**Universidade de Aveiro** Departamento de Matemática
**2010**

**Rui Alexandre Cardoso Ferreira**

# Cálculo das Variações em Escalas Temporais e Cálculo Fraccionário Discreto

# Calculus of Variations on Time Scales and Discrete Fractional Calculus

"When I have clarified and exhausted a subject, then
I turn away from it, in order to go into darkness again."

Carl Friedrich Gauss

**Universidade de Aveiro** Departamento de Matemática
**2010**

**Rui Alexandre Cardoso Ferreira**

# Cálculo das Variações em Escalas Temporais e Cálculo Fraccionário Discreto

# Calculus of Variations on Time Scales and Discrete Fractional Calculus

Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Doutor Delfim Fernando Marado Torres, Professor Associado do Departamento de Matemática da Universidade de Aveiro, e sob a co-orientação científica do Doutor Martin Bohner, Professor Catedrático do Departamento de Matemática e Estatística da Universidade de Ciência e Tecnologia de Missouri, EUA.

Thesis submitted to the University of Aveiro in fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics, under the supervision of Dr. Delfim Fernando Marado Torres, Associate Professor at the Department of Mathematics of the University of Aveiro, and co-supervision of Dr. Martin Bohner, Professor at the Department of Mathematics and Statistics of Missouri University of Science and Technology, Missouri, USA.

Apoio financeiro da FTC (Fundação para a Ciência e Tecnologia) através da bolsa de doutoramento com referência SFRH/BD/39816/2007.

Como sempre, dedico esta tese à minha família.

**o júri**

presidente

Prof. Doutor António Manuel Melo de Sousa Pereira
professor catedrático da Universidade de Aveiro
(por delegação do Reitor da Universidade de Aveiro)

Prof. Doutor Martin Bohner
professor catedrático da Missouri University of Science and Technology, Estados Unidos da América (co-orientador)

Prof. Doutor António Costa Ornelas Gonçalves
professor associado com agregação da Escola de Ciências e Tecnologia da Universidade de Évora

Prof. Doutor Delfim Fernando Marado Torres
professor associado da Universidade de Aveiro (orientador)

Prof. Doutora Maria Luísa Ribeiro dos Santos Morgado
professora auxiliar da Escola de Ciências e Tecnologia da Universidade Trás-os-Montes e Alto Douro

Prof. Doutora Natália da Costa Martins
professora auxiliar da Universidade de Aveiro

**agradecimentos**

**acknowledgements**

I am deeply grateful to my advisors, Delfim F. M. Torres and Martin Bohner, for their inestimable support during this thesis elaboration. I must emphasize the indisputable scientific support that I was provided, being a privilege to work with them. The hospitality of Martin Bohner during my stay in Rolla is also gratefully acknowledged.

I also thank *The Portuguese Foundation for Science and Technology* (FCT) for the financial support through the PhD fellowship SFRH/BD/39816/2007, without which this work wouldn't be possible.

My next thank goes to the people with whom I interacted at the University of Aveiro, Missouri University of Science and Technology and the Lusophone University of Humanities and Technologies, that in some way contributed to this work.

To all my friends I just want to say: "Thank you for all your encouragement words!".

Finally, and as always, I wish to thank and dedicate this dissertation to my family.

FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR Portugal

**palavras-chave**

Cálculo das variacões, condições necessárias de optimalidade do tipo de Euler–Lagrange e Legendre, controlo óptimo, derivada delta, derivada diamond, derivada fraccionária discreta, desigualdades integrais, escalas temporais.

**resumo**

Estudamos problemas do cálculo das variações e controlo óptimo no contexto das escalas temporais. Especificamente, obtemos condições necessárias de optimalidade do tipo de Euler–Lagrange tanto para lagrangianos dependendo de derivadas delta de ordem superior como para problemas isoperimétricos. Desenvolvemos também alguns métodos directos que permitem resolver determinadas classes de problemas variacionais através de desigualdades em escalas temporais. No último capítulo apresentamos operadores de diferença fraccionários e propomos um novo cálculo das variações fraccionário em tempo discreto. Obtemos as correspondentes condições necessárias de Euler–Lagrange e Legendre, ilustrando depois a teoria com alguns exemplos.

**keywords**

Calculus of variations, necessary optimality conditions of Euler–Lagrange and Legendre type, optimal control, delta-derivative, diamond-derivative, discrete fractional derivative, integral inequalities, time scales.

