2 and Proposition 5.3.2. □

The index formula where we "a priori" do not need to require that $\Phi_\ell \widetilde{\Phi_r^{-1}}$ admits a right AP factorization is now stated.

THEOREM 5.5.3. *Let* $\Phi \in \mathcal{G}PAPW^{N\times N}$, *and assume that* $\mathfrak{D}_\Phi$ *is Fredholm operator. Then*

$$\begin{aligned} \mathrm{Ind}\mathfrak{D}_\Phi &= -\sum_\ell \mathrm{ind}_\ell[\det \Xi] - \mathrm{ind}[\det \Theta] - \sum_{x\in\Delta_\Phi}\sum_{k=1}^{N}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg \xi_k(x)\right\}\right) \\ &\quad -\sum_{k=1}^{N}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{\pi}\arg \eta_k\right\}\right) \,, \end{aligned}$$

where $\Phi\widetilde{\Phi^{-1}} = \Theta\Xi$ *is a corresponding factorization in the sense of (5.3.1) for the invertible matrix-valued* $PAPW$ *function* $\Phi\widetilde{\Phi^{-1}}$ *which appears in the formula (1.7.1),* $\xi_k(x)$ *are the eigenvalues of the matrix function* $\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)$, *and* η_k *are the eigenvalues of the matrix* $\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})$.

Using this theorem and the fact that $PAPW$ is dense in PAP we can obtain a stronger result on the index formula for the Wiener-Hopf-Hankel operators with piecewise almost periodic symbols.

THEOREM 5.5.4. *If $\mathfrak{D}_\Phi$ is a Fredholm operator for some $\Phi \in \mathcal{G}PAP^{N\times N}$, then*

$$\begin{aligned}\mathrm{Ind}\mathfrak{D}_\Phi &= -\sum_{\ell} \mathrm{ind}_\ell[\det \Xi] - \mathrm{ind}[\det \Theta] - \sum_{x\in\Delta_\Phi}\sum_{k=1}^{N}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg \xi_k(x)\right\}\right) \\ &\quad - \sum_{k=1}^{N}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{\pi}\arg \eta_k\right\}\right),\end{aligned}$$

where $\Phi\widetilde{\Phi^{-1}} = \Theta\Xi$ is a corresponding factorization in the sense of (5.3.1) for the invertible matrix-valued PAP function $\Phi\widetilde{\Phi^{-1}}$ which appears in the formula (1.7.1), $\xi_k(x)$ are the eigenvalues of the matrix function $\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)$, and η_k are the eigenvalues of the matrix $\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})$.

Proof. The proof runs by analogy to the proof of Theorem 4.6.2. □

5.6 Examples

We will end this chapter with some examples. We will construct examples which show that a Wiener-Hopf operator may not be Fredholm although the Wiener-Hopf plus/minus Hankel operators with the same symbols are Fredholm.

5.6.1 Example 1

Let us consider the following matrix function.

$$\begin{aligned}\Phi(x) &= (1-u(x))\begin{bmatrix} e^{-i(1+\alpha)x} & 0 \\ e^{-i\alpha x} - 1 + e^{ix} & e^{i(1+\alpha)x}\end{bmatrix}\begin{bmatrix} a(x)h(x) & 0 \\ 0 & b(x)h(x)\end{bmatrix} \\ &\quad + u(x)\begin{bmatrix} e^{i(1+\alpha)x} & 0 \\ e^{i\alpha x} - 1 + e^{-ix} & e^{-i(1+\alpha)x}\end{bmatrix}\begin{bmatrix} a(x)h(x) & 0 \\ 0 & b(x)h(x)\end{bmatrix},\end{aligned}$$

where α and u are as in Section 4.7, $a, b \in \mathcal{G}C(\dot{\mathbb{R}})$ are any real-valued functions which satisfy the following conditions $a(\pm\infty) = b(\pm\infty) = 1$, but for more evidence we are going to set

$$a(x) = \frac{1}{2}(e^{-\frac{1}{x^2}} + 1) , \tag{5.6.1}$$

$$b(x) = 1 , \tag{5.6.2}$$

and

$$h(x) = \begin{cases} 1, & |x| \geq 1, \\ e^{i\pi(x+1)}, & -1 < x \leq 0, \\ (2-x)e^{i\pi(x-1)}, & 0 < x < 1. \end{cases}$$

Obviously Φ admits a factorization.

$$\Phi = FG , \tag{5.6.3}$$

where

$$F(x) \;=\; (1-u(x)) \begin{bmatrix} e^{-i(1+\alpha)x} & 0 \\ e^{-i\alpha x} - 1 + e^{ix} & e^{i(1+\alpha)x} \end{bmatrix} + u(x) \begin{bmatrix} e^{i(1+\alpha)x} & 0 \\ e^{i\alpha x} - 1 + e^{-ix} & e^{-i(1+\alpha)x} \end{bmatrix},$$

and

$$G(x) = \begin{bmatrix} a(x)h(x) & 0 \\ 0 & b(x)h(x) \end{bmatrix}.$$

Let us observe that Φ is invertible, because F and G are invertible matrix functions, which directly follows when checking their determinants. We have that Wiener-Hopf operator with symbol Φ is not Fredholm operator because the matrix function

$$\begin{bmatrix} e^{-i(1+\alpha)x} & 0 \\ e^{-i\alpha x} - 1 + e^{ix} & e^{i(1+\alpha)x} \end{bmatrix} \in \mathcal{G}AP$$

does not have a right AP factorization (cf. [47, pages 284-285]). However, if we consider Wiener-Hopf plus/minus Hankel operator it will be Fredholm. From the equality (5.6.3) one obtains:

$$\Phi\widetilde{\Phi}^{-1} = FG\widetilde{G}^{-1}\widetilde{F}^{-1} . \tag{5.6.4}$$

So, from (5.6.4) we have

$$\Phi_\ell \widetilde{\Phi_r^{-1}} = I_{2\times 2} \ .$$

Consequently

$$\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}}) = I_{2\times 2} \ .$$

Whence the eigenvalues are equal to $1 \notin i\mathbb{R}$. Now we will calculate

$$\mathrm{sp}[\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)], \quad x \in \mathbb{R} \ .$$

As we already know this expression coincides with ($x \in \mathbb{R}$)

$$\mathrm{sp}[G(-x+0)G^{-1}(x-0)G(x+0)G^{-1}(-x-0)] \ .$$

Obviously

$$G^{-1}(x) = \begin{bmatrix} a^{-1}(x)h^{-1}(x) & 0 \\ 0 & b^{-1}(x)h^{-1}(x) \end{bmatrix},$$

where

$$h^{-1}(x) = \begin{cases} 1, & |x| \geq 1, \\ e^{-i\pi(x+1)}, & -1 < x \leq 0, \\ \frac{e^{-i\pi(x-1)}}{2-x}, & 0 < x < 1. \end{cases}$$

Evidently G has only one point of discontinuity 0. Straightforward calculation shows that:

$$G(-x+0)|_{x=0} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}, \ G^{-1}(x-0)|_{x=0} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},$$

$$G(x+0)|_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \text{ and } \ G^{-1}(-x-0)|_{x=0} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}.$$

Consequently

$$G(-x+0)G^{-1}(x-0)G(x+0)G^{-1}(-x-0) = 2I_{2\times 2}.$$

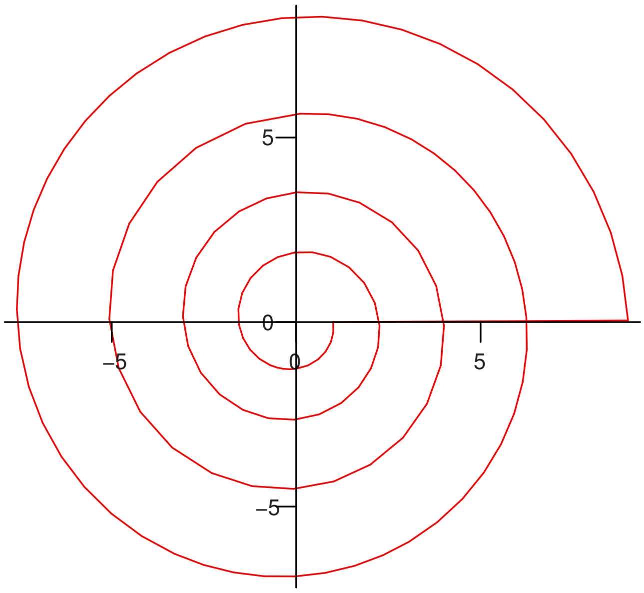

Figure 5.1: graph of $\det \Theta$

From here we have $\mathrm{sp}[G(-x+0)G^{-1}(x-0)G(x+0)G^{-1}(-x-0)] = \{2\} \notin (-\infty, 0]$. Thus we reached the last necessary condition for this Wiener-Hopf plus/minus Hankel operator to be Fredholm:

$$\mathrm{sp}[\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)] \cap (-\infty, 0] = \emptyset \,, \quad x \in \mathbb{R} \,.$$

For an index formula we need first to factorize $\Phi\widetilde{\Phi}^{-1}$ in the appropriate way. Taking into account that G is a diagonal matrix function we can ensure the following factorization:

$$\Phi\widetilde{\Phi}^{-1} = F\widetilde{F}^{-1}\Theta \,,$$

where

$$\Theta(x) = \begin{bmatrix} h(x)\widetilde{h(x)}^{-1} & 0 \\ 0 & h(x)\widetilde{h(x)}^{-1} \end{bmatrix} .$$

Now, let us analyze the determinants of $F\widetilde{F}^{-1}$ and Θ.

$$\det F\widetilde{F}^{-1} = \det F \det \widetilde{F}^{-1} = \det F (\det \widetilde{F})^{-1} = \det F (\det F)^{-1} = 1 \,.$$

Hence, we obtain $\operatorname{Ind}\det F\widetilde{F}^{-1} = 0$. Further

$$\det\Theta(x) = h^2(x)\widetilde{h(x)}^{-2} = \begin{cases} 1, & |x| \geq 1, \\ (x+2)^{-2}e^{4i\pi(x+1)}, & -1 < x \leq 0, \\ (x-2)^2 e^{4i\pi(x-1)}, & 0 < x < 1. \end{cases}$$

The graph of the function $\det\Theta$ is given in Figure 5.1 and it surrounds the origin four times in the positive (counter-clockwise) direction.

Now employing Euler's formula, one obtains:

$$\frac{e^{4i\pi(x+1)}}{(x+2)^2} = \frac{\cos(4\pi(x+1))}{(x+2)^2} + i\frac{\sin(4\pi(x+1))}{(x+2)^2} \ .$$

So, for $x \in (-1,0)$ we have:

$$\arg\det\Theta(x) = \arctan\left(\frac{(x+2)^2\sin(4\pi(x+1))}{(x+2)^2\cos(4\pi(x+1))}\right) = \arctan(\tan(4\pi(x+1))) = 4\pi(x+1) \ .$$

Analogously for $x \in (0,1)$ we will obtain

$$\arg\det\Theta(x) = 4\pi(x-1) \ .$$

Hence, the increment of the continuous argument of $\det\Theta$ on $(-1,0)$ is equal to 4π times 1 over 2π, so 2, and analogously the increment on $(0,1)$ is equal to 2. Consequently the first term on the right-hand side of (5.4.1) is equal to 4. Finally, from (5.4.1), we have

$$\operatorname{Ind}[W_\Phi + H_\Phi] + \operatorname{Ind}[W_\Phi - H_\Phi] = 4 \ .$$

5.6.2 Example 2

Now consider another example. Let

$$\Psi(x) = [(1-u(x))\Phi_\ell(x) + u(x)\Phi_r(x)]H(x)$$

where u is as above,

$$\Phi_\ell(x) = \begin{bmatrix} e^{-ix}+1 & e^{-ix} \\ e^{-ix} & e^{-ix}-1 \end{bmatrix}, \ \Phi_r(x) = \begin{bmatrix} 1-e^{-ix} & -e^{-ix} \\ e^{-ix} & 1+e^{-ix} \end{bmatrix},$$

$$H(x) = \begin{bmatrix} h(x) & 0 \\ 0 & h(x) \end{bmatrix},$$

and

$$h(x)=\begin{cases}1, & |x|\geq 1,\\ e^{i\pi(x+1)}, & -1<x\leq 0,\\ (x+2)e^{-i\pi(x-1)}, & 0<x<1.\end{cases}$$

Let us start with showing that Ψ is invertible. It is clear that H is invertible. We are left to show that $(1-u)\Phi_\ell+u\Phi_r$ is invertible. We need to calculate the determinant of this expression. Direct computation provides that:

$$\det[(1-u)\Phi_\ell+u\Phi_r]=4u(1-u)e^{-ix}+2u-1\ .$$

Observing that

$$4u(1-u)=1-\frac{4}{\pi^2}\arctan^2(x)\ ,$$

and

$$2u-1=\frac{2}{\pi}\arctan(x)\ ,$$

we will obtain:

$$\det[(1-u)\Phi_\ell+u\Phi_r]=e^{-ix}\left(1-\frac{4}{\pi^2}\arctan^2(x)\right)+\frac{2}{\pi}\arctan(x)=$$
$$\cos(x)\left(1-\frac{4}{\pi^2}\arctan^2(x)\right)+\frac{2}{\pi}\arctan(x)+i\sin(x)\left(\frac{4}{\pi^2}\arctan^2(x)-1\right).$$

From the last equality we have that

$$\det[(1-u)\Phi_\ell+u\Phi_r]=0$$

if and only if

$$\cos(x)\left(1-\frac{4}{\pi^2}\arctan^2(x)\right)+\frac{2}{\pi}\arctan(x)=0$$

and

$$\sin(x)\left(\frac{4}{\pi^2}\arctan^2(x)-1\right)=0\ .$$

Now our aim is to show that the last two equalities do not hold simultaneously. Indeed, we have that $\sin(x)(\frac{4}{\pi^2}\arctan^2(x)-1)=0$ if and only if $\frac{4}{\pi^2}\arctan^2(x)-1=0$ or $\sin(x)=0$. If $\frac{4}{\pi^2}\arctan^2(x)-1=0$, then $\arctan(x)=\pm\frac{\pi}{2}$, whence $\cos(x)(1-\frac{4}{\pi^2}\arctan^2(x))+\frac{2}{\pi}\arctan(x)=\pm 1$, and therefore in this case $\det[(1-u)\Phi_\ell+u\Phi_r]\neq 0$. Consider now the case when $\sin(x)=0$. From here we have that $x=\pi n$, where $n\in\mathbb{Z}$. So this leads us to the following criterion, that Ψ is not invertible if the following holds:

$$\pm\left(1-\frac{4}{\pi^2}\arctan^2(\pi n)\right)+\frac{2}{\pi}\arctan(\pi n)=0\,. \tag{5.6.5}$$

But we will show that (5.6.5) never holds for any integer number n. We have two quadratic equations with respect to $\arctan(\pi n)$. Whence we have four solutions, namely:

$$\arctan(\pi n)=\frac{(1\pm\sqrt{5})\pi}{4}\,,$$

and

$$\arctan(\pi n)=\frac{(-1\pm\sqrt{5})\pi}{4}\,.$$

This lead us to the following equalities:

$$\pi n=\frac{(1\pm\sqrt{5})\pi}{4}+k\pi\,,\quad k\in\mathbb{Z}\,,$$

or

$$\pi n=\frac{(-1\pm\sqrt{5})\pi}{4}+k\pi\,,\quad k\in\mathbb{Z}\,.$$

Therefore, we get a contradiction because the equality $n-k=(\pm 1\pm\sqrt{5})/4$ is not possible, due to the fact that $(\pm 1\pm\sqrt{5})/4$ is not an integer number. This contradiction proves that Ψ is invertible.

From the definition of Φ_ℓ and Φ_r it is readily seen that they are invertible almost-periodic "minus" class functions, and so they admit a canonical right AP factorization. However the Wiener-Hopf operator with symbol Ψ is not Fredholm, because

$$\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\,,$$

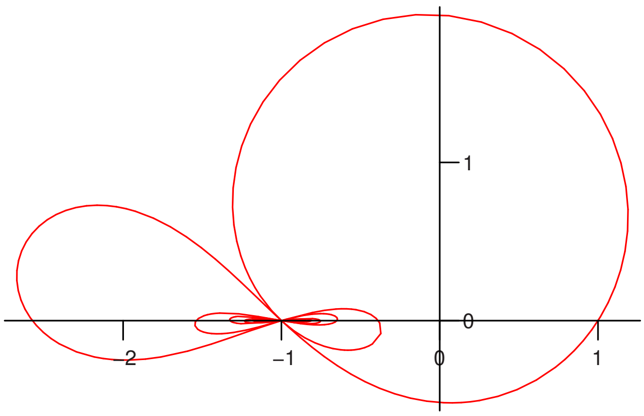

Figure 5.2: graph of Ξ

and therefore $\mathrm{sp}[\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)] = \{-1,1\} \cap (-\infty,0] \neq \emptyset$. Now we will show that the Wiener-Hopf plus Hankel operator with the same symbol is Fredholm. A straightforward computation shows that

$$\Phi_\ell \widetilde{\Phi_r}^{-1} = \begin{bmatrix} e^{ix}+e^{-ix}+1 & e^{ix}+e^{-ix} \\ e^{ix}+e^{-ix} & e^{ix}+e^{-ix}-1 \end{bmatrix}.$$

This matrix function admits a canonical right AP factorization:

$$\Phi_\ell \widetilde{\Phi_r}^{-1} = \begin{bmatrix} e^{-ix}+1 & e^{-ix} \\ e^{-ix} & e^{-ix}-1 \end{bmatrix} \begin{bmatrix} e^{ix}+1 & e^{ix} \\ -e^{ix} & 1-e^{ix} \end{bmatrix}.$$

From this factorization we also infer that

$$\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5.6.6}$$

Thus, $\mathrm{sp}[\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})] = \{-1,1\}$ and therefore $\mathrm{sp}[\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})] \cap i\mathbb{R} = \emptyset$. We are left to check the condition on the piecewise continuous function H. As far as H has exactly two points of discontinuity, namely 0 and 1, we have to calculate $H(-x-0)H^{-1}(x-0)H(x+0)H^{-1}(-x-0)$. Computations show that

$$H(-x-0)H^{-1}(x-0)H(x+0)H^{-1}(-x-0)|_{x=0} = 2I_{2\times 2},$$

and

$$H(-x-0)H^{-1}(x-0)H(x+0)H^{-1}(-x-0)|_{x=1} = \frac{1}{3}I_{2\times 2}.$$

Therefore $\mathrm{sp}[H(-x-0)H^{-1}(x-0)H(x+0)H^{-1}(-x-0)] = \{1/3, 2\} \cap (-\infty, 0] = \emptyset$, $(x \in \mathbb{R})$. Consequently we deduce that Wiener-Hopf plus/minus Hankel operator with symbol Ψ is Fredholm.

To calculate the sum of Fredholm indices of the operators Wiener-Hopf plus/minus Hankel with symbol Ψ we have to factorize $\Psi\widetilde{\Psi}^{-1}$. It is readily seen that

$$\Psi\widetilde{\Psi}^{-1} = \Theta\widetilde{\Theta}^{-1}H\widetilde{H}^{-1}\ ,$$

where $\Theta = (1-u)\Phi_\ell + u\Phi_r$, and recall that $H\widetilde{H}^{-1}$ is a diagonal function with equal entries on it. Let us consider the determinants of $\Theta\widetilde{\Theta}^{-1}$ and $H\widetilde{H}^{-1}$. Straightforwardly we see that

$$\Xi := \det \Theta\widetilde{\Theta}^{-1} = \frac{2u - 1 + 4u(1-u)e^{-ix}}{2\widetilde{u} - 1 + 4\widetilde{u}(1-\widetilde{u})e^{ix}}\ .$$

(The graph of Ξ is given in Figure 5.2). It is well-known that

$$\arg \Xi = \arctan\left(\frac{\Im m\Xi}{\Re e\Xi}\right), \tag{5.6.7}$$

where $\Im m\Xi$ stands for imaginary part of the complex quantity Ξ, and $\Re e\Xi$ – for the real part. Once again a straightforward calculation shows that

$$\begin{aligned}\frac{\Im m\Xi}{\Re e\Xi} \quad &= \quad \frac{\pi^4 - 4\pi^2\arctan^2(x) - 2\pi^4\cos^2(x) + 16\pi^2\cos^2(x)\arctan^2(x)}{\sin(2x)(4\arctan^2(x) - \pi^2)^2} \\ &\quad + \frac{16\arctan^4(x) - 32\cos^2(x)\arctan^4(x)}{\sin(2x)(4\arctan^2(x) - \pi^2)^2}\ .\end{aligned}$$

As we see the numerator of the fraction in (5.6.7) is an even and the denominator is an odd function, whence the fraction is an odd function. This means that $(\arg\Xi)(-x) =$

$-(\arg \Xi)(x)$. Employing one of the applications of the formula (4.3.1), one obtains:

$$\operatorname{ind}\det \Theta\widetilde{\Theta}^{-1} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}((\arg \Xi)(x) - (\arg \Xi)(-x))dx = \lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}(\arg \Xi)(x)dx = 0,$$

because the integral bounds are symmetric and the function under the integration is an odd one. As we have shown above (cf. (5.6.6)) the eigenvalues of the matrix $\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})$ are equal to -1 and 1. We need now the determinant of the matrix function $H\widetilde{H}^{-1}$. Calculating the determinant we have

$$Y(x) := \det[H(x)\widetilde{H}^{-1}(x)] = h^2(x)\widetilde{h}^{-2}(x) = \begin{cases} 1, & |x| \geq 1, \\ (x-2)^{-2}, & -1 < x < 0, \\ (x+2)^2, & 0 < x < 1\ . \end{cases}$$

Hence we get that Y is a positive real-valued function, whence the argument of it is equal to zero. Consequently from the formula (5.4.1) we have the following equality:

$$\operatorname{Ind}[W_\Psi + H_\Psi] + \operatorname{Ind}[W_\Psi - H_\Psi] = 0.$$

Chapter 6

Unitary and sectorial symbols

In this chapter we consider matrix Wiener-Hopf-Hankel operators acting between Lebesgue spaces on the real line with Fourier symbols presenting some even properties (which in particular include unitary matrix functions), and also with Fourier symbols which contain sectorial matrices. In both situations, different conditions are found to ensure the operators two-sided invertibility, one-sided invertibility, Fredholm property, and the n- and d-normal properties. These new results are assembled in Theorems 6.1.2 and 6.2.2. Although in this chapter we consider the theory for the matrix Wiener-Hopf-Hankel operators in the end we give examples for both the scalar and the matrix case.

6.1 Matrix Wiener-Hopf-Hankel operators with symmetry

Our goal is to obtain characterizations for the regularity properties of (5.0.1), in the cases where: (i) the Fourier symbol presents some even symmetry when combined with its conjugate transpose; (ii) the Fourier symbol is a matrix function which allows certain factorizations depending on sectorial elements. Therefore, we will generalize the results of [11], and will also consider other classes of Fourier symbols which were not treated in [11]. We recall that $\Phi \in [L^\infty(\mathbb{R})]^{N\times N}$ is called *unitary* if $\Phi\Phi^* = \Phi^*\Phi = I_{N\times N}$, where Φ^*

stands for the conjugate transpose of Φ.

In the Wiener-Hopf operators case there is a well-known theorem – due to Douglas and Sarason – about the regularity properties of this kind of operators and the distances from the Fourier symbols to certain spaces. More precisely, the theorem may be written in the following form.

