Universidade de Aveiro Departamento de Matemática
2006

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Famílias Normais e Crescimento de Funções Polimonogénicas

On Normal Families and Growth Behavior of Polymonogenic Functions

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tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica dos Professores Doutores Helmuth Robert Malonek, Professor Catedrático do Departamento de Matemática da Universidade de Aveiro e Gerhard Jank, Professor Catedrático, Lehrstuhl II fuer Mathematik, Rheinisch-Westfaelische Techniche Hochschule Aachen, Alemanha.

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palavras-chave

Análise de Clifford, famílias normais, critério de Marty, funções polymonogénicas, Teoria de Wiman-Valiron, crescimento assimptótico.

resumo

Este trabalho tem como objectivo contribuir para um estudo de famílias normais de funções meromórficas especiais assim como para o estudo do comportamento assimptótico de funções polimonogénicas no domínio da Análise Hipercomplexa.

Neste contexto, obtemos condições necessárias e/ou suficientes de normalidade para famílias de funções meromórficas especiais, nomeadamente a generalização do Teorema de Marty e a Lema de Zalcman.

Para a classe de funções polimonogénicas são demonstradas desigualdades do tipo de Cauchy e algumas generalizações de resultados da teoria de Wiman e Valiron. Consequentemente, são obtidas relações entre o máximo módulo da função, o termo máximo e índice central da sua respectiva série de Taylor-Almansi. Aplicam-se estes resultados ao crescimento assimptótico desta classe de funções.

Como aplicação, são obtidos teoremas sobre soluções assimptóticas de determinadas equações diferenciais de derivadas parciais e a classificação de algumas soluções das mesmas.

keywords

Clifford Analysis, normal families, Marty criteria, polymonogenic functions, Wiman-Valiron Theory, asymptotic growth.

abstract

The aim of this work is to provide some contributions to the study of normal family of special meromorphic functions as well as to the study of the asymptotic behaviour of polymonogenic functions in the framework of Hypercomplex Analysis.

In this context we have obtained necessary and/or sufficient normality conditions for families of special meromorphic functions, in particular, a generalization of Marty's criterion and also of Zalcman's lemma.

We prove inequalities of Cauchy-type estimates for a class of polymonogenic functions and also some generalizations of results of the Wiman-Valiron theory. Consequently, relations of the maximum modulus, the maximum term and the norm of the central index with respect to their Taylor-Almansi series expansion are obtained. These results are applied to the asymptotic growth behaviour of those functions classes.

As applications we establish theorems on the asymptotic of solutions of certain partial differential equations which allow us to provide a classification of some of such solutions.

Contents

Introduction

In this thesis we study normal families of Clifford-algebra-valued functions and the growth behavior of polymonogenic functions.

The numerous applications of normal families and growth estimates in the theory of holomorphic functions motivated us to perform an analogous study in the framework of Clifford Analysis.

Concrete applications arise, for instance, in complex dynamics, boundary valued problems and asymptotic behavior of solutions of partial differential equations or other fields of Physics and Engineering (see e.g. [8, 61] and also [33, 36, 37, 43, 70, 71]).

There exists manly two ways of generalizing the theory of functions of one complex variable. One is the function theory of several complex variables and the other way can be realized by using Clifford algebras which leads to hypercomplex function theory. In both theories the starting point can be the consideration of null-solutions of particular systems of first order partial differential equations in Euclidean spaces.

Hypercomplex function theory has several advantages compared with the theory of several complex variables. One advantage is that it does not depend on the dimension (even or odd) of the related real vector space. Another advantage is that higher order differential operators can be factorized into products of lower order operators (for example, the Laplace operator of several real variables can be factorized by two first order hypercomplex differential operators like in the plane case).

Hypercomplex analysis can be used synonymously with Clifford analysis [13, 21]. In turn, Clifford analysis is frequently considered as a generalization of quaternionic analysis (see [25, 28]). In this work we use both terms, Hypercomplex analysis as well as Clifford analysis. If we would like to stress the relationship to subjects of complex function theory we also use *hypercomplex function theory*.

The classes of functions which will be studied here are the class of solutions of the iterated Dirac equation $\mathcal{D}^k f = 0$, $k \in \mathbb{N}$, where $\mathcal{D} = \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$, and also the class of solutions of the iterated generalized Cauchy-Riemann equation $D^k f = 0$, where $D := e_0 \frac{\partial}{\partial x_0} + \mathcal{D}$. The elements of these classes are usually denoted as *polymonogenic* functions or also referred to as *k-monogenic* functions.

The starting points are 1-monogenic and, more general, meromorphic Clifford valued functions with the aim to extend the theory of normal families to these classes of functions. The notion of normal families was first introduced by Montel in [53] for holomorphic functions, and more generally, for meromorphic functions in 1927. Montel defines:

"*A family of meromorphic complex valued functions is called normal if every sequence of functions of the family contains a locally uniformly convergent subsequence.*"

In 1931, Marty gave a necessary and sufficient criterion for normality of families of meromorphic functions [48]. However this criterion is, in general, not easy to verify. Therefore, in 1975, Zalcman proves an equivalent criterion for normality [72]. Usually this is cited as Zalcman's lemma. Both results are the basic tools in the development of the theory of normal families. In this work we give a generalization of Marty's criterion as well as a generalization of Zalcman's lemma for special meromorphic functions in the hypercomplex setting. Special meromorphic functions are monogenic functions having at most isolated poles in which they converge to infinity. This is not true, in general, for meromorphic hypercomplex functions, and therefore different to the complex case.

The study of the growth behavior of polymonogenic functions is the main topic of the second part. In holomorphic function theory growth estimates have several applications to partial differential equations (see e.g. [33, 37, 43, 70, 71]).
The fundamentals in the study of the asymptotic growth of holomorphic and meromorphic functions have been established by Wiman [71], Valiron [70], Nevalinna [56], Clunie [16] and others.

Among other problems, Wiman and Valiron have considered questions like:

" Does a holomorphic function have the same growth behavior as its derivative? What is the relationship between the maximum modulus and the maximum of the coefficients in the power series?" Or "What is the relation between the growth of the function and the index of the maximum term in the power series, called central index?"

The results that we present in this thesis give an answer to this kind of questions in the context of polymonogenic functions.

We also obtain an explicit relation between special radial symmetric differential operators that act on polymonogenic functions. These include in particular the Euler operator $E := \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$ and the Gamma operator $\Gamma := \sum_{i,j=1, i<j}^{n} (x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}) e_i e_j$. These operators generalize the real part and imaginary part of the complex differential operator $z \frac{d}{dz}$, respectively.

For the case of iterates of the Dirac operator, we give an explicit relation between a polynomial of the Euler operator and the square of the Gamma operator applied to a polymonogenic function and the function itself. We also obtain a similar result for the case of the iterated generalized Cauchy-Riemann operator. Moreover, some applications to certain classes of partial differential equations are given.

The thesis is divided into five chapters. The outline of the contents of each chapter is as follows:

In Chapter 1 basic properties of special Clifford algebra of signature $(0, n)$ are given. We start by recalling the fundamental notions and results from hypercomplex function theory, which provides us with the basic tools for our analysis in the following chapters. Then a special class of Clifford algebra valued functions, named *special meromorphic* is introduced in the last section. Furthermore, a generalization of the spherical derivative is discussed.

In Chapter 2, using the one point compactification in $\mathbb{R}^s$ ($s \in \mathbb{N}$) the chordal distance and its properties are considered. Criteria of normality for families of Clifford valued monogenic functions and special meromorphic functions are studied. A generalized Marty's criteria is presented. Furthermore, a generalization of Zalcman's lemma is also obtained.

In Chapter 3 and Chapter 4 some rudiments of Wiman-Valiron's theory in the framework of hypercomplex function theory, are given.

More specifically:

In Chapter 3 we start by developing Cauchy type estimates for solutions of the iterated Dirac or iterated generalized Cauchy-Riemann equations. Relations between the generalized maximum modulus, the maximum term and the norm of the central index are obtained. In particular, we derive some Valiron type inequalities. In the last section of this chapter the growth behavior of a 1-monogenic function which maps the interior into the exterior of the unit ball is also studied.

In Chapter 4 some results on the asymptotic growth behavior of entire solutions of the iterated Dirac equations in $\mathbb{R}^n$ or iterated generalized Cauchy-Riemann equation are established. These are applied to obtain explicit asymptotic relations between the growth of these solutions and that of their iterated radial derivatives. We conclude Chapter 4 with remarks on functions classes which arise from applications of the iterated Gamma operator as well as iterated generalized Cauchy-Riemann operator.

In Chapter 5 some open problems for future research are stated.

Chapter 1

Some basic concepts of Clifford analysis

In this chapter we start by introducing the basic concepts of Clifford algebras and their associated function theory. For detailed information we refer, for instance to [13, 21, 67]. In the second part of this chapter we discuss a higher dimensional generalization of the spherical derivative and some of its properties.

1.1 Clifford algebras

The geometric properties induced by using complex numbers provided a strong motivation for Hamilton to look for higher dimensional in generalization of the complex number system. Searching for a three dimensional vector system he discovered the quaternions, in 1843, which is usually denoted by $\mathbb{H}$. Although, the associativity is obtained, another basic rule of arithmetics is lost, namely the commutativity.

For the standard basis system of the Hamiltonion quaternions, one often uses the notation $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. In this thesis we prefer to use $\{e_0, e_1, e_2, e_3\}$ instead. The basis elements satisfy the following multiplication rules:

$$\begin{array}{rclcrclcl} e_0^2 & = & e_0; & & e_ie_0 & = & e_0e_i & = & e_i, \qquad i = 1, 2, 3; \\ e_1^2 & = & e_2^2 & = -1; & e_1e_2 & = & -e_2e_1 & := & e_3. \end{array}$$

An element z of $\mathbb{H}$ is represented in the form

$$z = \mathrm{Sc}(z)e_0 + \mathrm{Vec}(z) := x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3,$$

where $\mathrm{Sc}(z) := x_0$ and $\mathrm{Vec}(z) := x_1e_1 + x_2e_2 + x_3e_3$ are the scalar and vector part of z, respectively. A pure quaternion

$$z = \mathrm{Vec}(z) = x_1e_1 + x_2e_2 + x_3e_3$$

can be identified with a vector in $\mathbb{R}^3$. In this sense, due to Gibbs, the product of two pure quaternions $z := \sum_{i=1}^{3} x_ie_i$ and $w := \sum_{i=1}^{3} w_ie_i$ is given by

$$zw := \langle z, w\rangle + z \times w,$$

where z, w are identified as vectors in $\mathbb{R}^3$. Also

$$\langle z, w\rangle := x_1w_1 + x_2w_2 + x_3w_3$$

and

$$z \times w := (x_2w_3 - x_3w_2)e_1 + (x_3w_1 - x_1w_3)e_2 + (x_1w_2 - x_2w_1)e_3,$$

represent the scalar (or inner) and vector (or cross) products, respectively.

Inspired by the work of Hamilton, in 1878, Clifford introduced an n-dimensional geometrical algebra in which the generalization of the scalar and vector product to higher dimensions are also obtained. This algebra is known as Clifford algebra. In 1844, Grassmann had already introduced the higher dimensional vector product - exterior product (or wedge product) - when he discovered the exterior algebra. For more information on the history of Clifford algebras we refer to [13, 44, 50].

A Clifford algebra is an associative but non-commutative algebra over the real or the complex field. In this work we consider the Clifford algebra of signature $(0, n)$ denoted by Cl_n and $\{e_0, e_1, e_2, \ldots, e_n\}$ stands for the canonical basis of the Euclidean vector space $\mathbb{R}^{n+1}$.

The basis elements satisfy the following multiplication rules

$$e_ie_j + e_je_i = -2\delta_{ij}e_0, \quad i, j = 1, \cdots, n,$$

where δ_{ij} is the Kronecker symbol.

A basis for the Clifford algebra Cl_n is given by the set $\{e_A : A \subseteq \{1, \cdots, n\}\}$ with $e_A = e_{l_1}e_{l_2}\cdots e_{l_r}$, where $1 \leq l_1 < \cdots < l_r \leq n$, $e_\emptyset := e_0 = 1$. Each element $a \in Cl_n$ can be written in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$. Notice that a is the direct sum of a scalar element, a vector element, a bi-vector element, ..., a k-vector element,..., a n-vector element, i.e.,

$$\begin{aligned}
Sc(a) &= a_\emptyset := a_0, \\
Vec(a) &= \sum_{|A|=1} a_A e_A := \sum_{i=1}^{n} a_i e_i, \\
Bi\text{-}Vec(a) &= \sum_{|A|=2} a_A e_A, \\
&\cdots \\
k\text{-}Vec(a) &= \sum_{|A|=k} a_A e_A, \\
&\cdots \\
n\text{-}Vec(a) &= a_{123\ldots n} e_{123\ldots n},
\end{aligned}$$

where $|A|$ means the cardinality of the set $A \subseteq \{1, \cdots, n\}$. Every k-vector can be interpreted geometrically as an oriented k-dimensional volume element.

Remark 1.1 *The associated complex Clifford algebra is represented by* $Cl_n \otimes_{\mathbb{R}} \mathbb{C}$, *where each element is represented by* $a := \sum_{A \subseteq \{1,\cdots,n\}} a_A e_A$, *with* $a_A := a_A^0 + i a_A^1$ *for* $a_A^0, a_A^1 \in \mathbb{R}$.

Let Cl_n^k be the subspace of k-vectors, i.e., the space spanned by the product of k different basis elements. Then the even subalgebra Cl_n^+ of the Clifford algebra Cl_n is defined by

$$Cl_n^+ = \bigoplus_{k\ even} Cl_n^k.$$

Some elementary involutions in the Clifford algebras Cl_n are:

$$\overline{a} := \sum_A a_A \overline{e}_A; \quad a^* := \sum_A (-1)^{|A|\frac{(|A|-1)}{2}} a_A e_A; \quad a' := \sum_A (-1)^{|A|} a_A e_A \tag{1.1}$$

where $\overline{e}_A = \overline{e}_{l_r}\overline{e}_{l_{r-1}}\cdots\overline{e}_{l_1}$, and $\overline{e}_j := -e_j$ for $j = 1, \cdots, n$, $\overline{e}_0 = e_0 = 1$. These are called conjugation, reversion and main involution, respectively. The conjugation and the reversion are anti-automorphism and the main involution is an automorphism.

Next we give some basic properties of these involutions. For $a, b \in Cl_n$, we have

$$\overline{ab} \;=\; \bar{b}\bar{a}; \qquad (ab)^* \;=\; b^*a^*; \qquad (ab)' \;=\; a'b'.$$

The scalar product between two Clifford numbers $a = \sum_A a_A e_A$ and $b = \sum_A b_A e_A$ is defined by:

$$\langle a, b\rangle := Sc(a\bar{b}) := \sum_A a_A b_A. \tag{1.2}$$

From the scalar product (1.2), the Clifford norm may be derived by

$$\|a\| := \sqrt{\langle a, a\rangle} = \sqrt{\left(\sum_A |a_A|^2\right)}.$$

Three important subspaces of the Clifford algebra Cl_n are the paravector space, the quaternion algebra and the field of the complex numbers.

The paravector space is the linear subspace defined by

$$\mathcal{A}_{n+1} := span_{\mathbb{R}}\{1, e_1, \cdots, e_n\} = \mathbb{R} \oplus \mathbb{R}^n \equiv \mathbb{R}^{n+1} \subset Cl_n$$

with elements of the form $z = x_0 + x_1e_1 + x_2e_2 + \cdots + x_ne_n$. Taking two vectors $a := \sum_{i=1}^{n} a_ie_i$ and $b := \sum_{i=1}^{n} b_ie_i$ we obtain the wedge product given by

$$a \wedge b := \frac{1}{2}(ab - ba).$$

Each non-zero paravector $z \in \mathcal{A}_{n+1}\backslash\{0\}$ has an inverse element given by $z^{-1} = \dfrac{\overline{z}}{\|z\|^2}$.

Another known subspace is the skew-field of real quaternions $\mathbb{H}$, which is identified with Cl_3^+. On the other hand, the Hamiltonian quaternions may be also identified with Cl_2. One also have the field of complex numbers which is identified with Cl_1.

Remark 1.2 *The involutions defined in (1.1), satisfy $\overline{a} = a'$ and $a^* = a$, for $a \in \mathcal{A}_{n+1}$ (or $\mathbb{H}$).*

1.2 Vahlen group and Möbius transformations

In the complex case, any planar Möbius transformation can be expressed by

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. This can also be extended to $\mathbb{C} \cup \{\infty\}$ by setting $f\left(\frac{-d}{c}\right) = \infty$ and $f(\infty) = \frac{a}{c}$. One convenient way to express a Möbius transformation is done by matrix notation. In 1902, using this process Vahlen treated Möbius transformation in higher dimensions [69]. Unfortunately, this paper had been forgotten. Only in 1949, Maass rediscovered and improved Vahlen's original approach. In 1923, Fueter rediscovered this representation for the quaternion case.

Further important contributions to the study of Möbius transformation in higher dimensional Euclidean spaces were also provide by Ahlfors [4, 5, 6], by Zöll in [74], among others.

From [4, 5] and [6] we recall:

Definition 1.1 *(Clifford group) The Clifford group, Γ_n, is defined as the set of elements $z \in Cl_n$ for which exist a natural number $k \in \mathbb{N}$ and elements $a_1, a_1, ..., a_k \in \mathcal{A}_{n+1} \setminus \{0\}$, such that $z = \prod_{i=1}^{k} a_i$.*
This group is also known as the Lipschitz group (see [32, p.118]).

One can verify that Γ_n is actually a group with respect to the Clifford multiplication. In the next proposition we recall some basic properties of the Clifford group.

Proposition 1.1 *Let $a, b \in \Gamma_n$ and $z \in \mathcal{A}_{n+1}$. Then*

(i) $\|a\|^2 = \overline{a}a$ *and* $\|ab\| = \|a\|\|b\|$,

(ii) $ab^{-1}, \ a^*b, \ b^{-1}a, \ ba^* \in \mathcal{A}_{n+1}$,

(iii) the map $h_a : \mathcal{A}_{n+1} \to \mathcal{A}_{n+1}$ defined by $h_a(z) = az(a')^{-1}$ is a bijective and sense--preserving isometry. In particular, if $a \in \mathcal{A}_{n+1}$ then $h_a(z) \in \Gamma_n$.

Based on the notion of the Clifford group one defines Vahlen matrix as follows:

Definition 1.2 *(Vahlen matrix) Consider the set of matrices defined by*

$$\Gamma_n^{2\times 2} = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \Gamma_n \cup \{0\} \right\}.$$

A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is a Vahlen matrix if the coefficients* a, b, c *and* d *satisfy the following conditions:*

$$\begin{array}{lll} ad^* - bc^* & \in \mathbb{R} \setminus \{0\} & \\ a^{-1}b & \in \mathcal{A}_{n+1}, & a \neq 0 \\ c^{-1}d & \in \mathcal{A}_{n+1}, & c \neq 0. \end{array} \tag{1.3}$$

The expression $ad^* - bc^*$ *is known as the pseudo-determinant of the matrix* A*.*

The set of Vahlen matrices is a group and it is denoted as the Vahlen group (see e.g. [32, p.119], [69]). This group is a generalization of the general linear group $GL(2,\mathbb{C})$. Using these matrices, it is possible to describe Möbius transformations in higher dimensional spaces in an analogous compact form as one can do in $\mathbb{R}^2$ using matrices from the general linear group $GL(2,\mathbb{C})$.

Definition 1.3 *The left, resp. the right representation of the Möbius transformation:*

$$M_{L_A} : \mathcal{A}_{n+1} \setminus \{-c^{-1}d\} \to Cl_n; \qquad M_{R_{A_1}} : \mathcal{A}_{n+1} \setminus \{-c_1^{-1}d_1\} \to Cl_n,$$

are defined, respectively, as:

$$\begin{array}{lll} M_{L_A}(z) & = & (az+b)(cz+d)^{-1}, \qquad a,b,c,d \in \Gamma_n \cup \{0\} \\ M_{R_{A_1}}(z) & = & (zc_1+d_1)^{-1}(za_1+b_1), \quad a_1,b_1,c_1,d_1 \in \Gamma_n \cup \{0\} \end{array}$$

where the associated matrices $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A_1 := \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ *belong to the Vahlen group.*

In the quaternionic case a Möbius transformation can be represented as:

Definition 1.4 *Denote*

$$M_{L_A} : \mathbb{H} \setminus \{-c^{-1}d\} \longrightarrow \mathbb{H}; \qquad M_{R_{A_1}} : \mathbb{H} \setminus \{-c_1^{-1}d_1\} \longrightarrow \mathbb{H},$$

as the representation of a left, resp. the right Möbius transformation, given explicitly by

$$\begin{array}{lll} M_{L_A}(z) & = & (az+b)(cz+d)^{-1}, \qquad a,b,c,d \in \mathbb{H} \\ M_{R_{A_1}}(z) & = & (zc_1+d_1)^{-1}(za_1+b_1), \quad a_1,b_1,c_1,d_1 \in \mathbb{H} \end{array}$$

where

$$\begin{aligned} &\|b - ac^{-1}d\|\|c\| \neq 0, \qquad c \neq 0 \quad or \quad \|ad\| \neq 0, \qquad c = 0 \\ &\|b_1 - a_1c_1^{-1}d_1\|\|c_1\| \neq 0, \quad c_1 \neq 0 \quad or \quad \|a_1d_1\| \neq 0, \quad c_1 = 0. \end{aligned} \tag{1.4}$$

In many context one considers, in particular, Vahlen matrices whose pseudo-determinant is equal to ± 1. This consideration leads to introduce:

Definition 1.5 *(Special Vahlen group) Consider a Vahlen matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *This matrix belongs to the special Vahlen group if the coefficients satisfy in addition to the condition (1.3), the normalization condition:*

$$ad^* - bc^* = \pm 1.$$

The special Vahlen group denoted as $SL(2, \Gamma_n)$ is a generalization of the special linear group $SL(2, \mathbb{C})$. In this context, Vahlen and Maass proved the following theorem. (see e.g. [6]).

Theorem 1.1 *The special Vahlen group,* $SL(2, \Gamma_n)$ *forms a group under matrix multiplication.*

Each matrix from $SL(2, \Gamma_n)$ *induces a Möbius transformations in* $\mathbb{R}^{n+1}$. *Conversely, every Möbius transformations is induced by* $SL(2, \Gamma_n)$.

Remark 1.3 *The Möbius transformations associated to the special Vahlen group satisfying* $ad^* - bc^* = 1$ *are the orientation preserving transformations, while* $ad^* - bc^* = -1$ *does not preserve the orientation.*

By Definition 1.3 and Definition 1.4 one has a left, resp. right representation for a Möbius transformations. The next result proves that, by means of the Vahlen matrix representation, a left Möbius transformations can be expressed by a right Möbius transformations (see [74]).

Theorem 1.2 *Any Möbius transformations can be represented equivalently by left coefficients and right coefficients.*

Proof. Consider the representation of a Möbius transformations with left coefficients $M_{L_A}(z) = (az+b)(cz+d)^{-1}$. Let us first suppose that $c = 0$. Then

$$\begin{aligned} M_{L_A}(z) &= (az+b)d^{-1} \\ &= (a^{-1})^{-1}(zd^{-1} + a^{-1}bd^{-1}) \;=\; M_{R_{A_1}}(z) \end{aligned}$$

where $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} d^{-1} & a^{-1}bd^{-1} \\ 0 & a^{-1} \end{pmatrix}$. Suppose $c \neq 0$, then

$$\begin{aligned} M_{L_A}(z) &= (az+b)(cz+d)^{-1} \\ &= (az+b)[(c^{-1})^{-1}(z+c^{-1}d)]^{-1} \\ &= (az+b-ac^{-1}d+ac^{-1}d)(z+c^{-1}d)^{-1}c^{-1} \\ &= [a(z+c^{-1}d)+b-ac^{-1}d](z+c^{-1}d)^{-1}c^{-1} \\ &= a(z+c^{-1}d)(z+c^{-1}d)^{-1}c^{-1} + (b-ac^{-1}d)(z+c^{-1}d)^{-1}c^{-1} \\ &= ac^{-1} + c^{-1}c(b-ac^{-1}d)(z+c^{-1}d)^{-1}c^{-1} \\ &= ac^{-1} + c^{-1}(cb-cac^{-1}d)(z+c^{-1}d)^{-1}c^{-1}. \end{aligned}$$

Denote $K := cb - cac^{-1}d$, the aim is to prove that $K \neq 0$.

Suppose that $a, b, c, d \in \mathbb{H}$. Then, in view of (1.4), we have that $\|b - ac^{-1}d\| \neq 0$. This implies that K has an inverse. If $a, b, c, d \in \Gamma_n \cup \{0\}$, then in view of condition (1.3), we have that $c^{-1}d = (c^{-1}d)^* = d^*(c^{-1})^* = d^*(c^*)^{-1}$. Hence:

$$K = cb - cac^{-1}d = c(b - ad^*(c^*)^{-1}) = c(bc^* - ad^*)(c^*)^{-1} \neq 0.$$

Since $K \neq 0$, we conclude that

$$\begin{aligned} M_{L_A}(z) &= ac^{-1} + c^{-1}K(z+c^{-1}d)^{-1}c^{-1} \\ &= ac^{-1} + (zK^{-1}c + c^{-1}dK^{-1}c)^{-1}c^{-1} \\ &= (zK^{-1}c + c^{-1}dK^{-1}c)^{-1}((zK^{-1}c + c^{-1}dK^{-1}c)ac^{-1} + c^{-1}) \\ &= (zK^{-1}c + c^{-1}dK^{-1}c)^{-1}(zK^{-1}cac^{-1} + (c^{-1}dK^{-1}cac^{-1} + c^{-1})) \\ &= M_{R_{A_1}}(z), \end{aligned}$$

where $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} K^{-1}cac^{-1} & c^{-1}dK^{-1}cac^{-1} + c^{-1} \\ K^{-1}c & c^{-1}dK^{-1}c \end{pmatrix}$. □

As a consequence of this theorem, there is no ambiguity in writing from now on $M(z) := M_{L_A}(z)$ for a Möbius transformation.

1.3 Clifford analysis

An important issue in Clifford analysis is to introduce the concept of derivative of a Clifford valued function. In the complex case, this can be defined by the following limit

$$\lim_{\Delta z \to 0} \left(\frac{f(z + \Delta z) - f(z)}{\Delta z} \right) = f'(z),$$

where f is a complex valued function in $\mathbb{C}$.
At the end of the 19th century some attempts were made to extend this definition to quaternionic valued functions in $\mathbb{H}$. However, the lack of commutativity, brought two cases into consideration:

$$\lim_{\Delta z \to 0} (f(z + \Delta z) - f(z))(\Delta z)^{-1} \text{ and } \lim_{\Delta z \to 0} (\Delta z)^{-1}(f(z + \Delta z) - f(z)).$$

However, the only quaternionic valued functions for which these limits exists have, respectively, the form

$$f(z) = az + b \text{ and } f(z) = za + b, \qquad a,\ b \in \mathbb{H}.$$

For more details we refer, for instance, [42, 49, 68].

Another possibility to generalize complex holomorphy is offered by following the Riemann approach. In this context we consider the differential operator

$$\mathcal{D} := \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} \tag{1.5}$$

which is the Dirac operator in $\mathbb{R}^n$, and the generalized Cauchy-Riemann operator, i.e.,

$$D := \frac{\partial}{\partial x_0} + \mathcal{D}. \tag{1.6}$$

The operator (1.6) is used when working in the paravector formalism, i.e., $\mathcal{A}_{n+1}$.

In the sense of the Riemann approach one introduces, cf. [21, p.138]

Definition 1.6 *(Monogenicity) Let U be an open subset of $\mathcal{A}_{n+1}$. A real differentiable function*

$$f : U \to Cl_n$$

is called left (right) monogenic or Clifford holomorphic in U if and only if

$$Df = 0 \quad (\text{or} \quad fD = 0).$$

Functions that are left (right) monogenic in the whole space are called left (right) entire monogenic.

Analogously one may define the notion of monogenicity in the context of the Dirac operator.

In contrast to the complex case, the composition of monogenic functions does not remain monogenic, in general. However, the generalized Cauchy-Riemann operator is quasi-invariant under the set of Möbius transformations.

More precisely, following for example [74, pp.45], we have

Theorem 1.3 *Let $\Omega \subset \mathcal{A}_{n+1}$ be a domain and $f : \Omega \to Cl_n$ be a left monogenic function. If $M(z) = (az+b)(cz+d)^{-1}$ is a Möbius transformation, then*

$$G(z) = \frac{\overline{(cz+d)}}{\|cz+d\|^{n+1}}(f \circ M)(z)$$

is also left monogenic in $M^{-1}(\Omega)$. In the case dealing with right monogenic functions we have that

$$G(z) = (f \circ M)(z)\frac{\overline{(zc^*+d^*)}}{\|zc^*+d^*\|^{n+1}}$$

is also right monogenic in $M^{-1}(\Omega)$.

The notion of left (right) monogenicity in $\mathcal{A}_{n+1}$ provides a powerful generalization of the concept of complex analyticity to Clifford analysis, since many classical theorems from complex analysis could be generalized to higher dimensions by this approach, we refer for instance [13, 21] and [25, 26, 27]. As for example the Cauchy integral theorem and Cauchy integral formula (cf. [13, pp.52]).

Theorem 1.4 *(Cauchy's integral theorem) Let $\Omega \subset \mathcal{A}_{n+1}$ be an open set and $S \subset \Omega$ be an $(n+1)$-dimensional compact differentiable and oriented manifold-with-boundary. Suppose that C is a $(n+1)$-chain on S. If f is left (right) monogenic in Ω then*

$$\int_{\partial C} d\sigma f = 0 \quad \left(resp. \int_{\partial C} f d\sigma = 0 \right),$$

where $d\sigma$ denotes the n-dimensional oriented Lebesgue surface measure.

Theorem 1.5 *(Cauchy's integral formula) Let $\Omega \subset \mathcal{A}_{n+1}$ be an open set and $S \subset \Omega$ be an $(n+1)$-dimensional compact differentiable and oriented manifold-with-boundary. If f is left (right) monogenic in Ω then*

$$\frac{1}{w_{n+1}} \int_{\partial S} q_{\mathbf{0}}(z-y) d\sigma_y f(y) = \begin{cases} f(z), & z \in S^\circ \\ 0, & z \in \Omega \setminus S. \end{cases} \tag{1.7}$$

Here, $q_{\mathbf{0}}(z-y) = \frac{\overline{z-y}}{\|z-y\|^{n+1}}$, S° is the open kernel of S and $w_{n+1} := 2\pi^{(n+1)/2} \frac{1}{\Gamma((n+1)/2)}$ (see. [24, p.75]) is the area of the unit hypersphere with $\Gamma(\cdot)$ the Gamma function.

Further generalizations of the classical theory are for instance:

Theorem 1.6 *(Maximum modulus theorem) Let f be a left (right) monogenic function in a domain Ω. If there exists a point $z_0 \in \Omega$ such that*

$$\|f(z)\| \leq \|f(z_0)\|,$$

for all $z \in \Omega$, then f must be a constant function in Ω.

Theorem 1.7 *(Maximum principle) Let Ω be a bounded open set in $\mathcal{A}_{n+1}$. If f is continuous in $\overline{\Omega}$ (the closure of Ω) and left (right) monogenic in Ω, then*

$$\sup_{z \in \overline{\Omega}} \|f(z)\| = \sup_{z \in \partial\Omega} \|f(z)\|.$$

Another important theorem is the Cauchy-Kowalewski extension theorem. This theorem establishes that any real-analytic function f in $\mathbb{R}^n$ can be uniquely extended to a monogenic function F in $\mathbb{R}^{n+1}$. First we introduce the notion of x_0-normal neighborhood, cf. [13, p.110].

Definition 1.7 *(x_0-normal neighborhood) Let $U \subset \mathbb{R}^n$ be open. Then an open neighborhood $V \subset \mathcal{A}_{n+1}$ of U is called a x_0-normal neighborhood if for each $z = x_0 + \mathbf{x}$ in V the line segment $\{z + t : t \in \mathbb{R}\} \cap U$ is connected and contains just one point of V.*

Following for example [21, p.151], we have:

Theorem 1.8 *(Cauchy-Kowalewski extension) Let $U \subset \mathbb{R}^n$ be open and connected. Suppose that $f : U \to Cl_n$ is a real-analytic function. Then the function F defined by*

$$F(z) := \sum_{k=0}^{+\infty} \frac{1}{k!}(-x_0)^k \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \right)^k f(\mathbf{x})$$

satisfies the left generalized Cauchy-Riemann equation $DF = 0$ in an open connected and x_0-normal neighborhood $V \subset \mathcal{A}_{n+1}$ of U. Furthermore, $F|_{x_0=0} = f$ in U. F is called the Cauchy-Kowalewski extension of f into the x_0-direction.

This extension allows to define the Cauchy-Kowalewski product (CK-product) which preserves the monogenicity of the factors, despite of the non-commutativity of the Clifford algebra ([13, p.68]).

In order to present the calculations in a more compact form, the following notations will be used. With $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}_0^n$ as a n-dimensional multi-index we denote:

$$\mathbf{x}^{\mathbf{m}} \ := \ x_1^{m_1} \cdots x_n^{m_n}, \ \ \mathbf{m}! \ := \ m_1! \cdots m_n!, \quad |\mathbf{m}| \ := \ m_1 + \cdots + m_n$$

where $\mathbf{x} \equiv (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. We also denote the multi-index $(m_1, \dots, m_n)$ with $m_j = \delta_{ij}$ for $1 \leq i, j \leq n$ by $\tau(i)$. From [21, p.173], we recall

Definition 1.8 *Let $\mathbf{m} \in \mathbb{N}_0^n \backslash \{\mathbf{0}\}$ and $p(\mathbf{x}) = \dfrac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!}$, then the Cauchy-Kowalewski extension of $p(\mathbf{x})$ is given by*

$$CK(p(\mathbf{x})) = \mathcal{P}_{\mathbf{m}}(z) := \frac{1}{|\mathbf{m}|!} \sum_{\pi \in perm(\mathbf{m})} z_{\pi(m_1)} \ldots z_{\pi(m_n)}, \tag{1.8}$$

where $perm(\mathbf{m})$ denotes the set of all permutations of the sequence $(m_1, ..., m_n)$ and $z_i := x_i - x_0 e_i$ for $i = 1, ..., n$. The functions $\mathcal{P}_{\mathbf{m}}(z)$ are the Fueter polynomials.

In [45, p.18], Malonek proved that all these functions take their values in $\mathcal{A}_{n+1}$. They can be written in the form of powers using permutational products. They also can be interpreted as generalized positive powers replacing the classic positive powers in many generalizations of the classical theorems. An estimate of these functions, is given by:

$$\|\mathcal{P}_{\mathbf{m}}(z)\| \leq \frac{\|z\|^{|\mathbf{m}|}}{\mathbf{m}!}. \tag{1.9}$$

These estimates were first established in the case of quaternionic valued functions by Fueter in [26], and for higher dimensions by Krauβhar in [38, p.19].

Using these functions instead of the powers of z^m for $m \in \mathbb{N}_0$, we obtain (e.g. cf. [13, pp.71]):

Theorem 1.9 *(Taylor expansion) Let $B(z_0, R) \subset \mathcal{A}_{n+1}$ be an open ball with center z_0 and radius R. Suppose that $f : B(z_0, R) \to Cl_n$ is a left (right) monogenic function. Then, for any $0 < r < R$, the function f has a unique Taylor series representation in $B(z_0, r)$ of the form*

$$f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} \mathcal{P}_{\mathbf{m}}(z - z_0) a_{\mathbf{m}}, \qquad \left(f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} a_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}(z - z_0) \right)$$

where $a_{\mathbf{0}} := f(z_0)$ and $a_{\mathbf{m}} := \dfrac{\partial^{|\mathbf{m}|} f}{\partial \mathbf{x}^{\mathbf{m}}}(z_0)$.

The analogue of the negative power functions were first introduced for the quaternion case by Fueter in [26] and for $\mathbb{R}^{n+1}$ by Delanghe in [23]. These generalized negative power functions arise from the Cauchy kernel function $q_{\mathbf{0}}$ and their partial derivatives

$$q_{\mathbf{0}}(z) := \frac{\overline{z}}{\|z\|^{n+1}}, \qquad q_{\mathbf{m}}(z) = \frac{\partial^{m_0+m_1+\cdots+m_n}}{\partial x_0^{m_0} \partial x_1^{m_1} \cdots \partial x_n^{m_n}} q_{\mathbf{0}}(z), \tag{1.10}$$

for $\mathbf{m} := (m_0, m_1, ..., m_n)$. In view of the monogenicity, one can restrict to multi-indices with $m_0 = 0$.

These functions are left and right monogenic in $\mathcal{A}_{n+1} \setminus \{0\}$ and take their values in $\mathcal{A}_{n+1}$. In [38, pp.24] the following recurrence formula is given:

$$\begin{aligned} \frac{\partial^{|\mathbf{m}|+1}}{\partial \mathbf{x}^{(\mathbf{m}+\tau(i))}} q_{\mathbf{0}}(z) &= \sum_{|\mathbf{j}|=0}^{|\mathbf{m}|} \binom{|\mathbf{m}|}{|\mathbf{j}|} |\mathbf{j}|! q_{\mathbf{m}-\mathbf{j}}(z) \times \left[\left(\frac{n-1}{2} \right) \prod_{k=1}^{|\mathbf{j}|+1} (\overline{z}^{-1} e_{t_k}) \right] \\ &+ \left. (-1)^{|\mathbf{j}|+1} \left(\frac{n-1}{2} \right) \prod_{k=1}^{|\mathbf{j}|+1} (e_{t_k} z^{-1}) \right] \end{aligned} \tag{1.11}$$

with $t_k \in \{1, ..., n\}$ not necessarily distinct numbers. From this representation one can easily derive the following estimate on these functions, see [20] and also [38, p.26]:

$$\left\| \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} q_{\mathbf{0}}(z) \right\| \leq \frac{n(n+1)...(n+|\mathbf{m}|-1)}{\|z\|^{n+|\mathbf{m}|}}. \tag{1.12}$$

These functions play the same role as the negative complex powers in the Laurent series expansion of an holomorphic function in an annular domain. The following theorem provides a representation of a monogenic function in an annular domain (see e.g. [23], [13, p.90]).

Theorem 1.10 *(Generalized Laurent expansion) Suppose that f is left (right) monogenic in $B(0,R)\setminus\overline{B(0,r)}$ where $0<r<R$. Then f has a unique Laurent series expansion of the form*

$$f(z)=\sum_{|\mathbf{m}|=0}^{+\infty}\mathcal{P}_{\mathbf{m}}(z)a_{\mathbf{m}}+\sum_{|\mathbf{m}|=0}^{+\infty}q_{\mathbf{m}}(z)b_{\mathbf{m}}, \tag{1.13}$$

$$\left(f(z)=\sum_{|\mathbf{m}|=0}^{+\infty}a_{\mathbf{m}}\mathcal{P}_{\mathbf{m}}(z)+\sum_{|\mathbf{m}|=0}^{+\infty}b_{\mathbf{m}}q_{\mathbf{m}}(z)\right)$$

where

$$a_{\mathbf{m}}=\frac{1}{w_{n+1}}\int_{\partial B(0,r)}q_{\mathbf{m}}(\xi)d\sigma(\xi)f(\xi),\quad b_{\mathbf{m}}=\frac{1}{w_{n+1}}\int_{\partial B(0,r)}\mathcal{P}_{\mathbf{m}}(\xi)d\sigma(\xi)f(\xi).$$

Notice that both series converge normally in $B(0,R)$, respectively in $\mathbb{R}^{n+1}\setminus\overline{B(0,r)}$.

Although a monogenic function in $\mathcal{A}_{n+1}$ can have singularities of manifolds of dimension $0,1,...,n-1$, in our study we focus on the singularities of manifolds of dimension 0. These are called isolated singularities. Notice that it is not possible to have singularities of manifolds of dimension n and $n+1$. This is a consequence of the generalized Cauchy--Riemann equation. For detailed information about singularities of monogenic functions we refer to [28, 29, 54, 55]. The following definitions are cited from [13, p.94] or [23].

Definition 1.9 *(Regular and Singular points) A point $z_0\in\mathcal{A}_{n+1}$ is called*

(i) a left (right) regular point of the Clifford valued function f, if there exists an open neighborhood $\mathcal{V}_{z_0}$ where f is left (right) monogenic;

(ii) a singular point of the Clifford valued function f, if there exists no open neighborhood $\mathcal{V}_{z_0}$ where f is left (right) monogenic;

(iii) an isolated singularity if it is a left (right) singular point of the Clifford valued function f and if there exists an open neighborhood $\mathcal{V}_{z_0}$ where f is left (right) monogenic in $\mathcal{V}_{z_0}\setminus\{z_0\}$.

Definition 1.10 *(Classification of isolated singularities) Let $\Omega \subset \mathcal{A}_{n+1}$ be an open set and $z_0 \in \Omega$. Suppose that $f : \Omega \setminus \{z_0\} \to Cl_n$ is left (right) monogenic and z_0 is a left (right) isolated singularity. Then z_0 is called a left (right)*

(i) isolated pole (or pole) of order $n + |\mathbf{m}|$, if the coefficients, $b_{\mathbf{k}}$ of the second series in the Laurent series expansion (1.13) are zero for $|\mathbf{k}| > |\mathbf{m}|$;

(ii) isolated essential singularity, if the cardinality of the set $\{\mathbf{k} \mid b_{\mathbf{k}} \neq 0\}$ is infinite, where $b_{\mathbf{k}}$ are the coefficients of the second series of the Laurent series expansion (1.13).

Definition 1.11 *(Meromorphic functions) Let Ω be an open subset of $\mathcal{A}_{n+1}$ and $f : \Omega \to Cl_n$. The function f is called left (right) meromorphic function in Ω if there exists a subset $S \subset \Omega$ such that:*

(i) S has no accumulation point in Ω;

(ii) f is left (right) monogenic in $\Omega \setminus S$;

(iii) f has a left (right) isolated pole at each point of S.

The condition (i) implies that no compact subset of Ω contains infinitely many points of S, i.e., S is at most countable (see [23, Lemma C]).

Remark 1.4 *In classical complex theory of one variable, if f is a meromorphic function and z_0 a pole of f then*

$$\lim_{z \to z_0} f(z) = \infty.$$

In case of meromorphic Clifford valued functions this is not always true. In order to present an example we first consider the representation formula of the generalized negative powers presented in [20].

In the formula of [20] the following notation is used: $\mathbf{p}, \mathbf{m} \in \mathbb{N}_0^n$ are defined as $\mathbf{p} = (p_1, p_2, ..., p_n)$, $\mathbf{m} = (m_1, m_2, ..., m_n)$:

$$\mathbf{p} \leq \mathbf{m} \quad :\Leftrightarrow \quad p_1 \leq m_1, ..., p_n \leq m_n.$$

Furthermore, we need the Pochhammer symbol $(k)_s := k(k+1)(k+2)...(k+s-1)$ where $k \in \mathbb{R},\ s \in \mathbb{N}$. From [20] we recall

$$\begin{aligned} q_{\mathbf{m}}(z) &= \frac{1}{n-1} \sum_{0 \leq 2\mathbf{p} \leq \mathbf{m}} a(n, \mathbf{m}, \mathbf{p}) \frac{(2Z)^{\mathbf{m}-2\mathbf{p}}}{\|z\|^{n-1+2|\mathbf{m}|-2|\mathbf{p}|}} \\ &\times \left(\frac{n-1+2|\mathbf{m}|-2|\mathbf{p}|}{z} + \sum_{q=1}^{n} \frac{m_q - 2p_q}{x_q} e_q \right) \end{aligned} \tag{1.14}$$

where $a(n,\mathbf{m},\mathbf{p}) := \frac{\mathbf{m}!}{(\mathbf{m}-2\mathbf{p})!\mathbf{p}!}\left(\frac{n-1}{2}\right)_{|\mathbf{m}|-|\mathbf{p}|}(-1)^{|\mathbf{m}|-|\mathbf{p}|}$, *and* $Z := x_1x_2...x_n$.

Next, we present an example which the function does not converge to infinity at the isolated pole. Consider $\mathbf{m} = \tau(i) + \tau(j)$ *for* $0 < i \neq j \leq n$, *using (1.14) we obtain the function* $q_{\tau(i)+\tau(j)}$ *defined as*

$$\begin{aligned} q_{\tau(i)+\tau(j)}(z) &= \frac{1}{n-1}a(n,\tau(i)+\tau(j),0)\frac{(2Z)^{\tau(i)+\tau(j)}}{\|z\|^{n+3}}\left(\frac{n+3}{z}+\sum_{q=1}^{n}\frac{m_q}{x_q}e_q\right) \\ &= \frac{1}{n-1}a(n,\tau(i)+\tau(j),0)\left((n+3)\frac{(2x_i)(2x_j)\overline{z}}{\|z\|^{n+5}}+\frac{(2x_i)e_j+(2x_j)e_i}{\|z\|^{n+3}}\right), \end{aligned}$$

where $z = 0$ *is an isolated pole. Substituting* $z := x_0e_0$ *we have*

$$q_{\tau(i)+\tau(j)}(x_0, 0, ..., 0) = 0.$$

This implies that the function in the direction of x_0 *will remain bounded when approaching the pole.*

In [64] Ryan proved, for left monogenic functions $f : \mathcal{A}_3 \setminus \{0\} \to \mathbb{H}$ with negative degree of homogeneity, that the set of lines radiating from the origin on which f vanishes has a finite cardinality.

In view of Remark 1.4 it becomes natural to think about isolated zeros regarded as points. Many questions related to value distribution theory in Clifford analysis are still not solved. However, Hempfling and Krau*ß*har in [35] obtained some results for some meromorphic functions in Clifford analysis. From [35] we take the following definition:

Definition 1.12 *Let* $f : \Omega \to \mathcal{A}_{n+1}$, *where* Ω *is an open set of* $\mathcal{A}_{n+1}$.

(i) Let $y \in \mathcal{A}_{n+1}$. *Then an element* $x \in \Omega$ *is called a y-point of* f *if* $f(x) = y$.

(ii) $x^* \in \Omega$ *is called an isolated y-point, if there exists* $\epsilon > 0$ *such that* $f(x) \neq y$ *for all* $x \in B(x^*, \epsilon) \setminus \{x^*\}$.

We also define ∞-point as follows.

Definition 1.13 *Let* $f : \Omega \to Cl_n$, *where* Ω *is an open set of* $\mathcal{A}_{n+1}$. $x^* \in \Omega$ *is called an* ∞*-point, if* x^* *is an isolated singularity of* f *and* $\lim_{x\to x^*} f(x) = \infty$, *independently from which path we approximate* x^*.

Next we define a special class of functions which will be studied later on.

Definition 1.14 *(Special meromorphic functions) Let f be a Clifford valued function defined in a domain $\Omega \subset \mathcal{A}_{n+1}$. f is called a left (right) special meromorphic function in Ω if f is left (right) meromorphic and each isolated singularity of f is an ∞-point.*

Some examples of this type of special meromorphic functions are presented next.

Example 1.1 *The Cauchy kernel $q_{\mathbf{0}}(z) := \dfrac{\overline{z}}{\|z\|^{n+1}}$ is an example of a special meromorphic function. We know that $q_{\mathbf{0}}(z)$ has an isolated pole of order n at the origin. Moreover 0 is an ∞-point, since $\lim_{z\to 0} q_{\mathbf{0}}(z) = \infty$.*
Furthermore, from [35] we also know that $q_{\mathbf{0}}$ has only isolated x-points.

