Universidade de Aveiro Departamento de Matemática
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José Alexandre Rocha Almeida

Espaços de Funções com Suavidade Generalizada e Integrabilidade Variável

Function Spaces with Generalized Smoothness and Variable Integrability

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Prof. Stefan G. Samko, Professor Catedrático da Faculdade de Ciências e Tecnologia da Universidade do Algarve e do Prof. António M. Caetano, Professor Associado com Agregação do Departamento de Matemática da Universidade de Aveiro.

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palavras-chave

Espaço de Besov, suavidade generalizada, interpolação com parâmetro função, decomposição em "wavelets", base de Schauder incondicional, espaço de Lebesgue, integrabilidade variável, espaço de Sobolev, expoente variável, potencial de Riesz, potencial de Bessel, integral hipersingular, função maximal, derivada fraccionária, imersão de Sobolev, espaço de Hölder, ordem variável.

resumo

Suavidade generalizada e integrabilidade variável são dois importantes tópicos de investigação na teoria de espaços de funções. Neste trabalho consideramos tanto espaços de Besov com suavidade generalizada como espaços de Lebesgue (e de Sobolev correspondentes) com parâmetro de integração variável. São estudadas no caso geral propriedades de interpolação com parâmetro função de espaços de Besov generalizados, as quais estendem resultados já conhecidos formulados no caso Banach. Além disso, obtemos decomposições em "wavelets" para estes espaços através do uso de técnicas de interpolação apropriadas, que por sua vez são usadas para obter novos resultados de interpolação.

Potenciais de Riesz e de Bessel são tratados no âmbito dos espaços de Lebesgue com expoente variável. Em particular, estudamos a inversão do operador potencial de Riesz e apresentamos uma caracterização, tanto para os espaços de potenciais de Riesz como para os espaços de potenciais de Bessel, em termos de convergência de integrais hipersingulares.

Também lidamos com desigualdades pontuais no quadro dos espaços de Sobolev com parâmetro de integração variável. Tais desigualdades são usadas para generalizar imersões de Sobolev clássicas ao contexto de expoentes variáveis, no caso em que, para além de condições naturais de regularidade, o expoente toma valores superiores à dimensão do espaço euclidiano. Por outro lado, são dados resultados acerca da limitação de operadores hipersingulares definidos em espaços de Sobolev variáveis em domínios limitados.

keywords Besov space, generalized smoothness, interpolation with function parameter, wavelet decomposition, unconditional Schauder basis, Lebesgue space, variable integrability, Sobolev space, variable exponent, Riesz potential, Bessel potential, hypersingular integral, maximal function, fractional derivative, Sobolev embedding, Hölder space, variable order.

abstract Generalized smoothness and variable integrability are two important research topics in the theory of function spaces. In this work, we consider both Besov spaces with generalized smoothness and Lebesgue (and corresponding Sobolev) spaces with variable exponent. Interpolation properties with function parameter of generalized Besov spaces are studied in the general case, which extends already known results stated for the Banach case. Moreover, we obtain wavelet decompositions for these spaces by using suitable interpolation techniques, which in turn are used to get new interpolation results.

Riesz and Bessel potentials are considered within the framework of the variable exponent Lebesgue spaces. In particular, we study the inversion of the Riesz potential operator and give a characterization both for the Riesz potential spaces and for the Bessel potential spaces, in terms of convergence of hypersingular integrals.

We also deal with pointwise inequalities in the context of the variable Sobolev spaces. Such inequalities are used to extend some classical Sobolev embeddings to the variable exponent setting, in the case when, besides natural assumptions, the exponent takes values greater than the dimension of the Euclidean space. Furthermore, boundedness results for hypersingular integral operators on variable Sobolev spaces over bounded domains are given.

Contents

Symbols

Basic notation:

$\emptyset$	empty set;
$\cdot$	takes the place of the variable with respect to which the function is evaluated, the (quasi-)norm is calculated, etc;
$:=$	equality, by definition;
$\sim$	equivalence (according to Section 1.1);
$f * g$	convolution product;
f^*	decreasing rearrangement of f;
$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers, $\mathbb{N} \cup \{0\}$;
$\mathbb{R}$	set of real numbers;
$\mathbb{C}$	set of complex numbers;
$\#$	cardinal of a countable set;
$[a]$	entire part of $a \in \mathbb{R}$;
$\mathbb{R}^n$	n-dimensional Euclidean space;
$\mathbb{Z}^n$	lattice of points in $\mathbb{R}^n$ with integer components;
$\beta!$	$\beta_1! \cdots \beta_n!$ for $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$;
$\lvert \cdot \rvert$	the Euclidean norm of an element of $\mathbb{R}^n$ or the length of a multi-index or the Lebesgue measure of a set of $\mathbb{R}^n$;
diam	diameter of a set in $\mathbb{R}^n$;
$x \cdot y$	scalar product in $\mathbb{R}^n$;
$B(x, r)$	open ball centered at $x \in \mathbb{R}^n$ and of radius $r > 0$;
$\overline{B}(x, r)$	corresponding closed ball;
χ_E	characteristic function of the set E;
supp	support of a function (or distribution);
Ω, $\partial\Omega$, $\overline{\Omega}$	domain in $\mathbb{R}^n$ (i.e., a non-empty open subset of $\mathbb{R}^n$), boundary of Ω, closure of Ω;
J_ν	Bessel function of the first kind, $\nu > 0$;
Γ	Gamma function;
$\binom{a}{k}$	binomial coefficient, given by $\frac{\Gamma(a+1)}{k!\,\Gamma(a+1-k)}$ with $a \in \mathbb{R}$, $k \in \mathbb{N}$;
D^β	classic or weak partial derivative of order $\beta \in \mathbb{N}_0^n$;
Δ, ∇	Laplacian, gradient;
Δ_y^ℓ	finite difference, with $\ell \in \mathbb{N}$, $y \in \mathbb{R}^n$;
$\hookrightarrow$	continuous embedding (between function spaces);
$\square$	end of proof.

Other symbols and pages where they are introduced:

Introduction

This dissertation is divided into two parts corresponding to the study of two important features in the theory of function spaces: generalized smoothness and variable integrability.

Function spaces with generalized smoothness have been considered by many authors from different points of view[1]. Important contributions were made by the former Soviet school, where approaches based on approximation by entire functions (Kalyabin and Lizorkin [75]) and modulus of continuity (Gold'man [59]) were developed. As far as applications are concerned, spaces of generalized smoothness have been successfully used in the resolution of some problems coming from stochastic processes and probability theory (see [52] for further details).

Part I deals with Besov spaces $B^{\phi}_{pq}(\mathbb{R}^n)$, where ϕ is a certain function representing a generalization of the usual smoothness parameter s. These spaces arise naturally as the interpolation result obtained by real interpolation with function parameter between Bessel potential spaces. This and other interpolation properties were investigated in the Banach case ($p, q \geq 1$) by Merucci [94] and afterwards by Cobos and Fernandez [23], following the Fourier-analytical approach (see Definition 2.1.2). Although the extension to the full range $0 < p, q \leq \infty$ has been announced in [23] as a forthcoming goal, the fact is that it has never been published (and does not even exist as a draft, to the best of our knowledge). This question is the starting point of the first part of our thesis.

In Part II we consider Lebesgue spaces with variable exponent, $L_{p(\cdot)}(\Omega)$, and the corresponding Sobolev spaces, where now the exponent $p = p(x)$ is a (measurable) function defined on Ω (this is why we write $p(\cdot)$ instead of p). Riesz and Bessel potentials, hypersingular integrals and pointwise inequalities are studied within the framework of these spaces.

The theory of variable exponent spaces have attracted many researchers during the last

[1]Function spaces of variable smoothness have also been studied by some authors. For instance, we mention the contributions by Besov [11, 10, 12] and by Leopold [86, 87].

years. The interest to these spaces comes not only from theoretical curiosity but also from their relevance in some applications. They appear in the modelling of some problems of fluid dynamics, elasticity theory and differential equations with the so-called non-standard growth conditions. We refer to [34], [111] and [1] as examples illustrating these applications. Recently, it was realized that variable exponent spaces are also very suitable to deal with problems related to image restoration (see [89]).

The spaces $L_{p(\cdot)}(\Omega)$ do not share all the properties of their classical analogues $L_p(\Omega)$ with constant exponent p. For instance, $L_{p(\cdot)}(\mathbb{R}^n)$ is not translation invariant, which brings some problems related to the convolution on these spaces. Nevertheless, after the boundedness of the Hardy-Littlewood maximal function has been proved by Diening [30], there was an evident progress in the development of operator theory on the variable exponent setting. Convolution, potential and singular type operators are examples of classic operators which have been intensively studied during the recent years (see Chapter 4 for more details). The work in the second part of our dissertation continues this line of investigation.

We now give a brief description of the topics treated in each chapter. Chapter 1 provides the necessary preliminaries concerning the basic notation used throughout the text. For future reference we also give a short review on classical function spaces.

The Chapters 2 and 3 form Part I. Chapter 2 is devoted to the study of interpolation properties of spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ in the case when p is fixed. Corresponding statements given in [23] for $p, q \geq 1$ are extended to the quasi-Banach case. We follow the same general principle, that is, we make use of the so-called retraction and co-retraction method to reduce the initial interpolation problem to the interpolation of appropriate sequence spaces. However, some changes have to be made in the general situation, since the retraction and the co-retraction constructed in [23] ((2.10 and (2.11), respectively), based on the Fourier transform acting in $L_p(\mathbb{R}^n)$, are meaningless when $0 < p < 1$. Such as remarked in [131] (Theorem 2.2.10), we show that it is possible to replace the space $L_p(\mathbb{R}^n)$ by the local Hardy space $h_p(\mathbb{R}^n)$, $0 < p < \infty$, in the definition of the Besov spaces (see equivalence (2.12)). In this way we overcome the difficulties with the Fourier transform. On the other hand, one proves that $B^{\phi}_{pq}(\mathbb{R}^n)$ is a retract of the sequence space $\ell^{\phi}_q(h_p(\mathbb{R}^n))$. We then get the desired interpolation formulas (see (2.26) and (2.27) below) from the corresponding ones obtained first for the spaces $\ell^{\phi}_q(h_p(\mathbb{R}^n))$.

The aim of Chapter 3 is twofold. The first is to obtain wavelet decompositions for spaces

$B^{\phi}_{pq}(\mathbb{R}^n)$, extending a recent result of Triebel [137] on wavelet bases in classical Besov spaces $B^{s}_{pq}(\mathbb{R}^n)$. The proof of the main statement (Theorem 3.1.7) is based on suitable interpolation techniques by making use of interpolation results studied in Chapter 2. In this way we avoid the usage of several tools (such as local means and maximal functions) as in [137]. The second objective of this chapter is to give further interpolation statements for spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ but now in the case when p is changed. Theorem 3.2.6 provides a possible alternative to the approach followed in [23], with the advantage of being valid in the quasi-Banach case. For its proof we base ourselves on wavelet representations obtained in Theorem 3.1.7 to construct a new retraction and a co-retraction. In general, our work in Chapter 3 emphasizes an interesting role played between interpolation properties and wavelet expansions.

Chapter 4 is an introduction to Part II. It provides both specific notation and important statements on variable exponent spaces which will be often used in the following chapters. This chapter gives also a brief overview on the present theory of these spaces.

In Chapter 5 we study mapping properties of the Riesz potential operator $\mathcal{I}^{\alpha}$ acting in Lebesgue spaces with variable exponent. The main statement (Theorem 5.3.5) shows that the Riesz fractional derivative $\mathbb{D}^{\alpha}$ provides the (left) inverse to the Riesz potential operator, under the assumption on the boundedness of the maximal operator on $L_{p(\cdot)}$. This generalizes previous results known for classic Lebesgue spaces (see [121]) to the variable exponent setting.

The study of the Riesz potential operator is continued in Chapter 6. A description of the Riesz potential spaces $\mathcal{I}^{\alpha}[L_{p(\cdot)}]$ in terms of convergence of hypersingular integrals is given when $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \frac{n}{\alpha}$ (see Theorem 6.1.4). Hence, we partially extend the known results for constant p (see [121]). We give also a characterization of the range $\mathcal{B}^{\alpha}[L_{p(\cdot)}]$, where $\mathcal{B}^{\alpha}$ is the Bessel potential operator. This statement is based on the previous description of $\mathcal{I}^{\alpha}[L_{p(\cdot)}]$ and on the study of two important convolution kernels defined by their Fourier transforms, which requires substantial efforts. Chapter 6 ends with a comparison of the Riesz and Bessel potential spaces with the variable Sobolev spaces $W^{m}_{p(\cdot)}(\mathbb{R}^n)$. In particular, we extend the well-known identity $\mathcal{B}^{m}[L_p] = W^{m}_{p}(\mathbb{R}^n)$ due to Calderón [19] to the variable exponent setting.

Finally, in Chapter 7 we study some pointwise inequalities within the context of the variable exponent Sobolev spaces. After recovering the known fact that the oscillation of Sobolev functions may be estimated by the fractional maximal operator of their gradient, we obtain Sobolev embeddings into Hölder spaces with variable order (see Theorem 7.3.7) and give boundedness results for hypersingular operators on spaces $W^m_{p(\cdot)}(\Omega)$ over bounded domains.

As far as possible, we have tried to give precise bibliographic references about the statements already known or the places where they can be found. Nevertheless, this is not an easy task due to the large number of people working on these topics. The numbers between braces in the list of references mean the pages where each paper or book is referred in this dissertation.

In order to facilitate the reading, all the notation and symbols used throughout the text are listed in an appropriate table. Moreover, we also provide a subject index in the end.

Almost all the results obtained in this thesis have been published in scientific journals. With exception of Section 3.2, the main statements in Chapters 2 and 3 are published in the papers

- A. Almeida, Wavelet bases in generalized Besov spaces, *J. Math. Anal. Appl.* **304**, No. 1 (2005), 198-211;

- A. Almeida, Wavelet representation in Besov spaces of generalized smoothness, In *"Function spaces, differential operators and nonlinear analysis"*. Proceedings of the conference FSDONA-04 dedicated to the 70th birthday of Prof. Alois Kufner, Milovy, Czech Republic, May 28 - June 2, 2004, P. Drábek and J. Rákosník (ed.). Prague: Math. Institute Acad. Sci. Czech Republic, pp. 7-18, 2005.

The content of Chapter 5 is published in the paper

- A. Almeida, Inversion of the Riesz potential operator on Lebesgue spaces with variable exponent, *Fract. Calc. Appl. Anal.* **6**, No. 3 (2003), 311-327.

The results in Chapters 6 and 7 are based on joint work with Stefan Samko and they can be found, respectively, in the papers

- A. Almeida and S. Samko, *Characterization of Riesz and Bessel potentials on variable Lebesgue spaces*, J. Funct. Spaces Appl. (to appear);

- A. Almeida and S. Samko, *Pointwise inequalities in variable Sobolev spaces and applications*, Z. Anal. Anwend. (to appear).

Chapter 1

Preliminaries

1.1 Basic notation

As usually, the set of natural numbers is denoted by $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We write $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ for the set of integer, real and complex numbers, respectively. By $\mathbb{R}^n$ we denote the n-dimensional real Euclidean space with $n \in \mathbb{N}$. A point $x \in \mathbb{R}^n$ is denoted by $x = (x_1, \dots, x_n)$ and its Euclidean norm by $|x|$. $\mathbb{Z}^n$ stands for the usual lattice consisting of all points in $\mathbb{R}^n$ with integer components, and $\mathbb{N}_0^n$ denotes the set of all multi-indices $\beta = (\beta_1, \dots, \beta_n)$, each component belonging to $\mathbb{N}_0$. For $\beta \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, we write $|\beta| = \beta_1 + \cdots + \beta_n$ (without risk of confusion), $\beta! = \beta_1! \cdots \beta_n!$ and $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.

Any reference with respect to measurability or to integrability should be always understood in the Lebesgue sense. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for its measure. The characteristic function over E is denoted by χ_E. We write $B(x,r)$ for the open ball centered at $x \in \mathbb{R}^n$ and radius $r > 0$, and $\overline{B}(x,r)$ for the corresponding closed ball.

In general, we are interested in function spaces of (measurable) complex-valued functions defined on $\mathbb{R}^n$. However, in Part II we shall deal with function spaces defined over other domains. Partial derivative operators are denoted by $\frac{\partial}{\partial x_j}$, $j = 1, \dots, n$, while higher order derivatives are given by $D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$ with $\beta \in \mathbb{N}_0^n$. The gradient of a function (with enough regularity) f on $\mathbb{R}^n$ is the vector $\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$ and the Laplacian is $\Delta f := \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$.

By $C^\infty(\mathbb{R}^n)$ we denote the class of all infinitely differentiable functions on $\mathbb{R}^n$, and by $C_0^\infty(\mathbb{R}^n)$ the class consisting of those functions in $C^\infty(\mathbb{R}^n)$ with compact support. We write

$\mathcal{S}(\mathbb{R}^n)$ for the Schwartz space of all rapidly decreasing functions in $C^\infty(\mathbb{R}^n)$, and $\mathcal{S}'(\mathbb{R}^n)$ for its topological dual, that is, the space of all tempered distributions on $\mathbb{R}^n$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $F\varphi$ (or $\widehat{\varphi}$) stands for the Fourier transform of φ,

$$(F\varphi)(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\,\varphi(x)\,dx, \quad \xi \in \mathbb{R}^n \tag{1.1}$$

($x \cdot \xi$ meaning scalar product), whereas $F^{-1}\varphi$ (or $\varphi^\vee$) denotes its related inverse Fourier transform. Both the Fourier transform and its inverse are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the usual way. The class of Fourier transforms of integrable functions will be denoted by $\mathcal{W}_0(\mathbb{R}^n)$.

Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X into Y is continuous. Furthermore, writing the equivalence "$\sim$" in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

means that there are two positive constants c_1 and c_2 such that

$$c_1\,a_k \le b_k \le c_2\,a_k \quad \text{or} \quad c_1\,\varphi(x) \le \psi(x) \le c_2\,\varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $\{a_k\}$, $\{b_k\}$ are non-negative sequences and φ, ψ are non-negative functions. We will also use the same symbol "$\sim$" to mean equivalence of quasi-norms.

All unimportant positive constants are denoted by C (or c) which may be different even in a single chain of inequalities. Sometimes, for convenience, we use additional subscripts $(c_1,\ c_2, ...)$ and we emphasize the dependence of the constants on certain parameters. For instance, when we write "$c(n)$" it means that the constant c depends on n and that other possible dependences are irrelevant in the context. In what follows "log" is always taken with respect to base 2.

Further notation will be properly introduced whenever needed.

1.2 Classical function spaces

We will deal with different generalizations of function spaces. By convention, in their notation we shall write the smoothness index as superscript, while the integrability index is written as subscript.

For $0 < p \le \infty$, $L_p(\mathbb{R}^n)$ denotes the well-known *Lebesgue space*, quasi-normed by

$$\|f \mid L_p(\mathbb{R}^n)\| := \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p}, \tag{1.2}$$

with the usual modification to $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|$ if $p = \infty$. We shall also use the notation $\|\cdot\|_p$ to denote the quasi-norm (1.2), mainly in the case of variable exponents p (writing $\|\cdot\|_{p(\cdot)}$ in that case; see Part II).

As usual, $L_{pq}(\mathbb{R}^n)$, $0 < p,\, q \le \infty$, denotes the classical *Lorentz space*, consisting of all (equivalence classes of) Lebesgue measurable functions f on $\mathbb{R}^n$ such that $\|f \mid L_{pq}(\mathbb{R}^n)\|$ is finite, with

$$\|f \mid L_{pq}(\mathbb{R}^n)\| := \left(\int_0^\infty \left[t^{\frac{1}{p}} f^*(t)\right]^q \frac{dt}{t}\right)^{1/q} \quad \text{if} \quad 0 < q < \infty,$$

$$\|f \mid L_{pq}(\mathbb{R}^n)\| := \sup_{t>0} \left[t^{\frac{1}{p}} f^*(t)\right] \quad \text{if} \quad q = \infty,$$

where f^* denotes the decreasing rearrangement of f, given by

$$f^*(t) = \inf\left\{\delta \ge 0 :\ |\{x \in \mathbb{R}^n :\ |f(x)| > \delta\}| \le t\right\}, \quad t > 0.$$

$C(\mathbb{R}^n)$ stands for the space of all uniformly continuous bounded functions in $\mathbb{R}^n$ and, for $r \in \mathbb{N}$,

$$C^r(\mathbb{R}^n) = \left\{f \in C(\mathbb{R}^n) :\ D^\beta f \in C(\mathbb{R}^n), \quad |\beta| \le r\right\}, \tag{1.3}$$

normed by

$$\|f \mid C^r(\mathbb{R}^n)\| := \sum_{|\beta|\le r} \|D^\beta f \mid L_\infty(\mathbb{R}^n)\|.$$

Let $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ be a system with the following properties:

$$\operatorname{supp} \varphi_0 \subset \{\xi \in \mathbb{R}^n :\ |\xi| \le 2\}; \tag{1.4}$$

$$\operatorname{supp} \varphi_j \subset \left\{\xi \in \mathbb{R}^n :\ 2^{j-1} \le |\xi| \le 2^{j+1}\right\}, \quad j \in \mathbb{N}; \tag{1.5}$$

$$\sup_{\xi\in\mathbb{R}^n} |D^\beta \varphi_j(\xi)| \le c(\beta)\, 2^{-j|\beta|}, \quad j \in \mathbb{N}_0, \quad \beta \in \mathbb{N}_0^n; \tag{1.6}$$

$$\sum_{j=0}^\infty \varphi_j(\xi) = 1, \quad \xi \in \mathbb{R}^n. \tag{1.7}$$

In other words, $\{\varphi_j\}_{j\in\mathbb{N}_0}$ forms a dyadic smooth partition of unity in $\mathbb{R}^n$. Examples of such systems can be constructed as follows: given $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp} \varphi_0 \subset \{\xi \in \mathbb{R}^n :\ |\xi| \le 2\} \quad \text{and} \quad \varphi_0(\xi) = 1 \ \text{if} \ |\xi| \le 1,$$

one defines, for each $j \in \mathbb{N}$,

$$\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi), \quad \xi \in \mathbb{R}^n. \tag{1.8}$$

If $f \in \mathcal{S}'(\mathbb{R}^n)$ then $(\varphi_j \widehat{f})^\vee$, $j \in \mathbb{N}_0$, makes sense pointwise since it is an analytic function on $\mathbb{R}^n$ by the Paley-Wiener-Schwartz theorem (cf. [132], p. 13). Moreover,

$$f = \sum_{j=0}^{\infty} (\varphi_j \widehat{f})^\vee \tag{1.9}$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$.

For $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$, the usual *Besov* and *Triebel-Lizorkin spaces* are defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f \,|\, B^s_{pq}(\mathbb{R}^n)\| := \left(\sum_{j=0}^{\infty} 2^{jsq} \, \|(\varphi_j \widehat{f})^\vee \,|\, L_p(\mathbb{R}^n)\|^q \right)^{1/q} \tag{1.10}$$

and

$$\|f \,|\, F^s_{pq}(\mathbb{R}^n)\| := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \, |(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{1/q} \,|\, L_p(\mathbb{R}^n) \right\|, \tag{1.11}$$

(with the usual modification if $q = \infty$, and assuming $p < \infty$ in the F-case) are finite, respectively. Together with the expressions (1.10), (1.11), they form quasi-Banach spaces (Banach spaces if $p \geq 1$, $q \geq 1$) and they are independent of the system $\{\varphi_j\}_{j \in \mathbb{N}_0}$ chosen according to (1.4)–(1.7), up to the equivalence of quasi-norms. We refer to [132] for a systematic theory on these spaces. Sometimes we add a comma to the notation (for example, instead of $B^s_{12}(\mathbb{R}^n)$ we would write $B^s_{1,2}(\mathbb{R}^n)$) to clarify the values that the parameters p and q are taking.

These scales contain some classical spaces as special cases. For instance,

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \quad s \in \mathbb{R}, \quad 1 < p < \infty,$$

are the *fractional Sobolev spaces* (also called *Bessel potential spaces*[1] or *Liouville spaces*). In particular, when $s \in \mathbb{N}$ then $F^s_{p,2}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n)$ are the classical *Sobolev spaces*, equivalently normed by

$$\|f \,|\, W^s_p(\mathbb{R}^n)\| := \sum_{|\beta| \leq s} \|D^\beta f \,|\, L_p(\mathbb{R}^n)\|, \qquad f \in W^s_p(\mathbb{R}^n),$$

[1]Later, in Chapter 6, we will denote the Bessel potential spaces in a different way. Our preference by another notation there intends to emphasize that these spaces are the range of the Bessel operator (acting in Lebesgue spaces with variable exponent p).

and $F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ are the Lebesgue spaces. Moreover,

$$F^0_{p,2}(\mathbb{R}^n) = h_p(\mathbb{R}^n), \quad 0 < p < \infty, \tag{1.12}$$

are the *local Hardy spaces* introduced by Goldberg [58] and

$$B^s_{\infty\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n), \quad s > 0, \tag{1.13}$$

are the *Hölder-Zygmund spaces* (see [132], Section 2.2.2, for precise definitions). Notice that all these identities should be interpreted in the sense of equivalence of quasi-norms.

We shall make use of general weighted sequence spaces as follows. If $\{X_i\}_{i\in I}$ is a countable family of quasi-Banach spaces, $\omega \equiv \{\omega_i\}_{i\in I}$ is a non-negative "sequence" and $0 < q \leq \infty$, then we will denote by $\ell_q(I, X_i, \omega_i)$ the "weighted sequence space" of all the families $a \equiv \{a_i\}_{i\in I}$, with $a_i \in X_i$, $i \in I$, such that $\|a \,|\, \ell_q(I, X_i, \omega_i)\|$ is finite, where

$$\|a \,|\, \ell_q(I, X_i, \omega_i)\| := \left(\sum_{i\in I} (\omega_i \, \|a_i \,|\, X_i\|)^q \right)^{1/q}, \quad 0 < q < \infty, \tag{1.14}$$

and

$$\|a \,|\, \ell_\infty(I, X_i, \omega_i)\| := \sup_{i\in I} \omega_i \, \|a_i \,|\, X_i\| \tag{1.15}$$

define quasi-norms. The right-hand side in (1.14) expresses the q-summability[2] of the non-negative family $\{\omega_i \, \|a_i \,|\, X_i\|\}_{i\in I}$ in $\mathbb{R}$.

We shall omit the index set I if it is clear from the context. When $X_i = X$ for every $i \in I$ then we shall write $\ell(I, X, \omega)$, where $\omega \equiv \{\omega_i\}_{i\in I}$. Moreover, in the particular case $X = \mathbb{C}$ we will also omit the "X" in the notation and we write only $\ell_q(I)$ if $\omega_i = 1$ for all $i \in I$.

[2]The expression "summable family" is used in the case when the set of indices is not ordered, at least *a priori*. Sometimes the summability is referred as the (uncondicional) convergence of the corresponding series (see Chapter 3 for further details).

Part I

Real Interpolation of Generalized Besov Spaces and Applications

Chapter 2

Real Interpolation of Besov Spaces with Generalized Smoothness

In this chapter we deal with Besov spaces with generalized smoothness, which will be denoted by $B^{\phi}_{pq}(\mathbb{R}^n)$. These spaces are defined as in (1.10) with $\phi(2^j)$ instead of 2^{js}, where ϕ is some "admissible" function.

Real interpolation with function parameter of spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ was first studied by Merucci in [94]. The investigation in [94] was then followed up by Cobos and Fernandez [23] leading to the extension of several classical results to spaces $B^{\phi}_{pq}(\mathbb{R}^n)$.

The statements both in [94] and in [23] were only formulated in the Banach case p, $q \geq 1$. Some of them can be extended to the quasi-Banach case although the same techniques are no longer available when $0 < p < 1$. This fact was observed in [23] (see Remark 5.4) based on a earlier remark from [131], Theorem 2.2.10. Since we did not find further information on this subject in the literature, we decided to give a detailed description how this question in the general case may be dealt with.

We start Section 2.1 with some historical remarks concerning Besov spaces. Then we introduce the generalized Besov spaces we are interested in and compare them to other known function spaces. In Section 2.2 we provide a review of basic tools on real interpolation with function parameter. Section 2.3 is devoted to the discussion of interpolation properties of $B^{\phi}_{pq}(\mathbb{R}^n)$ spaces in the general context mentioned above. The main statements provide interpolation formulas between these spaces in the case when the parameter p is not changed.

2.1 Besov spaces of generalized smoothness

2.1.1 A short note on Besov spaces

In a sense, Besov spaces B^s_{pq} appeared to fill the gaps between the Hölder-Zygmund spaces and the Sobolev spaces. Nikol'skii was the first who introduced the spaces $B^s_{p,\infty}$. Afterwards they were generalized to the case where q is different from ∞ by Besov [9] (with $s > 0$, $1 < p < \infty$, $1 \leq q < \infty$) at the end of the fifties of the last century, by means of the modulus of continuity. The Fourier-analytical characterization of these spaces was provided by Peetre in the sixties, based on the principle that every tempered distribution can be decomposed into a sum of entire analytic functions (see (1.9)). The same author extended the definition to the whole range $s \in \mathbb{R}$, $0 < p, q \leq \infty$ in the earlier seventies (see [103]). We refer to the monographs [104], [132] and [133] for further historical remarks on this subject.

Besov spaces are very useful from the applications point of view. They arise often in problems of signal processing and image compression. An important task is to get information on Besov functions from the study of the distribution of the associated wavelet coefficients. We mention the paper [73] as an example where such problems are treated. The case $0 < p < 1$ seems to be particularly useful to check the quality of the approximation of solutions of certain boundary value problems, obtained from numerical algorithms (see, for instance, [26] for further details).

Besov spaces (and other function spaces) with generalized smoothness have been considered by several authors from different approaches. They started to be studied in the 1970's mainly by the Russian school. For instance, we refer to [59] for an approach in terms of modulus of continuity and to [75] for a study based on approximation by entire functions of exponential type. Generalized Besov spaces were also studied from the interpolation point of view by Merucci [94] as the interpolation spaces obtained by real interpolation with function parameter between Bessel potential spaces. Further historical remarks and references can be found in [52].

We notice that function spaces of generalized smoothness have interesting applications to other fields. For example, they were recently used in some investigations in probability theory and stochastic processes (see [52]).

2.1.2 Definition and basic properties

Roughly speaking we obtain Besov spaces of generalized smoothness by replacing the usual regularity index s in (1.10) by a certain function fulfilling certain properties. The generalization we are interested in is based on the class $\mathfrak{B}$ defined as follows.

Definition 2.1.1. *We say that a function $\phi : (0,\infty) \to (0,\infty)$ belongs to the class $\mathfrak{B}$ if it is continuous, $\phi(1) = 1$, and*

$$\overline{\phi}(t) := \sup_{u>0} \frac{\phi(tu)}{\phi(u)} < \infty$$

for every $t > 0$.

We refer to [94] for details concerning this class. A basic example of a function belonging to $\mathfrak{B}$ is $\phi(t) = \phi_{s,0}(t) = t^s$, $s \in \mathbb{R}$. Other examples will be given later. As we will see in the sequel, the class $\mathfrak{B}$ is sufficiently wide in the sense that it allow us to cover the most common cases appearing in the literature.

We will follow the Fourier-analytic approach as in (1.10) to introduce the spaces we are interested in. First we need some auxiliary sequence spaces.

If E is a quasi-normed space, $0 < q \leq \infty$ and $\phi \in \mathfrak{B}$, we may consider the sequence spaces $\ell_q^{\phi}(E) := \ell_q(\mathbb{N}_0, E, \phi(2^j))$ equipped with the quasi-norms $\| \cdot \mid \ell_q(\mathbb{N}_0, E, \phi(2^j))\|$, according to (1.14) and (1.15) (with $I = \mathbb{N}_0$). When $\phi(t) = t^s$, $s \in \mathbb{R}$, we simply write $\ell_q^s(E)$ instead of $\ell_q^{\phi}(E)$ for short.

Definition 2.1.2. *Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a dyadic partition of unity with the properties (1.4)–(1.7) above. For $\phi \in \mathfrak{B}$, $0 < p \leq \infty$ and $0 < q \leq \infty$, we define $B^{\phi}_{pq}(\mathbb{R}^n)$ as the class of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\{(\varphi_j \widehat{f})^{\vee}\}_{j\in\mathbb{N}_0} \in \ell_q^{\phi}(L_p(\mathbb{R}^n))$ with the quasi-norm*

$$\|f \mid B^{\phi}_{pq}(\mathbb{R}^n)\| := \left\| \{(\varphi_j \widehat{f})^{\vee}\}_{j\in\mathbb{N}_0} \mid \ell_q^{\phi}(L_p(\mathbb{R}^n)) \right\|.$$

These spaces were studied by Merucci [94] as the result of real interpolation with function parameter between Sobolev spaces and afterwards by Cobos and Fernandez in [23]. Like in the classical case (according to (1.10)), they are quasi-Banach spaces and are independent of the system $\{\varphi_j\}_{j\in\mathbb{N}_0}$ chosen, up to the equivalence of quasi-norms. We point out that the spaces $B^s_{pq}(\mathbb{R}^n)$ can be obtained as a particular case of the spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ by taking $\phi(t) = t^s$, $s \in \mathbb{R}$. For convenience we shall refer to the spaces $B^s_{pq}(\mathbb{R}^n)$ as classical Besov spaces.

As mentioned above, Besov spaces with generalized smoothness have been considered and studied by many authors in different contexts. The paper [52] gives detailed information about literature concerning this subject. In [52] we can also find a general and unified approach for these spaces, as well as the counterpart for the Triebel-Lizorkin scale. As far as Besov spaces are concerned, it is possible to define generalized spaces $B^{\sigma}_{pq}(\mathbb{R}^n)$ by replacing $\phi(2^j)$ by σ_j, $j \in \mathbb{N}_0$, in Definition 2.1.2, where σ is a certain *admissible sequence* of positive real numbers (in the sense of [52]):

$$B^{\sigma}_{pq}(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \, \|f \,|\, B^{\sigma}_{pq}(\mathbb{R}^n)\| := \|\{\sigma_j \, (\varphi_j \widehat{f})^{\vee}\}_{j\in\mathbb{N}_0} \,|\, \ell_q(L_p(\mathbb{R}^n))\| < \infty \Big\}, \qquad (2.1)$$

where $\sigma \equiv \{\sigma_j\}_{j\in\mathbb{N}_0}$ satisfies the condition

$$d_0 \, \sigma_j \le \sigma_{j+1} \le d_1 \, \sigma_j, \quad \forall j \in \mathbb{N}_0, \qquad (2.2)$$

for some d_0, $d_1 > 0$. The definition given in [52] is even more general: it is introduced a fourth parameter $N \equiv \{N_j\}_{j\in\mathbb{N}_0}$ related to generalized decompositions of unity, namely allowing different sizes for the support of the functions φ_j. We restrict ourselves here to the standard decomposition, that is, to the case $N \equiv \{2^j\}_{j\in\mathbb{N}_0}$.

Some "other" generalized spaces of Besov type were introduced by Edmunds and Triebel [39], [40]. Those spaces, usually denoted by $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, are defined as in (1.10) with $2^{jsq}\,\Psi(2^{-j})^q$ in place of 2^{jsq}. The parameter Ψ here represents a perturbation on the smoothness index s and it satisfies certain conditions. We refer to [96] for a systematic study on spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$. In the particular case $\Psi(t) = (1 + |\log t|)^b$, $t \in (0, 1]$, $b \in \mathbb{R}$, one obtains the spaces $B^{s,b}_{pq}(\mathbb{R}^n)$ considered by Leopold [88].

The spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ are covered by the general formulation (2.1). In fact, as remarked in [52], the sequence $\{\sigma_j\}_{j\in\mathbb{N}_0}$ given by $\sigma_j = 2^{js}\,\Psi(2^{-j})$, $j \in \mathbb{N}_0$, is admissible. Since $\overline{\phi}(1/2)^{-1}\,\phi(2^j) \le \phi(2^{j+1}) \le \overline{\phi}(2)\,\phi(2^j)$, $j \in \mathbb{N}_0$ (see (2.9) below), the spaces $B^{\phi}_{pq}(\mathbb{R}^n)$, $\phi \in \mathfrak{B}$, are also a particular case of the spaces defined in (2.1). Nevertheless, we would like to stress that it suffices to consider the spaces $B^{\phi}_{pq}(\mathbb{R}^n)$. This fact may be justified by the following result, which was suggested to us by Caetano.

Proposition 2.1.3. *Let σ be an admissible sequence in the sense of (2.2) and $0 < p$, $q \le \infty$. Then there exists a function $\phi_\sigma \in \mathfrak{B}$ such that*

$$B^{\phi_\sigma}_{pq}(\mathbb{R}^n) = B^{\sigma}_{pq}(\mathbb{R}^n).$$

Proof. Let σ be admissible. First, we remark that one can always assume $\sigma_0 = 1$ without loss of generality. In fact, the sequence σ' defined as $\sigma'_0 = 1$ and $\sigma'_j = \sigma_j,\ j \in \mathbb{N}$, is equivalent to σ, so that $B^{\sigma}_{pq}(\mathbb{R}^n) = B^{\sigma'}_{pq}(\mathbb{R}^n)$.

We can construct a function $\phi_\sigma \in \mathfrak{B}$ as follows:

$$\phi_\sigma(t) = \begin{cases} \frac{\sigma_{j+1}-\sigma_j}{2^j}\,(t-2^j) + \sigma_j\,, & t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0, \\ \sigma_0 & , \quad t \in (0,1) \end{cases}$$

(cf. [18], Section 2.2). Hence, $\phi_\sigma(2^j) = \sigma_j$ for all $j \in \mathbb{N}_0$ and we get the result. $\square$

According to this proposition we will only consider generalized Besov spaces from Definition 2.1.2 in what follows.

We summarize now some basic embedding results which will be used later.

Proposition 2.1.4. (i) *Let $\phi \in \mathfrak{B},\ 0 < p \le \infty,\ 0 < q \le \infty$. Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{\phi}_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

(ii) *Let $\phi \in \mathfrak{B},\ 0 < p \le \infty,\ 0 < q_0 \le q_1 \le \infty$. Then*

$$B^{\phi}_{pq_0}(\mathbb{R}^n) \hookrightarrow B^{\phi}_{pq_1}(\mathbb{R}^n).$$

(iii) *Let $\phi,\ \psi \in \mathfrak{B},\ 0 < p \le \infty,\ 0 < q_0,\ q_1 \le \infty$. If $\left\{\frac{\phi(2^j)}{\psi(2^j)}\right\}_{j\in\mathbb{N}_0} \in \ell_{\min\{q_1,\,1\}}$ then*

$$B^{\psi}_{pq_0}(\mathbb{R}^n) \hookrightarrow B^{\phi}_{pq_1}(\mathbb{R}^n).$$

We do not give the proofs here since they are similar to the classical case treated in [132], p. 47-48. Embeddings (ii) and (iii) generalize properties (2) and (4) contained in Theorem 4.1 in [23].

2.2 On real interpolation with function parameter

By an *interpolation quasi-normed couple* $\{A_0, A_1\}$ we mean a couple of quasi-normed spaces A_0 and A_1 which are both continuously embedded in some Hausdorff topological vector space. In this case, the expressions

$$\|a \,|\, A_0 \cap A_1\| := \max\left(\|a \,|\, A_0\|,\ \|a \,|\, A_1\|\right), \quad a \in A_0 \cap A_1,$$

and

$$\|a \,|\, A_0 + A_1\| := \inf_{\substack{a=a_0+a_1 \\ a_0\in A_0,\, a_1\in A_1}} \left(\|a_0 \,|\, A_0\| + \|a_1 \,|\, A_1\|\right), \quad a \in A_0 + A_1$$

(where the infimum is taken over all possible decompositions of a as the sum of elements in A_0 and A_1), define quasi-norms in $A_0 \cap A_1$ and $A_0 + A_1$, respectively.

Roughly speaking, the *interpolation property* may be described as follows. Given a second interpolation quasi-normed couple $\{B_0, B_1\}$, we say that the spaces A and B ($A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$ and $B_0 \cap B_1 \hookrightarrow B \hookrightarrow B_0 + B_1$) are *interpolation spaces* (with respect to $\{A_0, A_1\}$ and $\{B_0, B_1\}$) if for every linear operator $T : A_0 + A_1 \to B_0 + B_1$ whose restrictions to A_i, $i = 0, 1$, are bounded linear operators from A_i into B_i, then its restriction to A is also a bounded linear operator from A into B. In the particular case when $A_0 = B_0$ and $A_1 = B_1$, then we shall say that A is an interpolation space (with respect to $\{A_0, A_1\}$).

There are several ways to construct interpolation spaces. We refer to the monographs [8] and [134] for a detailed presentation and references on interpolation theory.

We shall make use of the real method of interpolation based on the *K-functional* introduced by Peetre (this is why it is sometimes referred as the *K-method*). This functional is defined by

$$K(t, a) = K(t, a; A_0, A_1) = \inf_{\substack{a=a_0+a_1 \\ a_0\in A_0,\, a_1\in A_1}} \left(\|a_0 \,|\, A_0\| + t\,\|a_1 \,|\, A_1\|\right), \quad t > 0, \quad a \in A_0 + A_1. \tag{2.3}$$

For each $t \in (0, \infty)$, $K(t, \cdot; A_0, A_1)$ gives an equivalent quasi-norm in the linear sum $A_0 + A_1$. $K(\cdot, a)$ is an increasing function for each $a \in A_0 + A_1$. Furthermore, it admits the following estimate, which can be checked directly:

$$K(t, a; A_0, A_1) \leq c\, K_q(t, a; A_0, A_1), \tag{2.4}$$

where $c > 0$ is independent of $t > 0$ and $a \in A_0 + A_1$, and

$$K_q(t, a) = K_q(t, a; A_0, A_1) := \inf_{\substack{a=a_0+a_1 \\ a_0\in A_0,\, a_1\in A_1}} \left(\|a_0 \,|\, A_0\|^q + t^q\,\|a_1 \,|\, A_1\|^q\right)^{1/q}, \quad 0 < q < \infty. \tag{2.5}$$

We are interested in general interpolation spaces where the function parameter is taken from the class $\mathfrak{B}$ (recall Definition 2.1.1). Nevertheless, the usual statements of interpolation theory require additional hypothesis on the function parameter. These assumptions are usually expressed in terms of the *Boyd indices*. For a function $\phi \in \mathfrak{B}$, the *Boyd upper* and *lower indices*

$\alpha_{\overline{\phi}}$ and $\beta_{\overline{\phi}}$ are given by

$$\alpha_{\overline{\phi}} = \lim_{t\to\infty} \frac{\log \overline{\phi}(t)}{\log t} \qquad \text{and} \qquad \beta_{\overline{\phi}} = \lim_{t\to 0} \frac{\log \overline{\phi}(t)}{\log t},$$

respectively. They are real numbers, $\beta_{\overline{\phi}} \leq \alpha_{\overline{\phi}}$ and they satisfy the properties (see [14]):

$$\beta_{\overline{\phi}} > 0 \quad \text{if and only if} \quad \int_0^1 \frac{\overline{\phi}(t)}{t}\, dt < \infty \tag{2.6}$$

and

$$\alpha_{\overline{\phi}} < 0 \quad \text{if and only if} \quad \int_1^\infty \frac{\overline{\phi}(t)}{t}\, dt < \infty. \tag{2.7}$$

Example 2.2.1. ([94]) The functions $\phi_{a,b}$ given by

$$\phi_{a,b}(t) = t^a\,(1 + |\log t|)^b, \quad a,\, b \in \mathbb{R}, \tag{2.8}$$

belong to the class $\mathfrak{B}$. Moreover,

$$\overline{\phi}_{a,b}(t) = t^a\,(1 + |\log t|)^{|b|} \quad \text{and} \quad \alpha_{\overline{\phi}_{a,b}} = \beta_{\overline{\phi}_{a,b}} = a.$$

Other examples can be found in [91, 92].

The following proposition summarizes several useful properties of class $\mathfrak{B}$.