THEOREM 6.1.1 (Douglas and Sarason [32]). *If $\Phi \in [L^\infty(\mathbb{R})]^{N\times N}$ is unitary, then:*

(a) W_Φ *is two-sided invertible if and only if* $\operatorname{dist}(\Phi, \mathcal{G}[H^\infty_+(\mathbb{R})]^{N\times N}) < 1$ *if and only if* $\operatorname{dist}(\Phi, \mathcal{G}[H^\infty_-(\mathbb{R})]^{N\times N}) < 1$;

(b) W_Φ *is left-invertible if and only if* $\operatorname{dist}(\Phi, [H^\infty_+(\mathbb{R})]^{N\times N}) < 1$;

(b′) W_Φ *is right-invertible if and only if* $\operatorname{dist}(\Phi, [H^\infty_-(\mathbb{R})]^{N\times N}) < 1$;

(c) W_Φ *is Fredholm if and only if* $\operatorname{dist}(\Phi, \mathcal{G}[C(\dot{\mathbb{R}}) + H^\infty_+(\mathbb{R})]^{N\times N}) < 1$ *if and only if* $\operatorname{dist}(\Phi, \mathcal{G}[C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})]^{N\times N}) < 1$;

(d) W_Φ *is n-normal if and only if* $\operatorname{dist}(\Phi, [C(\dot{\mathbb{R}}) + H^\infty_+(\mathbb{R})]^{N\times N}) < 1$;

(d′) W_Φ *is d-normal if and only if* $\operatorname{dist}(\Phi, [C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})]^{N\times N}) < 1$.

The last theorem served as a motivation for obtaining such kind of result for our Wiener-Hopf-Hankel operators. However, it is clear that adding Hankel operators to the above Wiener-Hopf operators will give rise to several changes in the regularity properties of the resulting operators. In addition, we will work not only with unitary matrix functions but with the more general class which appears in the general assumption of the next result.

THEOREM 6.1.2. *Let $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ and assume that $\Phi^*\Phi$ is an even matrix-valued function.*

(a) $\mathfrak{D}_\Phi$ *is two-sided invertible if and only if* $\operatorname{dist}(\Phi\widetilde{\Phi^{-1}}, \mathcal{G}[H^\infty_+(\mathbb{R})]^{N\times N}) < 1$ *if and only if* $\operatorname{dist}(\Phi\widetilde{\Phi^{-1}}, \mathcal{G}[H^\infty_-(\mathbb{R})]^{N\times N}) < 1$;

(b) $\mathfrak{D}_\Phi$ *is left-invertible if and only if* $\mathrm{dist}(\Phi\widetilde{\Phi^{-1}}, [H^\infty_+(\mathbb{R})]^{N\times N}) < 1$;

(b′) $\mathfrak{D}_\Phi$ *is right-invertible if and only if* $\mathrm{dist}(\Phi\widetilde{\Phi^{-1}}, [H^\infty_-(\mathbb{R})]^{N\times N}) < 1$;

(c) $\mathfrak{D}_\Phi$ *is Fredholm if and only if* $\mathrm{dist}(\Phi\widetilde{\Phi^{-1}}, \mathcal{G}[C(\dot{\mathbb{R}}) + H^\infty_+(\mathbb{R})]^{N\times N}) < 1$ *if and only if* $\mathrm{dist}(\Phi\widetilde{\Phi^{-1}}, \mathcal{G}[C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})]^{N\times N}) < 1$;

(d) $\mathfrak{D}_\Phi$ *is n-normal if and only if* $\mathrm{dist}(\Phi\widetilde{\Phi^{-1}}, [C(\dot{\mathbb{R}}) + H^\infty_+(\mathbb{R})]^{N\times N}) < 1$;

(d′) $\mathfrak{D}_\Phi$ *is d-normal if and only if* $\mathrm{dist}(\Phi\widetilde{\Phi^{-1}}, [C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})]^{N\times N}) < 1$.

Proof. The proof is based on the notion of equivalence after extension relation (recall Chapter 1). From (1.7.1) we have that $\mathfrak{D}_\Phi$ is equivalent after extension with $W_{\Phi\widetilde{\Phi}^{-1}}$. Thus, we are now going to analyze the Fourier symbol $\Phi\widetilde{\Phi^{-1}}$.

Let us observe that for $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ the function $\Phi\widetilde{\Phi^{-1}}$ is unitary if and only if $\Phi^*\Phi$ is even. Indeed, suppose that $\Phi^*\Phi$ is even. By the definition we have:

$$\Phi^*\Phi = \widetilde{\Phi^*}\widetilde{\Phi} .$$

From here it directly follows that

$$\Phi\widetilde{\Phi^{-1}} = (\Phi^*)^{-1}\widetilde{\Phi^*} . \tag{6.1.1}$$

To simplify further arguments, let us introduce the new notation: $\Psi := \Phi\widetilde{\Phi^{-1}}$. To prove that Ψ is unitary we have to show that

$$\Psi\Psi^* = \Psi^*\Psi = I_{N\times N} .$$

Performing a direct substitution, we will have that

$$\Psi\Psi^* = \Phi\widetilde{\Phi^{-1}}(\widetilde{\Phi^{-1}})^*\Phi^*$$

Having in mind formula (6.1.1), from the last equality one obtains:

$$\Psi\Psi^* = (\Phi^*)^{-1}(\widetilde{\Phi^*})(\widetilde{\Phi^{-1}})^*\Phi^* = (\Phi^*)^{-1}(\widetilde{\Phi^*})(\widetilde{\Phi^*})^{-1}\Phi^* = (\Phi^*)^{-1}\Phi^* = I_{N\times N} .$$

Analogously, we have:

$$\Psi^*\Psi = (\widetilde{\Phi^{-1}})^*\Phi^*\Phi\widetilde{\Phi^{-1}} = (\widetilde{\Phi^{-1}})^*\Phi^*(\Phi^*)^{-1}\widetilde{\Phi^*} = I_{N\times N} .$$

To prove the above stated equivalence we are left to show that if $\Phi\widetilde{\Phi^{-1}}$ is an unitary matrix function, then $\Phi^*\Phi$ is even. If $\Phi\widetilde{\Phi^{-1}}$ is unitary, then we derive that

$$\Phi\widetilde{\Phi^{-1}}(\widetilde{\Phi^{-1}})^*\Phi^* = I_{N\times N} .$$

Consequently, we have:

$$\Phi\widetilde{\Phi^{-1}} = (\Phi^*)^{-1}\widetilde{\Phi^*} .$$

Hence, $\Phi^*\Phi = \widetilde{\Phi^*}\widetilde{\Phi}$ and we have shown the above announced equivalence.

From the above reasoning we have that $\Phi\widetilde{\Phi^{-1}}$ is unitary. We can now apply Theorem 6.1.1 to the operator $W_{\Phi\widetilde{\Phi}^{-1}}$ and obtain all the above stated conditions in terms of distances. Now, the result follows if we employ an equivalence after extension relation between $\mathfrak{D}_\Phi$ and $W_{\Phi\widetilde{\Phi}^{-1}}$, which allows the transfer of regularity properties from $W_{\Phi\widetilde{\Phi}^{-1}}$ to $\mathfrak{D}_\Phi$. □

REMARK 6.1.3. Note that the global assumption of the last theorem which requires that $\Phi^*\Phi$ is an even matrix-valued function is more general than assuming Φ to be an unitary matrix function.

6.2 Matrix Wiener-Hopf-Hankel operators with sectorial components

In the present section we will work out a different characterization for the regularity properties of matrix Wiener-Hopf-Hankel operators, and which is now based on the use of certain sectorial parts of the matrix Fourier symbols of the operators.

We recall that by $\mathcal{S}^{N\times N}$ is denoted the set of all sectorial matrix functions (in $[L^\infty(\mathbb{R})]^{N\times N}$). Once again, for matrix Wiener-Hopf operators with such kind of Fourier symbols a description of the possible regularity properties is known.

THEOREM 6.2.1. *[27] If $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$, then:*

(a) W_Φ *is two-sided invertible if and only if* $\Phi = Sh, S \in \mathcal{S}^{N\times N}, h \in \mathcal{G}[H^\infty_+(\mathbb{R})]^{N\times N}$ *if and only if* $\Phi = hS, S \in \mathcal{S}^{N\times N}, h \in \mathcal{G}[H^\infty_-(\mathbb{R})]^{N\times N}$;

(b) W_Φ *is left-invertible if and only if* $\Phi = Sh,\ S \in \mathcal{S}^{N\times N},\ h \in [H^\infty_+(\mathbb{R})]^{N\times N}$;

(b′) W_Φ *is right-invertible if and only if* $\Phi = hS,\ S \in \mathcal{S}^{N\times N},\ h \in [H^\infty_-(\mathbb{R})]^{N\times N}$;

(c) W_Φ *is Fredholm if and only if* $\Phi = Sh,\ S \in \mathcal{S}^{N\times N},\ h \in \mathcal{G}[C(\dot{\mathbb{R}}) + H^\infty_+(\mathbb{R})]^{N\times N}$ *if and only if* $\Phi = hS,\ S \in \mathcal{S}^{N\times N},\ h \in \mathcal{G}[C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})]^{N\times N}$;

(d) W_Φ *is n-normal if and only if* $\Phi = Sh, S \in \mathcal{S}^{N\times N},\ h \in [C(\dot{\mathbb{R}}) + H^\infty_+(\mathbb{R})]^{N\times N}$;

(d′) W_Φ *is d-normal if and only if* $\Phi = hS, S \in \mathcal{S}^{N\times N},\ h \in [C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})]^{N\times N}$.

We will now introduce a corresponding theorem for Wiener-Hopf-Hankel operators (i.e., for the operator $\mathfrak{D}_\Phi$).

THEOREM 6.2.2. *Let* $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$.

(a) $\mathfrak{D}_\Phi$ *is two-sided invertible if and only if*

$$\Phi\widetilde{\Phi^{-1}} = Sh,\ S \in \mathcal{S}^{N\times N},\ h \in \mathcal{G}[H^\infty_+(\mathbb{R})]^{N\times N}$$

if and only if

$$\Phi\widetilde{\Phi^{-1}} = hS,\ S \in \mathcal{S}^{N\times N},\ h \in \mathcal{G}[H^\infty_-(\mathbb{R})]^{N\times N};$$

(b) $\mathfrak{D}_\Phi$ *is left-invertible if and only if*

$$\Phi\widetilde{\Phi^{-1}} = Sh,\ S \in \mathcal{S}^{N\times N},\ h \in [H^\infty_+(\mathbb{R})]^{N\times N};$$

(c) $\mathfrak{D}_\Phi$ *is right-invertible if and only if*

$$\Phi\widetilde{\Phi^{-1}} = hS,\ S \in \mathcal{S}^{N\times N},\ h \in [H^\infty_-(\mathbb{R})]^{N\times N};$$

(d) $\mathfrak{D}_\Phi$ *is Fredholm if and only if*

$$\Phi\widetilde{\Phi^{-1}} = Sh,\ S \in \mathcal{S}^{N\times N},\ h \in \mathcal{G}[C(\dot{\mathbb{R}}) + H^\infty_+(\mathbb{R})]^{N\times N}$$

if and only if

$$\Phi\widetilde{\Phi^{-1}} = hS,\ S \in \mathcal{S}^{N\times N},\ h \in \mathcal{G}[C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})]^{N\times N}\,;$$

(e) $\mathfrak{D}_\Phi$ *is n-normal if and only if*

$$\Phi\widetilde{\Phi^{-1}} = Sh,\ S \in \mathcal{S}^{N\times N},\ h \in [C(\dot{\mathbb{R}}) + H^\infty_+(\mathbb{R})]^{N\times N}\,;$$

(f) $\mathfrak{D}_\Phi$ *is d-normal if and only if*

$$\Phi\widetilde{\Phi^{-1}} = hS,\ S \in \mathcal{S}^{N\times N},\ h \in [C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})]^{N\times N}\,.$$

Proof. To prove this result we perform the same reasoning as in the proof of Theorem 6.1.2, and make use of the fact that $\mathfrak{D}_\Phi$ is equivalent after extension with $W_{\Phi\widetilde{\Phi^{-1}}}$. This allows us to transfer the above mentioned regularity properties from the operator $W_{\Phi\widetilde{\Phi^{-1}}}$ to the operator $\mathfrak{D}_\Phi$ by using Theorem 6.2.1 and the indicated operator relation. □

6.3 Examples

We end with some examples showing the applicability of the last result. Let us consider the Wiener-Hopf plus Hankel operator

$$WH_{\varphi_p} : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)\,,$$

with the particular Fourier symbol

$$\varphi_p(x) = (2 + \sin(x))\, e^{i\alpha x}\,, \qquad x \in \mathbb{R}\,,$$

where $\alpha \in \mathbb{R}$ is a given parameter. Direct computations show that

$$\varphi_p(x)\, \widetilde{\varphi_p^{-1}}(x) = \frac{2+\sin(x)}{2-\sin(x)}\, e^{2i\alpha x}\,.$$

So, if we choose $s_p(x) = (2 + \sin(x))/(2 - \sin(x))$ and $h_p(x) = e^{2i\alpha x}$, we see that $s_p \in \mathcal{S}$. This occurs because $1/3 \leq s_p \leq 3$, and therefore the range of s_p is contained in the right half-plane (which boundary passes through the origin). Moreover, depending whether $\alpha \geq 0$ or $\alpha \leq 0$, we have $h_p \in H^\infty_+(\mathbb{R})$ or $h_p \in H^\infty_-(\mathbb{R})$, respectively. Therefore, applying Theorem 6.2.2 to

$$\varphi_p \, \widetilde{\varphi_p^{-1}} = s_p \, h_p \ ,$$

we conclude that:

(a) if $\alpha = 0$, then WH_{φ_p} is two-sided invertible;

(b) if $\alpha > 0$, then WH_{φ_p} is left-invertible;

(c) if $\alpha < 0$, then WH_{φ_p} is right-invertible.

Now we give an example with matrix operators. Let us consider the matrix Wiener-Hopf plus Hankel operator

$$WH_{\Phi_p} : [L^2_+(\mathbb{R})]^2 \to [L^2(\mathbb{R}_+)]^2 \, ,$$

with the particular Fourier symbol

$$\Phi_p(x) = \begin{pmatrix} 2+\sin(x) & 0 \\ \cos(x) & 1 \end{pmatrix} \begin{pmatrix} e^{i\alpha x} & 0 \\ 0 & 1 \end{pmatrix} , \qquad x \in \mathbb{R} \ ,$$

where $\alpha \in \mathbb{R}$ is a given parameter. Direct computations show that

$$\Phi_p(x) \, \widetilde{\Phi_p^{-1}}(x) = \begin{pmatrix} e^{2i\alpha x} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (2+\sin(x))(2-\sin(x))^{-1} & 0 \\ (e^{2i\alpha x} - 1)\cos(x)(2-\sin(x))^{-1} & 1 \end{pmatrix} =: h_p S_p \ ,$$

and

$$\Phi_p(x) \, \widetilde{\Phi_p^{-1}}(x) = \begin{pmatrix} (2+\sin(x))(2-\sin(x))^{-1} & 0 \\ (1 - e^{-2i\alpha x})\cos(x)(2-\sin(x))^{-1} & 1 \end{pmatrix} \begin{pmatrix} e^{2i\alpha x} & 0 \\ 0 & 1 \end{pmatrix} =: s_p H_p \, .$$

So, we have that s_p and S_p are sectorial matrix functions because the main minors of that matrices are positive definite and this is a particular case of sectorial matrix functions. Moreover, depending whether $\alpha \geq 0$ or $\alpha \leq 0$, we have $h_p, H_p \in [H^\infty_+(\mathbb{R})]^{2\times 2}$ or $h_p, H_p \in [H^\infty_-(\mathbb{R})]^{2\times 2}$, respectively. Therefore, applying Theorem 6.2.2, we conclude that:

(a) if $\alpha = 0$, then WH_{Φ_p} is two-sided invertible;

(b) if $\alpha > 0$, then WH_{Φ_p} is left-invertible;

(c) if $\alpha < 0$, then WH_{Φ_p} is right-invertible.

Chapter 7

Scalar Wiener-Hopf plus Hankel operators via odd factorizations

In this chapter the invertibility of Wiener-Hopf plus Hankel operators with essentially bounded Fourier symbols is described via certain factorization properties of the Fourier symbols. In addition, a Fredholm criterion for these operators is also obtained and the dimensions of the kernel and cokernel are described.

Once again, let us emphasize that in recent times algebraic combinations of Wiener-Hopf and Hankel operators have been receiving an increased attention in view of their invertibility and Fredholm properties. Part of this interest comes from certain applications where such combinations of operators arise. Recent works within this context can be found e.g. in [2], [3], [11], [19], [24], [26], [29], [51], [58], [60], [61]. Some of these works are devoted to certain asymmetric factorizations concepts which are helpful to look for in view of the invertibility properties of corresponding operators with symmetries. In coherence to these developments, in the present chapter we propose an odd asymmetric factorization concept which will be crucial to find out an invertibility and Fredholm characterization for Wiener-Hopf plus Hankel operators with essentially bounded Fourier symbols.

Thus, the main goal of the present chapter is to obtain invertibility and Fredholm criteria for Wiener-Hopf plus Hankel operators acting between L^2 Lebesgue spaces.

7.1 Odd factorizations on the real line and main invertibility result

We start by introducing a definition that will be later on completed by a corresponding stronger version which will have a central role in our invertibility and Fredholm criteria for the Wiener-Hopf plus Hankel operators. Thus, at the end of the present section it will be already possible to state the main invertibility criterion for WH_φ (cf. (1.4.5)).

DEFINITION 7.1.1. *A function $\varphi \in \mathcal{G}L^\infty(\mathbb{R})$ is said to admit a weak odd asymmetric factorization in $L^2(\mathbb{R})$ if it admits a representation*

$$\varphi(x) = \varphi_-(x) \left(\frac{x-i}{x+i}\right)^{\varkappa} \varphi_o(x) \, , \qquad x \in \mathbb{R} \, ,$$

such that $\varkappa \in \mathbb{Z}$, and

(i) $\frac{x}{(x-i)^2}\varphi_- \in H^2_-(\mathbb{R}) \, , \qquad \frac{1}{(x-i)^2}\varphi_-^{-1} \in H^2_-(\mathbb{R}) \, ,$

(ii) $\frac{1}{x^2+1}\varphi_o \in L^2_{odd}(\mathbb{R}) \, , \qquad \frac{|x|}{x^2+1}\varphi_o^{-1} \in L^2_{odd}(\mathbb{R}) \, .$

Here and in what follows $L^2_{odd}(X)$ stands for the class of odd functions from the space $L^2(X)$. The integer $\varkappa$ is called the index of a weak odd asymmetric factorization in $L^2(\mathbb{R})$.

Let us note that we have the uniqueness (up to a constant) of such type of factorizations. This last property is given in exact terms in the next theorem.

THEOREM 7.1.2. *Assume that $\varphi \in \mathcal{G}L^\infty(\mathbb{R})$ admits two weak odd asymmetric factorizations in $L^2(\mathbb{R})$:*

$$\varphi(x) = \varphi_-^{(1)}(x) \left(\frac{x-i}{x+i}\right)^{\kappa_1} \varphi_o^{(1)}(x) = \varphi_-^{(2)}(x) \left(\frac{x-i}{x+i}\right)^{\kappa_2} \varphi_o^{(2)}(x), \qquad x \in \mathbb{R} \, .$$

Then, we necessarily have $\kappa_1 = \kappa_2$, $\varphi_-^{(1)} = C\varphi_-^{(2)}$ and $\varphi_o = C^{-1}\varphi_o^{(2)}$, for some constant $C \in \mathbb{C} \setminus \{0\}$.

Proof. Let φ admit two weak odd asymmetric factorizations:

$$\varphi(x) = \varphi_-^{(1)}(x) \left(\frac{x-i}{x+i}\right)^{\kappa_1} \varphi_o^{(1)}(x) = \varphi_-^{(2)}(x) \left(\frac{x-i}{x+i}\right)^{\kappa_2} \varphi_o^{(2)}(x) \, , \quad x \in \mathbb{R} \qquad (7.1.1)$$

(where $\varphi_-^{(1)}$, $\varphi_-^{(2)}$ and $\varphi_o^{(1)}$, $\varphi_o^{(2)}$ have the corresponding properties of (i) and (ii) in Definition 7.1.1). From (7.1.1) we immediately have that

$$\varphi_-^{(1)}(x)(\varphi_-^{(2)}(x))^{-1}\left(\frac{x-i}{x+i}\right)^{\kappa_1-\kappa_2}=\varphi_o^{(2)}(x)(\varphi_o^{(1)}(x))^{-1}\,,\quad x\in\mathbb{R}. \tag{7.1.2}$$

We can assume without lost of generality that $\kappa:=\kappa_1-\kappa_2\leq 0$, since otherwise we would consider

$$\varphi_-^{(2)}(x)(\varphi_-^{(1)}(x))^{-1}\left(\frac{x-i}{x+i}\right)^{\kappa_2-\kappa_1}=\varphi_o^{(1)}(x)(\varphi_o^{(2)}(x))^{-1}\,,\quad x\in\mathbb{R}$$

instead of (7.1.2) (and from this last identity we are able to take the same conclusion and therefore proceed with the proof in a similar way).

Let us now consider the following auxiliary function:

$$\psi(x):=\frac{x}{(x-i)^4}\,\varphi_-^{(1)}(x)(\varphi_-^{(2)}(x))^{-1}\in H^1_-(\mathbb{R})\,. \tag{7.1.3}$$

A direct computation yields that

$$\widetilde{\psi}(x):=\frac{-x}{(x+i)^4}\,\widetilde{\varphi_-}^{(1)}(x)(\widetilde{\varphi_-}^{(2)}(x))^{-1}\in H^1_+(\mathbb{R})\,. \tag{7.1.4}$$

The right-hand side of (7.1.2) is an even function (since it is a product of two odd functions). Hence, from (7.1.2), we immediately obtain that

$$\varphi_-^{(1)}(x)(\varphi_-^{(2)}(x))^{-1}\left(\frac{x-i}{x+i}\right)^{2\kappa}=\widetilde{\varphi_-}^{(1)}(x)(\widetilde{\varphi_-}^{(2)}(x))^{-1}\,.$$

This identity together with (7.1.3) and (7.1.4), lead to the conclusion that

$$\psi(x)\left(\frac{x-i}{x+i}\right)^{2\kappa+4}=-\widetilde{\psi}(x)\,. \tag{7.1.5}$$

Due to the inclusions in (7.1.3) and (7.1.4), if $2\kappa+4\leq 0$ then from (7.1.5) we immediately obtain that $\psi=0$ is identically zero and hence we would have a contradiction. This means that it only remains the possibilities of $\kappa=-1$ and $\kappa=0$. Let us analyze the case where $\kappa=-1$. In the present case, (7.1.5) is reduced to the form

$$(x-i)^2\psi(x)=-(x+i)^2\widetilde{\psi}(x)\,.$$

Hence, using (7.1.3)–(7.1.4), we have a contradiction which shows that κ cannot be equal to -1. Thus, the only possibility which is left for κ is to be equal to zero. Therefore, in such a case, $\kappa_1 = \kappa_2$. In this case we will have that

$$\varphi_-^{(1)}(x)(\varphi_-^{(2)}(x))^{-1} = \widetilde{\varphi_-}^{(1)}(x)(\widetilde{\varphi_-}^{(2)}(x))^{-1} .$$

Consequently, $\varphi_-^{(1)}(x)(\varphi_-^{(2)}(x))^{-1} = C$ for a constant $C \in \mathbb{C} \setminus \{0\}$ (cf., e.g., [24, Theorem 4.2]). Thus $\varphi_-^{(1)} = C\varphi_-^{(2)}$ and $\varphi_o^{(1)} = C^{-1}\varphi_o^{(2)}$. □

The following definition may be viewed as a strong version of the previous introduced weak factorization and will play a crucial role in the main theorem below.