Example 1.2 *Consider the simply-periodic Clifford cotangent function associated with the lattice $2\mathbb{Z}e_j$ $(0 \leq j \leq p)$ $(1 < p < n)$ defined by*

$$cot^{(1)}(z, 2\mathbb{Z}e_j) := \sum_{m\in\mathbb{Z}} q_{\mathbf{0}}(z + 2me_j)$$

and the simply-periodic Clifford tangent function defined by

$$tan^{(1)}(z, 2\mathbb{Z}e_j) = -cot^{(1)}(z + e_j, 2\mathbb{Z}e_j),$$

from [35, 38]. The poles of these functions are the points of $2\mathbb{Z}e_j$ and $\mathbb{Z}e_j \setminus 2\mathbb{Z}e_j$, respectively. Consequently the poles are ∞-points. Hence, these functions are special meromorphic functions.

Example 1.3 *The function $q_{\tau(i)}(z) = \dfrac{\partial}{\partial x_i} q_{\mathbf{0}}(z)$, $\quad (i = 1, 2, ..., n)$ has an isolated pole at zero. Let us prove that these functions are special meromorphic.*

Using formula (1.11) these functions can be rewritten as

$$q_{\tau(i)}(z) = q_{\mathbf{0}}(z)\left[\frac{n-1}{2}(\overline{z}^{-1}e_i) - \frac{n+1}{2}(e_i z^{-1})\right].$$

Taking the norm, leads to

$$\begin{aligned}
\|q_{\tau(i)}(z)\| &= \|q_{\mathbf{0}}(z)\|\left[\left\|-\frac{n-1}{2}(\overline{z}^{-1}e_i) + \frac{n+1}{2}(e_i z^{-1})\right\|\right] \\
&\geq \|q_{\mathbf{0}}(z)\|\left[\frac{n+1}{2}\|(e_i z^{-1})\| - \frac{n-1}{2}\|(\overline{z}^{-1}e_i)\|\right] \\
&= \|q_{\mathbf{0}}(z)\|\left(\frac{n+1}{2} - \frac{n-1}{2}\right)\|z\|^{-1} \\
&= \frac{1}{\|z\|^{n+1}}. \qquad (1.15)
\end{aligned}$$

Applying inequality (1.12) and (1.15), we have:

$$\frac{1}{\|z\|^{n+1}} \leq \|q_{\tau(i)}(z)\| \leq \frac{n}{\|z\|^{n+1}}.$$

Taking the limit over the isolated singularity, it yields $\lim_{z\to 0}\|q_{\tau(i)}(z)\| = \infty$. *Hence, we conclude that* $q_{\tau(i)}$ *are special meromorphic.*

Relaying on the representation formula given in (1.14) we present the following example.

Example 1.4 *Let* $\mathbf{m} = 2\tau(i)$ *for* $0 < i \leq n$. *Then*

$$\begin{aligned} q_{2\tau(i)}(z) &= \frac{1}{n-1}\left(a(n,2\tau(i),0)\frac{Z^{2\tau(i)}}{\|z\|^{n+3}}\left(\frac{n+3}{z}+\frac{2}{x_i}e_i\right)+a(n,2\tau(i),\tau(i))(n+1)\frac{\overline{z}}{\|z\|^{n+3}}\right) \\ &= \frac{1}{n-1}\left(a(n,2\tau(i),0)\left(\frac{\overline{z}x_i^2(n+3)}{\|z\|^{n+5}}+\frac{2x_i\|z\|^2}{\|z\|^{n+5}}e_i\right)+a(n,2\tau(i),\tau(i))\frac{\overline{z}(n+1)}{\|z\|^{n+3}}\right). \end{aligned}$$

Taking the norm, one has:

$$\begin{aligned} \left\|q_{2\tau(i)}\right\|^2 &= \frac{(n+1)^2}{4\|z\|^{2n+8}}\left[\|z\|^4+4(n^2+4n+3)x_i^4-4(n+1)x_i^2\|z\|^2\right] \\ &= \frac{(n+1)^2}{4\|z\|^{2n+8}}(\|z\|^2-2(n+1)x_i^2)^2+8(n+1)x_i^2 \end{aligned}$$

where a solution of $(\|z\|^2-2(n+1)x_i^2)^2+8(n+1)x_i^2 = 0$ *is zero.*

Therefore, we obtain

$$\lim_{z\to 0} q_{2\tau(i)}(z) = \infty.$$

In the next example we consider functions which have at most isolated pole of order $n+1$.

Example 1.5 *Let* z_0 *be a pole of order* $n+1$ *of a left monogenic function* f. *Consider the following series expansion of* f *in a neighborhood of* z_0

$$f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} \mathcal{P}_{\mathbf{m}}(z-z_0)a_{\mathbf{m}} + \sum_{|\mathbf{m}|=0}^{1} q_{\mathbf{m}}(z-z_0)b_{\mathbf{m}}, \tag{1.16}$$

such that, for $|\mathbf{m}^*| = 1$, $b_{\mathbf{m}^*} \in \Gamma_n$ *and*

$$\|b_{\mathbf{m}^*}\| > n\sum_{\substack{|\mathbf{m}|=1 \\ \mathbf{m}\neq\mathbf{m}^*}} \|b_{\mathbf{m}}\|. \tag{1.17}$$

Let us prove that f *is also a left special meromorphic function.*

In order to do so, we start by proving that there exists two positive constants c_{inf} *and* c_{sup} *such that in a neighborhood of* z_0 *we have*

$$\frac{c_{inf}}{\|z-z_0\|^{n+1}} - o(1) \leq \|f(z)\| \leq \frac{c_{sup}}{\|z-z_0\|^{n+1}} + o(1). \tag{1.18}$$

Without loss of generality, assume that $z_0 = 0$*. Using formulas (1.12) and (1.9), we obtain*

$$\begin{aligned} \|f(z)\| &\leq \sum_{|\mathbf{m}|=0}^{+\infty} \|\mathcal{P}_{\mathbf{m}}(z)\|\|a_{\mathbf{m}}\| + \sum_{|\mathbf{m}|=0}^{1} \|q_{\mathbf{m}}(z)\|\|b_{\mathbf{m}}\| \\ &\leq \frac{1}{\|z\|^{n+1}} \left[n \sum_{|\mathbf{m}|=1} \|b_{\mathbf{m}}\| + \|z\|\|b_{\mathbf{0}}\| + \|z\|^{n+1} \sum_{|\mathbf{m}|=0}^{+\infty} \|a_{\mathbf{m}}\| \frac{\|z\|^{\mathbf{m}}}{\mathbf{m}!} \right] \\ &\leq \frac{1}{\|z\|^{n+1}} \left[n \sum_{|\mathbf{m}|=1} \|b_{\mathbf{m}}\| + o(1) \right]. \end{aligned} \tag{1.19}$$

Using the same arguments as in Example 1.3 and applying the condition (1.17) on the coefficients, leads to:

$$\begin{aligned} \|f(z)\| &\geq \frac{1}{\|z\|^{n+1}} \left[\left(\|b_{\mathbf{m}^*}\| - n \sum_{\substack{\mathbf{m}\neq\mathbf{m}^* \\ |\mathbf{m}|=1}} \|b_{\mathbf{m}}\| \right) - \|z\| \left(\|b_{\mathbf{0}}\| + \|z\|^n \left\| \sum_{|\mathbf{m}|=0}^{+\infty} V_{\mathbf{m}}(z) a_{\mathbf{m}} \right\| \right) \right] \\ &\geq \frac{1}{\|z\|^{n+1}} \left[\left(\|b_{\mathbf{m}^*}\| - n \sum_{\substack{\mathbf{m}\neq\mathbf{m}^* \\ |\mathbf{m}|=1}} \|b_{\mathbf{m}}\| \right) - o(1) \right]. \end{aligned} \tag{1.20}$$

Taking the limit towards the isolated singularity, in view of (1.19) and (1.20), we obtain

$$\lim_{z\to 0} \|f(z)\| = \infty.$$

This proves that f *is left special meromorphic.*

In order to proceed we define the Jacobian matrix of a Clifford valued function g defined as

$$\begin{aligned} g(z) &= \sum_{i=0}^{n} g_i(z) e_i + \sum_{\substack{i_1<i_2 \\ |(i_1,i_2)|=2}} g_{(i_1,i_2)}(z) e_{i_1 i_2} + \ldots \\ &+ \sum_{\substack{i_1<\ldots<i_n \\ |(i_1,\ldots,i_{n-1})|=n-1}} g_{(i_1,\ldots,i_{n-1})}(z) e_{i_1\ldots i_{n-1}} + g_{(1,2,\ldots,n)}(z) e_{12\ldots n}, \end{aligned}$$

where the real valued functions $g_{\mathbf{m}}$ ($\mathbf{m} := (m_1, m_2, \ldots, m_n)$ for $m_1 < \ldots < m_n$, and $m_i \in \{0, 1, 2, \cdots, n\}$, $i = 1, ..., n$) are denoted as the real component functions of g. Identifying this function with a vector in $\mathbb{R}^{2^n}$, the Jacobian matrix J_g is represented by

$$J_g(z) := \begin{bmatrix} \frac{\partial}{\partial x_0} g_0(z) & \frac{\partial}{\partial x_1} g_0(z) & \cdots & \frac{\partial}{\partial x_n} g_0(z) \\ \frac{\partial}{\partial x_0} g_1(z) & \frac{\partial}{\partial x_1} g_1(z) & \cdots & \frac{\partial}{\partial x_n} g_1(z) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_0} g_n(z) & \frac{\partial}{\partial x_1} g_n(z) & \cdots & \frac{\partial}{\partial x_n} g_n(z) \\ \frac{\partial}{\partial x_0} g_{(1,2)}(z) & \frac{\partial}{\partial x_1} g_{(1,2)}(z) & \cdots & \frac{\partial}{\partial x_n} g_{(1,2)}(z) \\ \frac{\partial}{\partial x_0} g_{(1,3)}(z) & \frac{\partial}{\partial x_1} g_{(1,3)}(z) & \cdots & \frac{\partial}{\partial x_n} g_{(1,3)}(z) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_0} g_{(1,n)}(z) & \frac{\partial}{\partial x_1} g_{(1,n)}(z) & \cdots & \frac{\partial}{\partial x_n} g_{(1,n)}(z) \\ \frac{\partial}{\partial x_0} g_{(2,3)}(z) & \frac{\partial}{\partial x_1} g_{(2,3)}(z) & \cdots & \frac{\partial}{\partial x_n} g_{(2,3)}(z) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_0} g_{(n-1,n)}(z) & \frac{\partial}{\partial x_1} g_{(n-1,n)}(z) & \cdots & \frac{\partial}{\partial x_n} g_{(n-1,n)}(z) \\ \frac{\partial}{\partial x_0} g_{(1,2,3)}(z) & \frac{\partial}{\partial x_1} g_{(1,2,3)}(z) & \cdots & \frac{\partial}{\partial x_n} g_{(1,2,3)}(z) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_0} g_{(n-2,n-1,n)}(z) & \frac{\partial}{\partial x_1} g_{(n-2,n-1,n)}(z) & \cdots & \frac{\partial}{\partial x_n} g_{(n-2,n-1,n)}(z) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_0} g_{(1,2,3,\ldots,n)}(z) & \frac{\partial}{\partial x_1} g_{(1,2,3,\ldots,n)}(z) & \cdots & \frac{\partial}{\partial x_n} g_{(1,2,3,\ldots,n)}(z) \end{bmatrix}_{2^n \times (n+1)} . \tag{1.21}$$

This can also be written in a more compact form as ∇g.

Remark 1.5 *Notice that for a $\mathcal{A}_{n+1}$-valued function f, the Jacobian matrix is a $(n+1) \times (n+1)$ matrix.*
If the determinant of the Jacobian matrix of the function f at an y-point of f, x^* is non zero (i.e., $\det J_f(x^*) \neq 0$) then x^* is an isolated y-point of f. This is a consequence of the implicit function theorem.

An interesting type of Clifford valued function is the Clifford group valued function (ClG-valued function) in a domain $\Omega \subset \mathcal{A}_{n+1}$. A Clifford group valued function g as the form

$$g(z) := \prod_{i=0}^{k} G_i(z), \qquad k \in \mathbb{N}$$

where G_i are paravector valued functions in Ω. Since these functions are products of paravector valued functions, then they also have an inverse with respect to the Clifford multiplication. This is

$$g^{-1}(z) := \frac{\overline{g(z)}}{\|g(z)\|^2} = \frac{\overline{G_k(z)}}{\|G_k(z)\|^2} \frac{\overline{G_{k-1}(z)}}{\|G_{k-1}(z)\|^2} \cdots \frac{\overline{G_0(z)}}{\|G_0(z)\|^2}.$$

If we assume that these functions belongs at least to the class $\mathcal{C}^1(\Omega')$ for $\Omega' \subset \Omega$ then we may establish the following result.

Proposition 1.2 *Let g be a left (right) monogenic Clifford group valued function in a domain $\Omega \subset \mathcal{A}_{n+1}$. If $g^{-1}(z) := \frac{\overline{g(z)}}{\|g(z)\|^2}$ belongs to the class $\mathcal{C}^1(\Omega')$ for $\Omega' \subset \Omega$, then*

$$J_{g^{-1}}(z) = \frac{1}{\|g(z)\|^2} \left[J_{\overline{g}}(z) - \frac{2}{\|g(z)\|^2} M J_{\overline{g}}(z) \right],$$

where $M := \Big[g_A g_B \Big]_{|A| \times |B|}$ for $|A|, |B| = 0, \ldots, n$, is at most $2^n \times 2^n$ matrix. Furthermore,

$$\|J_{g^{-1}}(z)\| \leq \frac{(2^n + 2)}{\|g(z)\|^2} \|J_g(z)\|.$$

In particular, if g is a paravector valued function in Ω and if g^{-1} belongs at least to $\mathcal{C}^1(\Omega')$ then

$$\|J_{g^{-1}}(z)\| \leq \frac{(n + 3)}{\|g(z)\|^2} \|J_g(z)\|.$$

Proof. We decompose g into its real components, i.e.,

$$g(z) := \sum_{|A|=0}^{n} g_A(z) e_A, \qquad A \subset \{1, \cdots, n\}.$$

This representation in turn can be identified with a vector in $\mathbb{R}^{2^n}$. Then the Jacobian matrix is given by

$$J_g(z) := \left[\frac{\partial g_A}{\partial x_i}(z) \right]_{2^n \times (n+1)} \qquad |A| = 0, \cdots, n, \quad i = 0, 1, \ldots, n.$$

Since the function g consists of a product of paravector valued functions, g has an inverse with respect to the Clifford multiplication which has the form

$$g^{-1}(z) := \frac{\overline{g(z)}}{\|g(z)\|^2} = \sum_{|A|=0}^{n} (-1)^{|A|} \frac{g_A(z)}{\|g(z)\|^2} e_A, \qquad |A| = 0, \cdots, n. \tag{1.22}$$

Identifying (1.22) as a vector in $\mathbb{R}^{2^n}$ and calculating the partial derivative of each component function of g^{-1}, one obtains

$$\frac{\partial}{\partial x_i}\left(\frac{g_A(z)}{\|g(z)\|^2}\right) := \frac{1}{\|g(z)\|^4}\left(\frac{\partial g_A}{\partial x_i}(z)\|g(z)\|^2 - 2g_A(z)\sum_{|B|=0}^{n}\frac{\partial g_B}{\partial x_i}(z) g_B(z)\right),$$

for $A, B \subseteq \{1, \cdots, n\}$ and $i = 0, 1, \ldots, n$. Therefore, the Jacobian matrix of g^{-1} has the following form

$$\begin{aligned} J_{g^{-1}}(z) &= \frac{1}{\|g(z)\|^2}\left(J_{\overline{g}}(z) - \frac{2}{\|g(z)\|^2} M J_{\overline{g}}(z)\right) \\ &= \frac{1}{\|g(z)\|^2}\left(I_{2^n\times 2^n} - \frac{2}{\|g(z)\|^2} M\right) J_{\overline{g}}(z), \end{aligned} \tag{1.23}$$

where

$$J_{\overline{g}}(z) := \left[(-1)^{|A|}\frac{\partial}{\partial x_i}\left(g_A(z)\right)\right]_{2^n\times(n+1)},$$

$I_{2^n\times 2^n}$ the identity matrix and the matrix M is defined as

$$\begin{bmatrix} g_0^2(z) & \cdots & -g_0(z)g_n(z) & -g_0(z)g_{(1,2)}(z) & \cdots & -g_0(z)g_{(1,2,3,\ldots,n)}(z) \\ -g_1(z)g_0(z) & \cdots & -g_1(z)g_n(z) & -g_1(z)g_{(1,2)}(z) & \cdots & -g_1(z)g_{(1,2,3,\ldots,n)}(z) \\ -g_2(z)g_0(z) & \cdots & -g_2(z)g_n(z) & -g_2(z)g_{(1,2)}(z) & \cdots & -g_2(z)g_{(1,2,3,\ldots,n)}(z) \\ \vdots & \vdots & \vdots & \vdots & & \\ -g_n(z)g_0(z) & \cdots & g_n^2(z) & -g_n(z)g_{(1,2)}(z) & \cdots & -g_n(z)g_{(1,2,3,\ldots,n)}(z) \\ \vdots & \vdots & \vdots & \vdots & & \\ -g_{(1,2)}(z)g_0(z) & \cdots & -g_{(1,2)}(z)g_n(z) & g_{(1,2)}^2(z) & \cdots & -g_{(1,2)}(z)g_{(1,2,3,\ldots,n)}(z) \\ \vdots & \vdots & \vdots & \vdots & & \\ -g_{(1,2,3,\ldots,n)}(z)g_0(z) & \cdots & -g_{(1,2,3,\ldots,n)}(z)g_n(z) & -g_{(1,2,3,\ldots,n)}(z)g_{(1,2)}(z) & \cdots & g_{(1,2,3,\ldots,n)}^2(z) \end{bmatrix}.$$

For simplicity, M is denoted by $M := \Big[g_A g_B\Big]_{|A|\times|B|}$ for $|A|, |B| = 0, \ldots, n$. Moreover, since $\|J_{\overline{g}}(z)\| = \|J_g(z)\|$ and $\|M\| = \|g(z)\|^2$ we infer that

$$\begin{aligned} \|J_{g^{-1}}(z)\| &\leq \frac{1}{\|g(z)\|^2} \left\| \left[I_{2^n \times 2^n} - \frac{2}{\|g(z)\|^2} M \right] \right\| \|J_{\overline{g}}(z)\| \\ &\leq \frac{(2^n+2)}{\|g(z)\|^2} \|J_g(z)\|. \end{aligned}$$

For the particular case where g is a paravector valued function then $\|I_{(n+1)\times(n+1)}\| = n+1$, consequently we obtain

$$\begin{aligned} \|J_{g^{-1}}(z)\| &\leq \frac{1}{\|g(z)\|^2} \left\| \left[I_{(n+1)\times(n+1)} - \frac{2}{\|g(z)\|^2} M \right] \right\| \|J_{\overline{g}}(z)\| \\ &\leq \frac{(n+3)}{\|g(z)\|^2} \|J_g(z)\|. \end{aligned}$$

□

We conclude this chapter introducing a generalization of the classical spherical derivative and discuss some of its properties.

Definition 1.15 *Let f be a special meromorphic Clifford valued function defined in a domain Ω of $\mathcal{A}_{n+1}$. We define $\Theta(f) : \Omega \to \mathbb{R}_0^+$ by*

$$\Theta(f)(z) := \frac{||J_f(z)||}{1 + \|f(z)\|^2}$$

whenever z is not an isolated pole of f, and

$$\Theta(f)(z) := \lim_{w \to z} \Theta(f)(w)$$

if z is an isolated pole of f.

Analogously as in the complex case we obtain the following property.

Proposition 1.3 *If z_0 is an isolated pole of a special meromorphic Clifford valued function f, then $\Theta(f)(z_0) = 0$.*

Proof. Without loss of generality, let $z = 0$ be a pole of order $n + p$. Since f is special meromorphic we know that the function has the following behavior

$$\frac{c_{inf}}{\|z\|^{n+p}} - o(1) \leq \|f(z)\| \leq \frac{c_{sup}}{\|z\|^{n+p}} + o(1), \tag{1.24}$$

in a neighborhood of 0, where c_{inf}, c_{sup} are positive constants, i.e., $\|f(z)\| \cong \frac{c}{\|z\|^{n+p}}$ where c is a positive constant.

Consider the series expansion of f:

$$f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} \mathcal{P}_{\mathbf{m}}(z) a_{\mathbf{m}} + \sum_{|\mathbf{m}|=0}^{p} q_{\mathbf{m}}(z) b_{\mathbf{m}},$$

with principal part $P_p(z) := \sum_{|\mathbf{m}|=0}^{p} q_{\mathbf{m}}(z) b_{\mathbf{m}}$ where

$$\frac{c_{inf}}{\|z\|^{n+p}} - o(1) \leq \|P_p(z)\| \leq \frac{c_{sup}}{\|z\|^{n+p}} + o(1)$$

for c_{inf}, c_{sup} two positive constants.

The next step is to obtain an upper bound estimate for $\|J_f\|$. Computing the partial derivatives of f we obtain:

$$\frac{\partial f}{\partial x_i}(z) = \sum_{|\mathbf{m}|=0}^{+\infty} \frac{\partial \mathcal{P}_{\mathbf{m}}}{\partial x_i}(z) a_{\mathbf{m}} + \sum_{|\mathbf{m}|=0}^{p} \frac{\partial q_{\mathbf{m}}}{\partial x_i}(z) b_{\mathbf{m}}. \tag{1.25}$$

Taking the norm of (1.25) and using (1.12) we obtain

$$\begin{aligned}
\left\| \frac{\partial f}{\partial x_i}(z) \right\| &\leq \left\| \sum_{|\mathbf{m}|=0}^{+\infty} \frac{\partial \mathcal{P}_{\mathbf{m}}}{\partial x_i}(z) a_{\mathbf{m}} \right\| + \sum_{|\mathbf{m}|=0}^{p} \left\| \frac{\partial q_{\mathbf{m}}}{\partial x_i}(z) \right\| \|b_{\mathbf{m}}\| \\
&\leq \left\| \sum_{|\mathbf{m}|=0}^{+\infty} \frac{\partial \mathcal{P}_{\mathbf{m}}}{\partial x_i}(z) a_{\mathbf{m}} \right\| + \sum_{|\mathbf{m}|=0}^{p} \frac{n(n+1)...(n+|\mathbf{m}|)}{\|z\|^{n+|\mathbf{m}|+1}} \|b_{\mathbf{m}}\| \\
&= \frac{1}{\|z\|^{n+p+1}} \left[n(n+1)...(n+p) \sum_{|\mathbf{m}|=p} \|b_{\mathbf{m}}\| \right. \\
&+ \left. \|z\|^p \left(\sum_{|\mathbf{m}|=0}^{p-1} \frac{n(n+1)...(n+|\mathbf{m}|)}{\|z\|^{|\mathbf{m}|}} \|b_{\mathbf{m}}\| + \|z\|^{n+1} \left\| \sum_{|\mathbf{m}|=0}^{+\infty} \frac{\partial \mathcal{P}_{\mathbf{m}}}{\partial x_i}(z) a_{\mathbf{m}} \right\| \right) \right] \\
&\leq \frac{b}{\|z\|^{n+p+1}} + o(1)
\end{aligned} \tag{1.26}$$

where $b := n(n+1)...(n+p) \sum_{|\mathbf{m}|=p} \|b_{\mathbf{m}}\|$ and $i = 0, ..., n$.

Using inequality (1.26), we obtain

$$\begin{aligned}
\|J_f(z)\|^2 &= \sum_{i=0}^{n} \left\| \frac{\partial f}{\partial x_i}(z) \right\|^2 \\
&\leq \sum_{i=0}^{n} \left(\frac{b}{\|z\|^{n+p+1}} + o(1) \right)^2 \\
&= (n+1) \left(\frac{b}{\|z\|^{n+p+1}} + o(1) \right)^2 . \qquad (1.27)
\end{aligned}$$

Applying inequalities (1.27) and (1.24), yields:

$$\begin{aligned}
\Theta(f)(z) &= \frac{\|J_f(z)\|}{1 + \|f(z)\|^2} \\
&\leq \sqrt{n+1} \frac{\left(\frac{b}{\|z\|^{n+p+1}} + o(1) \right)}{1 + \left(\frac{c_{inf}}{\|z\|^{n+p}} - o(1) \right)^2} \\
&= \sqrt{n+1} \frac{\|z\|^{(n+p-1)} \left(b + o(1)\|z\|^{n+p+1} \right)}{\|z\|^{2(n+p)} + \left(c_{inf} - o(1)\|z\|^{n+p} \right)^2} .
\end{aligned}$$

Since $c_{inf} \neq 0$ and $n + p > 1$, we get

$$\Theta(f)(0) := \lim_{z \to 0} \Theta(f)(z) = 0.$$

Hence, we conclude that the function $\Theta(f)(z_0) = 0$ for an arbitrary pole z_0 of f. □

Remark 1.6 *In the classical case the spherical derivative is defined as*

$$\Theta(f)(z) := \begin{cases} \dfrac{|f'(z)|}{1 + |f(z)|^2}, & z \text{ is not a pole} \\ \frac{1}{|Res(f,z)|}, & z \text{ is a pole of order } 1 \\ 0, & z \text{ is a pole of order } s \geq 2 \end{cases}$$

where $Res(f, z)$ is the residue of f at z.

Next we discuss some basic properties of the higher dimensional generalization.

Proposition 1.4 *Let f be a special meromorphic Clifford valued function in Ω. Then:*

(i) $\Theta(f)$ *is a continuous function;*

(ii) $\Theta(f)(z) < \infty$ *for all* $z \in \Omega$.

Proof. From the definition of $\Theta(f)$ and the property that f is special meromorphic, we obtain that $\Theta(f)$ is continuous.

It remains to prove *(ii)*. Since f is special meromorphic, it has only isolated singularities which are ∞-points. Let Σ be the set of all isolated singularities of f. If $z_0 \in \Sigma$ then, by Proposition 1.3, $\Theta(f)(z_0) = 0$.

If $z_0 \in \Omega \setminus \Sigma$ then there exists a positive constant $M < \infty$ such that $\Theta(f)(z_0) \leq M$. $\square$

Consider the following examples:

Example 1.6 *Consider the quaternion valued Cauchy kernel function*

$$q_{\mathbf{0}}(z) := \frac{\overline{z}}{\|z\|^4},$$

where $z = 0$ is a pole of order 3. The function $q_{\mathbf{0}}$ is a special meromorphic where each component function is given by

$$q_{\mathbf{0}_j}(z) = \overline{e_j}\frac{x_j}{\|z\|^4}, \quad j = 0, 1, 2, \ldots$$

and the partial derivatives by

$$\begin{aligned}
&\frac{\partial q_{\mathbf{0}_0}(z)}{\partial x_0} = \frac{\|z\|^4 - 4x_0x_i}{\|z\|^6}; \qquad \frac{\partial q_{\mathbf{0}_0}(z)}{\partial x_i} = \frac{-4x_0x_i}{\|z\|^6} \quad i = 1, 2, 3;\\
&\frac{\partial q_{\mathbf{0}_j}(z)}{\partial x_j} = -\frac{\|z\|^4 - 4x_jx_i}{\|z\|^6}; \qquad \frac{\partial q_{\mathbf{0}_j}(z)}{\partial x_i} = -\frac{-4x_jx_i}{\|z\|^6} \quad i \neq j \quad i, j = 1, 2, 3.
\end{aligned} \tag{1.28}$$

Therefore, $\|J_{q_{\mathbf{0}}}(z)\| = \dfrac{\sqrt{12}\|z\|^2}{\|z\|^6}$ *obtaining the following expression for*

$$\Theta(q_{\mathbf{0}})(z) = \frac{\sqrt{12}\|z\|^2}{1 + \|z\|^6}.$$

For $z = 0$ we have $\Theta(q_{\mathbf{0}})(0) = 0$.

In fact, for $\|z\| < r$ $(r > 0)$ we obtain the following upper bound estimate

$$\Theta(q_{\mathbf{0}})(z) < \sqrt{12}r^2.$$

Example 1.7 *Let us consider the generalized monogenic exponential function given in [13, p.117] by*

$$\begin{aligned} g(z) &= \exp(x_0, x_1, \ldots, x_n) \\ &= e^{x_1+\cdots+x_n}\left(\cos(x_0\sqrt{n}) - \frac{1}{\sqrt{n}}(e_1+\cdots+e_n)\sin(x_0\sqrt{n})\right). \end{aligned}$$

The component functions are given by

$$\begin{aligned} g_0(z) &= e^{x_1+\cdots+x_n}\cos(x_0\sqrt{n}); \\ g_i(z) &= -\frac{e^{x_1+\cdots+x_n}}{\sqrt{n}}\sin(x_0\sqrt{n}), \quad i=1,2,\ldots,n. \end{aligned}$$

and the partial derivatives by

$$\frac{\partial g_0}{\partial x_0}(z) = -\sqrt{n}e^{x_1+\cdots+x_n}\sin(x_0\sqrt{n}); \qquad \frac{\partial g_0}{\partial x_i}(z) = e^{x_1+\cdots+x_n}\cos(x_0\sqrt{n}),\ i=1,2,\ldots,n;$$

$$\frac{\partial g_i}{\partial x_0}(z) = -e^{x_1+\cdots+x_n}\cos(x_0\sqrt{n}); \qquad \frac{\partial g_i}{\partial x_j}(z) = -\frac{e^{x_1+\cdots+x_n}}{\sqrt{n}}\sin(x_0\sqrt{n}),\ j,i=1,2,\ldots,n.$$

Therefore, we obtain the following expression for

$$\Theta(g)(z) = \frac{e^{x_1+\cdots+x_n}\sqrt{n+1}}{1+e^{2(x_1+\cdots+x_n)}}.$$

Take $\|z\| < r$ *where* $r > 0$*. The function* $\Theta(g)(z)$ *is bounded by*

$$\Theta(g)(z) \le e^{x_1+\cdots+x_n}\sqrt{n+1} < e^{nr}\sqrt{n+1}.$$

Chapter 2

Marty's criterion and Zalcman's Lemma in Clifford analysis

The aim of this chapter is to develop some fundamentals of the theory of normal families in the framework of Clifford analysis. We provide a generalization of Marty's criterion which is one of the basic results in the classical theory. As an application we proved a generalization of Zalcman's lemma.

The concept of a normal family of holomorphic and meromorphic complex valued functions was introduced by Montel [53]. A necessary and sufficient condition of normality was obtained by Marty [48]. A proof from a more geometrical point of view of Marty's criterion was given by Ahlfors [3, pp.218]. An analytic proof was also given by Hayman [34, pp.158].

Although Marty's criterion gives a complete answer to the question " When is a family of functions normal?", in practice it is very difficult to check normality by this criterion. Based on Marty's criterion, Zalcman established a necessary and sufficient condition of normality in [72]. A refinement has been obtained in [73].

In 1982, Miniowitz generalized the Zalcman lemma for families of K-quasimeromorphic mappings[1] in the (n+1)-dimensional unit ball [51].

In this chapter, we give the basic notions, such as chordal distance in $\mathcal{A}_{n+1}$, and also normality of a family of left (right) monogenic and left (right) special meromorphic

[1] see e.g. [52]

functions defined in a fixed domain Ω.

Throughout this chapter $\mathbb{A}$ is to be denoted as either $\mathcal{A}_{n+1}$ or $\mathbb{H}$ when no ambiguity occurs.

2.1 Chordal distance

Since $\mathbb{A} \equiv \mathcal{A}_{n+1}$ is isomorphic to $\mathbb{R}^{n+1}$, a one point compactification of $\mathbb{R}^{n+1}$ is used. The extended space is denoted by $\overline{\mathbb{A}} = \mathbb{A} \cup \{\infty\}$.

As a model for $\overline{\mathbb{A}}$ we introduce a hypersphere in $\mathbb{R}^{n+2}$. The relation which links the coordinates in $\mathbb{A}$ and the rectangular coordinates in the image on the hypersphere is as follows.

Consider an orthonormal coordinate system defined by the standard basis $\{e_0, e_1, ..., e_n, e_{n+1}\}$ in $\mathbb{R}^{n+2}$ which can be identified with $\mathbb{R}^{n+2} = \mathbb{R} \oplus \mathbb{A}$. A point of $\mathbb{A}$ is denoted by

$$z = x_0 e_0 + x_1 e_1 + ... + x_n e_n$$

and identified with $(x_0, x_1, ..., x_n)$. The hypersphere $\mathrm{S}^{n+1}_{\frac{1}{2}}(0, \ldots, \frac{1}{2})$ is defined as:

$$\mathrm{S}^{n+1}_{\frac{1}{2}}(0, \ldots, \frac{1}{2}) := \left\{ \widetilde{y} = (y_0, y_1, ..., y_{n+1}) : \; y_0^2 + y_1^2 + ... + \left(y_{n+1} - \frac{1}{2} \right)^2 = \frac{1}{4} \right\}. \tag{2.1}$$

The relation between $y_0, y_1, ..., y_n, 1 - y_{n+1}$ for $(y_0, y_1, ..., y_n, 1 - y_{n+1}) \in \mathrm{S}^{n+1}_{\frac{1}{2}}(0, \ldots, \frac{1}{2})$ and the components of $z \in \mathbb{A}$ are obtained by

$$y_i = \alpha \, x_i, \qquad 1 - y_{n+1} = \alpha, \tag{2.2}$$

for $i = 0, 1, ..., n$ and α a positive real number. Substituting the system of equations given by (2.2) into $y_0^2 + y_1^2 + ... + \left(y_{n+1} - \frac{1}{2}\right)^2 = \frac{1}{4}$, one obtains the explicit expression $\alpha = \dfrac{1}{1 + \|x\|^2}$.

Using (2.2), the map $\phi : \mathrm{S}^{n+1}_{\frac{1}{2}}(0, \ldots, \frac{1}{2}) \to \overline{\mathbb{A}}$ is defined as

$$\phi(\widetilde{y}) \;\; := \;\; \begin{cases} \displaystyle\sum_{i=0}^{n} \frac{y_i}{1 - y_{n+1}} e_i, & \widetilde{y} \in \mathrm{S}^{n+1}_{\frac{1}{2}}(0, \ldots, \frac{1}{2}) \setminus \{(0, ..., 0, 1)\} \\ \infty, & \widetilde{y} = (0, ..., 0, 1), \end{cases} \tag{2.3}$$

and its inverse $\psi : \overline{\mathbb{A}} \to \mathrm{S}^{n+1}_{\frac{1}{2}}(0,\ldots,\frac{1}{2})$ as

$$\psi(z) \;:=\; \begin{cases} \left(\dfrac{x_0}{1+\|x\|^2},...,\dfrac{x_n}{1+\|x\|^2},\dfrac{\|x\|^2}{1+\|x\|^2}\right), & z \in \mathbb{A} \\ (0,...,0,1), & z = \infty, \end{cases} \tag{2.4}$$

where we identify $z := x_0e_0 + x_1e_1 + ..., +x_ne_n$ with $(x_0, x_1, ..., x_n)$, and use the expression for α. We obtain a relation between $\overline{\mathbb{A}}$ and $\mathrm{S}^{n+1}_{\frac{1}{2}}(0,\ldots,\frac{1}{2})$ in terms of ϕ and ψ. The next step is to obtain a metric on $\overline{\mathbb{A}}$.

As an example, consider the Euclidian distance between two elements of $\mathbb{A}$, $a := \sum_{i=0}^{n} a_ie_i$ and $b := \sum_{i=0}^{n} b_ie_i$, given by $\|a-b\|^2 = \sum_{i=0}^{n}(a_i-b_i)^2$. In this metric the ideal point $z = \infty$ plays an exceptional role. However, it is possible to introduce a metric avoiding this problem.

Let a, b denote two points in $\overline{\mathbb{A}}$ and let $\widetilde{a}, \widetilde{b}$ their corresponding points on the hypersphere $\mathrm{S}^{n+1}_{\frac{1}{2}}(0,\ldots,\frac{1}{2})$ induced by (2.1). The length of the line segment between $\widetilde{a}, \widetilde{b}$ is defined to be the chordal distance between a, b and is denoted by $d_{ch}[a, b]$. To set up a closed expression of the chordal distance, we distinguish three cases:

(i) Both elements are finite

$$\begin{aligned} d_{ch}[a,b]^2 &= \|\psi(a)-\psi(b)\|^2 \\ &= \sum_{i=0}^{n}\left(\frac{a_i}{1+\|a\|^2}-\frac{b_i}{1+\|b\|^2}\right)^2 + \left(\frac{\|a\|^2}{1+\|a\|^2}-\frac{\|b\|^2}{1+\|b\|^2}\right)^2 \\ &= \frac{\|a\|^2\,(1+\|a\|^2)}{(1+\|a\|^2)^2} + \frac{\|b\|^2\,(1+\|b\|^2)}{(1+\|b\|^2)^2} - 2\frac{\sum_{i=0}^{n} a_ib_i + \|a\|^2\|b\|^2}{(1+\|a\|^2)\,(1+\|b\|^2)} \\ &= \frac{\|a\|^2\,(1+\|b\|^2)+\|b\|^2\,(1+\|a\|^2)}{(1+\|a\|^2)\,(1+\|b\|^2)} - 2\frac{\sum_{i=0}^{n} a_ib_i + \|a\|^2\|b\|^2}{(1+\|a\|^2)\,(1+\|b\|^2)} \\ &= \frac{\|a-b\|^2}{(1+\|a\|^2)\,(1+\|b\|^2)}. \end{aligned}$$

Therefore, we obtain:

$$d_{ch}[a,b] = \frac{\|a-b\|}{\sqrt{(1+\|a\|^2)}\sqrt{(1+\|b\|^2)}}.$$

(ii) One of the elements is finite and the other one is infinity, for instance $b = \infty$. Using the same arguments as given in (i), we obtain:

$$\begin{aligned} d_{ch}[a,b]^2 &= \|\psi(a) - \psi(b)\|^2 \\ &= \sum_{i=0}^{n} \left(\frac{a_i}{1+\|a\|^2} \right)^2 + \left(\frac{\|a\|^2}{1+\|a\|^2} - 1 \right)^2 \\ &= \frac{\|a\|^2}{(1+\|a\|^2)^2} + \frac{1}{(1+\|a\|^2)^2} \\ &= \frac{1}{1+\|a\|^2}. \end{aligned}$$

(iii) If both $a = b = \infty$ then $d_{ch}[a,b] = 0$.

Consequently it makes sense to introduce:

Definition 2.1 *(Chordal distance) Let $d_{ch} : \mathcal{A}_{n+1} \to \mathbb{R}_0^+$ be defined by:*

$$d_{ch}[a,b] := \begin{cases} \dfrac{\|a-b\|}{(\sqrt{1+\|a\|^2})(\sqrt{1+\|b\|^2})}, & a, b \text{ finite} \\ \dfrac{1}{\sqrt{1+\|a\|^2}}, & b = \infty, a \text{ finite} \\ 0, & a = \infty,\ b = \infty. \end{cases} \tag{2.5}$$

$d_{ch}[a,b]$ is called the chordal distance since it measures the length of the chord between the corresponding points on the hypersphere.

Remark 2.1 *It is also possible to define the chordal distance for any two elements in the Clifford group, Γ_n. Let $a := \prod_{i=0}^{k_1} \alpha_i$ and $b := \prod_{j=0}^{k_2} \beta_j$ be two elements of the Clifford group, where $\alpha_i, \beta_j \in \mathcal{A}_{n+1} \backslash \{0\}$ for all $i = 0, 1, \ldots, k_1$ and $j = 0, 1, \ldots, k_2$ for $k_1, k_2 \in \mathbb{N}$. Further let us write $a := \sum_{|A|=0}^{k_1} a_A e_A$ and $b := \sum_{|A|=0}^{k_2} b_A e_A$. The elements a and b can be identified with elements of $\mathbb{R}^m$ where $m = \sum_{s=0}^{k} \binom{k}{s}$, for $k = \max\{k_1, k_2\}$. Moreover, when extending the Clifford group Γ_n to $\Gamma_n \cup \{0, \infty\}$ we denote $0^{-1} := \infty$ and $\infty^{-1} := 0$. For $a \in \Gamma_n$ we define $a + \infty = \infty + a := \infty$, $a\ 0 = 0\ a := 0$, $a\ \infty = \infty\ a := \infty$ and consequently*

$a\,0^{-1} := \infty,\ a\,\infty^{-1} := 0$. *We define* $d_{ch} : \Gamma_n \cup \{0,\ \infty\} \to \mathbb{R}_0^+$ *by*

$$d_{ch}[a,b] \;:=\; \begin{cases} \dfrac{\sqrt{\sum\limits_{|A|=0}^{k}(a_A-b_A)^2}}{\sqrt{1+\sum\limits_{|A|=0}^{k}a_A^2}\sqrt{1+\sum\limits_{|A|=0}^{k}b_A^2}}, & a,b \text{ finite} \\ \dfrac{1}{\sqrt{1+\sum\limits_{|A|=0}^{k}a_A^2}}, & b=\infty, a \text{ finite} \\ 0, & a=\infty,\ b=\infty. \end{cases} \tag{2.6}$$

An important invariance property of the chordal distance is stated in the following proposition.

Proposition 2.1 *If* $a,b \in \overline{\mathbb{A}}$ *then* $d_{ch}\left[\dfrac{\overline{a}}{\|a\|^2}, \dfrac{\overline{b}}{\|b\|^2}\right] = d_{ch}\left[a,b\right].$

Proof. If $a,b \in \mathbb{A}\setminus\{0\}$

$$\begin{aligned} \left(d_{ch}\left[\frac{\overline{a}}{\|a\|^2}, \frac{\overline{b}}{\|b\|^2}\right]\right)^2 &= \frac{\left\|\frac{\overline{a}}{\|a\|^2}-\frac{\overline{b}}{\|b\|^2}\right\|^2}{\left(1+\left\|\frac{\overline{a}}{\|a\|^2}\right\|^2\right)\left(1+\left\|\frac{\overline{b}}{\|b\|^2}\right\|^2\right)} \\ &= \frac{\|a\|^2\|b\|^2\left\|\frac{\overline{a}}{\|a\|^2}-\frac{\overline{b}}{\|b\|^2}\right\|^2}{\left(1+\|a\|^2\right)\left(1+\|b\|^2\right)} \\ &= \frac{\|a\|^2\|b\|^2\left(\left(\frac{a_0}{\|a\|^2}-\frac{b_0}{\|b\|^2}\right)^2+\sum_{k=1}^n\left(\frac{a_k}{\|a\|^2}-\frac{b_k}{\|b\|^2}\right)^2\right)}{\left(1+\|a\|^2\right)\left(1+\|b\|^2\right)} \\ &= \frac{\|a\|^2\|b\|^2\left(\frac{\|a\|^2}{\|a\|^4}+\frac{\|b\|^2}{\|b\|^4}-2\sum_{k=0}^n\frac{a_kb_k}{\|a\|^2\|b\|^2}\right)}{\left(1+\|a\|^2\right)\left(1+\|b\|^2\right)} \\ &= \frac{\|b\|^2+\|a\|^2-2\sum_{k=0}^n a_kb_k}{\left(1+\|a\|^2\right)\left(1+\|b\|^2\right)} \\ &= \frac{\|a-b\|^2}{\left(1+\|a\|^2\right)\left(1+\|b\|^2\right)} \\ &= \left(d_{ch}\left[a,b\right]\right)^2. \end{aligned}$$

If $a=\infty$, then $\dfrac{\overline{a}}{\|a\|^2} = a^{-1} = 0$ and

$$d_{ch}\left[0, \frac{\overline{b}}{\|b\|^2}\right] = \frac{1}{\sqrt{1+\frac{1}{\|b\|^2}}} = \frac{\|b\|}{\sqrt{1+\|b\|^2}} = d_{ch}\left[\infty, b\right].$$

An analogous result is obtained for $b = \infty$.

If $a = 0$, then $\dfrac{\overline{a}}{\|a\|^2} = a^{-1} = \infty$ and

$$d_{ch}\left[\infty, \frac{\overline{b}}{\|b\|^2}\right] = \frac{\|b\|}{\sqrt{1+\|b\|^2}} = \frac{1}{\sqrt{1+\frac{1}{\|b\|^2}}} = d_{ch}\left[0, b\right].$$

□

Remark 2.2 *This property also holds for any two arbitrary elements in the extended Clifford group,* $\Gamma_n \cup \{0, \infty\}$ *.*

Another property of the chordal distance, which is analogous to the one in the complex case (see [14, p.4]), is described in:

Proposition 2.2 *Consider* z, w *in* $\overline{\mathbb{A}}$*. Let* $A < B$ *be two positive numbers where* B *can also be* ∞*. Then there is a positive number* $\mu = \mu(A, B)$ *depending only on* A *and* B *such that for* $\|z\| \leq A$*,* $\|w\| \geq B$*,* $d_{ch}\left[z, w\right] \geq \mu$ *is obtained.*

Proof. If $\|z\| \leq A$, $\|w\| \geq B$ with $w \neq \infty$, then

$$\begin{aligned} d_{ch}\left[z, w\right] &= \frac{\|z-w\|}{\sqrt{1+\|z\|^2}\sqrt{1+\|w\|^2}} \geq \frac{\|w\|-\|z\|}{\sqrt{1+\|z\|^2}\sqrt{1+\|w\|^2}} \\ &\geq \frac{1-\frac{\|z\|}{\|w\|}}{\sqrt{1+\|z\|^2}\sqrt{1+\frac{1}{\|w\|^2}}} \geq \frac{1-\frac{A}{B}}{\sqrt{1+A^2}\sqrt{1+\left(\frac{1}{B}\right)^2}}. \end{aligned}$$

If $\|z\| \leq A$ and $w = \infty$, then

$$d_{ch}[z, w] = \frac{1}{\sqrt{1+\|z\|^2}} \geq \frac{1}{\sqrt{1+A^2}}.$$

Therefore, we obtain $d_{ch}[z, w] \geq \mu(A, B)$, where

$$\mu(A, B) := \begin{cases} \dfrac{1-\frac{A}{B}}{\sqrt{1+A^2}\sqrt{1+\frac{1}{B^2}}}, & \text{if } A, B \text{ finite} \\ \dfrac{1}{\sqrt{1+A^2}}, & \text{if } A \text{ finite, } B = \infty. \end{cases}$$

□

In the next section we study normal families of Clifford valued functions.

2.2 Normal families

The theory of normal families plays an important role in complex function theory and has a wide range of applications, as for example, it provides us with tools to prove extensions of Picard's theorem, Schottky's theorem, Landau's theorem (see e.g. [61]), and also to study for example, problems in complex dynamics as well as extremal problems (see e.g. [8]).

In this section we start to study convergence of sequences that consist of Clifford valued functions. In the sequel, some fundamental definitions and results are presented.

Definition 2.2 *Let Ω be a non-empty open subset of $\mathcal{A}_{n+1}$ and $\{f_m\}_{m\in\mathbb{N}}$ be a sequence of real valued functions in Ω.*

(i) A sequence $\{f_m\}_{m\in\mathbb{N}}$ converges uniformly in Ω to $f:\Omega\to\overline{\mathbb{R}}$, if for all $\varepsilon>0$ there exists $m_0\in\mathbb{N}$ such that for $m>m_0$

$$d_{ch}[f_m(z), f(z)] \; < \; \varepsilon, \qquad \forall z\in\Omega,$$

where d_{ch} denotes the chordal distance on $\overline{\mathbb{R}}$. We also say that $\{f_m\}_{m\in\mathbb{N}}$ converges uniformly with respect to the chordal distance to f.