Proposition 2.2.2. ([94])

(i) *If $\phi,\ \phi_0,\ \phi_1 \in \mathfrak{B}$ and $\delta > 0$, then also $\phi^\delta,\ \frac{1}{\phi},\ \phi_0\,\phi_1,\ \frac{\phi_0}{\phi_1} \in \mathfrak{B}$ with*

$$\overline{\phi^\delta}(t) = \overline{\phi}^{\delta}(t), \quad \overline{1/\phi}(t) = \overline{\phi}(1/t), \quad \overline{\phi_0\phi_1}(t) \leq \overline{\phi}_0(t)\,\overline{\phi}_1(t), \quad \overline{\phi_0/\phi_1}(t) \leq \overline{\phi}_0(t)\,\overline{\phi}_1(1/t),$$

and $\alpha_{\overline{\phi^\delta}} = \delta\,\alpha_{\overline{\phi}}$, $\quad \alpha_{\overline{\phi_0\phi_1}} = \alpha_{\overline{\phi}_0} + \alpha_{\overline{\phi}_1}$, being the same formulas valid for the lower index. Moreover, the connection between ϕ and $\overline{\phi}$ is made by

$$\frac{\phi(u)}{\overline{\phi}(1/t)} \leq \phi(u\,t) \leq \phi(u)\,\overline{\phi}(t), \quad u,\, t \in (0,\infty). \tag{2.9}$$

(ii) *For every $\psi \in \mathfrak{B}$ such that $\beta_{\overline{\psi}} > 0$ (resp. $\alpha_{\overline{\psi}} < 0$) there exists an increasing (resp. decreasing) function $\psi_0 \in \mathfrak{B}$ equivalent to ψ, that is, $\psi_0(t) \sim \psi(t)$.*

Definition 2.2.3. *Let $\{A_0, A_1\}$ be an interpolation couple of quasi-normed spaces. Let also $\gamma \in \mathfrak{B}$ and $0 < q \leq \infty$. The space $(A_0, A_1)_{\gamma,q}$ consist of all $a \in A_0 + A_1$ for which $\|a\|_{\gamma,q}$ is finite, where*

$$\|a\|_{\gamma,q} := \left(\int_0^\infty \left[\gamma(t)^{-1}\,K(t,a)\right]^q \frac{dt}{t}\right)^{1/q} \quad \text{if} \quad 0 < q < \infty,$$
$$\|a\|_{\gamma,q} := \sup_{t>0} \left[\gamma(t)^{-1}\,K(t,a)\right] \quad \text{if} \quad q = \infty,$$

are quasi-norms.

If $0 < \beta_{\overline{\gamma}} \le \alpha_{\overline{\gamma}} < 1$ then $(A_0, A_1)_{\gamma,q}$ is an interpolation quasi-normed space (with respect to $\{A_0, A_1\}$). Moreover, if A_0 and A_1 are quasi-Banach spaces then $(A_0, A_1)_{\gamma,q}$ is also a quasi-Banach space.

Example 2.2.4. In the particular case $\gamma(t) = \phi_{\theta,0}(t) = t^\theta$, $0 < \theta < 1$, then $(A_0, A_1)_{\gamma,q} = (A_0, A_1)_{\theta,q}$ is the usual interpolation space, quasi-normed by $\|\cdot\|_{\gamma,q} = \|\cdot\|_{\theta,q}$ (cf. [8], [134]).

An interesting way to describe interpolation spaces is by the so-called *method of retraction and co-retraction*. This method gives us the possibility to obtain unknown interpolation spaces from known ones with the aid of *retractions* and *co-retractions*. We shall make use of this procedure to obtain interpolation results for Besov spaces from interpolation formulas of suitable sequence spaces.

Let us give precise definitions (see [8], [134]).

Definition 2.2.5. *Let A and B be quasi-normed spaces. B is a retract of A if there are bounded linear operators $\mathcal{R} : A \to B$ (retraction) and $\mathcal{J} : B \to A$ (co-retraction), such that the composition $\mathcal{R}\mathcal{J}$ is the identity on B.*

The method of retraction and co-retraction is provided by the following theorem, whose proof may be derived from the basic properties of interpolation theory.

Theorem 2.2.6. *Let $\{A_0, A_1\}$ and $\{B_0, B_1\}$ be two interpolation couples of quasi-normed spaces. Let also $\gamma \in \mathfrak{B}$, with $0 < \beta_{\overline{\gamma}} \le \alpha_{\overline{\gamma}} < 1$, and $0 < q \le \infty$. If B_i is a retract of A_i, $i = 0, 1$, with mappings $\mathcal{R}$ and $\mathcal{J}$ as in Definition 2.2.5, then $(B_0, B_1)_{\gamma,q}$ is a retract of $(A_0, A_1)_{\gamma,q}$ with "the same mappings" $\mathcal{R}$ and $\mathcal{J}$.*

Proof. The hypothesis guarantees the existence of bounded linear operators $\mathcal{R} : A_i \to B_i$ and $\mathcal{J} : B_i \to A_i$ such that $\mathcal{R}\mathcal{J} = I_{B_i}$ (where I_{B_i} denotes the identity on B_i), $i = 0, 1$. Both "$\mathcal{R}$" and "$\mathcal{J}$" can be defined on the linear sums $A_0 + A_1$ and $B_0 + B_1$, resp., in the usual way. For instance, one defines $\mathcal{R}a := \mathcal{R}a_0 + \mathcal{R}a_1$, where $a = a_0 + a_1$ is a decomposition of a as the sum of elements $a_0 \in A_0$ and $a_1 \in A_1$ (note that this definition does not depend on the representation $a = a_0 + a_1$ taken). Moreover, simple calculations show that $\mathcal{R}\mathcal{J} = I_{B_0+B_1}$.

Now we conclude that $(B_0, B_1)_{\gamma,q}$ is a retract of $(A_0, A_1)_{\gamma,q}$ by using the interpolation property described above. $\square$

Remark 2.2.7. In a concrete case we may describe the unknown space $(B_0, B_1)_{\gamma,q}$ from the knowledge of the already known space $(A_0, A_1)_{\gamma,q}$ by using the equivalence

$$\|f \,|\, (B_0, B_1)_{\gamma,q}\| \sim \|\mathcal{J} f \,|\, (A_0, A_1)_{\gamma,q}\|,$$

which is a consequence of Theorem 2.2.6.

2.3 Interpolation of spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ with p fixed

This section concerns the discussion of interpolation properties for spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ in the case when p is fixed. The results given below, being valid in the quasi-Banach case, extend previous statements obtained by Cobos and Fernandez in the Banach case $p, q \geq 1$ (see Theorems 5.1 and 5.3 (1) of [23]).

The approach followed in [23] was based on interpolation properties of sequence spaces. Those properties were then transferred to the generalized Besov spaces by means of retractions and co-retractions as described in Section 2.2. The key point is that $B^{\phi}_{pq}(\mathbb{R}^n)$ is a retract of the sequence space $\ell^{\phi}_q(L_p(\mathbb{R}^n))$ when $p \geq 1$. Indeed, it is possible to show that, for any system $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ with the properties (1.4)–(1.7), the mapping

$$\mathcal{R}\{f_j\}_{j\in\mathbb{N}_0} := \sum_{j=0}^{\infty} F^{-1}(\widetilde{\varphi}_j \, F f_j), \quad \text{with} \quad \widetilde{\varphi}_j = \sum_{r=-1}^{1} \varphi_{j+r}, \tag{2.10}$$

(convergence in $\mathcal{S}'(\mathbb{R}^n)$) is a retraction from $\ell^{\phi}_q(L_p(\mathbb{R}^n))$ into $B^{\phi}_{pq}(\mathbb{R}^n)$ and

$$\mathcal{J} f := \{F^{-1}(\varphi_j \, F f)\}_{j\in\mathbb{N}_0} \tag{2.11}$$

is the corresponding co-retraction (see [8], Theorem 6.4.3, and [134], Section 2.3.2, for a discussion of the classical case $\phi(t) = t^s$). In (2.10) we assume that $\varphi_{-1} \equiv 0$.

The mappings in (2.10) and (2.11), based on the Fourier transform acting in $L_p(\mathbb{R}^n)$, are meaningless if $0 < p < 1$. Nevertheless, some interpolation statements obtained in [23] for the spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ remain valid in the quasi-Banach case as well, if we replace $L_p(\mathbb{R}^n)$ by the local Hardy space $h_p(\mathbb{R}^n)$ in the definition of the Besov spaces (see Remark 5.4 in [23]).

In the sequel we describe the main steps of the general procedure. Inspired by the proof of Theorem 2.2.10 in [131], we state the following result.

Proposition 2.3.1. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $0 < p < \infty$ and $\{\varphi_j\}_{j\in\mathbb{N}_0}$ satisfy the conditions (1.4)–(1.7). Then $F^{-1}(\varphi_j Ff) \in L_p(\mathbb{R}^n)$ if, and only if, $F^{-1}(\varphi_j Ff) \in h_p(\mathbb{R}^n)$, $j \in \mathbb{N}_0$. Moreover, there exist constants c_1, $c_2 > 0$ independent of f and j such that*

$$c_1 \, \|F^{-1}(\varphi_j Ff) \,|\, L_p(\mathbb{R}^n)\| \le \|F^{-1}(\varphi_j Ff) \,|\, h_p(\mathbb{R}^n)\| \le c_2 \, \|F^{-1}(\varphi_j Ff) \,|\, L_p(\mathbb{R}^n)\|. \tag{2.12}$$

Proof. Only the case $0 < p \le 1$ is of interest since $h_p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ when $1 < p < \infty$. Let $\{\psi_j\}_{j\in\mathbb{N}_0}$ be a dyadic smooth partition of unity satisfying conditions corresponding to (1.4)–(1.7). If we assume $\psi_{-1} \equiv 0$ then we have $\varphi_j = \varphi_j \sum_{r=-1}^{1} \psi_{j+r}$, for every $j \in \mathbb{N}_0$.
Hence, for fixed $f \in \mathcal{S}'(\mathbb{R}^n)$ and $j \in \mathbb{N}_0$, we get

$$\begin{aligned}
\left\|F^{-1}(\varphi_j Ff) \,|\, L_p(\mathbb{R}^n)\right\| &\le \left\| \sum_{r=-1}^{1} \left|F^{-1}(\psi_{j+r}\, \varphi_j Ff)\right| \,|\, L_p(\mathbb{R}^n)\right\| \\
&\le \sqrt{3} \left\| \left(\sum_{r=-1}^{1} \left|F^{-1}(\psi_{j+r}\, \varphi_j Ff)\right|^2 \right)^{1/2} \,|\, L_p(\mathbb{R}^n)\right\| \\
&= \sqrt{3} \left\| \left(\sum_{k=j-1}^{j+1} \left|F^{-1}(\psi_k\, \varphi_j Ff)\right|^2 \right)^{1/2} \,|\, L_p(\mathbb{R}^n)\right\| \\
&= \sqrt{3} \left\| \left(\sum_{k=0}^{\infty} \left|F^{-1}[\psi_k\, F(F^{-1}(\varphi_j Ff))]\right|^2 \right)^{1/2} \,|\, L_p(\mathbb{R}^n)\right\| \\
&= \sqrt{3} \left\|F^{-1}(\varphi_j Ff) \,|\, F^0_{p,2}(\mathbb{R}^n)\right\|,
\end{aligned}$$

which proves the left-hand side of (2.12) taking into account (1.12).

Conversely,

$$\begin{aligned}
\left\|F^{-1}(\varphi_j Ff) \,|\, h_p(\mathbb{R}^n)\right\| &\le c \left\| \left(\sum_{k=0}^{\infty} \left|F^{-1}(\psi_k\, \varphi_j Ff)\right|^2 \right)^{1/2} \,|\, L_p(\mathbb{R}^n)\right\| \\
&\le c_1 \sum_{r=-1}^{1} \left\|F^{-1}[\psi_{j+r}\, F(F^{-1}(\varphi_j Ff))] \,|\, L_p(\mathbb{R}^n)\right\| \\
&\le c_2 \sum_{r=-1}^{1} \left\|\psi_{j+r}(2^{j+1}\cdot) \,|\, W_2^m(\mathbb{R}^n)\right\| \, \left\|F^{-1}(\varphi_j Ff) \,|\, L_p(\mathbb{R}^n)\right\|
\end{aligned}$$

where the last inequality follows from estimate (13) in [132], p. 28, with $m \in \mathbb{N}$ such that $m > n\left(\frac{1}{p} - \frac{1}{2}\right)$. We observe that $\operatorname{supp} \psi_{j+r}(2^{j+1}\cdot) \subset \overline{B}(0,2)$. Moreover, property (1.6) yields

$$\left|D^\beta(\psi_{j+r}(2^{j+1}x))\right| \le c(\beta)\; 2^{|\beta|(1-r)}, \quad |x| \le 2.$$

Hence

$$\left\|\psi_{j+r}(2^{j+1}\cdot)\,|\,W_2^m(\mathbb{R}^n)\right\| \le c_3 \sum_{|\beta|\le m} \left\|D^\beta(\psi_{j+r}(2^{j+1}\cdot))\,|\,L_2(\mathbb{R}^n)\right\| \le C$$

for some $C>0$ not depending on j and f. This completes the proof. □

The replacement of $L_p(\mathbb{R}^n)$ by $h_p(\mathbb{R}^n)$ $(0<p<\infty)$ in Definition 2.1.2 is now clear from inequality (2.12). The situation $p=\infty$ does not bring any problem since, in that case, it is not necessary to replace $L_\infty(\mathbb{R}^n)$ for our purposes.

Lemma 2.3.2. *Let $\phi\in\mathfrak{B}$, $0<p<\infty$ and $0<q\le\infty$. Assume that $\{g_j\}_{j\in\mathbb{N}_0}\subset\mathcal{S}'(\mathbb{R}^n)$ fulfills the conditions*

$$\operatorname{supp} Fg_0 \subset \{x:\, |x|\le 2\} \quad \text{and} \quad \operatorname{supp} Fg_j \subset \{x:\, 2^{j-1}\le |x| \le 2^{j+1}\},\ j\in\mathbb{N}.$$

If $\left\|\left\{\phi(2^j)\,g_j\right\}_j\,|\,\ell_q(h_p(\mathbb{R}^n))\right\|<\infty$ then $\sum_{j=0}^{\infty} g_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$.

Proof. Some arguments used below are similar to those used in [140], where convergence problems were discussed in the case $\phi(t)=t^s$.

Let us consider

$$\psi(t) := \phi(t)\,t^{-\delta}, \quad t>0 \quad (\text{with } \delta>0). \tag{2.13}$$

Then $\psi\in\mathfrak{B}$ and $\left\{\frac{\psi(2^k)}{\phi(2^k)}\right\}_{k\in\mathbb{N}_0}\in\ell_1$. Moreover, for appropriate values of δ, we also have $\left\{\psi(2^k)\right\}_{k\in\mathbb{N}_0}\in\ell_1$. Indeed, if $\alpha_{\overline{\psi}}=\alpha_{\overline{\phi}}-\delta<0$ then, by Proposition 2.2.2, there exists a decreasing function $\psi_0\in\mathfrak{B}$ with $\psi_0(t)\sim\psi(t)$. Since $\psi_0(2^k)\le\overline{\psi}_0(2^k)$ and $\overline{\psi}_0$ is decreasing, it suffices to show that $\int_1^\infty \overline{\psi}_0(2^t)\,dt<\infty$. But

$$\int_1^\infty \overline{\psi}_0(2^t)\,dt \le c\int_2^\infty \frac{\overline{\psi}(u)}{u}\,du<\infty$$

according to (2.7). Hence, we shall choose $\delta>\max\left(0,\alpha_{\overline{\phi}}\right)$. For such a choice and the corresponding function ψ from (2.13), we have

$$g_j\in h_p(\mathbb{R}^n)=F^0_{p,2}(\mathbb{R}^n)\hookrightarrow B^0_{p\infty}(\mathbb{R}^n)\hookrightarrow B^{\psi}_{p\infty}(\mathbb{R}^n)$$

in view of Proposition 2.1.4, (*iii*).

We prove the convergence of the series above in $\mathcal{S}'(\mathbb{R}^n)$ by showing that $\left\{g^{[N]}\right\}_{N\in\mathbb{N}_0}$, with $g^{[N]}$ given by $\sum_{j=0}^{N} g_j$, is a Cauchy sequence in $B^{\psi}_{p\infty}(\mathbb{R}^n)$.

For $N>M$ one gets

$$
\begin{aligned}
\left\| g^{[N]} - g^{[M]} \,|\, B^{\psi}_{p\infty}(\mathbb{R}^n) \right\| &= \left\| \left\{ F^{-1}\varphi_k F \left(\sum_{j=M+1}^{N} g_j \right) \right\}_k \,|\, \ell^{\psi}_{\infty}(h_p(\mathbb{R}^n)) \right\| \\
&\leq \sum_{k=0}^{\infty} \psi(2^k) \left\| \sum_{j=M+1}^{N} F^{-1}(\varphi_k \, F g_j) \,|\, h_p(\mathbb{R}^n) \right\| \\
&\leq \sum_{k=M-1}^{N+2} \psi(2^k) \left\| \sum_{j=-2}^{2} F^{-1}(\varphi_k \, F g_{k+j}) \,|\, h_p(\mathbb{R}^n) \right\| \\
&\leq c(p) \sum_{j=-2}^{2} \sum_{k=M-1}^{N+2} \left\| \psi(2^k) \, F^{-1}(\varphi_k \, F g_{k+j}) \,|\, h_p(\mathbb{R}^n) \right\| \\
&\longrightarrow 0 \quad \text{as} \quad N, M \to \infty, \qquad (2.14)
\end{aligned}
$$

where the first inequality follows from the embedding $\ell_1 \hookrightarrow \ell_\infty$ and the second is obtained from the relation $\operatorname{supp} \varphi_k \cap \operatorname{supp} F g_j = \emptyset$ for $|k-j| > 2$ (under the assumption $\varphi_k = g_j \equiv 0$ if $k,\, j < 0$). The convergence in (2.14) follows from the convergence of the series on k for fixed j:

$$
\begin{aligned}
\sum_{k=0}^{\infty} \left\| \psi(2^k) \, F^{-1}[\varphi_k \, F g_{k+j}] \,|\, h_p(\mathbb{R}^n) \right\| &\leq \left\| \left\{ \frac{\psi(2^k)}{\phi(2^k)} \right\}_k \,|\, \ell_1 \right\| \sup_{k \in \mathbb{N}_0} \left\| \phi(2^k) \, F^{-1}[\varphi_k F g_{k+j}] \,|\, h_p(\mathbb{R}^n) \right\| \\
&\leq c \left\| \left\{ \phi(2^k) \, \left\| F^{-1}[\varphi_k \, F g_{k+j}] \,|\, h_p(\mathbb{R}^n) \right\| \right\}_k \,|\, \ell_q \right\| \\
&\leq c_1 \left\| \left\{ \phi(2^k) \, \| g_{k+j} \,|\, h_p(\mathbb{R}^n) \| \right\}_k \,|\, \ell_q \right\| \qquad (2.15) \\
&\leq c_1 \, \overline{\phi}(2^{-j}) \left\| \left\{ \phi(2^{k+j}) \, \| g_{k+j} \,|\, h_p(\mathbb{R}^n) \| \right\}_k \,|\, \ell_q \right\| \\
&\leq c_2 \, \overline{\phi}(2^{-j}) \left\| \left\{ \phi(2^k) \, g_k \right\}_k \,|\, \ell_q(h_p(\mathbb{R}^n)) \right\| < \infty.
\end{aligned}
$$

The second inequality is easily justified by the embedding $\ell_q \hookrightarrow \ell_\infty$, while in the fourth one we took into account the property (2.9). In (2.15) we have applied a Fourier multiplier theorem for $h_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n)$, namely Theorem 2.3.7 in [132], as follows:

$$
\left\| F^{-1}[\varphi_k \, F g_{k+j}] \,|\, h_p(\mathbb{R}^n) \right\| \leq c \, \|\!|\varphi_k|\!\|_N \, \| g_{k+j} \,|\, h_p(\mathbb{R}^n) \| \,, \qquad (2.16)
$$

with $c > 0$ not depending on k nor f, and

$$
\|\!|\varphi_k|\!\|_N := \sup_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n} \left(1 + |x|^2\right)^{\frac{|\beta|}{2}} \left| D^\beta \varphi_k(x) \right|, \qquad (2.17)
$$

where N is a large natural number ($N > \frac{3n}{\min(p,2)} + n + 2$). It remains to show that the quantity (2.17) may be estimated by an absolute constant with respect to k. For convenience, we may assume that $\{\varphi_k\}_{k\in\mathbb{N}_0}$ is the particular system given by (1.8), so that $\varphi_k = \varphi_1(2^{-k+1}\cdot)$, $k \in \mathbb{N}$. For $|\beta| \le N$, $x \in \mathbb{R}^n$ and $k \ge 1$,

$$\begin{aligned}\left(1+|x|^2\right)^{\frac{|\beta|}{2}} \left|D^\beta \varphi_k(x)\right| &= \left(1+|x|^2\right)^{\frac{|\beta|}{2}} 2^{(-k+1)|\beta|} \left|(D^\beta\varphi_1)(2^{-k+1}x)\right| \\ &= \left(2^{-2(k-1)} + |2^{-k+1}x|^2\right)^{\frac{|\beta|}{2}} \left|(D^\beta\varphi_1)(2^{-k+1}x)\right|.\end{aligned}$$

Since $\varphi_1 \in \mathcal{S}(\mathbb{R}^n)$ we easily obtain

$$|\!|\!|\varphi_k|\!|\!|_N \le \sup_{|\beta|\le N} \sup_{y\in\mathbb{R}^n} \left(1+|y|^2\right)^{\frac{N}{2}} \left|D^\beta\varphi_1(y)\right| \le c(N,\varphi_1).$$

This completes the proof. □

The interpolation statement given in [23], Theorem 5.3 (1), can be extended to the quasi-Banach case in view of the theorem below.

Theorem 2.3.3. *Let $\phi \in \mathfrak{B}$, $0 < p < \infty$ and $0 < q \le \infty$. Then $B^{\phi}_{pq}(\mathbb{R}^n)$ is a retract of $\ell^{\phi}_q(h_p(\mathbb{R}^n))$.*

Proof. We are to show that the mappings defined in (2.10) and (2.11) are, respectively, a retraction and a co-retraction acting on the corresponding spaces, now with $h_p(\mathbb{R}^n)$ in place of $L_p(\mathbb{R}^n)$.

First we prove that the series involved in the definition of $\mathcal{R}$ (see (2.10)) converges in $\mathcal{S}'(\mathbb{R}^n)$, namely the series $\sum_{j=0}^{\infty} F^{-1}[\widetilde{\varphi}_j\, F f_j]$, with $\widetilde{\varphi}_j = \varphi_{j-1}+\varphi_j+\varphi_{j+1}$, where now $\{f_j\}_{j\in\mathbb{N}_0} \in \ell^{\phi}_q(h_p(\mathbb{R}^n))$. According to Lemma 2.3.2, we have to show that

$$\left\|\left\{\phi(2^j)\; F^{-1}[\varphi_m\, F f_j]\right\}_j \mid \ell_q(h_p(\mathbb{R}^n))\right\| < \infty, \quad \text{for} \quad m = j-1, j, j+1 \tag{2.18}$$

(under the assumption $\varphi_k \equiv 0$ when $k < 0$). Let us take, for instance, $m = j$. An application of the Fourier multiplier theorem for $h_p(\mathbb{R}^n)$ as above (see (2.16)) yields

$$\left\|\left\{\phi(2^j)\; F^{-1}[\varphi_j\, F f_j]\right\}_j \mid \ell_q(h_p(\mathbb{R}^n))\right\| \le c \left\|\left\{\phi(2^j)\, f_j\right\}_j \mid \ell_q(h_p(\mathbb{R}^n))\right\| < \infty.$$

The statement (2.18) corresponding to the cases $m = j \mp 1$ can be shown by using similar arguments.

As regards the mapping properties of $\mathcal{R}$ and $\mathcal{J}$, only the boundedness of $\mathcal{R}$ requires long justifications, because the boundedness of $\mathcal{J}$ is clear from (2.12) and

$$\mathcal{R}\mathcal{J} f = \sum_{j=0}^{\infty} F^{-1}[\widetilde{\varphi}_j\, \varphi_j\, F f] = f$$

by (1.9) (note also that $\widetilde{\varphi}_j \equiv 1$ over supp φ_j). The continuity of the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ together with the properties of the system $\{\varphi_j\}_{j\in\mathbb{N}_0}$ yield (with $\varphi_l = Ff_l \equiv 0$ if $l<0$)

$$\begin{aligned}
\left\| \mathcal{R}\{f_j\}_j \,|\, B^{\phi}_{pq}(\mathbb{R}^n) \right\| &= \left\| \sum_{j=0}^{\infty} F^{-1}[\widetilde{\varphi}_j\, Ff_j] \,|\, B^{\phi}_{pq}(\mathbb{R}^n) \right\| \\
&= \left\| \left\{ F^{-1}\left[\sum_{j=k-2}^{k+2} \varphi_k\, \widetilde{\varphi}_j\, Ff_j \right] \right\}_k \,|\, \ell^{\phi}_q(h_p(\mathbb{R}^n)) \right\| \\
&\le c \left\| \left\{ \sum_{j=k-2}^{k+2} \phi(2^k)\, \left\| F^{-1}[\varphi_k\, \widetilde{\varphi}_j\, Ff_j] \,|\, h_p(\mathbb{R}^n) \right\| \right\}_k \,|\, \ell_q \right\| \\
&\le c_1 \sum_{j=-2}^{2} \sum_{m=-1}^{1} \left\| \left\{ \phi(2^k)\, \left\| F^{-1}[\varphi_k\, \varphi_{k+j+m}\, Ff_{k+j}] \,|\, h_p(\mathbb{R}^n) \right\| \right\}_k \,|\, \ell_q \right\| .
\end{aligned}$$

As before, we may apply the Fourier multiplier theorem for $h_p(\mathbb{R}^n)$, leading to the quantity $|||\varphi_k\, \varphi_{k+j+m}|||_N$ (with N large). This quantity can be estimated (by a constant independent of k), just by proceeding as above, with the aid of the Leibniz rule on the differentiation of the product. Hence, we obtain

$$\begin{aligned}
\left\| \mathcal{R}\{f_j\}_j \,|\, B^{\phi}_{pq}(\mathbb{R}^n) \right\| &\le c_2 \sum_{j=-2}^{2} \left\| \left\{ \phi(2^k)\, \| f_{k+j} \,|\, h_p(\mathbb{R}^n) \| \right\}_k \,|\, \ell_q \right\| \\
&\le c_2 \sum_{j=-2}^{2} \overline{\phi}(2^{-j}) \left\| \left\{ \phi(2^{k+j})\, \| f_{k+j} \,|\, h_p(\mathbb{R}^n) \| \right\}_k \,|\, \ell_q \right\| \\
&\le c_3 \left\| \{f_k\}_k \,|\, \ell^{\phi}_q(h_p(\mathbb{R}^n)) \right\| ,
\end{aligned}$$

where $c_3 > 0$ does not depend on $\{f_k\}_{k\in\mathbb{N}_0}$. □

Remark 2.3.4. When $p=\infty$ then $B^{\phi}_{pq}(\mathbb{R}^n)$ is a retract of $\ell^{\phi}_q(L_p(\mathbb{R}^n))$ (see Theorem 2.5 in [23]).

By Theorem 2.3.3 the interpolation problem of Besov spaces is now reduced to the study of interpolation properties of sequence spaces. Having in mind future applications, we give the proofs in a more general context. Hence, until the end of this chapter E will denote a quasi-normed space.

The general ideas below are inspired by the study of the classical case, which may be found in [8], mainly Section 5.6. Let us start with a somewhat technical result.

Lemma 2.3.5. *Let s_0, $s_1 \in \mathbb{R}$ and $0 < q < \infty$. Then*

$$K_q(t, \xi;\, \ell^{s_0}_q(E), \ell^{s_1}_q(E)) \sim \left(\sum_{j=0}^{\infty} \left[\min(2^{js_0}, 2^{js_1} t) \, \|\xi_j \,|\, E\| \right]^q \right)^{1/q} \tag{2.19}$$

where $\xi = \{\xi_j\}_{j \in \mathbb{N}_0} \in \ell^{s_0}_q(E) + \ell^{s_1}_q(E)$, $t > 0$ and K_q is the functional defined in (2.5).

Proof.

$$\begin{aligned}
K_q(t, \xi;\, \ell^{s_0}_q(E), \ell^{s_1}_q(E)) &= \inf_{\substack{\xi = \xi^0 + \xi^1 \\ \xi^i \in \ell^{s_i}_q(E)}} \left(\sum_{j=0}^{\infty} \left[(2^{js_0} \, \|\xi^0_j \,|\, E\|)^q + (2^{js_1} t \, \|\xi^1_j \,|\, E\|)^q \right] \right)^{1/q} \\
&= \left(\sum_{j=0}^{\infty} \inf_{\substack{\xi^0_j + \xi^1_j = \xi_j \\ \xi^i_j \in E}} \left[(2^{js_0} \, \|\xi^0_j \,|\, E\|)^q + (2^{js_1} t \, \|\xi^1_j \,|\, E\|)^q \right] \right)^{1/q} \\
&= \left(\sum_{j=0}^{\infty} 2^{js_0 q} \, \|\xi_j \,|\, E\|^q \inf_{\substack{\eta^0_j + \eta^1_j = \frac{\xi_j}{\|\xi_j | E\|} \\ \eta_j \in E,\ \xi_j \neq 0_E}} \left[\|\eta^0_j \,|\, E\|^q + \left(\frac{2^{js_1} t}{2^{js_0}} \right)^q \|\eta^1_j \,|\, E\|^q \right] \right)^{1/q} \\
&= \left(\sum_{j=0}^{\infty} 2^{js_0 q} \, \|\xi_j \,|\, E\|^q \, F_j \left(\left(\frac{2^{js_1} t}{2^{js_0}} \right)^q \right) \right)^{1/q},
\end{aligned}$$

where

$$F_j(u) := \inf_{\substack{a_0 + a_1 = \frac{\xi_j}{\|\xi_j | E\|} \\ a_i \in E}} \left(\|a_0 \,|\, E\|^q + u \, \|a_1 \,|\, E\|^q \right), \quad u > 0.$$

The result follows now from the equivalence $F_j(u) \sim \min(1, u)$ (with constants not depending on j and u). Indeed, simple calculations yield

$$F_j(u) \le \min(1, u) \le C_E^q \, \max(1, 2^{q-1}) \, F_j(u), \quad u > 0, \ \ j \in \mathbb{N}_0,$$

where $C_E \ge 1$ is the constant coming from the quasi-triangular inequality of E. $\square$

Theorem 2.3.6. *Let $\phi \in \mathfrak{B}$, $0 < q_0, q_1, q \le \infty$. Let also s_0, s_1 be real numbers such that $s_1 < \beta_{\overline{\phi}} \le \alpha_{\overline{\phi}} < s_0$. Then*

$$(\ell^{s_0}_{q_0}(E), \ell^{s_1}_{q_1}(E))_{\gamma, q} = \ell^{\phi}_q(E), \tag{2.20}$$

where

$$\gamma(t) = \frac{t^{\frac{s_0}{s_0 - s_1}}}{\phi\left(t^{\frac{1}{s_0 - s_1}} \right)}, \quad t \in (0, \infty). \tag{2.21}$$

Proof. Step 1. First we prove the embedding

$$(\ell^{s_0}_\infty(E), \ell^{s_1}_\infty(E))_{\gamma,q} \hookrightarrow \ell^{\phi}_q(E). \tag{2.22}$$

We start by observing that $\gamma \in \mathfrak{B}$, with the Boyd indices given by

$$0 < \frac{s_0 - \alpha_{\overline{\phi}}}{s_0 - s_1} = \beta_{\overline{\gamma}} \leq \alpha_{\overline{\gamma}} = \frac{s_0 - \beta_{\overline{\phi}}}{s_0 - s_1} < 1,$$

which can be shown by simple calculations.

Let $\xi \in (\ell^{s_0}_\infty(E), \ell^{s_1}_\infty(E))_{\gamma,q}$ and $0 < q < \infty$. A suitable change of variables yields

$$\|\xi\|^q_{\gamma,q} = (s_0 - s_1) \int_0^\infty K(u^{s_0 - s_1}, \xi)^q \left(\frac{\phi(u)}{u^{s_0}}\right)^q \frac{du}{u}.$$

Denoting $\phi_0(u) = u^{-s_0}$, $u > 0$, then we have $\phi\phi_0 \in \mathfrak{B}$. Moreover, since $\alpha_{\overline{\phi\phi_0}} = \alpha_{\overline{\phi}} - s_0 < 0$, there exists a decreasing function $\psi \in \mathfrak{B}$ such that $\psi \sim \phi\phi_0$ (see Proposition 2.2.2).

We also observe that if $\xi^0 + \xi^1 = \xi$, $\xi^i \in \ell^{s_i}_\infty$, $i = 0, 1$, is any decomposition of ξ, then

$$\|\xi^0 \,|\, \ell^{s_0}_\infty(E)\| + t\, \|\xi^1 \,|\, \ell^{s_1}_\infty(E)\| \geq c \sup_{j \in \mathbb{N}_0} \min(2^{js_0}, 2^{js_1} t)\, \|\xi_j \,|\, E\| \tag{2.23}$$

with $c > 0$ independent of t and ξ. Hence

$$\begin{aligned}
\|\xi\|^q_{\gamma,q} &\geq c \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} K(u^{s_0 - s_1}, \xi)^q\, \psi(u)^q \frac{du}{u} \\
&\geq c \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \left[\sup_{j \in \mathbb{N}_0} \min(2^{js_0}, 2^{k(s_0 - s_1)}\, 2^{js_1})\, \|\xi_j \,|\, E\|\right]^q \psi(u)^q \frac{du}{u} \\
&\geq c \sum_{k=0}^{\infty} \int_{2^k}^{2^{k+1}} \left[\min(2^{ks_0}, 2^{k(s_0 - s_1)}\, 2^{ks_1})\, \|\xi_k \,|\, E\|\right]^q \psi(u)^q \frac{du}{u} \\
&\geq c \sum_{k=0}^{\infty} \int_{2^k}^{2^{k+1}} 2^{ks_0 q}\, \|\xi_k \,|\, E\|^q\, \psi(2^{k+1})^q\, 2^{-(k+1)}\, du \\
&\geq c \sum_{k=0}^{\infty} \|\xi_k \,|\, E\|^q\, \phi(2^k)^q,
\end{aligned}$$

where we have made use of property (2.9) in last inequality. The case $q = \infty$ can be proved in a similar way, just by replacing the "sum" and the "integral" above by the corresponding "sup".

Step 2. Let us show now that

$$\ell^{\phi}_q(E) \hookrightarrow (\ell^{s_0}_r(E), \ell^{s_1}_r(E))_{\gamma,q} \quad \text{for all } \ 0 < r < q. \tag{2.24}$$

Since $\beta_{\overline{\gamma}} > 0$, Proposition 2.2.2 guarantees the existence of an increasing function $\eta \in \mathfrak{B}$ equivalent to γ. We prove (2.24) in the case $0 < q < \infty$.

$$
\begin{aligned}
\|\xi\|^q_{\gamma,q} &\le c \sum_{k=-\infty}^{\infty} \int_{2^{(k-1)(s_0-s_1)}}^{2^{k(s_0-s_1)}} K(t,\xi)^q\, \eta(t)^{-q}\, \frac{dt}{t} \\
&\le c \sum_{k=-\infty}^{\infty} \int_{2^{(k-1)(s_0-s_1)}}^{2^{k(s_0-s_1)}} K(2^{k(s_0-s_1)},\xi)^q\, \frac{2^{-(k-1)(s_0-s_1)}}{\eta(2^{(k-1)(s_0-s_1)})^q}\, dt \\
&\le c \sum_{k=-\infty}^{\infty} K_r(2^{k(s_0-s_1)},\xi)^q\, \gamma(2^{(k-1)(s_0-s_1)})^{-q} \qquad \text{(see (2.4))} \\
&\le c \sum_{k=-\infty}^{\infty} \left(\sum_{j=0}^{\infty} \Big[\min(2^{js_0}, 2^{k(s_0-s_1)}2^{js_1})\, \|\xi_j \mid E\|\Big]^r \right)^{q/r} \gamma(2^{(k-1)(s_0-s_1)})^{-q} \quad \text{(see (2.19))} \\
&= c \sum_{k=-\infty}^{\infty} \left(\sum_{\substack{j=-\infty \\ k+j\ge 0}}^{\infty} \Big[2^{(k+j)s_0} \min(1, 2^{-j(s_0-s_1)})\, \|\xi_{k+j} \mid E\|\Big]^r \right)^{q/r} \gamma(2^{(k-1)(s_0-s_1)})^{-q} \\
&= c \sum_{k=-\infty}^{\infty} \left(\sum_{\substack{j=-\infty \\ k+j\ge 0}}^{\infty} \left[\min(1, 2^{-j(s_0-s_1)})\, \phi(2^{k+j})\, \|\xi_{k+j} \mid E\| \frac{2^{(k+j)s_0}}{\phi(2^{k+j})\, \gamma(2^{(k-1)(s_0-s_1)})}\right]^r \right)^{q/r} \\
&\le c \sum_{k=-\infty}^{\infty} \left(\sum_{\substack{j=-\infty \\ k+j\ge 0}}^{\infty} \Big[\min(1, 2^{-j(s_0-s_1)})\, 2^{js_0}\, \overline{\phi}(2^{-j})\, \phi(2^{k+j})\, \|\xi_{k+j} \mid E\|\Big]^r \right)^{q/r}
\end{aligned}
$$

where the last inequality follows from Proposition 2.2.2, property (2.9). From Minkowski inequality, we also obtain

$$
\begin{aligned}
\|\xi\|_{\gamma,q} &\le c \left(\sum_{j=-\infty}^{\infty} \left(\sum_{\substack{k=-\infty \\ k+j\ge 0}}^{\infty} \Big[\min(1, 2^{-j(s_0-s_1)})\, 2^{js_0}\, \overline{\phi}(2^{-j})\, \phi(2^{k+j})\, \|\xi_{k+j} \mid E\|\Big]^q \right)^{r/q} \right)^{1/r} \\
&= c\, (A_r(\gamma))^{1/r}\, \|\xi \mid \ell^{\phi}_q(E)\|,
\end{aligned}
$$

where

$$
A_r(\gamma) := \sum_{j=-\infty}^{\infty} \Big[\min(1, 2^{-j(s_0-s_1)})\, \overline{\gamma}(2^{j(s_0-s_1)})\Big]^r .
$$

It remains to show that the quantity $A_r(\gamma)$ is finite. Since $\beta_{\overline{\gamma^r}} = r\, \beta_{\overline{\gamma}} > 0$ and $\alpha_{\overline{(\frac{\gamma}{t})^r}} = r\,(\alpha_{\overline{\gamma}} - 1) < 0$, then

$$
\int_0^{\infty} \left(\frac{\overline{\gamma}(t)}{\max(1,t)} \right)^r \frac{dt}{t} = \int_0^1 \overline{\gamma}(t)^r\, \frac{dt}{t} + \int_1^{\infty} \left(\frac{\overline{\gamma}(t)}{t} \right)^r \frac{dt}{t} < \infty, \tag{2.25}
$$

according to statements (2.6) and (2.7). On the other hand, for every $\sigma > 0$, we derive

$$\begin{aligned}
\int_0^\infty \left[\min(1,t^{-1})\,\overline{\gamma}(t)\right]^r \frac{dt}{t} &= \sum_{j=-\infty}^{\infty} \int_{2^{j\sigma}}^{2^{(j+1)\sigma}} \left[\min(1,t^{-1})\,\overline{\gamma}(t)\right]^r \frac{dt}{t} \\
&\geq c \sum_{j=-\infty}^{\infty} (2^{(j+1)\sigma} - 2^{j\sigma}) \left[\min(1,2^{-(j+1)\sigma})\,\overline{\eta}(2^{j\sigma})\right]^r 2^{-(j+1)\sigma} \\
&\geq c \sum_{j=-\infty}^{\infty} \left[\min(1,2^{-j\sigma})\,\overline{\gamma}(2^{j\sigma})\right]^r
\end{aligned}$$

where η is an increasing function in $\mathfrak{B}$ equivalent to γ as above. Thus, the finiteness of $A_r(\phi)$ is an immediate consequence of (2.25) together with the last estimate (with $\sigma = s_0 - s_1$).

The case $q = \infty$ can be proved analogously with the necessary modifications.

Step 3. The proof of the theorem can now be completed from a combination of (2.22) and (2.24), by choosing $0 < r < \min(q_0, q_1, q)$ in the latter:

$$\ell_q^\phi(E) \hookrightarrow (\ell_r^{s_0}(E), \ell_r^{s_1}(E))_{\gamma,q} \hookrightarrow (\ell_{q_0}^{s_0}(E), \ell_{q_1}^{s_1}(E))_{\gamma,q} \hookrightarrow (\ell_\infty^{s_0}(E), \ell_\infty^{s_1}(E))_{\gamma,q} \hookrightarrow \ell_q^\phi(E).$$

□

Following the discussion from Theorem 2.3.3 we are now able to formulate general interpolation results for generalized Besov spaces.

Theorem 2.3.7. *Let $\phi \in \mathcal{B}$, $0 < p \leq \infty$ and $0 < q_0, q_1, q \leq \infty$. Suppose that s_0, $s_1 \in \mathbb{R}$ satisfy $s_1 < \beta_{\overline{\phi}} \leq \alpha_{\overline{\phi}} < s_0$ and γ is given by (2.21). Then*

$$\left(B^{s_0}_{pq_0}(\mathbb{R}^n), B^{s_1}_{pq_1}(\mathbb{R}^n)\right)_{\gamma,q} = B^\phi_{pq}(\mathbb{R}^n) \tag{2.26}$$

Proof. Formula (2.26) follows from a combination of Theorems 2.3.3 (and Remark 2.3.4), 2.3.6 (with $E = h_p(\mathbb{R}^n)$ if $0 < p < \infty$ and $E = L_p(\mathbb{R}^n)$ if $p = \infty$) and Remark 2.2.7. Recall that the retraction and the corresponding co-retraction involved are given by (2.10) and (2.11), respectively. □

Theorem 2.3.7 will play a crucial role in the next chapter. It shows that generalized Besov spaces may be obtained by real interpolation of classical Besov spaces with a suitable function parameter. This fact was already observed in [16] for the spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ mentioned in Subsection 2.1.2.

Theorem 2.3.8. *Let ϕ_0, ϕ_1, $\gamma \in \mathfrak{B}$, $0 < p \leq \infty$ and $0 < q_0$, q_1, $q \leq \infty$. If $0 < \beta_{\overline{\gamma}} \leq \alpha_{\overline{\gamma}} < 1$ and $\beta_{\overline{(\frac{\phi_0}{\phi_1})}} > 0$ (or $\alpha_{\overline{(\frac{\phi_0}{\phi_1})}} < 0$) then*

$$\left(B^{\phi_0}_{pq_0}(\mathbb{R}^n), B^{\phi_1}_{pq_1}(\mathbb{R}^n)\right)_{\gamma,q} = B^{\phi}_{pq}(\mathbb{R}^n), \tag{2.27}$$

where $\phi \in \mathfrak{B}$ is given by

$$\phi(t) := \frac{\phi_0(t)}{\gamma\left(\frac{\phi_0(t)}{\phi_1(t)}\right)}. \tag{2.28}$$

Proof. The proof of the statement $\phi \in \mathfrak{B}$ may be found in [22], Theorem 5.3.