DEFINITION 7.1.3. *A function $\varphi \in \mathcal{G}L^\infty(\mathbb{R})$ is said to admit an odd asymmetric factorization in $L^2(\mathbb{R})$ if it admits a representation*

$$\varphi(x) = \varphi_-(x)\left(\frac{x-i}{x+i}\right)^{\varkappa}\varphi_o(x) , \qquad x \in \mathbb{R} , \tag{7.1.6}$$

such that $\varkappa \in \mathbb{Z}$, and

(i) $\frac{x}{(x-i)^2}\varphi_- \in H^2_-(\mathbb{R})$, $\qquad \frac{1}{(x-i)^2}\varphi_-^{-1} \in H^2_-(\mathbb{R})$,

(ii) $\frac{1}{(x^2+1)}\varphi_o \in L^2_{odd}(\mathbb{R})$, $\qquad \frac{|x|}{(x^2+1)}\varphi_o^{-1} \in L^2_{odd}(\mathbb{R})$,

(iii) the linear operator $\mathscr{S} := W^0_{\varphi_o^{-1}}(I-J)\ell_0 W_{\varphi_-^{-1}} : L^2(\mathbb{R}) \to L^2_{even}(\mathbb{R})$ *is bounded.*

The integer $\varkappa$ is called the index of the odd asymmetric factorization in $L^2(\mathbb{R})$. Please, note that in (iii) the "Wiener-Hopf operator" is acting on the full space $L^2(\mathbb{R})$.

We are now in a position to state the main result about the invertibility of our Wiener-Hopf plus Hankel operators with L^∞ symbols.

THEOREM 7.1.4. *Let $\varphi \in \mathcal{G}L^\infty(\mathbb{R})$. The operator WH_φ is invertible if and only if φ admits an odd asymmetric factorization in $L^2(\mathbb{R})$ with index $\varkappa = 0$.*

The proof of this theorem will be given in Section 7.3.

7.2 Odd factorizations on the unit circle

In the present section we will introduce some auxiliary notions which will be useful to work out some conclusions in the unit circle setting. The reader should recall that odd functions on the unit circle are exactly those which satisfy the condition given on page 3.

DEFINITION 7.2.1. *A function $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ is said to admit a weak odd asymmetric factorization in $L^2(\Gamma_0)$ if it admits a representation*

$$\phi(t) = \phi_-(t) t^k \phi_o(t) \ , \quad t \in \Gamma_0 \ ,$$

such that $k \in \mathbb{Z}$ and

(i) $(1 + t^{-1})\phi_- \in H^2_-(\Gamma_0) \,, \qquad (1 - t^{-1})\phi_-^{-1} \in H^2_-(\Gamma_0) \,,$

(ii) $|1 - t|\phi_o \in L^2_{odd}(\Gamma_0) \,, \qquad |1 + t|\phi_o^{-1} \in L^2_{odd}(\Gamma_0) \,.$

The integer k is called the index of an asymmetric factorization in $L^2(\Gamma_0)$.

Now we will present a theorem about the uniqueness of a weak odd asymmetric factorization in $L^2(\Gamma_0)$.

THEOREM 7.2.2. *(an analogue of [3, Proposition 3.1]) Assume that ϕ admits two weak odd asymmetric factorizations in $L^2(\Gamma_0)$:*

$$\phi(t) = \phi_-^{(1)}(t) t^{k_1} \phi_o^{(1)}(t) = \phi_-^{(2)}(t) t^{k_2} \phi_o^{(2)}(t), \quad t \in \Gamma_0 \,. \tag{7.2.1}$$

Then $k_1 = k_2$, $\phi_-^{(1)} = C\phi_-^{(2)}$, $\phi_o^{(1)} = C^{-1}\phi_o^{(2)}$ with $C \in \mathbb{C} \setminus \{0\}$.

Proof. Without lost of generality, we can assume that $\varkappa := k_1 - k_2 \leq 0$. From (7.2.1) we have that

$$(\phi_-^{(2)})^{-1}\phi_-^{(1)} t^\varkappa = \phi_o^{(2)}(\phi_o^{(1)})^{-1}. \tag{7.2.2}$$

Take $\psi := (1 - t^{-2})(\phi_-^{(2)})^{-1}\phi_-^{(1)}$. Obviously, we have that $\psi \in H^1_-(\Gamma_0)$. Formula (7.2.2) leads to

$$(1 - t^{-2})^{-1}\psi(t) t^\varkappa = \phi_o^{(2)}(t)(\phi_o^{(1)}(t))^{-1} \ ,$$

where the right-hand side is an even function (since it is a product of two odd functions). Therefore,

$$(1-t^{-2})^{-1}\psi(t)t^{\varkappa} = (1-t^{2})^{-1}\psi(t^{-1})t^{-\varkappa}\,,$$

and from here we have:

$$\psi(t)t^{2\varkappa+2} = -\psi(t^{-1}).$$

If we assume that $\varkappa \leq -1$ we would obtain that $\psi = 0$ (by observing the Fourier coefficients of ψ), which is a contradiction. Hence $\varkappa = 0$. In this case we have that $\psi(t) = C(1-t^{-2})$ with $C \neq 0$. From here finally we have: $k_1 = k_2$, $\phi_-^{(1)} = C\phi_-^{(2)}$, and $\phi_o^{(1)} = C^{-1}\phi_o^{(2)}$. □

Let $\mathfrak{R}$ stand for the linear space of all trigonometric polynomials. Suppose that $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ admits a weak odd asymmetric factorization in $L^2(\Gamma_0)$ with index $k = 0$. Hence $\phi = \phi_-\phi_o$. We will set [3]

$$\begin{aligned} X_1 &:= \{(1-t^{-1})f(t) : f \in \mathfrak{R}\}\,, \\ X_2 &:= \{(1+t^{-1})\phi_o^{-1}(t)f(t) : f \in \mathfrak{R}, f(t) = f(t^{-1})\}\,. \end{aligned}$$

We make the simple observation that X_1 is a dense subset of $L^2(\Gamma_0)$.

Consider the following complementary projections:

$$P_{J_{\Gamma_0}} := \frac{I + J_{\Gamma_0}}{2} : L^2(\Gamma_0) \to L^2(\Gamma_0), \qquad Q_{J_{\Gamma_0}} := I - P_{J_{\Gamma_0}}\,.$$

These projections decompose $L^2(\Gamma_0)$ into the direct sum: $L^2(\Gamma_0) = \mathrm{Im}P_{J_{\Gamma_0}} \oplus \mathrm{Im}Q_{J_{\Gamma_0}}$.

A natural strong version of the last definition is given next.

DEFINITION 7.2.3. *A function $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ is said to admit an odd asymmetric factorization in $L^2(\Gamma_0)$ if it admits a representation*

$$\phi(t) = \phi_-(t)t^k\phi_o(t), \qquad t \in \Gamma_0, \tag{7.2.3}$$

such that $k \in \mathbb{Z}$ and

(i) $(1+t^{-1})\phi_- \in H^2_-(\Gamma_0)$, $\qquad (1-t^{-1})\phi_-^{-1} \in H^2_-(\Gamma_0)$,

(ii) $|1-t|\phi_o \in L^2_{odd}(\Gamma_0)$, $\qquad |1+t|\phi_o^{-1} \in L^2_{odd}(\Gamma_0)$,

(iii) *the linear operator* $\mathscr{E} := L(\phi_o^{-1})(I+J_{\Gamma_0})P_{\Gamma_0}L(\phi_-^{-1})$ *acting from* X_1 *into* X_2 *extends to a linear bounded operator* $\widetilde{\mathscr{E}}$ *acting from* $L^2(\Gamma_0)$ *into* $\mathrm{Im}Q_{J_{\Gamma_0}}$.

As before, also in here k *is called the index of a weak odd asymmetric factorization in* $L^2(\Gamma_0)$.

An equivalent formulation of the condition (iii) is evidently the following:

(iii^*) the operator $\mathscr{E}$ is a bounded operator on $L^2(\Gamma_0)$.

We observe that Definition 7.2.3 is related with Definition 7.1.3 in the sense that a function $\phi : \Gamma_0 \to \mathbb{C}$ admits an odd asymmetric factorization in $L^2(\Gamma_0)$ if and only if the function $\varphi(x) := \phi\left(\frac{x-i}{x+i}\right)$, $x \in \mathbb{R}$, admits an odd asymmetric factorization in $L^2(\mathbb{R})$.

PROPOSITION 7.2.4. *A function* $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ *admits an odd asymmetric factorization in* $L^2(\Gamma_0)$ *with index* k *if and only if* $\varphi := (B_0^{-1}\phi) \in \mathcal{G}L^\infty(\mathbb{R})$ *admits an odd asymmetric factorization in* $L^2(\mathbb{R})$ *with index* k.

Proof. Let us assume that ϕ admits an odd asymmetric factorization in $L^2(\Gamma_0)$ with index k. Hence, we can write (cf. (7.2.3)):

$$\phi(t) = \phi_-(t)t^k\phi_o(t), \qquad t \in \Gamma_0, \tag{7.2.4}$$

with the properties (i)–(iii) on the factors stated in Definition 7.2.3. Performing the B_0^{-1} transformation in both sides of the equality (7.2.4), we obtain:

$$(B_0^{-1}\phi)(x) = (B_0^{-1}\phi_-)(x)(B_0^{-1}d)(x)(B_0^{-1}\phi_o)(x) ,$$

where d denotes the function $d(t) := t^k$. Now, if defining

$$\begin{aligned}
\varphi(x) &:= (B_0^{-1}\phi)(x) = \phi\left(\frac{x-i}{x+i}\right) , \\
\varphi_-(x) &:= (B_0^{-1}\phi_-)(x) = \phi_-\left(\frac{x-i}{x+i}\right) , \\
\varphi_o(x) &:= (B_0^{-1}\phi_o)(x) = \phi_o\left(\frac{x-i}{x+i}\right) ,
\end{aligned}$$

it follows

$$\varphi(x) = \varphi_-(x) \left(\frac{x-i}{x+i} \right)^k \varphi_o(x) \, .$$

I.e., formula (7.1.6) with $\varkappa$ taken to be equal to k. Thus, we are left to show that the corresponding conditions (i)–(iii) on the factors used in the factorizations of Definitions 7.2.3 and 7.1.3 are equivalent.

We have that

$$(1+t^{-1})\phi_- \in H^2_-(\Gamma_0)$$

if and only if

$$\frac{\sqrt{2}}{x-i} \left(1 + \frac{x+i}{x-i} \right) \varphi_- \in H^2_-(\mathbb{R}) \, .$$

Indeed, let $(1+t^{-1})\phi_- \in H^2_-(\Gamma_0)$, then $[B_0^{-1}(1+t^{-1})\phi_-](x) \in (x-i)H^2_-(\mathbb{R})$ (cf., e.g., [14, page 108 and in particular formula (6.3)]). That means

$$2\sqrt{2}\frac{x}{(x-i)^2}\varphi_-(x) = \frac{\sqrt{2}}{x-i} \left(1 + \frac{x+i}{x-i} \right) \varphi_-(x) \in H^2_-(\mathbb{R}) \, ,$$

and therefore we have the equivalence of the first propositions of conditions (i).

To prove the equivalence of the first proposition of (ii)–conditions we need to "compensate" the space with a particular even weight. Letting $|1-t|\phi_o \in L^2_{\mathrm{odd}}(\Gamma_0)$, then

$$B_0^{-1}(|1-t|\phi_o) \in B_0^{-1}(L^2_{\mathrm{odd}}(\Gamma_0)) \, . \tag{7.2.5}$$

Thus, to obtain from the last inclusion a new one where we will be dealing with the space $L^2_{\mathrm{odd}}(\mathbb{R})$ we just need to use in (7.2.5) the multiplication by the weight function $\frac{1}{\sqrt{x^2+1}}$ and therefore reach to

$$\frac{1}{\sqrt{x^2+1}}(B_0^{-1}(|1-t|\phi_o))(x) \in L^2_{\mathrm{odd}}(\mathbb{R}) \, .$$

Consequently, we have:

$$\frac{2}{x^2+1}\varphi_o(x) \in L^2_{\mathrm{odd}}(\mathbb{R}) \, .$$

Analogous arguments will give corresponding equivalences for the second inclusions of conditions (i) and (ii).

We will prove now the equivalence of conditions (iii). As far as the condition (iii) of Definition 7.2.3 can be written in the form of the condition (iii^*) cited after Definition 7.2.3, we will show that $\mathscr{E}$ is a bounded operator if and only if $\mathscr{S}$ is a bounded operator. Consider the following operator:

$$\mathcal{F}^{-1}B\mathscr{E}B^{-1}\mathcal{F} \, . \tag{7.2.6}$$

This operator is equivalent to $\mathscr{E}$ simply because it is obtained from $\mathscr{E}$ by multiplying from the left and from the right by invertible operators. Moreover, from (7.2.6) we have:

$$\begin{aligned}
\mathcal{F}^{-1}B\mathscr{E}B^{-1}\mathcal{F} &= \mathcal{F}^{-1}BL(\phi_o^{-1})(I+J_{\Gamma_0})P_{\Gamma_0}L(\phi_-^{-1})B^{-1}\mathcal{F} \\
&= \mathcal{F}^{-1}BL(\phi_o^{-1})\underbrace{B^{-1}\mathcal{F}\mathcal{F}^{-1}B}_{I}(I+J_{\Gamma_0})\underbrace{B^{-1}\mathcal{F}\mathcal{F}^{-1}B}_{I}P_{\Gamma_0} \\
&\quad \underbrace{B^{-1}\mathcal{F}\mathcal{F}^{-1}B}_{I}L(\phi_-^{-1})B^{-1}\mathcal{F} \\
&= \mathcal{F}^{-1}\varphi_o^{-1}\mathcal{F}(I-J)\ell_0 r_+\mathcal{F}^{-1}\varphi_-^{-1}\mathcal{F} \\
&= W^0_{\varphi_o^{-1}}(I-J)\ell_0 W_{\varphi_-^{-1}} = \mathscr{S} \, ,
\end{aligned}$$

where we employed formulas (1.8.2), (1.8.3) and (1.8.4). Finally this means that $\mathscr{E}$ and $\mathscr{S}$ are unitarily equivalent operators.

From the above reasoning it is clear that we can proceed in a "reverse" direction, i.e., starting from a factorization for the function φ and obtain a corresponding factorization to the function ϕ, which completes the proof. □

7.3 Proof of the main invertibility result

To prove the main invertibility result of the present chapter (i.e., Theorem 7.1.4) we need to prepare several auxiliary material, which in some cases –at a first look– may seem similar to some of the results of [3], but actually the present ones incorporate significant differences.

7.3.1 Auxiliary notions, operators, and results

We will relate Toeplitz minus Hankel operators with the following operators:

$$\begin{aligned} \Phi_\phi &:= P_{\Gamma_0} L(\phi) Q_{J_{\Gamma_0}} : \mathrm{Im} Q_{J_{\Gamma_0}} \to H^2_+(\Gamma_0), \\ \Psi_\psi &:= Q_{J_{\Gamma_0}} L(\psi) P_{\Gamma_0} : H^2_+(\Gamma_0) \to \mathrm{Im} Q_{J_{\Gamma_0}} \ , \end{aligned} \tag{7.3.1}$$

where $\psi(t) = \phi^{-1}(-t^{-1})$. It is readily seen that $2\Phi_\phi = (T_\phi - H_\phi)|_{\mathrm{Im} Q_{J_{\Gamma_0}}}$.

The following well-known lemma is of interest and will be needed to prove Proposition 7.3.3.

LEMMA 7.3.1. *Let X_1 and X_2 be linear spaces, $A : X_1 \to X_2$ be a linear and invertible operator, $P_1 : X_1 \to X_1$ and $P_2 : X_2 \to X_2$ be linear projections, and $Q_1 = I - P_1$ and $Q_2 = I - P_2$. Then $P_2 A P_1 : \mathrm{Im} P_1 \to \mathrm{Im} P_2$ is invertible if and only if $Q_1 A^{-1} Q_2 : \mathrm{Im} Q_2 \to \mathrm{Im} Q_1$ is invertible.*

The proof of Lemma 7.3.1 can be found e.g. in [70].

The next two propositions are essentially taken from [3].

PROPOSITION 7.3.2. *Let $\phi \in \mathcal{G}L^\infty(\Gamma_0)$. The operator $\Phi_\phi \in \mathcal{L}(\mathrm{Im} Q_{J_{\Gamma_0}}, H^2_+(\Gamma_0))$ (defined in (7.3.1)) is equivalent to the Toeplitz minus Hankel operator $T_\phi - H_\phi \in \mathcal{L}(H^2_+(\Gamma_0))$.*

Proof. Let us consider the operators

$$\begin{aligned} R_1 &:= (I - J_{\Gamma_0}) P_{\Gamma_0} : H^2_+(\Gamma_0) \to \mathrm{Im} Q_{J_{\Gamma_0}} \\ R_2 &:= \frac{1}{2} P_{\Gamma_0} (I - J_{\Gamma_0}) : \mathrm{Im} Q_{J_{\Gamma_0}} \to H^2_+(\Gamma_0) \ . \end{aligned}$$

These operators are inverses to one another and a direct computation yields that

$$\Phi_\phi R_1 = T_\phi - H_\phi$$

which shows explicitly the equivalence relation between the operators Φ_ϕ and $T_\phi - H_\phi$. □

PROPOSITION 7.3.3. *Let $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ and $\psi(t) = \phi^{-1}(-t^{-1})$, $t \in \Gamma_0$. The operator $\Psi_\psi : H^2_+(\Gamma_0) \to \mathrm{Im} Q_{J_{\Gamma_0}}$ is invertible if and only if $T_\phi - H_\phi : H^2_+(\Gamma_0) \to H^2_+(\Gamma_0)$ is invertible.*

Proof. We will make use of Lemma 7.3.1 by choosing $P_1 = Q_{J_{\Gamma_0}}$, $P_2 = P_{\Gamma_0}$, $Q_1 = P_{J_{\Gamma_0}}$ and $Q_2 = Q_{\Gamma_0}$. Thus, from Lemma 7.3.1 we derive that Φ_ϕ is invertible if and only if $P_{J_{\Gamma_0}} L(\phi^{-1}) Q_{\Gamma_0}$ is invertible. Multiplying from the left and the right in this last operator by J_{Γ_0}, we obtain

$$J_{\Gamma_0} P_{J_{\Gamma_0}} L(\phi^{-1}) Q_{\Gamma_0} J_{\Gamma_0} = P_{J_{\Gamma_0}} J_{\Gamma_0} L(\phi^{-1}) J_{\Gamma_0} P_{\Gamma_0} \,. \tag{7.3.2}$$

Now, to reach the operator Ψ_ψ, we will consider the operator $V_{\Gamma_0} : L^2(\Gamma_0) \to L^2(\Gamma_0)$, $(V_{\Gamma_0} f)(t) = f(-t)$, and use it in (7.3.2) in the way that:

$$V_{\Gamma_0} P_{J_{\Gamma_0}} J_{\Gamma_0} L(\phi^{-1}) J_{\Gamma_0} P_{\Gamma_0} V_{\Gamma_0} = Q_{J_{\Gamma_0}} \underbrace{V_{\Gamma_0} J_{\Gamma_0} L(\phi^{-1}) J_{\Gamma_0} V_{\Gamma_0}}_{L(\psi)} P_{\Gamma_0} = \Psi_\psi \,,$$

where $\psi(t) = \phi^{-1}(-t^{-1})$. $\square$

We assemble in the next corollary a direct consequence of the last two propositions.

COROLLARY 7.3.4. *(an analogue of [3, Proposition 2.4]) Let $\phi \in \mathcal{G}L^\infty(\Gamma_0)$. Then the following assertions are equivalent:*

(i) $T_\phi - H_\phi$ *is invertible in* $\mathcal{L}(H^2_+(\Gamma_0))$,

(ii) Φ_ϕ *is invertible in* $\mathcal{L}(\mathrm{Im} Q_{J_{\Gamma_0}}, H^2_+(\Gamma_0))$,

(iii) Ψ_ψ *is invertible in* $\mathcal{L}(H^2_+(\Gamma_0), \mathrm{Im} Q_{J_{\Gamma_0}})$, *where* $\psi(t) = \phi^{-1}(-t^{-1})$.

LEMMA 7.3.5. *(an analogue of [3, Lemma 4.1]) Assume that $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ admits a weak odd asymmetric factorization in $L^2(\Gamma_0)$ with index $k = 0$. Then the following assertions hold:*

(i) *the operator* $\mathscr{E} = L(\phi_o^{-1})(I + J_{\Gamma_0}) P_{\Gamma_0} L(\phi_-^{-1})$ *is a well-defined linear operator acting from* X_1 *into* X_2 ,

(ii) $\Phi_\phi \mathscr{E} = P_{\Gamma_0 | X_1}$,

(iii) $\mathrm{Ker} \Phi_\phi = \{0\}$.

Proof. (i) Let $f \in X_1$ and $\phi = \phi_- \phi_o$ (with ϕ_- and ϕ_o under the conditions of Definition 7.2.1). We will compute $\mathscr{E} f$. First, we write $f(t) = (1 - t^{-1}) f_1(t)$ with $f_1 \in \mathfrak{R}$. Multiplying both sides of the last equality by ϕ_-^{-1}, we have:

$$\phi_-^{-1}(t) f(t) = (1 - t^{-1}) \phi_-^{-1}(t) f_1(t) \ .$$

Hence, we can decompose $\phi_-^{-1} f$ in an unique way:

$$\phi_-^{-1}(t) f(t) = u_1(t) + p_1(t) \ , \tag{7.3.3}$$

where $u_1 \in t^{-1} H_-^2(\Gamma_0)$ and p_1 is a polynomial. From the last equality and from the assumption that $f \in X_1$ it also follows that f has the following form: $f(t) = \phi_-(t)(u_1(t) + p_1(t))$. Later on we will use this fact. Now, applying the Riesz projection to (7.3.3), we will have $P_{\Gamma_0}(\phi_-^{-1} f)(t) = p_1(t)$. Hence $(\mathscr{E} f)(t) = \phi_o^{-1}(t)(p_1(t) + t^{-1} p_1(t^{-1}))$. Since $p_1(t) + t^{-1} p_1(t^{-1})$ vanishes at $t = -1$, this expression is $(1 + t^{-1})$ times a trigonometrical polynomial f_2, such that $f_2(t) = f_2(t^{-1})$. Now it is clear that $\mathscr{E} f$ belongs to the space X_2.

(ii) Let us take again $f \in X_1$ and assume the existence of a weak odd asymmetric factorization of $\phi = \phi_- \phi_o$ in $L^2(\Gamma_0)$ (with index $k = 0$). From the part (i) of the proof we know that

$$f(t) = \phi_-(t)(u_1(t) + p_1(t)) \ ,$$

where u_1 and p_1 are as in the formula (7.3.3). Our aim is to compute $\Phi_\phi \mathscr{E} f$. We have already calculated $(\mathscr{E} f)(t) = \phi_o^{-1}(t)(p_1(t) + t^{-1} p_1(t^{-1}))$. From here we have that

$$\begin{aligned}(\Phi_\phi \mathscr{E} f)(t) &= (P_{\Gamma_0} L(\phi_-) L(\phi_o) \frac{I - J_{\Gamma_0}}{2} \phi_o^{-1}(p_1 + t^{-1} \widetilde{p_1}))(t) \\ &= P_{\Gamma_0}(\phi_-(p_1 + t^{-1} \widetilde{p_1}))(t) \ .\end{aligned}$$

In addition, we need to prove that $\Phi_\phi \mathscr{E} = P_{\Gamma_0 | X_1}$. To this end, we need to show that the following inclusion holds true: $\phi_-(p_1 + t^{-1} \widetilde{p_1}) - f \in t^{-1} H_-^1(\Gamma_0)$. We are left to note that the last inclusion was already deduced (even in a more general setting) in [3, Lemma 4.1]. This proves the part (ii) of the lemma.