(ii) A sequence $\{f_m\}_{m\in\mathbb{N}}$ converges locally uniformly to f if for each $z_0\in\Omega$ there exists a neighborhood $\mathcal{V}_{z_0}\subset\Omega$ such that $\{f_m\}_{m\in\mathbb{N}}$ converges uniformly on $\mathcal{V}_{z_0}$ to f.

(iii) A sequence of Clifford valued functions in Ω converges locally uniformly in Ω if every component function converges locally uniformly as defined in (ii).

From [21, p.149] an analogue to the classical Weierstrass' theorem is presented, for sequences of monogenic Clifford valued functions.

Theorem 2.1 *Let Ω be a non empty open subset of $\mathcal{A}_{n+1}$. If a sequence of Clifford valued left (right) monogenic functions $\{f_m\}_{m\in\mathbb{N}}$ in Ω converges locally uniformly to a Clifford valued function f, then*

(i) f is left (right) monogenic in Ω.

(ii) for each multi-index $\mathbf{s} := (s_0, s_1, ..., s_n) \in \mathbb{N}^{n+1}$, the sequence $\left\{\frac{\partial^{|\mathbf{s}|}}{\partial \mathbf{x}^{\mathbf{s}}} f_m\right\}_{m\in\mathbb{N}}$ converges locally uniformly to $\frac{\partial^{|\mathbf{s}|}}{\partial \mathbf{x}^{\mathbf{s}}} f$.

The next step is to study the limit function of a uniformly convergent sequence of special meromorphic $\mathbb{A}$-valued functions.

Since $\mathbb{A}$ is either $\mathbb{H}$ or $\mathcal{A}_{n+1}$, a non-zero element of $\mathbb{A}$ has an inverse with respect to the Clifford multiplication. Therefore, for each $\mathbb{A}$-valued function defined in a domain Ω there exists an inverse with respect to the Clifford multiplication of the following form

$$f^{-1}(z) := \frac{\overline{f(z)}}{\|f(z)\|^2}, \qquad z \in \Omega'$$

where $\Omega' := \{z \in \Omega : \ f(z) \neq 0\} \subseteq \Omega$.

The following result establishes a relation between uniform convergence with respect to the chordal distance and uniform convergence with respect to the Euclidean distance.

Theorem 2.2 *Let $\{f_m\}_{m\in\mathbb{N}}$ be a sequence of left (right) special meromorphic $\mathbb{A}$-valued functions in $B(z_0, r)$ for $r > 0$. If $\{f_m\}_{m\in\mathbb{N}}$ is uniformly convergent in $B(z_0, r)$ with respect to the chordal distance and if f is the limit function, then*

(i) if $f(z_0) \neq \infty$, there exists $r_0 > 0$, $r_0 < r$, such that for $z \in B(z_0, r_0)$ the functions f_m and f are left (right) monogenic, moreover

$$\lim_{m\to+\infty} \|f_m(z) - f(z)\| = 0,$$

uniformly in $B(z_0, r_0)$.

(ii) if $f(z_0) = \infty$, then $\{f_m^{-1}\}_{m\in\mathbb{N}}$ converges locally uniformly to f^{-1}, i.e.,

$$\lim_{m\to+\infty} \|f_m^{-1}(z) - f^{-1}(z)\| = 0,$$

uniformly in $B(z_0, r_0)$ where $0 < r_0 < r$.

In order to prove this theorem it is convenient to establish the following result which is analogous to the one given in [14, pp.8] for the complex case.

Lemma 2.1 *If f is a left (right) special meromorphic $\mathbb{A}$-valued (or ClG-valued) function in a domain Ω, then f is continuous in Ω with respect to the chordal distance.*

Proof. Consider a point $z_0 \in \Omega$ and $\mathcal{V}_{z_0} \subset \Omega$ an open neighborhood of z_0. First assume that $f(z_0) \neq \infty$. Applying Definition 2.1 (or (2.6)), for $z \in \mathcal{V}_{z_0}$, the following inequality holds $d_{ch}[f(z), f(z_0)] \leq \|f(z) - f(z_0)\|$. In view of the continuity of f in $\mathcal{V}_{z_0}$, it follows that

$$\lim_{z\to z_0} d_{ch}[f(z), f(z_0)] = 0.$$

For the case $f(z_0) = \infty$ it follows for $z \in \mathcal{V}_{z_0}$ that $\lim\limits_{z\to z_0} f(z) = f(z_0) = \infty$, since f is special meromorphic. Therefore, using the function f^{-1}, we obtain

$$\lim_{z\to z_0} d_{ch}\left[f^{-1}(z), f^{-1}(z_0)\right] = 0.$$

Applying Proposition 2.1 leads to

$$\lim_{z\to z_0} d_{ch}[f(z), f(z_0)] = \lim_{z\to z_0} d_{ch}\left[f^{-1}(z), f^{-1}(z_0)\right] = 0.$$

This proves that f is continuous in Ω with respect to the chordal distance. □

Proposition 2.2 and Lemma 2.1 enable us to prove Theorem 2.2.

Proof of Theorem 2.2.

(i) Assume that $f(z_0) \neq \infty$. Consider $k := d_{ch}[f(z_0), \infty] > 0$, define $A := \frac{2}{k}$, $B := \sqrt{3}(\frac{2}{k}+1)$ and let $\mu(A,B)$ be a positive number as defined in Proposition 2.2.

For $\varepsilon_0 := \min\left\{\frac{k}{6}, \mu(A,B)\right\}$, there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$

$$d_{ch}[f_m(z), f(z)] < \varepsilon_0, \qquad z \in B(z_0, r). \tag{2.7}$$

Since each f_m is special meromorphic then f_m are continuous with respect to the chordal distance, in view of Lemma 2.1. Therefore, there exist r_0 ($r > r_0 > 0$) such that

$$d_{ch}[f_m(z), f_m(z_0)] < \frac{k}{6}, \qquad z \in B(z_0, r_0). \tag{2.8}$$

Using the triangular inequality, the following holds

$$\begin{aligned} d_{ch}[f(z_0), \infty] &\leq d_{ch}[f(z_0), f_m(z_0)] + d_{ch}[f_m(z_0), f_m(z)] \\ &+ d_{ch}[f_m(z), f(z)] + d_{ch}[f(z), \infty]. \end{aligned} \tag{2.9}$$

Inserting (2.7) and (2.8) into (2.9), leads to

$$d_{ch}[f(z_0), \infty] < \frac{k}{2} + d_{ch}[f(z), \infty]. \tag{2.10}$$

Since $k := d_{ch}[f(z_0), \infty]$, we infer by inequality (2.10)

$$k - \frac{k}{2} < d_{ch}[f(z), \infty]. \tag{2.11}$$

Therefore, we obtain

$$\frac{k}{2} < d_{ch}[f(z), \infty] := \frac{1}{\sqrt{1 + \|f(z)\|^2}}, \tag{2.12}$$

and consequently $\|f(z)\|^2 \leq \left(\frac{2}{k}\right)^2 - 1 < \left(\frac{2}{k}\right)^2 =: A^2$.

On the other hand, since $d_{ch}[f(z), \infty] \leq d_{ch}[f_m(z), f(z)] + d_{ch}[f_m(z), \infty]$ and using (2.11), it follows

$$\frac{k}{2} < d_{ch}[f(z), \infty] \leq d_{ch}[f_m(z), f(z)] + d_{ch}[f_m(z), \infty].$$

In view of (2.7) we have

$$\frac{k}{2} < \frac{k}{6} + d_{ch}[f_m(z), \infty].$$

Therefore, we obtain

$$\frac{1}{\sqrt{1+\|f_m(z)\|^2}} =: d_{ch}[f_m(z), \infty] > \frac{k}{2} - \frac{k}{6} = \frac{k}{3}, \quad z \in B(z_0, r_0)$$

which implies that $\|f_m(z)\|^2 \leq \left(\frac{3}{k}\right)^2 - 1 < 3\left(\frac{2}{k}+1\right)^2 := B^2$.

Then, for all $m \geq m_0$ and $z \in B(z_0, r_0)$, we obtain

$$\begin{aligned} \|f_m(z) - f(z)\| &= d_{ch}[f_m(z), f(z)]\sqrt{1+\|f_m(z)\|^2}\sqrt{1+\|f(z)\|^2} \\ &\leq d_{ch}[f_m(z), f(z)]\sqrt{1+B^2}\sqrt{1+A^2}. \end{aligned} \tag{2.13}$$

Since $\{f_m\}_{m\in\mathbb{N}}$ converges locally uniformly with respect to the chordal distance and since $\{f_m\}_{m\in\mathbb{N}}$ satisfies condition (2.13), for $z \in B(z_0, r_0)$ we have

$$\lim_{m\to\infty} \|f_m(z) - f(z)\| = 0.$$

Furthermore, applying Theorem 2.1 the limit function f is left (right) monogenic in $B(z_0, r_0)$.

(ii) Consider $f(z_0) = \infty$. Applying the same arguments as in (i) to the functions $f_m^{-1}(z)$ and $f^{-1}(z)$ in $B(z_0, r_0)$, we obtain:

$$\|f^{-1}(z)\| < A \qquad \text{and} \qquad \|f_m^{-1}(z)\| < B.$$

Therefore, for all $m \geq m_0$ and $z \in B(z_0, r_0)$ $(r > r_0 > 0)$ it follows

$$\begin{aligned} \|f_m^{-1}(z) - f^{-1}(z)\| &= d_{ch}[f_m^{-1}(z), f^{-1}(z)]\sqrt{1+\|f_m^{-1}(z)\|^2}\sqrt{1+\|f^{-1}(z)\|^2} \\ &\leq d_{ch}[f_m^{-1}(z), f^{-1}(z)]\sqrt{1+B^2}\sqrt{1+A^2}. \end{aligned}$$

Since

$$d_{ch}[f_m^{-1}(z), f^{-1}(z)] = d_{ch}[f_m(z), f(z)],$$

it follows that

$$\|f_m^{-1}(z) - f^{-1}(z)\| \leq d_{ch}[f_m(z), f(z)]\sqrt{1+B^2}\sqrt{1+A^2}.$$

Since $\{f_m\}_{m\in\mathbb{N}}$ is locally uniformly convergent with respect to the chordal distance, we conclude that $\{f_m^{-1}\}_{m\in\mathbb{N}}$ converges locally uniformly, i.e.,

$$\lim_{m\to\infty} \|f_m^{-1}(z) - f^{-1}(z)\| = 0, \qquad z \in B(z_0, r_0).$$

$\square$

Theorem 2.3 *Let $\Omega \subset \mathcal{A}_{n+1}$ be a domain, $\{f_m\}_{m\in\mathbb{N}}$ be a sequence of left (right) special meromorphic $\mathbb{A}$-valued function and let $\Sigma \subset \Omega$ be a discrete set containing all poles of all functions f_m of the sequence. If $\{f_m\}_{m\in\mathbb{N}}$ converges locally uniformly in Ω with respect to the chordal distance, then the limit function f is left (right) special meromorphic.*

Proof. Since each function in the sequence $\{f_m\}_{m\in\mathbb{N}}$ is left (right) special meromorphic in Ω then the functions are left (right) monogenic in $\Omega\backslash\Sigma$. Theorem 2.1 implies that f is left (right) monogenic in $\Omega\backslash\Sigma$. It remains to prove that f is continuous for elements of Σ with respect to the chordal distance, i.e., elements of Σ are ∞-points of f. Let $z_0 \in \Sigma$. Since the functions of the sequence are left (right) special meromorphic then by Lemma 2.1 they are continuous with respect to the chordal distance. In view of the continuity of the functions and the fact that the sequence is locally uniformly convergent, it follows that f is continuous with respect to the chordal distance. Hence forth f is left (right) special meromorphic. $\square$

From now on $\mathcal{F}$ denotes a family of Clifford valued functions in a domain Ω, and the subscript indicates the set to which the values of the functions belong to. Hence, we have:

$$\mathcal{F}_{\mathbb{H}} := \{f|\ \ f:\Omega\to\mathbb{H}\}; \quad \mathcal{F}_{\mathcal{A}_{n+1}} := \{f|\ \ f:\Omega\to\mathcal{A}_{n+1}\}; \quad \mathcal{F}_{Cl_n} := \{f|\ \ f:\Omega\to Cl_n\}.$$

When no ambiguity occurs we write $\mathcal{F}_{\mathbb{A}}$ for the family of $\mathbb{A}$-valued functions, where $\mathbb{A}$ is either $\mathbb{H}$ or $\mathcal{A}_{n+1}$.

Let us recall the following definition:

Definition 2.3 *(i) Let $\mathcal{F}_{Cl_n}$ be a family of left (right) monogenic functions in Ω. The family $\mathcal{F}_{Cl_n}$ is called equi-continuous in $E \subset \Omega$ if for all $z_0 \in E$ and for all $\varepsilon > 0$ there exists $\delta > 0$, such that for all $f \in \mathcal{F}_{Cl_n}$*

$$\|f(z) - f(z_0)\| < \varepsilon, \quad \forall z \in B(z_0, \delta) \cap E.$$

(ii) Let $\mathcal{F}_{Cl_n}$ be a family of left (right) special meromorphic functions in Ω. The family $\mathcal{F}_{Cl_n}$ is called equi-continuous in $E \subset \Omega$ with respect to the chordal distance, if for all $z_0 \in E$ and for all $\varepsilon > 0$ there exists $\delta > 0$, such that for all $f \in \mathcal{F}_{Cl_n}$

$$d_{ch}[f(z) - f(z_0)] < \varepsilon, \quad \forall z \in B(z_0, \delta) \cap E.$$

The next proposition establishes a relation between equi-continuity and locally uniform convergence of a sequence. The proof of this proposition is analogous to the one given in [14, pp.15] for the classical case.

Proposition 2.3 *Let $\{f_m\}_{m\in\mathbb{N}}$ be a sequence of left (right) special meromorphic $\mathbb{A}$-valued functions in a domain Ω. If $\{f_m\}_{m\in\mathbb{N}}$ is locally uniformly convergent with respect to the chordal distance, then $\{f_m\}_{m\in\mathbb{N}}$ is an equi-continuous sequence with respect to the chordal distance.*

Next we prove a sufficient condition for equi-continuity:

Proposition 2.4 *Let $\mathcal{F}_{Cl_n}$ be a family of left (right) monogenic Clifford valued functions in a domain $\Omega \subset \mathcal{A}_{n+1}$. If the norm of the Jacobian matrix of f is locally bounded for all $f \in \mathcal{F}_{Cl_n}$, then $\mathcal{F}_{Cl_n}$ is equi-continuous.*

Proof. Let f be an element of $\mathcal{F}_{Cl_n}$ and $z^* \in \Omega$. For $r > 0$ let $B(z^*, r) \subset \Omega$. Consider $\gamma : [0,1] \to B(z^*, r)$ to be defined as $\gamma(t) := tz_1 + (1-t)z_0$, which we identify as an element of $\mathbb{R}^{n+1}$. Since $\|J_f(\gamma(t))\| \leq M$ for all $f \in \mathcal{F}_{Cl_n}$ and for all $t \in [0,1]$, with $\varphi(t) := f(\gamma(t))$

we obtain

$$\begin{aligned}\|f(z_1)-f(z_0)\| = \|\varphi(1)-\varphi(0)\| &= \left\| \int_0^1 J_f(\gamma(t)) \frac{d\gamma(t)}{dt} dt \right\| \\ &\leq \int_0^1 \|J_f(\gamma(t))\| \left\| \frac{d\gamma(t)}{dt} \right\| dt \\ &\leq M\|z_1 - z_0\|.\end{aligned}$$

Therefore, since M is independent of f, $\mathcal{F}_{Cl_n}$ is equi-continuous. □

Another property of families of functions is normality. Next we introduce the concept of normal families and give some necessary and/or sufficient conditions.

From [3] we recall the following definition.

Definition 2.4 *A family of functions $\mathcal{G}$ is called normal, if every sequence of functions from $\mathcal{G}$ contains a locally uniformly convergent subsequence.*

The next result which is analogous to the classical case, gives a necessary and sufficient condition for normality.

Theorem 2.4 *Let $\mathcal{F}_{\mathbb{A}}$ be a family of left (right) special meromorphic functions defined in a domain Ω. $\mathcal{F}_{\mathbb{A}}$ is normal if and only if the family $\mathcal{F}_{\mathbb{A}}$ is equi-continuous in Ω with respect to the chordal distance.*
Analogously this is obtained for families of left (right) monogenic Cl_n-valued functions.

Proof. Let us start by proving that equi-continuity is a sufficient condition for normality. Consider a sequence $\{z_m\}_{m\in\mathbb{N}}$ of points dense in Ω. Let $\mathcal{S} := \{f_m\}_{m\in\mathbb{N}}$ be a sequence of $\mathcal{F}_{\mathbb{A}}$. For $z_1 \in \Omega$ consider the sequence of points

$$\mathcal{S}_1 := \{f_m(z_1)\}_{m\in\mathbb{N}},$$

in $\overline{\mathbb{A}}$. Since $\overline{\mathbb{A}}$ is compact, there exists a subsequence $\{f_{m_j^1}(z_1)\}_{j\in\mathbb{N}}$ of $\mathcal{S}_1$ and a point $w_1 \in \overline{\mathbb{A}}$ such that

$$\lim_{j\to\infty} d_{ch}[f_{m_j^1}(z_1), w_1] = 0.$$

In the next step, consider the sequence of points

$$\mathcal{S}_2 := \{f_{m_j^1}(z_2)\}_{j\in\mathbb{N}}$$

for which it is possible to find a subsequence $\{f_{m_j^2}(z_2)\}_{j\in\mathbb{N}}$ of $\mathcal{S}_2$ and a point $w_2 \in \overline{\mathbb{A}}$ where

$$\lim_{j\to\infty} d_{ch}[f_{m_j^2}(z_2), w_2] = 0 \quad \text{and} \quad \lim_{j\to\infty} d_{ch}[f_{m_j^2}(z_1), w_1] = 0.$$

By repeating the above procedure, we obtain sequences $\{f_{m_j^i}(z_s)\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} d_{ch}[f_{m_j^i}(z_s), w_s] = 0, \quad s = 1, 2, \ldots, i, \ldots \quad i = 1, 2, \ldots. \tag{2.14}$$

where $w_s \in \overline{\mathbb{A}}$.

Consider the diagonal sequence

$$\left\{g_k(z_j) := f_{m_k^j}(z_j)\right\}_{k\in\mathbb{N}}$$

such that $\lim_{k\to\infty} d_{ch}[g_k(z_j), w_j] = 0$ for $j = 1, 2, \cdots$. Let us prove that the sequence $\{g_k(z)\}_{k\in\mathbb{N}}$ is locally uniformly convergent with respect to the chordal distance.

Since all $f \in \mathcal{F}_{\mathbb{A}}$ are special meromorphic then by Lemma 2.1, we observe that f is continuous with respect to the chordal distance, i.e., let $x \in \Omega$ then for all $\epsilon > 0$ there exist $\delta > 0$ such that for $y \in B(x, \delta) \subset \Omega$

$$d_{ch}[f(y), f(x)] < \frac{\epsilon}{5}.$$

Next, since the points $\{z_m\}_{m\in\mathbb{N}}$ are dense in Ω, there exists $z_j \in B(x, \delta)$. In view of pointwise convergence of the sequence $\{g_k(z_j) := f_{m_k^j}(z_j)\}_{k\in\mathbb{N}}$, there exists a positive integer k_x, such that for $k, k_1 > k_x$

$$d_{ch}[g_k(z_j), g_{k_1}(z_j)] < \frac{\epsilon}{5}.$$

Therefore, for $z \in B(x, \delta)$ and $k, k_1 > k_x$, the following is obtained

$$\begin{aligned} d_{ch}[g_k(z), g_{k_1}(z)] &\leq d_{ch}[g_k(z), g_k(x)] + d_{ch}[g_k(x), g_k(z_j)] \\ &+ d_{ch}[g_k(z_j), g_{k_1}(z_j)] + d_{ch}[g_{k_1}(z_j), g_{k_1}(x)] \\ &+ d_{ch}[g_{k_1}(x), g_{k_1}(z)] \\ &< \frac{5\epsilon}{5} = \epsilon. \end{aligned} \tag{2.15}$$

This proves uniformly convergence of $\{g_k\}_{k\in\mathbb{N}}$ in $B(x,\delta)$. Since x was chosen arbitrarily, we conclude that $\{g_k\}_{k\in\mathbb{N}}$ is locally uniformly convergent in Ω. Hence, the arbitrary chosen sequence $\{f_m\}_{m\in\mathbb{N}}$ has a locally uniformly convergent subsequence. Thus, the family $\mathcal{F}_{\mathbb{A}}$ is normal.

Next we prove that equi-continuity is a necessary condition for normality. Suppose that $\mathcal{F}_{\mathbb{A}}$ is not equi-continuous. Then it is possible to find a sequence $\mathcal{S} := \{f_m\}_{m\in\mathbb{N}}$ in $\mathcal{F}_{\mathbb{A}}$ and $z_0 \in \Omega$ such that for $\epsilon_0 > 0$ and a sequence δ_m of positive integers with $\lim\limits_{m\to\infty} \delta_m = 0$, holds:

$$\sup_{z\in \overline{B(z_0,\delta_m)}} d_{ch}[f_m(z), f_m(z_0)] \geq \epsilon_0. \tag{2.16}$$

Since $\mathcal{F}_{\mathbb{A}}$ is normal in Ω, there is a subsequence $\mathcal{S}_1 := \{f_{m_k}\}_{k\in\mathbb{N}}$ of $\mathcal{S}$ which converges locally uniformly with respect to the chordal distance.

The aim, is to prove that $\mathcal{S}_1$ is equi-continuous and consequently to obtain a contradiction.

Since $\{f_{m_k}\}_{k\in\mathbb{N}}$ is locally uniformly convergent with respect to the chordal distance, then for all $\epsilon > 0$ there exists $r_0 > 0$ and $k_0 > 0$ such that for $k, k_1 > k_0$

$$d_{ch}[f_{m_k}(z), f_{m_{k_1}}(z)] < \frac{\epsilon}{3}, \qquad z \in B(z_0, r_0) \cap \Omega.$$

Moreover, one has in view of continuity of $f_{m_{k_0}}$, for $r_1 > 0$ with $r_1 < r_0$

$$d_{ch}[f_{m_{k_0}}(z), f_{m_{k_0}}(z_0)] < \frac{\epsilon}{3}$$

for $z \in B(z_0, r_1) \cap \Omega$. Therefore, for $k \geq k_0$ and $z \in B(z_0, r_1) \cap \Omega$ we infer

$$\begin{aligned} d_{ch}[f_{m_k}(z), f_{m_k}(z_0)] &\leq d_{ch}[f_{m_k}(z), f_{m_{k_0}}(z)] + d_{ch}[f_{m_{k_0}}(z), f_{m_{k_0}}(z_0)] \\ &+ d_{ch}[f_{m_{k_0}}(z_0), f_{m_k}(z_0)] \\ &< \epsilon. \end{aligned}$$

For the finite number of continuous functions f_{m_k}, $k = 1, 2, ..., k_0$ there exists $r_2 > 0$ $(r_2 < r_1)$ such that

$$d_{ch}[f_{m_k}(z), f_{m_k}(z_0)] < \epsilon, \qquad z \in B(z_0, r_2) \cap \Omega.$$

Therefore if we choose $\epsilon < \epsilon_0$, we obtain for all $k \geq 1$

$$d_{ch}[f_{m_k}(z), f_{m_k}(z_0)] < \epsilon, \qquad z \in B(z_0, r_2) \cap \Omega, \tag{2.17}$$

which is a contradiction to condition (2.16). □

In the following sections, we establish some results which provide us with criteria of normality for families of left (right) monogenic and left (right) special meromorphic functions.

2.3 Normal families of monogenic functions

Throughout this section, we consider families of left (right) monogenic Clifford valued functions defined in a domain Ω of $\mathcal{A}_{n+1}$, i.e., $\mathcal{F}_{Cl_n}$.

To obtain a Montel-type criterion of normality for these families, the following definition is needed.

Definition 2.5 *$\mathcal{F}_{Cl_n}$ is called locally bounded in Ω if for each point in Ω there exists a neighborhood $\mathcal{V}$ in Ω and a positive constant C, such that for all $f \in \mathcal{F}_{Cl_n}$ the following inequality $\|f(z)\| \leq C$ holds in $\mathcal{V}$.*

Proposition 2.5 *Let $\mathcal{F}_{Cl_n}$ be a family of left (right) monogenic functions in Ω. If $\mathcal{F}_{Cl_n}$ is locally bounded, then $\mathcal{F}_{Cl_n}$ is normal.*

Proof. Let f be a left monogenic function of $\mathcal{F}_{Cl_n}$ and $r > 0$ such that $B(z, r) \subset \Omega$. Using Cauchy's integral formula (Theorem 1.5), we obtain:

$$f(z) = \frac{1}{w_{n+1}} \int\limits_{\|y-z\|=r} q_{\mathbf{0}}(y - z) d\sigma_y f(y). \tag{2.18}$$

Furthermore, the partial derivatives of f are given by:

$$\frac{\partial f}{\partial x_i}(z) = \frac{1}{w_{n+1}} \int\limits_{\|y-z\|=r} \frac{\partial q_{\mathbf{0}}}{\partial x_i}(y - z) d\sigma_y f(y), \qquad i = 0, 1, ..., n. \tag{2.19}$$

Applying the upper bound estimate on the generalized negative powers given in (1.12) it follows

$$\left\| \frac{\partial q_{\mathbf{0}}}{\partial x_i}(y-z) \right\| \le \frac{n}{\|y-z\|^{n-1}} \le \frac{n}{r^{n-1}}.$$

Hence, altogether and for $M := \max\limits_{\|y-z\|=r} \|f(y)\|$, yields:

$$\left\| \frac{\partial f}{\partial x_i}(z) \right\| \le M \frac{n}{r^{n-1}}, \qquad i = 0, 1, \ldots, n. \tag{2.20}$$

Next we consider the Jacobian matrix of the function f defined in (1.21). Using the inequality (2.20), we obtain:

$$\|J_f(z)\| \le \frac{(n+1)nM}{r^{n-1}}.$$

Applying Proposition 2.4 it follows that $\mathcal{F}_{Cl_n}$ is equi-continuous. Consequently, in view of (2.6), the family $\mathcal{F}_{Cl_n}$ is equi-continuous with respect to the chordal distance. Finally, as a consequence of Theorem 2.4, we conclude that the family $\mathcal{F}_{Cl_n}$ is normal. □

In this proposition the monogenicity property is involved when we use Cauchy's integral formula.

A direct conclusion is stated in the following corollary.

Corollary 2.1 *Let $\mathcal{F}_{Cl_n}$ be a family of left (right) monogenic functions in a domain Ω. If the family*

$$\mathcal{F}'_{Cl_n} := \left\{ \frac{\partial f}{\partial x_i} | \quad f \in \mathcal{F}_{Cl_n}, \quad i = 0, 1, ..., n \right\}$$

is locally bounded, then $\mathcal{F}_{Cl_n}$ is normal in Ω.

Proof. Let $z_0 \in \Omega$ and $\mathcal{V}_{z_0} \subset \Omega$ be a neighborhood of z_0. For $f \in \mathcal{F}_{Cl_n}$ and $i \in \{0, 1, \ldots, n\}$, there exists a positive constant c such that

$$\left\| \frac{\partial f}{\partial x_i}(z) \right\| \le c, \qquad z \in \mathcal{V}_{z_0}.$$

Moreover, for $f \in \mathcal{F}_{Cl_n}$ we obtain

$$\|J_f(z)\| \le nc, \qquad z \in \mathcal{V}_{z_0}.$$

Since f is an arbitrary element of $\mathcal{F}_{Cl_n}$ and z_0 is an arbitrary point of Ω, with Proposition 2.4, it follows that $\mathcal{F}_{Cl_n}$ is equi-continuous. According to the definition of the chordal distance, $\mathcal{F}_{Cl_n}$ is equi-continuous with respect to the chordal distance. Finally, applying Theorem 2.4, we conclude that the family $\mathcal{F}_{Cl_n}$ is normal. □

Next we give some examples of normal and non-normal families.

Example 2.1 *The family*

$$\mathcal{P}_{Cl_n} = \left\{ f_k | \; f_k(z) = \sum_{|\mathbf{m}|=k} \mathcal{P}_{\mathbf{m}}(z) a_{\mathbf{m}}, \quad k \in \mathbb{N}, \quad \|z\| < 1 \right\}$$

is normal. It is known that $\mathcal{P}_{Cl_n}$ is a family of left monogenic functions and for each $f_k \in \mathcal{P}_{Cl_n}$ we have

$$\|f_k(z)\| \leq \sum_{|\mathbf{m}|=k} \|\mathcal{P}_{\mathbf{m}}(z)\| \, \|a_{\mathbf{m}}\| \leq \|z\|^k \sum_{|\mathbf{m}|=k} \frac{\|a_{\mathbf{m}}\|}{\mathbf{m}!} < \sum_{|\mathbf{m}|=k} \|a_{\mathbf{m}}\| =: a.$$

Since $\mathcal{P}_{Cl_n}$ is locally bounded we obtain, applying Proposition 2.5, that $\mathcal{P}_{Cl_n}$ is normal. One can also observe that $\mathcal{P}_{Cl_n}$ is normal in any compact subset of $\mathcal{A}_{n+1}$.

Example 2.2 *Let $\mathbf{m} \in \mathbb{N}_0^n \backslash \{\mathbf{0}\}$ be a fixed multi-index. The family*

$$\mathcal{G}_{Cl_n} = \{ f_k | \; f_k(z) = k\mathcal{P}_{\mathbf{m}}(z); \quad \|z\| < 1 \}$$

is not a normal family. For $z = 0$ we obtain $\lim\limits_{k\to\infty} f_k(0) = 0$. Taking $z \neq 0$, the limit is $\lim\limits_{k\to\infty} f_k(z) = \infty$. Since the sequence and every subsequence are not locally uniformly convergent, we have that the family is not normal.

2.4 Normal families of special meromorphic functions

In this section we restrict to the class of left (right) special meromorphic functions with values in $\mathcal{A}_{n+1}$. A normality criterion of families of these functions is based on the classical Marty's criterion [48].

Theorem 2.5 *(Marty's criterion) A family $\mathcal{G}$ of holomorphic or meromorphic complex valued functions in a domain $D \subset \mathbb{C}$ is normal if and only if*

$$\frac{|f'(z)|}{1+|f(z)|^2}$$

is locally bounded in D.

In order to prove a generalization of Marty's criterion in the framework of hypercomplex function theory it is necessary to start by defining the set $\Sigma_{\mathcal{F}_{\mathcal{A}_{n+1}}}$ as the set of all isolated poles of the family of left (right) special meromorphic $\mathcal{A}_{n+1}$-valued functions in a domain $\Omega \subset \mathcal{A}_{n+1}$, $\mathcal{F}_{\mathcal{A}_{n+1}}$. We define

$$\mathcal{F}^*_{\mathcal{A}_{n+1}} := \left\{ f \in \mathcal{F}_{\mathcal{A}_{n+1}} \ : \ \Sigma_{\mathcal{F}_{\mathcal{A}_{n+1}}} \text{is discrete} \right\}. \tag{2.21}$$

For $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ we obtain the following result:

Theorem 2.6 *The family $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ defined in (2.21) is normal if and only if*

$$\frac{\|J_f(z)\|}{1+\|f(z)\|^2} \tag{2.22}$$

is locally bounded in Ω.

Proof. Let us start by proving that local boundedness of expression (2.22) is a sufficient condition for normality. First, we prove that the family $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ is equi-continuous. In order to do so it is necessary to prove a local estimate either for f or for f^{-1}.

Since expression (2.22) is locally bounded, for $z_0 \in \Omega$, there exists $\delta > 0$ such that $B(z_0, \delta) \subset \Omega$, and there exists a positive constant K such that for every $f \in \mathcal{F}^*_{\mathcal{A}_{n+1}}$

$$\frac{\|J_f(z)\|}{1+\|f(z)\|^2} \le K, \qquad \forall z \in B(z_0, \delta). \tag{2.23}$$

We define the curve $\gamma : [0,\delta] \to \mathcal{A}_{n+1}(\cong \mathbb{R}^{n+1})$, by $\gamma(t) = z_0 + tu$, where u is an element of $\mathcal{A}_{n+1}$ with $\|u\| = 1$.

On one hand we obtain, for $0 < r \leq \delta$:

$$\int_0^r \frac{\|J_f(\gamma(t))\|}{1+\|f\left(\gamma(t)\right)\|^2}dt \leq Kr \leq K\delta. \tag{2.24}$$

On the other hand, interpreting $\gamma(t)$ as a vector in $\mathbb{R}^{n+1}$ (which will be done during this proof) we have

$$\|J_f(\gamma(t))\|\|\gamma'(t)\| \geq \|J_f(\gamma(t))\gamma'(t)\|.$$

Therefore, since $\|\gamma'(t)\| = 1$ for $t \in [0,\delta]$, we obtain

$$\begin{aligned}
\int_0^r \frac{\|J_f(\gamma(t))\|}{1+\|f\left(\gamma(t)\right)\|^2}dt &= \int_0^r \frac{\|J_f(\gamma(t))\|\|\gamma'(t)\|}{1+\|f(\gamma(t))\|^2}dt \\
&\geq \int_0^r \frac{\|J_f(\gamma(t))\gamma'(t)\|}{1+\|f(\gamma(t))\|^2}dt \\
&= \int_\varphi \frac{\|df(z)\|}{1+\|f(z)\|^2} \qquad (2.25)
\end{aligned}$$

where $\varphi = f\circ\gamma$ with $\varphi(0) = f(z_0)$ and $\varphi(r) = f(\gamma(r)) = f(z)$. Hence, φ is an $\mathcal{A}_{n+1}$-valued function, which can be represented by the following spherical coordinates

$$\begin{aligned}
\varphi_0 &= R\cos\theta_1 \\
\varphi_1 &= R\sin\theta_1\cos\theta_2 \\
\varphi_2 &= R\sin\theta_1\sin\theta_2\cos\theta_3 \\
\varphi_3 &= R\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 \\
\vdots\ &\quad \vdots \\
\varphi_n &= R\sin\theta_1\sin\theta_2\sin\theta_3 ... \sin\theta_{n-1}\sin\theta_n.
\end{aligned}$$

where $0 < R < +\infty$, $0 < \theta_1, \theta_2, \cdots, \theta_{n-1} \leq \pi$ and $0 < \theta_n \leq 2\pi$. Rewriting φ in spherical coordinates, yields:

$$\varphi(t) = R(t)U(t), \tag{2.26}$$

where $R(t) := \|f(\gamma(t))\| = \|\varphi(t)\| \in \mathbb{R}^+$ and $U(t) := \sum_{i=0}^{n} e_i \frac{\varphi_i(t)}{\|\varphi(t)\|}$ satisfies $\|U(t)\| = 1$ (see [13, pp.48]). Therefore, if we substitute the expression (2.26) into (2.25), we get

$$\int_{\varphi} \frac{\|df(z)\|}{1+\|f(z)\|^2} = \int_0^r \frac{\|\varphi'(t)\|}{1+\|\varphi(t)\|^2} dt.$$

Since $\|df\|^2 = (dR)^2 + R^2(\text{angular differentials}) \geq (dR)^2$ we obtain:

$$\begin{aligned} \int_{\varphi} \frac{\|df(z)\|}{1+\|f(z)\|^2} &\geq \int_0^r \frac{|dR(t)|}{1+\|f(\gamma(t))\|^2} \\ &= \int_0^r \frac{|R'(t)|}{1+R^2(t)} dt \\ &\geq \left| \int_0^r \frac{R'(t)}{1+R^2(t)} \right| \\ &= \left| \arctan \|f(z)\| - \arctan \|f(z_0)\| \right|. \end{aligned}$$

Finally, after applying inequality (2.24), we obtain:

$$\left| \arctan \|f(z)\| - \arctan \|f(z_0)\| \right| \leq K\delta, \qquad z \in B(z_0, \delta). \tag{2.27}$$

Taking $\delta \leq \frac{\pi}{12K}$ the following cases occur:

(i) If $\|f(z_0)\| \leq 1$, then

$$\arctan \|f(z)\| \leq K\delta + \arctan \|f(z_0)\| \leq K\frac{\pi}{12K} + \arctan(1) = \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3}.$$

Therefore $\|f(z)\| \leq \sqrt{3}$ for all $z \in B(z_0, \delta)$. Moreover, in view of (2.23) we obtain:

$$\|J_f(z)\| \leq \left(1 + \|f(z)\|^2\right) K \leq \left(1 + (\sqrt{3})^2\right) K = 4K. \tag{2.28}$$

(ii) If $\|f(z_0)\| \geq 1$, then

$$\arctan \|f(z)\| \geq \arctan \|f(z_0)\| - K\delta \geq \frac{\pi}{4} - K\frac{\pi}{12K} = \frac{\pi}{4} - \frac{\pi}{12} = \frac{\pi}{6}.$$

Hence, we obtain $\|f(z)\| \geq \frac{\sqrt{3}}{3}$ for all $z \in B(z_0, \delta)$.

Therefore, using the inequality $\|J_{f^{-1}}(z)\| \leq \frac{n+3}{\|f(z)\|^2}\|J_f(z)\|$ from Proposition 1.2, condition (2.23) and the inequality $\|f(z)\| \geq \sqrt{3}$ we have:

$$\begin{aligned}
\|J_{f^{-1}}(z)\| &\leq \frac{1+\|f^{-1}(z)\|^2}{1+\|f^{-1}(z)\|^2}\frac{(n+3)}{\|f(z)\|^2}\|J_f(z)\| \\
&= \left(1+\|f^{-1}(z)\|^2\right)(n+3)\frac{\|J_f(z)\|}{1+\|f(z)\|^2} \\
&\leq 4(n+3)K. \qquad (2.29)
\end{aligned}$$

To proceed in the proof of equi-continuity of f or f^{-1}, we consider $z_1, z_2 \in B(z_0, \delta)$ for $B(z_0,\delta) \subset \mathcal{A}_{n+1}$ and the curve $\gamma : [0,1] \to \mathcal{A}_{n+1}$ defined as $\gamma(t) := tz_2 + (1-t)z_1$. Denote $\varphi(t) = f(\gamma(t))$ where $\varphi'(t) = J_f(\gamma(t))\gamma'(t)$.

If $\|f(z_0)\| \leq 1$ together with inequality (2.28) we obtain:

$$\begin{aligned}
\|f(z_2) - f(z_1)\| = \|\varphi(1) - \varphi(0)\| &= \left\|\int_0^1 J_f(\gamma(t))\gamma'(t)dt\right\| \\
&\leq \int_0^1 \|J_f(\gamma(t))\|\,\|\gamma'(t)\|\,dt \\
&\leq 4K\|z_2 - z_1\|, \qquad (2.30)
\end{aligned}$$

for all $z_1, z_2 \in B(z_0,\delta)$ and all functions $f \in \mathcal{F}^*_{\mathcal{A}_{n+1}}$, since K does not depend on the function.

If $\|f(z_0)\| \geq 1$, then we obtain analogously with $\varphi_1(t) = f^{-1}(\gamma(t))$ where γ is the rectifiable curve defined by $\gamma(t) = tz_2 + (1-t)z_1$ for $t \in [0,1]$ and $\varphi_1'(t) = J_{f^{-1}}(\gamma(t))\gamma'(t)$, together with inequality (2.29) the following

$$\begin{aligned}
\left\|f^{-1}(z_2) - f^{-1}(z_1)\right\| = \|\varphi_1(1) - \varphi_1(0)\| &= \left\|\int_0^1 J_{f^{-1}}(\gamma(t))\gamma'(t)dt\right\| \\
&\leq \int_0^1 \|J_{f^{-1}}(\gamma(t))\|\|\gamma'(t)\|dt \\
&\leq 4(n+3)K\|z_2 - z_1\|, \qquad (2.31)
\end{aligned}$$

for $z_1, z_2 \in B(z_0, \delta)$ and all functions $f \in \mathcal{F}^*_{\mathcal{A}_{n+1}}$.

Let $z_0 \in \Omega$, $z_1, z_2 \in B(z_0, \delta)$. If $\|z_2 - z_1\| \leq \frac{\varepsilon}{12(n+3)K}$ for $\varepsilon > 0$, then following Proposition 2.1 two cases occur.

For the case $\|f(z_0)\| \leq 1$, we have in view of Definition 2.1 and (2.30)

$$\begin{aligned} d_{ch}\left[f(z_2), f(z_1)\right] &:= \frac{\|f(z_2) - f(z_1)\|}{(1 + \|f(z_2)\|^2)^{\frac{1}{2}}\,(1 + \|f(z_1)\|^2)^{\frac{1}{2}}} \\ &\leq \|f(z_2) - f(z_1)\| \\ &< 4K\|z_2 - z_1\| \\ &\leq 4K\frac{\varepsilon}{12(n+3)K} \qquad \leq \varepsilon. \end{aligned}$$

In the case $\|f(z_0)\| \geq 1$, we obtain in view of the definition of the chordal distance and (2.31)

$$\begin{aligned} d_{ch}\left[f^{-1}(z_2), f^{-1}(z_1)\right] &:= \frac{\|f^{-1}(z_2) - f^{-1}(z_1)\|}{(1 + \|f^{-1}(z_2)\|^2)^{\frac{1}{2}}\,(1 + \|f^{-1}(z_1)\|^2)^{\frac{1}{2}}} \\ &\leq \|f^{-1}(z_2) - f^{-1}(z_1)\| \\ &< 4(n+3)K\|z_2 - z_1\| \\ &\leq 4(n+3)K\frac{\varepsilon}{12(n+3)K} \qquad \leq \varepsilon. \end{aligned}$$

Since this holds for all $z_0 \in \Omega$ and for all $f \in \mathcal{F}^*_{\mathcal{A}_{n+1}}$ the family is equi-continuous. In view of Theorem 2.4 we conclude that $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ is normal.

Let us prove the reciprocal statement. Suppose that

$$\Theta(f)(z) := \frac{\|J_f(z)\|}{1 + \|f(z)\|^2}$$

is not locally bounded. Then there exists a sequence of functions $\{f_m\}_{m\in\mathbb{N}}$ in $\mathcal{F}^*_{\mathcal{A}_{n+1}}$, a point z_0 and a sequence of points $\{z_m\}_{m\in\mathbb{N}}$ in Ω that converges to z_0 such that

$$\Theta(f_m)(z_m) = \frac{\|J_{f_m}(z_m)\|}{1 + \|f_m(z_m)\|^2} \to +\infty, \qquad m \to \infty. \tag{2.32}$$

Since $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ is normal, by assumption, for any sequence $\{f_m\}_{m\in\mathbb{N}}$ there exists a subsequence $\{f_{m_k}\}_{k\in\mathbb{N}}$ that converges locally uniformly with respect to the chordal distance.

Since $\{f_{m_k}\}_{k\in\mathbb{N}}$ is a sequence of special meromorphic functions with at most isolated poles in a discrete set $\Sigma_{\mathcal{F}_{\mathcal{A}_{n+1}}}$, by Theorem 2.3 the limit function is special meromorphic. Applying Theorem 2.2 we infer that

$$\frac{\|J_{f_m}(z)\|}{1+\|f_m(z)\|^2} \to \frac{\|J_f(z)\|}{1+\|f(z)\|^2}, \qquad m\to\infty,$$

locally uniformly in $\Omega\setminus\Sigma_{\mathcal{F}_{\mathcal{A}_{n+1}}}$. Let $z^*\in\Sigma_{\mathcal{F}_{\mathcal{A}_{n+1}}}$ such that $f_m(z^*)=\infty$ for a infinitely many $m\in\mathbb{N}$. (Otherwise f is monogenic in z^*. Hence $\Theta(f)$ is bounded in a neighborhood of z^*.) Then there exists a subsequence $\{f_{m_k}\}_{k\in\mathbb{N}}$ of $\{f_m\}_{m\in\mathbb{N}}$ such that in view of Proposition 1.3

$$\Theta(f_{m_k})(z^*) = \frac{\|J_{f_{m_k}}(z^*)\|}{1+\|f_{m_k}(z^*)\|^2} = 0,$$

for $k=1,2,\ldots$. Since f is special meromorphic it must have a pole at z^*. Therefore, we also have $\frac{\|J_f(z^*)\|}{1+\|f(z^*)\|^2}=0$. This reveals that

$$\lim_{k\to\infty}\frac{\|J_{f_{m_k}}(z)\|}{1+\|f_{m_k}(z)\|^2} = \frac{\|J_f(z)\|}{1+\|f(z)\|^2} < C,$$

where C is a positive real constant depends on z_0 and on δ for $B(z_0,\delta)\subset\Omega$. Since $\lim\limits_{k\to\infty} z_{m_k}=z_0$ and $\|z_{m_k}-z_0\|<\delta$ we obtain

$$\frac{\|J_{f_{m_k}}(z_{m_k})\|}{1+\|f_{m_k}(z_{m_k})\|^2}\le C,$$

for k sufficiently large. This contradicts

$$\lim_{m\to\infty}\frac{\|J_{f_m}(z_m)\|}{1+\|f_m(z_m)\|^2} = \lim_{k\to\infty}\frac{\|J_{f_{m_k}}(z_{m_k})\|}{1+\|f_{m_k}(z_{m_k})\|^2} = \infty,$$

for $\lim\limits_{m\to\infty} z_m = \lim\limits_{k\to\infty} z_{m_k} = z_0$. $\square$

Notice that this result is also true for families of real-analytic paravector valued functions.

In the next section we present a generalization of a famous result due to Zalcman.

2.5 Zalcman's Lemma

In 1975, Zalcman gave a necessary and sufficient condition for normality in [72], which is now known as Zalcman's lemma. Later, in 1998, a new version of this lemma was presented in [73]. The new version uses a parameter depending on the multiplicity of all zeros and all poles of the functions.

The following result is a generalization of Zalcman's lemma for a family $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ defined in (2.21).

Theorem 2.7 *Let $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ be a family of special meromorphic functions in the unit ball $B(0,1)$ with the same conditions as in (2.21). $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ is not normal in $B(0,1)$ if and only if there exists a*

(i) number $0 < r < 1$;

(ii) sequence of points $\{z_m\}_{m\in\mathbb{N}}$;

(iii) sequence of functions $\{f_m\}_{m\in\mathbb{N}}$ in $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ and

(iv) sequence of positive numbers ρ_m with $\lim\limits_{m\to\infty} \rho_m = 0$,

such that the sequence of functions defined by

$$g_m(\xi) := f_m\left(z_m + \rho_m \xi\right) \tag{2.33}$$

converges locally uniformly with respect to the chordal distance in $\mathcal{A}_{n+1}$ to a non-constant special meromorphic function g.