**abstract**

We study problems of the calculus of variations and optimal control within the framework of time scales. Specifically, we obtain Euler–Lagrange type equations for both Lagrangians depending on higher order delta derivatives and isoperimetric problems. We also develop some direct methods to solve certain classes of variational problems via dynamic inequalities. In the last chapter we introduce fractional difference operators and propose a new discrete-time fractional calculus of variations. Corresponding Euler–Lagrange and Legendre necessary optimality conditions are derived and some illustrative examples provided.

**2010 Mathematics Subject Classification:** 26D10, 26D15, 26E70, 34A40, 34N05, 39A12, 45J05, 49K05.

# Contents

# Introduction

This work started when the author first met his advisor Delfim F. M. Torres in July, 2006. In a preliminary conversation Delfim F. M. Torres introduced to the author the Time Scales calculus.

> "A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both."

E. T. Bell wrote this in 1937 and Stefan Hilger accomplished it in his PhD thesis [60] in 1988. Stefan Hilger defined a time scale as a nonempty closed subset of the real numbers and on such sets he defined a (delta) $\Delta$-derivative and a $\Delta$-integral. These delta derivative and integral are the classical derivative and Riemann integral when the time scale is the set of the real numbers, and become the forward difference and a sum when the time scale is the set of the integer numbers (cf. the definitions in Chapter 1). This observation highlights one of the purposes of Hilger's work: unification of the differential and difference calculus. But at the same time, since a time scale can be any other subset of the real numbers, an extension was also achieved (see Remark 49 pointing out this fact). As written in [24], the time scales calculus has a tremendous potential for applications. For example, it can model insect populations that are continuous while in season (and may follow a difference scheme with variable step-size), die out in (say) winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. Motivated by its theoretical and practical usefulness many researchers have dedicated their time to the development of the time scales calculus in the last twenty years (cf. [24, 25] and references therein).

Since Delfim F. M. Torres works mainly in Calculus of Variations and Optimal Control and in 2006 only three papers [14, 21, 61], one of them by Martin Bohner [21], within this topic were available in the literature, we decided to invite Martin Bohner, whose knowledge in time scales calculus is widely known, to be also an author's advisor and we began to make an attempt to develop the theory. This was indeed the main motivation for this work.

The two research papers [21] and [61] were published in 2004. In the first one, necessary conditions for weak local minima of the basic problem of the calculus of variations were established, among them the Euler–Lagrange equation, the Legendre necessary condition, the strengthened Legendre condition, and the Jacobi condition, while in the second one, a calculus of variations problem with variable endpoints was considered in the space of piecewise rd-continuously $\Delta$-differentiable functions. For this problem, the Euler-Lagrange equation, the transversality condition, and the accessory problem were derived as necessary conditions for weak local optimality. Moreover, assuming the coercivity of the second variation, a corresponding second order sufficiency criterion was established. The paper [14] was published in 2006 and there, were also obtained necessary optimality conditions for the basic problem of the calculus of variations but this time for the class of functions having continuous (nabla) $\nabla$-derivative (cf. Definition 5). The authors of [14] find this derivative most adequate to use in some problems arising in economics.

In a first step towards the development of the theory we decided to study a variant of the basic problem of the calculus of variations, namely, the problem in which the Lagrangian depends on higher order $\Delta$-derivatives. What seemed to be a natural consequence of the basic problem (as it is in the differential case) it turned out not to be! We only achieved partial results for this problem, being the time scales restricted to the ones with forward operator given by $\sigma(t) = a_1 t + a_0$ for some $a_1 \in \mathbb{R}^+$ and $a_0 \in \mathbb{R}$, $t \in [a, \rho(b)]_{\mathbb{T}}$ (see Section 4.1 for the discussions). Another subject that we studied was the isoperimetric problem and here we were able to prove the corresponding Euler–Lagrange equation on a general time scale (cf. Theorem 63). We were also interested in using direct methods to solve some calculus of variations and optimal control problems. This is particularly useful since even simple classes of problems lead to dynamic Euler–Lagrange equations for which methods to compute explicit solutions are very hard or not known. In order to accomplish this, some integral inequalities[1] needed to be proved on a general time scale. This is shown in Chapter 5.

Another subject that interests us and that we have been working in is the development of the discrete fractional calculus [76] and, in particular, of the discrete fractional calculus of variations. Recently, two research papers [15, 16] appeared in the literature and therein it is suggested that in the future a general theory of a *fractional time scales calculus* can exist (see the book [77] for an introduction to fractional calculus and its applications). We have made already some progresses in this field (specifically for the time scale $h\mathbb{Z}, h > 0$), namely in the calculus of variations, and we present our results in Chapter 7.