(iii) Let $f \in \mathrm{Ker}\Phi_\phi$. This means that $f \in \mathrm{Im}Q_{J_{\Gamma_0}}$ and

$$P_{\Gamma_0}(\phi f) = 0 \; . \tag{7.3.4}$$

Define $f_- := \phi f$. From the definition of P_{Γ_0} and (7.3.4) it follows that $f_- \in t^{-1}H^2_-(\Gamma_0)$. Consequently, we have

$$\phi_-^{-1} f_- = \phi_o f \; ,$$

and therefore

$$t(1 - t^{-1})\phi_-^{-1}(t) f_-(t) = (t-1)\phi_o(t) f(t) =: \psi(t) \; . \tag{7.3.5}$$

Additionally, we have that $(1 - t^{-1})\phi_-^{-1} \in H^2_-(\Gamma_0)$ and $tf_- \in H^2_-(\Gamma_0)$. Then it follows from (7.3.5) that $\psi \in H^1_-(\Gamma_0)$. Moreover, from the last identity in (7.3.5) we have $\widetilde{\psi} = -\psi$. In particular, this implies that $\psi = 0$ and consequently $f = 0$.

□

LEMMA 7.3.6. *(an analogue version of [3, Lemma 5.1] to the present case) Suppose that Φ_ϕ is invertible. Then there exists functions $f_- \neq 0$ and f_o such that*

$$f_-(t) = \phi(t) f_o(t) \; ,$$

and

$$(1 + t^{-1}) f_- \in H^2_-(\Gamma_0), \qquad |1 + t| f_o \in L^2_{odd}(\Gamma_0) \; .$$

Proof. If Φ_ϕ is invertible, then $\mathrm{Im}\Phi_\phi = H^2_+(\Gamma_0)$. Let us consider $h_o \in \mathrm{Im}Q_{J_{\Gamma_0}}$ such that $\Phi_\phi h_o = 1$. In addition, take $h_-(t) := \phi(t) h_o(t)$. From here we have that $h_- \in H^2_-(\Gamma_0)$ and $h_- \neq 0$. If defining

$$f_- := (1 + t^{-1})^{-1} h_-(t) \; ,$$

then f_- satisfies the required conditions and

$$f_-(t) = (1 + t^{-1})^{-1} \phi(t) h_o(t) \; .$$

Now, construct $f_o(t) := (1+t^{-1})^{-1}h_o(t)$. It is readily seen that $|1+t|f_o \in L^2(\Gamma_0)$ and f_o is an odd function. Indeed,

$$\begin{aligned} f_o(t^{-1}) &= (1+t)^{-1}h_o(t^{-1}) = -(1+t)^{-1}th_o(t) = -(1+t^{-1})^{-1}h_o(t) \\ &= -f_o(t) \ . \end{aligned}$$

Consequently, we have the desired "factorization"

$$f_- = \phi f_o \ .$$

□

LEMMA 7.3.7. *(an analogue of [3, Lemma 5.2]) If Ψ_ψ is an invertible operator, then there exists functions $g_- \neq 0$ and g_o such that*

$$g_-(t) = g_o(t)\phi^{-1}(t) \ ,$$

and

$$(1-t^{-1})g_- \in H^2_-(\Gamma_0), \qquad |1-t|g_o \in L^2_{odd}(\Gamma_0) \ .$$

Proof. If Ψ_ψ is invertible, then its adjoint operator $(\Psi_\psi)^* = \Phi_{\overline{\psi}}$ is also invertible. It follows from the previous lemma that there exist elements $f_- \neq 0$ and f_o such that

$$(1+t^{-1})f_- \in H^2_-(\Gamma_0) \ , \qquad |1+t|f_o \in L^2_{\mathrm{odd}}(\Gamma_0) \ ,$$

and

$$f_-(t) = \overline{\phi^{-1}(-t^{-1})}f_o(t).$$

Let us now pass to the complex conjugate and make the substitution $t \mapsto -t^{-1}$. Choosing $g_-(t) = \overline{f_-(-t^{-1})}$ and $g_o(t) = \overline{f_o(-t^{-1})}$, it follows that

$$(1-t^{-1})g_- \in H^2_-(\Gamma_0), \quad |1-t|g_o \in L^2_{\mathrm{odd}}(\Gamma_0) \ ,$$

and $g_-(t) = \phi^{-1}(t)g_o(t)$. □

REMARK 7.3.8. The results stated in Lemma 7.3.6 and Lemma 7.3.7 still hold true if we substitute the assumption about the two-sided invertibility of Φ_ϕ and Ψ_ψ by only the right and left invertibility of these operators, respectively.

THEOREM 7.3.9. *(an analogue of [3, Theorem 5.3]) Let $\phi \in \mathcal{G}L^\infty(\Gamma_0)$. The operator $T_\phi - H_\phi$ is invertible if and only if ϕ admits an odd asymmetric factorization in $L^2(\Gamma_0)$ with index $k = 0$.*

Proof. If $T_\phi - H_\phi$ is invertible, then by Corollary 7.3.4 it follows that the operators Φ_ϕ and Ψ_ψ are also invertible. Applying Lemma 7.3.6 and Lemma 7.3.7 we will obtain that $f_- = \phi f_o$ and $g_- = g_o\phi^{-1}$ for f_-, g_-, f_o and g_o enjoying the appropriate properties described in that lemmas. Multiplying the corresponding elements in the last two identities, we obtain $g_- f_- = g_o f_o$. Moreover, it follows that $g_- f_- = g_o f_o =: C$ is a nonzero constant (this can be proved in a similar way as in the proof of the uniqueness of weak odd asymmetric factorizations in $L^2(\Gamma_0)$).

Now we put $\phi_- = f_- = C g_-^{-1}$ and $\phi_o = f_o^{-1} = g_o C^{-1}$. Hence

$$\phi = \phi_- \phi_o \ ,$$

and we have shown that ϕ admits a weak odd asymmetric factorization in $L^2(\Gamma_0)$ (with index $k = 0$). Now we have to prove that $\mathscr{E}$ can be extended to a linear bounded operator which acts on $L^2(\Gamma_0)$. From Lemma 7.3.5 we have that $\mathscr{E}$ is well-defined. Assertion (ii) of the same lemma gives the following:

$$\mathscr{E} = \Phi_\phi^{-1} P_{\Gamma_0 | X_1}$$

(recall that Φ_ϕ is invertible due to the hypothesis on $T_\phi - H_\phi$). Obviously, this right-hand side can be extended by continuity to a linear bounded operator acting from $L^2(\Gamma_0)$ into $\mathrm{Im} Q_{J_{\Gamma_0}}$ (since that is a restriction of such an operator to the space X_1), and hence also $\mathscr{E}$ can be extended as well. Thus the "only if" part is proved.

Let us now assume that ϕ admits an odd asymmetric factorization in $L^2(\Gamma_0)$ with index $k = 0$ (and so the conditions (i)–(iii) of the Definition 7.2.3 are satisfied). By $\widetilde{\mathscr{E}}$ we will denote the continuous extension of the operator $\mathscr{E}$. As far as X_1 is dense in $L^2(\Gamma_0)$ we have that

$$\Phi_\phi \widetilde{\mathscr{E}} = P_{\Gamma_0} \ .$$

for operators defined in $L^2(\Gamma_0)$. In particular, this shows that $\widetilde{\mathscr{E}}_{|H^2_+(\Gamma_0)}$ is the right inverse of Φ_ϕ. Moreover, from the above identity we obtain

$$\Phi_\phi \widetilde{\mathscr{E}}_{|H^2_+(\Gamma_0)} \Phi_\phi = \Phi_\phi \ ,$$

and from here we have

$$\Phi_\phi (\widetilde{\mathscr{E}}_{|H^2_+(\Gamma_0)} \Phi_\phi - I) = 0 \ . \tag{7.3.6}$$

Recalling now that the kernel of Φ_ϕ is trivial (cf. Lemma 7.3.5 (*iii*)), it follows from (7.3.6) that $\widetilde{\mathscr{E}}_{|H^2_+(\Gamma_0)} \Phi_\phi = I$. Consequently, Φ_ϕ is invertible and its inverse is just $\widetilde{\mathscr{E}}_{|H^2_+(\Gamma_0)}$. In such a case, finally observe that from Corollary 7.3.4 we conclude that $T_\phi - H_\phi$ is also an invertible operator. □

7.3.2 Proof of Theorem 7.1.4

Finally, after all the previous auxiliary material, we are ready to give the proof of Theorem 7.1.4.

First of all recall that (cf. Chapter 1)

$$WH_\varphi : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)$$

is equivalent to

$$T_{(B_0\varphi)} - H_{(B_0\varphi)} : H^2_+(\Gamma_0) \to H^2_+(\Gamma_0).$$

Therefore, WH_φ is invertible if and only if $T_{(B_0\varphi)} - H_{(B_0\varphi)}$ is invertible, and the last mentioned property of $T_{(B_0\varphi)} - H_{(B_0\varphi)}$ happens if and only if $B_0\varphi$ admits an odd asymmetric factorization in $L^2(\Gamma_0)$ with index $k = 0$ (cf. Theorem 7.3.9). In addition, due to Proposition 7.2.4, we have that $B_0\varphi$ admits an odd asymmetric factorization in $L^2(\Gamma_0)$ with index

$k = 0$ if and only if $B_0^{-1}(B_0\varphi) = \varphi$ admits an odd asymmetric factorization in $L^2(\mathbb{R})$ with index $k = 0$. Finally, putting altogether, we have that WH_φ is invertible if and only if φ admits an odd asymmetric factorization in $L^2(\mathbb{R})$ with index $k = 0$.

7.4 Fredholm property

In the present section it will be obtained a Fredholm criterion for WH_φ. Besides this, other particular results will follow as direct consequences of this Fredholm criterion.

THEOREM 7.4.1. *Let $\varphi \in \mathcal{G}L^\infty(\mathbb{R})$. The operator $WH_\varphi : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)$ is a Fredholm operator if and only if φ admits an odd asymmetric factorization in $L^2(\mathbb{R})$. Moreover, if WH_φ is a Fredholm operator, then it holds*

$$\dim \mathrm{Ker} WH_\varphi = \max\{0, -k\}, \quad \dim \mathrm{Ker} WH^*_\varphi = \max\{0, k\}\ , \tag{7.4.1}$$

where k is the index of an odd asymmetric factorization of φ in $L^2(\mathbb{R})$.

Proof. Assume that WH_φ is a Fredholm operator with index $-k$. We will start by using the classical Wiener-Hopf technique to built a corresponding auxiliary invertible operator. For this purpose, let us consider the auxiliary function $\psi(x) := \left(\frac{x-i}{x+i}\right)^{-k} \varphi(x)$. It is well-known that an Hankel operator with a continuous symbol is compact. Therefore (since for $k \in \mathbb{Z}$ the element ζ^{-k} with $\zeta(x) := \left(\frac{x-i}{x+i}\right)$ is continuous in the compactified real line), by employing formula (1.6.5), it follows that

$$WH_\psi = WH_{\zeta^{-k}} \ell_0 WH_\varphi + K_1\ , \tag{7.4.2}$$

where K_1 is a compact operator. In addition, let us also observe that $WH_{\zeta^{-k}} = W_{\zeta^{-k}} + K_2$ (where K_2 is a compact operator). Thus, from the Fredholm theory of Wiener-Hopf operators we conclude that $WH_{\zeta^{-k}}$ is a Fredholm operator with Fredholm index k. Consequently, from identity (7.4.2) we conclude that WH_ψ is a Fredholm operator with index zero.

Let us now consider the Lebesgue measure zero set

$$\mathbb{V}_\psi := \{x \in \mathbb{R} : \psi(x) = \psi(-x) = 0\}$$

(note that $\psi \in \mathcal{G}L^\infty(\mathbb{R})$), and the corresponding characteristic function

$$\chi_{\mathbb{V}_\psi}(x) = \begin{cases} 1, & x \in \mathbb{V}_\psi; \\ 0, & x \notin \mathbb{V}_\psi. \end{cases}$$

Arguing in a similar way as in the Toeplitz plus Hankel case (see [3]), it follows that

$$\mathrm{Ker} WH_\psi \cong \mathrm{Im} WH_{\chi_{\mathbb{V}_\psi}} \;\text{ or }\; \mathrm{Ker} WH_\psi^* = \{0\}, \tag{7.4.3}$$

where $\cong$ denotes the existence of an isomorphically isomorphism between the related sets. Since $\mathbb{V}_\psi$ has the Lebesgue measure zero, and hence $\chi_{\mathbb{V}_\psi} = 0$ for almost all $x \in \mathbb{R}$, it follows that $\mathrm{Im} WH_{\chi_{\mathbb{V}_\varphi}} = \{0\}$. This combined with (7.4.3) it leads to $\mathrm{Ker} WH_\psi \cong \{0\}$ or $\mathrm{Ker} WH_\psi^* = \{0\}$. Thus $\mathrm{Ker} WH_\psi = \{0\}$ or $\mathrm{Ker} WH_\psi^* = \{0\}$. This means that WH_ψ is invertible (since we have already previously concluded that WH_ψ is a Fredholm operator with index zero).

Now, employing Theorem 7.1.4 we deduce that ψ admits an odd asymmetric factorization in $L^2(\mathbb{R})$ with index zero. Hence φ admits an odd asymmetric factorization in $L^2(\mathbb{R})$ with index k.

Now we will proceed with the reverse implication. Assume that φ admits an odd asymmetric factorization in $L^2(\mathbb{R})$ with index k. Consequently, we have a corresponding operator decomposition:

$$WH_\varphi = W_{\varphi_-} \ell_0 WH_{\zeta^k} \ell_0 WH_{\varphi_o} + K\ ,$$

where K is a compact operator. Thus, WH_φ is a Fredholm operator if and only if $W_{\varphi_-} \ell_0 WH_{\zeta^k} \ell_0 WH_{\varphi_o}$ is a Fredholm operator. However, the latter operator is equivalent to WH_{ζ^k}, because ℓ_0, W_{φ_-} and WH_{φ_o} are invertible operators. Therefore, as above, we simply have to notice that $WH_{\zeta^k} = W_{\zeta^k} + H_{\zeta^k}$ is a Fredholm operator with index $-k$.

Let us now turn to formulas (7.4.1). Under the Fredholm property we already know that WH_φ has a Fredholm index equal to $-k$. Thus, combining this fact with (7.4.3) it directly follows the presented formulas for the defect numbers. $\square$

As a direct consequence of the last result we collect the following interesting conclusions.

COROLLARY 7.4.2. *If WH_φ is a Fredholm operator, then WH_φ has a trivial kernel or a trivial cokernel.*

COROLLARY 7.4.3. *The Wiener-Hopf plus Hankel operator WH_φ is invertible if and only if WH_φ is Fredholm with index zero.*

In addition, it is also clear that Theorem 7.4.1 implies Theorem 7.1.4 but we would like to emphasize that to prove Theorem 7.4.1 we needed to use Theorem 7.1.4.

Chapter 8

Scalar Toeplitz plus Hankel operators with infinite index

Discontinuities of almost periodic type appeared for the first time in the work of Gohberg and Feld'man [40], [41], [42] when studying Wiener-Hopf equations. The paper by Coburn and Douglas [28] is also an important mark for the beginning of the study of integral operators with symbols which present such kind of discontinuities. Since then, the consideration of Toeplitz and singular integral operators with symbols and coefficients with discontinuities of almost periodic type were considered by a big number of authors (cf., e.g., [5], [35], [36], [43], [48], [53]). The book of Dybin and Grudsky [37] provides a comprehensive description of the known results for Toeplitz operators with infinite index originated by symbols with almost periodic discontinuities.

The present chapter is devoted to the study of Toeplitz plus Hankel operators (cf. [3], [61]) with a finite number of standard almost periodic discontinuities in their symbols. The operators are acting between L^2 spaces on the unit circle. The results (cf. Section 8.2) provide conditions under which the Toeplitz plus Hankel operators are right-invertible but with infinite dimensional kernel or left-invertible but with infinite dimensional cokernel or simply not normally solvable.

This chapter is organized as follows. In Section 8.1 (which is divided into two subsections) we present the auxiliary notions and some known results for Toeplitz and Toeplitz plus Hankel operators. In Section 8.2, new results for Toeplitz plus Hankel operators are proposed which lead to their one-sided invertibility although within the case of infinite index. In Section 8.3 we provide two concrete examples of Toeplitz plus Hankel operators which are characterized by the use of the results in Section 8.2.

8.1 Auxiliary notions and known results

8.1.1 Factorization and Fredholm theory

We start by recalling several types of factorizations.

DEFINITION 8.1.1. *[37, Section 2.4] A function $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ admits a generalized factorization with respect to $L^2(\Gamma_0)$, if it can be represented in the form*

$$\phi(t) = \phi_-(t)t^k\phi_+(t), \quad t \in \Gamma_0,$$

where k is an integer, called the index of the factorization, and the functions $\phi_\pm$ satisfy the following conditions:

(1) $(\phi_-)^{\pm 1} \in L^2_-(\Gamma_0) \oplus \mathbb{C}$, $(\phi_+)^{\pm 1} \in L^2_+(\Gamma_0)$,

(2) the operator $\phi_+^{-1} S_{\Gamma_0} \phi_-^{-1} I$ is bounded in $L^2(\Gamma_0)$.

The class of functions admitting a generalized factorization will be denoted by $\mathbb{F}$.

DEFINITION 8.1.2. *[3, Section 3] A function $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ is said to admit a weak even asymmetric factorization in $L^2(\Gamma_0)$ if it admits a representation*

$$\phi(t) = \phi_-(t)t^k\phi_e(t) , \quad t \in \Gamma_0 ,$$

such that $k \in \mathbb{Z}$ and

(i) $(1 + t^{-1})\phi_- \in H^2_-(\Gamma_0)$, $\quad (1 - t^{-1})\phi_-^{-1} \in H^2_-(\Gamma_0)$,

(ii) $|1 - t|\phi_e \in L^2_{even}(\Gamma_0)$, $\quad |1 + t|\phi_e^{-1} \in L^2_{even}(\Gamma_0)$,

where $L^2_{even}(\Gamma_0)$ *stands for the class of even functions from the space* $L^2(\Gamma_0)$. *The integer* k *is called the index of the weak even asymmetric factorization.*

DEFINITION 8.1.3. *[3, Section 3] A function* $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ *is said to admit a weak antisymmetric factorization in* $L^2(\Gamma_0)$ *if it admits a representation*

$$\phi(t) = \phi_-(t) t^{2k} \widetilde{\phi_-^{-1}}(t) \, , \quad t \in \Gamma_0 \, ,$$

such that $k \in \mathbb{Z}$ *and*

$$(1+t^{-1})\phi_- \in H^2_-(\Gamma_0), \quad (1-t^{-1})\phi_-^{-1} \in H^2_-(\Gamma_0).$$

Also in here the integer k *is called the index of a weak antisymmetric factorization.*

The next proposition relates weak even asymmetric factorizations with weak antisymmetric factorizations.

PROPOSITION 8.1.4. *[3, Proposition 3.2] Let* $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ *and consider* $\Phi := \phi\widetilde{\phi^{-1}}$.

(i) If ϕ *admits a weak even asymmetric factorization,* $\phi = \phi_- t^k \phi_e$, *then the function* Φ *admits a weak antisymmetric factorization with the same factor* ϕ_- *and the same index* k;

(ii) If Φ *admits a weak antisymmetric factorization,* $\Phi = \phi_- t^{2k} \widetilde{\phi_-^{-1}}$, *then* ϕ *admits a weak even asymmetric factorization with the same factor* ϕ_-, *the same index* k *and the factor* $\phi_e := t^{-k}\phi_-^{-1}\phi$.

DEFINITION 8.1.5. *[3, Section 5] A function* $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ *is said to admit an even asymmetric factorization in* $L^2(\Gamma_0)$ *if it admits a representation*

$$\phi(t) = \phi_-(t) t^k \phi_e(t) \, , \quad t \in \Gamma_0 \, ,$$

such that $k \in \mathbb{Z}$ *and*

(i) $(1+t^{-1})\phi_- \in H^2_-(\Gamma_0), \quad (1-t^{-1})\phi_-^{-1} \in H^2_-(\Gamma_0)$,

(ii) $|1-t|\phi_e \in L^2_{even}(\Gamma_0), \quad |1+t|\phi_e^{-1} \in L^2_{even}(\Gamma_0)$,

(iii) the linear operator $L(\phi_e^{-1})(I+J_{\Gamma_0})P_{\Gamma_0}L(\phi_-^{-1})$ *is bounded on* X_1,

where X_1 *is as in Section 7.2. The integer* k *is called the index of an even asymmetric factorization.*

The next theorem is a classical result which deals with the Fredholm property for the Toeplitz operators.

THEOREM 8.1.6. *Let* $\phi \in L^\infty(\Gamma_0)$. *The operator* T_ϕ *given by (1.5.1) is Fredholm in the space* $L^2_+(\Gamma_0)$ *if and only if* $\phi \in \mathbb{F}$.

The next two theorems were obtained by Basor and Ehrhardt (cf. [3]), and give an useful invertibility and Fredholm characterization for Toeplitz plus Hankel operators with essentially bounded symbols.

THEOREM 8.1.7. *[3, Theorem 5.3] Let* $\phi \in \mathcal{G}L^\infty(\Gamma_0)$. *The operator* TH_ϕ *is invertible if and only if* ϕ *admits an even asymmetric factorization in* $L^2(\Gamma_0)$ *with index* $k=0$.

THEOREM 8.1.8. *[3, Theorem 6.4] Let* $\phi \in \mathcal{G}L^\infty(\Gamma_0)$. *The operator* TH_ϕ *is a Fredholm operator if and only if* ϕ *admits an even asymmetric factorization in* $L^2(\Gamma_0)$. *In this case, it holds*

$$\dim \mathrm{Ker} TH_\phi = \max\{0,-k\}, \quad \dim \mathrm{Coker} TH_\phi = \max\{0,k\}\ ,$$

where k *is the index of the even asymmetric factorization.*

We will now turn to the generalized factorizations with infinite index.

DEFINITION 8.1.9. *[37, Section 2.7] A function* $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ *admits a generalized factorization with infinite index in the space* $L^2(\Gamma_0)$ *if it admits a representation*

$$\phi = \varphi h \quad \text{or} \quad \phi = \varphi h^{-1} \tag{8.1.1}$$

where

(1) $\varphi \in \mathbb{F}$,

(2) $h \in L^{\infty}_{+}(\Gamma_0) \cap \mathcal{G}L^{\infty}(\Gamma_0)$.

The class of functions admitting a generalized factorization with infinite index in $L^2(\Gamma_0)$ will be denoted by $\mathbb{F}_\infty$. We list here some known important properties of the class $\mathbb{F}_\infty$ (cf. [37, Chapter 2]):

1. $\mathbb{F} \subset \mathbb{F}_\infty$. Therefore (from this inclusion and Theorem 8.1.6), it follows that the class $\mathbb{F}_\infty$ contains symbols of Fredholm Toeplitz operators. However, in another way, the following condition excludes elements which generate Fredholm operators from this class: for any polynomial u with complex coefficients,
$$u/h \notin L^{\infty}_{+}(\Gamma_0). \tag{8.1.2}$$
More precisely, if condition (8.1.2) is not satisfied, then for a given h and all $\varphi \in \mathbb{F}$ the operators $T_{\varphi h}$ and $T_{\varphi h^{-1}}$ are Fredholm.