Proof. We start by proving that conditions (i) to (iv) are sufficient for non normality. If $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ is not normal in $B(0,1)$ then, in view of Theorem 2.6, there exists a number $0 < r_0 < 1$, a sequence $\{f_m\}_{m\in\mathbb{N}}$ in $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ and a sequence of points z^*_m in $B(0,r_0)$ tending to z_0, such that

$$\Theta(f_m)(z^*_m) := \frac{\|J_{f_m}(z^*_m)\|}{1 + \|f_m(z^*_m)\|^2} \to \infty, \quad m \to \infty. \tag{2.34}$$

Without loss of generality, let us assume that $z_0 = 0$. Take $r > 0$ fixed, such that $0 < r_0 < r < 1$ and $\|z\| < r$. In view of continuity of $\Theta(f_m)$ we define:

$$M_m := \max_{\|z\|\leq r} \left(1 - \frac{\|z\|^2}{r^2}\right) \Theta(f_m)(z) = \left(1 - \frac{\|z_m\|^2}{r^2}\right) \Theta(f_m)(z_m). \tag{2.35}$$

Since $\|z_m\| < r$, the expression $\left(1 - \frac{\|z_m\|^2}{r^2}\right)$ remains bounded. Moreover, using condition (2.34) for $\|z_m\| < r_0$ we obtain $\lim\limits_{m\to\infty} M_m = \infty$. Furthermore, let us define

$$\rho_m := \frac{1}{M_m}\left(1 - \frac{\|z_m\|^2}{r^2}\right). \tag{2.36}$$

We have

$$\lim_{m\to\infty} \rho_m = 0. \tag{2.37}$$

Moreover, we obtain

$$\frac{\rho_m}{r - \|z_m\|} = \frac{1}{M_m}\frac{(r + \|z_m\|)}{r^2} \leq \frac{2}{rM_m} \to 0, \qquad m \to \infty. \tag{2.38}$$

Consider the functions g_m defined by

$$g_m(\xi) := f_m(z_m + \rho_m \xi),$$

where $\xi \in B(0, R_m)$ for $R_m := \frac{r - \|z_m\|}{\rho_m}$. By (2.38) follows that $\lim\limits_{m\to\infty} R_m = \infty$. Evaluating, the Jacobian matrix of g_m we obtain:

$$J_{g_m}(\xi) = \left[\frac{\partial}{\partial \xi_i} g_m^j(\xi)\right]_{j,i} = \left[\rho_m \frac{\partial}{\partial u_m^i} f_m^j(u_m)\right]_{j,i} = \rho_m J_{f_m}(u_m) \tag{2.39}$$

where $0 \leq i, j \leq n$, $\xi := \sum\limits_{i=0}^{n} \xi_i e_i$ and $u_m := \sum\limits_{i=0}^{n} u_m^i e_i = z_m + \rho_m \xi$. Calculating $\Theta(g_m)(0)$ yields

$$\Theta(g_m)(0) = \left.\frac{\|J_{g_m}(\xi)\|}{1 + \|g_m(\xi)\|^2}\right|_{\xi=0} = \rho_m \left.\frac{\|J_{f_m}(z_m + \rho_m \xi)\|}{1 + \|f_m(z_m + \rho_m \xi)\|^2}\right|_{\xi=0},$$

which, in view of (2.36), for each m gives

$$\Theta(g_m)(0) = \rho_m \frac{\|J_{f_m}(z_m)\|}{1 + \|f_m(z_m)\|^2} = 1. \tag{2.40}$$

Taking a fixed R, such that $\|\xi\| < R < R_m$ and $\|z_m + \rho_m \xi\| < r$, it follows:

$$\begin{aligned} \Theta(g_m)(\xi) &= \rho_m \Theta(f_m)(z_m + \rho_m \xi) \\ &\leq \rho_m \frac{M_m}{1 - \frac{\|z_m + \rho_m \xi\|^2}{r^2}}. \end{aligned}$$

Using the definition of ρ_m (see (2.36)) and the fact that $\dfrac{r+\|z_m\|}{r+(\|z_m\|+\rho_m\|\xi\|)} \leq 1$, we obtain for $\|\xi\| < R$

$$\begin{aligned}\Theta(g_m)(\xi) &\leq \frac{(r^2-\|z_m\|^2)}{r^2-\|z_m+\rho_m\xi\|^2}\\ &= \frac{(r+\|z_m\|)}{(r+[\|z_m\|+\rho_m\|\xi\|])}\frac{(r-\|z_m\|)}{(r-[\|z_m\|+\rho_m\|\xi\|])}\\ &\leq \frac{r-\|z_m\|}{r-(\|z_m\|+\rho_m\|\xi\|)}\\ &< \frac{r-\|z_m\|}{r-(\|z_m\|+\rho_m R)}. \end{aligned} \tag{2.41}$$

Furthermore, in view of (2.37) we have $\lim_{m\to\infty} \dfrac{r-\|z_m\|}{r-(\|z_m\|+\rho_m R)} = 1$. Then, in view of (2.41) and for $\|\xi\| < R$ we conclude that $\Theta(g_m)(\xi)$ is bounded.

Applying Theorem 2.6 it follows that the sequence $\{g_m\}_{m\in\mathbb{N}}$ is normal in $\|\xi\| < R$. As a consequence of the normality, there exists a subsequence $\{g_{m_k}\}_{k\in\mathbb{N}}$ that converges locally uniformly with respect to the chordal distance to a function g. By Theorem 2.3 we conclude that g is special meromorphic. Thus, for $\xi = 0$

$$\Theta(g)(0) = \lim_{k\to\infty} \Theta(g_{m_k})(0) = 1. \tag{2.42}$$

If the limit function of $\{g_{m_k}\}_{k\in\mathbb{N}}$ was constant then for all $\xi \in \mathcal{A}_{n+1}$ we would have $\Theta(g)(\xi) = 0$, in particular for $\xi = 0$, but from (2.42) we get a contradiction. Therefore, we conclude that g is not constant.

Let us prove the reciprocal. Suppose that (i) to (iv) holds as well as

$$f_m(z_m + \rho_m\xi) \to g(\xi), \qquad m \to \infty. \tag{2.43}$$

If $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ is normal, then we infer by Theorem 2.6 that, there exists positive constants M, r such that for $z \in \{z : \|z\| \leq \frac{1+r}{2}\} \subset B(0,1)$ and for all $f_m \in \mathcal{F}^*_{\mathcal{A}_{n+1}}$ holds

$$\max_{\|z\|\leq\frac{1+r}{2}} \Theta(f_m)(z) \leq M.$$

Using (2.39) for a fixed $\xi \in \mathcal{A}_{n+1}$ we have

$$\Theta(g)(\xi) = \lim_{m\to\infty} \rho_m \Theta(f_m)(z_m + \rho_m\xi) = 0,$$

where $\|z_m + \rho_m \xi\| \leq \frac{1+r}{2}$. Since ξ is arbitrarily chosen, then g must be a constant. This gives a contradiction. □

In the next result we obtain a sufficient condition for normality. In order to do so it is necessary to start with the following definition:

Let f be a $\mathbb{A}$-valued function. We define

$$\mathcal{E}_f(a) := \{z \in \mathbb{A} : f(z) = a\}$$

as the set of elements where f has the value a.

We proceed with the proof of the following result.

Lemma 2.2 *Let f be a special meromorphic $\mathbb{A}$-valued function in the unit ball, $B(0,1)$, K a positive constant such that $\|J_f(z)\| \leq K$ and $\det(J_f(z)) \neq 0$ for $z \in \mathcal{E}_f(0)$. Let $-1 < \alpha \leq 1$. Suppose that there exists a point z^*, $\|z^*\| < r < 1$ such that*

$$\frac{\left(1 - \frac{\|z^*\|^2}{r^2}\right)^{1+\alpha} \|J_f(z^*)\|}{\left(1 - \frac{\|z^*\|^2}{r^2}\right)^{2\alpha} + \|f(z^*)\|^2} \geq K + 1. \tag{2.44}$$

Then there exists an element $z_0 \in B(0,1)$ with $\|z_0\| < r$ and $0 < t < 1$ such that

$$\sup_{\|z\|<r} \frac{\left(1 - \frac{\|z\|^2}{r^2}\right)^{1+\alpha} t^{1+\alpha} \|J_f(z)\|}{\left(1 - \frac{\|z\|^2}{r^2}\right)^{2\alpha} t^{2\alpha} + \|f(z)\|^2} = \frac{\left(1 - \frac{\|z_0\|^2}{r^2}\right)^{1+\alpha} t^{1+\alpha} \|J_f(z_0)\|}{\left(1 - \frac{\|z_0\|^2}{r^2}\right)^{2\alpha} t^{2\alpha} + \|f(z_0)\|^2} = K + 1. \tag{2.45}$$

Proof. Fixing α $(-1 < \alpha \leq 1)$ we define

$$F(z,t) := \frac{\left(1 - \frac{\|z\|^2}{r^2}\right)^{1+\alpha} t^{1+\alpha} \|J_f(z)\|}{\left(1 - \frac{\|z\|^2}{r^2}\right)^{2\alpha} t^{2\alpha} + \|f(z)\|^2},$$

which is continuous on the cylinder $\mathbf{C} = \{(z,t) : \|z\| < r, 0 < t \leq 1\}$.

Let us now, consider a sequence $\{z_m\}_{m\in\mathbb{N}}$ such that for all $m \in \mathbb{N}$ one has $\|z_m\| < r_0$ and $\lim_{m\to\infty} z_m = z_0$ for $\|z_0\| \leq r$. For $0 < t_m < 1$ we define

$$\rho_m := \left(1 - \frac{\|z_m\|^2}{r^2}\right) t_m, \tag{2.46}$$

where

$$\lim_{m\to\infty} \rho_m = 0. \tag{2.47}$$

Next we prove that

$$\limsup_{m\to\infty} F(z_m, t_m) \le K. \tag{2.48}$$

If $f(z_0) \neq 0$, then

$$\limsup_{m\to\infty} F(z_m, t_m) = \limsup_{m\to\infty} \frac{\rho_m^{1+\alpha} \|J_f(z_m)\|}{\rho_m^{2\alpha} + \|f(z_m)\|^2} \le \limsup_{m\to\infty} \rho_m^{1+\alpha} \frac{\|J_f(z_m)\|}{\|f(z_m)\|^2}.$$

Since f and its partial derivatives are continuous functions, thus $f(z_m)$ and $\|J_f(z_m)\|$ are pointwise convergent, this implies that

$$\lim_{m\to\infty} \frac{\|J_f(z_m)\|}{\|f(z_m)\|^2} = \frac{\|J_f(z_0)\|}{\|f(z_0)\|^2} = c, \tag{2.49}$$

where c is a positive constant. Using (2.47) and (2.49), it follows

$$\limsup_{m\to\infty} \rho_m^{1+\alpha} \frac{\|J_f(z_m)\|}{\|f(z_m)\|^2} = 0.$$

Therefore, for $f(z_0) \neq 0$, $\|z_m\| < r_0$ and $0 < t_m < 1$

$$\limsup_{m\to\infty} F(z_m, t_m) = 0.$$

For the case $f(z_0) = 0$ we have that $\det J_f(z_0) \neq 0$ which implies that z_0 is an isolated point. Therefore, for $-1 < \alpha < 1$

$$\begin{aligned} \limsup_{m\to\infty} F(z_m, t_m) &= \limsup_{m\to\infty} \frac{\left(1 - \frac{\|z_m\|^2}{r^2}\right)^{1+\alpha} t_m^{1+\alpha} \|J_f(z_m)\|}{\left(1 - \frac{\|z_m\|^2}{r^2}\right)^{2\alpha} t_m^{2\alpha} + \|f(z_m)\|^2} \\ &\le \lim_{m\to\infty} \rho_m^{1+\alpha} \rho_m^{-2\alpha} \|J_f(z_m)\| \\ &= 0. \end{aligned}$$

If $\alpha = 1$, and since $\|J_f(z_0)\| \le K$ we have

$$\limsup_{m\to\infty} F(z_m, t_m) = \lim_{m\to\infty} \frac{\rho_m^2 \|J_f(z_m)\|}{\rho_m^2 + \|f(z_m)\|^2} = \|J_f(z_0)\| \le K.$$

We conclude that

$$\limsup_{m\to\infty} F(z_m, t_m) \le K. \tag{2.50}$$

Let us now complete the proof. Using the inequality (2.44), there exists a point $\|z^*\| < r < 1$ such that $F(z^*, 1) > K + 1$. Consider

$$U = \{(z,t) \in \mathbf{C} : F(z,t) > K+1\} \quad \text{and} \quad t_0 = \inf\{t : (z,t) \in U\}. \tag{2.51}$$

Notice that $t_0 < 1$ since it is an infimum and $t_0 > 0$ in view of (2.50). Take z_0 such that $(z_0, t_0) \in \overline{U}$. Then as consequence of (2.50) we have $\|z_0\| < r$ and in view of the continuity of F in $\mathbf{C}$ we infer:

$$\sup_{\|z\|<r} F(z, t_0) = F(z_0, t_0) = K + 1.$$

□

Applying Lemma 2.2 we obtain.

Proposition 2.6 *Let $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ be a family of special meromorphic functions in the unit ball $B(0,1)$, as defined in (2.21). Suppose that there exists $K \geq 1$ such that $\|J_f(z)\| \leq K$ and $\det(J_f(z)) \neq 0$ for $z \in \mathcal{E}_f(0)$ for all f in $\mathcal{F}^*_{\mathcal{A}_{n+1}}$, and the same assumptions given in Lemma 2.2. If $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ is not normal, then there exist, for each $-1 < \alpha \leq 1$*

(i) a number $0 < r < 1$;

(ii) a sequence of points z_m, z_0 of $B(0,1)$ satisfying $\lim_{m\to\infty} z_m = z_0$;

(iii) sequence of functions $\{f_m\}_{m\in\mathbb{N}}$ in $\mathcal{F}^*_{\mathcal{A}_{n+1}}$;

(iv) a sequence of real positive numbers $\{\rho_m\}_{m\in\mathbb{N}}$ with $\lim_{m\to\infty} \rho_m = 0$,

such that the sequence $\left\{g_m(\xi) := \frac{f_m(z_m+\rho_m\xi)}{\rho_m^\alpha}\right\}_{m\in\mathbb{N}}$ converges locally uniformly with respect to the chordal distance to a non-constant special meromorphic function g in $\mathcal{A}_{n+1}$. Moreover, g satisfies

$$\Theta(g)(\xi) \leq \Theta(g)(0) = K + 1.$$

Proof. Since $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ is not normal, then by Theorem 2.6 there exists: $0 < r^* < 1$, a sequence of points $z^*_m \in \{z : \|z\| < r^*\}$ and a sequence of functions f_m in $\mathcal{F}^*_{\mathcal{A}_{n+1}}$ such that

$$\Theta(f_m)(z^*_m) = \frac{\|J_{f_m}(z^*_m)\|}{1 + \|f_m(z^*_m)\|^2} \to \infty, \qquad m \to \infty. \tag{2.52}$$

For a fixed r, $r^* < r < 1$ we have:

$$\frac{\left(1 - \|\frac{z^*_m}{r}\|^2\right)^{1+\alpha} \|J_{f_m}(z^*_m)\|}{\left(1 - \frac{\|z^*_m\|^2}{r^2}\right)^{2\alpha} + \|f_m(z^*_m)\|^2} \geq \left(1 - \frac{\|z^*_m\|^2}{r^2}\right)^{1+\alpha} \frac{\|J_{f_m}(z^*_m)\|}{1 + \|f_m(z^*_m)\|^2}. \tag{2.53}$$

Under condition (2.52) we assume that (2.53) is always greater then $K + 1$. Relying on Lemma 2.2, there exists a $m \in \mathbb{N}$ where $\|z_m\| < r$ and $0 < t_m < 1$ such that:

$$\sup_{\|z\|<r} \frac{\left(1 - \frac{\|z\|^2}{r^2}\right)^{1+\alpha} t_m^{1+\alpha} \|J_{f_m}(z)\|}{\left(1 - \frac{\|z\|^2}{r^2}\right)^{2\alpha} t_m^{2\alpha} + \|f_m(z)\|^2} = \frac{\left(1 - \frac{\|z_m\|^2}{r^2}\right)^{1+\alpha} t_m^{1+\alpha} \|J_{f_m}(z_m)\|}{\left(1 - \frac{\|z_m\|^2}{r^2}\right)^{2\alpha} t_m^{2\alpha} + \|f_m(z_m)\|^2} = K + 1. \tag{2.54}$$

Furthermore,

$$\begin{aligned} K + 1 &\geq \frac{\left(1 - \frac{\|z^*_m\|^2}{r^2}\right)^{1+\alpha} t_m^{1+\alpha} \|J_{f_m}(z^*_m)\|}{\left(1 - \frac{\|z^*_m\|^2}{r^2}\right)^{2\alpha} t_m^{2\alpha} + \|f_m(z^*_m)\|^2} \\ &\geq t_m^{1+\alpha} \frac{\left(1 - \frac{\|z^*_m\|^2}{r^2}\right)^{1+\alpha} \|J_{f_m}(z^*_m)\|}{\left(1 - \frac{\|z^*_m\|^2}{r^2}\right)^{2\alpha} + \|f_m(z^*_m)\|^2}. \end{aligned} \tag{2.55}$$

In view of (2.52), the expression (2.55) tends to infinity, hence t_m tends to 0. Setting $\rho_m := \left(1 - \frac{\|z_m\|^2}{r^2}\right) t_m$, one has

$$\lim_{m\to\infty} \frac{\rho_m}{r - \|z_m\|} = 0. \tag{2.56}$$

For $\|\xi\| < R_m$, where $R_m = \frac{r - \|z_m\|}{\rho_m}$ we define:

$$g_m(\xi) := \frac{f_m(z_m + \rho_m \xi)}{\rho_m^\alpha}. \tag{2.57}$$

Observe that g_m is defined in $\mathcal{A}_{n+1}$, since $\lim\limits_{m\to\infty} R_m = \infty$. Evaluating the Jacobian of g_m, we obtain the following relation

$$J_{g_m}(\xi) = \rho_m^{1-\alpha} J_{f_m}(z_m + \rho_m \xi). \tag{2.58}$$

Therefore, $\Theta(g_m)(\xi)$ is given by

$$\begin{aligned}\frac{\|J_{g_m}(\xi)\|}{1+\|g_m(\xi)\|^2} &= \frac{\rho_m^{1+\alpha}\|J_{f_m}(z_m+\rho_m\xi)\|}{\rho_m^{2\alpha}+\|f_m(z_m+\rho_m\xi)\|^2}\\ &= \frac{\left(1-\left(\frac{\|z_m\|}{r}\right)^2\right)^{1+\alpha} t_m^{1+\alpha}\|J_{f_m}(z_m+\rho_m\xi)\|}{\left(1-\left(\frac{\|z_m\|}{r}\right)^2\right)^{2\alpha} t_m^{2\alpha}+\|f_m(z_m+\rho_m\xi)\|^2}.\end{aligned}$$

Applying inequality (2.54) for $\xi=0$, it follows

$$\frac{\|J_{g_m}(0)\|}{1+\|g_m(0)\|^2} = \frac{\left(1-\left(\frac{\|z_m\|}{r}\right)^2\right)^{1+\alpha} t_m^{1+\alpha}\|J_{f_m}(z_m)\|}{\left(1-\left(\frac{\|z_m\|}{r}\right)^2\right)^{2\alpha} t_m^{2\alpha}+\|f_m(z_m)\|^2} = K+1. \tag{2.59}$$

For an arbitrary ξ, where $\|\xi\|<R<R_m$ we have

$$\|z_m\|-\rho_m R \le \|z_m+\rho_m\xi\| \le \|z_m\|+\rho_m R$$

which yields

$$\frac{r^2-\|z_m\|^2}{r^2-\|z_m\|+2\rho_m R-R^2} \le \frac{r^2-\|z_m\|^2}{r^2-\|z_m+\rho_m\xi\|} \le \frac{r^2-\|z_m\|^2}{r^2-\|z_m\|-(2\rho_m R+R^2)}.$$

Consequently, we have that

$$\frac{r^2-\|z_m\|^2}{r^2-(\|z_m\|+\rho_m R)^2} \le \frac{r^2-\|z_m\|^2}{r^2-\|z_m+\rho_m\xi\|^2} \le \frac{r^2-\|z_m\|^2}{r^2-(\|z_m\|-\rho_m R)^2}.$$

Using (2.56) we obtain

$$\lim_{m\to\infty}\frac{r^2-\|z_m\|^2}{r^2-\|z_m+\rho_m\xi\|^2}=1. \tag{2.60}$$

Using (2.58), (2.59) and (2.60) we obtain for $\varepsilon_1>0$ the following

$$\begin{aligned}\frac{\|J_{g_m}(\xi)\|}{1+\|g_m(\xi)\|^2} &\le (1+\varepsilon_1)\frac{\left(1-\left(\frac{\|z_m+\rho_m\xi\|}{r}\right)^2\right)^{1+\alpha} t_m^{1+\alpha}\|J_{f_m}(z_m+\rho_m\xi)\|}{\left(1-\left(\frac{\|z_m+\rho_m\xi\|}{r}\right)^2\right)^{2\alpha} t_m^{2\alpha}+\|f_m(z_m+\rho_m\xi)\|^2}\\ &\le (1+\varepsilon_1)(K+1).\end{aligned} \tag{2.61}$$

Using Theorem 2.6, one concludes that the sequence $\{g_m\}_{m\in\mathbb{N}}$ is normal. Then there exists a subsequence $\{g_{m_k}\}_{k\in\mathbb{N}}$ of $\{g_m\}_{m\in\mathbb{N}}$, which is locally uniformly convergent with respect to the chordal distance.

By Theorem 2.3 the subsequence converges to a special meromorphic function g. Therefore, using (2.59) and (2.61) it follows

$$\frac{\|J_g(\xi)\|}{1+\|g(\xi)\|^2} \leq K+1,$$

in particular,

$$\frac{\|J_g(0)\|}{1+\|g(0)\|^2} = K+1$$

which implies that g is a non-constant function. □

Chapter 3

On the growth of polymonogenic functions

In this chapter the growth of entire polymonogenic Clifford valued functions is studied. Generalizations of the order of growth, the maximum term and the central index are introduced. Relations between them, as for example some generalizations of Valiron's inequalities, are also established.

The first step is to obtain generalizations of the Cauchy estimate for solutions of iterated Dirac and also of iterated generalized Cauchy-Riemann equations. In the last section of this chapter is established a relation between $\|\overline{D}f\|$ and $\|f\|$, for a 1-monogenic function that maps the unit ball to the complement of the closed unit ball.

In one variable complex analysis much effort has been done in the study of the asymptotic growth of holomorphic and meromorphic functions during the last century, starting for example with the work of Wiman [71], Valiron [70], Nevalinna [56], Clunie [16] and others. Their asymptotic analysis provided powerful tools to study complex partial differential equations (see [34, 37, 43]). Therefore our aim is to establish some first rudiments of a generalized Wiman-Valiron theory in the context of hypercomplex analysis.

Throughout this chapter we consider real-analytic functions of the form:

(i) $f : \mathbb{R}^n \to Cl_n$, solutions of the iterated Dirac system, i.e., $\mathcal{D}^k f = 0$ for a positive integer $k \in \mathbb{N}$, where $\mathcal{D}$ is the Dirac operator (1.5), and

(ii) $f : \mathbb{R}^{n+1} \to Cl_n$, solutions of the iterated generalized Cauchy-Riemann system, i.e., $D^k f = 0$ for a positive integer $k \in \mathbb{N}$, where D is the generalized Cauchy-Riemann operator (1.6).

Both function classes differ essentially from each other, when $k \geq 2$. In the case of iterations of the Dirac operator, one gets for k even that $\mathcal{D}^k = (-1)^{k/2}\Delta^{k/2}$ where Δ is the Laplace operator, whereas in the case of the iterated generalized Cauchy-Riemann operator already $k = 2$ results in $D^2 = \left(\frac{\partial^2}{\partial x_0^2} - \Delta\right) + 2\frac{\partial}{\partial x_0}\mathcal{D}$.

However, both classes of functions are called *k-monogenic* functions or *polymonogenic* functions. In the case that they are solutions of these systems in the whole space ($\mathbb{R}^n$, resp. $\mathbb{R}^{n+1}$) they are called *entire k-monogenic* or *entire polymonogenic*.

To distinguish both cases more clearly, we write $\mathbf{x}$ when working in the vector formalism and z when working in the paravector formalism.

3.1 Cauchy estimates for solutions of iterated Dirac equations in $\mathbb{R}^n$

In this section Cauchy type estimates are established for entire polymonogenic functions $f : \mathbb{R}^n \to Cl_n$ with respect to the Dirac operator.

Recalling that for $k < n$ the fundamental solution of $\mathcal{D}^k f = 0$, is given by

$$\frac{1}{\omega_n}\mathbf{q_0}^{(k)}(\mathbf{x}) := \begin{cases} \dfrac{C_{n,k}\overline{\mathbf{x}}}{\|\mathbf{x}\|^{n+1-k}} & k \text{ odd with } k \leq n-1 \\ \dfrac{C_{n,k}}{\|\mathbf{x}\|^{n-k}} & k \text{ even with } k \leq n-1, \end{cases} \tag{3.1}$$

where ω_n is the measure of the unit hypersphere $S^n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| = 1\}$ and

$$C_{n,k} = \frac{(-1)^{k-1}}{2^{k-1}[\frac{k-1}{2}]!}\frac{(\frac{n}{2} - 1 - [\frac{k}{2}])!}{(\frac{n}{2} - 1)!}. \tag{3.2}$$

Notice that $|C_{n,k}| \leq 1$ for all n, k.

It is well known (see for instance [65]), that k-monogenic functions satisfy the following Green's integral formula:

$$f(\mathbf{x}) = \frac{1}{\omega_n} \sum_{j=0}^{k-1} \int_{\partial B(0,r)} \mathbf{q}_{\mathbf{0}}^{(j+1)}(\mathbf{y}-\mathbf{x}) d\sigma(\mathbf{y}) (\mathcal{D}^j f)(\mathbf{y}), \tag{3.3}$$

where we suppose that f is k-monogenic in a domain that contains the closed ball $B(0,r)$. A fundamental ingredient of this formula are the functions $\mathbf{q}_{\mathbf{0}}^{(k)}(\mathbf{x})$. These functions are denoted as the *Cauchy-Green's kernel.*

The partial derivatives of the Cauchy-Green's kernel $\mathbf{q}_{\mathbf{0}}^{(k)}(\mathbf{x})$ will be denoted by

$$\mathbf{q}_{\mathbf{m}}^{(k)}(\mathbf{x}) := \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} \mathbf{q}_{\mathbf{0}}^{(k)}(\mathbf{x}) = \frac{\partial^{m_1+\cdots+m_n}}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \mathbf{q}_{\mathbf{0}}^{(k)}(\mathbf{x}). \tag{3.4}$$

As proved in [39], they satisfy the following sharp estimates

$$\left\| \mathbf{q}_{\mathbf{m}}^{(k)}(\mathbf{x}) \right\| \leq \frac{|C_{n,k}|(n-k)(n+1-k)\cdots(n+|\mathbf{m}|-1-k)}{\|\mathbf{x}\|^{n+|\mathbf{m}|-k}}. \tag{3.5}$$

As a consequence of Green's integral formula, one notices that every Cl_n-valued function that is entire k-monogenic is real-analytic in $\mathbb{R}^n$. Hence, it can be represented as a normally convergent Taylor series of the form

$$f(\mathbf{x}) = \sum_{|\mathbf{m}|=0}^{+\infty} \mathbf{x}^{\mathbf{m}} a_{\mathbf{m}}. \tag{3.6}$$

The Clifford algebra valued coefficients $a_{\mathbf{m}}$ are given by

$$a_{\mathbf{m}} = \frac{1}{\mathbf{m}!} \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} f(\mathbf{x}) \Big|_{\mathbf{x}=\mathbf{0}}. \tag{3.7}$$

Using Green's integral formula and the estimates (3.5) of the functions $\mathbf{q}_{\mathbf{m}}^{(k)}(\mathbf{x})$, a first version of a Cauchy type estimates is obtained.

Proposition 3.1 *Let f be an entire k-monogenic function in $\mathbb{R}^n$, with the following Taylor series representation $f(\mathbf{x}) = \sum_{|\mathbf{m}|=0}^{+\infty} \mathbf{x}^{\mathbf{m}} a_{\mathbf{m}}$. Then*

$$\|a_{\mathbf{m}}\| \leq \sum_{j=0}^{k-1} A(n,\mathbf{m},j) \frac{1}{r^{|\mathbf{m}|-j}} M(r, \mathcal{D}^j f)$$

where $A(n,\mathbf{m},j) = \frac{|C_{n,j+1}|(n-1-j)(n-j)\cdots(n+|\mathbf{m}|-2-j)}{\mathbf{m}!}$ *and* $C_{n,j+1} = \frac{(-1)^j}{2^j [\frac{j}{2}]!} \frac{(\frac{n}{2}-1-[\frac{j+1}{2}])!}{(\frac{n}{2}-1)!}$.

Proof. Applying (3.3) on (3.7) it follows

$$a_{\mathbf{m}} = \frac{1}{\mathbf{m}!\omega_n} \sum_{j=0}^{k-1} \int_{\partial B(0,r)} \mathbf{q}_{\mathbf{m}}^{(j+1)}(\mathbf{y}) d\sigma(\mathbf{y}) (\mathcal{D}^j f)(\mathbf{y}).$$

Using the estimates given by the formula (3.5), one gets:

$$\begin{aligned} \|a_{\mathbf{m}}\| &\leq \frac{1}{\mathbf{m}!\omega_n} \sum_{j=0}^{k-1} \int_{\partial B(0,r)} \|\mathbf{q}_{\mathbf{m}}^{(j+1)}(\mathbf{y})\| \, \|d\sigma(\mathbf{y})\| \, \|(\mathcal{D}^j f)(\mathbf{y})\| \\ &\leq \frac{1}{\omega_n} \sum_{j=0}^{k-1} \left[\frac{|C_{n,j+1}|(n-1-j)(n-j)\cdots(n+|\mathbf{m}|-2-j)}{\mathbf{m}! r^{n+|\mathbf{m}|-(j+1)}} r^{n-1} \omega_n M(r, \mathcal{D}^j f) \right] \\ &= \sum_{j=0}^{k-1} A(n, \mathbf{m}, j) \frac{1}{r^{|\mathbf{m}|-j}} M(r, \mathcal{D}^j f), \end{aligned} \tag{3.8}$$

where

$$A(n, \mathbf{m}, j) = \frac{|C_{n,j+1}|(n-1-j)(n-j)\cdots(n+|\mathbf{m}|-2-j)}{\mathbf{m}!}, \tag{3.9}$$

and $C_{n,j+1} = \dfrac{(-1)^j}{2^j [\frac{j}{2}]!} \dfrac{(\frac{n}{2} - 1 - [\frac{j+1}{2}])!}{(\frac{n}{2} - 1)!}$. □

In the particular monogenic case $k = 1$ this inequality simplifies to

$$\|a_{\mathbf{m}}\| \leq \frac{A(n, \mathbf{m}, 0)}{r^{|\mathbf{m}|}} M(r, f) = \frac{(n-1)n\cdots(n-|\mathbf{m}|-2)}{\mathbf{m}! r^{|\mathbf{m}|}} M(r, f)$$

which is the sharp upper bound for the Taylor coefficients of 1-monogenic functions (see [20]).

The relation (3.8) describes an estimate of the Taylor coefficients $a_{\mathbf{m}}$ of the k-monogenic function f that appear in its Taylor series representation formula (3.6), which is valid for general real-analytic functions. The property of k-monogenicity is involved only later, when applying Green's integral formula (3.3).

Due to k-monogenicity, the function f *a priori* also admits a more specific kind of Taylor series representation involving the monogenic Fueter polynomials. To proceed in this direction, it is important to recall that if f is an entire k-monogenic function, then there exist k entire 1-monogenic functions, say $f_0, \ldots, f_{k-1}$, such that

$$f = f_0 + \mathbf{x} f_1 + \mathbf{x}^2 f_2 + \cdots + \mathbf{x}^{k-1} f_{k-1}. \tag{3.10}$$

The decomposition (3.10) is called Almansi type decomposition. In the case where f is a polynomial k-monogenic function this representation is also called Fischer decomposition (see for instance [47]). The term $\mathbf{x}^l f_l$ in this decomposition can in turn be recovered from the original k-monogenic function f. More precisely, there are described by the following projective formula.

Proposition 3.2 *Let f be a k-monogenic function and $f_0, f_1, \cdots, f_{k-1}$ 1-monogenic such that $f = f_0 + \mathbf{x} f_1 + \mathbf{x}^2 f_2 + \cdots + \mathbf{x}^{k-1} f_{k-1}$. Then*

$$P_l f = \mathbf{x}^l f_l. \tag{3.11}$$

with

$$P_l = \sum_{q=l}^{(+\infty)} a_{ql} \mathbf{x}^q \mathcal{D}^q \tag{3.12}$$

which is actually a finite sum, where

$$a_{ql} := \begin{cases} \dfrac{(-1)^{[\frac{l}{2}]+[\frac{q}{2}]+lq}}{2^q [\frac{l}{2}]! [\frac{q-l}{2}]!} \dfrac{(\frac{n}{2} + E - [\frac{l+q+1}{2}] - 1)!}{(\frac{n}{2} + E - [\frac{l}{2}] - 1)!}, & \textit{if} \quad q \geq l, \\ 0, & \textit{if} \quad q < l. \end{cases} \tag{3.13}$$

Here $E := \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ denotes the Euler operator.

The expression in (3.13) has to be understood symbolically.

To establish Proposition 3.2 we relay on the following proposition.

Proposition 3.3 *Let f be a 1-monogenic function. Then*

$$\mathbf{x}^m \mathcal{D}^m (\mathbf{x}^p f) := \begin{cases} \dfrac{(-1)^m 2^m [\frac{p}{2}]! (\frac{n}{2} + E - [\frac{p}{2}] - 1)!}{[\frac{p-m}{2}]! (\frac{n}{2} + E - [\frac{p+m}{2}] - 1)!} \mathbf{x}^p f, & \textit{if}\, p \geq m, \\ 0, & \textit{if}\, p < m. \end{cases} \tag{3.14}$$

In order to prove Proposition 3.3 the following result is needed.

Lemma 3.1 *Let f be a 1-monogenic function, $a \in \mathbb{R}$ and $p, s \in \mathbb{N}$. Then*

(i) for $p \geq s$, $\mathbf{x}^s E^m (\mathbf{x}^{p-s} f) = [E - s]^m (\mathbf{x}^p f)$;

(ii) $\mathcal{D}[(E + a)(\mathbf{x}^p f)] = (E + a + 1)[\mathcal{D}(\mathbf{x}^p f)]$.

Proof. To prove (i) we use mathematical induction with respect to m. Take $m = 1$, let us prove that

$$\mathbf{x}^s E(\mathbf{x}^{p-s} f) = [E - s](\mathbf{x}^p f). \tag{3.15}$$

Using $E(\mathbf{x}^p) = p\mathbf{x}^p$ the first term of (3.15) is rewritten in the form

$$\begin{aligned} \mathbf{x}^s E(\mathbf{x}^{p-s} f) &= \mathbf{x}^s (E(\mathbf{x}^{p-s}) f + \mathbf{x}^{p-s} E(f)) \\ &= (p-s)\mathbf{x}^p f + \mathbf{x}^p E(f), \end{aligned}$$

and the second term is given by

$$\begin{aligned} [E - s](\mathbf{x}^p f) &= E(\mathbf{x}^p) f + \mathbf{x}^p E(f) - s\mathbf{x}^p f \\ &= p\mathbf{x}^p f + \mathbf{x}^p E(f) - s\mathbf{x}^p f. \end{aligned}$$

Hence, (i) is established for $m = 1$.

Suppose that $\mathbf{x}^s E^m(\mathbf{x}^{p-s} f) = [E - s]^m(\mathbf{x}^p f)$ is true. Then we obtain

$$\begin{aligned} [E - s]^{m+1}(\mathbf{x}^p f) &= [E - s](\mathbf{x}^s E^m(\mathbf{x}^{p-s} f)) \\ &= E(\mathbf{x}^s E^m(\mathbf{x}^{p-s} f)) - s\mathbf{x}^s E^m(\mathbf{x}^{p-s} f) \\ &= s\mathbf{x}^s E^m(\mathbf{x}^{p-s} f) + \mathbf{x}^s E^{m+1}(\mathbf{x}^{p-s} f) - s\mathbf{x}^s E^m(\mathbf{x}^{p-s} f) \\ &= \mathbf{x}^s E^{m+1}(\mathbf{x}^{p-s} f). \end{aligned}$$

To prove (ii) one uses that

$$\mathcal{D}(E(\mathbf{x}^p f)) = [E + 1](\mathcal{D}(\mathbf{x}^p f)).$$

□

Next follows the **Proof of Proposition 3.3**

The proof is again made applying mathematical induction. Consider $p = 1$. From Lemma 3.1 we obtain

$$\begin{aligned} \mathbf{x}\mathcal{D}(\mathbf{x} f) &= \mathbf{x} \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}(\mathbf{x} f) \\ &= -n\mathbf{x} f + \mathbf{x} \sum_{i=1}^{n} e_i \mathbf{x} \frac{\partial}{\partial x_i} f. \end{aligned} \tag{3.16}$$

Since

$$\sum_{i=1}^{n} e_i \mathbf{x} \frac{\partial}{\partial x_i} f = -\mathbf{x}\mathcal{D}(f) - 2E(f), \tag{3.17}$$

and $\mathcal{D}(f) := 0$ it follows

$$\begin{aligned} \mathbf{x}\mathcal{D}(\mathbf{x}f) &= (-1)\mathbf{x}(n+2E)f \\ &= (-1)2\left(\frac{n}{2} + E - 1\right)\mathbf{x}f. \end{aligned} \tag{3.18}$$

Consider now the case $p > 1$. Notice that if $p = 2s$ for $s \in \mathbb{N}$ then $\mathbf{x}^{2s} = (-1)^s \|\mathbf{x}\|^s$, and $\mathcal{D}(\mathbf{x}^{2s}) = -2s\mathbf{x}^{2s-1}$. Consequently, using Lemma 3.1 and (3.18), we get

$$\begin{aligned} \mathbf{x}\mathcal{D}(\mathbf{x}^p f) &= \begin{cases} \mathbf{x}\mathcal{D}(\mathbf{x}^p)f + \mathbf{x}^p\mathbf{x}\mathcal{D}(f), & p \textit{ even} \\ \mathbf{x}\mathcal{D}(\mathbf{x}^{p-1})\mathbf{x}f + \mathbf{x}^{p-1}\mathbf{x}\mathcal{D}(\mathbf{x}f), & p \textit{ odd} \end{cases} \\ &= \begin{cases} \mathbf{x}\mathcal{D}(\mathbf{x}^p)f, & p \textit{ even} \\ \mathbf{x}\mathcal{D}(\mathbf{x}^{p-1})\mathbf{x}f - \mathbf{x}^{p-1}((n+2E-2)(\mathbf{x}f)), & p \textit{ odd} \end{cases} \\ &= \begin{cases} -p\mathbf{x}^p f, & p \textit{ even} \\ -(p-1)\mathbf{x}^{p-1}\mathbf{x}f - (n+2E-2(p-1)-2)\mathbf{x}^p f, & p \textit{ odd} \end{cases} \\ &= \begin{cases} (-1)p\mathbf{x}^p f, & p \textit{ even} \\ (-1)2\left(\frac{n}{2} + E - 1 - \frac{p-1}{2}\right)\mathbf{x}^p f, & p \textit{ odd} \end{cases} \\ &= \frac{(-1)2[\frac{p}{2}]!(\frac{n}{2} + E - [\frac{p}{2}] - 1)!}{[\frac{p-1}{2}]!(\frac{n}{2} + E - [\frac{p+1}{2}] - 1)!}\mathbf{x}^p f. \end{aligned} \tag{3.19}$$

Notice that for p even $[\frac{p}{2}] = [\frac{p+1}{2}] = [\frac{p-1}{2}] + 1$ and for p odd $[\frac{p-1}{2}] = [\frac{p}{2}] = [\frac{p+1}{2}] - 1$. Next we prove the formula (3.14) for $m = 2$. For $\mathcal{D}^2$ we have:

$$\begin{aligned} \mathcal{D}^2(\mathbf{x}^p f) &= \mathcal{D}\left(\mathcal{D}(\mathbf{x}^p f)\right) \\ &= \begin{cases} \mathcal{D}\Big(-p\mathbf{x}^{p-1}f\Big), & p \;\; \textit{even} \\ \mathcal{D}\Big(-2(\frac{n}{2} + E - \frac{p-1}{2})\mathbf{x}^{p-1}f\Big), & p \;\; \textit{odd.} \end{cases} \end{aligned}$$

Since $\mathcal{D}E = (E+1)(\mathcal{D})$, then we obtain

$$\mathcal{D}^2(\mathbf{x}^p f) = \begin{cases} -p(-2)(\frac{n}{2} + E - \frac{p-2}{2})\mathbf{x}^{p-2}f, & p \;\; \textit{even} \\ \\ -2(\frac{n}{2} + E + 1 - \frac{p-1}{2})(-(p-1))\mathbf{x}^{p-2}f, & p \;\; \textit{odd.} \end{cases}$$

In view of $\mathbf{x}^2 E(\mathbf{x}^{p-2} f) = [E-2](\mathbf{x}^p f)$, it follows

$$\mathbf{x}^2\mathcal{D}^2(\mathbf{x}^p f) = \begin{cases} (-1)^2\; 2\, p(\frac{n}{2}+E-2-\frac{p-2}{2})\mathbf{x}^p f, & p \;\; even \\ \\ (-1)^2\; 2(\frac{n}{2}+E-2+1-\frac{p-1}{2})(p-1)\mathbf{x}^p f, & p\;\; odd \end{cases}$$

$$= \frac{2^2[\frac{p}{2}]!(\frac{n}{2}+E-[\frac{p}{2}]-1)!}{[\frac{p-2}{2}]!(\frac{n}{2}+E-[\frac{p+2}{2}]-1)!}\mathbf{x}^p f.$$

For $s < p$ the following expression is obtained

$$\mathcal{D}^s(\mathbf{x}^p f) = \begin{cases} (-1)^s\; 2^{[\frac{s}{2}]} \prod_{i=0}^{[\frac{s-1}{2}]} \chi([\frac{s-1}{2}]-i)(p-2i) \prod_{j=1}^{[\frac{s}{2}]} \chi([\frac{s}{2}]-j)(\frac{n}{2}+E+s-j-\frac{p}{2})\mathbf{x}^{p-s} f, \\ \hfill p\;\; even \\ (-1)^s\; 2^{[\frac{s+1}{2}]} \prod_{i=1}^{[\frac{s}{2}]} \chi([\frac{s}{2}]-i)(p-2i+1)\times \\ \quad \prod_{j=1}^{[\frac{s+1}{2}]} \chi([\frac{s+1}{2}]-j)(\frac{n}{2}+E+s-j+1-\frac{p+1}{2})\mathbf{x}^{p-s} f, \qquad p\;\; odd, \end{cases}$$

where $\chi(a) \equiv 1$ for $a \geq 0$ and $\chi(a) \equiv 0$ otherwise. Furthermore,

$$\mathbf{x}^s\mathcal{D}^s(\mathbf{x}^p f) = \begin{cases} (-1)^s\; 2^{[\frac{s}{2}]} \prod_{i=0}^{[\frac{s-1}{2}]} \chi([\frac{s-1}{2}]-i)(p-2i)\times \\ \quad \prod_{j=0}^{[\frac{s}{2}]-1} \chi([\frac{s}{2}]-1-j)(\frac{n}{2}+E-1-(\frac{p}{2}+j))\mathbf{x}^{p-s} f, & p\;\; even \\ (-1)^s\; 2^{[\frac{s+1}{2}]} \prod_{i=1}^{[\frac{s}{2}]} \chi([\frac{s}{2}]-i)(p-2i+1)\times \\ \quad \prod_{j=0}^{[\frac{s-1}{2}]} \chi([\frac{s-1}{2}]-j)(\frac{n}{2}+E-1-(\frac{p-1}{2}+j))\mathbf{x}^{p-s} f, & p\;\; odd \end{cases}$$

$$= \frac{(-1)^s 2^s[\frac{p}{2}]!(\frac{n}{2}+E-1-[\frac{p}{2}])!}{[\frac{p-s}{2}]!(\frac{n}{2}+E-1-[\frac{p+s}{2}])!}\mathbf{x}^p f. \tag{3.20}$$

Finally, for the case where $s > p$ it follows

$$\mathbf{x}^s\mathcal{D}^s(\mathbf{x}^p f) = 0. \qquad \square$$

Using Proposition 3.3 we obtain ([17]) the **Proof of Proposition 3.2**.

The aim is to prove that

$$S_{ij} := \sum_{m=0}^{(+\infty)} a_{mi}\mathbf{x}^m \mathcal{D}^m(\mathbf{x}^j f_j) = \delta_{ij}.$$

Taking $S_{imj} := a_{mi}\mathbf{x}^m \mathcal{D}^m(\mathbf{x}^j f_j)$ and using Proposition 3.3 we obtain the following

$$S_{imj} := \frac{(-1)^{[\frac{i}{2}]+[\frac{m}{2}]+(i+1)m}\left(\frac{n}{2}+E-1-\left[\frac{m+i+1}{2}\right]\right)!}{[\frac{m-i}{2}]![\frac{j-m}{2}]!\left(\frac{n}{2}+E-1-\left[\frac{j+m}{2}\right]\right)!}\frac{[\frac{j}{2}]!\left(\frac{n}{2}+E-1-\left[\frac{j}{2}\right]\right)!}{[\frac{i}{2}]!\left(\frac{n}{2}+E-1-\left[\frac{i}{2}\right]\right)!}. \tag{3.21}$$

For the case where $j < i$ one has that $S_{ij} = 0$.