We choose s_0, $s_1 \in \mathbb{R}$ such that

$$s_1 < \min(\beta_{\overline{\phi_0}}, \beta_{\overline{\phi_1}}, \beta_{\overline{\phi}}) \leq \max(\alpha_{\overline{\phi_0}}, \alpha_{\overline{\phi_1}}, \alpha_{\overline{\phi}}) < s_0.$$

By Theorem 2.3.7 we have $B^{\phi_i}_{pq_i}(\mathbb{R}^n) = (B^{s_0}_{p,1}(\mathbb{R}^n), B^{s_1}_{p,1}(\mathbb{R}^n))_{\gamma_i,q_i}$, $i = 0, 1$, with $\gamma_i \in \mathfrak{B}$ defined by (2.21) with ϕ_i in place of ϕ. Now the reiteration theorem (see [94], Theorem 2) yields

$$(B^{\phi_0}_{pq_0}(\mathbb{R}^n), B^{\phi_1}_{pq_1}(\mathbb{R}^n))_{\gamma,q} = (B^{s_0}_{p,1}(\mathbb{R}^n), B^{s_1}_{p,1}(\mathbb{R}^n))_{\rho,q},$$

where

$$\rho(t) = \gamma_0(t)\,\gamma\left(\frac{\gamma_1(t)}{\gamma_0(t)}\right).$$

Direct calculations show that ρ may be expressed in terms of the function ϕ given by (2.28), namely $\rho(t) = \phi\left(t^{\frac{1}{s_0-s_1}}\right)$. Applying Theorem 2.3.7 once again we get

$$(B^{s_0}_{p,1}(\mathbb{R}^n), B^{s_1}_{p,1}(\mathbb{R}^n))_{\rho,q} = B^{\phi}_{pq}(\mathbb{R}^n),$$

which completes the proof. □

Remark 2.3.9. We did not go into details when mentioning the use of the reiteration argument in the proof above. However, it is not hard to check that all the assumptions required in Theorem 2 in [94] are satisfied.

Example 2.3.10. Let s_0, $s_1 \in \mathbb{R}$. If $\phi_0(t) = t^{s_0}$, $\phi_1(t) = t^{s_1}$ and $\gamma(t) = t^{\theta}$ $(0 < \theta < 1)$, then the formula corresponding to (2.27) is

$$(B^{s_0}_{pq_0}(\mathbb{R}^n), B^{s_1}_{pq_1}(\mathbb{R}^n))_{\gamma,q} = B^{s}_{pq}(\mathbb{R}^n),$$

$s := (1-\theta)s_0 + \theta\, s_1,\ s_0 \neq s_1$, which is the statement (i) given in Theorem 2.4.2 in [132]. In fact, we have $\phi(t) = \frac{t^{s_0}}{t^{\theta(s_0-s_1)}} = t^{(1-\theta)s_0+\theta\, s_1} = t^s$, where the restriction $s_0 \neq s_1$ comes from the assumption on the Boyd indices of $\frac{\phi_0}{\phi_1}$.

Further notes

Real interpolation with function parameter was developed by Peetre in [102] in the sixties of the last century. This theory has been also studied by other authors such as Gustavsson [61], Kalugina [74], Merucci [93, 94], and Persson [105, 106]. We dealt with function parameters belonging to the class $\mathfrak{B}$, which was very suitable according to our purposes. However, in a general theory, there are other possibilities for classes where the interpolation parameter may be taken from (see the references above).

Theorem 2.3.8 extends Theorem 5.3 (1) of Cobos and Fernandez [23] to the quasi-Banach case. As we have mentioned before, this was not a mere generalization of the earlier result. This fact was stressed by Cobos and Fernandez when they announced a new paper dealing with the case $0 < p, q \leq \infty$ (see the introduction in [23]). As far as we know, no paper has appeared in that direction.

In this chapter we described interpolation properties of spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ for p fixed. Interpolation spaces with change of p will be investigated later in Section 3.2.

Chapter 3

Wavelet Bases in Generalized Besov Spaces and Applications

Wavelet decompositions have applications to non-linear approximation and to numerical resolution of some partial differential equations. In general, the consideration of wavelets in numerical problems has the advantage of providing fast and efficient algorithms. Nevertheless, our interest here in wavelets comes from their relevance in the theory of function spaces. Wavelet bases give us the possibility of describing the elements of a function space in terms of basic and simple "building blocks". In general, an important point is that we can characterize the original (quasi-)norm by means of certain sums involving the wavelet coefficients. On the other hand, wavelet bases can be quite useful to study some intrinsic questions related to functions spaces. Recently, for example, they were successfully used to estimate entropy numbers of compact embeddings between weighted spaces (see [69] for details).

We refer to the monographs [28], [95], [139] and to the references therein for a detailed discussion on wavelets. We also mention the paper [56], where wavelet decompositions of anisotropic Besov spaces were provided in a multiresolution analysis framework.

Motivated by a recent work of Triebel [137] on wavelet bases in function spaces, we deal with wavelet representations in Besov spaces with generalized smoothness. In [137] it was proved, in particular, that compactly supported wavelets of Daubechies type form an unconditional Schauder basis in the classical Besov spaces $B^s_{pq}(\mathbb{R}^n)$. Our main aim in this chapter is to extend this statement to the spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ studied in Chapter 2, showing that the same wavelet system also provides an *unconditional Schauder basis* in these spaces.

We realize that it is possible to get the result without repeating the approach suggested in [137]. Hence, instead of making use of all those powerful tools (atomic decompositions, local means, maximal functions, duality theory), we try mainly to take advantage of the "classical case" by means of suitable interpolation techniques. We would like to remark that interpolation tools were recently used by Caetano [17] in order to get subatomic representations of Bessel potential spaces modelled on Lorentz spaces from the corresponding ones for the usual spaces $H^s_p(\mathbb{R}^n)$.

The content of this chapter is basically divided into two sections. In the first section we study the wavelet representation of Besov spaces. For completeness, we will contextualize the problem recalling what is already done in the "classical case", and then we formulate our main result as well as some of their consequences. In Section 3.2 we obtain new interpolation results for generalized Besov spaces from an appropriate usage of wavelet representations.

3.1 Wavelet representation for Besov spaces

The aim of this section is to obtain wavelet representations for the generalized Besov spaces under consideration. We will make use of the system considered in [137] and follow the same notation.

Let $L_j = L = 2^n - 1$ if $j \in \mathbb{N}$ and $L_0 = 1$. It is known that, for any $r \in \mathbb{N}$, there are real compactly supported functions[1]

$$\psi_0 \in C^r(\mathbb{R}^n), \quad \psi^l \in C^r(\mathbb{R}^n), \quad l = 1, \dots, L, \tag{3.1}$$

with

$$\int_{\mathbb{R}^n} x^\beta \, \psi^l(x) \, dx = 0, \quad \beta \in \mathbb{N}_0^n, \quad |\beta| \le r, \tag{3.2}$$

such that

$$\left\{ 2^{jn/2} \psi^l_{jm} \, : \, j \in \mathbb{N}_0, \ 1 \le l \le L_j, \ m \in \mathbb{Z}^n \right\} \tag{3.3}$$

with

$$\psi^l_{jm}(\cdot) = \begin{cases} \psi_0(\cdot - m) & , \quad j = 0, \ m \in \mathbb{Z}^n, \ l = 1, \\ \psi^l(2^{j-1} \cdot - m) \, , & j \in \mathbb{N}, \ m \in \mathbb{Z}^n, 1 \le l \le L, \end{cases} \tag{3.4}$$

[1] The function ψ_0 in (3.1) is often referred in the literature as *scaling function* or father wavelet, while functions ψ^l, with $l = 1, \dots, L$, are called (mother) *wavelets*.

is an orthonormal basis in $L_2(\mathbb{R}^n)$.

The existence of such a system was proved by Daubechies [27, 28] in the one-dimensional case. We refer to [95], Theorem 3, p. 96, or to [139], Theorem 4.7, for precise formulations and further details. The extension to the n-dimensional case follows by the standard procedure by means of tensor products.

Nowadays the systems $\{\psi^l_{jm}\}$ described above are known as *Daubechies wavelet bases.* Their properties are sufficiently good to provide unconditional bases in many classical function spaces.

Before passing to the discussion of concrete examples we briefly review important basis notation. Recall that a family $\{x_\lambda\}_{\lambda\in\Lambda}$ in a quasi-Banach space X (with Λ countable) is *summable*, with sum $x \in X$, if for every $\varepsilon > 0$ there exists a finite subset $J_0 \subset \Lambda$ such that $\left\| x - \sum_{\lambda\in J} x_\lambda \right\| < \varepsilon$, whenever J is a finite subset of Λ satisfying $J \supset J_0$. The summability of the family $\{x_\lambda\}_{\lambda\in\Lambda}$ (with sum x) expresses the convergence of the series $\sum_{j=1}^{\infty} x_{\sigma(j)}$ (to x) for every one-to-one and onto map $\sigma : \mathbb{N} \to \Lambda$. This means that we may take any order for the summation without changing the sum.

Definition 3.1.1. *A sequence $\{b_j\}_{j=1}^\infty$ in a (complex) quasi-Banach X is called a Schauder basis if every $x \in X$ admits a unique representation as*

$$x = \sum_{j=1}^{\infty} \mu_j\, b_j, \quad \mu_j \in \mathbb{C} \tag{3.5}$$

(with convergence in X). If, in addition, $\{b_{\sigma(j)}\}_{j=1}^\infty$ is again a Schauder basis for any rearrangement σ of $\mathbb{N}$ (i.e., σ is an one-to-one map of $\mathbb{N}$ onto itself), and

$$x = \sum_{j=1}^{\infty} \mu_{\sigma(j)}\, b_{\sigma(j)}, \tag{3.6}$$

then $\{b_j\}_{j=1}^\infty$ is called an unconditional Schauder basis.

A detailed discussion on unconditional bases within the framework of the Lebesgue spaces may be found in [139].

In the sequel Ψ_r, $r \in \mathbb{N}$, will stand for a Daubechies wavelet system $\{\psi^l_{jm}\}_{(l,j,m)\in I}$ with the properties (3.1)–(3.4) above, where $I = \{(l,j,m) :\ j \in \mathbb{N}_0,\ 1 \le l \le L_j,\ m \in \mathbb{Z}^n\}$. We also consider below $I' = \{(l,j) :\ j \in \mathbb{N}_0,\ 1 \le l \le L_j\}$.

It is known that Ψ_r provides an unconditional Schauder basis in the Bessel potential spaces

$$H^s_p(\mathbb{R}^n) \quad \text{if} \quad 1 < p < \infty, \quad r > |s|,$$

in particular in $L_p(\mathbb{R}^n)$ if $1 < p < \infty$ and $r \in \mathbb{N}$, and in the Besov spaces

$$B^s_{pq}(\mathbb{R}^n) \quad \text{if} \quad 1 \leq p,\, q < \infty, \quad r > |s|$$

(for details see [95], Chapter 6).

These examples show that the smoothness required on the wavelets in (3.1) should be large enough, depending on the regularity of the functions that we pretend to represent. This dependence will be also underlined in next subsections.

3.1.1 The classical case

The main aim in [137] was to extend the assertions above on unconditional bases in Sobolev spaces and some Besov spaces to the entire scales $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. For convenience, we recall here the main statement related to Besov spaces.

Theorem 3.1.2. *Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ and*

$$r(s,p) := \max\left(s, \frac{2n}{p} + \frac{n}{2} - s\right). \tag{3.7}$$

(i) Assume $r \in \mathbb{N}$ with $r > r(s,p)$ and let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{(l,j,m)\in I} \mu^l_{jm}\, \psi^l_{jm} \quad \textit{where} \quad \mu = \{\mu^l_{jm}\}_{(l,j,m)\in I} \in b^s_{pq}, \tag{3.8}$$

with summability in $\mathcal{S}'(\mathbb{R}^n)$ and in any space $B^t_{pu}(\mathbb{R}^n)$ if $t < s$. Moreover, the representation (3.8) is unique:

$$\mu = \mu(f) \quad \textit{with} \quad \mu^l_{jm}(f) := 2^{jn}\langle f,\, \psi^l_{jm}\rangle. \tag{3.9}$$

Furthermore, $f \mapsto \left\{2^{jn}\langle f,\, \psi^l_{jm}\rangle\right\}_{(l,j,m)\in I}$ defines an isomorphic map of $B^s_{pq}(\mathbb{R}^n)$ onto b^s_{pq} and

$$\left\| f \,|\, B^s_{pq}(\mathbb{R}^n)\right\| \sim \left\|\mu(f) \,|\, b^s_{pq}\right\| \tag{3.10}$$

(equivalent quasi-norms).

(ii) In addition, if $\max(p,q) < \infty$, then (3.8) with (3.9) is summable in $B^s_{pq}(\mathbb{R}^n)$ and $\{\psi^l_{jm}\}_{(l,j,m)\in I}$ provides an unconditional Schauder basis in $B^s_{pq}(\mathbb{R}^n)$.

Here b^s_{pq} is the space of all complex-valued "sequences" $\mu \equiv \{\mu^l_{jm}\}_{(l,j,m)\in I}$ such that

$$\|\mu \,|\, b^s_{pq}\| := \left(\sum_{(l,j)\in I'} 2^{j(s-n/p)q} \left(\sum_{m\in\mathbb{Z}^n} |\mu^l_{jm}|^p \right)^{q/p} \right)^{1/q} < \infty, \tag{3.11}$$

with standard modifications if $p = \infty$ and/or $q = \infty$.

The "weight" $2^{j(s-n/p)q}$ expresses some adaptation of the spaces b_{pq} from [135], (13.13), to the set of indices I. In this way, the normalization problem is shifted to the structure of the sequence spaces. Note that the atomic decomposition theorem is an important tool for the proof of Theorem 3.1.2.

Remark 3.1.3. The symbol $\langle f, \psi^l_{jm}\rangle$ in (3.9), with $f \in B^s_{pq}(\mathbb{R}^n)$ and $\psi^l_{jm} \in C^r(\mathbb{R}^n)$, should be properly interpreted since the functions ψ^l_{jm} are not in $\mathcal{S}(\mathbb{R}^n)$ in general. As remarked in [137], it makes sense if $r > -s + \sigma_p$ (with $\sigma_p := \max(n/p - n, 0)$), which is covered by condition $r > r(s,p)$. In fact, in that case, the function f may be interpreted as an element of the dual of a space to which ψ^l_{jm} belongs to. We take the opportunity to show how this can be derived.

The case $1 < p < \infty$ $(\sigma_p = 0)$: Since ψ^l_{jm} has compact support then $\psi^l_{jm} \in W^r_{p'}(\mathbb{R}^n) = F^r_{p',2}(\mathbb{R}^n)$, where p' denotes the usual conjugate exponent. Now we can show that $f \in F^r_{p',2}(\mathbb{R}^n)'$ using embedding arguments (and noting that $s > -r$):

$$B^s_{pq}(\mathbb{R}^n) \hookrightarrow B^{-r}_{p,\min(p,2)}(\mathbb{R}^n) \hookrightarrow F^{-r}_{p,2}(\mathbb{R}^n) = F^r_{p',2}(\mathbb{R}^n)'.$$

The case $p = \infty$ $(\sigma_p = 0)$: Taking into account the embedding $B^s_{\infty q}(\mathbb{R}^n) \hookrightarrow B^s_{\infty\infty}(\mathbb{R}^n)$, the duality $\langle f, \psi^l_{jm}\rangle$ makes sense if $\psi^l_{jm} \in B^{-s}_{1,1}(\mathbb{R}^n)$. But $\psi^l_{jm} \in W^r_1(\mathbb{R}^n) \hookrightarrow B^r_{1,\infty}(\mathbb{R}^n)$, which follows from the fact that the quantity $\sum_{|\beta|\le r} \|D^\beta f \,|\, B^0_{1,\infty}(\mathbb{R}^n)\|$ provides an equivalent norm in $B^r_{1,\infty}(\mathbb{R}^n)$, together with the embedding $L_1(\mathbb{R}^n) \hookrightarrow B^0_{1,\infty}(\mathbb{R}^n)$ (cf. [132], p. 89). It remains to observe that $B^r_{1,\infty}(\mathbb{R}^n) \hookrightarrow B^{-s}_{1,1}(\mathbb{R}^n)$ when $r > -s$.

The case $0 < p \le 1$: The smoothness of ψ^l_{jm} and the compactness of its support allow us to conclude that $\psi^l_{jm} \in \overset{\circ}{C}{}^r(\mathbb{R}^n)$, where $\overset{\circ}{C}{}^r(\mathbb{R}^n)$ denotes the completion of $\mathcal{S}(\mathbb{R}^n)$ in $C^r(\mathbb{R}^n)$. We also have

$$\overset{\circ}{C}{}^r(\mathbb{R}^n)' = B^{-r}_{1,1}(\mathbb{R}^n)$$

(cf. [137]). Hence it remains to observe that

$$B^s_{pq}(\mathbb{R}^n) \hookrightarrow B^{s-\sigma_p}_{1,q}(\mathbb{R}^n) \hookrightarrow B^{-r}_{1,1}(\mathbb{R}^n)$$

when $r > -s + \sigma_p$ (see also [132], Theorem 2.7.1).

The statements in Theorem 3.1.2 have been recently extended to weighted versions of $B-$ and F-spaces (see [69]).

3.1.2 The general case

The proof of Theorem 3.1.2 was based on atomic decompositions, characterizations by local means and duality theory (we refer to [135] and [136] for details on these properties). An important point there was that the Daubechies wavelets were simultaneously atoms and kernels of those local means. In [137] there was also commented the possibility of getting a similar result in the context of other scales of function spaces. To do that, it would be enough to have the same tools available. However, as we mentioned before, we will not follow this approach. Instead, we will consider a scheme based on interpolation techniques in order to take advantage of the already known wavelet expansions for the classical context.

Let us introduce some generalized sequence spaces according to our purposes.

Definition 3.1.4. *Let $\phi \in \mathfrak{B}$, $0 < p \leq \infty$, $0 < q \leq \infty$. The space b^{ϕ}_{pq} consists of all complex-valued sequences $\mu \equiv \{\mu^l_{jm}\}_{(l,j,m)\in I}$ such that*

$$\|\mu \,|\, b^{\phi}_{pq}\| := \left(\sum_{(l,j)\in I'} \left(\phi(2^j)\, 2^{-jn/p}\right)^q \left(\sum_{m\in\mathbb{Z}^n} |\mu^l_{jm}|^p \right)^{q/p} \right)^{1/q} \tag{3.12}$$

(with the usual modifications if $p = \infty$ and/or $q = \infty$) is finite.

It is not hard to see that b^{ϕ}_{pq} is a linear space quasi-normed by (3.12). When $\phi(t) = t^s$, $s \in \mathbb{R}$, then b^{ϕ}_{pq} coincides with the space b^s_{pq} from (3.11). We would like to remark that sequence spaces with this structure were introduced by Frazier and Jawerth [54], [55] in connection with atomic decompositions of (classical) Besov and Triebel-Lizorkin spaces and they have been used afterwards by many authors.

The interpolation property below will be very useful for proving our main result.

Proposition 3.1.5. *Let $\phi \in \mathfrak{B}$ and $0 < p,\, q,\, q_0,\, q_1 \leq \infty$. If s_0, s_1 are real numbers fulfilling $s_1 < \beta_{\overline{\phi}} \leq \alpha_{\overline{\phi}} < s_0$, then we have*

$$(b^{s_0}_{pq_0},\, b^{s_1}_{pq_1})_{\gamma,q} = b^{\phi}_{pq}\,, \tag{3.13}$$

where γ is defined as in (2.21).

Proof. First of all we note that the spaces $b^{s_0}_{pq_0}$ and $b^{s_1}_{pq_1}$ form an interpolation couple since they are continuously embedded in $b^{s_1}_{p\infty}$. We can interpret b^{ϕ}_{pq} as the sequence space $\ell^{\phi_1}_q(\ell_p(\mathbb{Z}^n))$ with $\phi_1(t) := \phi(t)\, t^{-n/p},\ t \in (0,\infty)$, since the main role in the first sum in (3.12) is played by j. Simple modifications show that formula (2.20) (with $E = \ell_p(\mathbb{Z}^n)$) remains valid for these spaces. On the other hand, the lower and upper Boyd indices of ϕ_1 are given by

$$\beta_{\overline{\phi}_1} = \beta_{\overline{\phi}} - \frac{n}{p} \quad \text{and} \quad \alpha_{\overline{\phi}_1} = \alpha_{\overline{\phi}} - \frac{n}{p}.$$

Taking $\sigma_i = s_i - \frac{n}{p}$, $i = 0, 1$, then we get $\sigma_1 < \beta_{\overline{\phi}_1} \le \alpha_{\overline{\phi}_1} < \sigma_0$ and

$$\gamma_1(t) := \frac{t^{\frac{\sigma_0}{\sigma_0 - \sigma_1}}}{\phi_1\left(t^{\frac{1}{\sigma_0-\sigma_1}}\right)} = \gamma(t), \quad t \in (0,\infty),$$

is the function given by (2.21). By formula (2.20) we obtain

$$\left(\ell^{\sigma_0}_{q_0}(\ell_p(\mathbb{Z}^n)),\ \ell^{\sigma_1}_{q_1}(\ell_p(\mathbb{Z}^n))\right)_{\gamma,q} = \ell^{\phi_1}_q(\ell_p(\mathbb{Z}^n))$$

which allows to arrive at (3.13). □

We would like to emphasize that the summability in (3.8) is not an additional assumption, but a consequence of $\mu \in b^s_{pq}$.

Lemma 3.1.6. *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. If $\{\mu^l_{jm}\}_{(l,j,m)\in I} \in b^s_{pq}$ and r is a natural number such that $r > \max(s, \sigma_p - s)$, then $\{\mu^l_{jm}\, \psi^l_{jm}\}_{(l,j,m)\in I}$ is summable in $B^s_{pq}(\mathbb{R}^n)$ if $q < \infty$ and in any $B^t_{pq}(\mathbb{R}^n)$, with $t < s$, if $q = \infty$.*

Proof. First we assume that $q < \infty$. Properties (3.1), (3.2) and (3.4) of the system Ψ_r show that, for each l, j, m, the functions $2^{-j(s-n/p)}\, \psi^l_{jm}$ are normalized 1_r–atoms ($j = 0$) or $(s,p)_{r,r}$–atoms ($j \in \mathbb{N}$) according to [135], Definition 13.3, ignoring constants which are independent of ℓ, j and m.

Let K be an arbitrary finite subset of I. For each l, the function $f_l := \sum\limits_j \sum\limits_m \mu^l_{jm}\, \psi^l_{jm}$ (with the sums running over all indices j and m such that $(l,j,m) \in K$) belongs to $B^s_{pq}(\mathbb{R}^n)$. In fact, we may write

$$f_l = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \widetilde{\mu}^l_{jm}\, \psi^l_{jm} \quad \text{where} \quad \widetilde{\mu}^l_{jm} = \begin{cases} \mu^l_{jm}\,, & \text{if } (l,j,m) \in K \\ 0\ , & \text{otherwise.} \end{cases}$$

On the other hand, $2^{-j(s-n/p)}\,\widetilde{\mu}^l_{jm} \in b_{pq}$ (see (13.13) in [135]). So the "Atomic Decomposition Theorem" (cf. [135], p. 75-76) yields

$$\left\| f_l \,|\, B^s_{pq}(\mathbb{R}^n) \right\|^q \leq c \sum_{j=0}^{\infty} 2^{j(s-n/p)q} \left(\sum_{m \in \mathbb{Z}^n} |\widetilde{\mu}^l_{jm}|^p \right)^{q/p} < \infty$$

with $c > 0$ not depending on l. Taking now the sum over l, we get the estimate

$$\left\| \sum_{(l,j,m)\in K} \mu^l_{jm}\, \psi^l_{jm} \,|\, B^s_{pq}(\mathbb{R}^n) \right\|^q \leq C \sum_{l,j} 2^{j(s-n/p)q} \left(\sum_m |\mu^l_{jm}|^p \right)^{q/p} \tag{3.14}$$

(the sums on the right-hand side run over all indices (l,j) and m such that $(l,j,m) \in K$), where $C > 0$ is independent of K. From this estimate and from the summability of the two families of positive real numbers involved in (3.12), we conclude that the partial sums on the left-hand side of (3.14) constitute a generalized Cauchy sequence in the complete space $B^s_{pq}(\mathbb{R}^n)$, so that it converges in this space.

The summability in $B^t_{p\infty}(\mathbb{R}^n)$ for $t < s$ can be deduced from the previous case. Indeed, one makes use of the atomic decomposition result as before (with t in place of s) and observe that $b^s_{p\infty} \hookrightarrow b^t_{pu}$ if $0 < u \leq \infty$. □

We are now prepared to give a wavelet decomposition statement for spaces $B^\phi_{pq}(\mathbb{R}^n)$ and discuss some of their consequences.

Theorem 3.1.7. *Let $\phi \in \mathfrak{B}$, $0 < p < \infty$ and $0 < q \leq \infty$. Consider the system $\{\psi^l_{jm}\}_{(l,j,m)\in I}$ as before. Then there exists $r(\phi,p)$ such that, for any $r \in \mathbb{N}$ with $r > r(\phi,p)$, the following holds:*
Given $f \in \mathcal{S}'(\mathbb{R}^n)$, then $f \in B^\phi_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{(l,j,m)\in I} \mu^l_{jm}\, \psi^l_{jm} \quad \textit{with} \quad \mu = \{\mu^l_{jm}\}_{(l,j,m)\in I} \in b^\phi_{pq} \tag{3.15}$$

(summability in $\mathcal{S}'(\mathbb{R}^n)$). Moreover, the wavelet coefficients μ^l_{jm} are uniquely determined by

$$\mu^l_{jm} = \mu^l_{jm}(f) := 2^{jn} \langle f, \psi^l_{jm} \rangle, \quad (l,j,m) \in I. \tag{3.16}$$

Further,

$$\left\| f \,|\, B^\phi_{pq}(\mathbb{R}^n) \right\| \sim \left\| \mu(f) \,|\, b^\phi_{pq} \right\| \tag{3.17}$$

(equivalent quasi-norms), where $\mu(f) \equiv \{\mu^l_{jm}(f)\}_{(l,j,m)\in I}$.

Proof. Step 1. Assume that $f \in \mathcal{S}'(\mathbb{R}^n)$ can be represented as

$$f = \sum_{(l,j,m)\in I} \mu^l_{jm}\,\psi^l_{jm} \qquad \text{(summability in } \mathcal{S}'(\mathbb{R}^n))$$

for some $\mu \in b^\phi_{pq}$. Let $s_0,\, s_1 \in \mathbb{R}$. According to Lemma 3.1.6 above we conclude that the operator

$$T:\ b^{s_0}_{p,1} + b^{s_1}_{p,1} \longrightarrow B^{s_0}_{p,1}(\mathbb{R}^n) + B^{s_1}_{p,1}(\mathbb{R}^n),$$

given by

$$T\,\lambda = \sum_{(l,j,m)\in I} \lambda^l_{jm}\,\psi^l_{jm} \qquad \text{(summability in } \mathcal{S}'(\mathbb{R}^n)),$$

is well-defined and linear if, for example, $r > \max(r(s_0,p),\, r(s_1,p))$, where $r(s_i,p)$, $i = 0,1$, is given by (3.7). Moreover, by Theorem 3.1.2 one concludes that the restriction of T to each $b^{s_i}_{p,1}$ is a bounded linear operator into $B^{s_i}_{p,1}(\mathbb{R}^n)$. Choosing $s_0,\, s_1$ above such that $s_1 < \beta_{\overline{\phi}} \le \alpha_{\overline{\phi}} < s_0$ and by the interpolation property, Theorem 2.3.7 and Proposition 3.1.5, we arrive at the conclusion that the restriction of T to b^ϕ_{pq} is also a bounded linear operator into $B^\phi_{pq}(\mathbb{R}^n)$. Thus, $f \in B^\phi_{pq}(\mathbb{R}^n)$ and

$$\|f \,|\, B^\phi_{pq}(\mathbb{R}^n)\| = \|T\,\mu \,|\, B^\phi_{pq}(\mathbb{R}^n)\| \le c\, \|\mu \,|\, b^\phi_{pq}\|$$

for some $c > 0$ independent of μ and f.

Step 2. Now let $f \in B^\phi_{pq}$. Assume that $s_0,\ s_1$ and r fulfill the same conditions as in Step 1. Consider the operator

$$S:\ B^{s_0}_{p,1}(\mathbb{R}^n) + B^{s_1}_{p,1}(\mathbb{R}^n) \longrightarrow b^{s_0}_{p,1} + b^{s_1}_{p,1}$$

defined by

$$S\,g = \mu(g) := \left\{2^{jn}\left(\langle g_0,\, \psi^l_{jm}\rangle + \langle g_1,\, \psi^l_{jm}\rangle\right)\right\}_{(l,j,m)\in I}, \tag{3.18}$$

where $g = g_0 + g_1$ with $g_i \in B^{s_i}_{p,1}(\mathbb{R}^n),\ i = 0,1$. Theorem 3.1.2 (and Remark 3.1.3) shows that S is well-defined, it is linear and its restriction to each $B^{s_i}_{p,1}(\mathbb{R}^n)$ is a bounded linear operator into $b^{s_i}_{p1}$. Taking into account the interpolation property as before, one concludes that the restriction of S to $B^\phi_{pq}(\mathbb{R}^n)$ is a bounded linear operator into b^ϕ_{pq} as well. Therefore,

$$\|S\,f \,|\, b^\phi_{pq}\| = \|\mu(f) \,|\, b^\phi_{pq}\| \le c\, \|f \,|\, B^\phi_{pq}(\mathbb{R}^n)\|, \tag{3.19}$$

where $c > 0$ does not depend on f. So, $\mu(f) \in b^\phi_{pq}$ and hence

$$g := \sum_{(l,j,m)\in I} \mu^l_{jm}(f)\,\psi^l_{jm} \qquad \text{(summability in } \mathcal{S}'(\mathbb{R}^n)) \tag{3.20}$$

belongs to the space $B^{\phi}_{pq}(\mathbb{R}^n)$ by Step 1. But Theorem 3.1.2 once again allows us to conclude that TS is the identity operator, so $g = f$. But we have (by Step 1),

$$\|f \,|\, B^{\phi}_{pq}(\mathbb{R}^n)\| \le c\, \|\mu(f) \,|\, b^{\phi}_{pq}\|, \tag{3.21}$$

$c > 0$ independent of f. Therefore, equivalence (3.17) follows from estimates (3.19) and (3.21). It remains to show that representation (3.15) is unique. We do this next.

Suppose that $f \in B^{\phi}_{pq}(\mathbb{R}^n)$ admits the representation

$$f = \sum_{(l,j,m)\in I} \mu^l_{jm}\, \psi^l_{jm} \quad \text{with} \quad \mu \in b^{\phi}_{pq} \quad (\text{summability in } \mathcal{S}'(\mathbb{R}^n)).$$

Since $B^{\phi}_{pq} \hookrightarrow B^{s_0}_{p,1}(\mathbb{R}^n) + B^{s_1}_{p,1}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p,1}(\mathbb{R}^n)$ and $b^{\phi}_{pq} \hookrightarrow b^{s_0}_{p,1} + b^{s_1}_{p,1} \hookrightarrow b^{s_1}_{p,1}$ (note that $s_0 > s_1$), then $f \in B^{s_1}_{p\,1}(\mathbb{R}^n)$ has the representation

$$f = \sum_{(l,j,m)\in I} \mu^l_{jm}\, \psi^l_{jm} \quad \text{with} \quad \mu \in b^{s_1}_{pq} \quad (\text{summability in } \mathcal{S}'(\mathbb{R}^n)).$$

which is unique by Theorem 3.1.2. The proof of the theorem is completed. □

Remark 3.1.8. We can choose s_0 and s_1 close enough to $\alpha_{\overline{\phi}}$ and $\beta_{\overline{\phi}}$, respectively, and take

$$r(\phi, p) := \max\left(\alpha_{\overline{\phi}},\, \frac{2n}{p} + \frac{n}{2} - \beta_{\overline{\phi}}\right) \tag{3.22}$$

in Theorem 3.1.7. In fact, in that case, it is possible to choose $\varepsilon > 0$ such that r is greater than $\beta_{\overline{\phi}}$, $\frac{2n}{p} + \frac{n}{2} - \beta_{\overline{\phi}} + \varepsilon$, $\alpha_{\overline{\phi}} + \varepsilon$ and $\frac{2n}{p} + \frac{n}{2} - \alpha_{\overline{\phi}}$. On the other hand, one can take s_0 and s_1 such that

$$\beta_{\overline{\phi}} - \varepsilon < s_1 < \beta_{\overline{\phi}} \le \alpha_{\overline{\phi}} < s_0 < \alpha_{\overline{\phi}} + \varepsilon.$$

In this way we have $r > \max\left(r(s_0, p),\, r(s_1, p)\right)$ as required in the proof above. In the particular case $\phi(t) = t^s$ one has $\alpha_{\overline{\phi}} = \beta_{\overline{\phi}} = s$, so that the quantity (3.22) coincides with the corresponding one obtained by Triebel [137].

Remark 3.1.9. We would like to remark also that we did not make a direct use of duality results about the spaces $B^{\phi}_{pq}(\mathbb{R}^n)$. According to (3.18), we have defined the symbol $\langle f,\, \psi^l_{jm}\rangle$ in (3.16) as the sum of two quantities interpreted as in Remark 3.1.3. Of course, taking into account the choice of s_0 and s_1 made above (which implies $B^{\phi}_{pq}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p,1}(\mathbb{R}^n)$), one can take $f_0 = 0$ and $f_1 = f$, so $\langle f,\, \psi^l_{jm}\rangle$ can be evaluated as described in that remark.

Corollary 3.1.10. *Let ϕ, p and q be as in Theorem 3.1.7. If $r \in \mathbb{N}$ is large enough and $q < \infty$, then $\{\psi^l_{jm}\}_{(l,j,m)\in I}$ provides an unconditional Schauder basis in $B^{\phi}_{pq}(\mathbb{R}^n)$.*

Proof. According to Theorem 3.1.7, all we need to do is to check that the family involved in (3.15) is summable in $B^{\phi}_{pq}(\mathbb{R}^n)$ (if p, $q < \infty$). We proceed as in the first part of the proof of Lemma 3.1.6: observe that $\phi(2^j)^{-1}2^{jn/p}\,\psi^l_{jm}$ are 1_r–N–atoms ($l = 1$, $j = 0$) or $(\sigma, p)_{r,r}$–N–atoms ($j \in \mathbb{N}$) according to Definition 4.4.1 in [52], with $\sigma = \{\phi(2^j)\}_{j\in\mathbb{N}_0}$ and $N = \{2^j\}_{j\in\mathbb{N}_0}$. Hence, it is possible to use the “Atomic Decomposition Theorem” from [52], Subsection 4.4.2, in order to get the counterpart of estimate (3.14), that is,

$$\left\| \sum_{(l,j,m)\in K} \mu^l_{jm}\,\psi^l_{jm} \mid B^{\phi}_{pq}(\mathbb{R}^n) \right\|^q \le c \sum_{l,j} \left(\phi(2^j)\,2^{-jn/p}\right)^q \left(\sum_m |\mu^l_{jm}|^p\right)^{q/p}$$

with $c > 0$ independent of K (K being an arbitrary finite subset of I). We also have to assume that $r > r(\phi, p)$ satisfies the conditions mentioned in that theorem restricted to our particular case. We conclude now as in Lemma 3.1.6. □

The wavelet expansion obtained in Theorem 3.1.7 gives the following information about the structure of the spaces $B^{\phi}_{pq}(\mathbb{R}^n)$.

Corollary 3.1.11. *Let ϕ, p, q and r be as in Theorem 3.1.7. Then*

$$\mathcal{I} :\ f \longmapsto \left\{2^{jn}\langle f,\, \psi^l_{jm}\rangle\right\}_{(l,j,m)\in I}$$

establishes a topological isomorphism from $B^{\phi}_{pq}(\mathbb{R}^n)$ onto b^{ϕ}_{pq} (with $\langle f,\, \psi^l_{jm}\rangle$ interpreted as in Remark 3.1.9 above).

Proof. This result follows at once from the properties of the operators T and S studied in the proof of Theorem 3.1.7. □

3.2 Interpolation properties of spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ with change of p

This section may be viewed as a continuation of Section 2.3, where we have discussed interpolation formulas for the spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ with p fixed. Contrarily to that case, the change of p leads to different interpolation spaces, which are not included in the scale given by Definition 2.1.2. This is not a surprise if we have in mind the classical situation corresponding to the smoothness $\phi(t) = t^s$ and to the interpolation function parameter $\gamma(t) = t^\theta$. Indeed, Theorem 2.4.1 in [134] shows that

$$(B^s_{p_0q}(\mathbb{R}^n), B^s_{p_1q}(\mathbb{R}^n))_{\theta,q} = B^s_{pq(q)}(\mathbb{R}^n), \tag{3.23}$$

$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ (with $s \in \mathbb{R}$, $1 < p_0 \neq p_1 < \infty$, $1 \leq q < \infty$ and $0 < \theta < 1$), where $B^s_{pq(q)}(\mathbb{R}^n)$ is the space obtained from (1.10) replacing $L_p(\mathbb{R}^n)$ by the Lorentz space $L_{pq}(\mathbb{R}^n)$.

The counterpart of (3.23) for the generalized spaces $B^\phi_{pq}(\mathbb{R}^n)$ was obtained in [23], leading to the introduction of more general spaces, defined as in Definition 2.1.2 but with some generalized Lorentz space in place of $L_p(\mathbb{R}^n)$.

Only the Banach case was considered in [23] (see Theorem 5.8) and no reference was made concerning the general situation. We realize that the method of retraction and co-retraction described in Section 2.2 will help to deal with the quasi-Banach case. However, the approach followed in Section 2.3 has to be changed, since the replacement of $L_p(\mathbb{R}^n)$ by the local Hardy space $h_p(\mathbb{R}^n)$ would require the knowledge of interpolation results with function parameter for spaces $h_p(\mathbb{R}^n)$ (compare with (3.34) below), which are not available.

Following a suggestion of Caetano, we propose a new approach based on wavelet decompositions given by Theorem 3.1.7. The key point will be the usage of those representations for functions in $B^\phi_{pq}(\mathbb{R}^n)$ to construct a new retraction (and the corresponding co-retraction).

First we need to introduce some auxiliary spaces. Following [22], [94], we consider Lorentz sequence spaces over $\mathbb{Z}^n$ as follows.

Definition 3.2.1. *Let $\varphi \in \mathfrak{B}$ and $0 < q \leq \infty$. The Lorentz space $\lambda_q(\varphi, \mathbb{Z}^n)$ consists of all bounded complex-valued sequences $a \equiv \{a_m\}_{m\in\mathbb{Z}^n}$ having a finite quasi-norm*

$$\|a \,|\, \lambda_q(\varphi, \mathbb{Z}^n)\| := \left(\sum_{k=1}^{\infty} \left[\varphi(k)\, a^*_{k-1}\right]^q \; k^{-1} \right)^{1/q} \quad \textit{if} \quad 0 < q < \infty,$$

$$\|a \,|\, \lambda_q(\varphi, \mathbb{Z}^n)\| := \sup_{k\in\mathbb{N}} \left[\varphi(k)\, a^*_{k-1}\right] \quad \textit{if} \quad q = \infty,$$

where $\{a^*_k\}_{k\in\mathbb{N}_0}$ is the non-increasing rearrangement of $\{a_m\}_{m\in\mathbb{Z}^n}$, given by

$$a^*_k = \inf\{\delta \geq 0 : \; \#\{m \in \mathbb{Z}^n : \; |a_m| > \delta\} \leq k\}, \quad k \in \mathbb{N}_0.$$

If $\varphi(t) = t^{1/p}$ with $0 < p \leq \infty$, then $\lambda_q(\varphi, \mathbb{Z}^n)$ is the Lorentz sequence space $\ell_{pq}(\mathbb{Z}^n)$, which in turn is the classical space $\ell_q(\mathbb{Z}^n)$ of q-summable sequences when $p = q$.

Remark 3.2.2. The normalizing condition $\varphi(1) = 1$ for functions φ belonging to $\mathfrak{B}$ plays only a technical role. There is no problem to consider spaces of the type $\lambda_q(\eta, \mathbb{Z}^n)$ with functions $\eta : (0,\infty) \to (0,\infty)$ for which $\eta(1)^{-1}\eta \in \mathfrak{B}$. In fact, they coincide with $\lambda_q(\eta(1)^{-1}\eta, \mathbb{Z}^n)$, up to equivalence of quasi-norms.

We also need to introduce a more general version of spaces b^{ϕ}_{pq} from Definition 3.1.4.

Definition 3.2.3. *Let $\phi \in \mathfrak{B}$ and $0 < p,\ q,\ \nu \leq \infty$. Let also $\Upsilon \equiv \{\rho_j\}_{j\in\mathbb{N}_0} \subset \mathfrak{B}$. We define $b^{\phi,(\nu)}_{\Upsilon,p,q}$ as the class of all complex-valued "sequences" $\mu = \{\mu^l_{jm}\}_{(l,j,m)\in I}$ such that*

$$\left\| \mu \,|\, b^{\phi,(\nu)}_{\Upsilon,p,q} \right\| := \left(\sum_{(l,j)\in I'} \left[\phi(2^j)\, 2^{-j\frac{n}{\nu}} \left\| \{\mu^l_{jm}\}_{m\in\mathbb{Z}^n} \,|\, \lambda_p(\rho_j, \mathbb{Z}^n) \right\| \right]^q \right)^{1/q} \tag{3.24}$$

(with the usual modifications if $q = \infty$) is finite.

Notice that (3.24) defines a quasi-norm in $b^{\phi,(\nu)}_{\Upsilon,p,q}$.

Example 3.2.4. If $\rho_j(t) = t^{1/p}$ for every $j \in \mathbb{N}_0$ and $\nu = p$, then $b^{\phi,(\nu)}_{\Upsilon,p,q} = b^{\phi}_{pq}$ is the sequence space given by (3.12).

When $p = q$ we shall write only $b^{\phi,(\nu)}_{\Upsilon,q}$ instead of $b^{\phi,(\nu)}_{\Upsilon,q,q}$ for short. This sequence space together with the Daubechies wavelet systems considered in the beginning of Section 3.1 allow us to introduce a generalized space of Besov type as follows.

Let $\Psi_r \equiv \{\psi^l_{jm}\}_{(l,j,m)\in I}$ be a wavelet system with the properties (3.1)–(3.4) (in particular, each wavelet is a real function in $C^r(\mathbb{R}^n)$). If $f \in B^{\phi}_{p_0q}(\mathbb{R}^n) + B^{\phi}_{p_1q}(\mathbb{R}^n)$ (with $\phi \in \mathfrak{B}$, $0 < p_0, p_1, q \leq \infty$), and $f = f_0 + f_1$, $f_i \in B^{\phi}_{p_iq}(\mathbb{R}^n)$, $i = 0, 1$ is any decomposition of f, then the quantities $\langle f, \psi^l_{jm}\rangle$, given by

$$\langle f, \psi^l_{jm}\rangle := \langle f_0, \psi^l_{jm}\rangle + \langle f_1, \psi^l_{jm}\rangle, \quad (l,j,m) \in I, \tag{3.25}$$

with $\langle f_i, \psi^l_{jm}\rangle$ interpreted as in Remark 3.1.9, are well-defined if we take $r \in \mathbb{N}$ large enough, e.g.

$$r > \max\left(r(\phi, p_0), r(\phi, p_1)\right), \tag{3.26}$$

where $r(\phi, p_i)$ is defined by (3.22).

Note that the quantity $\langle f, \psi^l_{jm}\rangle$ does not depend on the decomposition of f taken as the sum $f = f_0 + f_1$, with $f_i \in B^{\phi}_{p_iq}(\mathbb{R}^n)$, $i = 0, 1$.

Definition 3.2.5. *Let $\phi \in \mathfrak{B}$, $0 < p_0, p_1, q, \nu \leq \infty$ and $\Upsilon_{p_0,p_1} \equiv \{\rho_j\}_{j\in\mathbb{N}_0}$ be a family of functions in $\mathfrak{B}$ depending on p_0, p_1. Let also $\Psi_r \equiv \{\psi^l_{jm}\}_{(l,j,m)\in I}$ be a Daubechies wavelet system with the natural number r fixed according to (3.26). We define the space $B^{\phi,(\nu),[\Psi_r]}_{\Upsilon_{p_0,p_1},q}(\mathbb{R}^n)$ as*

$$B^{\phi,(\nu),[\Psi_r]}_{\Upsilon_{p_0,p_1},q}(\mathbb{R}^n) = \left\{ f \in B^{\phi}_{p_0q}(\mathbb{R}^n) + B^{\phi}_{p_1q}(\mathbb{R}^n) : \ \Psi_r\langle f\rangle \in b^{\phi,(\nu)}_{\Upsilon_{p_0,p_1},q} \right\}$$

where in $\Psi_r\langle f\rangle = \{2^{jn}\,\langle f, \psi^l_{jm}\rangle\}_{(l,j,m)\in I}$, $\langle f, \psi^l_{jm}\rangle$ is given by (3.25).