2. A generalized factorization with infinite index does not enjoy the uniqueness property.

3. Let $\phi \in \mathbb{F}_\infty$ and let condition (8.1.2) be satisfied. Then the function h in (8.1.1) can be chosen so that $\mathrm{ind}\varphi = 0$.

4. Let $\phi \in \mathbb{F}_\infty$. Then for the function h in (8.1.1) one can choose an inner function u (i.e., a function u from the Hardy space $H^{\infty}_{+}(\Gamma_0)$ and such that $|u(t)| = 1$ almost everywhere on Γ_0).

The proof of these facts can be found for example in [37, Section 2.7].

THEOREM 8.1.10. *[37, Theorem 2.6] Assume that* $\phi \in \mathbb{F}_\infty$, *condition (8.1.2) is satisfied, and* $\mathrm{ind}\varphi = 0$.

1. *If* $\phi = \varphi h^{-1}$, *then the operator* T_ϕ *is right-invertible in the space* $L^2_+(\Gamma_0)$, *and* $\dim \mathrm{Ker} T_\phi = \infty$.

2. *If* $\phi = \varphi h$, *then the operator* T_ϕ *is left-invertible in the space* $L^2_+(\Gamma_0)$, *and* $\dim \mathrm{Coker} T_\phi = \infty$.

8.1.2 One-sided invertibility of Toeplitz operators

A factorization theorem which is crucial for the theory of Toeplitz operators is now stated.

THEOREM 8.1.11. *[37, Theorem 4.12] Let the function $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ be continuous on the set $\Gamma_0 \setminus \{t_j\}_{j=1}^n$ and have standard almost periodic discontinuities in the points t_j. Then*

$$\phi(t) = \left(\prod_{j=1}^{n} \exp(\lambda_j (t - t_j)^{-1})\right) \varphi(t) ,$$

with $\varphi \in \mathbb{F}$ and

$$\lambda_j = \sigma_{t_j}(\phi)\ t_j$$

where the functional $\sigma_{t_j}(\phi)$ is defined by the formula (2.4.5) at the point t_j.

Let us always write the factorization of a function ϕ in the way of the non-decreasing order of the values of $\sigma_{t_j}(\phi)$. I.e., we will always assume that $\sigma_{t_1}(\phi) \leq \sigma_{t_2}(\phi) \leq \ldots \leq \sigma_{t_n}(\phi)$. This is always possible because we can always re-enumerate the points t_j to achieve the desired non-decreasing sequence.

The next result characterizes the situation of Toeplitz operators with a symbol having a finite number of standard almost periodic discontinuities, and it was our starting point motivation in view to obtain a corresponding description to Toeplitz plus Hankel operators.

THEOREM 8.1.12. *[37, Theorem 4.13] Suppose that $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ is continuous on the set $\Gamma_0 \setminus \{t_j\}_{j=1}^n$, has standard almost periodic discontinuities in the points t_j, and $\sigma_{t_j}(\phi) \neq 0$, $1 \leq j \leq n$.*

1. *If $\sigma_{t_j}(\phi) < 0$, $1 \leq j \leq n$, then the operator T_ϕ is right-invertible in $L^2_+(\Gamma_0)$ and* $\dim \mathrm{Ker} T_\phi = \infty$,

2. *If $\sigma_{t_j}(\phi) > 0$, $1 \leq j \leq n$, then the operator T_ϕ is left-invertible in $L^2_+(\Gamma_0)$ and* $\dim \mathrm{Coker} T_\phi = \infty$,

3. *If $\sigma_{t_j}(\phi) < 0$, $1 \leq j \leq m$, and $\sigma_{t_j}(\phi) > 0$, $m+1 \leq j \leq n$, then the operator T_ϕ is not normally solvable in $L^2_+(\Gamma_0)$ and* $\dim \mathrm{Ker} T_\phi = \dim \mathrm{Coker} T_\phi = 0$.

8.2 Toeplitz plus Hankel operators with $SAPD$ in their symbols

To achieve the Toeplitz plus Hankel version of Theorem 8.1.12, we will combine several techniques. We will make use of operator matrix identities (cf. [20], [21], [45]), and in particular of Δ-relation after extension (cf. Chapter 1).

For starting, we will consider functions defined on the Γ_0 which have three standard almost periodic discontinuities, namely in the points t_1, t_2 and t_3, and such that $t_1^{-1} = t_2$. As we shall see, this is the most representative case, and the general case can be treated in the same manner as this one. Assume therefore that ϕ has standard almost periodic discontinuities in the points t_1, t_2, t_3, with characteristics $(p_1, \omega_1), (p_2, \omega_2), (p_3, \omega_3)$. Considering $\widetilde{\phi}$, it is clear that $\widetilde{\phi}$ has standard almost periodic discontinuities in the points $t_1^{-1}(= t_2), t_2^{-1}(= t_1)$ and t_3^{-1} (cf. Remark 2.4.2). Moreover, it is useful to observe that $\widetilde{\phi^{-1}}$ will have standard almost periodic discontinuities in the points t_1, t_2 and t_3^{-1}.

Set $\Phi := \phi\widetilde{\phi^{-1}}$. From formula (2.4.5) we will have:

$$\begin{aligned}
\sigma_{t_j}(\Phi) &= \sigma_{t_j}(\phi\widetilde{\phi^{-1}}) = \lim_{\delta\to 0} \frac{\delta}{4}\left[\arg(\phi(t)\widetilde{\phi^{-1}}(t))\right]\Big|_{t=t''}^{t'} \\
&= \lim_{\delta\to 0} \frac{\delta}{4}\left[\arg \phi(t)\right]\big|_{t=t''}^{t'} + \lim_{\delta\to 0} \frac{\delta}{4}\left[\arg \widetilde{\phi^{-1}}(t)\right]\Big|_{t=t''}^{t'} \\
&= \sigma_{t_j}(\phi) - \lim_{\delta\to 0} \frac{\delta}{4}\left[\arg \widetilde{\phi}(t)\right]\Big|_{t=t''}^{t'} \\
&= \sigma_{t_j}(\phi) + \lim_{\delta_1\to 0} \frac{\delta_1}{4}\left[\arg \phi(t)\right]\big|_{t=(t')^{-1}}^{(t'')^{-1}} \\
&= \sigma_{t_j}(\phi) + \sigma_{t_j^{-1}}(\phi)\ ,
\end{aligned} \tag{8.2.1}$$

where $t_j \in \Gamma_0$ and $\delta_1 = |(t'')^{-1} - t_j^{-1}| = |(t')^{-1} - t_j^{-1}| = |t'' - t_j| = |t' - t_j| = \delta$.

On the other hand, it is also clear that $\sigma_{t_j^{-1}}(\Phi) = \sigma_{t_j}(\phi) + \sigma_{t_j^{-1}}(\phi)$. Thus, the points of symmetric standard almost periodic discontinuities (with respect to the xx's axis on the complex plane) fulfill formula (8.2.1). This is the main reason why we do not need to treat more than three points of the standard almost periodic discontinuities in order to understand the qualitative result for Toeplitz plus Hankel operators with a finite number of standard almost periodic discontinuities in their symbols.

We are now in a position to present the Toeplitz plus Hankel version of Theorem 8.1.12 for three points of discontinuity.

THEOREM 8.2.1. *Suppose that the function $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ is continuous on the set $\Gamma_0 \backslash \{t_j\}_{j=1}^3$, has standard almost periodic discontinuities in the points t_j, such that $t_1^{-1} = t_2$, and let $\sigma_{t_j}(\phi) \neq 0$, $1 \leq j \leq 3$.*

(i) If $\sigma_{t_1}(\phi) + \sigma_{t_2}(\phi) \leq 0$ and $\sigma_{t_3}(\phi) < 0$, then the operator TH_ϕ is right-invertible in $L^2_+(\Gamma_0)$ and $\dim \mathrm{Ker} TH_\phi = \infty$,

(ii) If $\sigma_{t_1}(\phi) + \sigma_{t_2}(\phi) \geq 0$ and $\sigma_{t_3}(\phi) > 0$, then the operator TH_ϕ is left-invertible in $L^2_+(\Gamma_0)$ and $\dim \mathrm{Coker} TH_\phi = \infty$,

(iii) If $(\sigma_{t_1}(\phi) + \sigma_{t_2}(\phi))\sigma_{t_3}(\phi) < 0$, then the operator TH_ϕ is not normally solvable in $L^2_+(\Gamma_0)$ and $\dim \mathrm{Ker} TH_\phi = \dim \mathrm{Coker} TH_\phi = 0$.

Proof. Let us work with $\Phi := \phi\widetilde{\phi^{-1}}$. It is clear that Φ can be considered (due to the invertibility of ϕ), and also that Φ is invertible in $L^\infty(\Gamma_0)$. As far as ϕ has three points of almost periodic discontinuities (namely t_1, t_2 and t_3), then Φ will have four points of almost periodic discontinuities (due to the reason that $t_1^{-1} = t_2$). The discontinuity points of Φ are the following ones: t_1, t_2, t_3 and t_3^{-1}. From formula (8.2.1), we will have that

$$\sigma_{t_1}(\Phi) = \sigma_{t_1}(\phi) + \sigma_{t_1^{-1}}(\phi) = \sigma_{t_1}(\phi) + \sigma_{t_2}(\phi) \, , \tag{8.2.2}$$

$$\sigma_{t_2}(\Phi) = \sigma_{t_2}(\phi) + \sigma_{t_2^{-1}}(\phi) = \sigma_{t_2}(\phi) + \sigma_{t_1}(\phi) \, , \tag{8.2.3}$$

$$\sigma_{t_3}(\Phi) = \sigma_{t_3}(\phi) + \sigma_{t_3^{-1}}(\phi) = \sigma_{t_3}(\phi) \, , \tag{8.2.4}$$

$$\sigma_{t_3^{-1}}(\Phi) = \sigma_{t_3^{-1}}(\phi) + \sigma_{t_3}(\phi) = \sigma_{t_3}(\phi) \, . \tag{8.2.5}$$

In the above formulas, it was used the fact that ϕ is a continuous function in the point t_3^{-1}. Now, employing Theorem 8.1.11, we can ensure a factorization of the function

Φ in the form:

$$\Phi(t) = \left(\prod_{j=1}^{4} \exp(\lambda_j (t - t_j)^{-1}) \right) \varphi(t) , \tag{8.2.6}$$

where $\varphi \in \mathbb{F}$.

Let us denote

$$h(t) = \prod_{j=1}^{4} \exp(\lambda_j (t - t_j)^{-1}) . \tag{8.2.7}$$

We will now deduce propositions (i)–(iii) in the following three points 1–3, respectively:

1. If the conditions in part (i) are satisfied, then we will have that $\sigma_{t_j}(\Phi) \leq 0$, $j = \overline{1,4}$ (cf. formulas (8.2.2)–(8.2.5)). Hence, the function h given by (8.2.7) belongs to $L^{\infty}_{-}(\Gamma_0)$. Moreover, relaying on Proposition 2.4.3 and Remark 2.4.6, we have that $h^{-1} \in \mathcal{U}$. Using the first part of Theorem 8.1.12, we can conclude that T_Φ is right-invertible. Then, the Δ-relation after extension allows us to state that TH_ϕ is right-invertible.

   We are left to deduce that $\dim \operatorname{Ker} TH_\phi = \infty$.

   Suppose that $\dim \operatorname{Ker} TH_\phi = k < \infty$. We will show that in the present situation this is not possible. In the case at hand we would have a Fredholm Toeplitz plus Hankel operator with symbol ϕ. Thus, by Theorem 8.1.8, ϕ admits an even asymmetric factorization:

   $$\phi = \phi_- t^k \phi_e ,$$

   with corresponding properties for ϕ_- and ϕ_e. Employing now Proposition 8.1.4 we will have that Φ admits a weak antisymmetric factorization:

   $$\Phi = \phi_- t^{2k} \widetilde{\phi_-^{-1}} . \tag{8.2.8}$$

   On the other hand (cf. (8.2.6)) we have that

   $$\Phi = \varphi_- t^m \varphi_+ h ,$$

where $\varphi_\pm$ have the properties as stated in Definition 8.1.1 and m is integer. From the last two equalities we derive:

$$\phi_- t^{2k} \widetilde{\phi_-^{-1}} = \varphi_- t^m \varphi_+ h \, .$$

From here one obtains:

$$\phi_- \widetilde{\phi_-^{-1}} = t^{m-2k} \varphi_- \varphi_+ h \, . \tag{8.2.9}$$

In the last equality performing the change of variable $t \to t^{-1}$, we get that

$$\widetilde{\phi_-} \phi_-^{-1} = t^{2k-m} \widetilde{\varphi_-} \widetilde{\varphi_+} \widetilde{h} \, .$$

Now, taking the inverse of both sides of the last formula, one obtains:

$$\phi_- \widetilde{\phi_-^{-1}} = t^{m-2k} \widetilde{\varphi_-^{-1}} \widetilde{\varphi_+^{-1}} \widetilde{h^{-1}} \, . \tag{8.2.10}$$

From the formulas (8.2.9) and (8.2.10) we have:

$$t^{m-2k} \varphi_- \varphi_+ h = t^{m-2k} \widetilde{\varphi_-^{-1}} \widetilde{\varphi_+^{-1}} \widetilde{h^{-1}} \, .$$

This leads us to the following equality:

$$\varphi_+ \widetilde{\varphi_-} h \widetilde{h} = \varphi_-^{-1} \widetilde{\varphi_+^{-1}} \, . \tag{8.2.11}$$

To our reasoning, the most important term in the last equality is now $h\widetilde{h}$. Therefore, let us understand better the structure of $h\widetilde{h}$.

Firstly, let us assume that $h\widetilde{h} \neq \text{const}$. Rewriting formula (8.2.7) we will have:

$$h(t) = \exp\left(\frac{\lambda_1}{t-t_1}\right) \exp\left(\frac{\lambda_2}{t-t_2}\right) \exp\left(\frac{\lambda_3}{t-t_3}\right) \exp\left(\frac{\lambda_4}{t-t_3^{-1}}\right) \, .$$

From here, we also have the following identity:

$$\widetilde{h}(t) = c_1 \exp\left(\frac{-\lambda_1 t_2^2}{t-t_2}\right) \exp\left(\frac{-\lambda_2 t_1^2}{t-t_1}\right) \exp\left(\frac{-\lambda_3 t_3^2}{t-t_3^{-1}}\right) \exp\left(\frac{-\lambda_4 t_3^{-2}}{t-t_3}\right) \, ,$$

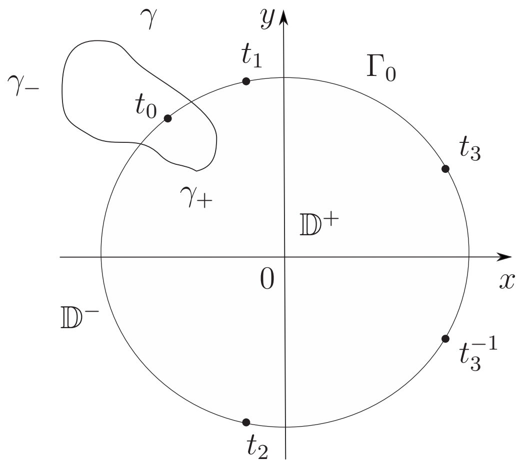

Figure 8.1: The unit circle Γ_0 intersected with a Jordan curve γ.

where c_1 is a certain nonzero constant which can be calculated explicitly (in fact, $c_1 = \exp\big(-\lambda_1 t_2 - \lambda_2 t_1 - \lambda_3 t_3^{-1} - \lambda_4 t_3\big)$). Performing the multiplication of the last two formulas, one obtains:

$$h(t)\widetilde{h}(t) = c_1 \exp\left(\frac{\lambda_1 - \lambda_2 t_1^2}{t - t_1}\right) \exp\left(\frac{\lambda_2 - \lambda_1 t_2^2}{t - t_2}\right)$$
$$\exp\left(\frac{\lambda_3 - \lambda_4 t_3^{-2}}{t - t_3}\right) \exp\left(\frac{\lambda_4 - \lambda_3 t_3^2}{t - t_3^{-1}}\right).$$

Hence, we have that

$$h(t)\widetilde{h}(t) = h_1(t)h_2(t)h_3(t)h_4(t)\ ,$$

where

$$h_1(t) = c_1 \exp\left(\frac{\lambda_1 - \lambda_2 t_1^2}{t - t_1}\right), \qquad h_2(t) = \exp\left(\frac{\lambda_2 - \lambda_1 t_2^2}{t - t_2}\right),$$
$$h_3(t) = \exp\left(\frac{\lambda_3 - \lambda_4 t_3^{-2}}{t - t_3}\right), \qquad h_4(t) = \exp\left(\frac{\lambda_4 - \lambda_3 t_3^2}{t - t_3^{-1}}\right).$$

If $h_1 \in L^\infty_-(\Gamma_0)$, then $h_2 \in L^\infty_+(\Gamma_0)$ (because $h_2 = c_2\widetilde{h}_1$, where c_2 is a certain nonzero constant). Of course, the same holds true for h_3 and h_4. At this point, we arrive at the fact that two of the four functions h_i, $1 \leq i \leq 4$, are from the minus class and two of them are from the plus class. Therefore, without lost of generality we can assume that h_1 and h_3 belong to $L^\infty_-(\Gamma_0)$, and h_2 and h_4 belong to $L^\infty_+(\Gamma_0)$. Consequently, we have a decomposition:

$$h\widetilde{h} = h_- h_+ \ ,$$

where $h_- := h_1 h_3$ and $h_+ := h_2 h_4$. From (8.2.11) we will have:

$$\varphi_+ \widetilde{\varphi_-} h_- h_+ = \varphi_-^{-1} \widetilde{\varphi_+^{-1}} \ . \tag{8.2.12}$$

Let us introduce the notation:

$$\Psi_+ := \varphi_+ \widetilde{\varphi_-} h_+, \quad H_- := h_-, \text{ and } \quad \Psi_- := \varphi_-^{-1} \widetilde{\varphi_+^{-1}}. \tag{8.2.13}$$

The identity (8.2.12) can be therefore presented in the following way:

$$H_- \Psi_+ = \Psi_- \ . \tag{8.2.14}$$

We will use now the same reasoning as in the proof of [37, Theorem 4.13, part (3)]. First of all let us observe that $\Psi_+ \in L^1_+(\Gamma_0)$ and $\Psi_- \in L^1_-(\Gamma_0)$. We claim that the functions $\Psi_\pm$ are analytic in the points of the curve Γ_0, except for the set $M_0 := \{t_1, t_3\}$. Let us take any point $t_0 \in \Gamma_0 \setminus M_0$ and surround it by a smooth contour γ, such that $\overline{\mathcal{D}^+_\gamma} \cap M_0 = \emptyset$ and such that the unit circle Γ_0 divides the domain $\mathcal{D}^+_\gamma$ into two simply connected domains bounded by closed curves γ_+ and γ_- with $\mathcal{D}^+_{\gamma_+} \subset \mathbb{D}^+$ and $\mathcal{D}^+_{\gamma_-} \subset \mathbb{D}^-$ (cf. Figure 8.1).

Let us make use of the function

$$\Psi(z) = \begin{cases} H_-(z)\Psi_+(z), & \text{if } z \in \mathbb{D}^+ \ , \\ \Psi_-(z), & \text{if } z \in \mathbb{D}^- \ , \end{cases}$$

which is defined on $\mathbb{C} \setminus \Gamma_0$ and has interior and exterior non-tangential limit values almost everywhere on Γ_0, which coincide due to equality (8.2.14).

We will now evaluate the integral

$$\int_\gamma \Psi(z) dz = \int_{\gamma_+} \Psi(z) dz + \int_{\gamma_-} \Psi(z) dz \ .$$

Since $\Psi_+ \in L^1_+(\Gamma_0)$, one can verify that $\Psi_+ \in L^1_+(\gamma_+)$ (by using the definition of the Smirnov space cf., e.g., [37, Section 2.3]). Therefore, $\Psi \in L^1_+(\gamma_+)$ (H_- is analytic in a neighborhood of the point t_0) and the integral along γ_+ is equal to zero (cf. [37, Proposition 1.1] for the Γ_0 case). Arguing in a similar way, one can also reach to the conclusion that the corresponding integral along γ_- is equal to zero. Thus,

$$\int_\gamma \Psi(z)dz = 0 \ ,$$

and the contour γ can be replaced by any closed rectifiable curve contained in $\mathcal{D}^+_\gamma$. By Morera's theorem, Ψ is analytic in $\mathcal{D}^+_\gamma$. Let us consider a neighborhood $\mathcal{O}(t_i)$ of any of the points $t_i = t_2$ or $t_i = t_3^{-1}$. Due to the identity

$$\varphi_+ \widetilde{\varphi_-} = h_+^{-1} \Psi_+ \ ,$$

where $\varphi_+ \widetilde{\varphi_-} \in L^1_+(\Gamma_0)$, we see that $h_+^{-1}\Psi_+ \in L^1_+(\Gamma_0)$. However, Ψ_+ is analytic in $\mathcal{O}(t_i)$, and the function $h_+^{-1}(z)$ grows exponentially when z approaches t_i non-tangentially, $z \in \mathbb{D}^+$. Since the function $(t - t_i)^n \exp(-\lambda_i (t - t_i)^{-1})$ does not belong to $L^1_+(\Gamma_0)$ for any choice of positive integer n, we conclude that $\Psi_+ = 0$, identically. This means that $\Psi_- = 0$, identically. From (8.2.13) we infer that φ_+ or φ_- must vanish on a set with positive Lebesgue measure, which gives that Φ is not invertible. Therefore, in this case we obtain a contradiction (due to the reason that Φ was taken to be invertible from the beginning).