If we take $j = i$, therefore for $i = m = j$ one has

$$S_{ii} = (-1)^{2[\frac{i}{2}]+i^2+i}\frac{\left(\frac{n}{2}+E+[\frac{2i+1}{2}]-1\right)!}{\left(\frac{n}{2}+E+[\frac{2i}{2}]-1\right)!} = (-1)^{2[\frac{i}{2}]+i^2+i} = 1.$$

Next, let us replace $m = i + m$ and $j = i + j$ in (3.21) where we obtain

$$S_{imj} := \mathcal{K}^{(1)}_{imj}\mathcal{K}^{(2)}_{ij} \tag{3.22}$$

where

$$\mathcal{K}^{(1)}_{imj} \;\; := \;\; \frac{(-1)^{[\frac{i+m}{2}]+(i+1)m}\left(\frac{n}{2}+E-1-i-\left[\frac{m+1}{2}\right]\right)!}{[\frac{m}{2}]![\frac{j-m}{2}]!\left(\frac{n}{2}+E-1-i-\left[\frac{j+m}{2}\right]\right)!} \tag{3.23}$$

and

$$\mathcal{K}^{(2)}_{ij} \;\; := \;\; \frac{(-1)^{[\frac{i}{2}]+(i+1)i}[\frac{j+i}{2}]!\left(\frac{n}{2}+E-1-\left[\frac{j+i}{2}\right]\right)!}{[\frac{i}{2}]!\left(\frac{n}{2}+E-1-\left[\frac{i}{2}\right]\right)!}. \tag{3.24}$$

For the expression S_{imj} one has the following four cases:

(i) if $m = 2m$ and $j = 2j$ then

$$S^{(1)}_{imj} = \frac{(-1)^{[\frac{i}{2}]+m+(i+1)2m}\left(\frac{n}{2}+E-1-i-m\right)!}{m!(j-m)!\left(\frac{n}{2}+E-1-i-(j+m)\right)!}\mathcal{K}^{(2)}_{ij}. \tag{3.25}$$

(ii) if $m = 2m$ and $j = 2j+1$ then

$$S^{(2)}_{imj} = \frac{(-1)^{[\frac{i}{2}]+m+(i+1)2m}\left(\frac{n}{2}+E-1-i-m\right)!}{m!(j-m)!\left(\frac{n}{2}+E-1-i-(j+m)\right)!}\mathcal{K}^{(2)}_{ij}. \tag{3.26}$$

(iii) if $m = 2m + 1$ and $j = 2j$ then

$$S_{imj}^{(3)} = \frac{(-1)^{[\frac{i+1}{2}]+m+(i+1)(1+2m)} \left(\frac{n}{2} + E - 1 - i - (m+1)\right)!}{m!(j-m-1)! \left(\frac{n}{2} + E - 1 - i - (j+m)\right)!} \mathcal{K}_{ij}^{(2)}. \tag{3.27}$$

(iv) if $m = 2m + 1$ and $j = 2j + 1$ then

$$S_{imj}^{(4)} = \frac{(-1)^{[\frac{i+1}{2}]+m+(i+1)(1+2m)} \left(\frac{n}{2} + E - 1 - i - (m+1)\right)!}{m!(j-m-1)! \left(\frac{n}{2} + E - 1 - i - (j+m)\right)!} \mathcal{K}_{ij}^{(2)}. \tag{3.28}$$

Taking the sum over m we have that $S_{imj}^{(s)}$ $(s = 1, 2, 3, 4)$ are the terms of an hypergeometric series (see e.g. [62]). Therefore, we obtain

$$\begin{aligned} S_{ij}^{(s)} &:= \sum_{m=0}^{j} S_{imj}^{(s)} \\ &= \frac{(-1)^{\left[\frac{i}{2}\right]} \Gamma(\frac{n}{2} + E - i)}{\Gamma(j+1)\Gamma(\frac{n}{2} + E - i - j)} {}_2F_1(-j, j - \frac{n}{2} - E + i, 1 - \frac{n}{2} - E + i, 1) \mathcal{K}_{ij}^{(2)}, \end{aligned}$$

for $s = 1, 2$ since $S_{imj}^{(1)} = S_{imj}^{(2)}$, and

$$\begin{aligned} S_{ij}^{(s)} &:= \sum_{m=0}^{j} S_{imj}^{(s)} \\ &= \frac{(-1)^{\left[\frac{i+1}{2}\right]+i+1} \Gamma(\frac{n}{2} + E - i - 1)}{\Gamma(j+1)\Gamma(\frac{n}{2} + E - 1 - i - j)} {}_2F_1(-j, j - \frac{n}{2} - E + i + 2, 2 - \frac{n}{2} - E + i, 1) \mathcal{K}_{ij}^{(2)} \end{aligned}$$

for $s = 3, 4$ since $S_{imj}^{(3)} = S_{imj}^{(4)}$ and where ${}_2F_1$ is an hypergeometric function.

Using the following inequalities from [30, 62], for $j \in \mathbb{N}$ and $\alpha := j - \frac{n}{2} - E + i$

$${}_2F_1(-j, \alpha, \alpha - j, 1) = \lim_{\delta \to 0} {}_2F_1(-j, \alpha, \alpha - j - \delta, 1) = \frac{\Gamma(\alpha - j)}{\Gamma(\alpha)} (-1)^j j!$$

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)},$$

one obtains

$$\begin{aligned} S_{ij}^{(1)} = S_{ij}^{(2)} &= \frac{(-1)^{\left[\frac{i}{2}\right]} \Gamma(\frac{n}{2} + E - i)}{\Gamma(j+1)\Gamma(\frac{n}{2} + E - i - j)} \frac{\Gamma(1 - \frac{n}{2} - E + i)}{\Gamma(1 - \frac{n}{2} - E + i + j)} (-1)^j j! \, \mathcal{K}_{ij}^{(2)} \\ &= (-1)^{\left[\frac{i}{2}\right]+j} \frac{\sin(\pi(\frac{n}{2} + E - i - j))}{\sin(\pi(\frac{n}{2} + E - i))} \, \mathcal{K}_{ij}^{(2)} \\ &= (-1)^{\left[\frac{i}{2}\right]+j} \, \mathcal{K}_{ij}^{(2)}, \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} S_{ij}^{(3)} = S_{ij}^{(4)} &= \frac{(-1)^{\left[\frac{i+1}{2}\right]+i+1}\Gamma(\frac{n}{2}+E-i-1)}{\Gamma(j+1)\Gamma(\frac{n}{2}+E-1-i-j)} \frac{\Gamma(2-\frac{n}{2}-E+i)}{\Gamma(2-\frac{n}{2}-E+i+j)}(-1)^j j!\, \mathcal{K}_{ij}^{(2)} \\ &= (-1)^{\left[\frac{i+1}{2}\right]+i+1+j} \frac{\sin(\pi(\frac{n}{2}+E-i-j-1))}{\sin(\pi(\frac{n}{2}+E-i-1))}\, \mathcal{K}_{ij}^{(2)} \\ &= (-1)^{\left[\frac{i+1}{2}\right]+i+1+j}\, \mathcal{K}_{ij}^{(2)}. \end{aligned} \tag{3.30}$$

Summarizing, we obtain

$$\begin{aligned} \sum_{s=1}^{4} S_{ij}^{(s)} &= 2\left((-1)^{\left[\frac{i}{2}\right]+j} + (-1)^{\left[\frac{i+1}{2}\right]+i+1+j}\right) \mathcal{K}_{ij}^{(2)} \\ &= 0. \end{aligned}$$

Therefore, we conclude that

$$P_i f := \sum_{m=0}^{(+\infty)} a_{mi} \mathbf{x}^m \mathcal{D}^m (\sum_{j=0}^{k-1} \mathbf{x}^j f_j) = \mathbf{x}^i f_i.$$

□

Next we need to recall the property that every monogenic component function f_j has a Taylor expansion of the form

$$\sum_{|\mathbf{m}|=0}^{+\infty} V_{\mathbf{m}}(\mathbf{x})\alpha_{\mathbf{m}} \tag{3.31}$$

where α are uniquely defined Clifford numbers, $V_{\mathbf{m}}(\mathbf{x}) := \mathbf{m}!\mathcal{P}_{\mathbf{m}}(\mathbf{x})$ and

$$\mathcal{P}_{m_2,\ldots,m_n}(\mathbf{x}) := \frac{1}{|\mathbf{m}|!} \sum (x_{\sigma(1)} + x_1 e_1 e_{\sigma(1)}) \ldots (x_{\sigma(|\mathbf{m}|)} + x_1 e_1 e_{\sigma(|\mathbf{m}|)}) \tag{3.32}$$

where $|\mathbf{m}| := m_2 + \cdots + m_n$, $\sigma(i) \in \{2, \ldots, n\}$. Here, the summation runs over all distinguished permutations of the expressions $(x_{\sigma(i)} + x_1 e_1 e_{\sigma(i)})$.

When applying the Almansi type decomposition formula (3.10) one may further infer that every k-monogenic function has the following Taylor-Almansi series representation

$$f(\mathbf{x}) := \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} \mathbf{x}^j V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},j}. \tag{3.33}$$

The associated Taylor-Almansi coefficients $a_{\mathbf{m},j}$ appearing in this series expansion satisfy the inequality

$$\|a_{\mathbf{m},j}\| = \left\| \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} f_j \right\| \leq \frac{(n-1)n\cdots(n+|\mathbf{m}|-2)}{\mathbf{m}! r^{|\mathbf{m}|}} M(r, f_j) \tag{3.34}$$

where we rely on Cauchy's inequality for 1-monogenic functions applied to the monogenic component function f_j.

It is possible to obtain a Cauchy type estimate in terms of the original k-monogenic function f and their Dirac derivatives, by applying the following proposition.

Proposition 3.4 *Let f be an entire k-monogenic function with the following Taylor--Almansi series representation* $f(\mathbf{x}) := \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{l=0}^{k-1} \mathbf{x}^l V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},l}$. *Then*

$$\|a_{\mathbf{m},l}\| \leq \frac{(n-1)n\cdots(n+|\mathbf{m}|-2)}{\mathbf{m}!r^{|\mathbf{m}|}} \sum_{q=l}^{k-1} \sum_{i=0}^{[\frac{q}{2}]} \|\gamma_{lqi}\| r^{q-l} \frac{M(r, \mathcal{D}^q f)}{\alpha(q,l,i,n)}. \tag{3.35}$$

where

$$\gamma_{lqi} = \binom{[\frac{l}{2}] + [\frac{q-l}{2}]}{[\frac{l}{2}]} \frac{(-1)^{q+[\frac{l+1}{2}]+1}}{2^q [\frac{l}{2}]! [\frac{q-1}{2}]!} (-1)^i \binom{[\frac{l+q-1}{2}] - [\frac{l}{2}]}{i}$$

and $\alpha(q,l,i,n) := q + \frac{n}{2} - 1 - i - \left[\frac{l}{2}\right] > 0$.

Proof. In order to prove this, we use the projection formula (3.11). It follows

$$M(r, f_l) \leq \sum_{q=l}^{k-1} M(r, \mathbf{x}^{-l} \beta_{lq}(E) \mathbf{x}^q \mathcal{D}^q f) \tag{3.36}$$

where

$$\beta_{lq}(E) = \binom{[\frac{l}{2}] + [\frac{q-l}{2}]}{[\frac{l}{2}]} \frac{(-1)^{q+[\frac{l+1}{2}]+1}}{2^q [\frac{l}{2}]! [\frac{q-1}{2}]!} \sum_{i=0}^{[\frac{q}{2}]} \frac{(-1)^i \binom{[\frac{l+q-1}{2}] - [\frac{l}{2}]}{i}}{\frac{n}{2} + E - 1 - i - [\frac{l}{2}]}, \tag{3.37}$$

for $l = 1, \ldots, k-1$, $q \geq l$ with $(l,q) \neq (0,0)$. For $(l,q) = (0,0)$ we have $\beta_{0,0} := 1$. For the sake of convenience and clarity we rewrite this expression in the following form

$$\beta_{lq}(E) := \sum_{i=0}^{[\frac{q}{2}]} \gamma_{lqi} \frac{1}{E + \frac{n}{2} - 1 - i - [\frac{l}{2}]}, \tag{3.38}$$

where

$$\gamma_{lqi} = \binom{[\frac{l}{2}] + [\frac{q-l}{2}]}{[\frac{l}{2}]} \frac{(-1)^{q+[\frac{l+1}{2}]+1}}{2^q [\frac{l}{2}]! [\frac{q-1}{2}]!} (-1)^i \binom{[\frac{l+q-1}{2}] - [\frac{l}{2}]}{i}. \tag{3.39}$$

Next one observes that

$$\mathbf{x}^{-l} \beta_{lq}(E) \mathbf{x}^q \mathcal{D}^q f = \mathbf{x}^{-l} \mathbf{x}^q \beta_{lq}(E+q) \mathcal{D}^q f = \mathbf{x}^{-l} \mathbf{x}^q \sum_{i=0}^{[\frac{q}{2}]} \gamma_{lqi} \frac{1}{E + \alpha(q,l,i,n)}$$

where $\alpha := \alpha(q,l,i,n) := q + \frac{n}{2} - 1 - i - \left[\frac{l}{2}\right] > 0$.

Relying on the relation

$$\left[\frac{1}{E+\alpha}\right] g(\mathbf{x}) = \int_0^1 t^{\alpha-1} g(t\mathbf{x})dt \leq \frac{M(r,g)}{\alpha} \tag{3.40}$$

where $\left[\frac{1}{E+\alpha}\right]$ is the inverse operator of $[E+\alpha]$, which holds for a general C^1-function g. Finally, one gets

$$M(r, \mathbf{x}^{-l}\beta_{lq}(E)\mathbf{x}^q\mathcal{D}^q f) \leq \sum_{i=0}^{[\frac{q}{2}]} \|\gamma_{lqi}\| r^{q-l} \frac{M(r,\mathcal{D}^q f)}{\alpha(q,l,i,n)}. \tag{3.41}$$

□

In the next section the analogous results for functions which are solutions of the iterated generalized Cauchy-Riemann equation are established.

3.2 Cauchy estimates for solutions of iterated Cauchy--Riemann equations in $\mathbb{R}^{n+1}$

In this section, D denotes the generalized Cauchy-Riemann operator

$$D := e_0 \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i},$$

acting on the Euclidean paravector space $\mathbb{R} \oplus \mathbb{R}^n \cong \mathbb{R}^{n+1}$ whose elements have the form $z = x_0 + \mathbf{x}$ with $\mathbf{x} \in \mathbb{R}^n$. The subject of study is the class of functions $f : \mathbb{R}^{n+1} \to Cl_n$ that satisfy $D^k f = 0$ for a positive integer $k \in \mathbb{N}$, with $k < n+1$.

As shown in [11, 12], k-monogenic functions satisfy the following Green's integral formula:

$$f(z) = \frac{1}{\omega_{n+1}} \sum_{j=0}^{k-1} \int_{\partial B(0,r)} (-1)^j g_{\mathbf{0}}^{(j+1)}(\zeta - z) d\sigma(\zeta)(D^j f)(\zeta), \tag{3.42}$$

where f is k-monogenic in a domain that contains the closed ball $B(0,r)$. The functions given by

$$\frac{1}{\omega_{n+1}} g_{\mathbf{0}}^{(k)}(z) = \frac{1}{\omega_{n+1}} q_{\mathbf{0}}(z) \frac{x_0^{k-1}}{(k-1)!}. \tag{3.43}$$

are denoted as Cauchy-Green's kernel functions for k-monogenic functions with respect to the iterated generalized Cauchy-Riemann operator.

An estimate of the partial derivatives of the kernel functions $g_{\mathbf{0}}^{(k)}(z)$ is needed.

Proposition 3.5 *Let* $\mathbf{m} = (m_0, m_1, \ldots, m_n) \in \mathbb{N}_0^{n+1} \backslash \{\mathbf{0}\}$. *Then*

$$\begin{aligned} g_{\mathbf{m}}^{(k)}(z) &:= \frac{\partial^{m_0+\cdots+m_n}}{\partial x_0^{m_0} \cdots \partial x_n^{m_n}} g_{\mathbf{0}}^{(k)}(z) \\ &= \sum_{j=0}^{m_0} \binom{m_0}{j} q_{m_0-j,m_1,\ldots,m_n}(z) \frac{\chi(k-1-j) x_0^{k-1-j}}{(k-1-j)!}, \end{aligned} \tag{3.44}$$

are the general partial derivatives of the kernel function $g_{\mathbf{0}}^{(k)}(z)$ *with the following estimates*

$$\|g_{\mathbf{m}}^{(k)}(z)\| \leq \sum_{j=0}^{m_0} \binom{m_0}{j} \frac{n(n+1)\cdots(n+|\widetilde{\mathbf{m}}|+m_0-j-1)}{\|z\|^{n+|\widetilde{\mathbf{m}}|+m_0-j}} \frac{\chi(k-1-j)|x_0|^{k-1-j}}{(k-1-j)!}, \tag{3.45}$$

where $\chi(a) \equiv 1$ *for* $a \geq 0$ *and* $\chi(a) \equiv 0$ *otherwise.*

Proof. For particular multi-indices of the form $\mathbf{m} = (0, m_1, \ldots, m_n)$ the following expression is true

$$g_{\mathbf{m}}^{(k)}(z) = q_{\mathbf{m}}(z) \frac{x_0^{k-1}}{(k-1)!}.$$

Therefore, using the estimates given in (1.12) we obtain

$$\begin{aligned} \|g_{\mathbf{m}}^{(k)}(z)\| &\leq \frac{n(n+1)\cdots(n+|\mathbf{m}|-1)}{\|z\|^{n+|\mathbf{m}|}} \frac{|x_0|^{k-1}}{(k-1)!} \\ &\leq \frac{n(n+1)\cdots(n+|\mathbf{m}|-1)}{(k-1)!\|z\|^{n+|\mathbf{m}|+1-k}}. \end{aligned}$$

To deduce an estimate for the general partial derivatives of $g_{\mathbf{0}}^{(k)}$, involving also derivations in the x_0-direction, first one observes that for $k \geq 2$:

$$\frac{\partial}{\partial x_0} g_{\mathbf{0}}^{(k)}(z) = q_{\tau(0)}(z) \frac{x_0^{k-1}}{(k-1)!} + q_{\mathbf{0}}(z) \frac{x_0^{k-2}}{(k-2)!}$$

where $\tau(0)$ stands for the index $(1, 0, \cdots, 0)$. The next differentiation step implies

$$\begin{aligned} \frac{\partial^2}{\partial x_0^2} g_{\mathbf{0}}^{(k)}(z) &= q_{2\tau(0)}(z) \frac{x_0^{k-1}}{(k-1)!} + 2q_{\tau(0)}(z)\chi(k-2) \frac{x_0^{k-2}}{(k-2)!} \\ &\quad + q_{\mathbf{0}}(z)\chi(k-3) \frac{x_0^{k-3}}{(k-3)!}, \end{aligned}$$

where $\chi(a) \equiv 1$ for $a \geq 0$ and $\chi(a) \equiv 0$ otherwise. By a direct induction argument one can establish that

$$\frac{\partial^{m_0}}{\partial x_0^{m_0}} g_{\mathbf{0}}^{(k)}(z) = \sum_{j=0}^{m_0} \binom{m_0}{j} q_{(m_0-j)\tau(0)}(z) \frac{\chi(k-1-j)x_0^{k-1-j}}{(k-1-j)!}.$$

As a consequence the following representation for the general partial derivatives of the kernel function is obtained:

$$\frac{\partial^{m_0+\cdots+m_n}}{\partial x_0^{m_0}\cdots\partial x_n^{m_n}} g_{\mathbf{0}}^{(k)}(z) = \sum_{j=0}^{m_0} \binom{m_0}{j} q_{m_0-j,m_1,\ldots,m_n}(z) \frac{\chi(k-1-j)x_0^{k-1-j}}{(k-1-j)!}.$$

Hence, for all $\mathbf{m} = (m_0, \widetilde{\mathbf{m}}) \in \mathbb{N}_0^{n+1}\backslash\{\mathbf{0}\}$ with $\widetilde{\mathbf{m}} = (m_1, \ldots, m_n)$ it follows

$$\|g_{\mathbf{m}}^{(k)}(z)\| \leq \sum_{j=0}^{m_0} \binom{m_0}{j} \frac{n(n+1)\cdots(n+|\widetilde{\mathbf{m}}|+m_0-j-1)}{\|z\|^{n+|\widetilde{\mathbf{m}}|+m_0-j}} \frac{\chi(k-1-j)|x_0|^{k-1-j}}{(k-1-j)!}, \quad (3.46)$$

where the estimates of $\|q_{m_0-j,m_1,\ldots,m_n}(z)\|$ given in (1.12) have been applied. □

Formulas (3.46) allows us to derive a Cauchy type estimate of the Taylor coefficients of a k-monogenic function that appear in the Taylor series expansion of the form

$$f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} x_0^{m_0}\cdots x_n^{m_n} a_{\mathbf{m}}.$$

This Taylor series representation is valid for general real-analytic functions. In fact, every k-monogenic function is real-analytic, which follows from Green's integral formula (3.42). Indeed,

$$\begin{aligned} a_{\mathbf{m}} &= \frac{1}{\mathbf{m}!} \frac{\partial^{m_0+\cdots+m_n}}{\partial x_0^{m_0}\cdots\partial x_n^{m_n}} f(z)|_{z=0} \\ &= \frac{1}{\mathbf{m}!\omega_{n+1}} \sum_{j=0}^{k-1} \int_{\partial B(0,r)} (-1)^j g_{\mathbf{m}}^{(j+1)}(\zeta) d\sigma(\zeta) D^j f(\zeta). \qquad (3.47) \end{aligned}$$

Next applying the estimate deduced in (3.45), the following inequality is established

$$\begin{aligned} \|a_{\mathbf{m}}\| &\leq \frac{1}{\mathbf{m}!\omega_{n+1}} \sum_{j=0}^{k-1}\sum_{l=0}^{m_0} \binom{m_0}{l} \frac{n(n+1)\cdots(n+|\widetilde{\mathbf{m}}|+m_0-l-1)}{r^{n+|\widetilde{\mathbf{m}}|+m_0-l}} \\ &\qquad \times \frac{\chi(j-l)r^{j-l}}{(j-l)!} r^n \omega_{n+1} M(r, D^j f) \\ &\leq \frac{1}{\mathbf{m}!} \sum_{j=0}^{k-1}\sum_{l=0}^{m_0} \binom{m_0}{l} \chi(j-l) \frac{n(n+1)\cdots(n+|\widetilde{\mathbf{m}}|+m_0-l-1)}{r^{|\widetilde{\mathbf{m}}|+m_0-j}(j-l)!} M(r, D^j f). \end{aligned}$$

Also for this class of k-monogenic function an analogue of the Almansi type decomposition into monogenic functions is valued. Actually, it has even a simpler form than in the case treated in the previous subsection. Following e.g. [12], if f is an entire solution of $D^k f = 0$, then there exist k 1-monogenic functions, say $f_0, f_1, \cdots, f_{k-1}$, such that

$$f = f_0 + x_0 f_1 + x_0^2 f_2 + \cdots + x_0^{k-1} f_{k-1}. \tag{3.48}$$

Moreover, one has the following result.

Proposition 3.6 *Let f be a k-monogenic function as defined in (3.48). The terms $x_0 f_0, \cdots, x_0^{k-1} f_{k-1}$ can be recovered by the following projective formula*

$$x_0^l f_l = P_l f \quad \text{with} \quad P_l = \sum_{q=l}^{(+\infty)} (-1)^{l-q} \frac{1}{l!(q-l)!} x_0^q D^q. \tag{3.49}$$

Proof. Using induction with respect to q it follows

$$x_0^q D^q (x_0^l f_l) = \begin{cases} l(l-1)\cdots(l-q+1) x_0^l f_l(z) + x_0^{l+q} D^q (f_l(z)), & q \leq l \\ 0, & q > l. \end{cases} \tag{3.50}$$

First we prove that $x_0^l f_l = P_l(f)$ for $f = f_0 + x_0 f_1 + x_0^2 f_2 + \cdots + x_0^{k-1} f_{k-1}$. For $l = 0$, using (3.50) we obtain

$$\begin{aligned}
P_0(f) &:= \sum_{q=0}^{k-1} (-1)^{-q} \frac{1}{q!} x_0^q D^q (f) \\
&= \sum_{l=0}^{k-1} \sum_{q=0}^{k-1} \frac{(-1)^q}{q!} x_0^q D^q (x_0^l f_l) \\
&= \sum_{l=0}^{k-1} \sum_{q=l}^{k-1} \frac{(-1)^q}{q!} x_0^q D^q (x_0^l f_l) \\
&= \sum_{l=0}^{k-1} \sum_{q=l}^{k-1} \frac{(-1)^q}{q!} l(l-1)\cdots(l-q+1) x_0^l f_l \\
&= f - \sum_{l=1}^{k-1} \sum_{q=l}^{k-1} \frac{(-1)^q}{q!} l(l-1)\cdots(l-q+1) x_0^l f_l - \cdots + (-1)^{k-1} x_0^{k-1} f_{k-1} \\
&= \sum_{l=0}^{k-1} \sum_{s=0}^{l} \begin{pmatrix} l \\ s \end{pmatrix} (-1)^s x_0^l f_l \\
&= f_0,
\end{aligned}$$

since $\sum_{s=0}^{l}(-1)^s\binom{l}{s} = 0$. Therefore, using the same arguments as before the following relation for any l $(0 \leq l \leq k-1)$ is true, i.e.,

$$\begin{aligned} P_l(f) &= \sum_{q=l}^{k-1}(-1)^{l-q}\frac{1}{l!(q-l)!}x_0^q D^q(f) \\ &= \sum_{s=0}^{k-1}\sum_{q=l}^{k-1}(-1)^{l-q}\frac{1}{l!(q-l)!}x_0^q D^q(x_0^s f_s) \\ &= \sum_{q=l}^{k-1}\sum_{s=q}^{k-1}(-1)^{l-q}\frac{1}{l!(q-l)!}s(s-1)\cdots(s-q+1)x_0^s f_s \\ &= x_0^l f_l + \sum_{p=0}^{l+1}(-1)^p\binom{l+1}{p}x_0^{l+1}f_{l+1} + \cdots + \sum_{p=0}^{k-1}(-1)^p\binom{k-1}{p}x_0^{k-1}f_{k-1} \\ &= x_0^l f_l. \end{aligned}$$ □

As a consequence of the Almansi type decomposition (3.48) one again has a Taylor--Almansi series representation for this class of k-monogenic functions involving the monogenic Fueter polynomials. Following, e.g. [11, 12], each entire k-monogenic function has the following Taylor-Almansi series representation

$$f(z) = \sum_{|\mathbf{m}|=0}^{+\infty}\sum_{j=0}^{k-1}x_0^j V_{\mathbf{m}}(z)a_{\mathbf{m},j}, \qquad a_{\mathbf{m},j} = \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}}f_j \tag{3.51}$$

where $V_{\mathbf{m}}(z) := \mathbf{m}!\mathcal{P}_{\mathbf{m}}(z)$ and $\mathcal{P}_{\mathbf{m}}(z)$ is given in (1.8).

Next a Cauchy estimate for the Taylor-Almansi coefficients $a_{\mathbf{m},j}$ in series expansion (3.51) for k-monogenic functions is established.

Proposition 3.7 *Consider f to be an entire k-monogenic function with series expansion given by (3.51). Then*

$$\begin{aligned} \|a_{\mathbf{m},j}\| &\leq \frac{n(n+1)\cdots(n+|\mathbf{m}|-1)}{\mathbf{m}!r^{|\mathbf{m}|}}M(r,f_j) \\ &\leq \frac{n(n+1)\cdots(n+|\mathbf{m}|-1)}{\mathbf{m}!r^{|\mathbf{m}|}}\sum_{q=l}^{k-1}\frac{1}{l!(q-l)!}r^{q-l}M(r,D^q f). \end{aligned} \tag{3.52}$$

Proof. Applying the classical Cauchy integral formula (Theorem 1.5) for 1-monogenic functions on the monogenic component functions f_l $(l = 0, \ldots, k-1)$, one obtains:

$$\|a_{\mathbf{m},j}\| = \left\| \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} f_j \right\| \leq \frac{n(n+1)\cdots(n+|\mathbf{m}|-1)}{\mathbf{m}! r^{|\mathbf{m}|}} M(r, f_j). \tag{3.53}$$

Relying on the projection formula (3.49), it yields

$$\begin{aligned} M(r, f_l) &\leq \sum_{q=l}^{k-1} \frac{1}{l!(q-l)!} M(r, x_0^{q-l} D^q f) \\ &\leq \sum_{q=l}^{k-1} \frac{1}{l!(q-l)!} M(r, r^{q-l} D^q f) \\ &\leq \sum_{q=l}^{k-1} \frac{1}{l!(q-l)!} r^{q-l} M(r, D^q f). \end{aligned}$$

□

3.3 Order of growth of polymonogenic functions

In this section we restrict mainly to treat the iterated Dirac equation in detail. The other function class can be treated rather analogously.

To analyze the growth behavior of functions belonging to these classes it is important to mention that the maximum principle (see Theorem 1.7) is only valid in its strict form for special subclasses. These classes are 1-monogenic ($k = 1$) and for $k = 2$ the solutions of the iterated Dirac operator, i.e. for harmonic functions. For the other cases $k \geq 3$ a strict maximum principle does not exist. Take for instance the function $f(x_1, x_2, x_3) = 1 - x_1^2 - x_2^2 - x_3^2$. This is 3-monogenic, but $\|f(0)\| = 1$ at the origin and $\|f(\mathbf{x})\| = 0$ for all $\|\mathbf{x}\| = 1$.

In what follows, by

$$M(r, f) := \max_{\|\mathbf{x}\|=r} \{\|f(\mathbf{x})\|\}$$

is denoted the maximum modulus of f on the boundary of the ball of radius r and by

$$\mathcal{M}(r, f) := \max_{\|\mathbf{x}\|\leq r} \{\|f(\mathbf{x})\|\}$$

its maximum modulus of f on the whole closed ball. When no ambiguity occurs we denote $M(r) := M(r,f)$ and also $\mathcal{M}(r) := \mathcal{M}(r,f)$.

However one still obtains a number of analogous properties for the function $M(r,f)$ and $\mathcal{M}(r,f)$, as in the classical complex case (see [37]).

Proposition 3.8 *Let f be a non-constant left (right) entire k-monogenic function. Then the functions $M(r,f)$ and $\mathcal{M}(r,f)$ are continuous functions.*

Proof. Let us prove that $M(r,f) := \max_{\|\mathbf{x}\|=r} \{\|f(\mathbf{x})\|\}$ is continuous. From [13, p.48] we express $\mathbf{x} := \sum_{i=1}^{n} e_i x_i$ by means of its spherical coordinates

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1\cos\theta_2 \\ \vdots &= \vdots \\ x_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-1} \end{aligned}$$

for $0 < \theta_1, \cdots, \theta_{n-2} \leq \pi$ and $0 < \theta_{n-1} \leq 2\pi$. We have $\mathbf{x} = rw$ with $r = \|\mathbf{x}\|$ and $w := \sum_{i=1}^{n} e_i \frac{x_i}{\|\mathbf{x}\|}$. Let

$$M(r,f) = \|f(\mathbf{x}_0)\| \quad \text{and} \quad M(R,f) = \|f(\mathbf{x}_1)\|$$

for $\mathbf{x}_0 := rw_0$ and $\mathbf{x}_1 := Rw_1$. Then for $|R-r| < \delta$ it follows

$$M(R,f) - M(r,f) = \|f(R\omega_1)\| - \|f(r\omega_0)\| < \|f(R\omega_1)\| - \|f(r\omega_1)\| < \varepsilon$$

and also

$$M(r,f) - M(R,f) = \|f(r\omega_0)\| - \|f(R\omega_1)\| < \|f(r\omega_0)\| - \|f(R\omega_0)\| < \varepsilon.$$

Therefore, for $|R-r| < \delta$ it follows $|M(R,f) - M(r,f)| < \varepsilon$. □

We continue by generalizing some classical results on the asymptotic of holomorphic polynomials to the context of polynomials that are in the kernel of iterated Dirac or iterated generalized Cauchy-Riemann operators.

Theorem 3.1 *Let*

$$P(\mathbf{x}) = \sum_{|\mathbf{m}|=0}^{N} \sum_{j=0}^{k-1} \mathbf{x}^j V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},j}, \quad \left(\textit{resp. } P(z) = \sum_{|\mathbf{m}|=0}^{N} \sum_{j=0}^{k-1} x_0^j V_{\mathbf{m}}(z) a_{\mathbf{m},j} \right)$$

be a k-monogenic polynomial of degree $N+k-1$ with $a_{\mathbf{m},j} \in Cl_n$. Then for arbitrary $\varepsilon > 0$ there exists an $r_0 > 0$ such that for all $\|\mathbf{x}\| = r > r_0$

$$\|P(\mathbf{x})\| \leq \left(\frac{k(n+N+k-1)!}{n!N!} + \varepsilon \right) \|a_{\mathbf{N},j^*}\| r^{N+k-1} \tag{3.54}$$

where $\mathbf{N}$ is a multi-index and $j^* \in \{0,1,\ldots,k-1\}$ such that $|\mathbf{N}| + j^* = N+k-1$ and $\|a_{\mathbf{N},j^*}\| \geq \|a_{\mathbf{m},j}\|$ for all multi-indices $\mathbf{m}$ and j satisfying $|\mathbf{m}| + j = N+k-1$.

Proof. The estimates $\|V_{\mathbf{m}}(\mathbf{x})\| \leq \|\mathbf{x}\|^{|\mathbf{m}|}$ leads to

$$\begin{aligned}
\|P(\mathbf{x})\| &\leq \sum_{|\mathbf{m}|=0}^{N} \sum_{j=0}^{k-1} \|\mathbf{x}\|^{|\mathbf{m}|+j} \|a_{\mathbf{m},j}\| \\
&\leq \|a_{\mathbf{N},j^*}\| \|\mathbf{x}\|^{N+k-1} \left(\sum_{|\mathbf{m}|+j=N+k-1} 1 + \sum_{|\mathbf{m}|+j=0}^{N+k-2} \frac{\|a_{\mathbf{m},j}\|}{\|a_{\mathbf{N},j^*}\|} \|\mathbf{x}\|^{|\mathbf{m}|+j-(N+k-1)} \right) \\
&\leq \|a_{\mathbf{N},j^*}\| \|\mathbf{x}\|^{N+k-1} \left(\frac{k(n+N+k-1)!}{n!N!} + r_{N+k-2}(\mathbf{x}) \right),
\end{aligned}$$

where

$$r_{N+k-2}(\mathbf{x}) = \sum_{|\mathbf{m}|+j \leq N+k-2} \frac{\|a_{\mathbf{m},j}\|}{\|a_{\mathbf{N},j^*}\|} \|\mathbf{x}\|^{|\mathbf{m}|+j-(N+k-1)}.$$

For a sufficiently large $r_0 > 0$ it follows

$$|r_{N+k-2}(\mathbf{x})| < \varepsilon, \qquad \forall \|\mathbf{x}\| > r_0.$$

Therefore, for all $\|\mathbf{x}\| > r_0$ it holds

$$\|P(\mathbf{x})\| \leq \left(\frac{k(n+N+k-1)!}{n!N!} + \varepsilon \right) \|a_{\mathbf{N},j^*}\| r^{N+k-1}.$$

□

The following theorem is a generalization of the classical Liouville theorem.

Theorem 3.2 *Suppose that $f : \mathbb{R}^n \to Cl_n$ (resp. $f : \mathbb{R}^{n+1} \to Cl_n$) is an entire k--monogenic function given in the Almasi type decomposition form*
$f(\mathbf{x}) = f_0(\mathbf{x})+\mathbf{x}f_1(\mathbf{x})+\cdots+\mathbf{x}^{k-1}f_{k-1}(\mathbf{x})$ (resp. $f(z) = f_0(z)+x_0f_1(z)+\cdots+x_0^{k-1}f_{k-1}(z)$). If there exist a non-negative integers $s \in \mathbb{N}_0$ and L_j for $j = 0, 1, ..., k-1$ such that

$$(i) \quad \liminf_{r\to\infty} \frac{\mathcal{M}(r, f_j)}{r^{s-j}} = L_j < \infty, \tag{3.55}$$

or

$$(ii) \quad \liminf_{r\to\infty} \frac{\mathcal{M}(r, \mathcal{D}^j f)}{r^{s-j}} = L_j < \infty, \quad \left(\text{resp. } \liminf_{r\to\infty} \frac{\mathcal{M}(r, D^j f)}{r^{s-j}} = L_j < \infty\right) \tag{3.56}$$

then f is a polynomial function of total degree s.

Proof. Consider $L = \max\limits_{0\le j\le k-1}\{L_j\}$ and take an arbitrary sequence $\{r_i\}_{i\in\mathbb{N}}$ with $r_i \to \infty$.

(i) By (3.55) holds

$$\frac{\mathcal{M}(r_i, f_j)}{r_i^{s-j}} \le L+1, \tag{3.57}$$

for all $j = 0, ..., k-1$. Applying Cauchy's inequality (3.34) (resp. (3.53)) on the Taylor-Almansi coefficients $a_{\mathbf{l},j}$ of the function f in combination with (3.57) leads to

$$\|a_{\mathbf{l},j}\| \le \frac{(n-1)n\cdots(n+|\mathbf{l}|-2)}{\mathbf{l}!}(L+1)r_i^{s-|\mathbf{l}|-j}$$

from which follows that $a_{\mathbf{l},j} = 0$ for all $(\mathbf{l}, j)$ with $|\mathbf{l}| + j > s$.

(ii) By (3.56) holds

$$\frac{\mathcal{M}(r_i, \mathcal{D}^j f)}{r_i^{s-j}} \le L+1, \quad \left(\textit{resp. } \frac{\mathcal{M}(r_i, D^j f)}{r_i^{s-j}} \le L+1\right) \tag{3.58}$$

for all $j = 0, ..., k-1$.

Using Proposition 3.4 (resp. Proposition 3.7) and (3.58) leads to

$$\|a_{\mathbf{l},j}\| \le (L+1)\frac{(n-1)n\cdots(n+|\mathbf{l}|-2)}{\mathbf{l}!}\sum_{q=j}^{k-1}\sum_{i=0}^{[\frac{q}{2}]}\frac{\|\gamma_{jqi}\|r_i^{s-j-|\mathbf{l}|}}{\alpha(q,j,i,n)}$$

$$\left(\text{resp.} \qquad \|a_{\mathbf{l},j}\| \le (L+1)\frac{n(n+1)\cdots(n+|\mathbf{l}|-1)}{\mathbf{l}!}\sum_{q=j}^{k-1}\frac{r_i^{s-j-|\mathbf{l}|}}{j!(q-j)!},\right)$$

from which follows that $a_{\mathbf{l},j} = 0$ for all $(\mathbf{l}, j)$ with $|\mathbf{l}| + j > s$. □

We recall from [37, pp.17] the definition of the plus-logarithm and some of its properties.

Definition 3.1 *Let $\alpha \geq 0$. Then the plus-logarithm is defined by*

$$\log^+(\alpha) := \max\{0, \log(\alpha)\}. \tag{3.59}$$

Proposition 3.9 *Let $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_s$ be non-negative real numbers. Then*

i) $\log(\alpha) \leq \log^+(\alpha)$;

ii) if $\alpha_1 \leq \alpha_2$ *then* $\log^+(\alpha_1) \leq \log^+(\alpha_2)$;

iii) $\log(\alpha) = \log^+(\alpha) - \log^+\left(\frac{1}{\alpha}\right)$;

iv) $|\log(\alpha)| = \log^+(\alpha) + \log^+\left(\frac{1}{\alpha}\right)$;

v) $\log^+\left(\prod_{i=1}^{s} \alpha_i\right) \leq \sum_{i=1}^{s} \log^+(\alpha_i)$;

vi) $\log^+\left(\sum_{i=1}^{s} \alpha_i\right) \leq \sum_{i=1}^{s} \log^+(\alpha_i) + \log(s)$.

In the same way as in the planar case (see [37]) one also may introduce the notion of *order of growth* for the hypercomplex case (see also [1, 2]).

Definition 3.2 *Let $f : \mathbb{R}^n \to Cl_n$ (resp. $f : \mathbb{R}^{n+1} \to Cl_n$) be a left (right) entire k--monogenic function. Then*

$$\rho(f) := \limsup_{r \to \infty} \frac{\log^+(\log^+ \mathcal{M}(r, f))}{\log(r)}, \quad 0 \leq \rho \leq \infty \tag{3.60}$$

is called the order of growth of the function f. Furthermore,

$$\lambda(f) := \liminf_{r \to \infty} \frac{\log^+(\log^+ \mathcal{M}(r, f))}{\log(r)}, \quad 0 \leq \lambda \leq \infty \tag{3.61}$$

is defined as the lower order of growth of f.

If $\rho = \lambda$, then we say that f is a function of regular growth. If $\rho > \lambda$ then f is called of irregular growth.

Let us discuss some particular examples of 1-monogenic functions which are solutions of the generalized Cauchy-Riemann equation.

Example 3.1 *Let $P(z)$ be an arbitrary left 1-monogenic polynomial, i.e., there exist Clifford numbers $a_{\mathbf{m}} \in Cl_n$ and $N \in \mathbb{N}_0$ such that $P(z) = \sum_{|\mathbf{m}|=0}^{N} V_{\mathbf{m}}(z)a_{\mathbf{m}}$. From Theorem 3.1 it is known, for an arbitrarily small $\varepsilon > 0$ and for r sufficiently large, that*

$$\|P(z)\| \leq \left(\frac{(n-1+N)!}{(n-1)!N!} + \varepsilon \right) \|a_{\mathbf{N}}\| r^N,$$

where $\mathbf{N}$ is the index of length N for which $\|a_{\mathbf{N}}\| \geq \|a_{\mathbf{m}}\|$ for all $|\mathbf{m}| = N$.

Hence, it follows with $C(N) := \left(\frac{(n-1+N)!}{(n-1)!N!} + \varepsilon \right) \|a_{\mathbf{N}}\|$ that

$$\lim_{r\to\infty} \frac{\log^+(\log^+(M(r,P))}{\log(r)} \leq \lim_{r\to\infty} \frac{\log^+(\log^+(C(N)r^N))}{\log(r)} = 0.$$

Thus, all 1-monogenic polynomials satisfy $\rho(P) = \lambda(P) = 0$, like holomorphic polynomials in the complex case.

In the classical case, the exponential function has growth order equal to 1. In the case of 1-monogenic function the different generalized exponential function considered in [13, 21, 31] turn out to have the same growth order.

Example 3.2 *The monogenic plane wave function from [21]:*

$$P(\mathbf{m}, z) := (1 + i\mathbf{m})e^{-x_0}e^{i<\mathbf{m},\mathbf{x}>}, \tag{3.62}$$

where $\mathbf{m}$ is an arbitrary fixed vector from the $(n-1)$-dimensional unit sphere S^n, is left entire and satisfies

$$\max_{\|z\|=r} \|P(\mathbf{m}, z)\| = \|1 + i\mathbf{m}\|e^r.$$

Hence, for $r > 1$ the following is obtained

$$\lim_{r\to\infty} \frac{\log^+(\log^+(M(r, P(\mathbf{m}, z)))}{\log(r)} = \lim_{r\to\infty} \frac{\log^+(\log^+(\|1 + i\mathbf{m}\|) + r)}{\log(r)} = 1,$$

i.e., for all $\mathbf{m} \in S^n$

$$\rho(P(\mathbf{m}, z)) = \lambda(P(\mathbf{m}, z)) = 1.$$

Example 3.3 *Also the previously introduced monogenic generalization from [13, p.117]*

$$\begin{aligned} g(z) &= \exp(x_0, x_1, \ldots, x_n) \\ &= e^{x_1+\cdots+x_n}\left(\cos(x_0\sqrt{n}) - \frac{1}{\sqrt{n}}(e_1+\cdots+e_n)\sin(x_0\sqrt{n})\right) \end{aligned}$$

satisfies $\|g(z)\| = e^{x_1+\cdots+x_n} \leq e^{nr}$. *On the other hand, there must be a positive real number* $0 < c \leq n$ *with*

$$\max_{\|z\|=r} \|g(z)\| \geq e^{cr}.$$

The constant c *needs to be positive, otherwise it would have* $\max_{\|z\|=r} e^{x_1+\cdots+x_n} = 1$ *which would be wrong. Hence,*

$$\lim_{r\to\infty} \frac{\log^+(\log^+(M(r,g))}{\log(r)} = \lim_{r\to\infty} \frac{\log^+(\log^+(e^{cr}))}{\log(r)} = 1,$$

with $0 < c \leq n$*, so that we again get* $\rho(g) = \lambda(g) = 1$*, analogously to the classical case dealing with the complex analytic exponential function.*

Example 3.4 *Consider the four-dimensional quaternionic* 3*-fold periodic exponential function from [31], which has the representation*

$$EXP(\underline{x}) = e_0 Exp_0(\underline{x}) + e_1 Exp_1(\underline{x}) + e_2 Exp_2(\underline{x}) + e_3 Exp_3(\underline{x}),$$

with $\underline{x} := (x_0, x_1, x_2, x_3)$ *and*

$$Exp_0(\underline{x}) = e^{x_0}(\cos(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) - \sin(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}})),$$

$$Exp_1(\underline{x}) = e^{x_0}\frac{\sqrt{3}}{3}(\sin(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) + \cos(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}})),$$

$$Exp_2(\underline{x}) = e^{x_0}\frac{\sqrt{3}}{3}(\cos(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) + \sin(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}})),$$

$$Exp_3(\underline{x}) = e^{x_0}\frac{\sqrt{3}}{3}(\sin(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) + \cos(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}})).$$

By a direct computation, one notices that

$$\frac{\sqrt{3}}{3}e^r \leq \max_{\|\underline{x}\|=r} \|EXP(\underline{x})\| \leq e^r,$$

hence, once more

$$1 = \lim_{r\to\infty} \frac{\log(\log(\frac{\sqrt{3}}{3}e^r))}{\log(r)} \leq \lim_{r\to\infty} \frac{\log^+(\log^+ M(r, EXP))}{\log(r)} \leq \lim_{r\to\infty} \frac{\log(\log(e^r))}{\log(r)} = 1.$$

As in the classical case, a refinement in the same class of functions with the same order can be classified by its *type*. Analogously to the complex case the definition of the *type* of an entire k-monogenic function is presented by (see also [1]).

Definition 3.3 *Let f be a left (right) entire k-monogenic function in $\mathbb{R}^{n+1}$ of order ρ with $0 < \rho < \infty$. Then*

$$\tau(f) := \limsup_{r\to\infty} \frac{\log^+(\mathcal{M}(r, f))}{r^\rho}$$

is called the type of f. When no ambiguity occurs we denote $\tau(f) = \tau$.

Notice that the Examples 3.2, 3.4 are all of *type* $\tau = 1$, while Example 3.3 has *type* $\tau(g) = n$.

In classical complex analysis, one has the result that the order of growth of a holomorphic function and that of its derivative is the same. For 1-monogenic functions a similar result was established (see [10]).

Theorem 3.3 *Let g be a left (right) entire 1-monogenic function in $\mathbb{R}^{n+1}$, g_i given by $g_i := \frac{\partial}{\partial x_i} g$ and $M_i(r) := \max_{\|z\|=r} \{\|g_i(z)\|\}$ where $r > 0$ and $i \in \{0, \ldots, n\}$. Then*

$$\rho(g) = \rho'(g) \quad \textit{and} \quad \lambda(g) = \lambda'(g), \tag{3.63}$$

where

$$\rho'(g) := \limsup_{r\to\infty} \frac{\log^+(\log^+(M'(r)))}{\log(r)} \quad \textit{and} \quad \lambda'(g) := \liminf_{r\to\infty} \frac{\log^+(\log^+(M'(r)))}{\log(r)},$$

for $M'(r) := \max_{i=0,1,\ldots,n} \{M_i(r)\}$.

In the more general context of entire k-monogenic functions ($k > 1$), the following result is obtained:

Theorem 3.4 *Let g be an entire k-monogenic function. Consider g_i denoted as the function $g_i := \frac{\partial}{\partial x_i} g$ and $\mathcal{M}_i(r) := \max\limits_{\|\mathbf{x}\| \leq r} \left\{ \|g_i(\mathbf{x})\| \right\}$ where $r > 0$ and $i \in \{1, \ldots, n\}$. Then*

$$\rho(g) \leq \rho'(g) \quad and \quad \lambda(g) \leq \lambda'(g), \tag{3.64}$$

and

$$\rho(g_i) \leq \rho^*(g) \quad and \quad \lambda(g_i) \leq \lambda^*(g), \tag{3.65}$$

where

$$\rho'(g) := \limsup_{r \to \infty} \frac{\log^+(\log^+(\mathcal{M}'(r)))}{\log(r)} \qquad \lambda'(g) := \liminf_{r \to \infty} \frac{\log^+(\log^+(\mathcal{M}'(r)))}{\log(r)},$$

and

$$\rho^*(g) := \limsup_{r \to \infty} \frac{\log^+(\log^+(\widetilde{\mathcal{M}}(r)))}{\log(r)} \qquad \lambda^*(g) := \liminf_{r \to \infty} \frac{\log^+(\log^+(\widetilde{\mathcal{M}}(r)))}{\log(r)},$$

for $\mathcal{M}'(r) := \max\limits_{1 \leq i \leq n} \{\mathcal{M}_i(r)\}$ and $\widetilde{\mathcal{M}}(r) := \max\limits_{0 \leq i \leq k-1} \left\{ \max\limits_{\|\mathbf{x}\| \leq r} \left\{ r^i \|\mathcal{D}^i g(\mathbf{x})\| \right\} \right\}$.