The space $B^{\phi,(\nu),[\Psi_r]}_{\Upsilon_{p_0,p_1},q}(\mathbb{R}^n)$ is naturally quasi-normed by

$$\left\| f \,|\, B^{\phi,(\nu),[\Psi_r]}_{\Upsilon_{p_0,p_1},q}(\mathbb{R}^n) \right\| := \left\| \Psi_r\langle f\rangle \,|\, b^{\phi,(\nu)}_{\Upsilon_{p_0,p_1},q} \right\|. \tag{3.27}$$

Now we are able to present the main statement of this section.

Theorem 3.2.6. *Let $\phi \in \mathfrak{B}$, $0 < p_0 \neq p_1 \leq \infty$, $0 < q < \infty$. Let also $\gamma \in \mathfrak{B}$ with $0 < \beta_{\overline{\gamma}} \leq \alpha_{\overline{\gamma}} < 1$. Then*

$$B^{\phi,(p_0),[\Psi_r]}_{\Upsilon_{p_0,p_1},q}(\mathbb{R}^n) \hookrightarrow (B^{\phi}_{p_0q}(\mathbb{R}^n), B^{\phi}_{p_1q}(\mathbb{R}^n))_{\gamma,q} \hookrightarrow B^{\phi,(p_0),[\Psi_r]}_{\Gamma_{p_0,p_1},q}(\mathbb{R}^n), \tag{3.28}$$

where $\Upsilon_{p_0,p_1} \equiv \{A_{\gamma,p_0,p_1}(j)\,\rho_j\}_{j\in\mathbb{N}_0}$, $\Gamma_{p_0,p_1} \equiv \left\{A_{\gamma,p_0,p_1}(j)^{-1}\,\rho_j\right\}_{j\in\mathbb{N}_0}$, with

$$\rho_j(t) = \frac{t^{\frac{1}{p_0}}}{\gamma\left(2^{-jn(\frac{1}{p_0}-\frac{1}{p_1})}\, t^{\frac{1}{p_0}-\frac{1}{p_1}}\right)}, \quad j \in \mathbb{N}_0, \tag{3.29}$$

and

$$A_{\gamma,p_0,p_1}(j) := \overline{\gamma}\left(2^{-jn(\frac{1}{p_0}-\frac{1}{p_1})}\right)\overline{\gamma}\left(2^{jn(\frac{1}{p_0}-\frac{1}{p_1})}\right), \quad j \in \mathbb{N}_0. \tag{3.30}$$

Proof. Step 1. By Theorem 3.1.7 we may construct a retraction $\mathcal{R}$ from $b^{\phi}_{p_iq}$ into $B^{\phi}_{p_iq}(\mathbb{R}^n)$, $i = 0, 1$, defined by

$$\mathcal{R}\{\mu^l_{jm}\} = \sum_{(l,j,m)\in I} \mu^l_{jm}\,\psi^l_{jm} \quad \text{(summability in } \mathcal{S}'(\mathbb{R}^n)) \tag{3.31}$$

and a corresponding co-retraction $\mathcal{J}$ from $B^{\phi}_{p_iq}(\mathbb{R}^n)$ into $b^{\phi}_{p_iq}$, given by

$$\mathcal{J}f = \{\mu^l_{jm}(f)\} \quad \text{with} \quad \mu^l_{jm}(f) = 2^{jn}\,\langle f, \psi^l_{jm}\rangle, \tag{3.32}$$

where the duality in (3.32) is interpreted as described above. Taking into account Theorem 2.2.6 (and Remark 2.2.7) we get

$$\left\| f \,|\, (B^{\phi}_{p_0q}(\mathbb{R}^n), B^{\phi}_{p_1q}(\mathbb{R}^n))_{\gamma,q} \right\| \sim \left\| \mathcal{J}f \,|\, (b^{\phi}_{p_0q}, b^{\phi}_{p_1q})_{\gamma,q} \right\|. \tag{3.33}$$

Step 2. Let us study the sequence interpolation space in the right-hand side of (3.33). We observe that $\{b^{\phi}_{p_0 q}, b^{\phi}_{p_1 q}\}$ forms an interpolation couple since both spaces are continuously embedded in $\ell_\infty\left(\ell_\infty(\mathbb{Z}^n), 2^{-j\frac{n}{\min(p_0,p_1)}}\phi(2^j)\right)$. Each space $b^{\phi}_{p_i q}$, $i=0,1$, may be interpreted as a suitable weighted sequence space, namely $\ell_q\left(\ell_{p_i}(\mathbb{Z}^n), \omega^i\right)$, where $\omega^i = \{\phi(2^j)\, 2^{-j\frac{n}{p_i}}\}_{(l,j)\in I'}$ (cf. the notation from Chapter 1). By Proposition 3.2 (and Section 5) in [106] we obtain

$$\left(\ell_q\left(\ell_{p_0}(\mathbb{Z}^n), \omega^0\right), \ell_q\left(\ell_{p_1}(\mathbb{Z}^n), \omega^1\right)\right)_{\gamma,q} = \ell_q\left((\ell_{p_0}(\mathbb{Z}^n), \ell_{p_1}(\mathbb{Z}^n))_{\gamma_j,q}, \omega^0_j\right), \tag{3.34}$$

where

$$\gamma_j(t) = \gamma\left(2^{-jn(\frac{1}{p_0}-\frac{1}{p_1})}t\right), \quad j\in\mathbb{N}_0.$$

If we put $\widetilde{\gamma}_j = \frac{\gamma_j}{\gamma_j(1)}$ then $\widetilde{\gamma}_j \in \mathfrak{B}$ for every $j\in\mathbb{N}_0$. Furthermore,

$$(\ell_{p_0}(\mathbb{Z}^n), \ell_{p_1}(\mathbb{Z}^n))_{\gamma_j,q} = \frac{1}{\gamma_j(1)}\,(\ell_{p_0}(\mathbb{Z}^n), \ell_{p_1}(\mathbb{Z}^n))_{\widetilde{\gamma}_j,q}$$

where $\frac{1}{\gamma_j(1)}\,(\ell_{p_0}(\mathbb{Z}^n), \ell_{p_1}(\mathbb{Z}^n))_{\widetilde{\gamma}_j,q}$ denotes the space $(\ell_{p_0}(\mathbb{Z}^n), \ell_{p_1}(\mathbb{Z}^n))_{\widetilde{\gamma}_j,q}$ equipped with the quasi-norm $\frac{1}{\gamma_j(1)}\,\|\cdot\|_{\widetilde{\gamma}_j,q}$. Now Theorem 3 in [94] yields

$$(\ell_{p_0}(\mathbb{Z}^n), \ell_{p_1}(\mathbb{Z}^n))_{\widetilde{\gamma}_j,q} = \lambda_q(\widetilde{\rho}_j, \mathbb{Z}^n), \tag{3.35}$$

where $\lambda_q(\widetilde{\rho}_j, \mathbb{Z}^n)$, $j\in\mathbb{N}_0$, are Lorentz sequence spaces (see Definition 3.2.1) and $\widetilde{\rho}_j = \gamma_j(1)\,\rho_j$, with ρ_j given by

$$\rho_j(t) = \frac{t^{\frac{1}{p_0}}}{\gamma_j\left(t^{\frac{1}{p_0}-\frac{1}{p_1}}\right)}, \quad j\in\mathbb{N}_0. \tag{3.36}$$

Notice that (3.35) may be written in the "non-normalized form" as

$$(\ell_{p_0}(\mathbb{Z}^n), \ell_{p_1}(\mathbb{Z}^n))_{\gamma_j,q} = \lambda_q(\rho_j, \mathbb{Z}^n), \quad j\in\mathbb{N}_0 \tag{3.37}$$

(recall also Remark 3.2.2).

Taking into account formula (3.34), we also need to known how the equivalence constants from (3.37) depend on j. By Proposition 2.2.2, the following statement holds:

$$\overline{\gamma}\left(2^{jn(\frac{1}{p_0}-\frac{1}{p_1})}\right)^{-1}\gamma(t) \le \gamma\left(2^{-jn(\frac{1}{p_0}-\frac{1}{p_1})}t\right) \le \overline{\gamma}\left(2^{-jn(\frac{1}{p_0}-\frac{1}{p_1})}\right)\gamma(t), \quad t>0, \quad j\in\mathbb{N}_0.$$

These inequalities together with (3.37) (in the case $j=0$) give

$$\frac{1}{c}\,\overline{\gamma}\left(2^{-jn(\frac{1}{p_0}-\frac{1}{p_1})}\right)^{-1}\|\cdot\mid\lambda_q(\rho_0,\mathbb{Z}^n)\| \le \|\cdot\|_{\gamma_j,q} \le c\,\overline{\gamma}\left(2^{jn(\frac{1}{p_0}-\frac{1}{p_1})}\right)\|\cdot\mid\lambda_q(\rho_0,\mathbb{Z}^n)\|,$$

with $c \geq 1$ not depending on j. From these estimates we arrive at

$$\frac{1}{c}\,(A_{\gamma,p_0,p_1}(j))^{-1}\,\|\cdot\mid\lambda_q(\rho_j,\mathbb{Z}^n)\| \leq \|\cdot\|_{\gamma_j,q} \leq c\,A_{\gamma,p_0,p_1}(j)\,\|\cdot\mid\lambda_q(\rho_j,\mathbb{Z}^n)\|,\quad j\in\mathbb{N}_0,$$

where $A_{\gamma,p_0,p_1}(j)$ is given by (3.30). Thus

$$\lambda_q(A_{\gamma,p_0,p_1}(j)\,\rho_j,\mathbb{Z}^n) \hookrightarrow (\ell_{p_0}(\mathbb{Z}^n),\ell_{p_1}(\mathbb{Z}^n))_{\gamma_j,q} \hookrightarrow \lambda_q(A_{\gamma,p_0,p_1}(j)^{-1}\,\rho_j,\mathbb{Z}^n),$$

where now the “embedding constants” are independent of j, which leads to

$$b^{\phi,(p_0)}_{\Upsilon_{p_0,p_1},q} \hookrightarrow (b^{\phi}_{p_0q},b^{\phi}_{p_1q})_{\gamma,q} \hookrightarrow b^{\phi,(p_0)}_{\Gamma_{p_0,p_1},q}, \tag{3.38}$$

with $\Upsilon_{p_0,p_1} = \{A_{\gamma,p_0,p_1}(j)\,\rho_j\}_{j\in\mathbb{N}_0}$ and $\Gamma_{p_0,p_1} = \left\{A_{\gamma,p_0,p_1}(j)^{-1}\,\rho_j\right\}_{j\in\mathbb{N}_0}$.

Step 3. Finally, let us derive the interpolation statement (3.28).

The inclusion $(B^{\phi}_{p_0q}(\mathbb{R}^n),B^{\phi}_{p_1q}(\mathbb{R}^n))_{\gamma,q} \subset B^{\phi,(p_0),[\Psi_r]}_{\Gamma_{p_0,p_1},q}(\mathbb{R}^n)$ follows immediately from (3.38). The remaining inclusion in (3.28) may be shown in the following way. If $f \in B^{\phi,(p_0),[\Psi_r]}_{\Upsilon_{p_0,p_1},q}(\mathbb{R}^n)$, then we have $f = f_0 + f_1$ for some $f_i \in B^{\phi}_{p_iq}(\mathbb{R}^n)$, $i = 0,1$, and $\mathcal{J}f \in b^{\phi,(p_0)}_{\Upsilon_{p_0,p_1},q} \subset (b^{\phi}_{p_0q},b^{\phi}_{p_1q})_{\gamma,q}$. Hence, we also have $\mathcal{R}\mathcal{J}f \in (B^{\phi}_{p_0q}(\mathbb{R}^n),B^{\phi}_{p_1q}(\mathbb{R}^n))_{\gamma,q}$. But

$$\mathcal{R}\mathcal{J}f = \mathcal{R}\mathcal{J}f_0 + \mathcal{R}\mathcal{J}f_1 = f_0 + f_1 = f,$$

so that $f \in (B^{\phi}_{p_0q}(\mathbb{R}^n),B^{\phi}_{p_1q}(\mathbb{R}^n))_{\gamma,q}$.

As regards the equivalence of the quasi-norms, it follows from (3.33) combined with definition (3.27). □

Remark 3.2.7. The restriction $p_0 \neq p_1$ arises when we apply Theorem 3 from [94], while the restriction $q < \infty$ comes from Proposition 3.2 in [106].

Remark 3.2.8. The quantities $A_{\gamma,p_0,p_1}(j)$ above cannot be uniformly estimated with respect to j in general (consider, for instance, the functions $\gamma = \phi_{a,b}$ from Example 2.2.1, with $a \in (0,1)$ and $b \neq 0$). All we can say is that $A_{\gamma,p_0,p_1}(j) \in [1,\infty)$, $j \in \mathbb{N}_0$, which follows from the properties of the class $\mathfrak{B}$.

In the classical case $\gamma(t) = t^{\theta}$ $(0 < \theta < 1)$, we have $A_{\gamma,p_0,p_1}(j) \equiv 1$, so that, in this case, Theorem 3.2.6 gives us an exact interpolation formula, that is, the corresponding endpoint spaces in (3.28) coincide (up to equivalence of quasi-norms).

Further notes

We have extended the wavelet representation from [137] to the Besov spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ by real interpolation with function parameter. In this way we avoided the usage of local means and maximal functions (according to [135], Section 2.4.6 and p. 109, formula (55), respectively). The statements in Theorem 3.1.2 were recently generalized to anisotropic spaces in ([138]) by using an anisotropic version of the arguments from [137].

Section 3.2 presents a new possible way to obtain interpolation results based on the construction of retractions from wavelet expansions. In a sense, this is an inverse approach to that one followed in Section 3.1.

Theorem 3.2.6 provides information about the interpolation spaces between spaces $B^{\phi}_{pq}(\mathbb{R}^n)$ in the case when p is changed. Since the sequence $\{A_{\gamma,p_0,p_1}(j)\}_{j\in\mathbb{N}_0}$, defined by (3.30), is not bounded in general, we can only "estimate" the space $(B^{\phi}_{p_0q}(\mathbb{R}^n), B^{\phi}_{p_1q}(\mathbb{R}^n))_{\gamma,q}$ from below and from above. In this sense, our result does not provide a direct generalization of Theorem 5.8 in [23]. Nevertheless, the statement (3.28) will become an exact interpolation formula if we find constants, playing the role of $A_{\gamma,p_0,p_1}(j)$, which could be absolutely estimated (with respect to j). In that case, the corresponding result would give an alternative description of the interpolation space when the range of parameters coincides with the corresponding range in Theorem 5.8 in [23].

In this chapter we discussed interpolation properties and wavelet decompositions for generalized Besov spaces. An interesting question would be to obtain corresponding statements for the generalized Triebel-Lizorkin spaces.

Part II

Riesz and Bessel Potential Spaces of Variable Exponent and Pointwise Inequalities on Variable Sobolev Spaces

Chapter 4

Lebesgue Spaces with Variable Exponent

4.1 A brief historical outline

In this section we give a brief description of the historical background for the variable exponent Lebesgue spaces. We make no attempt at completeness, so that we do not go into many details in general. Further information may be found in the surveys [33], [80] and [122].

Probably, Lebesgue spaces with variable exponent appeared in the literature already in 1931 in an implicit way. In a paper by Orlicz [101], some convergence problem seemed to have been discussed in connection with the Hölder inequality. These spaces appeared later in a book by Nakano [98] as an example illustrating the theory of the so-called modular spaces. We refer to [71], [72] and [97] for a detailed presentation on general modular spaces and further references.

Apparently, Lebesgue spaces of variable exponent (on the real line) were first studied as a special object by Sharapudinov [124]. In this paper, notable efforts to understand these spaces were made, starting with their topological structure and allowing general measurable exponent functions. Sharapudinov studied some other properties of these spaces, such as the existence of unconditional bases ([125]) and the boundedness of convolution operators ([126]).

A remarkable progress in variable exponent spaces was made in the beginning of the 1990's with the paper by Kováčik and Rákosník [85], where several basic properties were established. It should be noted, however, that the basic theory of these spaces has been independently developed by other authors. In this line, we also refer to the papers [42], [44], [51] and [120].

Roughly speaking, Lebesgue spaces with variable exponent are obtained from the classical L_p spaces by allowing the exponent p to vary from point to point. This simple procedure gives rise to some undesired effects. For instance, these "new" spaces, sometimes called *generalized* or *variable Lebesgue spaces*, are no longer translation invariant. At the first glance, the failure of essential properties restricted strongly the techniques available and often used in the context of the classical spaces. Nevertheless, the discovery of the appropriate smooth condition (the *log-Hölder continuity*) for managing varying exponents led to the beginning of a new stage. The boundedness of the maximal operator, proved for the first time by Diening [30], was an important breakthrough as far as operator theory is concerned. From this point, a multitude of results were derived, mainly concerned to the behavior of classical operators coming from harmonic analysis.

The increase of interest to variable Lebesgue and to the corresponding Sobolev spaces may be confirmed by the many papers published during the last years. Among them, we refer to [44], [112], where the denseness of smooth functions was considered, to [31], [34, 35, 36], [43], [41], [81, 83, 84], [118, 119], and to the recent preprints [20], [24], where various results on maximal, potential and singular operators in variable Lebesgue spaces were obtained. We also mention the papers [45], [46], [47], [65], [108] on Sobolev type embeddings, and [38], [53], [66], [67], [68], [78], [113, 114], as well as the recent preprint [32], for further results on the variable exponent framework.

The interest in these spaces is justified not only by theoretical reasons but also by their importance in some applications. Variable exponent spaces arise naturally in modelling problems of fluid dynamics, elasticity theory ([111]) and image restoration ([89]). Moreover, some mathematical models coming from these applications are related to the so-called variational integrals and differential equations with non-standard growth conditions. Nowadays there are many papers showing a strong connection between variable exponent spaces and variational problems of such a type. For instance, we refer to the papers [1, 2, 4], [15], [21], [48], [49], [50], [142], and to the former paper by Zhikov [141], where these problems started to be studied related to the so-called Lavrentiev phenomenon.

4.2 Definition and basic properties

Everywhere below Ω is assumed to be a domain in $\mathbb{R}^n$.

Let $p:\ \Omega \to [1,\infty)$ be a measurable bounded function, called a *variable exponent on* Ω, and denote

$$\overline{p}_\Omega := \operatorname{ess\,sup}_{x\in\Omega} p(x) \quad \text{and} \quad \underline{p}_\Omega := \operatorname{ess\,inf}_{x\in\Omega} p(x).$$

Definition 4.2.1. *The Lebesgue space with variable exponent $L_{p(\cdot)}(\Omega)$ is defined as the space of all measurable functions f on Ω for which the modular*

$$I_{p(\cdot),\Omega}(f) := \int_\Omega |f(x)|^{p(x)} dx$$

is finite.

This is a Banach space endowed with the norm

$$\|f\|_{p(\cdot),\Omega} := \inf \left\{ \lambda > 0 :\ I_{p(\cdot),\Omega}\left(\frac{f}{\lambda}\right) \leq 1 \right\}, \quad f \in L_{p(\cdot)}(\Omega). \tag{4.1}$$

When p is constant then $L_{p(\cdot)}(\Omega)$ coincides with the standard Lebesgue space $L_p(\Omega)$ (according to (1.2) with Ω in place of $\mathbb{R}^n$) and the norms in both spaces are equal. A detailed discussion on the properties of these spaces may be found in the papers [44], [51], [85], [119], [120] and [124]. We also refer to the papers [38] and [99], where discrete analogues of spaces $L_{p(\cdot)}(\Omega)$ were studied.

In order to emphasize that we are dealing with variable exponents, we shall write $p(\cdot)$ instead of p to denote an exponent function. In what follows, the omission of the Ω from the notation means that we are working with $\Omega = \mathbb{R}^n$. For example, we will then only write $\|\cdot\|_{p(\cdot)}$ instead of $\|\cdot\|_{p(\cdot),\mathbb{R}^n}$ to refer to the norm (4.1). We consider this assumption for short, and we will make use of it throughout the text with other notations.

The spaces $L_{p(\cdot)}(\Omega)$ inherit some properties from their classical analogues. In fact, under the additional assumption $\underline{p}_\Omega > 1$, they are uniformly convex, reflexive and its dual space is (isomorphic to) $L_{p'(\cdot)}(\Omega)$, where $p'(\cdot)$ is the natural conjugate exponent given by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} \equiv 1, \quad x \in \Omega. \tag{4.2}$$

Nevertheless, there are some basic properties of the classical Lebesgue spaces which are not transferred to the variable exponent case. For instance, the space $L_{p(\cdot)}(\mathbb{R}^n)$ is no longer translation invariant. As a consequence, Young's theorem and the so called *mean continuity property* fail in general (see [85], [53], [119] for details).

We notice that the assumption $\underline{p}_\Omega > 1$ is quite natural when one deals with the conjugate space of $L_{p(\cdot)}(\Omega)$. It often proves to be necessary within the framework of operator theory in these spaces, starting with the boundedness of the maximal operator (see Section 4.3 below). Hence, in what follows, we always consider exponents $p(\cdot)$ such that

$$1 < \underline{p}_\Omega \leq p(x) \leq \overline{p}_\Omega < \infty, \quad x \in \Omega. \tag{4.3}$$

We will often assume the uniform logarithmic assumption

$$|p(x) - p(y)| \leq \frac{A_0}{\ln \frac{1}{|x-y|}}, \quad x,\, y \in \Omega, \quad |x-y| \leq 1/2, \tag{4.4}$$

on the exponent ($A_0 > 0$ not depending on x). This local condition, usually called *log-Hölder continuity* or *Dini-Lipschitz continuity*, arises naturally when one deals with variable exponents, both as integrability index (see, for instance, [3], [107]) and as variable order of the Hölder condition (cf. [76, 77], [110]).

This assumption allows to prove a multitude of results in variable exponent spaces. Note that (4.4) implies

$$|p(x) - p(y)| \leq \frac{2NA_0}{\ln \frac{2N}{|x-y|}}, \quad x,\, y \in \Omega, \quad |x-y| \leq N, \tag{4.5}$$

where $N \in \mathbb{N}$.

In general, when we deal with unbounded domains, the exponent $p(\cdot)$ is also supposed to have a logarithmic decay at infinity, namely

$$|p(x) - p_\infty| \leq \frac{A_\infty}{\ln(e+|x|)}, \quad x \in \Omega, \tag{4.6}$$

where $p_\infty \geq 1$ and $A_\infty > 0$ are constants independent of x. This assumption was considered in [119] and it is equivalent to the uniform condition

$$|p(x) - p(y)| \leq \frac{C}{\ln(e+|x|)}, \quad x, y \in \Omega, \quad |y| \geq |x|. \tag{4.7}$$

Conditions (4.4) and (4.7) are optimal to get various results on operator theory in the context of the spaces $L_{p(\cdot)}(\Omega)$.

The Sobolev space $W^m_{p(\cdot)}(\Omega)$, $m \in \mathbb{N}_0$, is introduced as the space of all measurable functions f on Ω whose (weak) derivatives $D^\beta f$ up to order m are in $L_{p(\cdot)}(\Omega)$. The norm

$$\|f\|_{m,p(\cdot),\Omega} := \sum_{|\beta|\leq m} \|D^\beta f\|_{p(\cdot),\Omega}, \quad f \in W^m_{p(\cdot)}(\Omega), \tag{4.8}$$

makes $W^m_{p(\cdot)}(\Omega)$ a Banach space.

For completeness, we briefly state some essential properties of variable Lebesgue and Sobolev spaces. Their proofs can be found in the papers mentioned above.

Lemma 4.2.2. *(Hölder inequality) There exists $c = c(p) > 0$ such that*

$$\int_\Omega |f(x)g(x)|\, dx \leq c\, \|f\|_{p(\cdot),\Omega}\, \|g\|_{p'(\cdot),\Omega},$$

for all $f \in L_{p(\cdot)}(\Omega)$ and $g \in L_{p'(\cdot)}(\Omega)$, with $p'(\cdot)$ given by (4.2).

A central property of these spaces is that the convergence in norm is equivalent to the modular convergence.

Lemma 4.2.3. *Let $f \in L_{p(\cdot)}(\Omega)$. Then $\|f\|_{p(\cdot),\Omega} \leq c_1$ if and only if $I_{p(\cdot),\Omega}(f) \leq c_2$. Given $\{f_k\}_{k\in\mathbb{N}_0} \subset L_{p(\cdot)}(\Omega)$, then $\|f_k\|_{p(\cdot),\Omega} \to 0$ if and only if $I_{p(\cdot),\Omega}(f_k) \to 0$, as $k \to \infty$.*

As in the classical case with constant exponents the following embedding holds.

Lemma 4.2.4. *Let $p(\cdot)$ and $q(\cdot)$ be exponent functions such that $p(x) \leq q(x)$ almost everywhere in Ω. If Ω is bounded then $L_{q(\cdot)}(\Omega) \hookrightarrow L_{p(\cdot)}(\Omega)$ with*

$$\|f\|_{p(\cdot),\Omega} \leq (1 + |\Omega|)\, \|f\|_{q(\cdot),\Omega}.$$

Another important tool is the denseness of smooth functions in variable exponent spaces. The class $C_0^\infty(\Omega)$ consisting of all infinitely differentiable functions with compact support in Ω is dense in $L_{p(\cdot)}(\Omega)$. This was shown in [85], Theorem 2.11, together with the first basic properties of these spaces. However, the denseness of this class in the Sobolev spaces $W^m_{p(\cdot)}(\Omega)$ proved to be a more difficult problem and the solution required additional assumptions on the exponent. The following statement was proved by Samko [112], Theorem 3.

Theorem 4.2.5. *The class $C_0^\infty(\mathbb{R}^n)$ is dense in $W^m_{p(\cdot)}(\mathbb{R}^n)$ under the assumption (4.4) on the exponent $p(\cdot)$.*

The proof of this theorem was based on the uniform boundedness of dilation convolution operators. Other denseness results on general domains can be found in [30], [44], [51] and [70].

4.3 The Hardy-Littlewood maximal function in $L_{p(\cdot)}$ spaces

The *maximal function*, $\mathcal{M}_\Omega$, of a locally integrable function f is defined by

$$\mathcal{M}_\Omega f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap\Omega} |f(y)|\, dy, \quad x \in \Omega.$$

This is an important tool in the study of classical operators from harmonic analysis such as potential type operators and singular integral operators. While the behavior of $\mathcal{M}_\Omega$ on standard Lebesgue spaces $L_p(\Omega)$ is well-known, its boundedness on variable exponent spaces $L_{p(\cdot)}(\Omega)$ remained an open problem for a long time. It was first proved by Diening [30] over bounded domains, under the assumption that $p(\cdot)$ is log-Hölder continuous. He later extended the result to the case $\Omega = \mathbb{R}^n$ by supposing, in addition, that the exponent $p(\cdot)$ is constant outside some large fixed ball; see [31].

Diening's result was independently improved by Nekvinda [100] and Cruz-Uribe, Fiorenza and Neugebauer [25] by obtaining the boundedness of $\mathcal{M}_\Omega$ over general unbounded domains Ω, for exponents not necessarily constant at infinity. In the former, some integral condition was imposed, while in the second it was assumed that the exponent has a certain logarithmic decay at infinity. The statement in [25] runs as follows.

Theorem 4.3.1. *If the exponent $p(\cdot)$ satisfies the conditions (4.3), (4.4) and (4.6), then the maximal operator $\mathcal{M}_\Omega$ is bounded in $L_{p(\cdot)}(\Omega)$.*

The logarithmic Hölder continuity assumptions on the exponent are sufficient to derive several results based on the boundedness of the Hardy-Littlewood maximal operator. Furthermore, they are close to necessary. This is why conditions (4.4) and (4.6) are widely used in the theory of $L_{p(\cdot)}(\Omega)$ spaces nowadays. We mention the papers [25] and [107] for concrete counter-examples in $\mathbb{R}^1$.

We notice that weighted boundedness results for the maximal operator were studied in [84].

For simplicity, we shall denote by $\mathcal{P}(\Omega)$ the class of all exponents $p(\cdot)$, $1 < \underline{p}_\Omega \leq \overline{p}_\Omega < \infty$, such that $\mathcal{M}_\Omega$ is bounded in $L^{p(\cdot)}(\Omega)$. Recently, Diening [29] has given a necessary and sufficient condition for the exponent $p(\cdot)$ to be in $\mathcal{P}(\mathbb{R}^n)$ (see [29], Theorem 8.1).

4.4 Convolution operators in variable Lebesgue spaces

Since $L_{p(\cdot)}(\mathbb{R}^n)$ is not invariant with respect to translations, we should expect some problems related to convolutions. In general, the convolution operator does not behave well on these spaces, contrarily to what happens on Lebesgue spaces $L_p(\mathbb{R}^n)$ with constant exponent. Therefore, Young inequality does not hold for variable exponents if we take an arbitrary integrable kernel (see [118, 119] for details). In spite of the failure of this basic property, in [112] it was shown that it is possible to use the mollifier technique within the framework of variable Lebesgue spaces, if one assumes that the exponent satisfies the uniform logarithmic condition (4.4).

The approximation problem via mollifiers is closely related to the boundedness of the maximal operator (cf. [129], Section III.2.2). Diening [30] observed that the Stein theorem on convolutions with radial integrable dominants remains valid for the variable exponent setting. The statement in [30], Corollary 3.6, runs as follows.

Theorem 4.4.1. *Let $\varphi \in L_1(\mathbb{R}^n)$ and define $\varphi_\varepsilon(\cdot) = \varepsilon^{-n}\varphi(\cdot/\varepsilon)$, $\varepsilon > 0$. Suppose that the least decreasing radial majorant of φ is integrable, that is, $A := \int_{\mathbb{R}^n} \sup\limits_{|y|\geq|x|} |\varphi(y)|\, dx < \infty$. Then*
(i) $\sup\limits_{\varepsilon>0} |(f * \varphi_\varepsilon)(x)| \leq 2A\,(\mathcal{M}f)(x),\ f \in L_{p(\cdot)}(\mathbb{R}^n),\ x \in \mathbb{R}^n.$
If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then also
(ii) $\|f * \varphi_\varepsilon\|_{p(\cdot)} \leq c\,\|f\|_{p(\cdot)}$
(with c independent of ε and f) and, if in addition $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$, then
(iii) $f * \varphi_\varepsilon \to f$ *as $\varepsilon \to 0$ in $L_{p(\cdot)}(\mathbb{R}^n)$ and almost everywhere.*

Theorem 4.4.1 is an important tool which allows us to obtain boundedness of various concrete convolution operators, even in the case when they are defined by the Fourier transform of their kernel. Later we shall discuss some examples.

Having in mind further goals, we would like to remind the *Riesz transforms* given by

$$R_j\, f(x) = \lim_{\varepsilon\to 0} c_n \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}}\, f(x-y)\, dy, \qquad j = 1, 2, \ldots, n, \tag{4.9}$$

which are convolution type operators defined in the principal value sense. It is already known that R_j are bounded operators in $L_{p(\cdot)}(\mathbb{R}^n)$ as long as $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. This follows from more general statements given in [24] and [34] on the boundedness of singular integral operators in variable Lebesgue spaces.

Singular operators on spaces $L_{p(\cdot)}$ have been studied also in other papers. We refer to [82], [83], where weighted boundedness results were obtained and to [35], where Calderón-Zygmung singular operators were considered related to the half-space.

4.5 Fractional integrals and fractional maximal operators

The study of *fractional integral operators* on variable Lebesgue spaces is an important topic in our thesis. We recall here their definition.

Definition 4.5.1. *Let $0 < \alpha < n$ and $f \in L_1^{loc}(\Omega)$. The fractional integral of order α, $\mathcal{I}_\Omega^\alpha$, also known as the Riesz potential, is defined by*

$$\mathcal{I}_\Omega^\alpha f(x) := \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

The function f is sometimes called the *density* of the potential $\mathcal{I}_\Omega^\alpha f$, being this designation inspired by physical backgrounds.

Remark 4.5.2. It is also possible to consider Riesz potentials of variable order $\alpha = \alpha(x)$, $x \in \Omega$ (see [118], for instance). In that case, we shall write $\mathcal{I}_\Omega^{\alpha(\cdot)}$ instead of $\mathcal{I}_\Omega^\alpha$. We will make use of these "variable potentials" later, in Chapter 7.

The classical *Sobolev theorem* on fractional integrals says that $\mathcal{I}_\Omega^\alpha$ is a bounded operator from $L_p(\Omega)$ into $L_q(\Omega)$, where q is the *Sobolev exponent* given by $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, with $1 < p < \frac{n}{\alpha}$ (see, for example, [129], Section V.1.2). The boundedness of $\mathcal{I}_\Omega^\alpha$ on Lebesgue spaces with variable exponent was firstly considered by Samko [118], where a conditional result for bounded domains was proved. After Diening has proved the boundedness of the maximal operator, the conditional Sobolev theorem in [118] became an unconditional statement. For further goals, we recall here the statement in [118], which was stated for general potentials of variable order.

Theorem 4.5.3. *Let Ω be a bounded domain. Suppose that $p(\cdot)$ satisfies (4.4) and $\alpha = \alpha(x)$ satisfies the conditions*

$$\operatorname*{ess\,inf}_{x\in\Omega} \alpha(x) > 0, \qquad \operatorname*{ess\,sup}_{x\in\Omega} \alpha(x)p(x) < n, \tag{4.10}$$

and let $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$, $x \in \Omega$. Then there exists $c > 0$ such that

$$\left\| \mathcal{I}_\Omega^{\alpha(\cdot)} f \right\|_{q(\cdot)} \le c\, \|f\|_{p(\cdot)}, \qquad f \in L_{p(\cdot)}(\Omega). \tag{4.11}$$

Diening [31] proved the Sobolev theorem on $\mathbb{R}^n$ for $p(\cdot)$ satisfying the local logarithmic condition (4.4) and being constant at infinity. Some weighted version of the Sobolev theorem on $\mathbb{R}^n$ was obtained by Kokilashvili and Samko [81]. Weighted estimates were also given in [84] for potential type operators of variable order, associated with weighted estimates for the maximal operator. More recently, Capone, Cruz-Uribe and Fiorenza [20] proved the Sobolev theorem on arbitrary domains for exponents $p(\cdot)$ not necessarily constant at infinity. Their statement for the case of the whole space $\mathbb{R}^n$ runs as follows.

Theorem 4.5.4. *Let $0 < \alpha < n$ and let $1 < \underline{p} \leq \overline{p} < \frac{n}{\alpha}$. Assume also that $p(\cdot)$ satisfies the* log*-Hölder conditions (4.4) and (4.6). Then there exists $c > 0$ such that*

$$\|\mathcal{I}^\alpha f\|_{q(\cdot)} \leq c\,\|f\|_{p(\cdot)}, \qquad f \in L_{p(\cdot)}(\mathbb{R}^n), \tag{4.12}$$

where $q(\cdot)$ is the Sobolev exponent given by

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}, \quad x \in \mathbb{R}^n. \tag{4.13}$$

Remark 4.5.5. Theorem 4.5.4 is sometimes referred to in the literature as the "Hardy-Littlewood-Sobolev theorem", since its corresponding version for constant exponents was first proved by Hardy and Littlewood in $\mathbb{R}^1$ and then extended to the multidimensional case by Sobolev.

Closely related to the Riesz potential operator is the *fractional maximal function* defined by

$$\mathcal{M}_\Omega^\lambda f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\lambda}{n}}} \int_{B(x,r)\cap\Omega} |f(y)|\,dy, \quad 0 < \lambda < n, \tag{4.14}$$

where f is a locally integrable function. This is a classical tool in harmonic analysis, being also used to study the behavior of Sobolev functions.

Although the Hardy-Littlewood maximal operator may be regarded as a limiting case of $\mathcal{M}_\Omega^\lambda$ (by taking $\lambda = 0$), their mapping properties on the Lebesgue spaces are quite different. It is known that the behavior of the fractional maximal operator is similar to the behavior of the Riesz potential operator of the same order, in terms of norm inequalities. For classical results on this subject see the monographs [5], [37] and [129].

In Chapter 7 we will use generalized versions of maximal functions by allowing variable orders as well. For future use, we state here a boundedness statement obtained in [81].

Theorem 4.5.6. *Let* Ω *be a bounded domain. Let* $p(\cdot)$ *satisfy (4.4) and* $\lambda(x)$ *satisfy the conditions*

$$\operatorname*{ess\,inf}_{x\in\Omega} \lambda(x) > 0, \qquad \operatorname*{ess\,sup}_{x\in\Omega} \lambda(x)p(x) < n, \tag{4.15}$$

and let $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\lambda(x)}{n}$. *Then*

$$\left\| \mathcal{M}_\Omega^{\lambda(\cdot)} f \right\|_{q(\cdot),\Omega} \leq c \, \|f\|_{p(\cdot),\Omega}, \qquad f \in L_{p(\cdot)}(\Omega). \tag{4.16}$$

Observe that (4.16) follows from the well-known pointwise estimate

$$\mathcal{M}_\Omega^{\lambda(\cdot)} f(x) \leq c \, \mathcal{I}_\Omega^{\lambda(\cdot)}(|f|)(x), \quad x \in \Omega,$$

(see [5], p. 72) and from the Sobolev type Theorem 4.5.3.

The boundedness of the fractional maximal operator on general domains Ω was obtained in [20], Theorem 1.6, under the same assumptions on the exponents $p(\cdot)$ and $q(\cdot)$ as in Theorem 4.5.4, but now over Ω. As a consequence, the classical Sobolev embedding into Lebesgue spaces was extended to the variable exponent setting in the case $\Omega = \mathbb{R}^n$ (see [20], Theorem 1.7). We state it here for completeness.

Theorem 4.5.7. *Suppose that the exponent* $p(\cdot)$ *satisfies the uniform conditions (4.4) and (4.6). If* $1 < \underline{p} \leq \overline{p} < \frac{n}{m}$, *then*

$$W^m_{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{q(\cdot)}(\mathbb{R}^n),$$

where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{m}{n}$, $x \in \mathbb{R}^n$.

This may be proved as in the classical case (see [143], Remark 2.8.6) by making use of the denseness argument given in Theorem 4.2.5 above.

Remark 4.5.8. The first version of the Sobolev embedding theorem on $\mathbb{R}^n$ was obtained by Diening [31] under a stronger condition on $p(\cdot)$ at infinity, namely he assumed that the exponent was constant outside some large ball. In the same paper, he also obtained corresponding embeddings for bounded domains Ω with Lipschitz boundary, assuming only that $p(\cdot)$ is locally log-Hölder continuous. Diening's results generalized earlier statements given in [45], [47] and [85].

Chapter 5

Inversion of the Riesz Potential Operator on Variable Lebesgue Spaces

The behavior of the Riesz potential operator $\mathcal{I}^\alpha$ have been studied by many authors within the framework of the Lesbesgue spaces with variable exponent. Starting with earlier results from [118], several statements were given leading to the generalization of the Sobolev theorem (cf. Theorem 4.5.4). In general, the content of this chapter may be seen as a continuation of the study of the mapping properties of $\mathcal{I}^\alpha$ on $L_{p(\cdot)}$ spaces.

Our main aim is to show that *hypersingular integrals* provide an inverse operator to the Riesz potential operator acting in $L_{p(\cdot)}(\mathbb{R}^n)$. This is a known fact when we deal with densities from the classical Lebesgue spaces $L_p(\mathbb{R}^n)$; see [121] for details and references.

In Section 5.1 we prove the denseness of the Lizorkin space in the variable Lebesgue spaces. We notice that the Lizorkin space is the appropriate class to deal with the Riesz potential operator, since it is invariant with respect to this operator. Section 5.2 is devoted to the consideration of hypersingular integrals of functions in $L_{p(\cdot)}(\mathbb{R}^n)$. We shall formulate the inversion statement in Section 5.3. Finally, we pay some attention to the hypersingular integral operator acting in $L_{p(\cdot)}(\mathbb{R}^n)$ concerning its independence on the order of the finite differences used in its construction.

5.1 Denseness of the Lizorkin class in variable Lebesgue spaces

None of the classes $C_0^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is invariant with respect to Riesz potential operator, since, roughly speaking, the Riesz potential of a nonnegative $C_0^\infty(\mathbb{R}^n)$-function, not equal to zero identically, slowly vanishes at infinity. This is a known fact and the details can be found in [123], p. 493.

The appropriate class with the required property is the so-called *Lizorkin space.*

Definition 5.1.1. *One defines the Lizorkin space* $\Phi(\mathbb{R}^n)$ *via Fourier transform as*

$$\Phi(\mathbb{R}^n) = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : (D^\beta \widehat{\varphi})(0) = 0, \ \ \beta \in \mathbb{N}_0^n\}$$

where D^β *stands for the usual derivative.*

The proof of the following statement may be found in [121], p. 46.

Proposition 5.1.2. *The Lizorkin space* $\Phi(\mathbb{R}^n)$ *is invariant with respect to the Riesz potential operator* $\mathcal{I}^\alpha$. *Moreover,*

$$\mathcal{I}^\alpha \left[\Phi(\mathbb{R}^n)\right] = \Phi(\mathbb{R}^n), \quad 0 < \alpha < n.$$

Now we shall prove that $\Phi(\mathbb{R}^n)$ is dense in $L_{p(\cdot)}(\mathbb{R}^n)$ following ideas from [121], Chapter 2. Let us start with some convolution results.

For $k \in \mathcal{S}(\mathbb{R}^n)$ and $f \in L_{p(\cdot)}(\mathbb{R}^n)$ we put $k_N(x) := N^{-n}\, k(x/N),\ N \in \mathbb{N},\ x \in \mathbb{R}^n$. The convolution $k_N * f$ is then given by

$$k_N * f(x) = \int_{\mathbb{R}^n} k(y)\, f(x - Ny)\, dy, \quad x \in \mathbb{R}^n, \quad N \in \mathbb{N}.$$

In a certain sense, the lemma below can be regarded as an alternative to the Young inequality.

Lemma 5.1.3. *Let* $k \in \mathcal{S}(\mathbb{R}^n)$ *and* $f \in L_{p(\cdot)}(\mathbb{R}^n)$. *If* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, *then*

$$\|k_N * f\|_{p(\cdot)} \leq c\, \|f\|_{p(\cdot)}$$

with $c > 0$ *not depending on* N *and* f.

Proof. Having in mind results from Theorem 4.4.1, we observe that the least decreasing radial majorant of k is integrable. In fact, since $k \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \sup_{|y| \geq |x|} |k(y)|\, dx &= \int_{\mathbb{R}^n} \sup_{|y| \geq |x|} \left(1 + |y|^2\right)^n \frac{|k(y)|}{(1 + |y|^2)^n}\, dx \\ &\leq \int_{\mathbb{R}^n} \left[\sup_{y \in \mathbb{R}^n} \left(1 + |y|^2\right)^n |k(y)|\right] \frac{1}{(1 + |x|^2)^n}\, dx < \infty. \end{aligned}$$

Hence, the statement (*ii*) of 4.4.1 allow us to arrive at the uniform estimate

$$\|k_N * f\|_{p(\cdot)} \leq c\,\|f\|_{p(\cdot)},$$

with respect to N (and f as well). $\square$

Lemma 5.1.4. *Let k, f and $p(\cdot)$ be as in Lemma 5.1.3. Then*

$$\|k_N * f\|_{p(\cdot)} \longrightarrow 0 \quad as \quad N \to \infty.$$

Proof. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L_{p(\cdot)}(\mathbb{R}^n)$, it is sufficient to check the convergence for elements in this class. In fact, taking $\delta > 0$ arbitrary, there exists $\varphi_\delta \in C_0^\infty(\mathbb{R}^n)$, such that $\|f - \varphi_\delta\|_{p(\cdot)} < \delta$. Thus, by Lemma 5.1.3, we have

$$\begin{aligned} \|k_N * f\|_{p(\cdot)} &= \|k_N * (f - \varphi_\delta + \varphi_\delta)\|_{p(\cdot)} \\ &\leq \|k_N * (f - \varphi_\delta)\|_{p(\cdot)} + \|k_N * \varphi_\delta\|_{p(\cdot)} \\ &\leq C\,\|f - \varphi_\delta\|_{p(\cdot)} + \|k_N * \varphi_\delta\|_{p(\cdot)} \\ &< C\,\delta + \|k_N * \varphi_\delta\|_{p(\cdot)} \end{aligned}$$

for all $N \in \mathbb{N}$. Hence, if the statement of the lemma is valid for elements of $C_0^\infty(\mathbb{R}^n)$, we get

$$\lim_{N\to\infty} \|k_N * f\|_{p(\cdot)} \leq C\,\delta,$$

and obtain the desired result in view of the arbitrariness of δ and the independence of the constant C on N.