Let us now assume that $h\widetilde{h} = c_1' = \text{const} \neq 0$. From (8.2.11) we get that $\varphi_- = c_1'' \widetilde{\varphi_+^{-1}}$. Hence, $\Phi = c_1'' \varphi_- t^m \widetilde{\varphi_-^{-1}} h$. Combining this with (8.2.8), it yields

$$\phi_- t^{2k} \widetilde{\phi_-^{-1}} = c_1'' \varphi_- t^m \widetilde{\varphi_-^{-1}} h \ .$$

Rearranging the last equality, one obtains:

$$c_1'' \varphi_- \phi_-^{-1} t^{m-2k} h = \widetilde{\phi_-^{-1}} \widetilde{\varphi_-} \ . \tag{8.2.15}$$

We have that $(1 - t^{-1})\phi_-^{-1} \in H^2_-(\Gamma_0)$ and $(1 - t)\widetilde{\phi_-^{-1}} \in H^2_+(\Gamma_0)$ (cf. Definition 8.1.2). If we use the multiplication by $(1 - t)(1 - t^{-1})$ in both sides of formula (8.2.15), then we will obtain:

$$(1-t)(1-t^{-1})c_1''\varphi_-\phi_-^{-1}t^{m-2k}h = (1-t)(1-t^{-1})\widetilde{\phi_-^{-1}}\widetilde{\varphi_-}\ . \tag{8.2.16}$$

Let us denote $\Theta_- := (1-t^{-1})\phi_-^{-1}$. It is clear that $\Theta_- \in H^2_-(\Gamma_0)$, and that $\widetilde{\Theta}_- \in H^2_+(\Gamma_0)$. Rewriting formula (8.2.16) and having in mind the introduced notation, we get:

$$c_2\Theta_-\varphi_-t^{m-2k+1}h = \widetilde{\Theta}_-\widetilde{\varphi_-}\ , \tag{8.2.17}$$

where $c_2 := -c_1''$. Set $N := m-2k+1$. If $N \leq 0$, then we have a trivial situation. Therefore, let us assume that $N > 0$. In this case, we will rewrite the formula (8.2.17) in the following way:

$$c_2\Theta_-\varphi_-t^N = \widetilde{\Theta}_-\widetilde{\varphi_-}h^{-1}\ .$$

From the last equality we have that the right-hand side belongs to $L^1_+(\Gamma_0)$. Therefore, the left-hand side must also belong to $L^1_+(\Gamma_0)$. This means that t^N must "dominate" the term $\Theta_-\varphi_-$, which in its turn implies that:

$$\Theta_-\varphi_- = b_0 + b_{-1}t^{-1} + \cdots + b_{-N+\nu}t^{-\nu} + \cdots + b_{-N}t^{-N}, \quad 0 \leq \nu \leq N$$

(by observing the Fourier coefficients). In particular, this shows that we will not have terms with less exponent than $-N$. In addition, the last equality directly implies that

$$\widetilde{\Theta}_-\widetilde{\varphi_-} = b_0 + b_{-1}t + \cdots + b_{-N+\nu}t^{\nu} + \cdots + b_{-N}t^N\ .$$

From the last three equalities we obtain that:

$$h = c_2^{-1}\frac{b_0 + b_{-1}t + \cdots + b_{-N+\nu}t^{\nu} + \cdots + b_{-N}t^N}{b_{-N} + b_{-N+1}t + \cdots + b_{-N+\nu}t^{N-\nu} + \cdots + b_0t^N}\ .$$

We are left to observe that $h \in L^\infty_-(\Gamma_0)$. Due to its special form (cf. (8.2.7)), h cannot be represented as a fraction of two polynomial functions (since h is not a rational function). Hence, once again, we arrive at a contradiction.

Altogether, we reached to the conclusion that the dimension of the kernel of the Toeplitz plus Hankel operator with symbol ϕ cannot be equal to a finite number k. Therefore, in the present case, the Toeplitz plus Hankel operator has an infinite dimensional kernel.

2. Let the conditions of proposition (ii) be satisfied. Then, $\sigma_{t_j}(\Phi) \geq 0$ for $j = \overline{1,4}$. Now, by using the argument of passage to the adjoint operator in the last case 1., we can conclude that in the present position TH_ϕ is left-invertible and $\dim \operatorname{Coker} TH_\phi = \infty$.

3. If the conditions of proposition (iii) are satisfied, then $\sigma_{t_j}(\Phi)$ will have different signs (cf. formulas (8.2.2)–(8.2.5)). Therefore, by the Δ-relation after extension and Theorem 8.1.12, we will obtain that $\dim \operatorname{Ker} TH_\phi + \dim \operatorname{Ker}(T_\phi - H_\phi) = 0$, and that $\dim \operatorname{Coker} TH_\phi + \dim \operatorname{Coker}(T_\phi - H_\phi) = 0$. As far as the dimensions cannot be negative, we will have that both defect numbers of the operator TH_ϕ must vanish; hence, $\dim \operatorname{Ker} TH_\phi = \dim \operatorname{Coker} TH_\phi = 0$.

   We are left to prove that TH_ϕ is not normally solvable.

   Let us assume the contrary, i.e. let TH_ϕ be normally solvable. Then we immediately conclude that TH_ϕ is invertible, due to the triviality of the defect numbers. Hence (by Theorem 8.1.7) ϕ admits an even asymmetric factorization with index zero:

$$\phi = \phi_- \phi_e \,, \tag{8.2.18}$$

   where ϕ_- and ϕ_e have the appropriate properties (as stated in the Definition 8.1.2). From (8.2.18), we obtain that:

$$\phi \widetilde{\phi^{-1}} = \phi_- \widetilde{\phi_-^{-1}} \,. \tag{8.2.19}$$

   As far as Φ admits a factorization (cf. (8.2.6)), we have:

$$\Phi = h_1 h_2 \varphi_- \varphi_+ t^m \,, \tag{8.2.20}$$

   where h_1 and h_2 are exponential type functions such that $h_1 \in L_-^\infty(\Gamma_0)$ $(h_1^{-1} \in \mathcal{U})$, $h_2 \in L_+^\infty(\Gamma_0)$ $(h_2 \in \mathcal{U})$, $\varphi_-^{\pm 1} \in L_-^2(\Gamma_0) \oplus \mathbb{C}$, $\varphi_+^{\pm 1} \in L_+^2(\Gamma_0)$, and m is an integer (the reader should note that these h_1 and h_2 functions are independent from those used in the first part of the present proof). Combining (8.2.19) and (8.2.20), we obtain that

$$\phi_-\widetilde{\phi_-^{-1}} = h_1 h_2 \varphi_- \varphi_+ t^m \ . \tag{8.2.21}$$

From the last equality it also follows that

$$h_1 h_2 \varphi_- \varphi_+ t^m = \widetilde{h_1^{-1}}\widetilde{h_2^{-1}}\widetilde{\varphi_-^{-1}}\widetilde{\varphi_+^{-1}} t^m \ .$$

From here, rearranging the terms of the last equality, one obtains:

$$\varphi_+ \widetilde{\varphi_-} h_1 \widetilde{h_1} h_2 \widetilde{h_2} = \widetilde{\varphi_+^{-1}} \varphi_-^{-1} \ . \tag{8.2.22}$$

As it was shown in the proof of proposition (i), we can factorize the functions $h_1\widetilde{h_1}$ and $h_2\widetilde{h_2}$ in the following way (in case that $h_1\widetilde{h_1}h_2\widetilde{h_2} \neq \text{const}$):

$$\begin{aligned} h_1\widetilde{h_1} &= h_1^- h_1^+ \ , \\ h_2\widetilde{h_2} &= h_2^- h_2^+ \ . \end{aligned}$$

These equalities allow us to factorize $h_1\widetilde{h_1}h_2\widetilde{h_2}$ in the convenient way:

$$h_1\widetilde{h_1}h_2\widetilde{h_2} = h_- h_+ \ ,$$

where $h_- := h_1^- h_2^- \in L_-^\infty(\Gamma_0)$ $(h_-^{-1} \in \mathcal{U})$ and $h_+ := h_1^+ h_2^+ \in L_+^\infty(\Gamma_0)$ $(h_+ \in \mathcal{U})$ are the exponential type functions. Recalling formula (8.2.22) within this notation, we have:

$$\varphi_+ \widetilde{\varphi_-} h_+ h_- = \widetilde{\varphi_+^{-1}} \varphi_-^{-1} \ .$$

Let us also introduce the notation: $\Psi_+ := \varphi_+ \widetilde{\varphi_-} h_+ \in L_+^1(\Gamma_0)$, $H_- := h_-$ and $\Psi_- := \widetilde{\varphi_+^{-1}} \varphi_-^{-1} \in L_-^1(\Gamma_0)$. We will therefore have:

$$H_- \Psi_+ = \Psi_- \ .$$

Now we are in a very similar situation as in the proof of proposition (i) of the present theorem. Arguing in a very similar way as in the proof of part 1., we can obtain that

$\Psi_+ = \Psi_- = 0$, identically. This leads to the conclusion that Φ is not invertible – which is a contradiction. Consequently, in this case TH_ϕ is not a normally solvable operator.

Let us now consider the case when $h_1\widetilde{h_1}h_2\widetilde{h_2} = \text{const} \neq 0$.

Similarly as in the proof of the part 1., we have that $\varphi_+ = c_2\varphi_-^{-1}$, where c_2 is a nonzero constant. From the equality (8.2.21), we have that:

$$\phi_-\widetilde{\phi_-^{-1}} = h_1h_2c_2\varphi_-\widetilde{\varphi_-^{-1}}t^m \ .$$

In a very similar manner as in the part 1., we derive the equality:

$$c_2\Theta_-\varphi_-t^{m+1}h_1h_2 = \widetilde{\Theta}_-\widetilde{\varphi_-}$$

(with $\Theta_- := (1-t^{-1})\phi_-^{-1}$), and may rewriting it in the form:

$$t^{-m-1}h_1^{-1}h_2^{-1}\widetilde{\Theta}_-\widetilde{\varphi_-} = c_2\Theta_-\varphi_- \ .$$

Assume that $m > -1$. Then, by denoting $\Psi_+ := h_1^{-1}\widetilde{\Theta}_-\widetilde{\varphi_-} \in L^1_+(\Gamma_0)$, $\Psi_- := c_2\Theta_-\varphi_- \in L^1_-(\Gamma_0)$ and $H_- := t^{-m-1}h_2^{-1} \in L^\infty_-(\Gamma_0)$, we obtain:

$$H_-\Psi_+ = \Psi_- \ .$$

This is enough to reach to a contradiction (by arguing in the same way as above).

Let us now assume that $m \leq -1$. For this case, we will use the notation: $\Psi_+ := t^{-m-1}h_1^{-1}\widetilde{\Theta}_-\widetilde{\varphi_-} \in L^1_+(\Gamma_0)$, $\Psi_- := c_2\Theta_-\varphi_- \in L^1_-(\Gamma_0)$, and $H_- := h_2^{-1} \in L^\infty_-(\Gamma_0)$. Then, also in this case we will obtain a corresponding equality with the appropriate structure

$$H_-\Psi_+ = \Psi_-$$

which also leads us to a contradiction.

Therefore, we conclude that TH_ϕ is not normally solvable under the conditions of proposition (iii). □

We will present in the next theorem the general case of a symbol ϕ with $n \in \mathbb{N}$ points of standard almost periodic discontinuities.

THEOREM 8.2.2. *Suppose that the function* $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ *is continuous in the set* $\Gamma_0 \setminus \{t_j\}_{j=1}^n$, *and has standard almost periodic discontinuities at the points* t_j, $1 \le j \le n$. *In addition, assume that* $\sigma_{t_j}(\phi) \neq 0$ *for all* $j = \overline{1,n}$.

(i) If $\sigma_{t_j}(\phi) + \sigma_{t_j^{-1}}(\phi) = 0$ *for all* $j = \overline{1,n}$, *then the operator* TH_ϕ *is Fredholm.*

(ii) If $\sigma_{t_j}(\phi) + \sigma_{t_j^{-1}}(\phi) \le 0$ *for all* $j = \overline{1,n}$, *and there is at least one index* j *for which* $\sigma_{t_j}(\phi) + \sigma_{t_j^{-1}}(\phi) \neq 0$, *then the operator* TH_ϕ *is right-invertible in* $L^2_+(\Gamma_0)$ *and* $\dim \mathrm{Ker} TH_\phi = \infty$.

(iii) If $\sigma_{t_j}(\phi) + \sigma_{t_j^{-1}}(\phi) \ge 0$ *for all* $j = \overline{1,n}$, *and there is at least one index* j *for which* $\sigma_{t_j}(\phi) + \sigma_{t_j^{-1}}(\phi) \neq 0$, *then the operator* TH_ϕ *is left-invertible in* $L^2_+(\Gamma_0)$ *and* $\dim \mathrm{Coker} TH_\phi = \infty$.

(iv) If $(\sigma_{t_j}(\phi) + \sigma_{t_j^{-1}}(\phi))(\sigma_{t_l}(\phi) + \sigma_{t_l^{-1}}(\phi)) < 0$ *for at least two different indices* j *and* l, *then* $\dim \mathrm{Ker} TH_\phi = \dim \mathrm{Coker} TH_\phi = 0$ *and the operator* TH_ϕ *is not normally solvable.*

Since the proof of this theorem goes along the same methods as in the proof of Theorem 8.2.1, we will not present here the corresponding fully detailed proof but just the following sketch of proof.

Proof Sketch. For $\phi \in \mathcal{G}L^\infty(\Gamma_0)$, continuous in $\Gamma_0 \setminus \{t_j\}_{j=1}^n$, and with standard almost periodic discontinuities at the points t_j, $1 \le j \le n$, such that $\sigma_{t_j}(\phi) \neq 0$ for all $1 \le j \le n$, we will work with $\Phi := \phi\widetilde{\phi^{-1}}$ (as in the previous case of Theorem 8.2.1). In general, this Φ will have $2n$ points of standard almost periodic discontinuities. In addition, the formula

$$\sigma_{t_j}(\Phi) = \sigma_{t_j}(\phi) + \sigma_{t_j^{-1}}(\phi) \ ,$$

allows us to employ the Δ-relation after extension and to deduce the above stated right and left invertibility properties of TH_ϕ, as well as the triviality of the kernel and the cokernel of TH_ϕ, and the Fredholm property of TH_ϕ (under the corresponding different assumptions).

The propositions which deal with the dimensions of the kernel and the cokernel under the assumptions in (ii) and (iii), and also the issues about the normal solvability of TH_ϕ, are proved in the same manner as in the proof of three discontinuity points case. In fact, e.g., to prove the formula for the dimension of the kernel in case (ii), the method used in the proof of the part (i) of Theorem 8.2.1 also works here for the situation of n points of standard almost periodic discontinuities. In this situation, instead of the factorization (8.2.6), we will have

$$\Phi(t) = \left(\prod_{j=1}^{2n} \exp(\lambda_j (t - t_j)^{-1}) \right) \varphi(t) \ ,$$

and therefore we have to choose now

$$h(t) = \prod_{j=1}^{2n} \exp(\lambda_j (t - t_j)^{-1}) \ .$$

Thus, up to the equality (8.2.11), the reasoning will be the same. Analogously as in the proof of part (i) of Theorem 8.2.1, we would be able to decompose $h\widetilde{h}$ in the following convenient form (in the case of $h\widetilde{h} \neq \text{const}$):

$$h\widetilde{h} = h_- h_+ \ ,$$

with $h_- \in L^\infty_-(\Gamma_0)$, and $h_+ \in L^\infty_+(\Gamma_0)$. Moreover, h_- and h_+ have disjoint points of standard almost periodic discontinuities.

Let M_{h_-} stand for the points of standard almost periodic discontinuities of the function h_-. Continuing with the same reasoning as in the proof of part (i) of Theorem 8.2.1, we will arrive at an analogous equality as (8.2.14) where the corresponding H_- has now $2n$ points of standard almost periodic discontinuities. In this case, we consider a point t_0 such that $t_0 \in \Gamma_0 \setminus M_{h_-}$, and reach into a contradiction in the same manner as in the proof of Theorem 8.2.1. The case of $h\widetilde{h} = \text{const}$, runs also by using the same arguments as above.

Finally, note that the assumption in propositions (ii) and (iii) which requires that there is at least one index j for which $\sigma_{t_j}(\phi) + \sigma_{t_j^{-1}}(\phi) \neq 0$ was automatically fulfilled in Theorem 8.2.1, and is added in here only for the matter of excluding these cases to fall in the situation of present proposition (i). $\square$

REMARK 8.2.3. *Note that in the first case of the last theorem we will have that the Toeplitz operator T_Φ (with symbol $\Phi = \phi\widetilde{\phi^{-1}}$) has an invertible continuous symbol, and hence it is a Fredholm operator.*

As a direct conclusion from the last theorem, if we consider only one point with standard almost periodic discontinuity, we have the following result.

COROLLARY 8.2.4. *Let the function $\phi \in \mathcal{G}L^\infty(\Gamma_0)$ be continuous on the set $\Gamma_0 \setminus \{t_0\}$ and have a standard almost periodic discontinuity at the point t_0 with $\sigma_{t_0}(\phi) \neq 0$.*

(i) If $\sigma_{t_0}(\phi) < 0$, then the operator TH_ϕ is right-invertible in $L^2_+(\Gamma_0)$ and $\dim \mathrm{Ker} TH_\phi = \infty$.

(ii) If $\sigma_{t_0}(\phi) > 0$, then the operator TH_ϕ is left-invertible in $L^2_+(\Gamma_0)$ and $\dim \mathrm{Coker} TH_\phi = \infty$.

8.3 Examples

In this last section of this chapter we would like to present two simple examples for illustrating some of the above presented theory.

As for the first example, let us consider the Toeplitz operator $T_{\rho_1} : L^2_+(\Gamma_0) \to L^2_+(\Gamma_0)$, where

$$\rho_1(t) = \exp\left(\frac{i}{t-i}\right) \exp\left(\frac{i}{t+i}\right) \exp\left(\frac{1}{t-1}\right) , \quad t \in \Gamma_0 .$$

From the definition of ρ_1 it is clear that it is an invertible element. It is also clear that ρ_1 has three points of standard almost periodic discontinuities (namely, the points i, $-i$ and 1). A direct computation allows the conclusion that

$$\sigma_i(\rho_1) = 1 , \qquad \sigma_{-i}(\rho_1) = -1 , \qquad \sigma_1(\rho_1) = 1 .$$

Hence, T_{ρ_1} is not normally solvable and $\dim \operatorname{Ker} T_{\rho_1} = \dim \operatorname{Coker} T_{\rho_1} = 0$ (cf. Theorem 8.1.12, part 3).

Let us analyze the corresponding Toeplitz plus Hankel operator $TH_{\rho_1} : L^2_+(\Gamma_0) \to L^2_+(\Gamma_0)$, with symbol ρ_1. Direct computations lead us to the following equalities and inequality:

$$\begin{aligned} \sigma_i(\rho_1) + \sigma_{i^{-1}}(\rho_1) &= \sigma_i(\rho_1) + \sigma_{-i}(\rho_1) \;=\; 0\,, \\ \sigma_{-i}(\rho_1) + \sigma_{(-i)^{-1}}(\rho_1) &= \sigma_{-i}(\rho_1) + \sigma_i(\rho_1) \;=\; 0\,, \\ \sigma_1(\rho_1) + \sigma_{(1)^{-1}}(\rho_1) &= 2\sigma_1(\rho_1) \;=\; 2 \;>\; 0\,. \end{aligned}$$

Applying proposition (iii) of Theorem 8.2.2, we conclude that TH_{ρ_1} is a left-invertible operator with infinite dimensional cokernel.

As a second example, we will consider an adaptation of the first example in which a Toeplitz operator with a particular symbol will be not normally solvable but the Toeplitz plus Hankel operator with the same symbol will be two-sided invertible.

Let us work with the Toeplitz operator $T_{\rho_2} : L^2_+(\Gamma_0) \to L^2_+(\Gamma_0)$, where

$$\rho_2(t) = \exp\left(\frac{i}{t-i}\right) \exp\left(\frac{i}{t+i}\right) , \quad t \in \Gamma_0\,.$$

The symbol ρ_2 is invertible, and has standard almost periodic discontinuities only at the points i and $-i$. In particular, we have

$$\sigma_i(\rho_2) = 1\,, \qquad \sigma_{-i}(\rho_2) = -1\,. \tag{8.3.1}$$

Hence, T_{ρ_2} is not normally solvable (cf. Theorem 8.1.12, part 3).

Let us now look for corresponding properties of the Toeplitz plus Hankel operator $TH_{\rho_2} : L^2_+(\Gamma_0) \to L^2_+(\Gamma_0)$, with symbol ρ_2. It turns out that by using (8.3.1) and proposition (i) of Theorem 8.2.2 we conclude that TH_{ρ_2} is a Fredholm operator. Moreover, in this particular case, we can even reach into the stronger conclusion that TH_{ρ_2} is a two-sided invertible operator. Indeed, $\rho_2\widetilde{\rho_2^{-1}} = 1$ and therefore $T_{\rho_2\widetilde{\rho_2^{-1}}}$ is two-sided invertible (since it is the identity operator on $L^2_+(\Gamma_0)$). Thus, the Δ-relation after extension ensures in this case that TH_{ρ_2} is also a two-sided invertible operator on $L^2_+(\Gamma_0)$.

Chapter 9

Matrix Toeplitz plus Hankel operators with PAP symbols

The main goal of the present chapter is to present a Fredholm criterion for matrix Toeplitz plus Hankel operators TH_Φ, where Φ is a $N \times N$ matrix function with entries in the class of piecewise almost periodic functions.

The class of operators TH_Φ has an important role in the mathematical description of various applications. This is the case due to the combination of Toeplitz and Hankel operators which appear in the structure of operators TH_Φ. Several results are presently known for Fredholm characteristics of these operators when with symbols from smaller classes – like the piecewise continuous or the almost periodic matrix functions. Here we provide a Fredholm characterization of TH_Φ when the matrix Fourier symbol Φ is in the piecewise almost periodic class, and therefore allowing the two previously mentioned classes at the same time (cf. Theorem 9.3.1).

To reach this goal, in Section 9.1 we start by presenting some structures and results of significant importance in the so-called symbol calculus. Firstly, this is done within the framework of piecewise continuous elements (Subsection 9.1.1), and secondly for semi-almost periodic Fourier symbols (Subsection 9.1.2). In Section 9.2 we prepare the main result to be obtained in Section 9.3, by describing the conditions which ensure the Fredholm property of some auxiliary paired operators. As a natural result to be achieved after the Fredholm characterization of Section 9.3, in the last section a formula for the Fredholm index of TH_Φ is derived based on some approximating procedures which are applied to elements in the so-called Wiener subalgebra of piecewise almost periodic matrix functions.

9.1 Auxiliary results on symbol calculus

In the present section we will present a set of important results which will have direct consequences in our final result of the present chapter. In all them the so-called Allan-Douglas local principle plays a fundamental role, and so we will recall it now.

Let G be a Banach algebra with identity. A subalgebra Z of G is said to be a central subalgebra if $zg = gz$ for all $z \in Z$ and all $g \in G$.

THEOREM 9.1.1. *[18, Theorem 1.35(a)] Let G be a Banach algebra with unit e and let Z be a closed central subalgebra of G containing e. Let $\mathbf{M}(Z)$ be the maximal ideal space of Z, and for $\omega \in \mathbf{M}(Z)$, let J_ω refer to the smallest closed two-sided ideal of G containing the ideal ω. Then, an element g is invertible in G if and only if $g + J_\omega$ is invertible in the quotient algebra g/J_ω for all $\omega \in \mathbf{M}(Z)$.*

9.1.1 Symbol calculus for piecewise continuous symbols

The next two results are due to Duduchava (cf. [33], [34]).

LEMMA 9.1.2. *(a) If $a, b \in PC$ and $a(\pm\infty) = b(\pm\infty) = 0$, then the operators aW_b^0 and $W_b^0 aI$ are compact.*

(b) If $a \in C(\dot{\mathbb{R}})$, $b \in PC$ or if $a \in PC$, $b \in C(\dot{\mathbb{R}})$, then the commutator $aW_b^0 - W_b^0 aI$ is compact.

(c) If $a, b \in C(\overline{\mathbb{R}})$, then the commutator $aW_b^0 - W_b^0 aI$ is compact.

Let us consider the C^*-algebra

$$\mathfrak{A} := \mathrm{alg}(C(\bar{\mathbb{R}}), W^0(PC))$$

(generated by the operators aW_b^0 with $a \in C(\bar{\mathbb{R}})$ and $b \in PC$), and also

$$\mathfrak{D} := \mathrm{alg}(PC, W^0(PC)).$$

It is clear that both $\mathfrak{A}$ and $\mathfrak{D}$ contain the C^*-subalgebra $Z := \mathrm{alg}(C(\dot{\mathbb{R}}), W^0(C(\dot{\mathbb{R}})))$. Let in addition $\mathcal{K} := \mathcal{K}(L^2(\mathbb{R}))$ be the set of all compact operators on $L^2(\mathbb{R})$. One can show that $\mathcal{K} \subset Z$. Denote $\mathfrak{A}^\pi := \mathfrak{A}/\mathcal{K}$, $\mathfrak{D}^\pi := \mathfrak{D}/\mathcal{K}$, $Z^\pi := Z/\mathcal{K}$, and abbreviate the coset $A+\mathcal{K}$ to A^π.