Proof. To prove (3.64), consider an arbitrary rectifiable curve from the origin to $\mathbf{x}$, then

$$g(\mathbf{x}) = g(0) + \int_0^1 \sum_{i=1}^n x_i \; g_i(t\mathbf{x}) dt. \tag{3.66}$$

For $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\| \leq r$ the following is true

$$\begin{aligned} \|g(\mathbf{x})\| &\leq \|g(0)\| + r \sum_{i=1}^n \mathcal{M}_i(r) \\ &\leq \|g(0)\| + nr\mathcal{M}'(r). \end{aligned} \tag{3.67}$$

Therefore, it follows

$$\mathcal{M}(r) \leq \|g(0)\| + nr\mathcal{M}'(r).$$

Applying Proposition 3.9 ((v) and (vi)), we obtain

$$\log^+(\mathcal{M}(r)) \leq \log^+(\|g(0)\|) + \log^+(nr) + \log^+(\mathcal{M}'(r)) + \log(2),$$

which in turn leads to

$$\rho(g) \leq \rho'(g) \quad \text{and} \quad \lambda(g) \leq \lambda'(g). \tag{3.68}$$

Next let $0 < r < R < +\infty$. By Green's integral formula (3.3) holds

$$g_i(\mathbf{x}) = \frac{1}{w_n} \sum_{j=0}^{k-1} \int\limits_{\|\mathbf{y}-\mathbf{x}\|=R-r} \mathbf{q}_{\tau(i)}^{(j+1)}(\mathbf{y}-\mathbf{x}) d\sigma(\mathbf{y}) \mathcal{D}^j g(\mathbf{y}). \tag{3.69}$$

Applying the estimate (3.5) into (3.69) it follows

$$\begin{aligned} \|g_i(\mathbf{x})\| &= \frac{1}{\omega_n} \sum_{j=0}^{k-1} \int\limits_{\|\mathbf{y}-\mathbf{x}\|=R-r} \frac{|C_{n,j+1}|(n-1-j)}{(R-r)^{n+1-(j+1)}} \|d\sigma(\mathbf{y})\| \|\mathcal{D}^j g(\mathbf{y})\| \\ &\leq \sum_{j=0}^{k-1} \frac{|C_{n,j+1}|(n-1-j)}{(R-r)^{-j}} R^{-j} \max_{\|\mathbf{y}\|=R} \{R^j \|\mathcal{D}^j g(\mathbf{y})\|\}. \end{aligned}$$

Note that $|C_{n,k}| \geq |C_{n,j+1}|$ for all $j = 0, 1, \ldots, k-1$, i.e.,

$$\begin{aligned} \mathcal{M}_i(r) &\leq |C_{n,k}|(n-1) \sum_{j=0}^{k-1} \left(\frac{R-r}{R}\right)^j \widetilde{M}(R) \\ &\leq |C_{n,k}|(n-1) \sum_{j=0}^{k-1} \left(\frac{R-r}{R}\right)^j \widetilde{\mathcal{M}}(R). \end{aligned} \tag{3.70}$$

Inserting $R = 2r$ into (3.70) leads to

$$\begin{aligned} \mathcal{M}_i(r) &\leq |C_{n,k}|(n-1) \sum_{j=0}^{k-1} \left(\frac{1}{2}\right)^j \widetilde{\mathcal{M}}(2r) \\ &= |C_{n,k}| a(n-1) \widetilde{\mathcal{M}}(2r) \quad \text{with } a = \frac{2^k - 1}{2^{k-1}}. \end{aligned}$$

Thus, we have

$$\log^+ \mathcal{M}_i(r) \leq \log^+ \widetilde{\mathcal{M}}(2r) + \log^+ \Big(|C_{n,k}| a(n-1)\Big) + \log 2.$$

Hence, for $r > 1$ the following estimate is obtained

$$\begin{aligned} \frac{\log^+ \log^+ \mathcal{M}_i(r)}{\log r} &\leq \frac{\log^+ \log^+ \widetilde{\mathcal{M}}(2r)}{\log 2r} \frac{\log 2r}{\log r} \\ &+ \frac{\log^+ \log^+ (|C_{n,k}| a(n-1))}{\log r} + \frac{\log 2}{\log r}. \end{aligned}$$

Consequently, we have

$$\rho(g_i) \leq \rho^*(g) \quad \text{and} \quad \lambda(g_i) \leq \lambda^*(g).$$

□

Remark 3.1 *Dealing with entire solutions h of the iterated generalized Cauchy-Riemann equation the result is analogous. The only difference is given in the proof of*

$$\rho(h_i) \leq \rho^*(h) \quad \text{and} \quad \lambda(h_i) \leq \lambda^*(h),$$

where $h_i := \dfrac{\partial h}{\partial x_i}$ *for* $i = 0, \cdots, n$. *For this case we use Green's integral formula (3.42)*

$$h(z) = \frac{1}{\omega_{n+1}} \sum_{j=0}^{k-1} \int\limits_{\|\zeta - z\| = R-r} (-1)^j g_{\mathbf{0}}^{(j+1)}(\zeta - z) d\sigma(\zeta)(D^j h)(\zeta),$$

where

$$g_{\mathbf{0}}^{(k)}(z) = q_{\mathbf{0}}(z) \frac{x_0^{k-1}}{(k-1)!}. \tag{3.71}$$

In the cases $i \neq 0$, *one obtains*

$$\begin{aligned}
\|h_i(z)\| &\leq \frac{1}{\omega_{n+1}} \sum_{j=0}^{k-1} \int\limits_{\|\zeta - z\| = R-r} \|g_{\tau(i)}^{(j+1)}(\zeta - z)\| \|d\sigma(\zeta)\| \|(D^j h)(\zeta)\| \\
&\leq \frac{1}{\omega_{n+1}} \sum_{j=0}^{k-1} \int\limits_{\|\zeta - z\| = R-r} \frac{n}{\|\zeta - z\|^{n+1}} \frac{|\zeta_0 - z_0|^j}{j!} \|d\sigma(\zeta)\| R^{-j} \|R^j D^j h(\zeta)\| \\
&\leq \sum_{j=0}^{k-1} \frac{n}{j!} \left(\frac{R-r}{R} \right)^j \max_{\|\zeta\| = R} \{ R^j \|D^j h(\zeta)\| \} \\
&\leq n \sum_{j=0}^{k-1} \left(\frac{R-r}{R} \right)^j \widetilde{\mathcal{M}}(R).
\end{aligned}$$

In the remaining case, $i = 0$, *the following holds*

$$\begin{aligned}
\|h_0(z)\| \leq\ & \frac{1}{\omega_{n+1}} \int\limits_{\|\zeta - z\| = R-r} \Bigg[\|q_{\tau(0)}(\zeta - z)\| \|d\sigma(\zeta)\| \|h(\zeta)\| \\
+\ & \sum_{j=1}^{k-1} \|g_{\tau(0)}^{(j+1)}(\zeta - z)\| \|d\sigma(\zeta)\| \|D^j h(\zeta)\| \Bigg] \\
\leq\ & n\widetilde{\mathcal{M}}(R) + \frac{1}{\omega_{n+1}} \sum_{j=1}^{k-1} \int\limits_{\|\zeta - z\| = R-r} \Bigg[\left\| \frac{q_{\tau(0)}(\zeta - z)(\zeta_0 - z_0)^j}{j!} + \frac{q_{\mathbf{0}}(\zeta - z)(\zeta_0 - z_0)^{j-1}}{(j-1)!} \right\| \\
\times\ & \|d\sigma(\zeta)\| \|D^j h(\zeta)\| \Bigg]
\end{aligned}$$

$$
\begin{aligned}
\|h_0(z)\| &\leq n\widetilde{\mathcal{M}}(R) + \sum_{j=1}^{k-1} \left[\frac{n(R-r)^j}{(R-r)^{n+1}j!} + \frac{(R-r)^{j-1}}{(R-r)^n(j-1)!} \right] \frac{(R-r)^{n+1}}{R^j} \widetilde{\mathcal{M}}(R) \\
&= n\widetilde{\mathcal{M}}(R) + \sum_{j=1}^{k-1} \frac{1}{j!} \left(\frac{R-r}{R} \right)^j (n+j)\widetilde{\mathcal{M}}(R) \\
&\leq (n+k-1) \sum_{j=0}^{k-1} \left(\frac{R-r}{R} \right)^j \widetilde{\mathcal{M}}(R).
\end{aligned}
$$

By the same arguments one can prove that in the iterated generalized Cauchy-Riemann operator case one obtains

$$\rho(h_i) \leq \rho^*(h) \quad \textit{and} \quad \lambda(h_i) \leq \lambda^*(h).$$

3.4 The maximum term and central indices of polymonogenic functions

Both, for the class of null solutions of the iterated Dirac equation and of the iterated generalized Cauchy-Riemann equation it makes sense to introduce the notion of maximum term in the following way.

Let f be an entire k-monogenic function with the following Taylor-Almansi series expansion

$$f(\mathbf{x}) = \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} \mathbf{x}^j V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},j}.$$

The norm of the function satisfies

$$
\begin{aligned}
\|f(\mathbf{x})\| &\leq \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} \|\mathbf{x}^j\| \|V_{\mathbf{m}}(\mathbf{x})\| \|a_{\mathbf{m},j}\| \\
&\leq \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} \|\mathbf{x}\|^{|\mathbf{m}|+j} \|a_{\mathbf{m},j}\| \\
&\leq \sum_{s=0}^{+\infty} \|\mathbf{x}\|^s \binom{n-1+s}{s} d_s \\
&\leq \sum_{s=0}^{+\infty} \|\mathbf{x}\|^s (n-1)^s d_s, \qquad (3.72)
\end{aligned}
$$

where $d_s = \max_{|\mathbf{m}|+j=s} \{\|a_{\mathbf{m},j}\|\}$. If f is transcendental, then the series (3.72) converges. Therefore, if $\|\mathbf{x}\| = r$ is fixed, then

$$\lim_{s\to\infty} d_s \|\mathbf{x}\|^s (n-1)^s = 0,$$

i.e., there must exist a term in the sequence which is greater or equal than all the other terms of the sequence. This term will be denoted as the *maximum term* in the following way.

Definition 3.4 *(Maximum term)*

Let $f : \mathbb{R}^n \to Cl_n$ *(resp.* $f : \mathbb{R}^{n+1} \to Cl_n$*) be an entire solution of* $\mathcal{D}^k f = 0$ *(resp. of* $D^k f = 0$*) for a positive integer* k*. Let*

$$f(\mathbf{x}) = \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} \mathbf{x}^j V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},j},$$

$$(\text{resp.} \quad f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} x_0^j V_{\mathbf{m}}(z) a_{\mathbf{m},j},)$$

be its Taylor-Almansi series expansion. Then the maximum term in this series expansion is defined by

$$\mu(r,f) := \max_{|\mathbf{m}|,j} \left\{ \|a_{\mathbf{m},j}\| r^{|\mathbf{m}|+j} \right\}. \tag{3.73}$$

In case of no ambiguity we denote $\mu(r,f) = \mu(r)$ *or simply* μ*.*

Definition 3.5 *(Central indices)*

Let $f : \mathbb{R}^n \to Cl_n$ *(resp.* $f : \mathbb{R}^{n+1} \to Cl_n$*) be an entire solution of* $\mathcal{D}^k f = 0$ *(resp. of* $D^k f = 0$*) for a positive integer* k*. Let*

$$f(\mathbf{x}) = \sum_{|\mathbf{m}|=p}^{+\infty} \sum_{j=0}^{k-1} \mathbf{x}^j V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},j},$$

$$(\text{resp.} \quad f(z) = \sum_{|\mathbf{m}|=p}^{+\infty} \sum_{j=0}^{k-1} x_0^j V_{\mathbf{m}}(z) a_{\mathbf{m},j}).$$

For $r > 0$ the index (or the indices) $(\mathbf{m}, j)$ with maximal length $|\mathbf{m}| + j$ with $\mu(r) = \|a_{\mathbf{m},j}\| r^{|\mathbf{m}|+j}$ is (are) called central index (indices) and denoted by

$$\nu(r, f) := (\mathbf{m}, j).$$

When no ambiguity occurs we denote $\nu(r, f) = \nu(r)$ or simply ν. Denoting $\nu(0)$ as the indices $(\mathbf{m}, 0)$ which satisfy $|\mathbf{m}| = p$.

Remark 3.2 *In the particular case $k = 1$ the definitions of the maximum term and central indices coincide with those introduced in [10] for the 1-monogenic case.*

In a similar way these notions can also be introduced for k-monogenic polynomials. For a k-monogenic polynomial $P(z) = \sum_{|\mathbf{m}|=0}^{N} \sum_{j=0}^{k-1} \mathbf{x}^j V_{\mathbf{m}}(z) a_{\mathbf{m},j}$, the maximum term is given by $\mu(r, P) = \|a_{\mathbf{N},i}\| r^{|\mathbf{N}|+i}$ (where $(\mathbf{N}, i)$ is (are) the index (indices) of length $N + k - 1$ satisfying $\|a_{\mathbf{N},i}\| \geq \|a_{\mathbf{m},j}\|$ for all $|\mathbf{m}| + j = N + k - 1$) and $\nu(r, P) = (\mathbf{N}, i)$, provided r is sufficiently large.

The case of transcendental functions is more complicated. This will be studied now.

We start by proving

Theorem 3.5 *Assume that $f : \mathbb{R}^{n+1} \to Cl_n$ is an entire k-monogenic transcendental function. Then*

(i) $\mu(r)$ increases for $r \leq r_0$ strictly monotonically and $\lim_{r\to\infty} \mu(r) = \infty$.

(ii) $|\nu(r)|$ increases monotonically and $\lim_{r\to\infty} |\nu(r)| = \infty$. Furthermore, $|\nu(r)|$ is piecewise constant.

Proof.

(i) Since f is not a constant function, there exists an $r_0 > 0$ such that $|\nu(r)| \geq 1$ for $r \geq r_0$. From the definition follows, that for $R > r \geq r_0$:

$$\mu(r) = \|a_{\nu(r)}\| r^{|\nu(r)|} < \|a_{\nu(r)}\| R^{|\nu(r)|} \leq \|a_{\nu(R)}\| R^{|\nu(R)|} = \mu(R). \tag{3.74}$$

Thus, $\mu(r)$ is strictly monotonically increasing. Let $\mathbf{l} \in \mathbb{N}_0^n$. Then

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log(\mu(r))}{\log(r)} &\geq \liminf_{r\to\infty} \frac{\log(\|a_{\mathbf{l},j}\| r^{|\mathbf{l}|})}{\log(r)} \\ &= \liminf_{r\to\infty} \frac{\log \|a_{\mathbf{l},j}\| + (|\mathbf{l}|+j)\log(r)}{\log(r)} \\ &= \liminf_{r\to\infty} \left(\frac{\log \|a_{\mathbf{l},j}\|}{\log(r)} + |\mathbf{l}| + j \right) = |\mathbf{l}| + j \qquad \forall \mathbf{l} \in \mathbb{N}_0^n.\end{aligned}$$

Since f is a transcendental function, it follows that

$$\liminf_{r\to\infty} \frac{\log(\mu(r))}{\log(r)} = \infty, \tag{3.75}$$

which implies that $\mu(r)$ tends to infinity for $r \to \infty$.

(ii) Next we prove that $|\nu(r)|$ is monotonically increasing. For $r < R$ the following two estimates are given:

$$\begin{aligned}\|a_{\nu(R)}\| R^{|\nu(R)|} &\geq \|a_{\nu(r)}\| R^{|\nu(r)|} \\ \|a_{\nu(r)}\| \; r^{|\nu(r)|} &\geq \|a_{\nu(R)}\| r^{|\nu(R)|}\end{aligned}$$

from which is inferred

$$\left(\frac{R}{r}\right)^{|\nu(R)|} \geq \left(\frac{R}{r}\right)^{|\nu(r)|}. \tag{3.76}$$

Thus, $|\nu(r)|$ is monotonically increasing. Taking $d_s := \max_{|\mathbf{l}|+j=s} \{\|a_{\mathbf{l},j}\|\}$ it is known that $\lim_{s\to\infty} d_s = 0$. Hence, there exists a positive constant C such that

$$\mu(r) = \|a_{\nu(r)}\| r^{|\nu(r)|} \leq C r^{|\nu(r)|} \tag{3.77}$$

from which

$$\frac{\log(\mu(r))}{\log(r)} \leq \frac{\log(C)}{\log(r)} + |\nu(r)|. \tag{3.78}$$

Formula (3.75), i.e.,

$$\lim_{r\to\infty} \frac{\log(\mu(r))}{\log(r)} = \infty$$

means that $\lim_{r\to\infty} |\nu(r)| = \infty$. Since $\nu(r) \in \mathbb{N}_0^{n+1}$ and $|\nu(r)|$ tends monotonically to infinity, $|\nu(r)|$ has to be piecewise constant and has at most a countable number of discontinuities. □

Analogous to the classical case (see e.g. [37, p.34]) one proves the following result, simply relying on Theorem 3.5.

Proposition 3.10 *If f is a k-monogenic function, then*

(i) $|\nu(r,f)|$ is right continuous,

(ii) $\mu(r,f)$ is continuous.

Theorem 3.6 *Suppose that $f : \mathbb{R}^n \to Cl_n$ is an entire k-monogenic transcendental function with the property that its first Taylor-Almansi coefficient $a_{\mathbf{0},0} \neq 0$. Then*

$$\log(\mu(r)) - \log \|a_{\mathbf{0},0}\| = \int_0^r \frac{|\nu(t)|}{t} dt. \tag{3.79}$$

Proof. Assume, without loss of generality, that the first Taylor-Almansi coefficient $a_{\mathbf{0},0} = f(0) = 1$. If $0 = t_0 < t_1 < t_2 < \cdots$ are the discontinuities of $|\nu(r)|$, then for $t_j < t < t_{j+1}$ we infer

$$\mu(t) = \|a_{\mathbf{m}^*,j^*}\| t^{|\mathbf{m}^*|+j^*} \tag{3.80}$$

with a fixed $(\mathbf{m}^*, j^*) = \nu(t)$. Furthermore,

$$\mu'(t) = (|\mathbf{m}^*| + j^*)\|a_{\mathbf{m}^*,j^*}\| t^{|\mathbf{m}^*|+j^*-1} = \frac{|\nu(t)|}{t}\mu(t). \tag{3.81}$$

Thus, in an interval $[0, r]$ it holds with exception of a finite number of points

$$\frac{d}{dt}\left\{ \log(\mu(t)) \right\} = \frac{\mu'(t)}{\mu(t)} = \frac{|\nu(t)|}{t}. \tag{3.82}$$

Since $\mu(t)$ is a continuous function, we obtain

$$\begin{aligned} \log(\mu(r)) = \log(\mu(r)) - \log(\mu(0)) &= \int_0^r \frac{d}{dt}\Big(\log(\mu(t)) \Big) dt \\ &= \int_0^r \frac{|\nu(t)|}{t} dt. \end{aligned}$$

□

3.5 Generalized Valiron type theorems for polymonogenic functions

For the function class of entire k-monogenic functions the following Valiron type inequality is established:

Theorem 3.7 *Let $f : \mathbb{R}^n \to Cl_n$ be an entire k-monogenic function, then for all $r > 0$ such that $r < R$*

$$\mathcal{M}(r) \leq \mu(r)\Big[k|\nu(R)|(1+|\nu(R)|)^{n-2} + \frac{R(R^k - r^k)}{r^{k-1}(R-r)^2}\Big]. \tag{3.83}$$

Proof. Since f is entire k-monogenic, it can be represented in the form

$$f(\mathbf{x}) = \sum_{|\mathbf{l}|=0}^{+\infty}\sum_{j=0}^{k-1} \mathbf{x}^j V_{\mathbf{l}}(\mathbf{x}) a_{\mathbf{l},j}.$$

Taking $\nu(R) = (\mathbf{l}^*, j^*)$, for $0 < r < R$ holds

$$\begin{aligned}
\mathcal{M}(r) &\leq \sum_{|\mathbf{l}|=0}^{+\infty}\sum_{j=0}^{k-1} \|a_{\mathbf{l},j}\| r^{|\mathbf{l}|+j} \\
&= \sum_{|\mathbf{l}|=0}^{|\nu(R)|-j^*-1}\sum_{j=0}^{k-1} \|a_{\mathbf{l},j}\| r^{|\mathbf{l}|+j} + \sum_{|\mathbf{l}|=|\nu(R)|-j^*}^{+\infty}\sum_{j=0}^{k-1} \|a_{\mathbf{l},j}\| r^{|\mathbf{l}|+j} \\
&\leq k \sum_{|\mathbf{l}|=0}^{|\nu(R)|-j^*-1} \mu(r) + \sum_{|\mathbf{l}|=|\nu(R)|-j^*}^{+\infty}\sum_{j=0}^{k-1} \|a_{\mathbf{l},j}\| r^{|\mathbf{l}|+j}.
\end{aligned} \tag{3.84}$$

Notice that, we get

$$\begin{aligned}
\sum_{|\mathbf{l}|=0}^{|\nu(R)|-j^*-1} 1 &= \sum_{|\mathbf{l}|=0} 1 + \sum_{|\mathbf{l}|=1} 1 + \cdots + \sum_{|\mathbf{l}|=|\nu(R)|-j^*-1} 1 \\
&= 1 + \frac{((n-2)+1)!}{(n-2)!1!} + \cdots + \frac{[(n-2)+(|\nu(R)|-j^*-1)]!}{(n-2)!(|\nu(R)|-j^*-1)!} \\
&\leq (|\nu(R)|-j^*)\left[\frac{[(n-2)+|\nu(R)|-j^*-1]!}{(n-2)!(|\nu(R)|-j^*-1)!}\right].
\end{aligned}$$

This is obtained, relying on the inequality

$$\frac{(n-2+s)!}{(n-2)!s!} \leq \frac{(n-2+(s+1))!}{(n-2)!(s+1)!}, \qquad n \geq 2.$$

Furthermore, it follows

$$
\begin{aligned}
& (|\nu(R)| - j^*)\left[\frac{[(n-2)+|\nu(R)|-j^*-1]!}{(n-2)!(|\nu(R)|-j^*-1)!}\right] \\
= \; & \prod_{\beta=1}^{n-2}\left[\frac{|\nu(R)|-j^*+(\beta-1)}{\beta}\right] \\
\leq \; & (|\nu(R)| - j^*)\left[\underbrace{(1+\frac{|\nu(R)|-j^*}{n-2})}_{\leq 1+|\nu(R)|-j^*}\underbrace{(1+\frac{|\nu(R)|-j^*}{n-3})}_{\leq 1+|\nu(R)|-j^*}\cdots\cdots\underbrace{(1+\frac{|\nu(R)|-j^*}{1})}_{=1+|\nu(R)|-j^*}\right] \\
\leq \; & |\nu(R)|\left[(1+|\nu(R)|)^{n-2}\right].
\end{aligned}
$$

Concluding

$$
\begin{aligned}
\sum_{|\mathbf{l}|=|\nu(R)|-j^*}^{+\infty}\sum_{j=0}^{k-1}\|a_{\mathbf{l},j}\|r^{|\mathbf{l}|+j} \leq \; & \sum_{|\mathbf{l}|=|\nu(R)|-j^*}^{+\infty}\sum_{j=0}^{k-1}\|a_{\mathbf{l},j}\|r^{|\mathbf{l}|+j}\frac{\|a_{\nu(r)}\|r^{|\nu(r)|}R^{|\mathbf{l}|+j+|\nu(R)|-j^*}}{\|a_{\nu(R)}\|r^{|\nu(R)|}R^{|\mathbf{l}|+j+|\nu(R)|-j^*}} \\
= \; & \mu(r)\sum_{|\mathbf{l}|=|\nu(R)|-j^*}^{+\infty}\sum_{j=0}^{k-1}\frac{\|a_{\mathbf{l},j}\|R^{|\mathbf{l}|+j}}{\|a_{\nu(R)}\|R^{|\nu(R)|}}\frac{r^{|\mathbf{l}|+j}R^{|\nu(R)|-j^*}}{r^{|\nu(R)|}R^{|\mathbf{l}|+j-j^*}} \\
\leq \; & \mu(r)\sum_{|\mathbf{l}|=|\nu(R)|-j^*}^{+\infty}\sum_{j=0}^{k-1}\left(\frac{r}{R}\right)^{|\mathbf{l}|-|\nu(R)|+j^*}\frac{r^{j-j^*}}{R^{j-j^*}} \\
= \; & \mu(r)\sum_{|\mathbf{l}|=|\nu(R)|-j^*}^{+\infty}\left(\frac{r}{R}\right)^{|\mathbf{l}|-|\nu(R)|+j^*}\left(\frac{R}{r}\right)^{j^*}\sum_{j=0}^{k-1}\left(\frac{r}{R}\right)^{j} \\
\leq \; & \mu(r)\sum_{|\mathbf{l}|=|\nu(R)|-j^*}^{+\infty}\left(\frac{r}{R}\right)^{|\mathbf{l}|-|\nu(R)|+j^*}\left(\frac{R}{r}\right)^{k-1}\frac{R^k-r^k}{R^{k-1}(R-r)} \\
\leq \; & \mu(r)\frac{R(R^k-r^k)}{r^{k-1}(R-r)^2}.
\end{aligned}
$$

Applying this inequalities into (3.84) leads to

$$
\mathcal{M}(r) \leq \mu(r)\left[k|\nu(R)|(1+|\nu(R)|)^{n-2}+\frac{R(R^k-r^k)}{r^{k-1}(R-r)^2}\right].
$$

□

Remark 3.3 *Within the context of entire solutions of the iterated generalized Cauchy--Riemann equation in* $\mathbb{R}^{n+1}$ *the same result is obtained, simply replacing* n *by* $n+1$.

For the case $k=1$*, this inequality simplifies to*

$$M(r) = \mathcal{M}(r) \leq \mu(r)\Big[|\nu(R)|(1+|\nu(R)|)^{n-1} + \frac{R}{R-r}\Big].$$

This is the Valiron inequality for 1*-monogenic functions established in [10, Theorem* 5.2*].*

For 1-monogenic functions we also obtain, applying Cauchy's inequality, the following direct estimate

$$\mu(r) \leq M(r)\frac{n(n+1)\cdots(n-|\nu(r)|-1)}{\nu(r)!}, \tag{3.85}$$

where $\nu(r)$ is one central index (see [10]). Furthermore, applying Stirling's formula to (3.85) ([17]), one has

$$\mu(r) \leq M(r)\frac{n(n+1)\cdots(n+|\nu(r)|-1)}{\nu(r)!} \leq \frac{M(r)}{(n-1)!(\sqrt{2\pi})^{n-1}}|\nu(r)|^{\frac{n-1}{2}}n^{\frac{n}{2}+|\nu(r)|}.$$

For polymonogenic functions also holds:

Proposition 3.11 *For an entire* k*-monogenic function* $g:\mathbb{R}^n \to Cl_n$ $(g:\mathbb{R}^{n+1}\to Cl_n)$ *of order* ρ *and lower order* λ*, we set*

$$\rho_1 := \limsup_{r\to\infty}\frac{\log^+(\log^+\mu(r))}{\log(r)} \qquad \rho_2 := \limsup_{r\to\infty}\frac{\log^+(|\nu(r)|)}{\log(r)} \tag{3.86}$$

and

$$\lambda_1 := \liminf_{r\to\infty}\frac{\log^+(\log^+\mu(r))}{\log(r)} \qquad \lambda_2 := \liminf_{r\to\infty}\frac{\log^+(|\nu(r)|)}{\log(r)}. \tag{3.87}$$

Then $\rho \leq \rho_1 = \rho_2$ *and* $\lambda \leq \lambda_1 = \lambda_2$.

Proof. Although the proof of $\rho_1=\rho_2$ and $\lambda_1=\lambda_2$ can be done in analogy to the complex case presented in [37, Theorem 4.5], we present it here.

Let us start by proving $\rho_1 \leq \rho_2$. Suppose that g is a transcendental k-monogenic function with the following Taylor-Almansi series expansion

$$g(\mathbf{x}) = \sum_{|\mathbf{m}|=0}^{+\infty}\sum_{j=0}^{k-1}\mathbf{x}^j V_{\mathbf{m}}(\mathbf{x})a_{\mathbf{m},j}.$$

Since this series is convergent, we have for a sufficient large $r > 0$ and $c > 0$

$$\mu(r, g) = \|a_{\nu(r)}\| r^{|\nu(r)|} \leq c r^{|\nu(r)|}.$$

Furthermore, it yields

$$\log^+(\mu(r, g)) \leq |\nu(r)| \log^+(r) + \log^+(c).$$

Moreover, it follows

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log^+(\log^+(\mu(r,g)))}{\log r} &\leq \limsup_{r\to\infty} \frac{\log^+(|\nu(r)|) + \log^+(\log^+(r)) + \log^+(\log^+(c)) + \log 2}{\log r} \\ &\leq \limsup_{r\to\infty} \frac{\log^+(|\nu(r)|)}{\log r}. \end{aligned}$$

To prove that $\lambda_1 \leq \lambda_2$, we use the same argument as in the proof of $\rho_1 \leq \rho_2$. To prove that $\rho_1 \geq \rho_2$ we take $0 < r < R$ and infer

$$\left(\frac{R}{r}\right)^{|\nu(r)|} = \frac{\|a_{\nu(r)}\| R^{|\nu(r)|}}{\|a_{\nu(r)}\| r^{|\nu(r)|}} \leq \frac{\mu(R, g)}{\mu(r, g)}.$$

Moreover, we obtain

$$|\nu(r)| \log^+\left(\frac{R}{r}\right) \leq \log^+(\mu(R, g)) - \log^+(\mu(r, g)) \leq \log^+(\mu(R, g)).$$

Taking $R = 2r$, follows

$$\log^+(|\nu(r)|) + \log(\log 2) \leq \log^+(\log^+(\mu(2r, g))).$$

Furthermore, yields

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log^+(|\nu(r)|) + \log(\log 2)}{\log r} &\leq \limsup_{r\to\infty} \frac{\log^+(\log^+(\mu(2r,g)))}{\log(r)} \\ &\leq \limsup_{r\to\infty} \frac{\log^+(\log^+(\mu(2r,g)))}{\log(2r)} \frac{\log(2r)}{\log(r)}. \end{aligned}$$

Hence, $\rho_1 \geq \rho_2$. In a similar way we obtain $\lambda_1 \geq \lambda_2$.

Now we give the proof of $\rho \leq \rho_1$. Without loss of generality, it is sufficient to consider the case $\rho_1 < \infty$, since the assertion is true in the remaining case where $\rho_1 = \infty$.

Inserting in particular $r = R/2$ into Theorem 3.7, leads to

$$\mathcal{M}(r) \leq \mu(r)[k|\nu(2r)|[1 + |\nu(2r)|]^{n-2} + 2(2^k - 1)]. \tag{3.88}$$

In view of

$$\frac{\log|\nu(2r)|}{\log(2r)} \leq \rho_2 + \varepsilon, \qquad \varepsilon > 0$$

which equivalently reads as

$$|\nu(2r)| \leq e^{(\rho_2+\varepsilon)\log(2r)} = (2r)^{\rho_2+\varepsilon},$$

one concludes that for a sufficiently large r there is an $\varepsilon_1 > 0$ and a $\delta > 0$ such that

$$\begin{aligned}\mathcal{M}(r) &\leq \mu(r)\Bigg(|\nu(2r)|^n(k+\varepsilon_1)\Bigg)\\ &\leq \mu(r)((2r)^{n(\rho_2+\varepsilon)}(k+\varepsilon_1))\\ &\leq \mu(r)(2r)^{n\rho_2+n\varepsilon}(2r)^{\delta}.\end{aligned}$$

Hence with $\varepsilon_2 := n\varepsilon + \delta$ it follows

$$\mathcal{M}(r) \leq \mu(r)(2r)^{n\rho_2+\varepsilon_2}. \tag{3.89}$$

Therefore, we arrive to

$$\begin{aligned}\frac{\log^+(\log^+\mathcal{M}(r))}{\log(r)} &\leq \frac{\log^+\log^+[\mu(r)(2r)^{n\rho_2+\varepsilon_2}]}{\log(r)}\\ &\leq \frac{\log^+[\log^+\mu(r) + \log^+(2r)^{n\rho_2+\varepsilon_2}]}{\log(r)}\\ &\leq \frac{\log^+(\log^+(\mu(r))) + \log^+(\log^+((2r)^{n\rho_2+\varepsilon_2})) + \log(2)}{\log(r)}\\ &\leq \frac{\log^+(\log^+\mu(r)) + \log^+((n\rho_2+\varepsilon_2)\log^+(2r)) + \log(2)}{\log(r)}.\end{aligned}$$

Furthermore, we have

$$\limsup_{r\to\infty}\frac{\log^+(\log^+\mathcal{M}(r))}{\log(r)} \leq \limsup_{r\to\infty}\frac{\log^+(\log^+\mu(r))}{\log(r)} =: \rho_1.$$

It remains to prove that $\lambda \leq \lambda_2(=\lambda_1)$. If $\lambda_2 = \infty$, then the assertion is true. Assume without loss of generality that $\lambda_2 < \infty$. Then there exists a sufficiently large R such that

$$\frac{\log^+|\nu(R)|}{\log R} \leq \lambda_2 + \varepsilon,$$

which equivalently reads

$$|\nu(R)| \leq R^{\lambda_2+\varepsilon}. \tag{3.90}$$

Since the Taylor-Almansi series converges, follows for sufficiently large R and $\varepsilon_0 > 0$

$$\mu(R) = |a_{\nu(R)}| R^{|\nu(R)|} \leq R^{R^{\lambda_2+\varepsilon_0}}. \tag{3.91}$$

Inserting $r = R/2$ into Theorem 3.7 and applying (3.90) and (3.91) leads to

$$\begin{aligned} \mathcal{M}(r) &\leq R^{R^{\lambda_2+\varepsilon_0}} [R^{n\lambda_2+n\varepsilon_0}(k+\varepsilon')] \\ &\leq R^{R^{\lambda_2+\varepsilon_0}} R^{n\lambda_2+n\varepsilon_0+\delta} \\ &\leq R^{R^{\lambda_2+\varepsilon_0+\delta'}} = (2r)^{(2r)^{\lambda_2+\varepsilon^*}} \leq r^{r^{\lambda_2+\varepsilon_1}} \end{aligned}$$

with some appropriately chosen $\varepsilon', \varepsilon^*, \varepsilon_1, \delta, \delta' > 0$. Finally one gets

$$\liminf_{r\to\infty} \frac{\log^+(\log^+ \mathcal{M}(r))}{\log(r)} \leq \liminf_{r\to\infty} \frac{\log^+(r^{\lambda_2+\varepsilon_1}\log r)}{\log(r)} =: \lambda_2 = \lambda_1.$$

□

Remark 3.4 *In the two-dimensional complex case the inequality (3.85) corresponds to $\mu(r) \leq M(r)$, allowing to establish the stronger result $\rho = \rho_1 = \rho_2$ and $\lambda = \lambda_1 = \lambda_2$, as shown for instance in [37, Theorem 4.5].*

Proposition 3.12 *Let $f : \mathbb{R}^n \to Cl_n$ (resp. $f : \mathbb{R}^{n+1} \to Cl_n$) be an entire k-monogenic function with $\rho_1(f) = 0$. Then, for all $k, s \in \mathbb{N}$*

$$\lim_{r\to\infty} \frac{|\nu(r,f)|^k}{r^s} = 0. \tag{3.92}$$

Proof. As a consequence of Proposition 3.11

$$0 = \rho_1(f) = \rho_2(f) := \limsup_{r\to\infty} \frac{\log^+(|\nu(r,f)|)}{\log(r)},$$

we obtain

$$\log(|\nu(r,f)|^k) = k\log(|\nu(r,f)|) \leq k\log^+(|\nu(r,f)|) \leq k\varepsilon\log(r) = \log(r^{k\varepsilon}),$$

for all r sufficiently large and $\varepsilon > 0$ sufficiently small such that $k\varepsilon < 1$. Therefore we get

$$\frac{|\nu(r,f)|^k}{r^s} \leq \frac{|\nu(r,f)|^k}{r} \leq \frac{r^{k\varepsilon}}{r} = r^{k\varepsilon-1},$$

since $k\varepsilon < 1$. This concludes the proof. □

This proposition allows us to establish the following result.

Theorem 3.8 *Let $f : \mathbb{R}^n \to Cl_n$ (resp. $f : \mathbb{R}^{n+1} \to Cl_n$) be an entire k-monogenic function with $\rho_2(f) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log \mathcal{M}_i(r)}{\log \mu(r)} \leq 1 \tag{3.93}$$

where $\mathcal{M}_i(r) := \max_{\|\mathbf{x}\|\leq r} \left\{ \left\| \frac{\partial}{\partial x_i} f(\mathbf{x}) \right\| \right\}$ for $i = 2, \ldots, n$ (resp. $i = 1, \ldots, n$).

Proof. Since f is entire k-monogenic, we have

$$f(\mathbf{x}) = \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} \mathbf{x}^j V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},j}, \quad \left(resp. \quad f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} x_0^j V_{\mathbf{m}}(z) a_{\mathbf{m},j} \right)$$

in the whole space $\mathbb{R}^n$ (resp. $\mathbb{R}^{n+1}$). Hence, so is

$$\begin{aligned} f_i(\mathbf{x}) := \frac{\partial}{\partial x_i} f(\mathbf{x}) &= \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} \frac{\partial}{\partial x_i} \left(\mathbf{x}^j V_{\mathbf{m}}(\mathbf{x}) \right) a_{\mathbf{m},j} \\ &= \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} \left(\frac{\partial}{\partial x_i} (\mathbf{x}^j) V_{\mathbf{m}}(\mathbf{x}) + \mathbf{x}^j \frac{\mathbf{m}!}{(\mathbf{m} - \tau(i))!} V_{\mathbf{m}-\tau(i)}(\mathbf{x}) \right) a_{\mathbf{m},j} \end{aligned}$$

$$\begin{aligned} \text{(resp.} \quad f_i(z) := \frac{\partial}{\partial x_i} f(z) &= \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} x_0^j \frac{\partial}{\partial x_i} V_{\mathbf{m}}(z) a_{\mathbf{m},j} \\ &= \sum_{|\mathbf{m}|=1}^{+\infty} \sum_{j=0}^{k-1} x_0^j \frac{\mathbf{m}!}{(\mathbf{m} - \tau(i))!} V_{\mathbf{m}-\tau(i)}(z) a_{\mathbf{m},j}, \quad i \neq 0.) \end{aligned}$$

Since

$$\mathbf{x}^j = \begin{cases} (-1)^m \left(\sum_{i=1}^n x_i^2 \right)^m & j = 2m, \;\; m \in \mathbb{N} \\ (-1)^m \left(\sum_{i=1}^n x_i^2 \right)^m \mathbf{x} & j = 2m+1, \;\; m \in \mathbb{N}, \end{cases}$$

then one gets

$$\frac{\partial \mathbf{x}^j}{\partial x_i} = \begin{cases} -2m x_i \mathbf{x}^{2m-2} & j = 2m, \;\; m \in \mathbb{N} \\ -2m x_i \mathbf{x}^{2m-1} + \mathbf{x}^{2m} e_i & j = 2m+1, \;\; m \in \mathbb{N}, \end{cases}$$

obtaining for $\|\mathbf{x}\| \leq r$:

$$\begin{aligned}\|f_i(\mathbf{x})\| &\leq \sum_{|\mathbf{m}|=1}^{+\infty} \sum_{j=0}^{k-1} \left(j + \frac{\mathbf{m}!}{(\mathbf{m}-\tau(i))!} \right) r^{|\mathbf{m}|+j-1} \|a_{\mathbf{m},j}\| \\ &\leq \sum_{|\mathbf{m}|=1}^{+\infty} \sum_{j=0}^{k-1} (j + |\mathbf{m}|)\, r^{|\mathbf{m}|+j-1} \|a_{\mathbf{m},j}\|. \qquad (3.94)\end{aligned}$$

In what follows μ_i is the maximum term of f_i and, similarly, ν_i are the central indices of f_i. If

$$\mu(r) = \|a_{\mathbf{m}^*,j^*}\| r^{|\mathbf{m}^*|+j^*},$$

then $(\mathbf{m}^*, j^*) = \nu(r)$ and concluding that

$$\mu_i(r) \leq \|a_{\nu(r)}\| r^{|\nu(r)|-1} |\nu(r)| = \mu(r) \frac{1}{r} |\nu(r)|. \qquad (3.95)$$

In view of $|\nu(r)| = |\nu_i(r)| + 1$ the following is obtained

$$\limsup_{r\to\infty} \frac{\log^+(|\nu_i(r)|+1)}{\log(r)} = \limsup_{r\to\infty} \frac{\log^+ |\nu(r)|}{\log(r)} =: \rho_2.$$

Further, for $\varepsilon > 0$, yields

$$|\nu_i(r)| \leq |\nu_i(r)| + 1 \leq r^{\rho_2+\varepsilon}.$$

Applying the same arguments as in the proof of Proposition 3.11, using Theorem 3.7 as well as inequality (3.95), for $\varepsilon_1 > 0$ follows

$$\mathcal{M}_i(r) \leq \mu(r) r^{n(\rho_2+\varepsilon)+\varepsilon_1-1},$$

because of $|\nu(r)| < r^{\rho_2+\varepsilon}$. Finally, putting $\delta_1 := \varepsilon_1 + n\varepsilon$ this leads to

$$\log \mathcal{M}_i(r) \leq \log \mu(r) + [n\rho_2 + \delta_1 - 1] \log r.$$

This permits to conclude that

$$\limsup_{r\to\infty} \frac{\log \mathcal{M}_i(r)}{\log \mu(r)} \leq \limsup_{r\to\infty} \left(1 + \left(n\rho_2 + \delta_1 - 1 \right) \frac{\log r}{\log \mu(r)} \right) = 1.$$

□

Remark 3.5 *In the complex case, under the same hypothesis as given in Theorem 3.8 and in view of*

$$\mu(r) \leq M(r) \leq \mu(r)r^k \quad (k \in \mathbb{N}),$$

one obtains

$$\log M'(r) \sim \log M(r)$$

where $M'(r) = \max_{\|z\|=r}\{\|g'(z)\|\}$. *For more details, see [37, p.38].*

3.6 Monogenic functions mapping the interior to the exterior of the unit ball

In this section an estimate for a 1-monogenic Clifford valued function f, which maps the interior of the unit ball $B(0,1)$ to the complement of the closed unit ball is obtained. Applying Schwarz's lemma it is possible to estimate the norm of $\overline{D}f$ by the norm of f in the origin. Moreover, an estimate of f is obtained in the whole unit ball which describes the growth behavior of the function f. For this a generalization of lemma 6.5 of [34] is used, following an approach similar to the one given in [41].

Theorem 3.9 *Let* $\Omega \subset \mathcal{A}_{n+1}$ *be a domain that contains the unit ball* $B(0,1)$. *Let* f *be a left 1-monogenic function with the following Taylor series representation*

$$f(z) = \sum_{s=0}^{+\infty} \sum_{|\mathbf{m}|=s} V_{\mathbf{m}}(z) a_{\mathbf{m}}$$

such that

$$\limsup_{s\to\infty} \sqrt[s]{(s+1)^n d_s} = \limsup_{s\to\infty} \sqrt[s]{d_s} \leq 1, \tag{3.96}$$

where $d_s = \max_{|\mathbf{m}|=s}\{\|a_{\mathbf{m}}\|\}$. *If* $\|f(z)\| > 1$ *for* $\|z\| < 1$, *then*

(i) $\|\overline{D}f(0)\| \leq 4\|f(0)\| \log(\|f(0)\|)$ *and*

(ii) if f is a Clifford group valued function, $\|w_0\| < 1$ and $\|w\| < 1$, then

$$\|f(w)\| < (2^n \|f(w_0)\|)^{\frac{1+k}{1-k}},$$

where $k = \dfrac{\|w - w_0\|}{\|1 - \overline{w_0}w\|}$.

Proof. Let us prove (i). Since f is left 1-monogenic in Ω, then f has the following series expansion in $B(0,1)$

$$f(z) = a_{\mathbf{0}} + \sum_{i=1}^{n} z_i a_{\tau(i)} + \sum_{s=2}^{+\infty} \sum_{|\mathbf{m}|=s} V_{\mathbf{m}}(z) a_{\mathbf{m}}, \tag{3.97}$$

where $a_{\mathbf{m}} := \dfrac{1}{\mathbf{m}!} \dfrac{\partial^s f}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_n^{m_n}}(0)$.

For $t \in \mathbb{C}$ we formally define a power series g whose coefficients are the moduli of the Taylor coefficients of the function f, i.e.,

$$g(t) = \|a_{\mathbf{0}}\| + \sum_{i=1}^{n} \left\|a_{\tau(i)}\right\| t + \sum_{s=2}^{+\infty} \sum_{|\mathbf{m}|=s} \|a_{\mathbf{m}}\| \frac{t^s}{s!}. \tag{3.98}$$

As a consequence of condition (3.96) the power series g converges in $|t| < 1$.

Since $a_{\mathbf{0}} := f(0)$ and $\|f(0)\| \neq 0$, taking an appropriate branch of the complex logarithm of g, we define h as

$$h(t) := \log(g(t)) = u(t) + iv(t). \tag{3.99}$$

The function h is holomorphic and has the following Taylor series expansion

$$h(t) = h_0 + h_1 t + h_2 \frac{t^2}{2!} + ... \tag{3.100}$$

where $h_0 = \log(g(0)) = \log(\|f(0)\|) > 0$ since $\|f(0)\| > 1$. Now consider the complex valued function ψ defined as

$$\psi(t) := \frac{h(t) - h_0}{h(t) + h_0}. \tag{3.101}$$

The function ψ has the following properties $|\psi(t)|^2 = \dfrac{(u(t) - h_0)^2 + v^2(t)}{(u(t) + h_0)^2 + v^2(t)} < 1$ for $|t| < 1$ and $\psi(0) = 0$.

Since ψ satisfies the hypothesis of Schwarz's lemma (see [3, pp.135]), the following is obtained

$$|\psi'(0)| = \left|\frac{h_1}{2h_0}\right| \leq 1, \tag{3.102}$$

where $h_0 = \log(\|f(0)\|)$ and $h_1 = \dfrac{\sum_{i=1}^{n} \|a_{\tau(i)}\|}{\|f(0)\|}$ for $a_{\tau(i)} = \dfrac{\partial f}{\partial x_i}(0)$.