It remains to justify the passage to the limit for $f \in C_0^\infty(\mathbb{R}^n)$. In this case one has $k_N * f \in \mathcal{S}(\mathbb{R}^n)$ for all N, which implies $k_N * f \in L_{\underline{p}}(\mathbb{R}^n) \cap L_{\overline{p}}(\mathbb{R}^n)$. Moreover,

$$L_{\underline{p}}(\mathbb{R}^n) \cap L_{\overline{p}}(\mathbb{R}^n) \hookrightarrow L_{p(\cdot)}(\mathbb{R}^n)$$

with

$$\begin{aligned} \|k_N * f\|_{p(\cdot)} &\leq \max\Big(\|k_N * f\|_{\underline{p}}\,,\ \|k_N * f\|_{\overline{p}}\Big) \\ &\leq \|k_N * f\|_{\underline{p}} + \|k_N * f\|_{\overline{p}}\,. \end{aligned}$$

At this stage, we may proceed like in [121], p. 42, where the case of constant exponents $p(x) \equiv p$ was treated.

Let $q \in \{\underline{p}, \overline{p}\}$. If $q = 2$ then

$$\|k_N * f\|_2^2 = \int_{\mathbb{R}^n} |F(k_N)(y)\, Ff(y)|^2\, dy \;\to 0 \quad \text{as} \quad N \to \infty, \tag{5.1}$$

which follows from the Parseval identity and from the Lebesgue dominated convergence theorem. The case $q \neq 2$ may be reduced to the previous case as follows. Let $r > 1$ be a number such that q is located between 2 and r. Since $f \in \mathcal{S}(\mathbb{R}^n)$, then we may apply the Hölder inequality and get

$$\|k_N * f\|_q \leq \|k_N * f\|_r^{1-\lambda}\, \|k_N * f\|_2^{\lambda},$$

with $\lambda \in (0,1)$ given by $\lambda = \frac{2}{q}\frac{q-r}{2-r}$. Now we derive the convergence $\|k_N * f\|_q \to 0$ (as $N \to \infty$) in view of the uniform estimate $\|k_N * f\|_r \leq \|k\|_1\, \|f\|_r$ combined with (5.1). □

We are able now to formulate the main statement of this section.

Theorem 5.1.5. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then the Lizorkin class $\Phi(\mathbb{R}^n)$ is dense in the variable Lebesgue space $L_{p(\cdot)}(\mathbb{R}^n)$.*

Proof. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L_{p(\cdot)}(\mathbb{R}^n)$ (see [85]), it is sufficient to approximate each element in $\mathcal{S}(\mathbb{R}^n)$ by elements in $\Phi(\mathbb{R}^n)$, in the norm of $L_{p(\cdot)}(\mathbb{R}^n)$.

Let $f \in \mathcal{S}(\mathbb{R}^n)$. Consider $\mu \in C^\infty([0,\infty))$, such that $\mu(r) \equiv 1$ for $r \geq 2$, $\mu(r) \equiv 0$ for $0 \leq r \leq 1$ and $0 \leq \mu(r) \leq 1$. We put

$$\psi_N(x) := \mu(N|x|)(F^{-1}f)(x), \quad x \in \mathbb{R}^n, \quad N \in \mathbb{N}.$$

Then we have $\psi_N \in \mathcal{S}(\mathbb{R}^n)$ with $\psi_N(x) \equiv 0$ when $|x| \leq 1/N$. If we define $f_N := F\psi_N$ then $f_N \in \Phi(\mathbb{R}^n)$. Moreover, considering the notations $v(\cdot) := \mu(|\cdot|)$ and $v^N(\cdot) := v(N\cdot)$, and making use of the properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$, we successively have

$$\begin{aligned}
f(x) - (2\pi)^{-n}\int_{\mathbb{R}^n} F[1-v](y)f(x-Ny)\,dy &= f(x) - (2\pi)^{-n}\int_{\mathbb{R}^n} N^{-n}F[1-v](z/N)f(x-z)\,dz \\
&= f(x) - (2\pi)^{-n}\int_{\mathbb{R}^n} F[1-v^N](z)f(x-z)\,dz \\
&= f(x) - (2\pi)^{-n}F\left((2\pi)^n[1-v^N]\cdot F^{-1}f\right)(x) \\
&= f(x) - F\,F^{-1}f(x) + F\left(v^N\cdot F^{-1}f\right)(x) \\
&= f_N(x).
\end{aligned}$$

Taking the kernel $k(y) = (2\pi)^{-n}\mathcal{F}[1-v](y)$ in Lemma 5.1.4, one obtains

$$\lim_{N\to\infty} \|f - f_N\|_{p(\cdot)} \;=\; \lim_{N\to\infty} \|k_N * f\|_{p(\cdot)} \;=\; 0,$$

which completes the proof. $\square$

5.2 Hypersingular integrals on $L_{p(\cdot)}$ spaces

An important fact concerning hypersingular integrals is their application to the inversion of potential-type operators. There are many papers on this subject, but we only refer to the monographs [121] and [123], where several references and historical remarks may be found.

A typical hypersingular integral has the form

$$\frac{1}{d_{n,\ell}(\alpha)} \int_{\mathbb{R}^n} \frac{\left(\Delta_y^\ell f\right)(x)}{|y|^{n+\alpha}}\, dy, \qquad \alpha > 0, \tag{5.2}$$

where $\Delta_y^\ell f$ denotes the non-centered finite difference of order $\ell \in \mathbb{N}$ of the function f,

$$(\Delta_y^\ell f)(x) := \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} f(x - ky), \quad y \in \mathbb{R}^n, \quad \ell \in \mathbb{N}, \tag{5.3}$$

and $d_{n,\ell}(\alpha)$ is a certain normalizing constant, which is chosen so that the construction in (5.2) does not depend on ℓ. The precise value of $d_{n,\ell}(\alpha)$ is not important for our purposes. We refer to [121], Chapter 3, for full details concerning the normalizing constants.

It is known that the integral (5.2) exists (for each $x \in \mathbb{R}^n$), for instance, for functions $f \in \mathcal{S}(\mathbb{R}^n)$ with $\ell > \alpha$. In fact, in that case, from the Taylor's formula with the remainder in the integral form (see Lemmma 3.7 in [121]), we derive $|(\Delta_y^\ell f)(x)| \le C(\ell, f)\, |y|^\ell$. Hence,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\left|\left(\Delta_y^\ell f\right)(x)\right|}{|y|^{n+\alpha}}\, dy &= \int_{|y|\le 1} \frac{\left|\left(\Delta_y^\ell f\right)(x)\right|}{|y|^{n+\alpha}}\, dy + \int_{|y|>1} \frac{\left|\left(\Delta_y^\ell f\right)(x)\right|}{|y|^{n+\alpha}}\, dy \\ &\le C(\ell, f) \int_{|y|\le 1} \frac{dy}{|y|^{n+\alpha-\ell}} + C(\ell)\, \|f\|_\infty \int_{|y|>1} \frac{dy}{|y|^{n+\alpha}} \\ &< \infty. \end{aligned}$$

Remark 5.2.1. For simplicity, in this chapter we consider only non-centered finite differences in the construction of the hypersingular integral. Nevertheless, centered differences are also admitted if we introduce necessary modifications. The important fact here is that the order ℓ should be chosen according to the following rule (as stated in [121], p. 65), which will be always assumed from now on:

1) *in the case of a non-centered difference we take* $\ell > 2\left[\frac{\alpha}{2}\right]$ *with the obligatory choice* $\ell = \alpha$ *for* α *odd;*

2) *in the case of a centered difference we take* ℓ *even and* $\ell > \alpha > 0$.

We remind that the centered differences are defined as in (5.3), but with $x - ky$ replaced by $x + \left(\frac{\ell}{2} - k\right) y$ in the right-hand side. In Chapter 6 we shall make use of centered differences in an explicit way.

In general, the integral in (5.2) may be divergent, and hence it needs to be properly interpreted. Let us start with the following definition.

Definition 5.2.2. *Let* $f \in L_1^{loc}(\mathbb{R}^n)$ *and* $\alpha > 0$. *The truncated hypersingular integral operator of order* α *is given by*

$$\mathbb{D}^\alpha_{\ell,\varepsilon} f(x) = \frac{1}{d_{n,\ell}(\alpha)} \int_{|y|>\varepsilon} \frac{\left(\Delta^\ell_y f\right)(x)}{|y|^{n+\alpha}}\, dy\,, \qquad \varepsilon > 0. \tag{5.4}$$

The operators $\mathbb{D}^\alpha_{\ell,\varepsilon}$ behave on $L_{p(\cdot)}(\mathbb{R}^n)$ as follows.

Proposition 5.2.3. *If* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, *then the truncated hypersingular integral operator* $\mathbb{D}^\alpha_{\ell,\varepsilon}$ *is bounded in* $L_{p(\cdot)}(\mathbb{R}^n)$, *for every* $\varepsilon > 0$.

Proof. We have

$$\begin{aligned} \left|\mathbb{D}^\alpha_{\ell,\varepsilon} f(x)\right| &\leq \frac{1}{d_{n,\ell}(\alpha)} \int_{|y|>\varepsilon} \frac{\left|\left(\Delta^\ell_y f\right)(x)\right|}{|y|^{n+\alpha}}\, dy \\ &\leq C(\alpha,\ell) \left(\int_{|y|>\varepsilon} \frac{|f(x)|}{|y|^{n+\alpha}}\, dy + \sum_{k=1}^{\ell} \binom{\ell}{k} \int_{|y|>\varepsilon} \frac{|f(x-ky)|}{|y|^{n+\alpha}}\, dy \right) \\ &= C(\alpha,\ell) \left(C(\varepsilon)\, |f(x)| + \sum_{k=1}^{\ell} \binom{\ell}{k} Q_k f(x) \right) \end{aligned}$$

where the convolutions

$$Q_k f(x) := \int_{|y|>\varepsilon} \frac{|f(x-ky)|}{|y|^{n+\alpha}}\, dy = k^\alpha \int_{|z|>k\varepsilon} \frac{|f(x-z)|}{|z|^{n+\alpha}}\, dz$$

generate bounded operators in the space $L_{p(\cdot)}(\mathbb{R}^n)$ by Theorem 4.4.1 (they are convolution operators with radial decreasing integrable kernels).

To estimate $\|\mathbb{D}^\alpha_{\ell,\varepsilon} f\|_{p(\cdot)}$, it suffices to evaluate $I_{p(\cdot)}(\mathbb{D}^\alpha_{\ell,\varepsilon} f)$ assuming that $\|f\|_{p(\cdot)} \leq 1$. Taking estimate (i) of Theorem 4.4.1 into account, we obtain

$$\begin{aligned} I_{p(\cdot)}\left(\mathbb{D}^\alpha_{\ell,\varepsilon} f\right) &\leq C(\alpha,\ell)\int_{\mathbb{R}^n}\left(C(\varepsilon)|f(x)|+\sum_{k=1}^{\ell}\binom{\ell}{k}C(\varepsilon,\alpha)\,\mathcal{M}f(x)\right)^{p(x)} dx \\ &\leq C(\varepsilon,\alpha,\ell)\int_{\mathbb{R}^n}\left(|f(x)|^{p(x)}+(\mathcal{M}f(x))^{p(x)}\right)dx \\ &= C(\varepsilon,\alpha,\ell)\left(I_{p(\cdot)}(f)+I_{p(\cdot)}(\mathcal{M}f)\right) \\ &\leq C(\varepsilon,\alpha,\ell), \end{aligned}$$

where the value of $C(\varepsilon,\alpha,\ell)$ may change in the chain. Recall that $\mathcal{M}f \in L_{p(\cdot)}(\mathbb{R}^n)$ because we assume $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$.

Now, since $\|f\|_{p(\cdot)}\leq 1$ implies $I_{p(\cdot)}(f)\leq 1$, then $I_{p(\cdot)}(\mathcal{M}f)\leq C$ for some constant $C>1$. Therefore $I_{p(\cdot)}\left(\frac{\mathbb{D}^\alpha_{\ell,\varepsilon}f}{C(\varepsilon,\alpha,\ell)}\right)\leq 1$, which allows us to arrive at the inequality

$$\left\|\mathbb{D}^\alpha_{\ell,\varepsilon}f\right\|_{p(\cdot)}\leq C(\varepsilon,\alpha,\ell) \qquad \text{(see (4.1))}.$$

□

Remark 5.2.4. When $p(x)\equiv p$ is constant, the proof is easier. In fact, in that case, it suffices to apply Minkowski inequality and then to make use of the property $\left\|\Delta^\ell_y f\right\|_p \leq c\,\|f\|_p$, where $c>0$ does not depend on y.

Remark 5.2.5. From Proposition 5.2.3 and Theorem 4.4.1 one concludes that the compositions $\mathbb{D}^\alpha_{\ell,\varepsilon}\mathcal{I}^\alpha$ are well defined on the space $L_{p(\cdot)}(\mathbb{R}^n)$ when $\overline{p}<\frac{n}{\alpha}$.

The *hypersingular operators* we are interested in are operators defined by a suitable limiting process. More precisely, we intend to deal with operators of the form

$$\mathbb{D}^\alpha_\ell f := \lim_{\varepsilon\to 0}\mathbb{D}^\alpha_{\ell,\varepsilon}f, \qquad \alpha>0. \tag{5.5}$$

According to our purposes, the limit in (5.5) will be taken in the norm of $L_{p(\cdot)}(\mathbb{R}^n)$. This makes sense in view of the Proposition 5.2.3 above.

We recall that the order ℓ of the finite differences is chosen taking into account the rule from Remark 5.2.1, so that "many" possibilities are allowed. However, it is possible to ensure the independence of the hypersingular integral on ℓ. We will return to this question in Section 5.4.

Sometimes $\mathbb{D}^\alpha_\ell$ is also called the *Riesz fractional derivative* since it can be interpreted as a certain positive fractional power of the Laplacian operator (see Section 5.3 below).

5.3 The inversion theorem

The Riesz potential operator $\mathcal{I}^\alpha$ may be seen as a negative fractional power of the Laplacian, namely

$$\mathcal{I}^\alpha f = (-\Delta)^{-\frac{\alpha}{2}} f, \quad 0 < \alpha < n, \tag{5.6}$$

which can be shown for sufficiently smooth functions f from basic manipulations with Fourier transforms (cf. [129], p. 117). On the other hand, the Riesz derivative $\mathbb{D}^\alpha_\ell$ realizes the positive fractional powers of the same differential operator,

$$\mathbb{D}^\alpha_\ell f = (-\Delta)^{\frac{\alpha}{2}} f, \quad \alpha > 0, \tag{5.7}$$

as is shown in [121], p. 57, for functions $f \in \Phi(\mathbb{R}^n)$. Identities (5.6) and (5.7) suggest that the hypersingular operator generates an inverse operator to the Riesz potential operator $\mathcal{I}^\alpha$, being it clear over the Lizorkin class $\Phi(\mathbb{R}^n)$. However, this statement holds also on the whole domain of $\mathcal{I}^\alpha$ within the framework of the standard spaces $L_p(\mathbb{R}^n)$ (see [121] for details). Our aim here is to extend the inversion results from [121] to the Lebesgue spaces of variable exponent.

Before dealing with the inversion problem we need to introduce some basic notation.

Definition 5.3.1. *We define the Riesz kernel of order α, $0 < \alpha < n$, as*

$$k_\alpha(x) := \frac{1}{\gamma_n(\alpha)} |x|^{\alpha - n},$$

where $\gamma_n(\alpha) = \frac{2^\alpha \pi^{n/2} \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}$ is the well-known normalizing constant.

We make use of the kernels $k_{\ell,\alpha}$ and $\mathcal{K}_{\ell,\alpha}$ as in [121], Section 3.2:

$$k_{\ell,\alpha}(x) := \left(\Delta^\ell_{e_1} k_\alpha\right)(x) = \frac{1}{\gamma_n(\alpha)} \sum_{r=0}^{\ell} (-1)^r \binom{\ell}{r} |x - r\, e_1|^{\alpha - n},$$

where $e_1 = (1, 0, \ldots, 0)$, and

$$\mathcal{K}_{\ell,\alpha}(x) := \frac{1}{d_{n,\ell}(\alpha)|x|^n} \int_{|y|<|x|} k_{\ell,\alpha}(y)\, dy. \tag{5.8}$$

The kernel (5.8) has the following property:

Lemma 5.3.2. *Let $0 < \alpha < n$. Then there exists $C > 0$, such that*

$$|\mathcal{K}_{\ell,\alpha}(x)| \le C \begin{cases} |x|^{\alpha - n}, & |x| \le 1 \\ |x|^{\alpha - n - \ell^*}, & |x| > 1 \end{cases}$$

where $\ell^* = \ell + 1$ if ℓ is odd and $\ell^* = \ell$ otherwise.

The proof is essentially technic and it may be found in [121], p. 68.

A key step in the inversion of the operator $\mathcal{I}^\alpha$ on classic spaces $L_p(\mathbb{R}^n)$ is the following representation formula for potentials:

$$\mathbb{D}^\alpha_{\ell,\varepsilon} f(x) = \mathcal{K}^\varepsilon_{\ell,\alpha} * \varphi(x), \quad x \in \mathbb{R}^n, \quad \varepsilon > 0, \tag{5.9}$$

where $\mathcal{K}^\varepsilon_{\ell,\alpha}(x) = \frac{1}{\varepsilon^n}\,\mathcal{K}_{\ell,\alpha}\left(\frac{x}{\varepsilon}\right)$ and $f = \mathcal{I}^\alpha\varphi$, with $\varphi \in L_p(\mathbb{R}^n)$, $1 \le p < n/\alpha$ and $0 < \alpha < n$.

First of all, we intend to obtain this kind of representation but now in the context of the spaces of variable integrability. We cannot extend this identity to the spaces $L_{p(\cdot)}(\mathbb{R}^n)$ directly, but its validity on $L_{p(\cdot)}(\mathbb{R}^n)$ may be obtained by considering the linear sum $L_{\overline{p}}(\mathbb{R}^n) + L_{\underline{p}}(\mathbb{R}^n)$ as follows.

Theorem 5.3.3. *Let $0 < \alpha < n$ and $1 < \underline{p} \le \overline{p} < \frac{n}{\alpha}$. Then for any $\varphi \in L_{\overline{p}}(\mathbb{R}^n) + L_{\underline{p}}(\mathbb{R}^n)$, and consequently for any $\varphi \in L_{p(\cdot)}(\mathbb{R}^n)$, the following representation formula holds:*

$$\mathbb{D}^\alpha_{\ell,\varepsilon}\mathcal{I}^\alpha\varphi(x) = \mathbb{K}^\varepsilon_{\ell,\alpha}\varphi(x), \quad x \in \mathbb{R}^n, \quad \varepsilon > 0, \tag{5.10}$$

where $\mathbb{K}^\varepsilon_{\ell,\alpha}\varphi := \mathcal{K}^\varepsilon_{\ell,\alpha} * \varphi$.

Proof. Let $\varphi \in L_{p(\cdot)}(\mathbb{R}^n)$ and $\varepsilon > 0$ be fixed. Recall that

$$\mathbb{D}^\alpha_{\ell,\varepsilon}\mathcal{I}^\alpha\varphi(x) = \frac{1}{d_{n,\ell}(\alpha)} \int_{|y|>\varepsilon} \frac{1}{|y|^{n+\alpha}} \left(\triangle^\ell_y\left(\mathcal{I}^\alpha\varphi\right)\right)(x)\, dy$$

and

$$\mathbb{K}^\varepsilon_{\ell,\alpha}\varphi(x) = \int_{\mathbb{R}^n} \varphi(x-y)\,\mathcal{K}^\varepsilon_{\ell,\alpha}(y)\, dy. \tag{5.11}$$

From (5.9) we have $\mathbb{D}^\alpha_\varepsilon\mathcal{I}^\alpha : L_{\underline{p}}(\mathbb{R}^n) \longrightarrow L_{\underline{p}}(\mathbb{R}^n)$ and $\mathbb{D}^\alpha_\varepsilon\mathcal{I}^\alpha : L_{\overline{p}}(\mathbb{R}^n) \longrightarrow L_{\overline{p}}(\mathbb{R}^n)$. Hence, we can define $\mathbb{D}^\alpha_\varepsilon\mathcal{I}^\alpha$ on $L_{\underline{p}}(\mathbb{R}^n) + L_{\overline{p}}(\mathbb{R}^n)$ in the natural way. On the other hand,

$$L_{p(\cdot)}(\mathbb{R}^n) \subset L_{\overline{p}}(\mathbb{R}^n) + L_{\underline{p}}(\mathbb{R}^n),$$

since we can split φ into $\varphi = \varphi_0 + \varphi_1$, with $\varphi_0 \in L_{\underline{p}}(\mathbb{R}^n)$ and $\varphi_1 \in L_{\overline{p}}(\mathbb{R}^n)$, where

$$\varphi_0(x) = \begin{cases} \varphi(x)\,, & |\varphi(x)| > 1 \\ 0\,, & \text{otherwise.} \end{cases}$$

Making use of representation (5.9) for each term, we get

$$\mathbb{D}^\alpha_{\ell,\varepsilon}\mathcal{I}^\alpha\varphi = \mathbb{D}^\alpha_{\ell,\varepsilon}\mathcal{I}^\alpha\varphi_0 + \mathbb{D}^\alpha_{\ell,\varepsilon}\mathcal{I}^\alpha\varphi_1 = \mathbb{K}^\varepsilon_{\ell,\alpha}\varphi_0 + \mathbb{K}^\varepsilon_{\ell,\alpha}\varphi_1 = \mathbb{K}^\varepsilon_{\ell,\alpha}\varphi.$$

$\square$

Corollary 5.3.4. *Let $0<\alpha<n$ and $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ with $\overline{p}<\frac{n}{\alpha}$. Then the compositions $\mathbb{D}^\alpha_{\ell,\varepsilon}\mathcal{I}^\alpha$ are uniformly bounded in $L_{p(\cdot)}(\mathbb{R}^n)$ with respect to ε, that is,*

$$\left\|\mathbb{D}^\alpha_{\ell,\varepsilon}\mathcal{I}^\alpha\varphi\right\|_{p(\cdot)}\le c\;\|\varphi\|_{p(\cdot)}\,,\quad \forall\varphi\in L_{p(\cdot)}(\mathbb{R}^n),$$

where $c>0$ does not depend on $\varepsilon>0$.

Proof. The proof follows from (5.10) and from the uniform estimate given in statement (ii) of Theorem 4.4.1. Note that the kernel $\mathcal{K}_{\ell,\alpha}$ has an integrable decreasing radial majorant in view of the bounds given in Lemma 5.3.2. □

The inversion result below is basically a consequence of the arguments discussed above.

Theorem 5.3.5. *(Inversion theorem) Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $\overline{p}<\frac{n}{\alpha}$. Then*

$$\mathbb{D}^\alpha_\ell\mathcal{I}^\alpha\varphi=\varphi,\qquad \varphi\in L_{p(\cdot)}(\mathbb{R}^n),$$

where the hypersingular operator $\mathbb{D}^\alpha_\ell$ is taken in the sense of convergence in $L_{p(\cdot)}(\mathbb{R}^n)$.

Proof. Since the least decreasing radial majorant of $\mathcal{K}_{\ell,\alpha}$ is integrable (cf. Lemma 5.3.2) and $\mathcal{K}_{\ell,\alpha}$ has the averaging property $\int_{\mathbb{R}^n}\mathcal{K}_{\ell,\alpha}(x)\,dx=1$ (which is a consequence of the choice of the normalizing constant), then we can pass to the limit in (5.10) in the norm $\|\cdot\|_{p(\cdot)}$ (see (iii) of Theorem 4.4.1), and write

$$\lim_{\varepsilon\to0}\mathbb{D}^\alpha_{\ell,\varepsilon}\mathcal{I}^\alpha\varphi=\lim_{\varepsilon\to0}\mathcal{K}^\varepsilon_{\ell,\alpha}*\varphi=\varphi.$$

□

Remark 5.3.6. From Theorem 4.4.1, we may conclude that the convergence holds almost everywhere as well, under the same conditions:

$$\lim_{\varepsilon\to0}\mathbb{D}^\alpha_{\ell,\varepsilon}\mathcal{I}^\alpha\varphi(x)=\varphi(x),$$

for almost all $x\in\mathbb{R}^n$.

5.4 On the independence of the hypersingular integral on the order of finite differences

Let us return to the independence problem introduced at the end of Section 5.2. As discussed there, any order of finite difference is admissible in the sense of the rule stated in

Remark 5.2.1. We intend to prove the independence of $\mathbb{D}^\alpha_\ell f$ with respect to ℓ within the framework of the variable exponent Lebesgue spaces.

Let us start with the following identity, which is the analogue of Lemma 3.26 in [121].

Lemma 5.4.1. *Let $\varepsilon > 0$, $0 < \alpha < n$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then for any $f \in L_{p(\cdot)}(\mathbb{R}^n)$,*

$$\mathbb{K}^\varepsilon_{\ell_0,\alpha}\mathbb{D}^\alpha_{\ell_1,\varepsilon} f = \mathbb{K}^\varepsilon_{\ell_1,\alpha}\mathbb{D}^\alpha_{\ell_0,\varepsilon} f, \tag{5.12}$$

where $\mathbb{K}^\varepsilon_{\ell_j,\alpha}$ is the convolution operator (5.11), $j = 0, 1$.

Proof. The proof is basically the same as in [121], Lemma 3.26. For completeness, we briefly point out the main steps.

First, one obtains (5.12) for f in the Lizorkin space $\Phi(\mathbb{R}^n)$. For this, we note that we may write $f = \mathcal{I}^\alpha\varphi$ for some $\varphi \in \Phi(\mathbb{R}^n)$ (see Proposition 5.1.2), then apply equality (5.10) and use properties of the Fourier transform on convolutions. After that we extend the equality to the whole space $L_{p(\cdot)}(\mathbb{R}^n)$ by continuity. Note that $\Phi(\mathbb{R}^n)$ is dense in $L_{p(\cdot)}(\mathbb{R}^n)$ because $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, by Theorem 5.1.5. Moreover, the operators involved in (5.12) are bounded: boundedness of $\mathbb{K}^\varepsilon_{\ell_j,\alpha}$ follows from statement (ii) of Theorem 4.4.1, while the boundedness of $\mathbb{D}^\alpha_{\ell_j,\varepsilon}$ was proved in Proposition 5.2.3. □

We are able to deal with the independence problem as follows.

Theorem 5.4.2. *Let $f \in L_{p(\cdot)}(\mathbb{R}^n)$ and $0 < \alpha < n$. Assume also that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Suppose that the hypersingular integral*

$$\mathbb{D}^\alpha_{\ell_0} f = \lim_{\varepsilon\to 0} \mathbb{D}^\alpha_{\ell_0,\varepsilon} f$$

exists for some ℓ_0 (in the sense of convergence in the $L_{p(\cdot)}$ norm). Then for any other ℓ for which the derivative $\mathbb{D}^\alpha_\ell f$ exists (in the same sense), we have

$$\mathbb{D}^\alpha_\ell f = \mathbb{D}^\alpha_{\ell_0} f.$$

Proof. Let $f \in L_{p(\cdot)}(\mathbb{R}^n)$ and $\varepsilon > 0$. All we need to do is to justify the passage to the limit in the equality

$$\mathbb{K}^\varepsilon_{\ell_0,\alpha}\mathbb{D}^\alpha_{\ell,\varepsilon} f = \mathbb{K}^\varepsilon_{\ell,\alpha}\mathbb{D}^\alpha_{\ell_0,\varepsilon} f \tag{5.13}$$

given in Lemma 5.4.1.

Assume that $\mathbb{D}^{\alpha}_{\ell_0,\varepsilon} f$ converges to $\mathbb{D}^{\alpha}_{\ell_0} f$ in $L_{p(\cdot)}(\mathbb{R}^n)$. Then

$$\begin{aligned}\left\|\mathbb{K}^{\varepsilon}_{\ell,\alpha}\mathbb{D}^{\alpha}_{\ell_0,\varepsilon} f - \mathbb{D}^{\alpha}_{\ell_0} f\right\|_{p(\cdot)} &= \left\|\mathbb{K}^{\varepsilon}_{\ell,\alpha}\left(\mathbb{D}^{\alpha}_{\ell_0,\varepsilon} f - \mathbb{D}^{\alpha}_{\ell_0} f\right) + \mathbb{K}^{\varepsilon}_{\ell,\alpha}\mathbb{D}^{\alpha}_{\ell_0} f - \mathbb{D}^{\alpha}_{\ell_0} f\right\|_{p(\cdot)} \\ &\leq \left\|\mathbb{K}^{\varepsilon}_{\ell,\alpha}\left(\mathbb{D}^{\alpha}_{\ell_0,\varepsilon} f - \mathbb{D}^{\alpha}_{\ell_0} f\right)\right\|_{p(\cdot)} + \left\|\mathbb{K}^{\varepsilon}_{\ell,\alpha}\mathbb{D}^{\alpha}_{\ell_0} f - \mathbb{D}^{\alpha}_{\ell_0} f\right\|_{p(\cdot)} \\ &\leq C\ \left\|\mathbb{D}^{\alpha}_{\ell_0,\varepsilon} f - \mathbb{D}^{\alpha}_{\ell_0} f\right\|_{p(\cdot)} + \left\|\mathbb{K}^{\varepsilon}_{\ell,\alpha}\mathbb{D}^{\alpha}_{\ell_0} f - \mathbb{D}^{\alpha}_{\ell_0} f\right\|_{p(\cdot)}\end{aligned}$$

where the constant C is independent of ε, because the operators $\mathbb{K}^{\varepsilon}_{\ell,\alpha}$ are uniformly bounded in $L_{p(\cdot)}(\mathbb{R}^n)$ with respect to ε. But the kernel $\frac{1}{\varepsilon^n}\mathcal{K}_{\ell,\alpha}\left(\frac{\cdot}{\varepsilon}\right)$ constitutes an identity approximation, so that we can pass to the limit as $\varepsilon \to 0$ in last inequality (see statement (iii) of Theorem 4.4.1). So, we arrive at the conclusion that the right-hand side of (5.13) converges to $\mathbb{D}^{\alpha}_{\ell_0} f$ in $L_{p(\cdot)}(\mathbb{R}^n)$. On the other hand, taking into account that $\mathbb{D}^{\alpha}_{\ell,\varepsilon} f$ converges to $\mathbb{D}^{\alpha}_{\ell} f$, we conclude also, using similar arguments, that the left-hand side converges to $\mathbb{D}^{\alpha}_{\ell} f$ in $L_{p(\cdot)}(\mathbb{R}^n)$. Hence the limits on both sides are equal, that is, $\mathbb{D}^{\alpha}_{\ell} f = \mathbb{D}^{\alpha}_{\ell_0} f$. □

This shows that the order of finite differences that we choose is irrelevant to the value of the Riesz derivative. As a consequence, we can omit the parameter ℓ in $\mathbb{D}^{\alpha}_{\ell} f$ and simply write $\mathbb{D}^{\alpha} f$ for functions f belonging to variable Lebesgue spaces, with $0 < \alpha < n$.

Remark 5.4.3. As mentioned before, the independence of the hypersingular integral on the order of finite differences is a consequence of the choice of the normalizing constants. In the case of functions in the Lizorkin class, the independence was already clear, according to identity (5.7). Note also that we could extract similar information for functions of potential type from the inversion Theorem 5.3.5.

Further notes

In general, we dealt with non-centered differences in this chapter. However, centered differences could be also considered as long as necessary technical modifications were made (we refer to [121] where full details are given).

The inversion of the Riesz potential operator was first investigated by Samko [115, 117] in the context of the classical Lebesgue spaces. Theorem 5.3.5 above may be seen as one more step in the knowledge of the mapping properties of the operator $\mathcal{I}^{\alpha}$ within the framework of the Lebesgue spaces with variable exponent.

The Lizorkin class $\Phi(\mathbb{R}^n)$ was intensively investigated by Lizorkin in connection with function spaces of fractional smoothness (see [90], for instance). Its invariance with respect to the Riesz potential operator is particulary important to deal with Riesz potentials in the distributional sense (cf. [121]).

The denseness of $\Phi(\mathbb{R}^n)$ in the classical Lebesgue spaces $L_p(\mathbb{R}^n)$ was originally proved by Lizorkin. An alternative proof of the same result was given by Samko in 1976 (see [121], Theorem 2.7, for details). Our denseness statement given in Theorem 5.1.5 above was based on Samko's approach, under the assumption that the maximal operator is bounded in $L_{p(\cdot)}(\mathbb{R}^n)$. An interesting problem would be to obtain the same result under weaker conditions on the exponent.

Riesz fractional differentiation $\mathbb{D}^\alpha$ was introduced by Stein [128] in the case $0 < \alpha < 2$ to describe the space of Bessel potentials (see Chapter 6 below). The generalization to the case of arbitrary $\alpha > 0$ is due to Lizorkin [90]. We refer to [121] and [123] where a systematic and comprehensive approach to hypersingular integrals and their applications may be found.

Chapter 6

Characterization of Riesz and Bessel Potentials on $\mathbf{L_{p(\cdot)}}$ Spaces

The generalization of the classical Sobolev theorem (see Theorem 4.5.4) to the variable exponent setting was an important step in the operator theory on variable Lebesgue spaces. An obvious consequence is that the range of the Riesz potential operator $\mathcal{I}^\alpha$, acting on $L_{p(\cdot)}$ spaces, is contained into the corresponding Lebesgue space $L_{q(\cdot)}$ whose exponent $q(\cdot)$ is determined by (4.13). As in the classical case it is possible to obtain further information, giving an exact description of the Riesz potentials of densities in $L_{p(\cdot)}(\mathbb{R}^n)$ in terms of convergence of hypersingular integrals. Such a characterization will be provided in Section 6.1, which partially extends the known results from [121] for constant exponents $p(x) \equiv p$.

In Section 6.2 we deal with function spaces of fractional smoothness defined in terms of hypersingular integrals, and analyze its connection with the Riesz potential spaces.

Section 6.3 is devoted to the consideration of Bessel potentials on spaces $L_{p(\cdot)}(\mathbb{R}^n)$. Mapping properties of the Bessel potential operator acting in these spaces and a description of the range of this operator in terms of Riesz fractional derivatives are provided. A special attention will be paid to the study of the properties of two convolution kernels, which allow us to derive important results and connections.

Section 6.4 contains a comparison of the spaces of Riesz and Bessel potentials on $L_{p(\cdot)}(\mathbb{R}^n)$ with the Sobolev spaces $W^m_{p(\cdot)}(\mathbb{R}^n)$ of variable exponent.

The last section may be seen as a short note where an alternative proof of the Sobolev embedding theorem on $\mathbb{R}^n$ is discussed.

In this chapter, we shall consider both non-centered finite differences and centered ones in the construction of the hypersingular integrals. However, when we write Δ_y^ℓ without any specification we mean a non-centered difference (as we did in Chapter 5). We point out that we still assume the rule stated in Remark 5.2.1.

6.1 Description of the Riesz potential spaces

When $\overline{p} < \frac{n}{\alpha}$ then the Riesz potentials of densities in $L_{p(\cdot)}(\mathbb{R}^n)$ are defined pointwise. Let us introduce the following definition.

Definition 6.1.1. *The space of Riesz potentials on $L_{p(\cdot)}(\mathbb{R}^n)$ is defined in a natural way as*

$$\mathcal{I}^\alpha[L_{p(\cdot)}] = \left\{ f \in L_1^{loc}(\mathbb{R}^n) : \; f = \mathcal{I}^\alpha \varphi, \quad \varphi \in L_{p(\cdot)}(\mathbb{R}^n) \right\}, \qquad \overline{p} < \frac{n}{\alpha}.$$

Following approaches in [121], we will show below that the space $\mathcal{I}^\alpha[L_{p(\cdot)}]$ can be described in terms of convergence of hypersingular integrals. First we give two auxiliary statements.

Let

$$B_{a,b,c}(x) := \int_{\mathbb{R}^n} \frac{dy}{|y|^a (1+|y|)^b |x-y|^c}.$$

The following statement is contained in Lemma 1.38 in [121].

Lemma 6.1.2. *Let $a < n, b \geq 0, c < n$ and $a+b+c > n$. Then*

$$B_{a,b,c}(x) \leq C \left(1 + |x|^{-\max(a+c-n,0)} \right) \qquad as \qquad |x| \to 0,$$

if $a + c \neq n$, and

$$B_{a,b,c}(x) \leq \frac{C}{(1+|x|)^{\min(a+b+c-n,c)}} \qquad as \qquad |x| \to \infty,$$

if $a + b \neq n$.

Lemma 6.1.3. *Let $1 < \underline{p} \leq p(x) \leq \overline{p} < \frac{n}{\alpha}$, $x \in \mathbb{R}^n$, with $0 < \alpha < n$. Let $p'(\cdot)$ be the usual conjugate exponent and $q(\cdot)$ be the Sobolev limiting exponent given by $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$. If $p(\cdot)$ satisfies the logarithmic conditions (4.4) and (4.6), then so do $p'(\cdot)$ and $q(\cdot)$.*

Proof. The proof follows at once from the inequalities

$$|p'(x) - p'(y)| \leq \frac{1}{(\underline{p}-1)} \, |p(x) - p(y)|, \qquad x, y \in \mathbb{R}^n,$$

and

$$|q(x) - q(y)| \leq \frac{n^2}{(n - \alpha\overline{p})^2}\, |p(x) - p(y)|, \qquad x,\, y \in \mathbb{R}^n.$$

$\square$

Theorem 6.1.4. *Let* $0 < \alpha < n$, $1 < \underline{p} \leq \overline{p} < \frac{n}{\alpha}$, $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$ *and let* f *be a locally integrable function. Assume also that* $p(\cdot)$ *satisfies the* log-*Hölder continuity conditions (4.4) and (4.6). Then* $f \in \mathcal{I}^\alpha[L_{p(\cdot)}]$ *if and only if* $f \in L_{q(\cdot)}(\mathbb{R}^n)$ *and there exists the Riesz derivative* $\mathbb{D}^\alpha f$ *(in the sense of convergence in* $L_{p(\cdot)}(\mathbb{R}^n)$*).*

Proof. First, assume that $f \in \mathcal{I}^\alpha[L_{p(\cdot)}]$. The fact that $f \in L_{q(\cdot)}(\mathbb{R}^n)$ follows from Theorem 4.5.4. On the other hand, as $f = \mathcal{I}^\alpha\varphi$ for some $\varphi \in L_{p(\cdot)}(\mathbb{R}^n)$, then we have

$$\mathbb{D}^\alpha f = \lim_{\varepsilon\to 0} \mathbb{D}^\alpha_\varepsilon \mathcal{I}^\alpha \varphi = \varphi \tag{6.1}$$

according to Theorem 5.3.5, with the convergence taken in $L_{p(\cdot)}(\mathbb{R}^n)$.

Conversely, let $f \in L_{q(\cdot)}(\mathbb{R}^n)$ and suppose that its Riesz derivative $\mathbb{D}^\alpha f$ exists. Our aim is to prove that $f = \mathcal{I}^\alpha\mathbb{D}^\alpha f$ and hence that $f \in \mathcal{I}^\alpha[L_{p(\cdot)}]$.

Both f and $\mathcal{I}^\alpha\mathbb{D}^\alpha f$ can be regarded as elements of $\Phi'(\mathbb{R}^n)$, the dual space of the Lizorkin class $\Phi(\mathbb{R}^n)$. Let us show that they coincide in this sense.

For all $\phi \in \Phi(\mathbb{R}^n)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{I}^\alpha\mathbb{D}^\alpha f(x)\,\phi(x)\,dx &= \int_{\mathbb{R}^n} \mathbb{D}^\alpha f(y) \left(\int_{\mathbb{R}^n} \frac{\phi(x)}{|x-y|^{n-\alpha}}\,dx \right) dy \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \left(\int_{|z|>\varepsilon} \frac{(\Delta^\ell_z f)(y)}{d_{n,\ell}(\alpha)\,|z|^{n+\alpha}}\,dz \right) \mathcal{I}^\alpha\phi(y)\,dy \\
&= \lim_{\varepsilon\to 0} \int_{|z|>\varepsilon} \left(\int_{\mathbb{R}^n} \frac{\sum\limits_{k=0}^{\ell} (-1)^k \binom{\ell}{k} f(u)\, \mathcal{I}^\alpha\phi(u+kz)}{d_{n,\ell}(\alpha)\,|z|^{n+\alpha}}\,du \right) dz \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} f(u) \left(\int_{|z|>\varepsilon} \frac{(\Delta^\ell_{-z}\mathcal{I}^\alpha\phi)(u)}{d_{n,\ell}(\alpha)\,|z|^{n+\alpha}}\,dz \right) du \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} f(u)\, \mathbb{D}^\alpha_\varepsilon \mathcal{I}^\alpha\phi(u)\,du \\
&= \int_{\mathbb{R}^n} f(u)\,\phi(u)\,du.
\end{aligned}$$

We give some justifications about the arguments used in the chain above. The first equality follows from the Fubini theorem since the double integral converges absolutely. In fact, since $|\phi(y)| \leq \frac{c}{(1+|y|)^N}$ with an arbitrary large N, then by Lemma 6.1.2, the Riesz potential $\mathcal{I}^\alpha(|\phi|)(x)$ is bounded and $\mathcal{I}^\alpha(|\phi|)(x) \leq \frac{c}{(1+|x|)^{n-\alpha}}$ as $|x| \to \infty$. Thus

$$I_{p'(\cdot)}(\mathcal{I}^\alpha(|\phi|)) = \int_{\mathbb{R}^n} [\mathcal{I}^\alpha(|\phi|)(x)]^{p'(x)} dx \leq c_1 + c_2 \int_{|x|>1} \frac{dx}{(1+|x|)^{(n-\alpha)p'(x)}} < \infty,$$

because $\inf\limits_{x\in\mathbb{R}^n} (n-\alpha)p'(x) > n$. Hence, using Hölder inequality we arrive at

$$\int_{\mathbb{R}^n} |\mathbb{D}^\alpha f(y)|\, \mathcal{I}^\alpha(|\phi|)(y)\, dy \leq c\, \|\mathbb{D}^\alpha f\|_{p(\cdot)}\, \|\mathcal{I}^\alpha(|\phi|)\|_{p'(\cdot)} < \infty.$$

In the second equality, we notice that the convergence with respect to the $L_{p(\cdot)}(\mathbb{R}^n)$ norm implies weak convergence in $\Phi'(\mathbb{R}^n)$ (note that $\mathcal{I}^\alpha\phi \in \Phi(\mathbb{R}^n)$ since this class is invariant with respect to $\mathcal{I}^\alpha$).

The third equality follows from similar arguments to those used in the first one, and then by the change of variables $y \to u+kz$, with fixed z. We only observe now that $\mathbb{D}^\alpha_\varepsilon f \in L_{q(\cdot)}(\mathbb{R}^n)$ (cf. Proposition 5.2.3 and Lemma 7.8).

Finally, the last passage is obtained by making use of the Lebesgue theorem. Indeed, since $\phi \in L_{q'(\cdot)}(\mathbb{R}^n)$, then $\mathbb{D}^\alpha_\varepsilon \mathcal{I}^\alpha \phi \in L_{q'(\cdot)}(\mathbb{R}^n)$ (cf. Proposition 5.2.3 and Lemma 7.8 again), so that $f(\cdot)\,(\mathbb{D}^\alpha_\varepsilon \mathcal{I}^\alpha\phi)(\cdot) \in L_1(\mathbb{R}^n)$. Then the result follows from the inversion theorem (Theorem 5.3.5).