Lemma 9.1.2 implies that Z^π is a central C^*-algebra of $\mathfrak{D}^\pi$. The maximal ideal space of Z^π can be identified with

$$M := (\dot{\mathbb{R}} \times \dot{\mathbb{R}}) \setminus (\mathbb{R} \times \mathbb{R}) = (\mathbb{R} \times \{\infty\}) \cup (\{\infty\} \times \mathbb{R}) \cup \{(\infty, \infty)\}\ .$$

We also consider the set

$$\mathcal{M} := (\mathbb{R} \times \{\infty\} \times [0,1]) \cup (\{\infty\} \times \mathbb{R} \times [0,1]) \cup ((\infty, \infty) \times \{0,1\})\ .$$

We equip M with the Gelfand topology and $\mathcal{M}$ with the discrete topology. For $A = aW_b^0$ $(a, b \in PC)$ and $(t, x, \mu) \in \mathcal{M}$, a matrix A is defined by

$$A(t,x,\mu) = \begin{pmatrix} a(t+0)(b(x+0)\mu + b(x-0)(1-\mu)) & a(t+0)(b(x+0)-b(x-0))\sqrt{\mu(1-\mu)} \\ a(t-0)(b(x+0)-b(x-0))\sqrt{\mu(1-\mu)} & a(t-0)(b(x-0)\mu + b(x+0)(1-\mu)) \end{pmatrix} \tag{9.1.1}$$

where by convention, $a(\infty \pm 0) = a(\mp\infty)$, $b(\infty \pm 0) = b(\mp\infty)$, and $\nu(\mu) := \sqrt{\mu(1-\mu)}$ denotes any function $\nu : [0,1] \to \mathbb{R}$ such that $\nu^2(\mu) = \mu(1-\mu)$ for all $\mu \in [0,1]$, and additionally $\nu(1/2) = -1/2$. Let $BC(\mathcal{M}, \mathbb{C}^{2\times 2})$ stand for the bounded continuous functions of $\mathcal{M}$ into $\mathbb{C}^{2\times 2}$.

THEOREM 9.1.3. *The map*

$$\mathrm{Sym} : \{A = aW_b^0 : a, b \in PC\} \to BC(\mathcal{M}, \mathbb{C}^{2\times 2})$$

(associating the matrix function in (9.1.1) with the operator A) extends (in a unique way) to a C^*-algebra homomorphism

$$\mathrm{Sym} : \mathfrak{D} \to BC(\mathcal{M}, \mathbb{C}^{2\times 2}) ,$$

whose kernel is $\mathcal{K}$.

In what follows we simply write A instead of Sym$A$. We denote by a_{ij} the (ij)-entry of A. Since

$$a(-\infty)b(\pm\infty) = \lim_{x\to\pm\infty} a_{11}(\infty, x, 0), \qquad a(+\infty)b(\mp\infty) = \lim_{x\to\pm\infty} a_{22}(\infty, x, 0),$$

Theorem 9.1.3 remains valid with $\mathcal{M}$ replaced by its subset

$$\mathcal{M}^0 := (\mathbb{R} \times \{\infty\} \times [0,1]) \cup (\{\infty\} \times \mathbb{R} \times [0,1]) .$$

Furthermore, when considering the C^*-subalgebra $\mathfrak{A} := \mathrm{alg}(C(\bar{\mathbb{R}}), W^0(PC))$ of $\mathfrak{D}$, the form of the symbol $A(t, x, \mu)$ can be simplified at the points $t \in \mathbb{R}$ for the generating operators $A = aW_b^0$ $(a \in C(\bar{\mathbb{R}}), b \in PC)$. Namely, we can put

$$A(t, \infty, \mu) = \begin{pmatrix} a(t)b(-\infty) & 0 \\ 0 & a(t)b(+\infty) \end{pmatrix}, \quad \mu \in [0,1].$$

9.1.2 Symbol calculus for SAP

Let

$$\mathfrak{S} := \mathrm{alg}(S_{\mathbb{R}}, [C(\bar{\mathbb{R}})]^{N\times N})$$

be the C^*-algebra generated by the singular integral operators $aI + bS_{\mathbb{R}}$ with coefficients $a, b \in [C(\bar{\mathbb{R}})]^{N\times N}$. We denote by $\mathcal{H}_\infty$ the closed two-sided ideal of $\mathfrak{S}$ that is generated by all the commutators $uS_{\mathbb{R}} - S_{\mathbb{R}}uI$ with $u \in C(\bar{\mathbb{R}})$, and $\mathcal{H}_{\mathbb{R}}$ is the closed two-sided ideal of the algebra $\mathfrak{S}$ which is generated by the commutators $cS_{\mathbb{R}} - S_{\mathbb{R}}cI$, where $c \in PC$ and with $c(+\infty) = c(-\infty)$.

The following algebras are also of interest:

$$\mathfrak{C} := \mathrm{alg}(S_{\mathbb{R}}, AP^{N\times N}), \quad \mathfrak{B} := \mathrm{alg}(S_{\mathbb{R}}, SAP^{N\times N})\ .$$

Let us first start with the C^*-algebra $\mathfrak{A} := \mathrm{alg}(C(\bar{\mathbb{R}}), W^0(PC))$ generated by the operators $aW^0(b)$ with $a \in C(\bar{\mathbb{R}})$ and $b \in PC$. It is readily seen that

$$\mathfrak{B} = \mathrm{alg}(\mathfrak{A}, u_{\mathbb{R}})\ ,$$

where $u_{\mathbb{R}} : \mathbb{R} \to \mathcal{L}(L^2(\mathbb{R}))$ is the unitary representation of the discrete group $\mathbb{R}$ given by $u_{\mathbb{R}} : \lambda \mapsto e_\lambda I$ (cf. [15]).

Note that due to

$$e_\lambda aI = ae_\lambda, \qquad e_\lambda W_b^0 = W_{b_\lambda}^0 e_\lambda I$$

(where $b_\lambda(x) = b(x+\lambda)$) the algebra $\mathfrak{B}$ is the $\mathcal{L}(L^2(\mathbb{R}))$ closure of the set $\mathfrak{B}^0$ of all operators of the form

$$B = \sum_\lambda A_\lambda e_\lambda I$$

where $A_\lambda \in \mathfrak{A}$ and λ ranges over arbitrary finite subsets of $\mathbb{R}$. Note however that for e_λ we continue with the previous (usual) notation of $e_\lambda(x) = e^{i\lambda x}$, $x \in \mathbb{R}$.

Let $\ell^2(\mathbb{R}, \mathbb{C}^2)$ denote the collection of all functions $f : \mathbb{R} \to \mathbb{C}^2$ for which the set $\{\lambda \in \mathbb{R} : f(\lambda) \neq 0\}$ is at most countable and

$$||f||^2_{\ell^2(\mathbb{R},\mathbb{C}^2)} := \sum_\lambda ||f(\lambda)||^2 < \infty\ ,$$

where $||f(\lambda)||$ denotes the usual norm in $\mathbb{C}^2$, i.e., if we have $f = (f_1, f_2)$, then $||f|| = (|f_1|^2 + |f_2|^2)^{\frac{1}{2}}$.

Let $\widetilde{\mathbb{R}}$ be the set resulting from $\dot{\mathbb{R}}$ by blowing up the point ∞ to the segment $[0,1]$:

$$\widetilde{\mathbb{R}} := \mathbb{R} \cup (\{\infty\} \times [0,1])\ .$$

We associate with each point of $\widetilde{\mathbb{R}}$ a representation of $\mathfrak{B}$. For $t \in \mathbb{R}$, let Π_t be the representation (cf. [15])

$$\Pi_t : \mathfrak{B} \to \mathcal{L}(\mathbb{C}^2)\,, \qquad \Pi_t\left(\sum_\lambda A_\lambda e_\lambda I\right) = \sum_\lambda A_\lambda(t,\infty,1)e^{i\lambda t}I,$$

and for $\mu \in [0,1]$, we define Π_∞, μ as the representation

$$\Pi_\infty, \mu : \mathfrak{B} \to \mathcal{L}(\ell^2(\mathbb{R},\mathbb{C}^2))\,, \ \left[\Pi_{\infty,\mu}\left(\sum_\lambda A_\lambda e_\lambda I\right) f\right](x) = \sum_\lambda A_\lambda(\infty, x, \mu) f(x+\lambda)\,, \tag{9.1.2}$$

where $x \in \mathbb{R}$ and $f \in \ell^2(\mathbb{R},\mathbb{C}^2)$. Also, for $B \in \mathfrak{B}$ we consider the operator-valued function $\tilde{B}$ given on $\tilde{\mathbb{R}}$ by

$$\tilde{B}(t) = \Pi_t(B) \quad \text{for } t \in \mathbb{R}, \qquad \tilde{B}(\infty,\mu) = \Pi_{\infty,\mu}(B) \quad \text{for } \mu \in [0,1].$$

The set $\tilde{\mathfrak{B}}$ is a C^*-algebra with pointwise operations and the norm

$$||\tilde{\mathfrak{B}}|| := \max\left\{\sup_{t\in\mathbb{R}} ||\Pi_t(B)||, \sup_{\mu\in[0,1]} ||\Pi_{\infty,\mu}(B)||\right\}\,.$$

THEOREM 9.1.4. *[15, Theorem 5.3] The map Φ defined by*

$$\Phi : \mathfrak{B}^\pi \to \tilde{\mathfrak{B}}, \ \ B^\pi \mapsto \tilde{B}$$

is a well-defined C^*-algebra isomorphism, where $\mathfrak{B}^\pi := \mathfrak{B}/\mathcal{K}$.

The next theorem is a key result for studying the Fredholm property for the operators from the algebra $\mathfrak{B}$.

THEOREM 9.1.5. *[15, Theorem 5.4] An operator $B \in \mathfrak{B} = \operatorname{alg}(S_\mathbb{R}, SAP)$ is Fredholm in $L^2(\mathbb{R})$ if and only if*

(a) the diagonal 2×2 matrices $\Pi_t(B)$ are invertible for all $t \in \mathbb{R}$,

(b) the operators $\Pi_{\infty,\mu}(B)$ are invertible in $\ell^2(\mathbb{R},\mathbb{C}^2)$ for all $\mu \in [0,1]$.

REMARK 9.1.6. This theorem is also valid for the matrix case.

9.2 Auxiliary operators

Let us now consider the following auxiliary operator which will be fundamental for our purposes

$$A := aP_{\mathbb{R}} + bQ_{\mathbb{R}} + cH_+ + H_- dI : [L^2(\mathbb{R})]^N \to [L^2(\mathbb{R})]^N \, , \tag{9.2.1}$$

where $a, b, c, d \in SAP^{N\times N}$, $Q_{\mathbb{R}} = I - P_{\mathbb{R}}$, and $H_\pm \in \mathcal{H}_\infty$.

By Sarason's result (cf. Theorem 2.2.2) we can decompose $SAP^{N\times N}$ matrix functions in the following way:

$$a = a_\ell(1-u) + a_r u + a_0 \, ,$$

where

$$a_\ell = \sum_\lambda a_\lambda^\ell e_\lambda, \quad a_r = \sum_\lambda a_\lambda^r e_\lambda \quad (a_\lambda^\ell, a_\lambda^r \in \mathbb{C}^{N\times N})$$

are the almost periodic representatives of a at $-\infty$ and $+\infty$ (and the series are to be understood in a formal sense or as converging in the Besicovitch space (recall Chapter 2)), $a_0 \in [C_0(\dot{\mathbb{R}})]^{N\times N}$, $u \in C(\bar{\mathbb{R}})$, $u(-\infty) = 0$, $u(+\infty) = 1$. Analogously, for b, c and d we have

$$b = b_\ell(1-u) + b_r u + b_0 \, ,$$
$$c = c_\ell(1-u) + c_r u + c_0 \, ,$$
$$d = d_\ell(1-u) + d_r u + d_0 \, ,$$

where $b_0, c_0, d_0 \in [C_0(\dot{\mathbb{R}})]^{N\times N}$ and the almost periodic representatives of b, c, d at $-\infty$ and $+\infty$ are given by

$$\begin{aligned} b_\ell &= \sum_\lambda b_\lambda^\ell e_\lambda, \quad b_r = \sum_\lambda b_\lambda^r e_\lambda \quad (b_\lambda^\ell, b_\lambda^r \in \mathbb{C}^{N\times N}) \, , \\ c_\ell &= \sum_\lambda c_\lambda^\ell e_\lambda, \quad c_r = \sum_\lambda c_\lambda^r e_\lambda \quad (c_\lambda^\ell, c_\lambda^r \in \mathbb{C}^{N\times N}) \, , \\ d_\ell &= \sum_\lambda d_\lambda^\ell e_\lambda, \quad d_r = \sum_\lambda d_\lambda^r e_\lambda \quad (d_\lambda^\ell, d_\lambda^r \in \mathbb{C}^{N\times N}) \, . \end{aligned}$$

It is known that

$$P_{\mathbb{R}} = \mathcal{F}\chi_+\mathcal{F}^{-1} : L^2(\mathbb{R}) \to H^2_+(\mathbb{R}), \text{ and } \ Q_{\mathbb{R}} = \mathcal{F}\chi_-\mathcal{F}^{-1} : L^2(\mathbb{R}) \to H^2_-(\mathbb{R}) \, ,$$

where $\chi_\pm$ are the characteristic functions of $\mathbb{R}_\pm$. Using standard arguments, we will rewrite these operators in a more convenient form for our purposes. Due to the formula (9.1.1) we will deduce the last two operators to the convolution form:

$$\begin{aligned} P_{\mathbb{R}} &= \mathcal{F}\chi_+\mathcal{F}^{-1} = \mathcal{F}J\chi_-J\mathcal{F}^{-1} = \mathcal{F}^{-1}\chi_-\mathcal{F} \,, \\ Q_{\mathbb{R}} &= \mathcal{F}\chi_-\mathcal{F}^{-1} = \mathcal{F}J\chi_+J\mathcal{F}^{-1} = \mathcal{F}^{-1}\chi_+\mathcal{F} \,, \end{aligned}$$

where we are using the facts that $\mathcal{F}^{-1} = \mathcal{F}J = J\mathcal{F}$ and $J\chi_\pm J = \chi_\mp$.

Having in mind the last two formulas, the operator A can be represented in the form

$$A = a\mathcal{F}^{-1}\chi_-\mathcal{F} + b\mathcal{F}^{-1}\chi_+\mathcal{F} + cH_+ + H_-dI$$

with $H_\pm \in \mathcal{H}_\infty$. In addition, let us consider the following operator:

$$B = \sum_\lambda A_\lambda e_\lambda I \,, \tag{9.2.2}$$

where

$$\begin{aligned} A_0 &= (a_0^\ell(1-u) + a_0^r u + a_0)\mathcal{F}^{-1}\chi_-\mathcal{F} + (b_0^\ell(1-u) + b_0^r u + b_0)\mathcal{F}^{-1}\chi_+\mathcal{F} \\ &\quad + (c_0^\ell(1-u) + c_0^r u + c_0)H_+ + H_-(d_0^\ell(1-u) + d_0^r u + d_0)I \,, \\ A_\lambda &= (a_\lambda^\ell(1-u) + a_\lambda^r u)e_\lambda\mathcal{F}^{-1}\chi_-\mathcal{F}e_{-\lambda} + (b_\lambda^\ell(1-u) + b_\lambda^r u)e_\lambda\mathcal{F}^{-1}\chi_+\mathcal{F}e_{-\lambda} \\ &\quad + (c_\lambda^\ell(1-u) + c_\lambda^r u)e_\lambda H_+ e_{-\lambda} + H_-(d_\lambda^\ell(1-u) + d_\lambda^r u)I \,. \end{aligned}$$

The operator B in (9.2.2) is understood as a uniform limit of operators B_k (cf. [15]) of the same form but with λ running through finite subsets of $\mathbb{R}$.

Using now (9.1.2) and (9.1.1), a family of operators $\Pi_{\infty,\mu}(B)$ (with $\mu \in [0,1]$) will be defined (cf. [15]) by:

$$(\Pi_{\infty,\mu}(B)f)(x) = \sum_\lambda A_\lambda(\infty, x, \mu)f(x+\lambda), \quad x \in \mathbb{R},\ f \in [\ell^2(\mathbb{R}, \mathbb{C}^2)]^{N\times N} \,, \tag{9.2.3}$$

with

$$\begin{aligned} A_0(\infty, x, \mu) &= \begin{pmatrix} a_0^\ell & 0 \\ 0 & a_0^r \end{pmatrix} P(\infty, x, \mu) + \begin{pmatrix} b_0^\ell & 0 \\ 0 & b_0^r \end{pmatrix} Q(\infty, x, \mu) \\ &\quad + \begin{pmatrix} c_0^\ell & 0 \\ 0 & c_0^r \end{pmatrix} H_+(\infty, x, \mu) + H_-(\infty, x, \mu) \begin{pmatrix} d_0^\ell & 0 \\ 0 & d_0^r \end{pmatrix} \end{aligned} \tag{9.2.4}$$

and for $\lambda \neq 0$, we have

$$\begin{aligned} A_\lambda(\infty, x, \mu) &= \begin{pmatrix} a_\lambda^\ell & 0 \\ 0 & a_\lambda^r \end{pmatrix} P(\infty, x+\lambda, \mu) + \begin{pmatrix} b_\lambda^\ell & 0 \\ 0 & b_\lambda^r \end{pmatrix} Q(\infty, x+\lambda, \mu) \\ &+ \begin{pmatrix} c_\lambda^\ell & 0 \\ 0 & c_\lambda^r \end{pmatrix} H_+(\infty, x+\lambda, \mu) + H_-(\infty, x+\lambda, \mu) \begin{pmatrix} d_\lambda^\ell & 0 \\ 0 & d_\lambda^r \end{pmatrix} . \end{aligned} \tag{9.2.5}$$

Here $P(\infty, x, \mu)$ and $Q(\infty, x, \mu)$ stand for the symbols of the operators $P_\mathbb{R}$ and $Q_\mathbb{R}$, respectively. A direct computation provides that $P(\infty, x, \mu)$ is equal to

$$P(\infty, x, \mu) = \begin{pmatrix} (\chi_-(x+0)\mu + \chi_-(x-0)(1-\mu))I_{N\times N} & (\chi_-(x+0) - \chi_-(x-0))\nu(\mu) I_{N\times N} \\ (\chi_-(x+0) - \chi_-(x-0))\nu(\mu) I_{N\times N} & (\chi_-(x-0)\mu + \chi_-(x+0)(1-\mu))I_{N\times N} \end{pmatrix} \tag{9.2.6}$$

and $Q(\infty, x, \mu)$ is given by

$$Q(\infty, x, \mu) = \begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix} - P(\infty, x, \mu) . \tag{9.2.7}$$

The following theorem follows immediately from Theorem 9.1.5 and Remark 9.1.6.

THEOREM 9.2.1. *[15] If $a, b, c, d \in SAP^{N\times N}$, then the operator A defined in (9.2.1) is Fredholm on $[L^2(\mathbb{R})]^N$ if and only if:*

(a) $a, b \in \mathcal{G}SAP^{N\times N}$;

(b) for every $\mu \in [0,1]$ the operators $A_\mu := \Pi_{\infty,\mu}(A)$ given by (9.2.3)-(9.2.7) are invertible in $[\ell^2(\mathbb{R}, \mathbb{C}^2)]^N$.

Using this theorem and considering the operators A^*A and AA^*, one obtains the following result.

THEOREM 9.2.2. *[15] Let $a, b, c, d \in SAP^{N\times N}$. The operator (9.2.1) is n-normal (resp. d-normal) on $[L^2(\mathbb{R})]^N$ if and only if $a, b \in \mathcal{G}SAP^{N\times N}$ and for every $\mu \in [0,1]$ the* *operators $A_\mu = \Pi_{\infty,\mu}(A)$ given by (9.2.3)-(9.2.7) are left-invertible (resp. right-invertible) in $[\ell^2(\mathbb{R},\mathbb{C}^2)]^N$.*

We will now consider a second auxiliary operator with the form:

$$B := aP_{\mathbb{R}} + bQ_{\mathbb{R}} + H_{\mathbb{R}} + cH_\infty : [L^2(\mathbb{R})]^N \to [L^2(\mathbb{R})]^N \tag{9.2.8}$$

where a, b, c are $N \times N$ matrix functions in PAP, $H_{\mathbb{R}} \in \mathcal{H}_{\mathbb{R}}$ and $H_\infty \in \mathcal{H}_\infty$. In what follows, we will be able to transfer some properties of operator B defined in (9.2.8) to the operator

$$\widehat{B} := \widehat{a}P_{\mathbb{R}} + \widehat{b}Q_{\mathbb{R}} + \widehat{c}H_\infty : [L^2(\mathbb{R})]^N \to [L^2(\mathbb{R})]^N \ , \tag{9.2.9}$$

where the $N \times N$ matrix functions $\widehat{a}, \widehat{b}, \widehat{c}$ belong to SAP, and have the same AP representatives $a_\pm, b_\pm, c_\pm$ at $\pm\infty$ as the matrix functions $a, b, c \in PAP$, respectively.

We will need to use the continuous arc $a^{\#}$, which is obtained from $\det a$ by filling in with line segments the eventual gaps generated by discontinuity jumps of $\det a$. In the same way, we will need $b^{\#}$. If the origin does not belong to $a^{\#}$ and $b^{\#}$, then we can construct $\widehat{a}$ and $\widehat{b}$ to be invertible at every point $x \in \mathbb{R}$. Furthermore, we will have that if the operator (9.2.8) is Fredholm on the space $[L^2(\mathbb{R})]^N$, then by the Allan-Douglas local principle, the operator $\widehat{B}$ in (9.2.9) is also Fredholm on $[L^2(\mathbb{R})]^N$. More precisely, if B is a Fredholm operator on $[L^2(\mathbb{R})]^N$, then it is locally Fredholm at ∞, and consequently $\widehat{B}$ is also locally Fredholm at ∞ (because in this case $H_{\mathbb{R}}$ has no contribution in the Fredholm property for the operator $\widehat{B}$). Moreover, $\widehat{B}$ is locally compact at every point $x \in \mathbb{R}$ simultaneously with the invertible operator $\widehat{a}(x)P_{\mathbb{R}} + \widehat{b}(x)Q_{\mathbb{R}}$ having nonzero constant coefficients.

Note that the operator $\widehat{B}$ obviously belongs to the algebra $\mathfrak{B}$, and therefore is under the conditions of Theorem 9.1.5.

9.3 Fredholm property of Toeplitz plus Hankel operators

Let us now turn to the main object of the present chapter, and consider the Toeplitz plus Hankel operator TH_Φ (defined in (1.5.4)) with symbol Φ in the class of piecewise almost periodic matrix functions ($PAP^{N\times N}$). The main goal of the present chapter is achieved in the next theorem where we will be using the operator

$$(Nf)(x) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(\tau)}{\tau + x} d\tau\,, \quad x \in \mathbb{R}.$$

THEOREM 9.3.1. *Let $\Phi \in PAP^{N\times N}$, and consider $\mathcal{T} := \mathfrak{a}P_{\mathbb{R}} + \mathfrak{b}Q_{\mathbb{R}} + \mathfrak{c}V$, where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in PAP^{2N\times 2N}$ are given by*

$$\mathfrak{a} := \begin{pmatrix} \chi_+\Phi + \chi_- & 0 \\ \chi_+\widetilde{\Phi} & I \end{pmatrix}, \mathfrak{b} := \begin{pmatrix} I & \chi_+\Phi \\ 0 & \chi_+\widetilde{\Phi} + \chi_- \end{pmatrix}, \mathfrak{c} := \frac{1}{2}\begin{pmatrix} \chi_+\Phi & \chi_+(I-\Phi) \\ \chi_+(\widetilde{\Phi} - I) & -\chi_+\widetilde{\Phi} \end{pmatrix}, \tag{9.3.1}$$

and $V := \chi_+ N \chi_+ I + \chi_- N \chi_- I$.