This work is divided in two major parts. The first part has three chapters in which we provide some preliminaries about time scales calculus, calculus of variations, and integral inequalities. The second part is divided in four chapters and in each we present our original work. Specifically, in Section 4.1, the Euler–Lagrange equation for problems of the calculus of variations depending on the $\Delta$-derivatives up to order $r \in \mathbb{N}$ of a function is obtained. In Section 4.2, we prove a necessary optimality condition for isoperimetric problems. In Sections

[1]We have also applied some integral inequalities to solve some initial value problems on time scales (cf. Chapter 6).

5.1 and 5.2, some classes of variational problems are solved directly with the help of some integral inequalities. In Section 6.1 some extensions of Gronwall's type inequalities are obtained while in Section 6.2, using topological methods, we prove that certain integrodynamic equations have $C^1[\mathbb{T}]$ solutions. In Section 6.3, we prove Hölder, Cauchy–Schwarz, Minkowski and Jensen's type inequalities in the more general setting of $\Diamond_\alpha$-integrals. In Section 6.4, we obtain some Gronwall–Bellman–Bihari type inequalities for functions depending on two time scales variables. Chapter 7 is devoted to a preliminary study of the calculus of variations using discrete fractional derivatives. In Section 7.2, we obtain some results that in turn will be used in Section 7.3 to prove the main results, namely, an Euler–Lagrange type equation (cf. Theorem 151) and a Legendre's necessary optimality condition (cf. Theorem 154).

Finally, we write our conclusions in Chapter 8 as well as some projects for the future development of some subjects herein studied.

# Part I

# Synthesis

# Chapter 1

# Time Scales Calculus: definitions and basic results

A nonempty closed subset of $\mathbb{R}$ is called a *time scale* and is denoted by $\mathbb{T}$. The two most basic examples of time scales are the sets $\mathbb{R}$ and $\mathbb{Z}$. It is clear that closed subsets of $\mathbb{R}$ may not be connected, e.g., $[0,1] \cup [2,3]$. The following operators are massively used within the theory:

**Definition 1.** The mapping $\sigma : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum $M$) is called *forward jump operator*. Accordingly we define the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ with $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum $m$). The symbol $\emptyset$ denotes the empty set.

The following classification of points is used within the theory: A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* and *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively.

Let us define the sets $\mathbb{T}^{\kappa^n}$, inductively: $\mathbb{T}^{\kappa^1} = \mathbb{T}^{\kappa} = \{t \in \mathbb{T} : t \text{ non-maximal or left-dense}\}$ and $\mathbb{T}^{\kappa^n} = (\mathbb{T}^{\kappa^{n-1}})^{\kappa}$, $n \geq 2$. Also, $\mathbb{T}_{\kappa} = \{t \in \mathbb{T} : t \text{ non-minimal or right-dense}\}$ and $\mathbb{T}_{\kappa^n} = (\mathbb{T}_{\kappa^{n-1}})_{\kappa}$, $n \geq 2$. We denote $\mathbb{T}^{\kappa} \cap \mathbb{T}_{\kappa}$ by $\mathbb{T}^{\kappa}_{\kappa}$.

The functions $\mu, \nu : \mathbb{T} \to [0, \infty)$ are defined by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$.

**Example 2.** If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and $\mu(t) = \nu(t) = 0$. If $\mathbb{T} = [0,1] \cup [2,3]$, then

$$\sigma(t) = \begin{cases} t & \text{if } t \in [0,1) \cup [2,3]; \\ 2 & \text{if } t = 1, \end{cases}$$

while,

$$\rho(t) = \begin{cases} t & \text{if } t \in [0,1] \cup (2,3]; \\ 1 & \text{if } t = 2. \end{cases}$$

Also, we have that,

$$\mu(t) = \begin{cases} 0 & \text{if } t \in [0,1) \cup [2,3]; \\ 1 & \text{if } t = 1, \end{cases}$$

and,

$$\nu(t) = \begin{cases} 0 & \text{if } t \in [0,1] \cup (2,3]; \\ 1 & \text{if } t = 2. \end{cases}$$

For two points $a, b \in \mathbb{T}$, the time scales interval is defined by

$$[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

We now state the two definitions of differentiability on time scales. Throughout we will frequently write $f^\sigma(t) = f(\sigma(t))$ and $f^\rho(t) = f(\rho(t))$.

**Definition 3.** We say that a function $f : \mathbb{T} \to \mathbb{R}$ is $\Delta$-*differentiable* at $t \in \mathbb{T}^\kappa$ if there is a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood $U$ of $t$ (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the $\Delta$-*derivative* of $f$ at $t$.