To finish the proof, we observe that since both f and $\mathcal{I}^\alpha\mathbb{D}^\alpha f$ are tempered distributions, then $\mathcal{I}^\alpha\mathbb{D}^\alpha f = f + P$, where P is a polynomial (see Proposition 2.5 in [121]). Therefore $f + P \in L_{q(\cdot)}(\mathbb{R}^n)$, which implies $P \in L_{q(\cdot)}(\mathbb{R}^n)$. Thus we should have $P \equiv 0$, which means $\mathcal{I}^\alpha\mathbb{D}^\alpha f(x) = f(x)$ for almost all $x \in \mathbb{R}^n$. □

We generalize another characterization which is contained in Theorem 7.11 in [121].

Theorem 6.1.5. *In Theorem 6.1.4 above one can replace the assertion on the existence of the Riesz derivative of f by the following uniform boundedness condition: there exists $C > 0$ such that*

$$\|\mathbb{D}^\alpha_\varepsilon f\|_{p(\cdot)} \leq C \tag{6.2}$$

for all $\varepsilon > 0$.

Proof. If $f = \mathcal{I}^\alpha\varphi$ for some $\varphi \in L_{p(\cdot)}(\mathbb{R}^n)$, then (6.2) is immediate by Corollary 5.3.4.

Conversely, if $\sup\limits_{\varepsilon>0} \|\mathbb{D}^\alpha_\varepsilon f\|_{p(\cdot)} < \infty$ then there exists a subsequence of $\{\mathbb{D}^\alpha_\varepsilon f\}_{\varepsilon>0}$, say $\{\mathbb{D}^\alpha_{\varepsilon_k} f\}_{k\in\mathbb{N}}$, which converges weakly in $L_{p(\cdot)}(\mathbb{R}^n)$ (note that $L_{p(\cdot)}(\mathbb{R}^n)$ is a reflexive Banach space under conditions $1 < \underline{p} \le \overline{p} < \infty$). Let us denote its limit by $g \in L_{p(\cdot)}(\mathbb{R}^n)$, and let $\phi \in \Phi(\mathbb{R}^n)$. As in the proof of Theorem 6.1.4, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{I}^\alpha g(x)\, \phi(x)\, dx &= \int_{\mathbb{R}^n} g(y)\, \mathcal{I}^\alpha \phi(y)\, dy \\
&= \lim_{k\to+\infty} \int_{\mathbb{R}^n} \mathbb{D}^\alpha_{\varepsilon_k} f(y)\, \mathcal{I}^\alpha \phi(y)\, dy \\
&= \lim_{k\to+\infty} \int_{\mathbb{R}^n} f(z)\, (\mathbb{D}^\alpha_{\varepsilon_k} \mathcal{I}^\alpha \phi)(z)\, dz \\
&= \int_{\mathbb{R}^n} f(z)\, \phi(z)\, dz
\end{aligned}$$

The second equality follows directly from the weak convergence in $L_{p(\cdot)}(\mathbb{R}^n)$ by noticing that $\mathcal{I}^\alpha \phi \in L_{p'(\cdot)}(\mathbb{R}^n)$, while the last one is justified by the convergence of $\mathbb{D}^\alpha_{\varepsilon_k} \mathcal{I}^\alpha \phi$ to ϕ in $L_{q'(\cdot)}(\mathbb{R}^n)$ (taking into account the inversion theorem and the Lemma 7.8 once again) and from the fact that $f \in (L_{q'(\cdot)}(\mathbb{R}^n))' = L_{q(\cdot)}(\mathbb{R}^n)$. Hence, as previously, one gets $f = \mathcal{I}^\alpha g$, so that $f \in \mathcal{I}^\alpha[L_{p(\cdot)}]$. $\square$

6.2 Function spaces defined by fractional derivatives

Hypersingular integrals can also be used to construct function spaces of fractional smoothness. Similarly to the classical case let us consider the space

$$L^\alpha_{p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L_{p(\cdot)}(\mathbb{R}^n) : \ \mathbb{D}^\alpha f \in L_{p(\cdot)}(\mathbb{R}^n) \right\}, \quad \alpha > 0,$$

where the fractional derivative $\mathbb{D}^\alpha$ is treated in the usual way as convergent in the $L_{p(\cdot)}(\mathbb{R}^n)$ norm. We remark that this space does not depend on the order of the finite differences (chosen according to the rule from Remark 5.2.1), at least when $0 < \alpha < n$ (see Section 5.4), which is the case we are interested in. $L^\alpha_{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f \,|\, L^\alpha_{p(\cdot)}(\mathbb{R}^n)\| := \|f\|_{p(\cdot)} + \|\mathbb{D}^\alpha f\|_{p(\cdot)}.$$

These spaces will be shown to be the same as the spaces of Bessel potentials. They are connected with the spaces of Riesz potentials in the following way.

Theorem 6.2.1. *Assume that the exponent* $p(\cdot)$ *satisfies the usual logarithmic conditions (4.4) and (4.6). Let also* $0 < \alpha < n$ *and* $1 < \underline{p} \le \overline{p} < \frac{n}{\alpha}$. *Then*

$$L^\alpha_{p(\cdot)}(\mathbb{R}^n) = L_{p(\cdot)}(\mathbb{R}^n) \cap \mathcal{I}^\alpha[L_{p(\cdot)}].$$

Proof. By Theorem 6.1.4 we only need to prove the embedding $L^\alpha_{p(\cdot)}(\mathbb{R}^n) \subset L_{p(\cdot)}(\mathbb{R}^n) \cap \mathcal{I}^\alpha[L_{p(\cdot)}]$. So, let $f \in L^\alpha_{p(\cdot)}(\mathbb{R}^n)$. As in the proof of Theorem 6.1.4 (but here under the assumption that $f \in L_{p(\cdot)}(\mathbb{R}^n)$ instead of $f \in L_{q(\cdot)}(\mathbb{R}^n)$ as there), we have $f(x) = \mathcal{I}^\alpha \mathbb{D}^\alpha f(x)$ almost everywhere, so that $f \in \mathcal{I}^\alpha[L_{p(\cdot)}]$. □

Remark 6.2.2. Theorem 6.2.1 also holds if one takes centered differences (everything in the proof of Theorem 6.1.4 works in a similar way). Hence, from this theorem we conclude that the space $L^\alpha_{p(\cdot)}(\mathbb{R}^n)$ does not depend on the type of finite differences used to construct the fractional derivative $\mathbb{D}^\alpha$, at least when $\overline{p} < \frac{n}{\alpha}$, with $0 < \alpha < n$.

For further goals, we will show that functions from $L^\alpha_{p(\cdot)}(\mathbb{R}^n)$ can be approximated by $C_0^\infty(\mathbb{R}^n)$ functions. We start with a preliminary denseness result as follows. By $W^\infty_{p(\cdot)}(\mathbb{R}^n)$ we denote the Sobolev space of all functions of $L_{p(\cdot)}(\mathbb{R}^n)$ for which all their (weak) derivatives are also in $L_{p(\cdot)}(\mathbb{R}^n)$.

Proposition 6.2.3. *The set $C^\infty(\mathbb{R}^n) \cap W^\infty_{p(\cdot)}(\mathbb{R}^n)$ is dense in $L^\alpha_{p(\cdot)}(\mathbb{R}^n)$ for all $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.*

Proof. For simplicity, we split the proof into two parts.
Step 1: Let us show that $C^\infty(\mathbb{R}^n) \cap W^\infty_{p(\cdot)}(\mathbb{R}^n) \subset L^\alpha_{p(\cdot)}(\mathbb{R}^n)$, which is not clear at first glance in the case of variable exponents. If $f \in C^\infty(\mathbb{R}^n) \cap W^\infty_{p(\cdot)}(\mathbb{R}^n)$ then we already know that

$$\int\limits_{|y|>\varepsilon} \frac{(\Delta^\ell_y f)(x)}{|y|^{n+\alpha}}\, dy \in L_{p(\cdot)}(\mathbb{R}^n) \tag{6.3}$$

for any $\varepsilon > 0$ (see Proposition 5.2.3). On the other hand, we have

$$\left\| \int\limits_{|y|\le\delta} \frac{(\Delta^\ell_y f)(x)}{|y|^{n+\alpha}}\, dy \right\|_{p(\cdot)} \longrightarrow 0 \quad \text{as} \quad \delta \to 0. \tag{6.4}$$

To prove (6.4), we use the representation

$$(\Delta^\ell_y f)(x) = r \sum_{|j|=r} \sum_{k=1}^{\ell} \frac{y^j}{j!} (-1)^{r-k} k^r \binom{\ell}{k} \int_0^1 (1-t)^{r-1} (D^j f)(x - kty)\, dt, \tag{6.5}$$

(see [121], formula (3.31)) with the choice $\ell \ge r > \alpha$. Hence

$$\int\limits_{|y|\le\delta} \frac{(\Delta^\ell_y f)(x)}{|y|^{n+\alpha}}\, dy = \sum_{|j|=r} \sum_{k=1}^{\ell} c_{r,j,k} \int_0^1 (1-t)^{r-1} \left(\int\limits_{|y|\le\delta} \frac{y^j}{|y|^{n+\alpha}}\, (D^j f)(x - kty)\, dy \right) dt.$$

The change of variables $y \to \delta z$ yields

$$\int\limits_{|y|\le\delta} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\,dy = \delta^{r-\alpha} \sum_{|j|=r} \sum_{k=1}^{\ell} c_{r,j,k} \int_0^1 (1-t)^{r-1} \left(\frac{1}{\delta_k(t)^n} K_j\left(\frac{\cdot}{\delta_k(t)}\right) * D^j f \right)(x)\,dt \tag{6.6}$$

where K_j are given by

$$K_j(z) = \frac{z^j}{|z|^{n+\alpha}} \quad \text{when} \quad |z| \le 1 \quad \text{and} \quad K_j(z) = 0 \quad \text{otherwise,}$$

and $\delta_k(t) = k\delta t$. Since $|j| = r > \alpha$, the kernel K_j has a decreasing radial integrable dominant, so that Theorem 4.4.1 is applicable and we have

$$\left| \int\limits_{|y|\le\delta} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\,dy \right| \le \delta^{r-\alpha} \sum_{|j|=r} \sum_{k=1}^{\ell} |c_{r,j,k}| \int_0^1 (1-t)^{r-1}\, c\,\mathcal{M}(D^j f)(x)\,dt, \tag{6.7}$$

where $c > 0$ is independent of $\delta_k(t)$. Hence,

$$I_{p(\cdot)}\left(\int\limits_{|y|\le\delta} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\,dy \right) \le c\,\delta^{(r-\alpha)\underline{p}} \sum_{|j|=r} I_{p(\cdot)}(\mathcal{M}(D^j f)) \longrightarrow 0 \quad \text{as} \quad \delta \to 0. \tag{6.8}$$

We remind that the convergence in norm is equivalent to the modular convergence (see Lemma 4.2.3). Further, under the present assumptions on $p(\cdot)$, the maximal operator $\mathcal{M}$ maps $L_{p(\cdot)}(\mathbb{R}^n)$ into itself.

From (6.3) and (6.7) we see that the integral $\int\limits_{\mathbb{R}^n} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\,dy$ converges absolutely for all x and defines a function belonging to $L_{p(\cdot)}(\mathbb{R}^n)$. Moreover, by (6.8), it coincides with the Riesz derivative:

$$\left\| \int\limits_{\mathbb{R}^n} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\,dy - \int\limits_{|y|>\varepsilon} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\,dy \right\|_{p(\cdot)} = \left\| \int\limits_{|y|\le\varepsilon} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\,dy \right\|_{p(\cdot)} \longrightarrow 0 \quad \text{as } \varepsilon \to 0,$$

so that $\mathbb{D}^\alpha f \in L_{p(\cdot)}(\mathbb{R}^n)$.

We would like to remark that some modification is needed if α is odd. Indeed, in that case, we should choose $\ell = \alpha$ so that we cannot proceed from (6.5) above (recall the rule from Remark 5.2.1). Nevertheless, if we consider centered differences then $\ell > \alpha$ already, and we can proceed in a similar way.

Step 2: We use the standard approximation by using mollifiers (as in [121], Lemma 7.14). Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi \ge 0$ and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$ with supp $\varphi \subset \overline{B}(0,1)$. Put $\varphi_m(x) :=$

$m^n\,\varphi(mx)$, $m \in \mathbb{N}$. Then $\varphi_m \in C_0^\infty(\mathbb{R}^n)$ and supp $\varphi_m \subset \overline{B}(0, 1/m)$. Given $f \in L^\alpha_{p(\cdot)}(\mathbb{R}^n)$ let us define

$$f_m(x) := \varphi_m * f(x) = \int_{\mathbb{R}^n} \varphi(y)\; f\left(x - \frac{y}{m}\right)\, dy.$$

Hence $f_m \in C^\infty(\mathbb{R}^n)$. Moreover, we also have $f_m \in L_{p(\cdot)}(\mathbb{R}^n)$ by Theorem 4.4.1 and $D^j f_m = D^j(\varphi_m) * f \in L_{p(\cdot)}(\mathbb{R}^n)$. In the case of fractional derivatives we have, for each $\varepsilon > 0$ and $m \in \mathbb{N}$,

$$\mathbb{D}^\alpha_\varepsilon f_m = (\mathbb{D}^\alpha_\varepsilon f)_m,$$

that is,

$$\mathbb{D}^\alpha_\varepsilon(\varphi_m * f) = \varphi_m * \mathbb{D}^\alpha_\varepsilon f,$$

which can be proved by Fubini theorem. Hence

$$\mathbb{D}^\alpha f_m = \lim_{\varepsilon \to 0}\, (\varphi_m * \mathbb{D}^\alpha_\varepsilon f) = \varphi_m * \mathbb{D}^\alpha f = (\mathbb{D}^\alpha f)_m,$$

where the second equality follows from the continuity of the convolution operator. In particular, one concludes that $\mathbb{D}^\alpha f_m \in L_{p(\cdot)}(\mathbb{R}^n)$.

It remains to show that the functions f_m approximate the function f in the $L^\alpha_{p(\cdot)}(\mathbb{R}^n)$ norm. Of course, $\|f - f_m\|_{p(\cdot)} \to 0$ as $m \to \infty$ by Theorem 4.4.1 once again. On the other hand, using similar arguments, we have

$$\|\mathbb{D}^\alpha(f - f_m)\|_{p(\cdot)} = \|\mathbb{D}^\alpha f - \mathbb{D}^\alpha f_m\|_{p(\cdot)} = \|\mathbb{D}^\alpha f - (\mathbb{D}^\alpha f)_m\|_{p(\cdot)} \longrightarrow 0$$

as $m \to \infty$, since $\mathbb{D}^\alpha f \in L_{p(\cdot)}(\mathbb{R}^n)$. $\square$

Theorem 6.2.4. *If $p(\cdot)$ is as in Proposition 6.2.3 with $\overline{p} < \frac{n}{\alpha}$, then the class $C_0^\infty(\mathbb{R}^n)$ is dense in $L^\alpha_{p(\cdot)}(\mathbb{R}^n)$.*

Proof. By Proposition 6.2.3, it is sufficient to show that every function $f \in C^\infty(\mathbb{R}^n) \cap W^\infty_{p(\cdot)}(\mathbb{R}^n)$ can be approximated by functions in $C_0^\infty(\mathbb{R}^n)$ in the norm $\|\cdot \mid L^\alpha_{p(\cdot)}(\mathbb{R}^n)\|$. As in the case of constant p, we will use the "smooth truncation" of functions.

Let $\mu \in C_0^\infty(\mathbb{R}^n)$ with $\mu(x) = 1$ if $|x| \le 1$, supp $\mu \subset \overline{B}(0, 2)$ and $0 \le \mu(x) \le 1$ for every x. Define $\mu_m(x) := \mu\left(\frac{x}{m}\right)$, $x \in \mathbb{R}^n$, $m \in \mathbb{N}$. We are to show that the sequence of truncations $\{\mu_m f\}_{m \in \mathbb{N}}$ converges to f in $L^\alpha_{p(\cdot)}(\mathbb{R}^n)$.

The passage to the limit $\lim\limits_{m\to\infty} \|f - \mu_m f\|_{p(\cdot)} = 0 \iff \lim\limits_{m\to\infty} I_{p(\cdot)}(f - \mu_m f) = 0$ is directly checked by means of the Lebesgue dominated convergence theorem. It remains to show that $I_{p(\cdot)}(\mathbb{D}^\alpha(f - \mu_m f))$ also tends to zero as $m \to \infty$.

Taking Remark 6.2.2 into account, we may consider centered differences in the fractional derivative (under the choice $\ell > \alpha$ with ℓ even). For brevity we denote $\nu_m = 1 - \mu_m$. As centered differences can be written in terms of non-centered ones, we have

$$\begin{aligned}\mathbb{D}^\alpha(\nu_m f)(x) &= \frac{1}{d_{n,\ell}(\alpha)} \sum_{k=0}^{\ell} \binom{\ell}{k} \int_{\mathbb{R}^n} \frac{(\Delta_y^k \nu_m)(x+\frac{\ell}{2}y)\,(\Delta_y^{\ell-k} f)(x+(\frac{\ell}{2}-k)y)}{|y|^{n+\alpha}}\,dy \\ &=: \frac{1}{d_{n,\ell}(\alpha)} \sum_{k=0}^{\ell} \binom{\ell}{k} A_{m,k} f(x).\end{aligned}$$

So we need to show that $I_{p(\cdot)}(A_{m,k}f) \to 0$ as $m \to \infty$, for $k = 0, 1, \ldots, \ell$. We separately treat the cases $k = 0, k = \ell$ and $1 \le k \le \ell - 1$.

The case $k = 0$: we have

$$A_{m,0} f(x) = d_{n,\ell}(\alpha)\, \nu_m(x)\, \mathbb{D}^\alpha f(x) + B_m f(x) \tag{6.9}$$

where

$$\begin{aligned}B_m f(x) &= \int_{\mathbb{R}^n} \frac{[\nu_m(x+\frac{\ell}{2}y) - \nu_m(x)](\Delta_y^\ell f)(x+\frac{\ell}{2}y)}{|y|^{n+\alpha}}\,dy \\ &= \int_{\mathbb{R}^n} \frac{[\mu_m(x) - \mu_m(x+\frac{\ell}{2}y)](\Delta_y^\ell f)(x+\frac{\ell}{2}y)}{|y|^{n+\alpha}}\,dy\end{aligned}$$

The convergence of the first term in (6.9) is clear, so that it remains to prove that $I_{p(\cdot)}(B_m f) \to 0$ as $m \to \infty$. Put

$$B_m f(x) = \int_{|y|\le 1} (\cdots)\,dy + \int_{|y|>1} (\cdots)\,dy := B_m^0 f(x) + B_m^1 f(x).$$

To estimate the term $B_m^0 f$, we make use of the Taylor formula (of order 1) with the remainder in the integral form and obtain

$$\mu_m\left(x + \frac{\ell}{2}y\right) - \mu_m(x) = \frac{\ell}{2m} \sum_{j=1}^{n} y_j \int_0^1 \frac{\partial \mu}{\partial x_j}\left(\frac{x + \frac{\ell t}{2}y}{m}\right) dt.$$

Hence

$$\left|\mu_m\left(x + \frac{\ell}{2}y\right) - \mu_m(x)\right| \le \frac{c}{m}\,|y|, \tag{6.10}$$

where $c > 0$ does not depend on x, y and m.

As in the proof of Proposition 6.2.3, we can estimate $B_m^0 f$ in terms of the convolution of the derivatives of f with a "good kernel" in the sense of Theorem 4.4.1. In fact, taking (6.10) and (3.31) in [121] into account, we get

$$\begin{aligned}
|B_m^0 f(x)| \;\; &\leq \;\; \frac{c}{m}\sum_{|j|=r}\sum_{\nu=\frac{\ell}{2}}^{\ell}\Bigg[\int_0^{\frac{\ell}{2\nu}}(1-t)^{r-1}\left(\frac{1}{(-\theta_\nu(t))^n}K\left(\frac{\cdot}{-\theta_\nu(t)}\right)*|D^j f|\right)(x)\,dt \\
&\quad + \int_{\frac{\ell}{2\nu}}^1 (1-t)^{r-1}\left(\frac{1}{\theta_\nu(t)^n}K\left(\frac{\cdot}{\theta_\nu(t)}\right)*|D^j f|\right)(x)\,dt\Bigg] \\
&\quad + \frac{c}{m}\sum_{|j|=r}\sum_{\nu=1}^{\frac{\ell}{2}-1}\int_0^1(1-t)^{r-1}\left(\frac{1}{(-\theta_\nu(t))^n}K\left(\frac{\cdot}{-\theta_\nu(t)}\right)*|D^j f|\right)(x)\,dt \qquad (6.11)
\end{aligned}$$

where K is given by

$$K(z) = |z|^{r+1-n-\alpha} \quad \text{if} \quad |z| < 1 \quad \text{and} \quad K(z) = 0 \quad \text{otherwise},$$

with $\theta_\nu(t) = \nu t - \frac{\ell}{2}$ and under the choice $r > \alpha - 1$. So

$$I_{p(\cdot)}(B_m^0 f) \leq \frac{c}{m}\sum_{|j|=r} I_{p(\cdot)}\left[\mathcal{M}(|D^j f|)\right] \longrightarrow 0 \quad \text{as } m \to \infty.$$

For the term $B_m^1 f$ we may proceed as follows. Since μ is infinitely differentiable and compactly supported, then it satisfies the Hölder continuity condition of any order. Hence, for an arbitrary $\varepsilon \in (0,1]$, there exists $c = c(\varepsilon) > 0$ not depending on x and y, such that

$$\left|\mu_m\left(x + \frac{\ell}{2}y\right) - \mu_m(x)\right| \leq \frac{c}{m^\varepsilon}\,|y|^\varepsilon.$$

When $\alpha > 1$, we may proceed as previously by considering $r < \alpha < \ell$. Putting all these things together, one estimates $B_m^1 f(x)$ as in (6.11) with the corresponding kernel K given by

$$K(y) = \frac{|y|^r}{|y|^{n+\alpha-\varepsilon}} \quad \text{when} \quad |y| > 1 \quad \text{and} \quad K(y) = 0 \quad \text{otherwise.}$$

Under the choice $0 < \varepsilon < \min(1, \alpha - r)$, the kernel K above has an integrable radial decreasing dominant, so that we can apply Stein's theorem once more and arrive at the conclusion that

$$I_{p(\cdot)}(B_m^1 f) \leq \frac{c}{m^\varepsilon}\sum_{|j|=r} I_{p(\cdot)}(|D^j f|) \longrightarrow 0 \quad \text{as} \quad m \to \infty.$$

The case $0 < \alpha \leq 1$ can be treated without passing to the derivatives of f. In fact, in this case, we may take $\ell = 2$, and hence

$$|B_m^1 f(x)| \leq \frac{c}{m^\varepsilon}\left(\int_{|y|>1}\frac{|f(x+y)|}{|y|^{n+\alpha-\varepsilon}}\,dy + \int_{|y|>1}\frac{|f(x)|}{|y|^{n+\alpha-\varepsilon}}\,dy + \int_{|y|>1}\frac{|f(x-y)|}{|y|^{n+\alpha-\varepsilon}}\,dy\right).$$

Each term can be managed by using similar arguments as above but now with the choice $0 < \varepsilon < \alpha$.

<u>The case $k = \ell$</u>: let

$$
\begin{aligned}
A_{m,\ell} f(x) &= \int_{\mathbb{R}^n} \frac{(\Delta_y^\ell \nu_m)(x + \frac{\ell}{2} y)\, f(x - \frac{\ell}{2} y)}{|y|^{n+\alpha}}\, dy \\
&= \int_{|y| \le 1} (\cdots)\, dy + \int_{|y|>1} (\cdots)\, dy \\
&=: B^0_{m,\ell} f(x) + B^1_{m,\ell} f(x).
\end{aligned}
$$

Notice that $(\Delta_y^\ell \nu_m)(z) = -\left(\Delta_y^\ell \mu_m\right)(z) = -\left(\Delta^\ell_{\frac{y}{m}} \mu\right)\left(\frac{z}{m}\right)$. So, according to (6.5), one gets the estimate

$$
\left| (\Delta_y^\ell \nu_m)\left(x + \frac{\ell}{2} y\right) \right| = \left| (\Delta^\ell_{\frac{y}{m}} \mu) \left(\frac{x + \frac{\ell}{2} y}{m} \right) \right| \le c \left(\frac{|y|}{m} \right)^r \sum_{|j|=r} \|D^j \mu\|_\infty \le \frac{c}{m^r}\, |y|^r
$$

(with $\ell \ge r > \alpha$). Hence,

$$
|B^0_{m,\ell} f(x)| \le \frac{c}{m^r}\, (K * |f|)(x)
$$

where K is now given by

$$
K(y) = \frac{1}{|y|^{n+\alpha-r}} \quad \text{if} \quad |y| \le \frac{\ell}{2} \quad \text{and} \quad K(y) = 0 \quad \text{otherwise.}
$$

Since $r > \alpha$, the kernel K is under the assumptions of Theorem 4.4.1. As before, we get $\|B^0_{m,\ell} f\|_{p(\cdot)} \to 0$ as $m \to \infty$.

As far as the term $B^1_{m,\ell} f$ is concerned, when $\alpha > 1$ we may choose $\ell > \alpha > r$ and proceed in a similar way as in the case $k = 0$ above. When $0 < \alpha \le 1$ we may take $\ell = 2$ and get

$$
\begin{aligned}
&|B^1_{m,\ell} f(x)| \le \int_{|y|>1} \frac{\left| \left(\Delta^2_{\frac{y}{m}} \mu \right) \left(\frac{x+y}{m} \right) \right| |f(x-y)|}{|y|^{n+\alpha}}\, dy \\
&= \int_{|y|>1} \frac{\left| \mu\left(\frac{x+y}{m}\right) - 2\,\mu\left(\frac{x}{m}\right) + \mu\left(\frac{x-y}{m}\right) \right| |f(x-y)|}{|y|^{n+\alpha}}\, dy \\
&\le \int_{|y|>1} \frac{\left| \mu\left(\frac{x+y}{m}\right) - \mu\left(\frac{x}{m}\right) \right| |f(x-y)|}{|y|^{n+\alpha}}\, dy + \int_{|y|>1} \frac{\left| \mu\left(\frac{x-y}{m}\right) - \mu\left(\frac{x}{m}\right) \right| |f(x-y)|}{|y|^{n+\alpha}}\, dy \\
&\le \frac{c}{m^\varepsilon} \int_{|y|>1} \frac{|y|^\varepsilon\, |f(x-y)|}{|y|^{n+\alpha}}\, dy
\end{aligned}
$$

for any $\varepsilon \in (0, 1]$ (with $c > 0$ independent of m). Thus we arrive at the desired conclusion by taking $\varepsilon < \alpha$.

The case $k \in \{1, 2, \ldots, \ell - 1\}$: as in the previous case, we have

$$\begin{aligned} A_{m,k}f(x) &= \int_{\mathbb{R}^n} \frac{(\Delta_y^k \nu_m)(x + \frac{\ell}{2}y)\ (\Delta_y^{\ell-k} f)(x + (\frac{\ell}{2} - k)y)}{|y|^{n+\alpha}}\, dy \\ &= \int_{|y|\le 1} (\cdots)\, dy + \int_{|y|>1} (\cdots)\, dy \\ &=: B^0_{m,k}f(x) + B^1_{m,k}f(x). \end{aligned}$$

We may estimate the term $B^0_{m,k}f$ by noticing that

$$\left|(\Delta_y^k \nu_m)(x + \frac{\ell}{2}y)\right| \le c\ \left(\frac{|y|}{m}\right)^k$$

and then by proceeding as above with an appropriate choice of r.

For the term $B^1_{m,k}$ we first consider the case $\alpha > 1$. Since $\binom{k}{l} = \binom{k}{k-l}$, for $l = 0, 1, \ldots, k$, we may write

$$(\Delta^k_{\frac{y}{m}} \mu)\left(\frac{x + \frac{\ell}{2}y}{m}\right) = \sum_{l=0}^{\frac{k-1}{2}} \binom{k}{l} \left[\mu\left(\frac{x + \frac{\ell}{2}y}{m} - l\frac{y}{m}\right) - \mu\left(\frac{x + \frac{\ell}{2}y}{m} - (k-l)\frac{y}{m}\right)\right]$$

if k is odd. When k is even, we can also represent our finite difference as the sum of the first order differences of two appropriate terms since $\sum_{l=0}^{k}(-1)^l\binom{k}{l} = 0$. In both situations we may again make use of the Hölder continuity (of order ε) of the function μ. Finally, we shall arrive at the desired estimate by using arguments as above, but under the assumption $0 < \varepsilon < \min(1, \alpha - 1)$. The case $0 < \alpha \le 1$ can be easily solved by taking $\ell = 2$. In that way, we have $k = \ell - k = 1$ and hence

$$\begin{aligned} |B^1_{m,k}f(x)| &\le \int_{|y|>1} \frac{\left|(\Delta^1_{\frac{y}{m}}\mu)\left(\frac{x+y}{m}\right)\right|\ |(\Delta^1_y f)(x)|}{|y|^{n+\alpha}}\, dy \\ &\le \int_{|y|>1} \frac{\left(\frac{|y|}{m}\right)^\varepsilon\ |f(x)|}{|y|^{n+\alpha}}\, dy + \int_{|y|>1} \frac{\left(\frac{|y|}{m}\right)^\varepsilon\ |f(x-y)|}{|y|^{n+\alpha}}\, dy, \end{aligned}$$

so that we can proceed as in the previous cases. □

6.3 Bessel potentials on variable Lebesgue spaces

The main aim of this section is to describe the range of the Bessel potential operator on $L_{p(\cdot)}(\mathbb{R}^n)$ in terms of convergence of hypersingular integrals. This is known in the case of constant p, see [121], Section 7.2, or [123], Section 27.3, and references therein. Here we consider only the case $\overline{p} < \frac{n}{\alpha}$ with $0 < \alpha < n$.

6.3.1 The Bessel potential operator: basic properties

The *Bessel kernel* G_α can be introduced in terms of Fourier transform by

$$\widehat{G}_\alpha(x) = (1 + |x|^2)^{-\alpha/2}, \quad x \in \mathbb{R}^n, \quad \alpha > 0.$$

It is known that G_α admits the integral representation

$$G_\alpha(x) = c(\alpha) \int_0^\infty e^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}} \, t^{\frac{\alpha-n}{2}} \, \frac{dt}{t}, \quad x \in \mathbb{R}^n,$$

where $c(\alpha)$ is a certain constant (see, for example, [129], Section V.3.1), so that G_α is a non-negative, radially decreasing function. Moreover, G_α is integrable with $\|G_\alpha\|_1 = \widehat{G}_\alpha(0) = 1$.

The Bessel kernel behaves at the origin and at infinity as follows ([123], Lemma 27.1):

$$G_\alpha(x) \sim \begin{cases} c(\alpha, n)\, |x|^{\alpha-n}, & \text{if } 0 < \alpha < n \\ c(n)\, \ln\left(\frac{1}{|x|}\right), & \text{if } \alpha = n \\ c(\alpha, n), & \text{if } \alpha > n. \end{cases} \tag{6.12}$$

as $|x| \to 0$, and

$$G_\alpha(x) \sim c(\alpha, n)\, |x|^{\frac{\alpha-n-1}{2}} \, e^{-|x|} \tag{6.13}$$

as $|x| \to \infty$.

This shows that G_α is similar to the Riesz kernel at the origin and it has an exponential decay at infinity.

We recall also the estimate

$$0 \le G_\alpha(x) \le c\, k_\alpha(x) \tag{6.14}$$

for $0 < \alpha < n$, where k_α is the Riesz kernel from Definition 5.3.1.

Definition 6.3.1. *The Bessel potential of order $\alpha > 0$ of the density φ is defined by*

$$\mathcal{B}^\alpha \varphi(x) = \int_{\mathbb{R}^n} G_\alpha(x - y)\, \varphi(y)\, dy. \tag{6.15}$$

For convenience, we also denote $\mathcal{B}^0 \varphi = \varphi$.

Theorem 6.3.2. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ then the Bessel potential operator $\mathcal{B}^\alpha$ is bounded in $L_{p(\cdot)}(\mathbb{R}^n)$.*

Proof. The boundedness of the operator $\mathcal{B}^\alpha$ follows from the properties of the kernel G_α described above. Taking into account Theorem 4.4.1, there exists a constant $c > 0$ such that

$$\|\mathcal{B}^\alpha \varphi\|_{p(\cdot)} = \|G_\alpha * \varphi\|_{p(\cdot)} \le c\, \|\varphi\|_{p(\cdot)},$$

for all $\varphi \in L_{p(\cdot)}(\mathbb{R}^n)$. □

We are interested in Bessel potentials with densities in $L_{p(\cdot)}(\mathbb{R}^n)$.

Definition 6.3.3. *One defines the space of Bessel potentials*[1] *as the range of the Bessel potential operator,*

$$\mathcal{B}^\alpha[L_{p(\cdot)}] = \left\{ f \in L_1^{loc}(\mathbb{R}^n) : \quad f = \mathcal{B}^\alpha \varphi, \quad \varphi \in L_{p(\cdot)}(\mathbb{R}^n) \right\}, \qquad \alpha \geq 0.$$

According to Theorem 6.3.2, the space $\mathcal{B}^\alpha[L_{p(\cdot)}]$, also called sometimes *Liouville space of fractional smoothness*, is well defined, being a subspace of $L_{p(\cdot)}(\mathbb{R}^n)$ if the maximal operator is bounded in $L_{p(\cdot)}(\mathbb{R}^n)$, in particular, if the exponent $p(\cdot)$ is log-Hölder continuous both locally and at infinity.

$\mathcal{B}^\alpha[L_{p(\cdot)}]$ is a Banach space endowed with the norm

$$\|f \,|\, \mathcal{B}^\alpha[L_{p(\cdot)}]\| := \|\varphi\|_{p(\cdot)}, \tag{6.16}$$

where φ is the density from (6.15). Note that (6.16) provides a consistent definition, since $\mathcal{B}^\alpha \varphi = \mathcal{B}^\alpha \psi$ implies $\varphi = \psi$. This can be shown as for the classical case (see [129], p. 135).

Proposition 6.3.4. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha > \gamma \geq 0$, then $\mathcal{B}^\alpha[L_{p(\cdot)}] \hookrightarrow \mathcal{B}^\gamma[L_{p(\cdot)}]$.*

Proof. The proof follows immediately from the properties of the Bessel kernel and from the boundedness of the Bessel potential operator. Indeed, if $f = \mathcal{B}^\alpha \varphi$ for some $\varphi \in L_{p(\cdot)}(\mathbb{R}^n)$ then one can write $f = \mathcal{B}^\gamma(\mathcal{B}^{\alpha-\gamma}\varphi)$. Thus $f \in \mathcal{B}^\gamma[L_{p(\cdot)}]$ by Theorem 6.3.2. Furthermore,

$$\|f \,|\, \mathcal{B}^\gamma[L_{p(\cdot)}]\| := \|\mathcal{B}^{\alpha-\gamma}\varphi\|_{p(\cdot)} \leq c\,\|\varphi\|_{p(\cdot)} =: c\,\|f \,|\, \mathcal{B}^\alpha[L_{p(\cdot)}]\|.$$

□

6.3.2 On two important convolution kernels

The comparison of the ranges of the Bessel and Riesz potential operators is naturally made via the convolution type operator whose symbol is the ratio of the Fourier transforms of the Riesz and Bessel kernels. This operator is the sum of the identity operator and the convolution operator with a radial integrable kernel. Keeping in mind the application of Theorem 4.4.1, we have to show more, namely that this kernel has an integrable decreasing dominant.

[1] As observed in Chapter 1, we adopt here a different notation for the Bessel potential spaces just to emphasize that they are the range of the Bessel potential operator acting in $L_{p(\cdot)}(\mathbb{R}^n)$.

We have to show the existence of integrable decreasing dominants for two important kernels g_α and h_α defined by (6.17) and (6.18), respectively.

Let g_α and h_α be the functions defined via the following Fourier transforms

$$\frac{|x|^\alpha}{(1+|x|^2)^{\frac{\alpha}{2}}} = 1 + \widehat{g}_\alpha(x), \quad \alpha > 0, \quad x \in \mathbb{R}^n, \tag{6.17}$$

$$\frac{(1+|x|^2)^{\frac{\alpha}{2}}}{1+|x|^\alpha} = 1 + \widehat{h}_\alpha(x), \quad \alpha > 0, \quad x \in \mathbb{R}^n. \tag{6.18}$$

Observe that

$$\frac{1+|x|^\alpha}{(1+|x|^2)^{\frac{\alpha}{2}}} = \widehat{G}_\alpha(x) + \widehat{g}_\alpha(x) + 1. \tag{6.19}$$

It is known that g_α and h_α are integrable (see, for example, Lemma 1.25 in [121]).

The following two lemmas are crucial for our further goals.

Lemma 6.3.5. *The function g_α defined in (6.17) has an integrable and radially decreasing dominant.*

Proof. If we denote $\rho = (1+|x|^2)^{1/2}$, then we have

$$\frac{|x|^\alpha}{(1+|x|^2)^{\frac{\alpha}{2}}} - 1 = (1-\rho^{-2})^{\alpha/2} - 1.$$

Taking the expansion into the binomial series we get

$$(1-\rho^{-2})^{\frac{\alpha}{2}} - 1 = \sum_{k=0}^{\infty} \binom{\alpha/2}{k} \left(-\rho^{-2}\right)^k - 1 = \sum_{k=1}^{\infty} (-1)^k \binom{\alpha/2}{k} \rho^{-2k}, \quad \rho > 1.$$

Hence, for each $x \neq 0$,

$$\frac{|x|^\alpha}{(1+|x|^2)^{\frac{\alpha}{2}}} - 1 = \sum_{k=1}^{\infty} (-1)^k \binom{\alpha/2}{k} \widehat{G}_{2k}(x) := \sum_{k=1}^{\infty} c(\alpha,k)\, \widehat{G}_{2k}(x),$$

where G_{2k} is the Bessel kernel of order $2k$. Therefore

$$g_\alpha(x) = \sum_{k=1}^{\infty} c(\alpha,k)\, G_{2k}(x), \quad x \in \mathbb{R}^n. \tag{6.20}$$

Now, we stress that

$$m_\alpha(x) := \sum_{k=1}^{\infty} |c(\alpha,k)|\, G_{2k}(x)$$

defines a radial decreasing dominant of g_α, which is integrable,

$$\|m_\alpha\|_1 \leq \sum_{k=1}^{\infty} \left| \binom{\alpha/2}{k} \right| < \infty,$$

since $\left|\binom{\alpha/2}{k}\right| \leq \frac{c}{k^{1+\alpha/2}}$ as $k \to \infty$ (cf. [123], p. 14). $\square$

Lemma 6.3.6. *The kernel h_α given by (6.18) admits the bounds*

$$|h_\alpha(x)| \leq \frac{c}{|x|^{n-a}} \quad as \quad |x| < 1, \qquad a = \min\{1, \alpha\} \tag{6.21}$$

and

$$|h_\alpha(x)| \leq \frac{c}{|x|^{n+\alpha}} \quad as \quad |x| \geq 1. \tag{6.22}$$

where $c > 0$ is a constant not depending on x.

Proof. We split the proof into two different parts.

Step 1 (*proof of (6.21)*): Let us start by representing the function $\widehat{h}_\alpha$ as a finite sum of Fourier transforms of Bessel kernels plus an integrable function. To this end, we denote $t = \frac{1}{1+|x|^2}$. Then $\widehat{h}_\alpha(x) = \frac{1}{t^\beta+(1-t)^\beta} - 1$, with $\beta = \frac{\alpha}{2}$. But

$$\frac{1}{t^\beta + (1-t)^\beta} - 1 = \frac{1}{(1-t)^\beta} \cdot \frac{1}{1 + \left(\frac{t}{1-t}\right)^\beta} - 1 = \frac{1}{(1-t)^\beta} \sum_{k=0}^{\infty} (-1)^k \left(\frac{t}{1-t}\right)^{k\beta} - 1$$

where the series converges if $\frac{t}{1-t} < 1$, that is, if $t < \frac{1}{2}$ or $|x| > 1$. Since $\frac{1}{1-t} = \frac{1+|x|^2}{|x|^2}$ and $\frac{t}{1-t} = \frac{1}{|x|^2}$, we get

$$\widehat{h}_\alpha(x) = \frac{(1+|x|^2)^{\frac{\alpha}{2}}}{|x|^\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k}{|x|^{\alpha k}} - 1, \qquad |x| > 1.$$

For each natural number N, we can write

$$\widehat{h}_\alpha(x) = \left(1+|x|^2\right)^{\frac{\alpha}{2}} \sum_{k=0}^{N} \frac{(-1)^k}{|x|^{\alpha(k+1)}} - 1 + A_N(x), \qquad |x| > 1, \tag{6.23}$$

where $|A_N(x)| \leq \frac{c}{|x|^{\alpha N}}$. Indeed, since $\frac{1}{|x|^{\alpha k}} \to 0$ as $k \to +\infty$ (recall that $|x| > 1$), we have

$$|A_N(x)| = \left| \frac{(1+|x|^2)^{\frac{\alpha}{2}}}{|x|^\alpha} \sum_{k=N+1}^{\infty} \frac{(-1)^k}{|x|^{\alpha k}} \right| \leq \frac{(1+|x|^2)^{\frac{\alpha}{2}}}{|x|^{2\alpha}} \frac{1}{|x|^{\alpha N}} \leq \frac{2^\alpha}{|x|^{\alpha N}}. \tag{6.24}$$

Now it remains to represent the powers $\frac{1}{|x|^{\alpha(k+1)}}$ in terms of the powers $\frac{1}{\sqrt{1+|x|^2}}$. We observe that for any $\gamma > 0$, taking $\rho = \sqrt{1+|x|^2}$, we have

$$\frac{1}{|x|^\gamma} = \rho^{-\gamma} \left(1 - \frac{1}{\rho^2}\right)^{-\gamma/2} = \rho^{-\gamma} \left(\sum_{j=0}^{M} (-1)^j \binom{-\gamma/2}{j} \rho^{-2j} + \phi_M(\rho) \right), \tag{6.25}$$

where $M \in \mathbb{N}$ and

$$\phi_M(\rho) = \sum_{j=M+1}^{\infty} (-1)^j \binom{-\gamma/2}{j} \rho^{-2j}$$

converges absolutely for $\rho > 1$, that is, for $x \neq 0$. But

$$\left|\frac{\phi_M(\rho)}{\rho^\gamma}\right| \le c \sum_{j=M+1}^{\infty} \frac{1}{j^{1-\frac{\gamma}{2}}} \frac{1}{\rho^{2j+\gamma}} \le \frac{c}{\rho^{M+1}} \sum_{j=M+1}^{\infty} \frac{1}{j^{1-\frac{\gamma}{2}}} \frac{1}{2^{\frac{j+\gamma}{2}}} \tag{6.26}$$

where we took into account that $|x| > 1$, that is, $\rho \ge \sqrt{2}$. Hence

$$\left|\frac{\phi_M(\rho)}{\rho^\gamma}\right| \le \frac{c_1}{\rho^{M+1}} \le \frac{c_2}{\rho^M}. \tag{6.27}$$

Then from (6.25) and (6.27) we get

$$\frac{1}{|x|^\gamma} = \sum_{j=0}^{M} \frac{(-1)^j \binom{-\gamma/2}{j}}{(1+|x|^2)^{j+\gamma/2}} + B_M^\gamma(x), \tag{6.28}$$

where

$$|B_M^\gamma(x)| \le \frac{C}{|x|^{2M}} \qquad \text{as} \quad |x| > 1. \tag{6.29}$$

Substituting (6.28) into (6.23) (with $\gamma = \alpha(k+1)$), and taking $M = N$, we arrive at

$$\widehat{h}_\alpha(x) = \sum_{\substack{k,j=0\\k+j\neq 0}}^{N} \frac{(-1)^{k+j}\binom{-\alpha(k+1)/2}{j}}{(1+|x|^2)^{j+\alpha k}} + r_N(x), \tag{6.30}$$

where the function

$$r_N(x) = A_N(x) + \left(1+|x|^2\right)^{\frac{\alpha}{2}} \sum_{k=0}^{N} B_N^{\alpha(k+1)}(x) \tag{6.31}$$

satisfies the estimate

$$|r_N(x)| \le \frac{c}{|x|^\mu}, \qquad \mu = N\min(2,\alpha),$$

for all $|x| > 1$ according to (6.24) and (6.29). Hence, we only have to choose $N > \frac{n}{\min(2,\alpha)}$ in order to get the integrability of r_N at infinity.