The operator TH_Φ is Fredholm on $[H^2_+(\mathbb{R})]^N$ if and only if:

(a) $\Phi \in \mathcal{G}PAP^{N\times N}$*;*

(b) $\det \mathcal{T}(t, x, \mu) \neq 0$, *for* $(t, x, \mu) \in \mathbb{R} \times \dot{\mathbb{R}} \times [0, 1]$, *where $\mathcal{T}(t, x, \mu)$ is given by the formula (9.1.1);*

(c) for every $\mu \in [0, 1]$ the operators $\mathcal{T}_\mu$ calculated from (9.2.3)–(9.2.7) for the operator $\mathcal{T}$ are invertible in $[\ell^2(\mathbb{R}, \mathbb{C}^2)]^{2N}$.

Proof. We start by noticing that the operator TH_Φ acting on $[H^2_+(\mathbb{R})]^N$ is equivalent after extension [1] (recall also Remark 1.7.2) with

$$\Phi(I + J)P_{\mathbb{R}} + Q_{\mathbb{R}} : [L^2(\mathbb{R})]^N \to [L^2(\mathbb{R})]^N\,.$$

Let now $\Psi : [L^2(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^{2N}$ be the isomorphism defined by

$$(\Psi f)(x) = \begin{pmatrix} f(x) \\ f(-x) \end{pmatrix}, \qquad x \in \mathbb{R}_+.$$

The inverse of Ψ is provided by the formula:

$$(\Psi^{-1} f)(t) = \begin{cases} f_1(t), & \text{if } t > 0, \\ f_2(-t), & \text{if } t < 0, \end{cases}$$

where $f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \in [L^2(\mathbb{R}_+)]^{2N}$.

From the previous equivalence after extension relation, and from the computation

$$\begin{aligned} \Psi[\Phi(I+J)P_{\mathbb{R}} + Q_{\mathbb{R}}]\Psi^{-1} &= \begin{pmatrix} \Phi & 0 \\ \widetilde{\Phi} & I \end{pmatrix} P_{\mathbb{R}_+} + \begin{pmatrix} I & \Phi \\ 0 & \widetilde{\Phi} \end{pmatrix} Q_{\mathbb{R}_+} \\ &\quad + \frac{1}{2} \begin{pmatrix} \Phi & I - \Phi \\ \widetilde{\Phi} - I & -\widetilde{\Phi} \end{pmatrix} N_{\mathbb{R}_+} : [L^2(\mathbb{R}_+)]^{2N} \to [L^2(\mathbb{R}_+)]^{2N}, \end{aligned}$$

where

$$\begin{aligned} P_{\mathbb{R}_+} &= (I + S_{\mathbb{R}_+})/2 \;, \;\; Q_{\mathbb{R}_+} = I - P_{\mathbb{R}_+} \;, \\ (S_{\mathbb{R}_+}\Phi)(x) &= \frac{1}{\pi i} \int_{\mathbb{R}_+} \frac{\Phi(\tau)}{\tau - x} d\tau \;, \\ (N_{\mathbb{R}_+}\Phi)(x) &= \frac{1}{\pi i} \int_{\mathbb{R}_+} \frac{\Phi(\tau)}{\tau + x} d\tau \;, \text{ and } \; x \in \mathbb{R}_+ \;, \end{aligned}$$

we derive that TH_Φ is equivalent after extension with

$$\mathcal{T}_0 := \begin{pmatrix} \Phi & 0 \\ \widetilde{\Phi} & I \end{pmatrix} P_{\mathbb{R}_+} + \begin{pmatrix} I & \Phi \\ 0 & \widetilde{\Phi} \end{pmatrix} Q_{\mathbb{R}_+} + \frac{1}{2} \begin{pmatrix} \Phi & I - \Phi \\ \widetilde{\Phi} - I & -\widetilde{\Phi} \end{pmatrix} N_{\mathbb{R}_+} \,. \tag{9.3.2}$$

Now, using the technique of extension by identity in the framework of paired operators (cf., e.g., [70]), together with the corresponding direct sum decomposition $[L^2(\mathbb{R})]^{2N} = \chi_+[L^2(\mathbb{R})]^{2N} \oplus \chi_-[L^2(\mathbb{R})]^{2N}$, it follows that the operator $\mathcal{T}_0$ in (9.3.2) is equivalent after extension with

$$[\chi_+ \begin{pmatrix} \Phi & 0 \\ \widetilde{\Phi} & I \end{pmatrix} + \chi_- I]P_{\mathbb{R}} + [\chi_+ \begin{pmatrix} I & \Phi \\ 0 & \widetilde{\Phi} \end{pmatrix} + \chi_- I]Q_{\mathbb{R}}$$
$$+\frac{1}{2} \begin{pmatrix} \chi_+\Phi & \chi_+(I-\Phi) \\ \chi_+(\widetilde{\Phi}-I) & -\chi_+\widetilde{\Phi} \end{pmatrix} V.$$

Therefore, putting altogether and using the transitivity property of equivalence after extension relation, we have reached to the conclusion that TH_Φ is equivalent after extension with

$$\mathcal{T} = \mathfrak{a}P_{\mathbb{R}} + \mathfrak{b}Q_{\mathbb{R}} + \mathfrak{c}V.$$

In addition, it is known that V can be decomposed in $V = H_0 + H_\infty$, where $H_0 \in \mathcal{H}_0$ and $H_\infty \in \mathcal{H}_\infty$. In a more detailed way (cf. [4, Lemma 3.3]), the operator V belongs to the ideal $\mathcal{H}_0 + \mathcal{H}_\infty$. Therefore, we have the possibility to rewrite the operator $\mathcal{T}$ in the form

$$\mathcal{T} = \mathfrak{a}P_{\mathbb{R}} + \mathfrak{b}Q_{\mathbb{R}} + \mathfrak{c}H_0 + \mathfrak{c}H_\infty, \tag{9.3.3}$$

with $H_0 \in \mathcal{H}_0$ and $H_\infty \in \mathcal{H}_\infty$, and where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in PAP^{2N\times 2N}$ are given in (9.3.1).

In particular, the just presented operator relations imply that TH_Φ is a Fredholm operator if and only if $\mathcal{T}$ (in (9.3.3)) is a Fredholm operator.

Employing the method of [4, page 45] (in view of using here the Allan-Douglas local principle), we can associate with $\mathcal{T}$ the new operator

$$\widehat{\mathcal{T}} := \widehat{\mathfrak{a}}P_{\mathbb{R}} + \widehat{\mathfrak{b}}Q_{\mathbb{R}} + \widehat{\mathfrak{c}}H_\infty$$

with coefficients $\widehat{\mathfrak{a}}, \widehat{\mathfrak{b}}, \widehat{\mathfrak{c}} \in SAP^{2N\times 2N}$ such that $\widehat{\mathfrak{a}}, \widehat{\mathfrak{b}}, \widehat{\mathfrak{c}}$ have the same local representatives at $\pm\infty$ as $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in PAP^{2N\times 2N}$, and additionally $\widehat{\mathfrak{a}}$ and $\widehat{\mathfrak{b}}$ are chosen to be invertible, in a way that

$$\mathcal{T} = \mathcal{A}\widehat{\mathcal{T}} + K\ , \tag{9.3.4}$$

where $\mathcal{A} \in \mathfrak{D}$ is locally equivalent to $I_{2N\times 2N}$ at ∞, and K is a compact operator. We are now in a position to take profit of the fact that $\widehat{\mathcal{T}}$ belongs to $\mathfrak{B}$ and $\mathcal{A} \in \mathfrak{D}$ and prove the desired equivalence.

If TH_Φ is a Fredholm operator, then $\mathcal{T}$ is also a Fredholm operator. Moreover, we have that $\Phi \in \mathcal{G}PAP^{N\times N}$ (cf. Theorem 1.9.2 or [38, Proposition 2.6]). From the fact that $\mathcal{T}$ is a Fredholm operator it follows from (9.3.4) that $\mathcal{A}$ and $\widehat{\mathcal{T}}$ are also Fredholm operators. This implies that $\widehat{\mathcal{T}}_\mu$ are invertible operators, and from the equality $\widehat{\mathcal{T}}_\mu = \mathcal{T}_\mu$ we therefore derive that $\mathcal{T}_\mu$ are invertible operators. Once again, relaying on the fact that $\widehat{\mathcal{T}}$ and $\mathcal{A}$ are Fredholm operators, we obtain that $\det \widehat{\mathcal{T}}(t,x,\mu) \neq 0$ and $\det \mathcal{A} \neq 0$, for $(t,x,\mu) \in \mathbb{R} \times \dot{\mathbb{R}} \times [0,1]$. This can be done using the localization technique (cf. [4, page 46]). Hence $\det \mathcal{T}(t,x,\mu) \neq 0$. This gives the necessary part of the statement.

Let us now prove that the conditions (a)–(c) are sufficient for the Fredholm property of TH_Φ. From formula (9.3.4), it follows that

$$\det \mathcal{T}(t,x,\mu) = \det \mathcal{A}(t,x,\mu) \det \widehat{\mathcal{T}}(t,x,\mu), \quad (t,x,\mu) \in \mathcal{M} .$$

Since by the condition (b) we have that $\det \mathcal{T}(t,x,\mu) \neq 0$ we will have that $\det \mathcal{A}(t,x,\mu) \neq 0$. This and the fact that $\mathcal{A}(\infty,x,\mu) = I_{2N\times 2N}$ allows us to conclude that $\mathcal{A}$ is a Fredholm operator. Employing once again the equality (9.3.4), we obtain that $\mathcal{T}_\mu = \widehat{\mathcal{T}}_\mu$ since $\mathcal{A}$ is locally equivalent with $I_{2N\times 2N}$ at ∞. In addition, from conditions (a) and (c) and from Theorem 9.2.1, it follows that $\widehat{\mathcal{T}}$ is a Fredholm operator. Altogether, we have that under the conditions (a)–(c) both operators $\widehat{\mathcal{T}}$ and $\mathcal{A}$ are Fredholm. This means that $\mathcal{T}$ is a Fredholm operator, and hence TH_Φ is a Fredholm operator. □

9.4 Index formula

In this section we give an index formula for the operator TH_Φ with $\Phi \in PAP^{N\times N}$ under the assumption that TH_Φ is a Fredholm operator. Our reasoning relays on a certain approximating strategy. Namely, we will first give an index formula for TH_Φ with $\Phi \in PAPW^{N\times N}$, and then use the fact that $PAPW$ is dense in PAP and also certain stability properties (which occur under small perturbations). It should be mentioned that the index formula for TH_Φ with $\Phi \in PAPW^{N\times N}$ follows easily from the results obtained in [4].

As far as in this section our goal is to obtain a Fredholm index formula for Toeplitz plus Hankel operators with piecewise almost periodic functions, we will proceed by simplifying some symbols of the operators. Namely, here we will not need the symbol calculus for the operators with three variables and it is sufficient to consider the symbol calculus with two variables. For instance, for a function $f \in PC^{N\times N}$, and the Cauchy singular integral operator $S_{\mathbb{R}}$, the symbols are given by the formulas (cf., e.g., [4]):

$$(fI)(x,\mu) = \begin{bmatrix} f(x+0) & 0 \\ 0 & f(x-0) \end{bmatrix} \tag{9.4.1}$$

and

$$S_{\mathbb{R}}(x,\mu) = \begin{bmatrix} (2\mu-1) & 2\nu(\mu) \\ 2\nu(\mu) & (1-2\mu) \end{bmatrix}, \tag{9.4.2}$$

where $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$ and $\nu(\mu)$ is as in Section 9.1.

For starting let us analyze the singular integral operators with the form (9.3.3), but having coefficients from the Wiener subalgebra of $PAP^{N\times N}$. Now we will reproduce the constructions developed in [4, pages 47-48]. To this end let us denote by the $\mathcal{T}(x,\mu)$ the symbol of the operator $\mathcal{T}$ governed by the formulas (9.4.1) and (9.4.2). The necessary and sufficient conditions for the operator $\mathcal{T}$ to be Fredholm are given in [4, Theorem 5.2]. Furthermore from the same theorem [4, Theorem 5.2 (ii)–(iii)] we infer that for any fixed $\mu \in [0,1]$ the function $\mathcal{T}(\cdot,\mu) \in \mathcal{G}PAPW^{N\times N}$ and its almost periodic representatives $\det(b_\pm^{-1}a_\pm)$ at $\pm\infty$ admit canonical right APW factorizations. Further details on such kind of factorizations and related topics can be found e.g. in [14].

Consequently the function

$$\psi_\mu(x) := \arg \mathcal{T}(x,\mu)$$

belongs to $PAPW$ for every fixed $\mu \in [0,1]$, and hence the value

$$\mathrm{Ind}_{\mathbb{R}}\mathcal{T}^{\#} := \frac{1}{2\pi} \lim_{t\to+\infty} \frac{1}{t}\int_0^t (\psi_\mu(x) - \psi_\mu(-x))dx \tag{9.4.3}$$

exists, is finite and does not depend on the choice of $\mu \in [0,1]$. The fundamental properties of the quantity $\mathrm{Ind}_{\mathbb{R}}\mathcal{T}^{\#}$ given by the formula (9.4.3) are stated in [4, Lemma 5.3].

Let the operator TH_Φ with symbol $\Phi \in PAPW^{N\times N}$ be Fredholm operator. Then the operator $\mathcal{T}$ defined in Theorem 9.3.1 is also Fredholm and reasoning in a similar way as in [4, page 55] we can give another meaning to the quantity $\mathrm{Ind}_{\mathbb{R}}\mathcal{T}^{\#}$. More precisely in this situation we have (cf. [4, formula (6.2)])

$$\mathrm{Ind}_{\mathbb{R}}\mathcal{T}^{\#} = \frac{1}{2\pi}\lim_{t\to+\infty}\frac{1}{t}\int_0^t \psi_\mu(x)dx \ , \tag{9.4.4}$$

where $\psi_\mu(x) = \arg\mathcal{T}(x,\mu)$ and for $x<0$, $\psi_\mu(x) = 1$.

THEOREM 9.4.1. *(equivalent to [4, Theorem 6.4]) If the operator TH_Φ with symbol $\Phi \in PAPW^{N\times N}$ is Fredholm, then its index is given by the formula*

$$\mathrm{Ind}\mathcal{T} = -\mathrm{Ind}_{\mathbb{R}}\mathcal{T}^{\#} - \frac{1}{2\pi}\{\arg\det\widetilde{\mathcal{T}}(\infty,\mu)\}_{\mu\in[0,1]} \ ,$$

with $\mathrm{Ind}_{\mathbb{R}}\mathcal{T}^{\#}$ *provided by (9.4.4) and*

$$\widetilde{\mathcal{T}}(\infty,\mu) = \mathbf{d}\left(\begin{bmatrix} 0 & -\Phi_r\widetilde{\Phi}_\ell^{-1} \\ I & \widetilde{\Phi}_\ell^{-1}\end{bmatrix}\right)\begin{bmatrix}(1-\mu)I_n & i\nu(\mu)I_n \\ i\nu(\mu)I_n & (1-\mu)I_n\end{bmatrix} + \begin{bmatrix}\mu I_n & -i\nu(\mu)I_n \\ -i\nu(\mu)I_n & \mu I_n\end{bmatrix}, \tag{9.4.5}$$

where Φ_r and Φ_ℓ are the local representatives of the matrix function Φ (cf. Proposition 2.2.5). Here $\mathcal{T}$ is as in Theorem 9.3.1.

REMARK 9.4.2. *The proof of the next theorem reveals the structure of $\widetilde{\mathcal{T}}(\infty,\mu)$.*

The main result of this section is now stated.

THEOREM 9.4.3. *If TH_Φ is a Fredholm operator with symbol $\Phi \in PAP^{N\times N}$, then its Fredholm index is given by the formula:*

$$\mathrm{Ind}TH_\Phi = -\mathrm{Ind}_{\mathbb{R}}\mathcal{T}^{\#} - \frac{1}{2\pi}\{\arg\det\widetilde{\mathcal{T}}(\infty,\mu)\}_{\mu\in[0,1]} \ , \tag{9.4.6}$$

where $\mathrm{Ind}_{\mathbb{R}}\mathcal{T}^{\#}$ is given by (9.4.4) and $\widetilde{\mathcal{T}}(\infty,\mu)$ have the form of (9.4.5) for a $\mathcal{T}$ as in Theorem 9.3.1.

Proof. Let us take $\Phi_n \in PAPW^{N\times N}$ such that $\lim_{n\to\infty} ||\Phi_n - \Phi|| = 0$. Here and in what follows we are considering the supremum norms. Let us consider functions $\mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{c}_n \in PAPW^{2N\times 2N}$ such that

$$\lim_{n\to\infty} ||\mathfrak{a}_n - \mathfrak{a}|| = 0, \ \lim_{n\to\infty} ||\mathfrak{b}_n - \mathfrak{b}|| = 0, \ \lim_{n\to\infty} ||\mathfrak{c}_n - \mathfrak{c}|| = 0,$$

by taking into consideration Φ_n in the corresponding place of Φ in Theorem 9.3.1; cf. (9.3.1).

This allow us to consider the operator

$$\mathcal{T}_n := \mathfrak{a}_n P_{\mathbb{R}} + \mathfrak{b}_n Q_{\mathbb{R}} + \mathfrak{c}_n V \ .$$

If TH_Φ with symbol $\Phi \in PAP^{N\times N}$ is Fredholm, then the operator $\mathcal{T}$ defined in Theorem 9.3.1 is also Fredholm and their Fredholm indices coincide. Employing the fact that small perturbations preserve the Fredholm property and the Fredholm index, we conclude that the operators TH_Φ and $\mathcal{T}_n$ are Fredholm only simultaneously and their Fredholm indices coincide for sufficiently large values of n. Additionally, the Fredholm index for $\mathcal{T}_n$ is given by the next formula (cf. [4, Theorem 6.4])

$$\mathrm{Ind}\mathcal{T}_n = -\mathrm{Ind}_{\mathbb{R}}\mathcal{T}_n^{\#} - \frac{1}{2\pi}\{\arg\det\widetilde{\mathcal{T}}_n(\infty,\mu)\}_{\mu\in[0,1]} \ ,$$

where $\mathrm{Ind}_{\mathbb{R}}\mathcal{T}_n^{\#}$ is given by formula (9.4.4) and

$$\widetilde{\mathcal{T}}_n(\infty,\mu) = \mathbf{d}((\mathfrak{b}_n)_r^{-1}(\mathfrak{a}_n)_r)\begin{bmatrix} (1-\mu)I_n & i\nu(\mu)I_n \\ i\nu(\mu)I_n & (1-\mu)I_n \end{bmatrix} + \begin{bmatrix} \mu I_n & -i\nu(\mu)I_n \\ -i\nu(\mu)I_n & \mu I_n \end{bmatrix}.$$

Here $(\mathfrak{a}_n)_r$ and $(\mathfrak{b}_n)_r$ stand for the local representatives at $+\infty$ of the matrix functions $\mathfrak{a}_n$ and $\mathfrak{b}_n$. Having now in mind the stability of the geometrical mean value, we get the formula (9.4.6), since a direct computation provides that

$$(\mathfrak{b}_n)_r^{-1}(\mathfrak{a}_n)_r = \begin{bmatrix} 0 & -(\Phi_n)_r(\widetilde{\Phi}_n)_\ell^{-1} \\ I & (\widetilde{\Phi}_n)_\ell^{-1} \end{bmatrix}.$$

Using here the passage to the limit when $n \to \infty$, it turns clear the structure of $\widetilde{\mathcal{T}}(\infty,\mu)$ and the desired Fredholm index formula (9.4.6). Note that in general the limiting matrix belongs to the Besicovitch space (cf., e.g., [14] or [15]), but employing the known result about the stability of the geometrical mean value (see e.g., [14, Corollary 21.8], [15, Corollary 2.8], [46, Corollary 8]) we can here ensure that

$$\lim_{n\to\infty} \mathbf{d}((\mathfrak{b}_n)_r^{-1}(\mathfrak{a}_n)_r) = \mathbf{d}\left(\begin{bmatrix} 0 & -\Phi_r\widetilde{\Phi}_\ell^{-1} \\ I & \widetilde{\Phi}_\ell^{-1} \end{bmatrix}\right).$$

$\square$

Conclusion

The study of Wiener-Hopf-Hankel operators is important not only by pure theoretical reasons but also due to its appearance in various types of applications. This last interest comes e.g. from the Mathematical Physics, Statistics, Control Theory and many other areas of Mathematics. In this thesis we considered Wiener-Hopf-Hankel operators with symbols from the almost periodic, semi-almost periodic, piecewise almost periodic function classes and moreover with symbols associated in a certain way with unitary and sectorial matrix functions. The attention was paid also to Toeplitz plus Hankel operators with matrix piecewise almost periodic symbols and the Toeplitz plus Hankel operators with scalar symbols having n points of standard almost periodic discontinuities.

Chapters 1 and 2 were only of introductory nature and there were given very shortly the necessary background information for the development of the next chapters.

In Chapter 3 we considered the Wiener-Hopf plus Hankel operators with symbols from the algebra of matrix almost periodic functions. To deduce the one-sided or two-sided invertibility theory for Wiener-Hopf plus Hankel operators with AP matrix symbols we introduced the notion of an AP asymmetric factorization. In this framework were given sufficient conditions for the one-sided or two-sided invertibility of the Wiener-Hopf plus Hankel operators with matrix AP symbols. For such kind of operators were also exhibited generalized inverses for all the possible cases.

In Chapters 4 and 5 we obtained new results concerned a Fredholm property and a formula for the sum of the Fredholm indices of Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators with matrix SAP and PAP symbols, respectively. Anyway, the problem to obtain necessary and sufficient conditions for the Fredholm property of

Wiener-Hopf plus Hankel operators with SAP or PAP matrix symbols remains open.

In Chapter 6 we gave a corresponding version of the classical theorem by Douglas and Sarason for Toeplitz operators with sectorial and unitary symbols. Also in here the necessary and sufficient conditions for the Wiener-Hopf plus Hankel operators with the just mentioned symbols to be Fredholm operator remains open.

The main result of Chapter 7 was a necessary and sufficient condition for the Wiener-Hopf plus Hankel operators with L^∞ symbols to be Fredholm, or invertible. To obtain such a result we dealt with an odd asymmetric factorization with not "usual" weights. The corresponding theory with an even asymmetric factorization and the theory for the matrix case are open.

In Chapter 8 we found conditions under which Toeplitz plus Hankel operators generated by symbols which have n points of standard almost periodic discontinuities are right-invertible and with infinite dimensional kernel, left-invertible and with infinite dimensional cokernel or simply not normally solvable. In this direction there are an huge amount of open problems. E.g., the corresponding theory for matrix operators, the Fredholm index formula, and the consideration of other kinds of points of discontinuity (such as discontinuities of whirl points of power type) is still open.

In Chapter 9 were provided necessary and sufficient conditions for matrix Toeplitz plus Hankel operators with piecewise almost periodic symbols to have the Fredholm property. It is worth to mention that this result is highly theoretical, and the necessary and sufficient conditions for Toeplitz plus Hankel operators with matrix PAP symbols, which would be easily and effectively verifiable in practical problems remains open.

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