Since $\left(D + \overline{D}\right) f(z) \mid_{z=0} = 2\frac{\partial f}{\partial x_0}(0)$ and f is left 1-monogenic, we have $2\frac{\partial f}{\partial x_0}(0) = \overline{D}f(0)$. From (3.102) we infer that

$$\left\|\frac{1}{2}\overline{D}f(0)\right\| = \left\|\frac{\partial f}{\partial x_0}(0)\right\| \leq \sum_{i=1}^{n} \left\|\frac{\partial f}{\partial x_i}(0)\right\| \leq 2\|f(0)\| \log(\|f(0)\|). \tag{3.103}$$

To prove (ii), we use the same functions defined in (3.98), (3.100) and (3.101) as well as Schwarz's lemma. Let $z \in \mathcal{A}_{n+1}$ such that $\|z\| = |t| = k$. Then $|\psi(t)| \leq k$ and by definition of ψ it follows

$$\left|\frac{h(t) - h_0}{h(t) + h_0}\right| \leq k. \tag{3.104}$$

This implies that

$$|h(t)| - |h_0| \leq |h(t) - h_0| \leq k\ (|h(t)| + |h_0|), \tag{3.105}$$

with $h_0 = h(0)$ this results in

$$|h(t)| \leq |h(0)|\frac{1+k}{1-k}. \tag{3.106}$$

Furthermore, one has

$$|g(t)| = |e^{h(t)}| \leq e^{|h(t)|} = \|f(0)\|^{\frac{1+k}{1-k}}. \tag{3.107}$$

Since the coefficients of the series expansion of g are non-negative, one gets

$$\max_{\theta\in[0,2\pi[} \left|g\left(|t|e^{i\theta}\right)\right| = g\left(|t|\right). \tag{3.108}$$

Therefore, using (3.97), (3.98), (3.107) and (3.108) the following inequality is obtained

$$\|f(z)\| \leq g\left(|t|\right) = \max_{\theta\in[0,2\pi[} |g\left(t\right)| \leq \|f(0)\|^{\frac{1+k}{1-k}}. \tag{3.109}$$

Define the function $F(z)$ by

$$F(z) := 2^n\overline{(1 + \overline{w_0}z)}\|1 + \overline{w_0}z\|^{-(n+1)}f((z + w_0)(1 + \overline{w_0}z)^{-1}). \tag{3.110}$$

Then, from Theorem 1.3 it follows that the function F is again left monogenic in $B(0,1)$, since the Möbius transformation that we applied is an endomorphism of the $B(0,1)$. Moreover,

$$\|F(z)\| = 2^n \|1+\overline{w_0}z\|^{-n} \|f((z+w_0)(1+\overline{w_0}z)^{-1})\| > 1 \tag{3.111}$$

for $\|z\|<1$ and $\|w_0\|<1$, since also F maps the interior to the exterior of $B(0,1)$. The previous inequality holds in view of

$$\|f((z+w_0)(1+\overline{w_0}z)^{-1})\| > 1$$

and $\|1+\overline{w_0}z\| < 2$.

By applying (3.109) we obtain the inequality

$$\|F(z)\| \leq \|F(0)\|^{\frac{1+k}{1-k}} = \left(2^n \|f(w_0)\|\right)^{\frac{1+k}{1-k}}. \tag{3.112}$$

Therefore, from (3.111) it follows that

$$\|F(z)\| \geq \left\|f((z+w_0)(1+\overline{w_0}z)^{-1})\right\| \tag{3.113}$$

and together with (3.112) we obtain

$$\|f(w)\| \leq \left(2^n \|f(w_0)\|\right)^{\frac{1+k}{1-k}},$$

where

$$w = (z+w_0)(1+\overline{w_0}z)^{-1}.$$

Consequently,

$$z = (w-w_0^*)(1-\overline{w_0^*}w)^{-1} = (w-w_0)(1-\overline{w_0}w)^{-1}$$

since w_0 is a paravector $w_0^* = w_0$. Moreover, one has $\dfrac{\|w-w_0\|}{\|1-\overline{w_0}w\|} = \|z\| = k$. □

Chapter 4

Asymptotic growth of polymonogenic function

In this chapter we analyze the behavior of growth of entire polymonogenic Clifford algebra valued functions.

In the first part we establish some preparatory results which will be used in the following sections. Some estimates between the maximum modulus, the maximum term and the norm of the central index are obtained. In the last two sections we establish a relation on the asymptotic behavior between solutions of iterated generalized Cauchy-Riemann and iterated Euler operators. We also obtain a relation on the asymptotic behavior between solutions of iterated Dirac and polynomials in the Euler operator and in the Gamma operator.

4.1 Some fundamental results

In this section we denote the standard Euler operator in $\mathbb{R}^n$ (and in $\mathbb{R}^{n+1}$) by

$$E := \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \quad (E := \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}).$$

The Gamma operator in $\mathbb{R}^n$ (see e.g. [21]) is given by

$$\Gamma := \sum_{i,j=1,i<j}^{n} (x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}) e_i e_j. \tag{4.1}$$

Notice that

$$\Gamma = \mathbf{x}\mathcal{D} + E,$$

when working in the vector formalism. These operators generalize the real part and imaginary part of the complex differential operator $z\frac{d}{dz}$ to higher dimensional equations.

Under this point of view it seems natural to regard polynomial expressions in E and Γ as generalizations of iterations of the $z\frac{d}{dz}$ operator.

To establish a relation between the asymptotic behavior of the maximum term of a k-monogenic function and that of their polynomial expressions in terms of E and Γ it turns out to be convenient to prove first some preparatory propositions.

For simplicity we use the notation $\nu := \nu(r,g)$ for the central index of an entire k-monogenic function g if no ambiguity occurs. Let us assume that $\nu = (\mathbf{m}^*, j^*)$.

Proposition 4.1 *Let g be a transcendental entire k-monogenic function in $\mathbb{R}^n$ (in $\mathbb{R}^{n+1}$). Then*

$$\mathcal{M}(r,g) \leq \mu(r)L(r), \tag{4.2}$$

where

$$\begin{aligned} L(r) \quad &:= \quad \Bigg[A(n,\mathbf{m}^*) + \sum_{|\mathbf{m}|=|\mathbf{m}^*|}\left(\sum_{j=0}^{j^*-1}\frac{\|a_{\mathbf{m},j}\|}{\|a_{\mathbf{m}^*,j^*}\|}r^{j-j^*} + \sum_{j=j^*+1}^{k-1}\frac{\|a_{\mathbf{m},j}\|}{\|a_{\mathbf{m}^*,j^*}\|}r^{j-j^*}\right) \\ &+ \sum_{|\mathbf{m}|=0}^{|\mathbf{m}^*|-1}\sum_{j=0}^{k-1}\frac{\|a_{\mathbf{m},j}\|}{\|a_{\mathbf{m}^*,j^*}\|}r^{|\mathbf{m}|+j-|\nu|} + \sum_{|\mathbf{m}|=|\mathbf{m}^*|+1}^{+\infty}\sum_{j=0}^{k-1}\frac{\|a_{\mathbf{m},j}\|}{\|a_{\mathbf{m}^*,j^*}\|}r^{|\mathbf{m}|+j-|\nu|}\Bigg], \end{aligned}$$

and $A(n,\mathbf{m}^*) := \frac{(n-2+|\mathbf{m}^*|)!}{(n-2)!|\mathbf{m}^*|!}$ (resp. $A(n,\mathbf{m}^*) := \frac{(n-1+|\mathbf{m}^*|)!}{(n-1)!|\mathbf{m}^*|!}$).

Proof. Since g is a transcendental entire k-monogenic function it has a Taylor-Almansi expansion of the form

$$g(\mathbf{x}) = \sum_{|\mathbf{m}|=0}^{+\infty}\sum_{j=0}^{k-1}\mathbf{x}^j V_{\mathbf{m}}(\mathbf{x})a_{\mathbf{m},j},$$

(see also (3.51)). For $\|\mathbf{x}\| \leq r$ we have

$$\|g(\mathbf{x})\| \leq \sum_{|\mathbf{m}|=0}^{+\infty}\sum_{j=0}^{k-1} r^{j+|\mathbf{m}|}\|a_{\mathbf{m},j}\|.$$

By the same reason, there exists $(\mathbf{m}^*, j^*) \in \mathbb{N}_0^n$ (resp. $(\mathbf{m}^*, j^*) \in \mathbb{N}_0^{n+1}$) such that $a_{\mathbf{m}^*,j^*} \neq 0$. Then

$$\begin{aligned}\|g(\mathbf{x})\| \leq \|a_{\mathbf{m}^*,j^*}\| r^{|\mathbf{m}^*|+j^*} \Bigg[& \sum_{|\mathbf{m}|=|\mathbf{m}^*|} 1 + \sum_{|\mathbf{m}|=|\mathbf{m}^*|} \sum_{j=0}^{j^*-1} \frac{\|a_{\mathbf{m},j}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{j-j^*} \\ & + \sum_{|\mathbf{m}|=|\mathbf{m}^*|} \sum_{j=j^*+1}^{k-1} \frac{\|a_{\mathbf{m},j}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{j-j^*} + \sum_{|\mathbf{m}|=0}^{|\mathbf{m}^*|-1} \sum_{j=0}^{k-1} \frac{\|a_{\mathbf{m},j}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{|\mathbf{m}|+j-|\mathbf{m}^*|-j^*} \\ & + \sum_{|\mathbf{m}|=|\mathbf{m}^*|+1}^{+\infty} \sum_{j=0}^{k-1} \frac{\|a_{\mathbf{m},j}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{|\mathbf{m}|+j-|\mathbf{m}^*|-j^*} \Bigg].\end{aligned}$$

In view of $\sum_{|\mathbf{m}|=|\mathbf{m}^*|} 1 \leq A(n, \mathbf{m}^*)$, and if we take in particular the central index as $(\mathbf{m}^*, j^*) = \nu$ then the maximum term is given by $\mu(r) = \|a_{\mathbf{m}^*,j^*}\| r^{|\mathbf{m}^*|+j^*}$, one arrives at the stated result. □

In order to proceed we recall the notion of logarithmic measure (see e.g. [37, 70]), which shall be used later on.

Definition 4.1 *We denote a set F to be of finite logarithmic measure if*

$$\int_F \frac{dr}{r} < \infty.$$

The following proposition provides an estimate for the function $L(r)$ of (4.2).

Proposition 4.2 *Let g be a transcendental entire k-monogenic function in $\mathbb{R}^n$, with the following Taylor-Almansi series expansion*

$$g(\mathbf{x}) = \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} \mathbf{x}^j V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},j}.$$

Let $(P_k)_{k\in\mathbb{N}}$ be a sequence of real positive numbers satisfying

$$1 < P_1 < P_2 \cdots \qquad \textit{and} \qquad \lim_{k\to\infty} P_k = P < \infty.$$

Then there exists a real $r > 0$ such that

$$\frac{\|a_{\mathbf{m}^*-\mathbf{m},j^*-j}\|r^{|\nu|-|\mathbf{m}|-j}}{\|a_\nu\|r^{|\nu|}} \leq \frac{\prod_{i=|\nu|-|\mathbf{m}|-j+1}^{|\nu|} P_i}{P_{|\nu|}^{|\mathbf{m}|+j}}, \quad |\mathbf{m}|+j = 1,\cdots,|\nu| \tag{4.3}$$

$$\frac{\|a_{\mathbf{m}^*+\mathbf{m},j^*+j}\|r^{|\nu|+|\mathbf{m}|+j}}{\|a_\nu\|r^{|\nu|}} < \frac{P_{|\nu|}^{|\mathbf{m}|+j}}{\prod_{i=|\nu|+1}^{|\nu|+|\mathbf{m}|+j} P_i}, \qquad |\mathbf{m}|+j = 1,2,3,\ldots \tag{4.4}$$

where $\nu = (\mathbf{m}^*, j^*)$ and $\mu(r) = \|a_\nu\|r^{|\nu|}$ are respectively the central index and maximum term of g.

Proof. Consider

$$H(\mathbf{x}) = \sum_{|\mathbf{m}|=0}^{+\infty}\sum_{j=0}^{k-1}\mathbf{x}^j V_{\mathbf{m}}(\mathbf{x})a_{\mathbf{m},j}\left(\prod_{i=1}^{|\mathbf{m}|+j} P_i\right) := \sum_{|\mathbf{m}|=0}^{+\infty}\sum_{j=0}^{k-1}\mathbf{x}^j V_{\mathbf{m}}(\mathbf{x})b_{\mathbf{m},j}.$$

In view of $P_i < P$, we infer:

$$\begin{aligned}\|H(\mathbf{x})\| &\leq \sum_{|\mathbf{m}|=0}^{+\infty}\sum_{j=0}^{k-1}\|\mathbf{x}\|^{|\mathbf{m}|+j}\|a_{\mathbf{m},j}\|\left(\prod_{i=1}^{|\mathbf{m}|+j} P_i\right)\\ &\leq \sum_{|\mathbf{m}|=0}^{+\infty}\sum_{j=0}^{k-1}\|\mathbf{x}\|^{|\mathbf{m}|+j}\|a_{\mathbf{m},j}\|P^{|\mathbf{m}|+j}\\ &= \sum_{|\mathbf{m}|=0}^{+\infty}\sum_{j=0}^{k-1}(\|\mathbf{x}\|P)^{|\mathbf{m}|+j}\|a_{\mathbf{m},j}\|.\end{aligned} \tag{4.5}$$

For $\varrho := \|\mathbf{x}\| > 0$ we obtain

$$\|b_{\mathbf{m},j}\|\varrho^{|\mathbf{m}|+j} \leq \mu(\varrho, H) = \|b_{\mathbf{m}^*,j^*}\|\varrho^{|\mathbf{m}^*|+j^*}, \qquad |\mathbf{m}|+j = 0,1,...,|\mathbf{m}^*|+j^*$$

$$\|b_{\mathbf{m},j}\|\varrho^{|\mathbf{m}|+j} < \mu(\varrho, H) = \|b_{\mathbf{m}^*,j^*}\|\varrho^{|\mathbf{m}^*|+j^*}, \qquad |\mathbf{m}|+j > |\mathbf{m}^*|+j^*.$$

Since $\|b_{\mathbf{m},j}\| = \|a_{\mathbf{m},j}\|\left(\prod_{i=1}^{|\mathbf{m}|+j} P_i\right)$, we obtain

$$\|a_{\mathbf{m},j}\|\left(\prod_{i=1}^{|\mathbf{m}|+j} P_i\right)\varrho^{|\mathbf{m}|+j} \leq \|a_{\mathbf{m}^*,j^*}\|\left(\prod_{i=1}^{|\mathbf{m}^*|+j^*} P_i\right)\varrho^{|\mathbf{m}^*|+j^*}, \quad |\mathbf{m}|+j = 0,1,...,|\mathbf{m}^*|+j^* \tag{4.6}$$

and also

$$\|a_{\mathbf{m},j}\|\left(\prod_{i=1}^{|\mathbf{m}|+j} P_i\right)\varrho^{|\mathbf{m}|+j} < \|a_{\mathbf{m}^*,j^*}\|\left(\prod_{i=1}^{|\mathbf{m}^*|+j^*} P_i\right)\varrho^{|\mathbf{m}^*|+j^*}, \quad |\mathbf{m}|+j > |\mathbf{m}^*|+j^*. \quad (4.7)$$

Furthermore, for (4.6) and (4.7) we have

$$\frac{\|a_{\mathbf{m},j}\|\varrho^{|\mathbf{m}|+j}}{\|a_{\mathbf{m}^*,j^*}\|\varrho^{|\mathbf{m}^*|+j^*}} \leq \prod_{i=1+|\mathbf{m}|+j}^{|\mathbf{m}^*|+j^*} P_i, \qquad |\mathbf{m}|+j = 0,1,...,|\mathbf{m}^*|+j^*$$

$$\frac{\|a_{\mathbf{m},j}\|\varrho^{|\mathbf{m}|+j}}{\|a_{\mathbf{m}^*,j^*}\|\varrho^{|\mathbf{m}^*|+j^*}} < \frac{1}{\prod\limits_{i=1+|\mathbf{m}^*|+j^*}^{|\mathbf{m}|+j} P_i}, \qquad |\mathbf{m}|+j > |\mathbf{m}^*|+j^*.$$

Taking $\mathbf{m} := \mathbf{m}^* - \mathbf{l}$, $j := j^* - l$ for the first inequality and $\mathbf{m} := \mathbf{m}^* + \mathbf{l}$, $j := j^* + l$ for the second inequality ($\mathbf{l} \in \mathbb{N}_0^n$ and $l \in \mathbb{N}$) we have for $\nu := \nu(\varrho, H)$

$$\begin{aligned} &\frac{\|a_{\mathbf{m}^*-\mathbf{l},j^*-l}\|\varrho^{|\mathbf{m}^*-\mathbf{l}|+j^*-l}}{\|a_{\mathbf{m}^*,j^*}\|\varrho^{|\mathbf{m}^*|+j^*}} \leq \prod_{i=1}^{|\mathbf{l}|+l} P_{|\nu|-i+1}, \qquad |\mathbf{l}|+l = 0,1,...,|\nu| \\ &\frac{\|a_{\mathbf{m}^*+\mathbf{l},j^*+l}\|\varrho^{|\mathbf{m}^*+\mathbf{l}|+j^*+l}}{\|a_{\mathbf{m}^*,j^*}\|\varrho^{|\mathbf{m}^*|+j^*}} < \frac{1}{\prod\limits_{i=1}^{|\mathbf{l}|+l} P_{|\nu|+i}}, \qquad |\mathbf{l}|+l = 0,1,.... \end{aligned} \quad (4.8)$$

For each $\varrho > 0$ we define $r := \varrho P_{|\nu(\varrho,H)|} > 0$. Using (4.8) we obtain

$$\begin{aligned} &\frac{\|a_{\mathbf{m}^*-\mathbf{l},j^*-l}\|r^{|\mathbf{m}^*-\mathbf{l}|+j^*-l}}{\|a_{\mathbf{m}^*,j^*}\|r^{|\mathbf{m}^*|+j^*}} \leq \frac{\prod\limits_{i=1}^{|\mathbf{l}|+l} P_{|\nu|-i+1}}{P_{|\nu|}^{|\mathbf{l}|+l}} \leq 1, \qquad |\mathbf{l}|+l = 0,1,...,|\nu| \\ &\frac{\|a_{\mathbf{m}^*+\mathbf{l},j^*+l}\|r^{|\mathbf{m}^*+\mathbf{l}|+j^*+l}}{\|a_{\mathbf{m}^*,j^*}\|r^{|\mathbf{m}^*|+j^*}} < \frac{P_{|\nu|}^{|\mathbf{l}|+l}}{\prod\limits_{i=1}^{|\mathbf{l}|+l} P_{|\nu|+i}} < 1, \qquad |\mathbf{l}|+l = 0,1,.... \end{aligned} \quad (4.9)$$

The inequalities (4.9) imply that

$$\begin{aligned} &\|a_{\mathbf{m}^*-\mathbf{l},j^*-l}\|r^{|\mathbf{m}^*-\mathbf{l}|+j^*-l} \leq \|a_{\mathbf{m}^*,j^*}\|r^{|\mathbf{m}^*|+j^*}, \qquad |\mathbf{l}|+l = 0,1,...,|\nu| \\ &\|a_{\mathbf{m}^*+\mathbf{l},j^*+l}\|r^{|\mathbf{m}^*+\mathbf{l}|+j^*+l} < \|a_{\mathbf{m}^*,j^*}\|r^{|\mathbf{m}^*|+j^*}, \qquad |\mathbf{l}|+l = 0,1,... \end{aligned} \quad (4.10)$$

which means that $\|a_{\mathbf{m}^*,j^*}\|r^{|\mathbf{m}^*|+j^*}$ is the maximum term of g and $\nu := (\mathbf{m}^*, j^*)$ its central index.

The inequalities (4.3) and (4.4) are valid for each of those $r := \varrho P_{|\nu(\varrho,H)|}$, for $\varrho > 0$.

Next we prove that (4.3) and (4.4) are valid for all r except for a set of finite logarithmic measure.

Let $\varrho_1, \varrho_2, ...$ be the discontinuity points of $|\nu(\varrho, H)|$. Consider $|\nu(\varrho, H)|$ to be the constant ν_m for $\varrho_m \leq \varrho \leq \varrho_{m+1}$.

The inequalities (4.8) are satisfied as well as (4.9) for all r with

$$S_m := \varrho_m P_{\nu_m} \leq r \leq \varrho_{m+1} P_{\nu_m} =: S_m^1. \tag{4.11}$$

For the case where r belongs to the interval $[S_m^1, S_{m+1}[$ it can happen that the inequalities are not true. For that reason we define a set $F := \bigcup\limits_{m=1}^{\infty} [S_m^1, S_{m+1}[$. A calculation gives

$$\begin{aligned}
\int_F \frac{dr}{r} &= \sum_{m=1}^{\infty} \log\left(\frac{S_{m+1}}{S_m^1}\right) \\
&= \sum_{m=1}^{\infty} \log\left(\frac{\varrho_{m+1} P_{\nu_{m+1}}}{\varrho_{m+1} P_{\nu_m}}\right) \\
&= \lim_{k\to\infty} \log(P_{\nu_{k+1}}) - \log(P_{\nu_1}) \\
&= \log(P) - \log(P_{\nu_1}) < \infty,
\end{aligned}$$

and we may conclude that F has finite logarithmic measure. □

Remark 4.1 *The proof is analogous to the classical one, see Proposition 21.1 from [37, pp.189]. The auxiliary function $H(z)$, given in [37] is obtained by taking $k = 1$.*

The proof can directly be adapted to the case of solutions of the iterated generalized Cauchy-Riemann equation.

For our purpose we also need an asymptotic lower estimate of $|\nu(r)|$.

Proposition 4.3 *Let*

$$L^*(r) \quad := \quad \sum_{\substack{|\mathbf{m}|=0 \\ (\mathbf{m},j)\,\neq\,(\mathbf{m}^*,j^*)}}^{+\infty} \sum_{j=0}^{k-1} \frac{\|a_{\mathbf{m}^*-\mathbf{m},j^*-j}\| r^{|\nu|-|\mathbf{m}|-j}}{\|a_\nu\| r^{|\nu|}},$$

then for $\varepsilon > 0$

$$L^*(r) < |\nu(r)|^{\frac{1}{2}+\varepsilon}, \tag{4.12}$$

where $r \notin F$ *and* F *is a set of finite logarithmic measure.*

Proof. We consider

$$\begin{aligned} L(r) \quad := \quad & A(n,\mathbf{m}^*) + \sum_{|\mathbf{m}|=|\mathbf{m}^*|} \left(\sum_{l=0}^{j^*-1} \frac{\|a_{\mathbf{m},l}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{l-j^*} + \sum_{l=j^*+1}^{k-1} \frac{\|a_{\mathbf{m},l}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{l-j^*} \right) \\ & + \sum_{|\mathbf{w}|=0}^{|\mathbf{m}^*|-1} \sum_{l=0}^{k-1} \frac{\|a_{\mathbf{w},l}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{|\mathbf{w}|+l-|\nu|} + \sum_{|\mathbf{w}|=|\mathbf{m}^*|+1}^{+\infty} \sum_{l=0}^{k-1} \frac{\|a_{\mathbf{w},l}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{|\mathbf{w}|+l-|\nu|}. \end{aligned} \tag{4.13}$$

Now we take a suitable substitutions of $\mathbf{w}, l$ and apply inequalities (4.3) and (4.4).

Substituting $l := j^* + j$ in the following sum, one obtains

$$\begin{aligned} \sum_{|\mathbf{m}|=|\mathbf{m}^*|} \sum_{l=0}^{j^*-1} \frac{\|a_{\mathbf{m},l}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{l-j^*} \quad & \leq \quad \sum_{|\mathbf{m}|=|\mathbf{m}^*|} \sum_{j=1}^{j^*} \frac{\|a_{\mathbf{m},j^*-j}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{-(j-|\mathbf{m}|)} \\ & \leq \quad \sum_{|\mathbf{m}|=|\mathbf{m}^*|} \sum_{j=1}^{|\mathbf{m}|} \frac{\prod_{i=|\nu|-j+|\mathbf{m}|+1}^{|\nu|} P_i}{P_{|\nu|}^{j-|\mathbf{m}|}} \\ & + \quad \sum_{|\mathbf{m}|=|\mathbf{m}^*|} \sum_{j=|\mathbf{m}|+1}^{j^*} \frac{P_{|\nu|}^{j-|\mathbf{m}|}}{\prod_{i=|\nu|-j+|\mathbf{m}|+1}^{|\nu|} P_i}. \end{aligned} \tag{4.14}$$

Furthermore, after substituting $l := j^* - j$, one gets

$$\begin{aligned} \sum_{|\mathbf{m}|=|\mathbf{m}^*|} \sum_{l=j^*+1}^{k-1} \frac{\|a_{\mathbf{m},l}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{l-j^*} \quad & = \quad \sum_{|\mathbf{m}|=|\mathbf{m}^*|} \sum_{j=1}^{k-1-j^*} \frac{\|a_{\mathbf{m},j^*+j}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{j+|\mathbf{m}|} \\ & < \quad \sum_{|\mathbf{m}|=|\mathbf{m}^*|} \sum_{j=1}^{k-1-j^*} \frac{P_{|\nu|}^{j+|\mathbf{m}|}}{\prod_{i=|\nu|+1}^{|\nu|+j+|\mathbf{m}|} P_i}. \end{aligned} \tag{4.15}$$

If we replace $\mathbf{w} := \mathbf{m}^* - \mathbf{m}$ and $l := j^* - j$, then

$$\sum_{|\mathbf{w}|=0}^{|\mathbf{m}^*|-1} \sum_{l=0}^{j^*} \frac{\|a_{\mathbf{w},l}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{|\mathbf{w}|+l-|\nu|} = \sum_{|\mathbf{m}|=1}^{|\mathbf{m}^*|} \sum_{j=0}^{j^*} \frac{\|a_{\mathbf{m}^*-\mathbf{m},j^*-j}\|}{\|a_{\mathbf{m}^*,j^*}\|} \frac{r^{|\nu|-(|\mathbf{m}|+j)}}{r^{|\nu|}}$$
$$\leq \sum_{|\mathbf{m}|=1}^{|\mathbf{m}^*|} \sum_{j=0}^{j^*} \frac{\prod_{i=|\nu|-(|\mathbf{m}|+j)+1}^{|\nu|} P_i}{P_{|\nu|}^{|\mathbf{m}|+j}}. \tag{4.16}$$

For the next sum we replace $\mathbf{w} := \mathbf{m}^* - \mathbf{m}$ and $l := j^* - j$

$$\sum_{|\mathbf{w}|=|\mathbf{m}^*|-1}^{+\infty} \sum_{l=j^*}^{k-1} \frac{\|a_{\mathbf{w},l}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{|\mathbf{w}|+l-|\nu|} = \sum_{|\mathbf{m}|=1}^{+\infty} \sum_{j=0}^{k-1-j^*} \frac{\|a_{\mathbf{m}^*+\mathbf{m},j^*+j}\|}{\|a_{\mathbf{m}^*,j^*}\|} \frac{r^{|\nu|+(|\mathbf{m}|+j)}}{r^{|\nu|}}$$
$$< \sum_{|\mathbf{m}|=1}^{+\infty} \sum_{j=0}^{k-1-j^*} \frac{P_{|\nu|}^{|\mathbf{m}|+j}}{\prod_{i=|\nu|+1}^{|\nu|+|\mathbf{m}|+j} P_i}. \tag{4.17}$$

For the other terms of (4.13) we substitute $\mathbf{w} := \mathbf{m}^* - \mathbf{m}$, $l := j^* + j$ and $\mathbf{w} := \mathbf{m}^* + \mathbf{m}$, $l := j^* - j$, respectively

$$\sum_{|\mathbf{w}|=0}^{|\mathbf{m}^*|-1} \sum_{l=j^*}^{k-1} \frac{\|a_{\mathbf{w},l}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{|\mathbf{w}|+l-|\nu|} = \sum_{|\mathbf{m}|=1}^{|\mathbf{m}^*|} \sum_{j=0}^{k-1-j^*} \frac{\|a_{\mathbf{m}^*-\mathbf{m},j^*+j}\|}{\|a_{\mathbf{m}^*,j^*}\|} \frac{r^{|\nu|-(|\mathbf{m}|-j)}}{r^{|\nu|}}$$
$$< \sum_{|\mathbf{m}|=1}^{|\mathbf{m}^*|} \sum_{j=0}^{\min\{k-1-j^*,|\mathbf{m}|\}} \frac{\prod_{i=|\nu|-|\mathbf{m}|+j+1}^{|\nu|} P_i}{P_{|\nu|}^{|\mathbf{m}|-j}}$$
$$+ \sum_{|\mathbf{m}|=1}^{|\mathbf{m}^*|} \sum_{j=\min\{k-1-j^*,|\mathbf{m}|\}+1}^{k-1-j^*} \frac{P_{|\nu|}^{|\mathbf{m}|-j}}{\prod_{i=|\nu|+1}^{|\nu|+|\mathbf{m}|-j} P_i} \tag{4.18}$$

and also

$$\sum_{|\mathbf{w}|=|\mathbf{m}^*|+1}^{+\infty} \sum_{l=0}^{j^*} \frac{\|a_{\mathbf{w},l}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{|\mathbf{w}|+l-|\nu|} = \sum_{|\mathbf{m}|=1}^{+\infty} \sum_{j=0}^{j^*} \frac{\|a_{\mathbf{m}^*+\mathbf{m},j^*-j}\|}{\|a_{\mathbf{m}^*,j^*}\|} \frac{r^{|\nu|+|\mathbf{m}|-j}}{r^{|\nu|}}$$
$$< \sum_{|\mathbf{m}|=1}^{+\infty} \sum_{j=0}^{\min\{j^*,|\mathbf{m}|\}} \frac{P_{|\nu|}^{|\mathbf{m}|-j}}{\prod_{i=|\nu|+1}^{|\nu|+|\mathbf{m}|-j} P_i}$$
$$+ \sum_{|\mathbf{m}|=1}^{+\infty} \sum_{j=\min\{j^*,|\mathbf{m}|\}+1}^{j^*} \frac{\prod_{i=|\nu|-|\mathbf{m}|+j+1}^{|\nu|} P_i}{P_{|\nu|}^{|\mathbf{m}|-j}}. \tag{4.19}$$

To proceed we consider

$$\log(P_1) := 1, \qquad \log(P_i) := 1 + \frac{1}{1^\alpha} + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + ... + \frac{1}{(i-1)^\alpha} \tag{4.20}$$

for $i = 2, 3, \ldots$ and $\alpha = 1 + \delta$, $\delta > 0$. Taking $m \in \mathbb{N}$ and $\log(P_i)$ $(i \in \mathbb{N})$ as defined in (4.20), we get the following:

$$\begin{aligned}
\log\left(\frac{\prod_{i=1}^{m-1} P_{|\nu|-i}}{P_{|\nu|}^{m-1}}\right) &= \sum_{s=1}^{m-1} \log(P_{|\nu|-s}) - (m-1)\log(P_{|\nu|}) \\
&= \sum_{s=1}^{m-1}\left(1 + \sum_{\kappa=1}^{|\nu|-s-1} \frac{1}{\kappa^\alpha}\right) - (m-1)\left(1 + \sum_{\kappa=1}^{|\nu|-1} \frac{1}{\kappa^\alpha}\right) \\
&= -\sum_{s=1}^{m-1}(m-s)\frac{1}{(|\nu|-s)^\alpha} \\
&< \frac{-m(m-1)}{2(|\nu|+m-1)^\alpha},
\end{aligned} \tag{4.21}$$

and also

$$\begin{aligned}
\log\left(\frac{P_{|\nu|}^m}{\prod_{i=1}^{m} P_{|\nu|+i}}\right) &= m\log(P_{|\nu|}) - \sum_{s=1}^{m} \log(P_{|\nu|+s}) \\
&= m\left(1 + \sum_{\kappa=1}^{|\nu|-1} \frac{1}{\kappa^\alpha}\right) - \sum_{s=1}^{m}\left(1 + \sum_{\kappa=1}^{|\nu|+s-1} \frac{1}{\kappa^\alpha}\right) \\
&= -\sum_{s=0}^{m-1}(m-s)\frac{1}{(|\nu|+s)^\alpha} \\
&< \frac{-m^2}{2(|\nu|+m)^\alpha}.
\end{aligned} \tag{4.22}$$

Applying the inequalities (4.21) and (4.22) in (4.14)-(4.19), one obtains

$$\sum_{|\mathbf{m}|=|\mathbf{m}^*|}\left(\sum_{l=0}^{j^*-1}\frac{\|a_{\mathbf{m},l}\|}{\|a_{\mathbf{m}^*,j^*}\|}r^{l-j^*}+\sum_{l=j^*+1}^{k-1}\frac{\|a_{\mathbf{m},l}\|}{\|a_{\mathbf{m}^*,j^*}\|}r^{l-j^*}\right)+\sum_{|\mathbf{w}|=0}^{|\mathbf{m}^*|-1}\sum_{l=0}^{k-1}\frac{\|a_{\mathbf{w},l}\|}{\|a_{\mathbf{m}^*,j^*}\|}r^{|\mathbf{w}|+l-|\nu|}$$

$$+\sum_{|\mathbf{w}|=|\mathbf{m}^*|+1}^{+\infty}\sum_{l=0}^{k-1}\frac{\|a_{\mathbf{w},l}\|}{\|a_{\mathbf{m}^*,j^*}\|}r^{|\mathbf{w}|+l-|\nu|}$$

$$\leq \sum_{|\mathbf{m}|=|\mathbf{m}^*|}\sum_{j=1}^{|\mathbf{m}|}e^{\frac{-(j-|\mathbf{m}|)(j-|\mathbf{m}|-1)}{2(|\nu|+j-|\mathbf{m}|-1)^\alpha}}+\sum_{|\mathbf{m}|=|\mathbf{m}^*|}\sum_{j=|\mathbf{m}|+1}^{j^*}e^{\frac{-(j-|\mathbf{m}|)^2}{2(|\nu|+j-|\mathbf{m}|)^\alpha}}+\sum_{|\mathbf{m}|=|\mathbf{m}^*|}\sum_{j=1}^{k-1-j^*}e^{\frac{-(j+|\mathbf{m}|)^2}{2(|\nu|+j+|\mathbf{m}|)^\alpha}}$$

$$+\sum_{|\mathbf{m}|=1}^{|\mathbf{m}^*|}\sum_{j=1}^{j^*}e^{\frac{-(|\mathbf{m}|+j)(|\mathbf{m}|+j-1)}{2(|\nu|+|\mathbf{m}|+j-1)^\alpha}}+\sum_{|\mathbf{m}|=1}^{+\infty}\sum_{j=1}^{k-1-j^*}e^{\frac{-(|\mathbf{m}|+j)^2}{2(|\nu|+|\mathbf{m}|+j)^\alpha}}+\sum_{|\mathbf{m}|=1}^{|\mathbf{m}^*|}\sum_{j=0}^{\min\{k-1-j^*,|\mathbf{m}|\}}e^{\frac{-(|\mathbf{m}|-j)(|\mathbf{m}|-j-1)}{2(|\nu|+|\mathbf{m}|-j-1)^\alpha}}$$

$$+\sum_{|\mathbf{m}|=1}^{|\mathbf{m}^*|}\sum_{j=\min\{k-1-j^*,|\mathbf{m}|\}+1}^{k-1-j^*}e^{\frac{-(|\mathbf{m}|-j)^2}{2(|\nu|+|\mathbf{m}|-j)^\alpha}}+\sum_{|\mathbf{m}|=1}^{+\infty}\sum_{j=0}^{\min\{j^*,|\mathbf{m}|\}}e^{\frac{-(|\mathbf{m}|-j)^2}{2(|\nu|+|\mathbf{m}|-j)^\alpha}}$$

$$+\sum_{|\mathbf{m}|=1}^{+\infty}\sum_{j=\min\{j^*,|\mathbf{m}|\}+1}^{j^*}e^{\frac{-(|\mathbf{m}|-j)(|\mathbf{m}|-j-1)}{2(|\nu|+|\mathbf{m}|-j-1)^\alpha}}.$$

Moreover, one obtains the following upper estimate for $L(r)$.

$$\begin{aligned}L(r) &\leq A(n,\mathbf{m}^*)+5\sum_{s=1}^{+\infty}e^{\frac{-s(s-1)}{2(|\nu|+s)^\alpha}}+5\sum_{s=0}^{+\infty}e^{\frac{-s^2}{2(|\nu|+s-1)^\alpha}}\\ &\leq A(n,\mathbf{m}^*)+10+10\sum_{s=1}^{+\infty}e^{\frac{-s^2}{2(|\nu|+s)^\alpha}}.\end{aligned}\tag{4.23}$$

Furthermore, one has

$$\begin{aligned}L(r) &\leq A(n,\mathbf{m}^*)+10+10\int_0^{+\infty}e^{\frac{-x^2}{2(|\nu|+x)^\alpha}}dx\\ &\leq A(n,\mathbf{m}^*)+10+10\left(\int_0^{|\nu|}e^{\frac{-x^2}{2(2|\nu|)^\alpha}}dx+\int_{|\nu|}^{+\infty}e^{\frac{-x^{2-\alpha}}{2^{1+\alpha}}}dx\right).\end{aligned}$$

Using that $\alpha < 2$, then we obtain by applying the substitution $t^2 = \frac{x^2}{2^{1+\alpha}|\nu|^\alpha}$

$$\begin{aligned}\int_0^{|\nu|} e^{\frac{-x^2}{2(2|\nu|)^\alpha}} dx &\leq \sqrt{2^{1+\alpha}|\nu|^\alpha} \int_0^{+\infty} e^{-t^2} dt \\ &= \sqrt{2^{1+\alpha}|\nu|^\alpha} \frac{\sqrt{\pi}}{2}\end{aligned}$$

and

$$\int_{|\nu|}^{+\infty} e^{\frac{-x^{2-\alpha}}{2^{1+\alpha}}} dx \leq (2^{-1-\alpha})^{\frac{1}{\alpha-2}} \Gamma\left(1 + \frac{1}{2-\alpha}\right),$$

where $\Gamma(\cdot)$ is the Gamma function.

Summarizing, one has

$$L(r) \leq A(n, \mathbf{m}^*) + 10\left(C + \frac{\sqrt{\pi}}{2}\sqrt{2^{1+\alpha}}|\nu|^{\frac{\alpha}{2}}\right)$$

for $A(n, \mathbf{m}^*) = \frac{(n-1+|\mathbf{m}^*|)!}{(n-1)!(|\mathbf{m}^*|)}$ and $C := \Gamma\left(1 + \frac{1}{2-\alpha}\right)\left(2^{-(1+\alpha)}\right)^{\frac{1}{\alpha-2}}$.

Therefore,

$$L^*(r) \leq C_0|\nu|^{\frac{\alpha}{2}}$$

for C_0 a positive constant. Taking $\alpha := 1 + \delta$ for $\delta < 2\varepsilon$, $\varepsilon > 0$ we arrive at the desired inequality. □

One also obtains the following asymptotic upper bound estimate of $|\nu(r)|$.

Proposition 4.4 *For a given $\varepsilon > 0$ we have*

$$|\nu(r)| < (\log \mu(r))^{1+\varepsilon}, \tag{4.24}$$

for $r \notin F$, where F denotes a set of finite logarithmic measure.

Proof. In view of Proposition 4.2 and taking $\nu := (\mathbf{m}^*, j^*)$, we obtain

$$\frac{\|a_{\mathbf{m}^*-\mathbf{m},j^*-j}\|r^{|\nu|-|\mathbf{m}|-j}}{\|a_\nu\|r^{|\nu|}} \leq \frac{\prod_{i=|\nu|-|\mathbf{m}|-j+1}^{|\nu|} P_i}{P_{|\nu|}^{|\mathbf{m}|+j}}, \quad |\mathbf{m}| + j = 1, \cdots, |\nu|.$$

For $\mathbf{m} = \mathbf{m}^*$ and $j = j^*$, it follows that

$$\frac{\|a_{\mathbf{0},0}\| r^0}{\|a_{\mathbf{m}^*,j^*}\| r^{|\mathbf{m}^*|+j^*}} \leq \frac{\prod_{i=1}^{|\nu|} P_i}{P_{|\nu|}^{|\nu|}},$$

where

$$\mu(r) \geq \|a_{\mathbf{0},0}\| \frac{P_{|\nu|}^{|\nu|}}{\prod_{i=1}^{|\nu|} P_i}.$$

Next we use the same arguments as in [37, p.193]. Applying inequality (4.22) for $\|a_{\mathbf{0},0}\| \neq 0$, one gets

$$\log^+\left(\frac{\mu(r)}{\|a_{\mathbf{0},0}\|}\right) \geq \log\left(\frac{P_{|\nu|}^{|\nu|}}{\prod_{i=1}^{|\nu|} P_i}\right) = \sum_{i=1}^{|\nu|-1} \frac{i}{i^\alpha}. \tag{4.25}$$

Furthermore, one obtains

$$\log^+\left(\frac{\mu(r)}{\|a_{\mathbf{0},0}\|}\right) \geq \int_1^{|\nu|} x^{1-\alpha} dx = \frac{|\nu|^{2-\alpha}}{2-\alpha} - \frac{1}{2-\alpha}. \tag{4.26}$$

Taking $\alpha := 1 + \varepsilon_1$, for $0 < \varepsilon_1 < 1$ we obtain the following estimate

$$|\nu|^{1-\varepsilon_1} < (-\epsilon \log^+(\|a_{\mathbf{0},0}\|) + 1)(1 - \varepsilon_1) \log^+(\mu(r))$$

for $0 < \epsilon < 1$. Furthermore, for $\varepsilon > 0$

$$|\nu(r)| < \left[\log^+((\mu(r)))\right]^{1+\varepsilon}.$$

$\square$

In order to proceed, the following proposition is needed. Here, for $\mathbf{m} \in \mathbb{Z}^n$ we denote $[\mathbf{m}]$ for the expression $[\mathbf{m}] = \sum_{i=1}^{n} m_i$.

Proposition 4.5 *For $\varepsilon > 0$, $p', l \in \mathbb{N}, p, s \in \mathbb{N}_0$ and $\mathbf{l} \in \mathbb{N}_0^n$ (or $\mathbf{l} \in \mathbb{N}_0^{n-1}$), one has*

$$\sum_{[\mathbf{m}]=-|\mathbf{m}^*|}^{+\infty} \sum_{j=-j^*}^{k-1-j^*} \left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'} \|a_{[\mathbf{m}]+\mathbf{m}^*,j+j^*}\| r^{[\mathbf{m}]+j+|\nu(r)|} < \mu(r)|\nu(r)|^{\frac{1-p'}{2}+\varepsilon}, \tag{4.27}$$

$$\sum_{[\mathbf{m}]=[\mathbf{l}-\mathbf{m}^*]}^{+\infty} \sum_{j=s-j^*}^{k-1-j^*} \frac{|[\mathbf{m}]+j|^p}{|\nu(r)|^{p+l}} \|a_{\mathbf{m}^*+\mathbf{m},j^*+j}\| r^{[\mathbf{m}]+j+|\nu(r)|} < \mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon}. \tag{4.28}$$

where $|\nu(r)| := |\mathbf{m}^*| + j^*$ and $r \notin F$ for F a set of finite logarithmic measure.

Notice that the first indices $\mathbf{m}$ appearing in this sum are elements from $-\mathbb{N}_0^n$. The expression $[\mathbf{m}]$ coincides with the previously introduced length of an index for all $\mathbf{m} \in \mathbb{N}_0^n$.

Proof. We first prove the inequality (4.27). Since $\nu(r) := (\mathbf{m}^*, j^*)$ we obtain

$$\sum_{[\mathbf{m}]=-|\mathbf{m}^*|}^{+\infty} \sum_{j=-j^*}^{k-1-j^*} \left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'} \|a_{[\mathbf{m}]+\mathbf{m}^*,j+j^*}\| r^{[\mathbf{m}]+j+|\nu(r)|}$$

$$\begin{aligned}
&= \|a_{\mathbf{m}^*,j^*}\| r^{|\mathbf{m}^*|+j^*} \left[\sum_{[\mathbf{m}]=-|\mathbf{m}^*|}^{+\infty} \sum_{j=-j^*}^{k-1-j^*} \left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'} \frac{\|a_{[\mathbf{m}]+\mathbf{m}^*,j+j^*}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{[\mathbf{m}]+j} \right] \\
&\le \|a_{\mathbf{m}^*,j^*}\| r^{|\mathbf{m}^*|+j^*} \left[\sum_{[\mathbf{m}]=-|\mathbf{m}^*|}^{-1} \sum_{j=-j^*}^{0} \left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'} \frac{\|a_{[\mathbf{m}]+\mathbf{m}^*,j+j^*}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{[\mathbf{m}]+j} \right. \\
&+ \sum_{[\mathbf{m}]=0}^{+\infty} \sum_{j=1}^{k-1} \left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'} \frac{\|a_{[\mathbf{m}]+\mathbf{m}^*,j+j^*}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{[\mathbf{m}]+j} \\
&+ \sum_{[\mathbf{m}]=-|\mathbf{m}^*|}^{-1} \sum_{j=0}^{k-1-j^*} \left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'} \frac{\|a_{[\mathbf{m}]+\mathbf{m}^*,j+j^*}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{[\mathbf{m}]+j} \\
&+ \left. \sum_{[\mathbf{m}]=0}^{+\infty} \sum_{j=-j^*}^{0} \left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'} \frac{\|a_{[\mathbf{m}]+\mathbf{m}^*,j+j^*}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{[\mathbf{m}]+j} \right] \\
&\le \|a_{\mathbf{m}^*,j^*}\| r^{|\mathbf{m}^*|+j^*} \left[\sum_{[\mathbf{m}]=1}^{|\mathbf{m}^*|} \sum_{j=0}^{j^*} \left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'} \frac{\|a_{[\mathbf{m}]-\mathbf{m}^*,j-j^*}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{-([\mathbf{m}]+j)} \right. \\
&+ \sum_{[\mathbf{m}]=0}^{+\infty} \sum_{j=1}^{k-1} \left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'} \frac{\|a_{[\mathbf{m}]+\mathbf{m}^*,j+j^*}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{[\mathbf{m}]+j} \\
&+ \sum_{[\mathbf{m}]=0}^{|\mathbf{m}^*|} \sum_{j=j^*}^{k-1} \left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'} \frac{\|a_{[\mathbf{m}],j}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{[\mathbf{m}]+j-|\nu|} \\
&+ \left. \sum_{[\mathbf{m}]=|\mathbf{m}^*|}^{+\infty} \sum_{j=0}^{j^*} \left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'} \frac{\|a_{[\mathbf{m}],j}\|}{\|a_{\mathbf{m}^*,j^*}\|} r^{[\mathbf{m}]+j-|\nu|} \right]. \qquad (4.29)
\end{aligned}$$

Applying Proposition 4.2 in the expression (4.29) we obtain

$$
\begin{aligned}
&\sum_{[\mathbf{m}]=-|\mathbf{m}^*|}^{+\infty}\sum_{j=-j^*}^{k-1}\left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'}\|a_{[\mathbf{m}]+\mathbf{m}^*,j+j^*}\|r^{[\mathbf{m}]+j+|\nu(r)|}\\
<\ &\|a_{\mathbf{m}^*,j^*}\|r^{|\mathbf{m}^*|+j^*}\left[5\sum_{[\mathbf{m}]+j=1}^{|\nu|}\left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'}\frac{\prod_{i=|\mathbf{m}|+j}^{|\nu|}P_{|\nu|-i}}{P_{|\nu|}^{|\mathbf{m}|+j}}\right.\\
+\ &\left.5\sum_{[\mathbf{m}]+j=|\nu|+1}^{+\infty}\left(\frac{|[\mathbf{m}]+j|}{|\nu(r)|}\right)^{p'}\frac{P_{|\nu|}^{|\mathbf{m}|+j}}{\prod_{i=1}^{|\mathbf{m}|+j}P_{|\nu|+i}}\right].\\
=:\ &\|a_{\mathbf{m}^*,j^*}\|r^{|\mathbf{m}^*|+j^*}5\Psi_{|\mathbf{m}|+j}(r).
\end{aligned}
\tag{4.30}
$$

Applying the estimates (4.21) and (4.22) into (4.30), we obtain

$$
\begin{aligned}
\Psi_{|\mathbf{m}|+j}(r)\ &\leq\ \sum_{[\mathbf{m}]+j=1}^{|\nu|}\left(\frac{|[\mathbf{m}]+j|}{|\nu|}\right)^{p'}e^{\frac{-(|\mathbf{m}|+j)(|\mathbf{m}|+j-1)}{2(|\nu|+|\mathbf{m}|+j-1)^{\alpha}}}\\
&+\ \sum_{[\mathbf{m}]+j=|\nu|+1}^{+\infty}\left(\frac{|[\mathbf{m}]+j|}{|\nu|}\right)^{p'}e^{\frac{-(|\mathbf{m}|+j)^2}{2(|\nu|+|\mathbf{m}|+j)^{\alpha}}}\\
&\leq\ \frac{(n+1)}{|\nu|^{p'}}+\sum_{[\mathbf{m}]+j=2}^{|\nu|}\left(\frac{|[\mathbf{m}]+j|}{|\nu|}\right)^{p'}e^{\frac{-(|\mathbf{m}|+j)^2}{2(|\nu|+|\mathbf{m}|+j-1)^{\alpha}}}\\
&+\ \sum_{[\mathbf{m}]+j=|\nu|+1}^{+\infty}\left(\frac{|[\mathbf{m}]+j|}{|\nu|}\right)^{p'}e^{\frac{-(|\mathbf{m}|+j)^2}{2(|\nu|+|\mathbf{m}|+j)^{\alpha}}}.
\end{aligned}
\tag{4.31}
$$

For $x := |\mathbf{m}| + j$ we obtain that

$$
\Psi_{|\mathbf{m}|+j}(r) \leq \frac{(n+1)}{|\nu|^{p'}} + 2\int_0^{+\infty}\left(\frac{x}{|\nu|}\right)^{p'} e^{\frac{-x^2}{2(|\nu|+x)^{\alpha}}}\,dx. \tag{4.32}
$$

Using the substitution $y^2 := \frac{x^2}{2(|\nu|+x)^{\alpha}}$, one obtains

$$\int_0^{+\infty}\left(\frac{x}{|\nu|}\right)^{p'} e^{\frac{-x^2}{2(|\nu|+x)^\alpha}}dx \;\leq\; C_1|\nu(r)|^{\frac{\alpha}{2}(p'+1)-p'}\int_0^{+\infty}(y)^{p'}e^{-y^2}dy$$
$$= C_1|\nu(r)|^{\frac{\alpha}{2}(p'+1)-p'}\frac{1}{2}\Gamma\left(\frac{1+p'}{2}\right)$$
$$= C_0|\nu(r)|^{\frac{1-p'}{2}+\varepsilon},$$

where C_0 is an adequately chosen real positive constant. This completes the proof of (4.27).