The estimate at infinity was given for $|x| > 1$, but the equality (6.30) itself may be written for all $x \in \mathbb{R}^n$, just by defining r_N as

$$r_N(x) := \widehat{h}_\alpha(x) - \sum_{\substack{k,j=0\\k+j\neq 0}}^{N} c(k,j)\, \widehat{G}_{2j+\alpha k}(x), \quad x \in \mathbb{R}^n, \quad N > \frac{n}{\min(2,\alpha)},$$

where $G_{2j+\alpha k}$ are Bessel kernels and $c(k,j) := (-1)^{k+j}\binom{-\alpha(k+1)/2}{j}$.

So, we have $r_N \in \mathcal{W}_0(\mathbb{R}^n)$. In particular, r_N is a bounded continuous function. Also, r_N is integrable at infinity in view of the estimate above and hence $r_N \in \mathcal{W}_0(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)$. On the other hand, we have also $F^{-1}r_N \in \mathcal{W}_0(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)$, since r_N is radial. Thus, $F^{-1}r_N$ is a bounded continuous function too. So, there exists $C>0$ such that

$$|h_\alpha(x)| \leq \sum_{\substack{k,j=0\\ k+j\neq 0}}^{N} |c(k,j)|\,|G_{2j+\alpha k}(x)| + |F^{-1}r_N(x)| \leq \sum_{\substack{k,j=0\\ k+j\neq 0}}^{N} |c(k,j)|\,|G_{2j+\alpha k}(x)| + C, \quad x \in \mathbb{R}^n.$$

Attending to the behavior of the Bessel kernel given in (6.12), we derive

$$G_{2j+\alpha k}(x) \sim \frac{1}{|x|^{n-2j-\alpha k}}(x) \leq \frac{c}{|x|^{n-\min(1,\alpha)}} \quad \text{as} \quad |x|<1,$$

when $2j+\alpha k<n$. Thus, we arrive at (6.21) with $a=\min(1,\alpha)$. In the case $2j+\alpha k>n$ we arrive at the same estimate since

$$G_{2j+\alpha k}(x) \sim C(2j+\alpha k), \quad |x|<1.$$

For the case $2j+\alpha k=n$ we have the following logarithmic behavior:

$$G_{2j+\alpha k}(x) \sim \ln\left(\frac{1}{|x|}\right), \quad |x|<1.$$

But $\ln\left(\frac{1}{|x|}\right) \leq \frac{1}{|x|^{n-a}}$ for any $a\in(0,n)$. The proof of (6.21) is completed.

Step 2 (*proof of (6.22)*): To obtain (6.22), we transform the Bochner formula for the Fourier transform of radial functions via integration by parts and arrive at the formula

$$F^{-1}\widehat{h}_\alpha(x) = \frac{c}{|x|^{\frac{n}{2}+m-1}} \int_0^\infty \psi_\alpha^{(m)}(t)\, t^{\frac{n}{2}}\, J_{\frac{n}{2}+m-1}(t|x|)\,dt, \qquad x\neq 0, \tag{6.32}$$

where $\psi_\alpha(t)=\frac{(1+t^2)^{\frac{\alpha}{2}}}{1+t^\alpha}$, $t>0$, and m is an arbitrary integer such that $m>1+\frac{n}{2}$.

To justify formula (6.32), we make use of the standard regularization of the integral (cf. [130]):

$$\begin{aligned}
F^{-1}\widehat{h}_\alpha(x) &= (2\pi)^{-n} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} e^{-\varepsilon|y|}\, e^{-ix\cdot y}\, \widehat{h}_\alpha(|y|)\,dy \\
&= (2\pi)^{-n} \lim_{\varepsilon\to 0} \frac{(2\pi)^{n/2}}{|x|^{n/2-1}} \int_0^\infty e^{-\varepsilon t}\, \widehat{h}_\alpha(t)\, t^{n/2}\, J_{n/2-1}(t|x|)\,dt \\
&= \frac{(2\pi)^{-\nu}}{|x|^{\nu-1}} \lim_{\varepsilon\to 0} \int_0^\infty f_\varepsilon(t)\, t^\nu\, J_{\nu-1}(t|x|)\,dt \\
&= \frac{(2\pi)^{-\nu}}{|x|^{\nu-1}} \lim_{\varepsilon\to 0} \frac{(-1)^m}{|x|^m} \int_0^\infty f_\varepsilon^{(m)}(t)\, t^\nu\, J_{\nu+m-1}(t|x|)\,dt
\end{aligned} \tag{6.33}$$

where $J_{\nu-1}(t)$ denotes the *Bessel function of the first kind*, $\nu=\frac{n}{2}$, $m\in\mathbb{N}$ and

$$f_\varepsilon(t) := e^{-\varepsilon t}\, \widehat{h}_\alpha(t), \quad t>0, \quad \varepsilon>0.$$

The second equality above follows from the Bochner formula for Fourier transforms of radial functions, while the last is obtained via integration by parts and the relation

$$\frac{d}{du}[u^\nu\, J_\nu(u)] = u^\nu\, J_{\nu-1}(u)$$

(see (8.133) in [121]). Here we assumed that some quantities vanish, namely

$$f_\varepsilon^{(k)}(t)\, t^\nu\, J_{\nu+k}(t|x|)\Big|_0^\infty = 0, \quad k = 0, 1, \ldots, m-1. \tag{6.34}$$

To check this for $\psi_\alpha(t) = \varphi_\alpha(t)\cdot\phi_\alpha(t)$ with $\varphi_\alpha(t) = (1+t^2)^{\frac{\alpha}{2}}$ and $\phi_\alpha(t) = \frac{1}{1+t^\alpha}$, we observe that

$$\varphi_\alpha^{(k)}(t) = \varphi_\alpha(t)\sum_{j=0}^{[k/2]} c_j(\alpha)\,\frac{t^{k-2j}}{(1+t^2)^{k-j}}, \qquad k = 0, 1, \ldots$$

and

$$\phi_\alpha^{(k)}(t) = \phi_\alpha(t)\, t^{-k}\sum_{j=1}^{k} d_j(\alpha)\,\frac{t^{j\alpha}}{(1+t^\alpha)^j}, \qquad k = 1, 2, \ldots$$

where the constants $c_j(\alpha)$ and $d_j(\alpha)$ may vanish (but not all simultaneously), which may be proved by direct calculations. For $k \geq 1$, we have

$$\begin{aligned}
\psi_\alpha^{(k)}(t) &= \sum_{r=0}^{k}\binom{k}{r}\varphi_\alpha^{(r)}(t)\,\phi_\alpha^{(k-r)}(t) \\
&= \psi_\alpha(t)\sum_{r=0}^{k-1}\binom{k}{r}\left(\sum_{j=0}^{[r/2]}\frac{c_j(\alpha)\,t^{r-2j}}{(1+t^2)^{r-j}}\right)\left(t^{-(k-r)}\sum_{j=1}^{k-r}\frac{d_j(\alpha)\,t^{j\alpha}}{(1+t^\alpha)^j}\right) \\
&+ \psi_\alpha(t)\left(\sum_{j=0}^{[k/2]}\frac{c_j(\alpha)\,t^{k-2j}}{(1+t^2)^{k-j}}\right).
\end{aligned}$$

Since $f_\varepsilon(t) = e^{-\varepsilon t}\,(\psi_\alpha(t)-1)$, then

$$f_\varepsilon^{(k)}(t) = \sum_{j=0}^{k}\binom{k}{j}(-\varepsilon)^{k-j}\, e^{-\varepsilon t}\,\psi_\alpha^{(j)}(t) - (-\varepsilon)^k\, e^{-\varepsilon t}. \tag{6.35}$$

Let $k \in \{0, 1, 2, \ldots, m-1\}$. Taking into account that $J_{\nu+k}(u)$ behaves like $u^{\nu+k}$ for small values of u, we obtain

$$f_\varepsilon^{(k)}(t)\, t^\nu\, J_{\nu+k}(t|x|) \longrightarrow 0 \quad \text{as} \quad t \to 0.$$

On the other hand, $J_{\nu+k}(u)$ behaves like $\frac{1}{\sqrt{u}}$ for large values of u. Since $e^{-\varepsilon t}\, t^{\nu-1/2}$ goes to zero as $t\to\infty$ and $\left|\psi_\alpha^{(j)}(t)\right|$ behaves like a constant (0 or 1, if $j > 0$ or $j = 0$, respectively) when $t \to \infty$, then

$$f_\varepsilon^{(k)}(t)\, t^\nu\, J_{\nu+k}(t|x|) \longrightarrow 0 \quad \text{as} \quad t \to \infty,$$

which completes the verification of (6.34).

To derive (6.32) from (6.33), we notice that the functions $f_\varepsilon^{(m)}(t)\, t^\nu\, J_{\nu+m-1}(t|x|)$, $\varepsilon > 0$, are integrable in $(0,\infty)$. In fact, the integrability at the origin follows from the asymptotic behavior of the Bessel function, while its integrability at infinity follows from the definition of the Gamma function.

We notice now that $f_\varepsilon^{(m)}(t) \longrightarrow \psi_\alpha^{(m)}(t)$ as $\varepsilon \to 0$, by (6.35), and that the passage to the limit in (6.33) is possible if one assumes $m > 1 + \frac{n}{2}$, which yields (6.32).

To obtain (6.22) from (6.32), we observe that the following estimates hold:

$$\left|\psi_\alpha^{(m)}(t)\right| \le \frac{c}{t^m} \quad \text{as} \quad t \ge 1 \tag{6.36}$$

and

$$\left|\psi_\alpha^{(m)}(t)\right| \le c\left(t^{\alpha-m} + t^{m-2\left[\frac{m}{2}\right]}\right) \quad \text{as} \quad t < 1. \tag{6.37}$$

So, we have

$$\begin{aligned}
\frac{1}{|x|^{\nu+m-1}}\int_0^1 |\psi_\alpha^{(m)}(t)|\, t^\nu\, |J_{\nu+m-1}(t|x|)|\, dt &\le \frac{c}{|x|^{\nu+m-1}}\int_0^1 t^{\alpha-m+\nu}\, |J_{\nu+m-1}(t|x|)|\, dt \\
&\le \frac{c}{|x|^{n+\alpha}}\int_0^{|x|} t^{\alpha-m+\nu}\, |J_{\nu+m-1}(t)|\, dt \\
&\le \frac{c}{|x|^{n+\alpha}}\int_0^{\infty} t^{\alpha-m+\nu}\, |J_{\nu+m-1}(t)|\, dt \\
&= \frac{c_1}{|x|^{n+\alpha}}
\end{aligned}$$

if $m > 1 + \nu + \alpha$, which guarantees the convergence of the last integral at infinity. On the other hand,

$$\frac{1}{|x|^{\nu+m-1}}\int_1^\infty |\psi_\alpha^{(m)}(t)|\, t^\nu\, |J_{\nu+m-1}(t|x|)|\, dt \le \frac{c}{|x|^{\nu+m-1}}\int_1^\infty t^{\nu-m}\, dt \le \frac{c_1}{|x|^{n+\alpha}}$$

under the same assumption on m. The proof is completed. □

6.3.3 Characterization of the Bessel potentials in terms of hypersingular integrals

Lemmas 6.3.5 and 6.3.6 allow us to describe the range of the Bessel potential operator through the Riesz fractional derivatives. Before to formulate the main result of this section, we prove the following two statements.

Proposition 6.3.7. *Let $0 < \alpha < n$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $1 < \underline{p} \leq \overline{p} < \frac{n}{\alpha}$. Then every $\varphi \in L^{\alpha}_{p(\cdot)}(\mathbb{R}^n)$ can be represented as*

$$\varphi = \mathcal{B}^{\alpha}(I + U_{\alpha})(\varphi + \mathbb{D}^{\alpha}\varphi), \tag{6.38}$$

where I denotes the identity operator and U_{α} is the convolution operator with the kernel h_{α}.

Proof. Identity (6.38) holds for functions $\varphi \in C_0^{\infty}(\mathbb{R}^n)$. This follows immediately from equality (6.18) above through basic operations with Fourier transforms (see (7.39) in [121]). The denseness of $C_0^{\infty}(\mathbb{R}^n)$ in $L^{\alpha}_{p(\cdot)}(\mathbb{R}^n)$ (stated in Theorem 6.2.4) allows us to write (6.38) for all functions in $L^{\alpha}_{p(\cdot)}(\mathbb{R}^n)$. To this end, we observe that both operators $\mathcal{B}^{\alpha}$ and U_{α} are continuous in $L_{p(\cdot)}(\mathbb{R}^n)$. In fact, the boundedness of $\mathcal{B}^{\alpha}$ was proved in Theorem 6.3.2. On the other hand, the convolution operator U_{α} is bounded since its kernel has a radially decreasing and integrable dominant by Lemma 6.3.6. □

Proposition 6.3.8. *Let $0 < \alpha < n$ and let $1 < \underline{p} \leq \overline{p} < \frac{n}{\alpha}$. Then*

$$\mathcal{B}^{\alpha}\psi = \mathcal{I}^{\alpha}(I + K_{\alpha})\,\psi, \tag{6.39}$$

for all $\psi \in L_{\underline{p}}(\mathbb{R}^n) + L_{\overline{p}}(\mathbb{R}^n)$, where I is the identity operator and K_{α} is the convolution operator with the kernel g_{α}.

Proof. Representation (6.39) holds for densities belonging to classical Lebesgue spaces (see, for instance, (7.38) in [121]), where the kernel of K_{α} is precisely the function g_{α} from (6.17). By the Sobolev theorem one concludes that either $\mathcal{B}$ or $\mathcal{I}^{\alpha}(I + K_{\alpha})$ are linear operators from $L_{\underline{p}}(\mathbb{R}^n)$ into $L_{q(\underline{p})}(\mathbb{R}^n)$, with $\frac{1}{q(\underline{p})} = \frac{1}{\underline{p}} - \frac{\alpha}{n}$, and from $L_{\overline{p}}(\mathbb{R}^n)$ into $L_{q(\overline{p})}(\mathbb{R}^n)$, with $\frac{1}{q(\overline{p})} = \frac{1}{\overline{p}} - \frac{\alpha}{n}$. So, we can define these operators on the linear sum $L_{\underline{p}}(\mathbb{R}^n) + L_{\overline{p}}(\mathbb{R}^n)$ in the usual way. Hence, if $\psi = \psi_0 + \psi_1$, with $\psi_0 \in L_{\underline{p}}(\mathbb{R}^n)$ and $\psi_1 \in L_{\overline{p}}(\mathbb{R}^n)$, then we may make use of the already known representation for each term and then arrive at equality (6.39). □

Now we are able to give the main statement. The following theorem in the case of constant exponents $p(x) \equiv p$, $1 < p < \infty$, is due to Stein [128] when $0 < \alpha < 1$ and to Lizorkin [90] in the general case $0 < \alpha < \infty$ (see also the proof for constant p in [121], p. 186).

Theorem 6.3.9. *Let $0 < \alpha < n$. If $1 < \underline{p} \leq \overline{p} < \frac{n}{\alpha}$ and $p(\cdot)$ satisfies the* log*-Hölder continuity conditions (4.4) and (4.6), then $\mathcal{B}^{\alpha}[L_{p(\cdot)}] = L^{\alpha}_{p(\cdot)}(\mathbb{R}^n)$ with equivalent norms: there* *are constants* $c_1,\ c_2 > 0$ *such that*

$$c_1\, \|f \,|\, L^\alpha_{p(\cdot)}(\mathbb{R}^n)\| \le \|f \,|\, \mathcal{B}^\alpha[L_{p(\cdot)}]\| \le c_2\, \|f \,|\, L^\alpha_{p(\cdot)}(\mathbb{R}^n)\|, \quad \forall f \in \mathcal{B}^\alpha[L_{p(\cdot)}].$$

Proof. Assume first that $f \in \mathcal{B}^\alpha[L_{p(\cdot)}]$. Then $f \in L_{p(\cdot)}(\mathbb{R}^n)$ by Theorem 6.3.2. It remains to show that its Riesz derivative also belongs to $L_{p(\cdot)}(\mathbb{R}^n)$. Since $f = \mathcal{B}^\alpha \varphi$ for some $\varphi \in L_{p(\cdot)}(\mathbb{R}^n)$ and $L_{p(\cdot)}(\mathbb{R}^n) \subset L_{\underline{p}}(\mathbb{R}^n) + L_{\overline{p}}(\mathbb{R}^n)$, then by Proposition 6.3.8 one gets the representation

$$\mathcal{B}^\alpha \varphi = \mathcal{I}^\alpha (I + K_\alpha)\, \varphi.$$

Lemma 6.3.5 combined with Theorem 4.4.1, allow us to conclude that K_α is bounded in $L_{p(\cdot)}(\mathbb{R}^n)$, and hence $f \in \mathcal{I}^\alpha[L_{p(\cdot)}]$. So, according to the characterization given in Theorem 6.1.4, the Riesz derivative $\mathbb{D}^\alpha f$ exists in the sense of convergence in $L_{p(\cdot)}(\mathbb{R}^n)$. Therefore, $f \in L^\alpha_{p(\cdot)}(\mathbb{R}^n)$. Moreover,

$$\begin{aligned}
\|f \,|\, L^\alpha_{p(\cdot)}(\mathbb{R}^n)\| &= \|\mathcal{B}^\alpha \varphi\|_{p(\cdot)} + \|\mathbb{D}^\alpha \mathcal{B}^\alpha \varphi\|_{p(\cdot)} \\
&= \|\mathcal{B}^\alpha \varphi\|_{p(\cdot)} + \|\mathbb{D}^\alpha \mathcal{I}^\alpha (I + K_\alpha)\, \varphi\|_{p(\cdot)} \\
&= \|\mathcal{B}^\alpha \varphi\|_{p(\cdot)} + \|(I + K_\alpha)\, \varphi\|_{p(\cdot)} \\
&\le c\, \|\varphi\|_{p(\cdot)} = c\, \|f \,|\, \mathcal{B}^\alpha[L^\alpha_{p(\cdot)}]\|.
\end{aligned}$$

The third equality follows from the inversion Theorem 5.3.5, while the inequality is obtained from Theorem 6.3.2 and from the boundedness of K_α.

Conversely, suppose that $f \in L^\alpha_{p(\cdot)}(\mathbb{R}^n)$. Proposition 6.3.7 yields the representation

$$f = \mathcal{B}^\alpha (I + U_\alpha)(f + \mathbb{D}^\alpha f).$$

Taking into account Lemma 6.3.6 and Theorem 4.4.1, we arrive at the conclusion that $f \in \mathcal{B}^\alpha[L_{p(\cdot)}]$ and

$$\|f \,|\, \mathcal{B}^\alpha[L_{p(\cdot)}]\| = \|(I + U_\alpha)(f + \mathbb{D}^\alpha f)\|_{p(\cdot)} \le c\, (\|f\|_{p(\cdot)} + \|\mathbb{D}^\alpha f\|_{p(\cdot)}) = c\, \|f \,|\, L^\alpha_{p(\cdot)}(\mathbb{R}^n)\|.$$

□

Corollary 6.3.10. *If the exponent* $p(\cdot)$ *is under the conditions of Theorem 6.3.9, then* $C_0^\infty(\mathbb{R}^n)$ *is dense in the Bessel potential space* $\mathcal{B}^\alpha[L_{p(\cdot)}]$.

6.4 Comparison of the Riesz and Bessel potential spaces with the variable Sobolev spaces

The identification of the spaces of Bessel potentials of integer smoothness with Sobolev spaces is a well-known result within the framework of the classical Lebesgue spaces. The result is due to Calderón [19] and states that $\mathcal{B}^m[L_p] = W_p^m(\mathbb{R}^n)$, if $m \in \mathbb{N}_0$ and $1 < p < \infty$, with equivalent norms. We extend this to the variable exponent setting. The proof will follow mainly the case of constant p, which can be found, for instance, in [129], Sections V.3.3-4. In particular, we will make use of the Riesz transforms R_j, $j = 1, \ldots, n$, defined in (4.9).

The key point is the following characterization.

Theorem 6.4.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and let $\alpha \geq 1$. Then $f \in \mathcal{B}^\alpha[L_{p(\cdot)}]$, if and only if $f \in \mathcal{B}^{\alpha-1}[L_{p(\cdot)}]$ and $\frac{\partial f}{\partial x_j} \in \mathcal{B}^{\alpha-1}[L_{p(\cdot)}]$ for every $j = 1, \ldots, n$. Furthermore, there exist positive constants c_1 and c_2 such that*

$$c_1 \, \|f \,|\, \mathcal{B}^\alpha[L_{p(\cdot)}]\| \leq \|f \,|\, \mathcal{B}^{\alpha-1}[L_{p(\cdot)}]\| + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \,|\, \mathcal{B}^{\alpha-1}[L_{p(\cdot)}] \right\| \leq c_2 \, \|f \,|\, \mathcal{B}^\alpha[L_{p(\cdot)}]\|. \tag{6.40}$$

Proof. Suppose first that $f = \mathcal{B}^\alpha \varphi$ for some $\varphi \in L_{p(\cdot)}(\mathbb{R}^n)$. Then for each $j = 1, 2, \ldots, n$, we have

$$\frac{\partial f}{\partial x_j} = \mathcal{B}^{\alpha-1}[-R_j(I + K_1)\varphi], \tag{6.41}$$

where I is the identity operator and K_1 is the convolution operator whose kernel is g_1, given by (6.17) with $\alpha = 1$. This identity may be seen as a refinement of that in [129], p. 136, and it is known to be valid for $\varphi \in L_p(\mathbb{R}^n)$ when p is constant. Thus, it is also valid for variable $p(\cdot)$, since $L_{p(\cdot)}(\mathbb{R}^n) \subset L_{\overline{p}}(\mathbb{R}^n) + L_{\underline{p}}(\mathbb{R}^n)$.

The right-hand side inequality in (6.40) follows from (6.41) and from the mapping properties of the Bessel potential operator on spaces $L_{p(\cdot)}(\mathbb{R}^n)$.

The proof of the left-hand side inequality follows the known scheme for constant exponents. However, we need to refine the connection with the Riesz transforms and the derivatives, in order to overcome the difficulties associated to the convolution operators in the variable exponent setting. We write here the main steps of the proof for the completeness of the presentation.

Assume that both f and $\frac{\partial f}{\partial x_j}$ belong to $\mathcal{B}^{\alpha-1}[L_{p(\cdot)}]$. If $f = \mathcal{B}^{\alpha-1}\varphi$ with $\varphi \in L_{p(\cdot)}(\mathbb{R}^n)$, then the first order derivatives of φ exist in the weak sense and belong to $L_{p(\cdot)}(\mathbb{R}^n)$ (see [129], p.

137, for details). Moreover, $\frac{\partial f}{\partial x_j} = \mathcal{B}^{\alpha-1}\left(\frac{\partial \varphi}{\partial xj}\right)$. Since $\varphi \in W^1_{p(\cdot)}(\mathbb{R}^n)$ there exists a sequence of infinitely differentiable and compactly supported functions $\{\varphi_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \varphi_k = \varphi$ and $\lim_{k\to\infty} \frac{\partial \varphi_k}{\partial x_j} = \frac{\partial \varphi}{\partial x_j}$ in $L_{p(\cdot)}(\mathbb{R}^n)$, $j = 1, 2, \dots, n$. This follows from the denseness of $C_0^\infty(\mathbb{R}^n)$ in the Sobolev space $W^1_{p(\cdot)}(\mathbb{R}^n)$ (see Theorem 4.2.5), which holds under the assumptions on the exponent.

Since $\mathcal{B}^1$ maps $\mathcal{S}(\mathbb{R}^n)$ onto itself, then, for each k, there exists $\psi_k \in \mathcal{S}(\mathbb{R}^n)$ such that $\varphi_k = \mathcal{B}^1\psi_k$. Now, from the fact

$$1 = (1+|x|^2)^{-1/2}(1+\widehat{h}_1(x))(1+|x|), \quad x \in \mathbb{R}^n,$$

given by (6.18) (with $\alpha = 1$), one arrives at the identity

$$\psi_k = (I+U_1)\left(\varphi_k + \sum_{j=1}^n R_j\left(\frac{\partial \varphi_k}{\partial x_j}\right)\right),$$

where U_1 is the convolution operator as in Proposition 6.3.7. Now we make use of the boundedness of the involved operators and arrive at the left-hand side inequality in (6.40). This completes the proof. $\square$

Corollary 6.4.2. *Let $p(\cdot)$ be as in Theorem 6.4.1 and let $m \in \mathbb{N}_0$. Then*

$$\mathcal{B}^m[L_{p(\cdot)}] = W^m_{p(\cdot)}(\mathbb{R}^n),$$

up to the equivalence of the norms.

Proof. The identity above is obvious when $m = 0$. It can be extended to the case $m \geq 1$ from Theorem 6.4.1. $\square$

The theorem below provides a connection of the spaces of Riesz potentials with the Sobolev spaces. It partially extends the facts known for constant exponents p (see, for instance, [121], p. 181) to the variable exponent setting.

Theorem 6.4.3. *Let $p(\cdot)$ be* log*-Hölder continuous both locally and at infinity, and suppose that $1 < \underline{p} \leq \overline{p} < \frac{n}{\alpha}$. Then we have*

$$W^m_{p(\cdot)}(\mathbb{R}^n) \subset L_{p(\cdot)}(\mathbb{R}^n) \cap \mathcal{I}^\alpha[L_{p(\cdot)}] \tag{6.42}$$

if $0 < \alpha < \min(m, n)$, $m \in \mathbb{N}$, *and*

$$W^m_{p(\cdot)}(\mathbb{R}^n) = L_{p(\cdot)}(\mathbb{R}^n) \cap \mathcal{I}^m[L_{p(\cdot)}] \tag{6.43}$$

when $0 < m < n$.

Proof. Let us prove (6.43) first. Let $f \in W^m_{p(\cdot)}(\mathbb{R}^n)$. From Corollary 6.4.2, Proposition 6.3.4 and Theorem 6.3.9, we derive that not only $f \in L_{p(\cdot)}(\mathbb{R}^n)$, but also that $\mathbb{D}^m f \in L_{p(\cdot)}(\mathbb{R}^n)$. On the other hand, the Sobolev theorem states that $f \in L_{q(\cdot)}(\mathbb{R}^n)$, where $q(\cdot)$ is the usual Sobolev exponent. Then by Theorem 6.1.4 one concludes that f is a Riesz potential.

Reciprocally, if $f \in \mathcal{I}^m[L_{p(\cdot)}]$ then the application of Theorem 6.1.4 shows that $\mathbb{D}^m f$ exists in $L_{p(\cdot)}(\mathbb{R}^n)$, which implies $f \in L^m_{p(\cdot)}(\mathbb{R}^n)$. As previously, one gets $f \in W^m_{p(\cdot)}(\mathbb{R}^n)$.

The embedding (6.42) can be proved following similar arguments observing in addition that $\mathcal{B}^m[L_{p(\cdot)}] \hookrightarrow \mathcal{B}^\alpha[L_{p(\cdot)}]$ when $m > \alpha$. □

6.5 A note on the Sobolev embedding theorem

The identity between the Sobolev spaces and the Bessel potential spaces given in Corollary 6.4.2 allows to derive a different proof for the embedding

$$W^m_{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{q(\cdot)}(\mathbb{R}^n) \tag{6.44}$$

with $1 < \underline{p} \leq \overline{p} < \frac{n}{m}$ and $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{m}{n}$, $x \in \mathbb{R}^n$ (see Theorem 4.5.7).

This can be done by noticing that the Bessel potential can be estimated in terms of the Riesz potential.

Let us assume that $f \in W^m_{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{m,p(\cdot)} \leq 1$. We need to show that $\|f\|_{q(\cdot)} \leq C$, which is equivalent to prove that $I_{q(\cdot)}(f) \leq C$ since $q(\cdot)$ is bounded in view of the assumption $m\,\overline{p} < n$.

Since $W^m_{p(\cdot)}(\mathbb{R}^n) = \mathcal{B}^m[L_{p(\cdot)}]$ (with equivalent norms), then $f = G_m * \varphi$ for some function $\varphi \in L_{p(\cdot)}(\mathbb{R}^n)$. But

$$|f(x)| = |G_m * \varphi(x)| \leq c\,\mathcal{I}^m(|\varphi|)(x),$$

with $c > 0$ independent of x, which follows from (6.14).

Now we have $I_{q(\cdot)}(\mathcal{I}^m(|\varphi|)) \leq C$ since

$$\|\mathcal{I}^m(|\varphi|)\|_{q(\cdot)} \leq c\,\|\varphi\|_{p(\cdot)} = c\,\|f\,|\,\mathcal{B}^m[L_{p(\cdot)}] \leq C\,\|f\|_{m,p(\cdot)} \leq C.$$

Therefore

$$\int_{\mathbb{R}^n} |f(x)|^{q(x)} dx \leq c_1 \int_{\mathbb{R}^n} [\mathcal{I}^m(|\varphi|)(x)]^{q(x)}\, dx \leq C,$$

with $C > 0$ not depending on φ and f.

This provides another proof for (6.44). Note that we have assumed that the exponent satisfies the logarithmic assumptions (4.4) and (4.6), since we have used the Sobolev theorem.

Further notes

The characterization of function spaces of fractional smoothness realizes another application of the hypersingular integrals (recall that they have been used in the inversion of the Riesz potential operator).

In general, the statements presented in this chapter extend classical results on the description of the Riesz and Bessel potentials on spaces $L_p(\mathbb{R}^n)$. We restricted ourselves to the case where the Riesz potentials from $\mathcal{I}^\alpha[L_{p(\cdot)}]$ are defined pointwise (that is, when $\overline{p} < \frac{n}{\alpha}$). The characterization of $\mathcal{I}^\alpha[L_{p(\cdot)}]$ in the case $\overline{p} \geq \frac{n}{\alpha}$, and consequently the extension of Theorem 6.3.9 to this case remains an open question. As in the classical framework, it should require a different approach since the Riesz potentials need to be considered in the distributional sense over the Lizorkin space $\Phi(\mathbb{R}^n)$, being it possible in view of the invariance of $\Phi(\mathbb{R}^n)$ with respect to $\mathcal{I}^\alpha$ (see [121] for further details). We also note that the restrictions on the smoothness parameters α and m in Theorem 6.4.3 are due to the initial assumption $\overline{p} < \frac{n}{\alpha}$.

This is the first time that Bessel potentials are considered in the context of the Lebesgue spaces with variable exponent. After our results have been obtained, this subject was studied in the paper [60]. However, only the coincidence of the Bessel potential spaces with the Sobolev spaces was given, without any characterization and comparison with the Riesz potentials.

Historically, the Bessel potentials were introduced by Aronszajn and Smith [7], and Calderón [19]. Concerning Riesz potentials, they were first considered by Riesz [109], but their classical L_p inequalities are due to Hardy and Littlewood [64] in the one-dimensional case, and to Sobolev [127] in the general case.

We notice that the characterization of the space of Riesz potentials in terms of Riesz derivatives was investigated by Samko [116], [117] within the framework of the usual Lebesgue spaces. As remarked in the end of Chapter 5, the description of the Bessel potentials on these spaces is due to Stein [128] and Lizorkin [90].

Chapter 7

Pointwise Inequalities on Variable Sobolev Spaces and Applications

In this chapter we deal with pointwise type inequalities in Sobolev spaces with variable exponent. We recover the known statement that the oscillation of Sobolev functions may be estimated in terms of the fractional maximal function of its gradient (see, for instance, [13], [62], [63], [79]), and use it to study the pointwise behavior of functions in variable exponent Sobolev spaces. More precisely, we study Sobolev embeddings into Hölder spaces with variable order and consider hypersingular integrals of variable Sobolev functions. Our statements are based on pointwise estimates discussed in Section 7.2.

Sobolev embeddings on variable exponent Sobolev spaces have been studied by many authors, mainly in the case when the exponent is less than the dimension (see [31], [45], [46], [47], [108], where Sobolev-type theorems were discussed). The case when the exponent is greater than n was less studied. It was firstly treated in [45], where embeddings into Hölder spaces with variable order have been obtained (see Theorems 5.4 and 5.5 in [45]). More recently, this case was considered in [65] where the capacity approach was used to get embeddings either into the space of continuous functions or into the space of bounded functions.

In Section 7.3, we prove slightly different embeddings into Hölder spaces with variable exponent from those obtained in [45], providing other proofs. We base ourselves on estimation of the oscillation of Sobolev functions f by fractional maximal functions of ∇f. We show, in particular, that Sobolev functions coincide almost everywhere with Hölder continuous functions of variable order, this statement being valid on bounded domains whose boundary is locally the graph of a Lipschitz continuous function (see Theorem 7.3.7 below).

We also consider hypersingular integrals of variable Sobolev functions defined over bounded domains. In Section 7.4, we will derive boundedness and pointwise results for the hypersingular operator of variable order $\alpha = \alpha(x)$ acting in $W^{1,p(\cdot)}(\Omega)$ into an appropriate variable Lebesgue space, in the case when Ω is bounded with Lipschitz boundary. The results obtained here are new, even in the particular case when the exponents are constant.

7.1 On Hölder spaces of variable order

Hölder spaces may been seen as a refinement of the spaces of continuous functions. They are usually denoted by $C^{0,\alpha}(\Omega)$, $\alpha \in (0,1]$, and their elements f are characterized by the *Hölder condition of order* α, that is, there exists $c > 0$ such that

$$|f(x) - f(y)| \leq c\,|x-y|^{\alpha}, \quad x,\, y \in \Omega.$$

These spaces have an important role in the study of regularity properties in the framework of variational calculus and partial differential equations. We will consider an interesting generalization by allowing the order α vary from point to point.

We recall that $C(\Omega)$ stands for the space of all bounded uniformly continuous functions on Ω equipped with the "sup" norm (according to the notation from Chapter 1).

Definition 7.1.1. *Let $\alpha : \Omega \to (0,1]$ be a measurable function. By $C^{0,\alpha(\cdot)}(\Omega)$ we denote the space of all functions f in $C(\Omega)$ for which there exists $c > 0$ such that*

$$|f(x+h) - f(x)| \leq c\,|h|^{\alpha(x)}, \qquad x,\, x+h \in \Omega.$$

This is a natural generalization of the standard Hölder spaces $C^{0,\alpha}(\Omega)$ with constant α. $C^{0,\alpha(\cdot)}(\Omega)$ is a Banach space with respect to the norm

$$\|f \,|\, C^{0,\alpha(\cdot)}(\Omega)\| = \|f\|_{\infty,\Omega} + [f]_{\alpha(\cdot),\Omega}, \tag{7.1}$$

where

$$[f]_{\alpha(\cdot),\Omega} := \sup_{\substack{x,x+h\in\Omega \\ 0<|h|\leq 1}} \frac{|f(x+h) - f(x)|}{|h|^{\alpha(x)}}.$$

As for variable exponent Lebesgue spaces, we shall write $\alpha(\cdot)$ instead of α to emphasize that we are dealing with a variable order of regularity. We observe that for $0 < \alpha(x) \leq \lambda(x) \leq 1$, $x \in \Omega$, one gets

$$C^{0,\lambda(\cdot)}(\Omega) \hookrightarrow C^{0,\alpha(\cdot)}(\Omega) \hookrightarrow C(\Omega).$$

We stress that Hölder spaces with variable order were already considered in the papers [77] and [110], related to mapping properties of fractional integration operators. As we will see below (Section 7.3) they prove to be the natural target spaces when we deal with certain Sobolev embeddings of variable exponent.

7.2 Pointwise inequalities

The oscillation of Sobolev functions can be controlled by the fractional maximal function of its gradient. We refer to [13], [62], [63], [79], where this argument was used to derive important properties of functions in Sobolev spaces within the classical setting. This will be extended to the case of variable exponents (see Proposition 7.2.3 below).

The proof of the following inequality may be found in [57], p. 162.

Lemma 7.2.1. *Let B be a ball in $\mathbb{R}^n$. If $g \in W_1^1(B)$, then*

$$|g(x) - g_B| \leq c(n) \int_B \frac{|\nabla g(z)|}{|x-z|^{n-1}}\, dz$$

almost everywhere in B, where $g_B := \frac{1}{|B|}\int_B g(z)\, dz$ denotes the average of g over B.

Now we recall a classical statement due to Hedberg (see [5]) on the estimation of Riesz potentials through the maximal function. We also give its proof for completeness of the presentation.

Lemma 7.2.2. *Let $D \subset \mathbb{R}^n$ be an open bounded set, $0 < \alpha < n$ and $0 \leq \lambda < \alpha$. Then there exists $c > 0$, not depending on f, x and λ, such that*

$$\int_D \frac{|f(z)|\, dz}{|x-z|^{n-\alpha}} \leq \frac{c}{\alpha - \lambda}\, (diam(D))^{\alpha-\lambda}\, \mathcal{M}_D^{\lambda} f(x), \tag{7.2}$$

for almost all $x \in D$, and for every $f \in L_1(D)$, where it is admitted that λ may depend on x.

Proof. Let $d = diam(D)$. We have

$$\begin{aligned}
\int_D \frac{|f(z)|\,dz}{|x-z|^{n-\alpha}} &= \sum_{k=0}^{\infty} \int_{D\cap\ B(x,\frac{d}{2^k})\setminus B(x,\frac{d}{2^{k+1}})} \frac{|f(z)|}{|x-z|^{n-\alpha}}\,dz \\
&\leq \sum_{k=0}^{\infty} \left(\frac{2^{k+1}}{d}\right)^{n-\alpha} \int_{D\cap B(x,\frac{d}{2^k})} |f(z)|\,dz \\
&= 2^{n-\alpha} \sum_{k=0}^{\infty} \left(\frac{2^k}{d}\right)^{\lambda-\alpha} \left(\frac{d}{2^k}\right)^{\lambda-n} \int_{D\cap B(x,\frac{d}{2^k})} |f(z)|\,dz \\
&\leq c(n) \sum_{k=0}^{\infty} \left(\frac{2^k}{d}\right)^{\lambda-\alpha} \mathcal{M}_D^{\lambda} f(x),
\end{aligned}$$

from which (7.2) follows, since $\frac{1}{2^{\alpha-\lambda}-1} \leq \frac{c}{\alpha-\lambda}$ for some $c > 0$ independent of α and λ. □

Proposition 7.2.3. *Let Ω be a bounded domain with Lipschitz boundary or let $\Omega = \mathbb{R}^n$. Then for every $f \in W^1_{1,loc}(\Omega)$ and almost all $x, y \in \Omega$ there holds*

$$|f(x) - f(y)| \leq c \left[\frac{|x-y|^{1-\lambda}}{1-\lambda} \mathcal{M}_\Omega^{\lambda}(|\nabla f|)(x) + \frac{|x-y|^{1-\mu}}{1-\mu} \mathcal{M}_\Omega^{\mu}(|\nabla f|)(y)\right] \tag{7.3}$$

where $\lambda, \mu \in [0, 1)$ and the constant $c > 0$ does not depend on f, x, y, λ and μ and Ω, and it is admitted that λ and μ may depend on x and y.

Proof. For bounded domains estimate (7.3) can be proved as in [63], Lemma 4. For the case $\Omega = \mathbb{R}^n$ the arguments are similar: we observe that for all x, $y \in \mathbb{R}^n$, $x \neq y$, there exists a ball $B_{x,y}$ containing these points such that $diam(B_{x,y}) \leq 2\,|x-y|$. Then we write

$$|f(x) - f(y)| \leq |f(x) - f_{B_{x,y}}| + |f(y) - f_{B_{x,y}}|$$

and it remains to make use of Lemma 7.2.1 and afterwards Lemma 7.2.2 with $\alpha = 1$. □

Recall that a domain Ω has Lipschitz boundary if, roughly speaking, its boundary is locally the graph of a Lipschitz continuous function (see, for example, [6] for precise definitions).

Estimate (7.3) slightly differs from the usual formulation, since it allows different orders for the maximal functions on the right-hand side as well as the dependence of those orders on the variables. We shall see below that the consideration of maximal functions of variable order is very useful to deal with embeddings of Sobolev spaces of variable exponent.

7.3 Sobolev embeddings with variable exponent

The main aim of this section is to show that functions from $W^1_{p(\cdot)}(\Omega)$ are Hölder continuous everywhere where $p(x) > n$.

First we need some auxiliary statements.

Lemma 7.3.1. *([31]) Let $p : \mathbb{R}^n \to [1, \infty)$ be a continuous exponent. Then $p(\cdot)$ satisfies the log-Hölder condition (4.4) if and only if there exists a constant $C > 0$ such that*

$$|B|^{\inf\limits_{z\in B} \frac{1}{p(z)} - \sup\limits_{z\in B} \frac{1}{p(z)}} \leq C, \tag{7.4}$$

for all open ball B in $\mathbb{R}^n$.

For simplicity, given a ball B in $\mathbb{R}^n$ and a bounded exponent $p : \mathbb{R}^n \to [1, \infty)$, we will denote by $\frac{1}{p_B}$ the average of the function $\frac{1}{p}$ over B, that is,

$$\frac{1}{p_B} := \frac{1}{|B|} \int_B \frac{dz}{p(z)}.$$

Lemma 7.3.2. *([31]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then for every ball B there holds*

$$\|\chi_B\|_{p(\cdot)} \leq c(p)\, |B|^{\frac{1}{p_B}}. \tag{7.5}$$

We will also make use of the following statement, in which

$$\Pi_{p,\Omega} := \{x \in \Omega : p(x) > n\}. \tag{7.6}$$

Lemma 7.3.3. *Let Ω be a bounded domain and let $f \in L_{p(\cdot)}(\Omega)$, where $p(\cdot)$ satisfies condition (4.4). Assume also that the set $\Pi_{p,\Omega}$ is non-empty. Then*

$$\mathcal{M}_\Omega^{\frac{n}{p(x)}} f(x) \leq c\, \|f\|_{p(\cdot),\Omega}, \qquad x \in \Pi_{p,\Omega}, \tag{7.7}$$

with $c > 0$ not depending on x nor f.

Proof. First we observe that the exponent $p(\cdot)$ may be extended to the whole space $\mathbb{R}^n$ with the preservation of its continuity modulus. In fact, since $p(\cdot)$ is uniformly continuous (and bounded) on Ω then it extends to a continuous function on $\overline{\Omega}$. By a known extension result described in [129], Chapter 6, Section 2, there exists an extension $\tilde{p} : \mathbb{R}^n \to [1, \infty)$ satisfying a corresponding condition to (4.4) on $\mathbb{R}^n$ (possibly with a different constant). From $\tilde{p}(\cdot)$ we may construct another extension $\widetilde{\widetilde{p}}(\cdot) : \mathbb{R}^n \to [1, \infty)$, also preserving the continuity modulus, in such a way that $\widetilde{\widetilde{p}}(\cdot)$ is constant outside some large ball (see [31], Theorem 4.2 and Corollary 4.3, for details). In particular, we have $\widetilde{\widetilde{p}}(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

Let $B = B(x,r)$ be any ball centered at $x \in \Pi_{p,\Omega}$. By the Hölder inequality we obtain

$$\frac{1}{|B|^{1-\frac{1}{p(x)}}} \int_{B\cap\Omega} |f(z)|\, dz \leq \frac{c(p)}{|B|^{1-\frac{1}{p(x)}}} \, \|f\|_{\widetilde{\widetilde{p}}(\cdot)} \, \|\chi_B\|_{\widetilde{\widetilde{p}}'(\cdot)},$$

where $\widetilde{\widetilde{p}}'(\cdot)$ is the usual conjugate exponent, $\frac{1}{\widetilde{\widetilde{p}}(\cdot)} + \frac{1}{\widetilde{\widetilde{p}}'(\cdot)} = 1$, and it is assumed that f is continued as zero beyond Ω.

Since $\widetilde{\widetilde{p}}(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then the maximal operator is also bounded in $L^{\widetilde{\widetilde{p}}'(\cdot)}(\mathbb{R}^n)$ (see [29], Lemma 8.1). Therefore, Lemma 7.3.2 is applicable which yields $\|\chi_B\|_{\widetilde{\widetilde{p}}'(\cdot)} \leq c_1(p)\; |B|^{\frac{1}{\widetilde{\widetilde{p}}'_B}}$. Hence,

$$\frac{1}{|B|^{1-\frac{1}{p(x)}}} \int_{B\cap\Omega} |f(z)|\, dz \leq c\, \|f\|_{p(\cdot),\Omega} \, |B|^{\frac{1}{p(x)} - \frac{1}{\widetilde{\widetilde{p}}_B}}. \tag{7.8}$$

If $|B| \leq 1$ then Lemma 7.3.1 provides the estimate $|B|^{\frac{1}{p(x)} - \frac{1}{\widetilde{\widetilde{p}}_B}} \leq C$, for some $C > 0$ independent of B. Suppose now that $|B| > 1$. Notice that if $r > diam(\Omega)$ then $|B| > |\Omega|$, so that

$$\frac{1}{|B|^{1-\frac{1}{p(x)}}} \int_{B\cap\Omega} |f(z)|\, dz \leq \frac{1}{|\Omega|^{1-\frac{1}{p(x)}}} \int_{\Omega} |f(z)|\, dz \leq c(\Omega)\, \|f\|_{p(\cdot),\Omega}.$$

Hence, only the case $r \leq diam(\Omega)$ is of interest according to our purposes. In that case, the right-hand side in (7.8) may be estimated as follows:

$$|B|^{\frac{1}{p(x)} - \frac{1}{\widetilde{\widetilde{p}}_B}} \leq |B|^{1-\frac{1}{\widetilde{\widetilde{p}}}} \leq C\, diam(\Omega)^n.$$

This completes the proof of (7.7). □

Now we are able to give an important pointwise inequality.

Theorem 7.3.4. *Let Ω be a bounded domain with Lipschitz boundary and suppose that $p(\cdot)$ satisfies the local logarithmic condition (4.4) and has a non-empty set $\Pi_{p,\Omega}$. If $f \in W^1_{p(\cdot)}(\Omega)$, then*

$$|f(x) - f(y)| \leq C(x,y)\, \||\nabla f|\|_{p(\cdot),\Omega}\, |x-y|^{1-\frac{n}{\min[p(x),p(y)]}} \tag{7.9}$$

for all $x, y \in \Pi_{p,\Omega}$ such that $|x-y| \leq 1$, where

$$C(x,y) = \frac{c}{\min[p(x),p(y)] - n}$$

with $c > 0$ not depending on f, x and y.

Proof. After modifying f on a set of zero measure, we make use of (7.3) with $\lambda = \frac{n}{p(x)} \in (0,1)$ and $\mu = \frac{n}{p(y)} \in (0,1)$ and get

$$|f(x) - f(y)| \leq \frac{c\,|x-y|^{1-\frac{n}{\min[p(x),p(y)]}}}{\min[p(x)-n, p(y)-n]} \left[\mathcal{M}_\Omega^{\frac{n}{p(x)}}(|\nabla f|)(x) + \mathcal{M}_\Omega^{\frac{n}{p(y)}}(|\nabla f|)(y)\right] \tag{7.10}$$

for all $x, y \in \Pi_{p,\Omega}$. Hence, (7.9) immediately follows from (7.7). □

Remark 7.3.5. Let D be a subset in $\Pi_{p,\Omega}$. Under the assumption $\inf\limits_{x\in D} p(x) > n$, one may take a constant in (7.9) not depending on x, y when x and y run the set D. In particular, if $\underline{p}_\Omega > n$, estimate (7.9) is valid for the whole Ω with an absolute constant.

Corollary 7.3.6. *Let Ω be a bounded domain with Lipschitz boundary and let $p(\cdot)$ be under the assumptions of Theorem 7.3.4. If $f \in W^1_{p(\cdot)}(\Omega)$, then estimate (7.9) may be written in the form*

$$|f(x) - f(x+h)| \leq \frac{c}{\min[p(x), p(x+h)] - n}\, \||\nabla f|\|_{p(\cdot),\Omega}\, |h|^{1-\frac{n}{p(x)}}, \tag{7.11}$$

where x, $x+h \in \Pi_{p,\Omega}$ and $|h| \leq 1$, with $c > 0$ not depending on x, h, and f.

Proof. It suffices to observe that for x and y belonging to a bounded set we have

$$|x-y|^{\frac{n}{p(x)}} \sim |x-y|^{\frac{n}{p(y)}} \tag{7.12}$$

thanks to the log-condition for $p(\cdot)$. Indeed, inequality (7.12) is equivalent to

$$\frac{1}{C} \leq |x-y|^{\frac{1}{p(x)} - \frac{1}{p(y)}} \leq C, \tag{7.13}$$

with $C > 1$ not depending on x and y. But (7.13) means that

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \ln \frac{1}{|x-y|} \leq C_1, \qquad C_1 = \ln C,$$

if $0 < |x-y| \leq 1$, or

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \ln |x-y| \leq C_1, \qquad C_1 = \ln C,$$

in the case $1 < |x-y| \leq diam(\Omega)$. It remains to observe that these inequalities are immediately verified, since $p(\cdot)$ satisfies (4.4):

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq |p(x) - p(y)| \leq \frac{2\,diam(\Omega)\,A_0}{\ln \frac{2\,diam(\Omega)}{|x-y|}}.$$

□

Theorem 7.3.4 suggests that functions in $W^1_{p(\cdot)}(\Omega)$ admit a Hölder continuous representative of variable order.

Theorem 7.3.7. *Let Ω be a bounded domain and suppose that $p(\cdot)$ satisfies the logarithmic condition (4.4). If $\inf_{x\in\Omega} p(x) > n$, then the estimate*

$$|f(x)| \le C\left[\frac{\|f\|_{p(\cdot),\Omega}}{[dist(x,\partial\Omega)]^{\frac{n}{p(x)}}} + \||\nabla f|\|_{p(\cdot),\Omega}\right] \tag{7.14}$$

is valid, with $C > 0$ independent of $x \in \Omega$ and $f \in W^1_{p(\cdot)}(\Omega)$.
If, in addition, Ω has Lipschitzian boundary, then we have

$$W^1_{p(\cdot)}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p(\cdot)}}(\Omega). \tag{7.15}$$

Proof. Fix $x \in \Omega$ and let B_x be a ball containing x. According to Lemma 7.2.1, estimate (7.2) with $\alpha = 1$ and $\lambda = \frac{n}{p(x)}$, and inequality (7.7), for $f \in W^1_{p(\cdot)}(\Omega)$ we have

$$|f(x) - f_{B_x}| \le c\, diam(B_x)^{1-\frac{n}{p(x)}}\, \mathcal{M}_{\Omega}^{\frac{n}{p(x)}}(|\nabla f|)(x) \le c\, diam(B_x)^{1-\frac{n}{p(x)}}\, \||\nabla f|\|_{p(\cdot),\Omega}, \tag{7.16}$$

where it is assumed that the radius r of the ball B_x is sufficiently small, say $r = \frac{1}{2}\, dist(x, \partial\Omega)$. For f_{B_x} we may proceed as in the proof of Lemma 7.3.3. Hence, the Hölder inequality combined with (7.5) yield the estimate

$$|f_{B_x}| \le c(p)\, |B_x|^{-\frac{1}{p_{B_x}}}\, \|f\|_{p(\cdot),\Omega}.$$

Since $|B_x|^{-\frac{1}{p_{B_x}}} \le c\, |B_x|^{-\frac{1}{p(x)}}$, we also have

$$|f_{B_x}| \le c(p)\, |B_x|^{-\frac{1}{p(x)}}\, \|f\|_{p(\cdot),\Omega}. \tag{7.17}$$

Thus, having in mind the value of r above, we arrive at (7.14) from (7.16) and (7.17).

If Ω has Lipschitz boundary, then we may derive the embedding

$$W^1_{p(\cdot)}(\Omega) \hookrightarrow L_\infty(\Omega). \tag{7.18}$$

In fact, in that case, it is known (see [31]) that there exists a bounded linear extension operator

$$\mathcal{E} : W^1_{p(\cdot)}(\Omega) \to W^1_{\widetilde{p}(\cdot)}(\mathbb{R}^n)$$

such that $\mathcal{E}f|_{\Omega} = f$ almost everywhere, where $\widetilde{p}(\cdot)$ is the extension of $p(\cdot)$ used in the proof of Lemma 7.3.3. Similarly to (7.16), there holds

$$|f(x) - (\mathcal{E}f)_{B_x}| \le c\, diam(B_x)^{1-\frac{n}{p(x)}}\, \||\nabla f|\|_{p(\cdot),\Omega},$$

where now we suppose that the ball B_x is arbitrary (containing x). Moreover, we have

$$|(\mathcal{E}f)_{B_x}| \leq c\,|B_x|^{-\frac{1}{\tilde{\tilde{p}}_{B_x}}}\,\|\mathcal{E}f\|_{\tilde{\tilde{p}}(\cdot)} \leq c_1\,|B_x|^{-\frac{1}{\tilde{\tilde{p}}_{B_x}}}\,\|f\|_{p(\cdot),\Omega}.$$

Taking a ball such that $|B_x| = 1$, we get

$$|f(x)| \leq |f(x) - f_{B_x}| + |f_{B_x}| \leq C(p)\,\|f\|_{1,p(\cdot),\Omega},$$

which implies (7.18). The embedding (7.15) follows then from (7.11) and (7.18). □

In the particular case when the exponent is constant, $p(x) \equiv p > n$, we recover the classic Sobolev embedding into the standard Hölder spaces.

7.4 Hypersingular operators on $W^1_{p(\cdot)}(\Omega)$

In this section, we consider hypersingular integral operators of variable order $\alpha = \alpha(x)$, $0 < \alpha(x) < 1$, $x \in \Omega$, given by

$$\mathcal{D}^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(x) - f(y)}{|x-y|^{n+\alpha(x)}}\,dy, \quad x \in \Omega, \tag{7.19}$$

(see also (5.2). We recall that detailed information about hypersingular integrals of functions defined in $\mathbb{R}^n$ can be found in [121].

Theorem 7.4.1. *Let $0 < \alpha_0 \leq \alpha(x) \leq \alpha_1 < 1$ and let Ω be a bounded domain with Lipschitz boundary. Assume also that $p(\cdot)$ satisfies (4.4) and*

$$\operatorname*{ess\,sup}_{x\in\Omega} p(x)[1-\alpha(x)] < n. \tag{7.20}$$

Then the operator $\mathcal{D}^{\alpha(\cdot)}$ is bounded from $W^1_{p(\cdot)}(\Omega)$ into $L_{q(\cdot)}(\Omega)$ for any exponent $q(\cdot)$ such that $1 < \underline{q}_\Omega \leq \overline{q}_\Omega < \infty$ and

$$\operatorname*{ess\,sup}_{x\in\Omega} \left[\frac{1}{p(x)} - \frac{1}{q(x)} + \frac{\alpha(x)}{n}\right] < \frac{1}{n}, \tag{7.21}$$

the latter being equivalent to

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\lambda(x)}{n}, \qquad \textit{where} \qquad \operatorname*{ess\,sup}_{x\in\Omega} [\lambda(x) + \alpha(x)] < 1. \tag{7.22}$$

Proof. We may assume that $q(x) \geq p(x)$ since the domain is bounded and one has the imbedding $\|f\|_{q(\cdot)} \leq c\|f\|_{q^*(\cdot)}$ where $q^*(x) = \max\{q(x), p(x)\}$.

By Proposition 7.2.3, we have

$$\begin{aligned}\left|\mathcal{D}^{\alpha(\cdot)}f(x)\right| &\leq \int_\Omega \frac{|f(x)-f(y)|}{|x-y|^{n+\alpha(x)}}\,dy \\ &\leq \frac{c}{1-\lambda(x)}\int_\Omega \frac{\mathcal{M}_\Omega^{\lambda(\cdot)}(|\nabla f|)(x)+\mathcal{M}_\Omega^{\lambda(\cdot)}(|\nabla f|)(y)}{|x-y|^{n+\alpha(x)+\lambda(x)-1}}\,dy\end{aligned}$$

for almost all $x \in \Omega$, with $c > 0$ not depending on x and f, where $\lambda(x)$ may be an arbitrary function such that $0 \leq \lambda(x) < 1$.

Put $\beta(x) = 1-\alpha(x)-\lambda(x)$. Then $0 < \beta(x) < 1$ under the choice $\lambda(x) < 1-\alpha(x)$. We choose $\lambda(x)$ so that

$$\lambda(x) \geq 0 \qquad \text{and} \qquad \operatorname*{ess\,sup}_{x\in\Omega}\,[\lambda(x)+\alpha(x)] < 1, \tag{7.23}$$

which is possible, since $\operatorname*{ess\,sup}_{x\in\Omega}\alpha(x) \leq \alpha_1 < 1$. Then

$$\operatorname*{ess\,inf}_{x\in\Omega}\beta(x) > 0. \tag{7.24}$$

We have

$$\begin{aligned}\left|\mathcal{D}^{\alpha(\cdot)}f(x)\right| &\leq c\int_\Omega \frac{\mathcal{M}_\Omega^{\lambda(\cdot)}(|\nabla f|)(x)}{|x-y|^{n-\beta(x)}}\,dy + c\int_\Omega \frac{\mathcal{M}_\Omega^{\lambda(\cdot)}(|\nabla f|)(y)}{|x-y|^{n-\beta(x)}}\,dy \\ &\leq c\,|\Omega|^{\frac{\beta(x)}{n}}\,\mathcal{M}_\Omega^{\lambda(\cdot)}(|\nabla f|)(x) + c\,\mathcal{I}_\Omega^{\beta(\cdot)}\left[\mathcal{M}_\Omega^{\lambda(\cdot)}(|\nabla f|)\right](x),\end{aligned} \tag{7.25}$$

where $\mathcal{I}_\Omega^{\beta(\cdot)}$ denotes the variable Riesz potential as before. Since this inequality holds pointwise (almost everywhere), we may take the Lebesgue norms in both sides and write

$$\left\|\mathcal{D}^{\alpha(\cdot)}f\right\|_{q(\cdot),\Omega} \leq c\,\left\|\mathcal{M}_\Omega^{\lambda(\cdot)}(|\nabla f|)\right\|_{q(\cdot),\Omega} + c\,\left\|\mathcal{I}_\Omega^{\beta(\cdot)}\left[\mathcal{M}_\Omega^{\lambda(\cdot)}(|\nabla f|)\right]\right\|_{q(\cdot),\Omega}. \tag{7.26}$$

By condition (7.24) and the boundedness of the domain Ω, the operator $\mathcal{I}_\Omega^{\beta(\cdot)}$ is bounded in the space $L_{q(\cdot)}(\Omega)$ so that

$$\left\|\mathcal{D}^{\alpha(\cdot)}f\right\|_{q(\cdot),\Omega} \leq c\,\left\|\mathcal{M}_\Omega^{\lambda(\cdot)}(|\nabla f|)\right\|_{q(\cdot),\Omega}. \tag{7.27}$$

By Theorem 4.5.6 we then have

$$\left\|\mathcal{D}^{\alpha(\cdot)}f\right\|_{q(\cdot),\Omega} \leq c\,\||\nabla f|\|_{p(\cdot),\Omega} \leq c\,\|f\|_{1,p(\cdot),\Omega}, \qquad \frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\lambda(x)}{n},$$

that theorem being applicable since

$$\operatorname*{ess\,sup}_{x\in\Omega}\lambda(x)p(x) < \operatorname*{ess\,sup}_{x\in\Omega}\,[1-\alpha(x)]p(x) < n,$$

according to (7.23) and (7.20).

Thus the boundedness of $\mathcal{D}^{\alpha(\cdot)}$ from $W^1_{p(\cdot)}(\Omega)$ into $L_{q(\cdot)}(\Omega)$ has been proved for exponents $q(\cdot)$ of the form (7.22). It remains to show the equivalence of (7.22) to (7.21).

Condition (7.21) is easily obtained from (7.22). Conversely, condition (7.21) implies the existence of $\delta \in \left(0, \frac{1}{n}\right)$ such that

$$\frac{1}{p(x)} - \frac{1}{q(x)} + \frac{\alpha(x)}{n} \leq \frac{1}{n} - \delta \tag{7.28}$$

for almost all $x \in \Omega$. Let us define

$$\lambda(x) := n\left[\frac{1}{p(x)} - \frac{1}{q(x)}\right], \quad x \in \Omega.$$

Then according to (7.28) we obtain

$$\lambda(x) + \alpha(x) = n\left[\frac{1}{p(x)} - \frac{1}{q(x)} + \frac{\alpha(x)}{n}\right] \leq n\left(\frac{1}{n} - \delta\right) = 1 - n\delta$$

almost everywhere. Therefore, one gets

$$\operatorname*{ess\,sup}_{x\in\Omega}\,[\lambda(x) + \alpha(x)] < 1.$$

□

Remark 7.4.2. In (7.25) we made use of the estimation of Riesz potentials of constant densities. Let us note that this can be easily obtained as follows:

$$\int_\Omega \frac{dy}{|x-y|^{n-\beta(x)}} \leq \int_{|x-y|\leq diam(\Omega)} \frac{dy}{|x-y|^{n-\beta(x)}} = \frac{c(n)}{\beta(x)}\,[diam(\Omega)]^{\beta(x)},$$

which allows to arrive at (7.25) taking into account assumption (7.24).

For constant exponents the following holds.

Corollary 7.4.3. *Let α and Ω be as in Theorem 7.4.1 and suppose that $1 < p < \frac{n}{1-\alpha}$. Then there exists $c > 0$ such that*

$$\|\mathcal{D}^\alpha f\|_{q,\Omega} \leq c\,\|f\|_{1,p,\Omega}, \quad f \in W^1_p(\Omega),$$

for any exponent q fulfilling

$$p \leq q < \frac{np}{n-(1-\alpha)p}. \tag{7.29}$$

Theorem 7.3.4 (and Corollary 7.3.6) allows us to make some conclusions about the pointwise convergence of the hypersingular integral. More precisely, the following statement may be derived.

Proposition 7.4.4. *Let Ω be a bounded domain with Lipschitz boundary. Under the assumption (4.4) on $p(\cdot)$, the hypersingular integral $\mathcal{D}^{\alpha(\cdot)}$, with $0 < \alpha_0 \leq \alpha(x) < 1$, $x \in \Omega$, of functions in $W^1_{p(\cdot)}(\Omega)$ converges at all those points $x \in \Omega$ for which $p(x)(1-\alpha(x)) > n$.*

Proof. The pointwise convergence of the hypersingular integral is an immediate consequence of (7.9). We only observe that the assumption $p(x)(1-\alpha(x)) > n$ implies $\operatorname{ess\,inf}_{x\in\Omega} p(x) > n$. $\square$

Further notes

There are many papers in the literature dealing with Sobolev embeddings starting from the original work due to Sobolev [127]. Our results contribute to the development of this topic within the framework of the variable exponent spaces. We recall that the case studied was already considered in [45]. However, we followed a different approach based on generalized maximal functions of variable order. Also, we dealt explicitly with variable Hölder spaces.

The statements above on boundedness and pointwise convergence of hypersingular integrals are new even in the case when the exponents $p(\cdot)$ and $\alpha(\cdot)$ are constant. They were obtained under the usual log-condition on the integrability exponent. We stress, however, that no special assumption was required to the regularity exponent $\alpha(\cdot)$.

We worked over smooth domains having Lipschitz boundary. An interesting question would be to check the possibility to derive similar results over more general domains.

Bibliography

[1] E. Acerbi and G. Mingione, *Functionals with $p(x)$ growth*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. **11** (2000), no. 3, 169–174. {2,54}

[2] ______, *Regularity results for a class of functionals with non-standard growth*, Arch. Ration. Mech. Anal. **156** (2001), no. 2, 121–140. {54}

[3] ______, *Regularity results for stationary electrorheological fluids*, Arch. Ration. Mech. Anal. **164** (2002), no. 3, 213–259. {56}

[4] ______, *Gradient estimates for the $p(x)$-Laplacean system*, J. Reine Angew. Math. **584** (2005), 117–148. {54}

[5] D. Adams and L.I. Hedberg, *Function spaces and potential theory*, Springer-Verlag, 1996. {61,62,105}

[6] R. Adams and J. Fournier, *Sobolev spaces (2nd ed.)*, Elsevier, 2003. {106}

[7] N. Aronszajn and K.T. Smith, *Theory of Bessel potentials, I*, Ann. Inst. Fourier **11** (1961), 385–475. {102}

[8] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976. {18,20,21,26}

[9] O. Besov, *On a family of function spaces. Embedding theorems and extensions (Russian)*, Dokl. Akad. Nauk. SSSR **126** (1959), no. 1, 1163–1165. {14}

[10] O.V. Besov, *Embeddings of spaces of differentiable functions of variable smoothness*, Proc. Steklov Inst. Math. 214, 19-53 (1996); translation from Tr. Mat. Inst. Steklova 214, 25-58 , 1997. {1}

[11] ______, *Interpolation of spaces of differentiable functions on a domain*, Proc. Steklov Inst. Math. 214, 54-76 (1996); translation from Tr. Mat. Inst. Steklova 214, 59-82 , 1997. {1}

[12] ______, *Equivalent normings of spaces of functions of variable smoothness*, Function spaces, approximations, and differential equations. Collected papers dedicated to Oleg Vladimirovich Besov on his 70th birthday. Transl. from the Russian. Moscow: Maik Nauka/Interperiodika. Proceedings of the Steklov Institute of Mathematics 243, 80-88 (2003); translation from Tr. Mat. Inst. Steklova 243, 87-95 , 2003. {1}

[13] B. Bojarski and P. Hajłasz, *Pointwise inequalities for Sobolev functions and some applications*, Stud. Math. **106** (1993), no. 1, 77–92. {103,105}

[14] D.W. Boyd, *The Hilbert transform on rearrangement-invariant spaces*, Canadian J. Math. **19** (1967), 599–616. {19}

[15] A. Cabada and R.L. Pouso, *Existence theory for functional p-Laplacian equations with variable exponents*, Nonlinear Anal. **52** (2003), no. 2, 557–572. {54}

[16] A. Caetano and S.D. Moura, *Local growth envelopes of spaces of generalized smoothness: the sub-critical case*, Math. Nachr. **273**, 43–57. {30}

[17] A.M. Caetano, *Subatomic representation of Bessel potential spaces modelled on Lorentz spaces*, the J. A. Sampaio Martins Anniversary Volume, Textos de Matemática, Série B, 34, University of Coimbra, 2004, pp. 37–47. {34}

[18] A.M. Caetano and W. Farkas, *Local growth envelopes of Besov spaces of generalized smoothness*, Z. Anal. Anwend. (to appear). {17}

[19] A.P. Calderón, *Lebesgue spaces of differentiable functions and distributions*, Proc. Sympos. Pure Math. **4** (1961), 33–49. {3,99,102}

[20] C. Capone, D. Cruz-Uribe, and A. Fiorenza, *The fractional maximal function on variable L^p spaces*, Preprint of Istituto per le Applicazioni del Calcolo "Mauro Picone" - Sezione di Napoli (2004), RT 281/04, 22p. (http://www.iam.na.cnr.it). {54,61,62}

[21] V. Chiadò Piat and A. Coscia, *Hölder continuity of minimizers of functionals with variable growth exponent*, Manuscr. Math. **93** (1997), no. 3, 283–299. {54}

[22] F. Cobos, *On the Lorentz-Marcinkiewicz operator ideal*, Math. Nachr. **126** (1986), 281–300. {31,44}

[23] F. Cobos and D.L. Fernandez, *Hardy-Sobolev spaces and Besov spaces with a function parameter*, Lecture Notes in Math., vol. 1302, Springer, 1988, pp. 158–170. {1,2,3,13,15,17,21,25,26,32,44,49}

[24] D. Cruz-Uribe, A. Fiorenza, J.M. Martell, and C. Pérez, *The boundedness of classic operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. (to appear). {54,59}

[25] D. Cruz-Uribe, A. Fiorenza, and C.J. Neugebauer, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238. {58}

[26] S. Dahlke and R. DeVore, *Besov regularity for elliptic boundary value problems*, Commun. Partial Differ. Equations **22** (1997), no. 1-2, 1–16. {14}

[27] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Commun. Pure Appl. Math. **41** (1988), no. 7, 909–996. {35}

[28] ———, *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1992. {33,35}

[29] L. Diening, *Maximal function on Musiełak-Orlicz spaces and generalized Lebesgue spaces*, Preprint, Math. Fak., Albert-Ludwigs-Universitat Freiburg (2003), 21/2003, 46p. {58,108}

[30] ———, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7** (2004), no. 2, 245–253. {2,54,57,58,59}

[31] ———, *Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr. **268** (2004), 31–43. {54,58,61,62,103,107,108,110}

[32] L. Diening and P. Hasto, *Variable exponent trace spaces*, Preprint, Math. Fak., Albert-Ludwigs-Universitat Freiburg (2005), 05/2005, 12p. {54}

[33] L. Diening, P. Hasto, and A. Nekvinda, *Open problems in variable exponent Lebesgue and Sobolev spaces*, Drábek, P. and Rákosník, J. (ed.), Proceedings of the conference "Function Spaces, Differential Operators and Nonlinear Analysis" dedicated to the 70th

birthday of Alois Kufner, Milovy, Czech Republic, May 28 - June 2, 2004. Prague: Mathematical Institute, Academy of Sciences of the Czech Republic, 38-58, 2005. {53}

[34] L. Diening and M. Růžička, *Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Angew. Math. **563** (2003), 197–220. {2,54,59}

[35] ______, *Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$*, J. Math. Anal. Appl. **298** (2004), no. 2, 559–571. {54,60}

[36] ______, *Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, II*, J. Math. Anal. Appl. **298** (2004), no. 2, 572–588. {54}

[37] J. Duoandikoetxea, *Fourier Analysis*, American Mathematical Society, Providence, 2001. {61}

[38] D. Edmunds and A. Nekvinda, *Averaging operators on $l^{\{p_n\}}$ and $L^{p(x)}$*, Math. Inequal. Appl. **5** (2002), no. 2, 235–246. {54,55}

[39] D. Edmunds and H. Triebel, *Spectral theory for isotropic fractal drums*, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), 1269–1274. {16}

[40] ______, *Eigenfrequencies of isotropic fractal drums*, Rossmann, Jürgen (ed.) et al., The Maz'ya anniversary collection. Vol. 2: Rostock conference on functional analysis, partial differential equations and applications, Rostock, Germany, August 31-September 4, 1998. Basel: Birkhäuser. Oper. Theory, Adv. Appl. 110, 81-102 , 1999. {16}

[41] D.E. Edmunds, V. Kokilashvili, and A. Meskhi, *A trace inequality for generalized potentials in Lebesgue spaces with variable exponent*, J. Funct. Spaces Appl. **2** (2004), no. 1, 55–69. {54}

[42] D.E. Edmunds, J. Lang, and A. Nekvinda, *On $L^{p(x)}$ norms*, Proc. R. Soc. Lond., Ser. A **455** (1999), 219–225. {54}

[43] D.E. Edmunds and A. Meshki, *Potential-type operators in $L^{p(x)}$ spaces*, Z. Anal. Anwend. **21** (2002), no. 3, 681–690. {54}

[44] D.E. Edmunds and J. Rákosník, *Density of smooth functions in $W^{k,p(x)}(\Omega)$*, Proc. R. Soc. Lond., Ser. A **437** (1992), 229–236. {54,55,57}

[45] ———, *Sobolev embeddings with variable exponent*, Stud. Math. **143** (2000), no. 3, 267–293. {54,62,103,114}

[46] ———, *Sobolev embeddings with variable exponent, II*, Math. Nachr. **246/247** (2002), 53–67. {54,103}

[47] X. Fan, J. Shen, and D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl. **262** (2001), no. 2, 749–760. {54,62,103}

[48] X. Fan, Q. Zhang, and D. Zhao, *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl. **302** (2005), no. 2, 306–317. {54}

[49] X. Fan and Q.-H. Zhang, *Existence of solutions for p(x)-Laplacian Dirichlet problem*, Nonlinear Anal. **52** (2003), no. 8, 1843–1852. {54}

[50] X. Fan and D. Zhao, *Regularity of minimizers of variational integrals with continuous $p(x)$-growth conditions*, Chin. Ann. Math., Ser. A **17** (1996), no. 4, 327–336. {54}

[51] ———, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), no. 2, 424–446. {54,55,57}

[52] W. Farkas and H.-G. Leopold, *Characterizations of function spaces of generalized smoothness*, Annali Math. Pura Appl. **185** (2006), no. 1, 1–62. {1,14,16,43}

[53] A. Fiorenza, *A mean continuity type result for certain Sobolev spaces with variable exponent*, Commun. Contemp. Math. **4** (2002), no. 3, 587–605. {54,56}

[54] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), 777–799. {38}

[55] ———, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), 34–170. {38}

[56] G. Garrigós and A. Tabacco, *Wavelet decompositions of anisotropic Besov spaces*, Math. Nachr. **239/240** (2002), 80–102. {33}

[57] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 2001. {105}

[58] D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J. **46** (1979), no. 1, 27–42. {9}

[59] M.L. Gol'dman, *Description of the trace space for functions of a generalized Liouville class*, Dokl. Akad. Nauk. SSSR **231** (1976), 525–528. {1,14}

[60] P. Gurka, P. Harjulehto, and A. Nekvinda, *Bessel potential spaces with variable exponent*, Preprint, Dept. of Mathematics and Statistics, University of Helsinki (2005), 12p. (http://www.math.helsinki.fi/analysis/varsobgroup/). {102}

[61] J. Gustavsson, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand. **42** (1978), 289–305. {32}

[62] P. Hajłasz, *Boundary behaviour of Sobolev mappings*, Proc. Am. Math. Soc. **123** (1995), no. 4, 1145–1148. {103,105}

[63] P. Hajłasz and O. Martio, *Traces of Sobolev functions on fractal type sets and characterization of extension domains*, J. Funct. Anal. **143** (1997), 221–246. {103,105,106}

[64] G.H. Hardy and J.E. Littlewood, *Some properties of fractional integrals I*, Math. Z. **27** (1927), 565–606, *II* (ibid), **34**, (1932), 403-439. {102}

[65] P. Harjulehto and P. Hasto, *A capacity approach to the Poincaré inequality and Sobolev imbeddings in variable exponent Sobolev spaces*, Rev. Mat. Complut. **17** (2004), no. 1, 129–146. {54,103}

[66] ———, *Lebesgue points in variable exponent spaces*, Ann. Acad. Sci. Fenn. Math. **29** (2004), no. 2, 295–306. {54}

[67] P. Harjulehto, P. Hasto, M. Koskenoja, and S. Varonen, *Sobolev capacity on the space* $W^{1,p(\cdot)}(\mathbb{R}^n)$, J. Funct. Spaces Appl. **1** (2003), no. 1, 17–33. {54}

[68] P. Harjulehto, P. Hasto, and M. Pere, *Variable exponent Lebesgue spaces on metric spaces: the Hardy-Littlewood maximal operator*, Real Anal. Exchange **30** (2004/05), no. 1, 87–103. {54}

[69] D.D. Haroske and H. Triebel, *Wavelet bases and entropy numbers in weighted function spaces*, Math. Nachr. **278** (2005), no. 1-2, 108–132. {33,38}

[70] P. Hasto, *On the density of continuous functions in variable exponent Sobolev spaces*, Preprint, Dept. of Mathematics and Statistics, University of Helsinki (2004), 16p. (http://www.math.helsinki.fi/analysis/varsobgroup/). {57}

[71] H. Hudzik, *The problems of separability, duality, reflexivity and of comparison for generalized Orlicz-Sobolev spaces* $W^k_M(\Omega)$, Commentat. Math. **21** (1979), 315–324. {53}

[72] ———, *Uniform convexity of Musielak-Orlicz spaces with Luxemburg norm*, Ann. Soc. Math. Pol., Ser. I, Commentat. Math. **23** (1983), 21–32. {53}

[73] S. Jaffard, *Beyond Besov spaces I: distributions of wavelet coefficients*, J. Fourier Anal. Appl. **10** (2004), no. 3, 221–246. {14}

[74] T.F. Kalugina, *Interpolation of Banach spaces with a functional parameter. The reiteration theorem*, Moscow Univ. math. Bull. **30** (1975), no. 6, 108–116. {32}

[75] G.A. Kalyabin and P.I. Lizorkin, *Spaces of functions of generalized smoothness*, Math. Nachr. **133** (1987), 7–32. {1,14}

[76] N.K. Karapetyants and A.I. Ginzburg, *Fractional integrals and singular integrals in the Hölder classes of variable order*, Integral Transforms Spec. Funct. **2** (1994), no. 2, 91–106. {56}

[77] ———, *Fractional integrodifferentiation in Hölder classes of arbitrary order*, Georgian Math. J. **2** (1995), no. 2, 141–150. {56,105}

[78] A. Karlovich and A. Lerner, *Commutators of singular integrals on generalized* L^p *spaces with variable exponent*, Publ. Mat. **49** (2005), 111–125. {54}

[79] J. Kinnunen and O. Martio, *Hardy's inequalities for Sobolev functions*, Math. Res. Lett. **4** (1997), no. 4, 489–500. {103,105}

[80] V. Kokilashvili, *On a Progress in the Theory of Integral Operators in Weighted Banach Function Spaces*, Drábek, P. and Rákosník, J. (ed.), Proceedings of the conference "Function Spaces, Differential Operators and Nonlinear Analysis" dedicated to the 70th

birthday of Alois Kufner, Milovy, Czech Republic, May 28 - June 2, 2004. Prague: Mathematical Institute, Academy of Sciences of the Czech Republic, 152-175, 2005. {53}

[81] V. Kokilashvili and S. Samko, *On Sobolev theorem for Riesz-type potentials in Lebesgue spaces with variable exponent*, Z. Anal. Anwend. **22** (2003), no. 4, 899–910. {54,61}

[82] ______, *Singular integrals and potentials in some Banach function spaces with variable exponent*, J. Funct. Spaces Appl. **1** (2003), no. 1, 45–59. {60}

[83] ______, *Singular integrals in weighted Lebesgue spaces with variable exponent*, Georgian Math. J. **10** (2003), no. 1, 145–156. {54,60}

[84] ______, *Maximal and fractional operators in weighted $L^{p(x)}$ spaces*, Rev. Mat. Iberoamericana **20** (2004), 493–515. {54,58,61}

[85] O. Kovácik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czech. Math. J. **41(416)** (1991), no. 4, 592–618. {53,55,56,57,62,66}

[86] H.-G. Leopold, *On Besov spaces of variable order of differentiation*, Z. Anal. Anwend. **8** (1989), no. 1, 69–82. {1}

[87] ______, *On function spaces of variable order of differentiation*, Forum Math. **3** (1991), no. 1, 1–21. {1}

[88] ______, *Embeddings and entropy numbers in Besov spaces of generalized smoothness*, Hudzik, Henryk (ed.) et al., Function spaces. Proceedings of the 5th international conference, Poznan, Poland, August 28-September 3, 1998. New York, NY: Marcel Dekker. Lect. Notes Pure Appl. Math. 213, 323-336 , 2000. {16}

[89] S. Levine, Y. Chen, and J. Stanich, *Image restoration via nonstandard diffusion*, Preprint, Dept. of Mathematics and Computer Science, Duquesne University (2004), TR 04–1, 13p. (http://www.mathcs.duq.edu/ sel/). {2,54}

[90] P.I. Lizorkin, *Generalized Liouville differentiation and the functional spaces $L_p^r(E_n)$. Imbedding theorems (Russian)*, Math. Sb. **60** (1963), 325–353. {75,97,102}

[91] L. Maligranda, *Indices and interpolation*, Dissert. Math. **234** (1985), 1–49. {19}

[92] ______, *Orlicz Spaces and Interpolation*, Seminars in Mathematics 5, Univ. Estadual de Campinas, Campinas SP, Brazil, 1989. {19}

[93] C. Merucci, *Interpolation réelle avec fonction paramètré: réiteration et applications aux espaces $\Lambda^p(\phi)$ $(0 < p \leq +\infty)$*, C. R. Acad. Sci. Paris, Sér. I **295** (1982), 427–430. {32}

[94] ______, *Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces*, Lecture Notes in Math., vol. 1070, Springer, 1984, pp. 183–201. {1,13,14,15,19,31,32,44,47,48}

[95] Y. Meyer, *Wavelets and operators*, Cambridge Univ. Press, Cambridge, 1992. {33,35,36}

[96] S. Moura, *Function spaces of generalised smoothness*, Dissert. Math. **398** (2001), 1–87. {16}

[97] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1034. Springer-Verlag, Berlin, 1983. {53}

[98] H. Nakano, *Topology and topological linear spaces*, Maruzen Co., Ltd, Tokyo, 1951. {53}

[99] A. Nekvinda, *Equivalence of $\ell^{\{p_n\}}$ norms and shift operators*, Math. Inequal. Appl. **5** (2002), no. 4, 711–723. {55}

[100] ______, *Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$*, Math. Inequal. Appl. **7** (2004), no. 2, 255–265. {58}

[101] W. Orlicz, *Über konjugierte Exponentenfolgen (German)*, Studia Math. **3** (1931), 200–211. {53}

[102] J. Peetre, *A theory of interpolation of normed spaces*, Notas Mat. **39** (1968), 86 p. {32}

[103] ______, *Remarques sur les espaces de Besov. Le cas $0 < p < 1$*, C. R. Acad. Sci. Paris, Sér. A-B **277** (1973), 947–950. {14}

[104] ______, *New thoughts on Besov spaces*, Duke University Mathematics Series, 1976. {14}

[105] L. Persson, *Interpolation with a parameter function*, Math. Scand. **59** (1986), 199–222. {32}

[106] ______, *Real interpolation between cross-sectional L^p-spaces in quasi-Banach bundles*, Technical Report 1, Univ. of Luleå(1986), 199–222. {32,47,48}

[107] L. Pick and M. Růžička, *An example of a space of $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded*, Expo. Math. **19** (2001), no. 4, 369–371. {56,58}

[108] J. Rákosník, *Sobolev inequality with variable exponent*, Function spaces, differential operators and nonlinear analysis. Proceedings of the conference FSDONA-99, Syöte, Finland, June 10-16, 1999. Prague: Math. Institute, Acad. Sci. Czech Republic. 220-228, 2000. {54,103}

[109] M. Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy.*, Acta Math. **81** (1949), 1–223. {102}

[110] B. Ross and S. Samko, *Fractional integration operator of variable order in the Hölder spaces $H^{\lambda(x)}$*, Int. J. Math. Sci. **18** (1995), no. 4, 777–788. {56,105}

[111] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Lect. Notes Math., 1748. Springer-Verlag, Berlin, 2000. {2,54}

[112] S. Samko, *Denseness of $C_0^\infty(\mathbb{R}^N)$ in the generalized Sobolev spaces $W^{M,P(X)}(\mathbb{R}^N)$*, Direct and inverse problems of mathematical physics. Paper presented at special sessions of the ISAAC'97 congress, University of Delaware, Newark, June 2-7, 1997. Kluwer Academic Publishers, Int. Soc. Anal. Appl. Comput. 5, 333-342 , 2000. {54,57,59}

[113] ______, *Hardy inequality in the generalized Lebesgue spaces*, Fract. Calc. Appl. Anal. **6** (2003), no. 4, 355–362. {54}

[114] ______, *Hardy-Littlewood-Stein-Weiss inequality in the Lebesgue spaces with variable exponent*, Fract. Calc. Appl. Anal. **6** (2003), no. 4, 421–440. {54}

[115] S.G. Samko, *Spaces of Riesz potentials (Russian)*, Izv. Akad. Nauk SSSR, Ser. Mat. **40** (1976), no. 5, 1143–1172. {74}

[116] ______, *On the description of the range $I^\alpha(L_p)$ of Riesz potentials (Russian)*, Izv. Akad. Nauk Arm. SSR, Mat. **12** (1977), 329–334. {102}

[117] ———, *The spaces $L^{\alpha}_{p,r}(R^n)$ and hypersingular integrals (Russian)*, Stud. Math. **61** (1977), no. 3, 193–230. {74,102}

[118] ———, *Convolution and potential type operators in $L^{p(x)}(\mathbb{R}^n)$*, Integral Transforms Spec. Funct. **7** (1998), no. 3-4, 261–284. {54,59,60,63}

[119] ———, *Convolution type operators in $L^{p(x)}$*, Integral Transforms Spec. Funct. **7** (1998), no. 1-2, 123–144. {54,55,56,59}

[120] ———, *Differentiation and integration of variable order and the spaces $L^{p(x)}$*, Proceedings of Conference "Operator theory for complex and hypercomplex analysis", December 12-17, 1994, Mexico City. Contemp. Math. 212, 203-219, 1998. {54,55}

[121] ———, *Hypersingular integrals and their applications*, Analytical Methods and Special Functions. Taylor & Francis, 2002. {3,63,64,65,67,70,71,73,74,75,77,78,80,82,83,86,88,91,95,97,100,102,111}

[122] ———, *On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators*, Integral Transforms Spec. Funct. **16** (2005), no. 5-6, 461–482. {53}

[123] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, New York, 1993. {64,67,75,88,89,91}

[124] I.I. Sharapudinov, *Topology of the space $\mathcal{L}^{p(t)}([0,1])$*, Math. Notes **26** (1979), 796–806. {53,55}

[125] ———, *On the basis property of the Haar system in the space $\mathcal{L}^{p(t)}([0,1])$ and the principle of localization in the mean*, Math. USSR, Sb. **58** (1987), 279–287. {53}

[126] ———, *Uniform boundedness in L^p $(p = p(x))$ of some families of convolution operators*, Math. Notes **59** (1996), no. 2, 205–212. {53}

[127] S.L. Sobolev, *On a theorem of functional analysis*, Am. Math. Soc. Transl. **34** (1963), 39–68, Transl. Mat. Sb. (N. S.) **4(46)** (1938), 471-497. {102,114}

[128] E. Stein, *The characterization of functions arising as potentials, I*, Bull. Amer. Math. Soc. **67** (1961), no. 1, 102–104. {75,97,102}

[129] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970. {59,60,61,70,89,90,99,107}

[130] E.M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, 1971. {94}

[131] H. Triebel, *Spaces of Besov-Hardy-Sobolev type*, Teubner, Leipzig, 1978. {2,13,21}

[132] ______, *Theory of function spaces*, Birkhäuser, Basel, 1983. {8,9,14,17,22,24,31,37,38}

[133] ______, *Theory of function spaces II*, Birkhäuser, Basel, 1992. {14}

[134] ______, *Interpolation theory, function spaces, differential operators*, second ed., Johann Ambrosius Barth, Heidelberg, 1995. {18,20,21,44}

[135] ______, *Fractals and spectra*, Birkhäuser, Basel, 1997. {37,38,39,40,49}

[136] ______, *The structure of functions*, Birkhäuser, Basel, 2001. {38}

[137] ______, *A note on wavelet bases in function spaces*, Ciesielski, Zbigniew (ed.) et al., Orlicz centenary volume. Proceedings of the conferences "The Wladyslaw Orlicz centenary conference" and "Function spaces VII", Poznan, Poland, July 21-25, 2003. Volume I: Plenary lectures. Warsaw: Polish Academy of Sciences, Institute of Mathematics. Banach Center Publications 64, 193-206, 2004. {3,33,34,36,37,38,42,49}

[138] ______, *Wavelet bases in anisotropic function spaces*, Drábek, P. and Rákosník, J. (ed.), Proceedings of the conference "Function Spaces, Differential Operators and Nonlinear Analysis" dedicated to the 70th birthday of Alois Kufner, Milovy, Czech Republic, May 28 - June 2, 2004. Prague: Mathematical Institute, Academy of Sciences of the Czech Republic, 370-387, 2005. {49}

[139] P. Wojtaszczyk, *A mathematical introduction to wavelets*, Cambridge Univ. Press, Cambridge, 1997. {33,35}

[140] M. Yamazaki, *A quasihomogeneous version of paradifferential operators, I. Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. **33** (1986), no. 1, 131–174. {23}

[141] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR-Izv **29** (1987), no. 1, 33–66. {54}

[142] ———, *On some variational problems*, Russ. J. Math. Phys. **5** (1997), no. 1, 105–116. {54}

[143] W.P. Ziemer, *Weakly differentiable functions*, Springer-Verlag, Berlin, 1989. {62}

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