To prove inequality (4.28) we use mathematical induction with respect to p. Let us first consider $p = 0$. Applying inequality (4.12) one obtains a lower bound estimate for $|\nu(r)|$:

$$\sum_{[\mathbf{m}]=[\mathbf{l}-\mathbf{m}^*]}^{+\infty}\sum_{j=s-j^*}^{k-1}\frac{1}{|\nu|^l}\|a_{\mathbf{m}^*+\mathbf{m},j^*+j}\|r^{[\mathbf{m}]+j+|\nu|}$$
$$\leq \frac{\mu(r)}{|\nu|}\sum_{[\mathbf{m}]=-[\mathbf{m}^*]}^{+\infty}\sum_{j=-j^*}^{k-1}\frac{\|a_{\mathbf{m}^*+\mathbf{m},j^*+j}\|}{\|a_{\mathbf{m}^*,j^*}\|}r^{[\mathbf{m}]+j}$$
$$\leq \frac{\mu(r)}{|\nu|}L^*(r) < \mu(r)|\nu|^{-\frac{1}{2}+\varepsilon}.$$

The proof for the case $p \geq 1$ is done by using the estimate (4.27). □

These propositions will be useful in the study of the asymptotic behavior between some special iterated operators applied to entire k-monogenic functions and the function itself.

4.2 Asymptotic growth of solutions of iterated Cauchy--Riemann equations in $\mathbb{R}^{n+1}$

In this section we will prove a relation between the asymptotic behavior of the maximum term of a k-monogenic function and that of their iterated "generalized" radial derivatives. Such derivatives arise from the application of iterated Euler-type operators.

Theorem 4.1 *Let g be a transcendental entire k-monogenic function. Then for all $\kappa \in \mathbb{N}$ holds asymptotically*

$$\left\| \frac{1}{|\nu(r)|^\kappa}[E^\kappa]g(z) - g(z) \right\| \leq C\mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon}, \quad r \notin F \tag{4.33}$$

where $E := \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$ is the Euler operator on $\mathbb{R}^{n+1}$, C is a real positive constant, $\varepsilon > 0$ and F is a set of finite logarithmic measure.

Proof. Using induction with respect to κ one obtains the following relation

$$E^\kappa(x_0^j V_{\mathbf{m}}(z)) = (|\mathbf{m}| + j)^\kappa \, x_0^j V_{\mathbf{m}}(z). \tag{4.34}$$

Therefore, applying (4.34) we have

$$\begin{aligned} & \frac{1}{|\nu|^\kappa}\left[\sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}\right]^\kappa g(z) - g(z) \\ = \; & \frac{1}{|\nu|^\kappa}\left[\sum_{|\mathbf{m}|=1}^{+\infty}\sum_{j=0}^{k-1} (|\mathbf{m}| + j)^\kappa \, x_0^j V_{\mathbf{m}}(z) a_{\mathbf{m},j}\right] - \sum_{|\mathbf{m}|=0}^{+\infty}\sum_{j=0}^{k-1} x_0^j V_{\mathbf{m}}(z) a_{\mathbf{m},j}. \end{aligned}$$

Considering $\nu := (\mathbf{m}^*, j^*)$ and making the index substitution $|\mathbf{m}| = [\mathbf{n}] + |\mathbf{m}^*|$, $j = i + j^*$ together with the binomial expansion leads to

$$\begin{aligned} & \frac{1}{|\nu|^\kappa}\left[\sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}\right]^\kappa g(z) - g(z) \\ = \; & -a_{\mathbf{0},0} + \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \sum_{i=-j^*}^{k-1-j^*} \frac{([\mathbf{n}] + i + |\nu|)^\kappa - |\nu|^\kappa}{|\nu|^\kappa} x_0^{i+j^*} V_{\mathbf{n}+\mathbf{m}^*}(z) a_{\mathbf{n}+\mathbf{m}^*, i+j^*} \\ = \; & -a_{\mathbf{0},0} + \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \sum_{i=-j^*}^{k-1-j^*} \sum_{s=1}^{\kappa} ([\mathbf{n}] + i)^s |\nu|^{-s} \binom{\kappa}{s} x_0^{i+j^*} V_{\mathbf{n}+\mathbf{m}^*}(z) a_{\mathbf{n}+\mathbf{m}^*, i+j^*} \\ = \; & -a_{\mathbf{0},0} + \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \sum_{i=-j^*}^{k-1-j^*} \sum_{s=1}^{\kappa} |\nu|^{-s} P_s([\mathbf{n}] + i) x_0^{i+j^*} V_{\mathbf{n}+\mathbf{m}^*}(z) a_{\mathbf{n}+\mathbf{m}^*, i+j^*}, \end{aligned}$$

where $P_s([\mathbf{n}] + i) := ([\mathbf{n}] + i)^s \binom{\kappa}{s}$.

Let us define

$$S_s(z) := \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \sum_{i=-j^*}^{k-1-j^*} |\nu|^{-s} P_s([\mathbf{n}] + i) x_0^{i+j^*} V_{\mathbf{n}+\mathbf{m}^*}(z) a_{\mathbf{n}+\mathbf{m}^*, i+j^*}. \tag{4.35}$$

Taking $\|z\| = r > 0$ we estimate $\|S_s(z)\|$ by

$$\begin{aligned}
\|S_s(z)\| &\leq \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \sum_{i=-j^*}^{k-1-j^*} |\nu|^{-s} |P_s([\mathbf{n}]+i)| \|x_0\|^{i+j^*} \|V_{\mathbf{n}+\mathbf{m}^*}(z)\| \|a_{\mathbf{n}+\mathbf{m}^*,i+j^*}\| \\
&\leq \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \sum_{i=-j^*}^{k-1-j^*} |\nu|^{-s} P_s(|[\mathbf{n}]+i|) \|a_{\mathbf{n}+\mathbf{m}^*,i+j^*}\| r^{[\mathbf{n}]+|\mathbf{m}^*|+i+j^*} \\
&= \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \sum_{i=-j^*}^{k-1-j^*} \left(\frac{|[\mathbf{n}]+i|}{|\nu|} \right)^s \frac{\kappa!}{(\kappa-s)!s!} \|a_{\mathbf{n}+\mathbf{m}^*,i+j^*}\| r^{[\mathbf{n}]+i+|\nu|}.
\end{aligned}$$

We thus have

$$\|S_s(z)\| \leq \frac{\kappa!}{(\kappa-s)!s!} \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \sum_{i=-j^*}^{k-1-j^*} \left(\frac{|[\mathbf{n}]+i|}{|\nu|} \right)^s \|a_{\mathbf{n}+\mathbf{m}^*,i+j^*}\| r^{[\mathbf{n}]+i+|\nu|}. \tag{4.36}$$

Applying inequality (4.28) of Proposition 4.5, in the particular case $l = 0$, $s = 0$ and $|\mathbf{l}| = 1$, to the previous line, one has

$$\|S_s(z)\| \leq \frac{\kappa!}{(\kappa-s)!s!} \mu(r) |\nu(r)|^{-\frac{1}{2}+\varepsilon}, \qquad r \notin F. \tag{4.37}$$

Summarizing, we obtain for r sufficiently large

$$\begin{aligned}
&\left\| \frac{1}{|\nu|^\kappa} \left[\sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i} \right]^\kappa g(z) - g(z) \right\| \\
\leq\ & \|a_{\mathbf{0},0}\| + \sum_{s=1}^{\kappa} \|S_s(z)\| \\
\leq\ & \|a_{\mathbf{0},0}\| + \sum_{s=1}^{\kappa} \frac{\kappa!}{(\kappa-s)!s!} \mu(r) |\nu(r)|^{-\frac{1}{2}+\varepsilon}, \quad r \notin F \\
=\ & \|a_{\mathbf{0},0}\| + (2^\kappa - 1) \mu(r) |\nu(r)|^{-\frac{1}{2}+\varepsilon}, \quad r \notin F, \\
\leq\ & C \mu(r) |\nu(r)|^{-\frac{1}{2}+\varepsilon}, \quad r \notin F,
\end{aligned}$$

where C is a positive real constant. □

As an application one has the following.

Proposition 4.6 *Let* $0 < \delta < \frac{1}{2}$ *and* z *such that for* $\|z\| = r$*, the relation*

$$\|g(z)\| > \mu(r)|\nu(r)|^{-\frac{1}{2}+\delta}, \quad r \notin F \tag{4.38}$$

for z *in a neighborhood* $\mathcal{V}_{z_0}$ *of* z_0 *such that* $\|g(z_0)\| = \max_{\|z\|=r}\{\|g(z)\|\}$ *is satisfied. Then for all* $k \in \mathbb{N}$ *holds asymptotically*

$$\frac{1}{|\nu(r)|^k}[E^k]g(z) - g(z) = o(1)g(z) \quad z \in \mathcal{V}_{z_0}. \tag{4.39}$$

Proof. Let us now suppose that $\|z\| = r \notin F$. In view of condition (4.38) and Theorem 4.1, one has

$$\begin{aligned} & \frac{1}{\|g(z)\|}\left\|\frac{1}{|\nu|^k}\left[\sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}\right]^k g(z) - g(z)\right\| \\ \leq \ & \frac{C}{\mu(r)}|\nu(r)|^{\frac{1}{2}-\delta}\mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon} \\ = \ & C|\nu(r)|^{\varepsilon-\delta} \end{aligned} \tag{4.40}$$

which tends to zero if one chooses a ε sufficiently small (i.e. $\varepsilon < \delta$). In other words, one gets

$$\frac{1}{|\nu|^k}[E^k]g(z) - g(z) = o(1)g(z)$$

under the given condition. □

Remark 4.2 *This statement provides us with an analogy in the context of Clifford analysis of the classical Theorem 21.3 from [37] which states that entire complex analytic functions which satisfy*

$$\|g(z)\| > M(r,g)[\nu(r)]^{-\frac{1}{4}+\delta}$$

for $0 < \delta < \frac{1}{4}$ *and* z *such that* $\|z\| = r$*, have the asymptotic behavior*

$$g^{(m)}(z) = \left(\frac{\nu(r)}{z}\right)^m (1 + o(1))g(z).$$

In the Clifford analysis setting, one thus obtains a similar asymptotic result when substituting the complex operator $z\frac{d}{dz}$ *by the higher dimensional Euler operator* E.

For the particular case of a 1-monogenic entire function one obtains the following result.

Theorem 4.2 *Let g be an entire 1-monogenic ClG-valued function of finite order $\rho_2 < \infty$ and for z such that $\|z\| = r$ sufficiently large, the relation*

$$\|g(z)\| > \mu(r)|\nu(r)|^{-\frac{1}{2}+\delta}, \quad r \notin F$$

for z in a neighborhood $\mathcal{V}_{z_0}$ of z_0 such that $\|g(z_0)\| := \max_{\|z\|=r}\{\|g(z)\|\}$ is satisfied. Let

$$M_j[g] = a_j \prod_{i=0}^{k} (E^i(g))^{n_i},$$

where a_j are polynomials of degree j, and $M_j[g]$ has degree $\gamma_{M_j} = \sum_{i=0}^{k} n_i$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k} in_i$. Let

$$Q[g] = \sum_{j=0}^{s} M_j[g]$$

be of degree γ_Q and weight Γ_Q. If $\gamma_Q = \gamma_{M_0}$ then the differential equation $Q[g] = 0$ has no transcendental entire solutions.

Proof. If $Q[g] = 0$, then $M_0[g] = -\sum_{j=1}^{s} M_j[g]$. From the definition of M_j it follows that

$$a_0 \left[\prod_{i=0}^{k} (E^i(g))^{n_i}\right]_{M_0} = -\sum_{j=1}^{s} \left[a_j \prod_{i=0}^{k} (E^i(g))^{n_i}\right]_{M_j}.$$

Applying Proposition 4.6 we obtain that

$$\|a_0\||\nu(r)|^{\Gamma_{M_0}} \|g(z)\|^{\gamma_{M_0}} \leq \sum_{j=1}^{s} \left(\|a_j\||\nu(r)|^{\Gamma_{M_j}} \|g(z)\|^{\gamma_{M_j}} \right).$$

Since a_0 is a non zero constant and a_j are polynomials of degree j, taking the maximum over the norm, and applying Theorem 3.1 leads to

$$\begin{aligned} |\nu(r)|^{\Gamma_{M_0}} M(r,g)^{\gamma_{M_0}} &\leq |\nu(r)|^{\Gamma_Q} M(r,g)^{\gamma_Q - 1} \sum_{j=1}^{s} \max_{\|z\|=r} \frac{\|a_j\|}{\|a_0\|} \\ &\leq |\nu(r)|^{\Gamma_Q} M(r,g)^{\gamma_Q - 1} r^{\alpha}. \end{aligned} \tag{4.41}$$

Therefore, in view of $\gamma_Q = \gamma_{M_0}$ one has

$$M(r,g) \leq |\nu(r)|^{\Gamma_Q - \Gamma_{M_0}} r^{\alpha}. \tag{4.42}$$

For $\Gamma_Q - \Gamma_{M_0} < 0$ it follows

$$\liminf_{r\to\infty} \frac{M(r,g)}{r^{\alpha}} \leq \liminf_{r\to\infty} |\nu(r)|^{\Gamma_Q - \Gamma_{M_0}} = 0$$

which implies that g is a polynomial.

Let us now consider the case where $\Gamma_Q - \Gamma_{M_0} > 0$. Since $\rho_2 < \infty$, we have that $|\nu(r)| < r^{\rho_2+\epsilon}$ for $\epsilon > 0$. Therefore, there exists a $\beta > (\Gamma_Q - \Gamma_{M_0})(\rho_2 + \epsilon)$ such that

$$\begin{aligned}\liminf_{r\to\infty} \frac{M(r,g)}{r^{\beta+\alpha}} &\leq \liminf_{r\to\infty} \frac{|\nu(r)|^{\Gamma_Q - \Gamma_{M_0}}}{r^{\beta}} \\ &\leq \liminf_{r\to\infty} r^{(\Gamma_Q - \Gamma_{M_0})(\rho_2+\epsilon)-\beta} = 0\end{aligned}$$

which implies that g is a polynomial. □

4.3 Asymptotic growth of solutions of iterated Dirac equations in $\mathbb{R}^n$

In this section we establish an explicit asymptotic relation between the growth of solutions of the iterated Dirac equation and that of their iterated "generalized" radial derivatives resulting from the application of the Euler operator

$$E := \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

and from the application of even iterates of the Gamma operator from [21],

$$\Gamma := \sum_{i,j=1,i<j}^{n} (x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}) e_i e_j.$$

More generally, one also establish asymptotic relations between the growth of solutions of the iterated Dirac equation and that from applying on them a polynomial expression consisting of the radial symmetric operators E and Γ^2.

Theorem 4.3 *Let $s, d, N \in \mathbb{N}_0$ and $f : \mathbb{R}^n \to Cl_n$ be an entire k-monogenic function, and $\gamma_{s,d}$ some arbitrary real numbers. Then f satisfies asymptotically*

$$\left\| \sum_{s,d=0}^{N} \gamma_{s,d} \frac{E^s[\Gamma^{2d}[f(\mathbf{x})]]}{|\nu|^{s+2d}} - \sum_{s,d=0}^{N} \gamma_{s,d} f(\mathbf{x}) \right\| \leq C\mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon}, \quad r \notin F, \tag{4.43}$$

where C is a real constant, $\varepsilon > 0$ and F is a set of finite logarithmic measure.

Proof. Relying on formula (3.14) one gets

$$\mathbf{x}\mathcal{D}(\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x})) = \begin{cases} (-p)\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}), & p \text{ even} \\ (-n-2|\mathbf{m}|-p+1)\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}), & p \text{ odd.} \end{cases} \tag{4.44}$$

In view of

$$\frac{\partial \mathbf{x}^j}{\partial x_i} = \begin{cases} -2mx_i\mathbf{x}^{2m-2}, & j = 2m, \quad m \in \mathbb{N} \\ -2mx_i\mathbf{x}^{2m-1} + \mathbf{x}^{2m}e_i, & j = 2m+1, \quad m \in \mathbb{N}, \end{cases} \tag{4.45}$$

and using (4.45) one gets

$$\begin{aligned} E(\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x})) &= \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} (\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x})) \\ &= \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} (\mathbf{x}^p) V_{\mathbf{m}}(\mathbf{x}) + \mathbf{x}^p \left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} V_{\mathbf{m}}(\mathbf{x}) \right) \\ &= p\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}) + |\mathbf{m}|\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}) \\ &= (p+|\mathbf{m}|)\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}). \end{aligned} \tag{4.46}$$

Applying (4.44) and (4.46) we obtain, in view of $\Gamma := \mathbf{x}\mathcal{D} + E$

$$\Gamma(\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x})) = \begin{cases} |\mathbf{m}|\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}), & p \text{ even} \\ (-n+1-|\mathbf{m}|)\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}), & p \text{ odd.} \end{cases}$$

By induction, it follows that for any $s \in \mathbb{N}$:

$$\begin{aligned} E^s(\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x})) &= (p+|\mathbf{m}|)^s \mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}), \\ \Gamma^s(\mathbf{x}^p V_{\mathbf{m}}(\mathbf{x})) &= \begin{cases} |\mathbf{m}|^s \mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}), & p \text{ even,} \\ (1-n-|\mathbf{m}|)^s \mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}), & p \text{ odd.} \end{cases} \end{aligned} \tag{4.47}$$

Considering the composition of the operators E and Γ^2 for an arbitrary $s, d \in \mathbb{N}_0$ and using (4.47) one obtains:

$$E^s[\Gamma^{2d}[f(\mathbf{x})]] = \sum_{|\mathbf{m}|=0}^{+\infty} \left[\sum_{p=0, p\ even}^{k-1} |\mathbf{m}|^{2d}(p+|\mathbf{m}|)^s \mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},p} + \sum_{p=1, p\ odd}^{k-1} (1-n-|\mathbf{m}|)^{2d}(p+|\mathbf{m}|)^s \mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},p} \right].$$

Performing the index substitutions $|\mathbf{m}| := [\mathbf{n}] + |\mathbf{m}^*|$ and $p := i + j^*$, we thus get

$$\begin{aligned}
&\frac{E^s[\Gamma^{2d}[f(\mathbf{x})]]}{|\nu|^{s+2d}} - f(\mathbf{x}) \\
=\ & -a_{\mathbf{0},0} + \sum_{|\mathbf{m}|=1}^{+\infty} \left[\sum_{\substack{p=0,\\ p\ even}}^{k-1} \frac{|\mathbf{m}|^{2d}(p+|\mathbf{m}|)^s - |\nu|^{s+2d}}{|\nu|^{s+2d}} \mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},p} \right. \\
&+ \left. \sum_{\substack{p=1,\\ p\ odd}}^{k-1} \frac{(1-n-|\mathbf{m}|)^{2d}(p+|\mathbf{m}|)^s - |\nu|^{s+2d}}{|\nu|^{s+2d}} \mathbf{x}^p V_{\mathbf{m}}(\mathbf{x}) a_{\mathbf{m},p} \right] \\
=\ & -a_{\mathbf{0},0} \\
&+ \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \left[\sum_{\substack{i=-j^*,\\ i+j^*\ even}}^{k-1-j^*} \frac{([\mathbf{n}]+|\mathbf{m}^*|)^{2d}(i+[\mathbf{n}]+|\nu|)^s - |\nu|^{s+2d}}{|\nu|^{s+2d}} \mathbf{x}^{i+j^*} V_{[\mathbf{n}]+\mathbf{m}^*} a_{[\mathbf{n}]+\mathbf{m}^*, i+j^*} \right. \\
&+ \sum_{\substack{i=-j^*,\\ i+j^*\ odd}}^{k-1-j^*} \frac{(1-n-[\mathbf{n}]-|\mathbf{m}^*|)^{2d}(i+[\mathbf{n}]+|\nu|)^s - |\nu|^{s+2d}}{|\nu|^{s+2d}} \\
&\times \left. \mathbf{x}^{i+j^*} V_{[\mathbf{n}]+\mathbf{m}^*} a_{[\mathbf{n}]+\mathbf{m}^*, i+j^*} \right].
\end{aligned} \tag{4.48}$$

In view of $|\nu| = |\mathbf{m}^*| + j^*$ and applying the binomial expansion, we can rewrite the expression appearing in the first sum of (4.48) as follows:

$$\begin{aligned}
& \Big(([\mathbf{n}]+|\mathbf{m}^*|)^{2d}(i+[\mathbf{n}]+|\nu|)^s-|\nu|^{s+2d}\Big) \\
= & \Big(([\mathbf{n}]+|\nu|-j^*)^{2d}(i+[\mathbf{n}]+|\nu|)^s-|\nu|^{s+2d}\Big) \\
= & \left[\sum_{p=1}^{2d}\binom{2d}{p}([\mathbf{n}]-j^*)^p|\nu|^{2d-p}+|\nu|^{2d}\right]\times\left[\sum_{L=1}^{s}\binom{s}{L}([\mathbf{n}]+i)^L|\nu|^{s-L}+|\nu|^s\right]-|\nu|^{s+2d} \\
= & \sum_{p=1}^{2d}\sum_{L=1}^{s}\binom{2d}{p}\binom{s}{L}([\mathbf{n}]-j^*)^p([\mathbf{n}]+i)^L|\nu|^{2d+s-(L+p)}+\sum_{p=1}^{2d}\binom{2d}{p}([\mathbf{n}]-j^*)^p|\nu|^{2d+s-p} \\
+ & \sum_{L=1}^{s}\binom{s}{L}([\mathbf{n}]+i)^L|\nu|^{2d+s-L}+|\nu|^{s+2d}-|\nu|^{s+2d} \\
= & \sum_{p=1}^{2d}\sum_{L=1}^{s}\binom{2d}{p}\binom{s}{L}([\mathbf{n}]-j^*)^p([\mathbf{n}]+i)^L|\nu|^{2d+s-(L+p)}+\sum_{p=1}^{2d}\binom{2d}{p}([\mathbf{n}]-j^*)^p|\nu|^{2d+s-p} \\
+ & \sum_{L=1}^{s}\binom{s}{L}([\mathbf{n}]+i)^L|\nu|^{2d+s-L}. \qquad (4.49)
\end{aligned}$$

The second sum of (4.48) can be expressed as follows:

$$\begin{aligned}
& (1-n-[\mathbf{n}]-|\mathbf{m}^*|)^{2d}(i+[\mathbf{n}]+|\nu|)^s-|\nu|^{2d+s} \\
= & (1-n-[\mathbf{n}]+j^*-j^*-|\mathbf{m}^*|)^{2d}(i+[\mathbf{n}]+|\nu|)^s-|\nu|^{2d+s} \\
= & (1-n-[\mathbf{n}]+j^*-|\nu|)^{2d}(i+[\mathbf{n}]+|\nu|)^s-|\nu|^{2d+s} \\
= & \left(\sum_{p_1=0}^{2d}\binom{2d}{p_1}(-1)^{2d-p_1}(1-n-[\mathbf{n}]+j^*)^{p_1}|\nu|^{2d-p_1}\right) \\
\times & \left(\sum_{L_1=0}^{s}\binom{s}{L_1}([\mathbf{n}]+i)^{L_1}|\nu|^{s-L_1}\right)-|\nu|^{s+2d} \\
= & \left(\sum_{p_1=1}^{2d}\binom{2d}{p_1}(-1)^{2d-p_1}(1-n-[\mathbf{n}]+j^*)^{p_1}|\nu|^{2d-p_1}+(-1)^{2d}|\nu|^{2d}\right) \\
\times & \left(\sum_{L_1=1}^{s}\binom{s}{L_1}([\mathbf{n}]+i)^{L_1}|\nu|^{s-L_1}+|\nu|^s\right)-|\nu|^{s+2d}
\end{aligned}$$

(4.50)

$$\begin{aligned}
& (1-n-[\mathbf{n}]-|\mathbf{m}^*|)^{2d}(i+[\mathbf{n}]+|\nu|)^s-|\nu|^{2d+s} \\
= & \left(\sum_{p_1=1}^{2d}\binom{2d}{p_1}(-1)^{2d-p_1}(1-n-[\mathbf{n}]+j^*)^{p_1}|\nu|^{2d-p_1}\right) \\
\times & \left(\sum_{L_1=1}^{s}\binom{s}{L_1}([\mathbf{n}]+i)^{L_1}|\nu|^{s-L_1}\right)+(-1)^{2d}|\nu|^{2d}\left(\sum_{L_1=1}^{s}\binom{s}{L_1}([\mathbf{n}]+i)^{L_1}|\nu|^{s-L_1}\right) \\
+ & |\nu|^s\left(\sum_{p_1=1}^{2d}\binom{2d}{p_1}(-1)^{2d-p_1}(1-n-[\mathbf{n}]+j^*)^{p_1}|\nu|^{2d-p_1}\right) \\
+ & (-1)^{2d}|\nu|^{s+2d}-|\nu|^{s+2d} \\
= & \sum_{p_1=0}^{2d}\sum_{L_1=1}^{s}\binom{2d}{p_1}\binom{s}{L_1}(-1)^{-p_1}([\mathbf{n}]+i)^{L_1}(1-n-[\mathbf{n}]+j^*)^{p_1}|\nu|^{2d+s-(p_1+L_1)} \\
+ & \sum_{p_1=1}^{2d}\binom{2d}{p_1}(-1)^{-p_1}(1-n-[\mathbf{n}]+j^*)^{p_1}|\nu|^{2d+s-p_1}, \qquad (4.51)
\end{aligned}$$

the last equality is based on the fact that the number of iterations of the Gamma operator is even and equal to $2d$. With (4.49) and (4.51) we thus have arrived at

$$\begin{aligned}
& \frac{([\mathbf{n}]+|\mathbf{m}^*|)^{2d}(i+[\mathbf{n}]+|\nu|)^s-|\nu|^{s+2d}}{|\nu|^{s+2d}} \qquad (4.52) \\
= & \sum_{p=0}^{2d}\sum_{L=1}^{s}\binom{2d}{p}\binom{s}{L}[([\mathbf{n}]+i)^L([\mathbf{n}]-j^*)^p]|\nu|^{-(L+p)}+\sum_{p=1}^{2d}\binom{2d}{p}([\mathbf{n}]-j^*)^p|\nu|^{-p},
\end{aligned}$$

and

$$\begin{aligned}
& \frac{(1-n-[\mathbf{n}]-|\mathbf{m}^*|)^{2d}(i+[\mathbf{n}]+|\nu|)^s-|\nu|^{2d+s}}{|\nu|^{2d+s}} \\
= & \sum_{p_1=0}^{2d}\sum_{L_1=1}^{s}\binom{2d}{p_1}\binom{s}{L_1}(-1)^{-p_1}([\mathbf{n}]+i)^{L_1}(1-n-[\mathbf{n}]+j^*)^{p_1}|\nu|^{-(L_1+p_1)} \\
+ & \sum_{p_1=0}^{2d}\binom{2d}{p_1}(-1)^{-p_1}(1-n-[\mathbf{n}]+j^*)^{p_1}|\nu|^{-p_1}, \qquad (4.53)
\end{aligned}$$

respectively. Applying the norm in the expression (4.48) and using (4.52) and (4.53), we obtain for $\|\mathbf{x}\| = r$ the following estimates

$$\begin{aligned}
&\left\| \frac{E^s[\Gamma^{2d}[f(\mathbf{x})]]}{|\nu|^{s+2d}} - f(\mathbf{x}) \right\| \\
\leq\ & \|a_{\mathbf{0},0}\| \\
+ & \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|} \left[\sum_{i=-j^*, i+j^* \ even}^{k-1-j^*} \left[\sum_{p=0}^{2d} \sum_{L=1}^{s} \binom{2d}{p} \binom{s}{L} |([\mathbf{n}]+i)|^L \underbrace{|[\mathbf{n}]-j^*|^p}_{(I)} |\nu|^{-(L+p)} \right.\right. \\
+ & \left. \sum_{p=1}^{2d} \binom{2d}{p} \underbrace{|[\mathbf{n}]-j^*|^p}_{(I)} |\nu|^{-p} \right] r^{i+[\mathbf{n}]+|\nu|} \|a_{[\mathbf{n}]+\mathbf{m}^*, i+j^*}\| \\
+ & \sum_{i=-j^*, i+j^* \ odd}^{k-1-j^*} \left[\sum_{p_1=0}^{2d} \sum_{L_1=1}^{s} \binom{2d}{p_1} \binom{s}{L_1} |[\mathbf{n}]+i|^{L_1} \underbrace{|1-n-[\mathbf{n}]+j^*|^{p_1}}_{(II)} |\nu|^{-(L_1+p_1)} \right. \\
+ & \left.\left. \sum_{p_1=0}^{2d} \binom{2d}{p_1} \underbrace{|1-n-[\mathbf{n}]+j^*|^{p_1}}_{(II)} |\nu|^{-p_1} \right] r^{i+[\mathbf{n}]+|\nu|} \|a_{[\mathbf{n}]+\mathbf{m}^*, i+j^*}\| \right]. \qquad (4.54)
\end{aligned}$$

In order to proceed we estimate the expressions (I) and (II) in (4.54) by again applying the binomial expansion. For (I) we get

$$\begin{aligned}
|[\mathbf{n}]-j^*|^p &= |[\mathbf{n}]+i-i-j^*|^p \\
&\leq (|[\mathbf{n}]+i| + |i+j^*|)^p \\
&= \sum_{\alpha=0}^{p} \binom{p}{\alpha} |[\mathbf{n}]+i|^{p-\alpha} |i+j^*|^{\alpha} \\
&\leq (2(k-1))^p \sum_{\alpha=0}^{p} \binom{p}{\alpha} |[\mathbf{n}]+i|^{p-\alpha} \qquad (4.55)
\end{aligned}$$

the last inequality is obtain by $|i+j^*| \leq 2(k-1)$. For (II) one has

$$\begin{aligned}
|1-n-[\mathbf{n}]+j^*|^{p_1} &= |1-n-[\mathbf{n}]+i-i+j^*|^{p_1} \\
&\leq (|[\mathbf{n}]+i| + |1-n+i+j^*|)^{p_1} \\
&= \sum_{\beta=0}^{p_1} \binom{p_1}{\beta} |[\mathbf{n}]+i|^{p_1-\beta} |1-n+i+j^*|^{\beta} \\
&\leq ((n-1)+2(k-1))^{p_1} \sum_{\beta=0}^{p_1} \binom{p_1}{\beta} |[\mathbf{n}]+i|^{p_1-\beta}, \qquad (4.56)
\end{aligned}$$

since $|1-n+i+j^*| \leq |1-n| + |i+j^*| \leq (n-1)+2(k-1)$.

Notice that one also has for (4.55) the same estimate as the one given in (4.56), since

$$\begin{aligned} |[\mathbf{n}] - j^*|^p &\leq (2(k-1))^p \sum_{\alpha=0}^{p} \binom{p}{\alpha} |[\mathbf{n}] + i|^{p-\alpha} \\ &\leq ((n-1) + 2(k-1))^p \sum_{\alpha=0}^{p} \binom{p}{\alpha} |[\mathbf{n}] + i|^{p-\alpha}, \quad n \geq 1. \end{aligned} \tag{4.57}$$

Substituting (4.57) and (4.56) in the expression (4.54) then leads to

$$\begin{aligned} &\left\| \frac{E^s[\Gamma^{2d}[f(\mathbf{x})]]}{|\nu|^{s+2d}} - f(\mathbf{x}) \right\| \\ \leq\ & \|a_{\mathbf{0},0}\| \\ +\ & \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \left[\sum_{i=-j^*, i+j^*\ even}^{k-1-j^*} \left[\sum_{\substack{0 \leq L \leq s,\\ 0 \leq p \leq 2d,\\ L+p \neq 0}} \binom{2d}{p} \binom{s}{L} ((n-1) + 2(k-1))^p \right.\right. \\ \times\ & \left. \sum_{\alpha=0}^{p} \binom{p}{\alpha} |[\mathbf{n}] + i|^{L+p-\alpha} |\nu|^{-(L+p)} \right] r^{i+[\mathbf{n}]+|\nu|} \|a_{[\mathbf{n}]+\mathbf{m}^*, i+j^*}\| \\ +\ & \sum_{i=-j^*, i+j^*\ odd}^{k-1-j^*} \left[\sum_{\substack{0 \leq L_1 \leq s,\\ 0 \leq p_1 \leq 2d,\\ L_1+p_1 \neq 0}} \binom{2d}{p_1} \binom{s}{L_1} ((n-1) + 2(k-1))^{p_1} \right. \\ \times\ & \left.\left. \sum_{\beta=0}^{p_1} \binom{p_1}{\beta} |[\mathbf{n}] + i|^{L_1+p_1-\beta} |\nu|^{-(L_1+p_1)} \right] r^{i+[\mathbf{n}]+|\nu|} \|a_{[\mathbf{n}]+\mathbf{m}^*, i+j^*}\| \right]. \end{aligned} \tag{4.58}$$

Since the terms of the sum running over the even $i + j^*$ and odd $i + j^*$ are equal, one obtains the following relation

$$\begin{aligned} &\left\| \frac{E^s[\Gamma^{2d}[f(\mathbf{x})]]}{|\nu|^{s+2d}} - f(\mathbf{x}) \right\| \\ \leq\ & \|a_{\mathbf{0},0}\| + \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \sum_{i=-j^*}^{k-1-j^*} \left[\sum_{\substack{0 \leq L \leq s,\\ 0 \leq p \leq 2d,\\ L+p \neq 0}} \binom{2d}{p} \binom{s}{L} ((n-1) + 2(k-1))^p \right. \\ \times\ & \left. \sum_{\alpha=0}^{p} \binom{p}{\alpha} |[\mathbf{n}] + i|^{L+p-\alpha} |\nu|^{-(L+p)} \right] r^{i+[\mathbf{n}]+|\nu|} \|a_{[\mathbf{n}]+\mathbf{m}^*, i+j^*}\|. \end{aligned} \tag{4.59}$$

Let us now take

$$S_{L,p}(r) \quad := \quad \sum_{[\mathbf{n}]=1-|\mathbf{m}^*|}^{+\infty} \sum_{i=-j^*}^{k-1-j^*} \frac{|[\mathbf{n}]+i|^{L+p-\alpha} r^{i+[\mathbf{n}]+|\nu|}}{|\nu|^{L+p}} \|a_{[\mathbf{n}]+\mathbf{m}^*, i+j^*}\|. \tag{4.60}$$

Applying inequality (4.28) for the particular case $l=\alpha$, $s=0$ and $|\mathbf{l}|=1$ to the previous line, leads to

$$S_{L,p}(r) \leq \mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon}, \qquad r \notin F. \tag{4.61}$$

Therefore, one gets

$$\begin{aligned} & \left\| \frac{E^s[\Gamma^{2d}[f(\mathbf{x})]]}{|\nu|^{s+2d}} - f(\mathbf{x}) \right\| \\ \leq \; & \|a_{\mathbf{0},0}\| + \mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon} \sum_{\substack{0 \leq L \leq s, \\ 0 \leq p \leq 2d, \\ L+p \neq 0}} \binom{2d}{p}\binom{s}{L}((n-1)+2(k-1))^p \sum_{\alpha=0}^{p} \binom{p}{\alpha} \\ \leq \; & \|a_{\mathbf{0},0}\| + \left(((n-1)+2(k-1))^{2d} \sum_{L=0}^{s} \binom{s}{L} \sum_{p=0}^{2d} \binom{2d}{p} \sum_{\alpha=0}^{p} \binom{p}{\alpha} \right) \mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon} \end{aligned}$$

and, using $\sum\limits_{p=0}^{2d} \binom{2d}{p} \leq 2^{2d}$, one obtains

$$\begin{aligned} & \left\| \frac{E^s[\Gamma^{2d}[f(\mathbf{x})]]}{|\nu|^{s+2d}} - f(\mathbf{x}) \right\| \\ \leq \; & \|a_{\mathbf{0},0}\| + \left(((n-1)+2(k-1))^{2d} \sum_{L=0}^{s} \binom{s}{L} \sum_{p=0}^{2d} \binom{2d}{p} 2^p \right) \mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon} \\ \leq \; & \|a_{\mathbf{0},0}\| + \left(((n-1)+2(k-1))^{2d} 2^{4d+s} \right) \mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon} \\ \leq \; & c_{s,d}\mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon}. \end{aligned} \tag{4.62}$$

Applying (4.62) it follows

$$\begin{aligned} & \left\| \sum_{s,d=0}^{N} \gamma_{s,d} \frac{E^s[\Gamma^{2d}[f(\mathbf{x})]]}{|\nu|^{s+2d}} - \sum_{s,d=0}^{N} \gamma_{s,d} f(\mathbf{x}) \right\| \\ \leq \; & \sum_{s,d=0}^{N} |\gamma_{s,d}| \left\| \frac{E^s[\Gamma^{2d}[f(\mathbf{x})]]}{|\nu|^{s+2d}} - f(\mathbf{x}) \right\| \\ \leq \; & \sum_{s,d=0}^{N} |\gamma_{s,d}| c_{s,d}\mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon} \\ = \; & C\mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon}. \end{aligned} \tag{4.63}$$

Here C is a proper positive real constant. □

As special cases we obtain particularly.

Corollary 4.1 *Let $d \in \mathbb{N}$ be an arbitrary positive integer and $f : \mathbb{R}^n \to Cl_n$ be an entire k-monogenic function. Then f satisfies asymptotically*

$$\left\| \frac{\Gamma^{2d}[f(\mathbf{x})]}{|\nu|^{2d}} - f(\mathbf{x}) \right\| \leq C\mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon}, \quad r \notin F, \tag{4.64}$$

where C is a real constant, $\varepsilon > 0$ and F is a set of finite logarithmic measure.

Corollary 4.2 *Let $s \in \mathbb{N}$ be an arbitrary positive integer and $f : \mathbb{R}^n \to Cl_n$ be an entire k-monogenic function. Then f satisfies asymptotically*

$$\left\| \frac{E^s[f(\mathbf{x})]}{|\nu|^s} - f(\mathbf{x}) \right\| \leq C\mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon}, \quad r \notin F, \tag{4.65}$$

where C is a real constant, $\varepsilon > 0$ and F is a set of finite logarithmic measure.

The following result gives an explicit asymptotic relation between the growth of solutions of the iterated Dirac and that of the polynomial in E and Γ^2

Proposition 4.7 *Let f be a k-monogenic function, $0 < \delta < \frac{1}{2}$ and $\|\mathbf{x}\|$ be sufficiently large such that for $\|\mathbf{x}\| = r$, the relation*

$$\|f(\mathbf{x})\| > \mu(r)|\nu(r)|^{-\frac{1}{2}+\delta}, \quad r \notin F \tag{4.66}$$

for $\mathbf{x}$ in a neighborhood $\mathcal{V}_{\mathbf{x}_0}$ such that $\|f(\mathbf{x}_0)\| := \max_{\|\mathbf{x}\|=r}\{\|f(\mathbf{x})\|\}$ is satisfied. Let

$$\mathcal{L}[f(\mathbf{x})] := \sum_{s,d=0}^{N} \gamma_{s,d} \frac{E^s[\Gamma^{2d}[f(\mathbf{x})]]}{|\nu|^{s+2d}}$$

be a polynomial in E and Γ^2 with real coefficients. Then we get the asymptotic estimate

$$\mathcal{L}[f(\mathbf{x})] - \sum_{s,d=0}^{N} \gamma_{s,d} f(\mathbf{x}) = o(1)f(\mathbf{x}), \quad r \in \mathcal{V}_{\mathbf{x}_0}. \tag{4.67}$$

Proof. Let us now suppose that $\|\mathbf{x}\| = r \notin F$. In view of (4.66) and Theorem 4.3 we have

$$\begin{aligned} & \frac{1}{\|f(\mathbf{x})\|}\left\| \sum_{s,d=0}^{N} \gamma_{s,d} \frac{E^s[\Gamma^{2d}[f(\mathbf{x})]]}{|\nu|^{s+2d}} - \sum_{s,d=0}^{N} \gamma_{s,d} f(\mathbf{x}) \right\| \\ \leq \; & \frac{C}{\mu(r)} |\nu(r)|^{\frac{1}{2}-\delta} \mu(r) |\nu(r)|^{-\frac{1}{2}+\varepsilon} \\ \leq \; & C|\nu(r)|^{\varepsilon-\delta}, \end{aligned} \tag{4.68}$$

which tends to zero, when choosing ε sufficiently small (i.e., $\varepsilon < \delta$). We then obtain

$$\mathcal{L}[f(\mathbf{x})] - \sum_{s,d=0}^{N} \gamma_{s,d} f(\mathbf{x}) = o(1) f(\mathbf{x}).$$

□

In the next theorem one obtains a classification of the solution of special type of partial differential equation of 1-monogenic paravector valued function. The proof can be done analogously to the one given in Theorem 4.2.

Theorem 4.4 *If g is an entire 1-monogenic paravector valued function with order $\rho_2 < \infty$ and*

$$\|g(\mathbf{x})\| > \mu(r)|\nu(r)|^{-\frac{1}{2}+\delta}, \; r \notin F$$

for $\mathbf{x}$ in a neighborhood $\mathcal{V}_{\mathbf{x}_0}$ of $\mathbf{x}_0$ such that $\|g(\mathbf{x}_0)\| := \max_{\|\mathbf{x}\|=r}\{\|g(\mathbf{x})\|\}$ is satisfied. Let

$$M_j[g] = a_j \prod_{\substack{i = s+2d \\ i=0}}^{k} [E^s(\Gamma^{2d}(g)))]^{n_i}$$

where a_j are polynomials of degree j, and $M_j[g]$ has degree $\gamma_{M_j} = \sum_{i=0}^{k} n_i$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k} i n_i$. Let

$$Q[g] = \sum_{j=0}^{s} M_j[g]$$

be of degree γ_Q and weight Γ_Q. If $\gamma_Q = \gamma_{M_0}$, then the differential equation $Q[g] = 0$ has no transcendental entire solution.

Chapter 5

Open problems

We conclude this work by presenting some open problems for future research.

In the construction of normality criteria (given in Chapter 2) the possible occurrence of singularities of dimension 0, 1, ..., $n-1$ in the context of Clifford valued functions, caused great difficulties. One also has that the behavior in the neighborhood of an isolated singularity is very irregular, when comparing it to the behavior of the meromorphic functions in the complex case. Consequently, the following questions can be posed:

Question 1: Which type of compactification of the space can be found, such that a general Clifford valued function has a regular behavior in a neighborhood of the singularities?

Question 2: Is it possible to obtain similar criteria for general Clifford valued meromorphic functions, or even for the general case dealing with Clifford valued functions?

Regarding sufficient conditions for normality, for instance, the known classical results:

Due to Montel [53]:

Montel's Theorem: *Let $\mathcal{F}$ be a family of meromorphic complex valued functions defined in the domain D. If there exists three points w_1, w_2, w_3 on the Riemann sphere such that $w_i \notin f(D)$ $(i = 1, 2, 3)$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

Due to X. Pang and L. Zalcman [57]:

Theorem: *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc D, with the property that all their zeros are of multiplicity (at least) k. If there exist $b \neq 0$ and a positive constant c such that for every $f \in \mathcal{F}$, $\overline{\mathcal{E}}_f(0) = \overline{\mathcal{E}}_{f^{(k)}}(b)$ and $0 < \|f^{(k+1)}(z)\| \leq c$ whenever $z \in \overline{\mathcal{E}}_f(0) := \{z \in D : \; f(z) = 0\}$, then $\mathcal{F}$ is a normal family on D.*

It is natural to ask:

Question 3: Is it possible to establish generalizations of these types of results to general Clifford valued functions?

In the analysis of the growth behavior we observed relations between the maximum modulus, the maximum term and the norm of the central index in the framework of polymonogenic functions.

In the complex case, due to Borel [7], one has a relation between the growth of an entire function and the growth of its real part, i.e.,

Borel's Theorem: *Let g be an entire transcendental function, $M(r) := \max_{\|z\|=r} \|g(z)\|$ and $A(r) := \max_{\|z\|=r} |Re(g(z))|$. Then, for $0 < r < R$*

$$M(r) \leq \frac{R}{R-r}[4A(R) + 2\|g(0)\|].$$

Question 4: In the context of polymonogenic functions, can one estimate the growth of an entire function by the growth of one of its component functions?

Another question arise when observing that the different generalized exponential functions in Example 3.2, Example 3.4 and Example 3.3 have the same order of growth but different *type*.

Question 5: What can one say about the growth behavior of other elementary examples of Clifford valued functions? Which general relations do exist between the *type* and the order of growth of polymonogenic functions?

Since this work is based on polymonogenic functions it is natural to ask:

Question 6: Is it possible to establish similar results for meromorphic functions ?

In this work we studied entire solutions of the iterated Dirac equation as well as entire solutions of the iterated generalized Cauchy-Riemann equation. One can also analyze the asymptotic growth behavior concerning, for example:

- entire solutions to higher dimension polynomial Cauchy-Riemann equation;
- entire paravector valued solution to the hypermonogenic equation.

Furthermore, one could also look for normality criteria for these classes of functions.

List of principal symbols

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