*Remark* 4. We note that if the number $f^\Delta(t)$ of Definition 3 exists then it is unique (see [58, 59], and also the recent paper [27] written in Portuguese).

The $\Delta$-derivative of order $n \in \mathbb{N}$ of a function $f$ is defined by $f^{\Delta^n}(t) = \left(f^{\Delta^{n-1}}(t)\right)^\Delta$, $t \in \mathbb{T}^{\kappa^n}$, provided the right-hand side of the equality exists, where $f^{\Delta^0} = f$.

**Definition 5.** We say that a function $f : \mathbb{T} \to \mathbb{R}$ is $\nabla$-*differentiable* at $t \in \mathbb{T}_\kappa$ if there is a number $f^\nabla(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood $U$ of $t$ (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f^\rho(t) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \varepsilon|\rho(t) - s|, \quad \text{for all } s \in U.$$

We call $f^\nabla(t)$ the $\nabla$-*derivative* of $f$ at $t$.

Some basic properties will now be given for the $\Delta$-derivative. We refer the reader to the works presented in [24] for results on $\nabla$-derivatives.

*Remark* 6. Our results are mainly proved using the $\Delta$-derivative. We point out that it is immediate to get the analogous ones for the $\nabla$-derivative. This is done in an elegant way using the recent duality theory of C. Caputo [29].

**Theorem 7.** *[24, Theorem 1.16] Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following:*

1. *If $f$ is $\Delta$-differentiable at $t$, then $f$ is continuous at $t$.*

2. *If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is $\Delta$-differentiable at $t$ with*

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}. \tag{1.1}$$

*3. If $t$ is right-dense, then $f$ is $\Delta$-differentiable at $t$ if and only if the limit*

$$\lim_{s\to t}\frac{f(s)-f(t)}{s-t}$$

*exists as a finite number. In this case,*

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(s)-f(t)}{s-t}.$$

*4. If $f$ is $\Delta$-differentiable at $t$, then*

$$f^{\sigma}(t)=f(t)+\mu(t)f^{\Delta}(t). \tag{1.2}$$

It is an immediate consequence of Theorem 7 that if $\mathbb{T}=\mathbb{R}$, then the $\Delta$-derivative becomes the classical one, i.e., $f^{\Delta}=f'$ while if $\mathbb{T}=\mathbb{Z}$, the $\Delta$-derivative reduces to the *forward difference* $f^{\Delta}(t)=\Delta f(t)=f(t+1)-f(t)$.

**Theorem 8.** *[24, Theorem 1.20] Assume $f,g:\mathbb{T}\to\mathbb{R}$ are $\Delta$-differentiable at $t\in\mathbb{T}^{\kappa}$. Then:*

1. *The sum $f+g:\mathbb{T}\to\mathbb{R}$ is $\Delta$-differentiable at $t$ and $(f+g)^{\Delta}(t)=f^{\Delta}(t)+g^{\Delta}(t)$.*

2. *For any number $\xi\in\mathbb{R}$, $\xi f:\mathbb{T}\to\mathbb{R}$ is $\Delta$-differentiable at $t$ and $(\xi f)^{\Delta}(t)=\xi f^{\Delta}(t)$.*

3. *The product $fg:\mathbb{T}\to\mathbb{R}$ is $\Delta$-differentiable at $t$ and*

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f^{\sigma}(t)g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g^{\sigma}(t). \tag{1.3}$$

The next lemma will be used later in this text, specifically, in Section 4.1. We give a proof here.

**Lemma 9.** *Let $t\in\mathbb{T}^{\kappa}$ ($t\neq\min\mathbb{T}$) satisfy the property $\rho(t)=t<\sigma(t)$. Then, the jump operator $\sigma$ is not continuous at $t$ and therefore not $\Delta$-differentiable also.*

*Proof.* We begin to prove that $\lim_{s\to t^-}\sigma(s)=t$. Let $\varepsilon>0$ and take $\delta=\varepsilon$. Then for all $s\in(t-\delta,t)$ we have $|\sigma(s)-t|\leq|s-t|<\delta=\varepsilon$. Since $\sigma(t)>t$, this implies that $\sigma$ is not continuous at $t$, hence not $\Delta$-differentiable by 1. of Theorem 7. □

Now we turn to $\Delta$-*integration on time scales*.

**Definition 10.** A function $f:\mathbb{T}\to\mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points and if the left-sided limit exists at left-dense points.

*Remark* 11. We denote the set of all rd-continuous functions by $\mathrm{C}_{\mathrm{rd}}$ or $\mathrm{C}_{\mathrm{rd}}(\mathbb{T})$, and the set of all $\Delta$-differentiable functions with rd-continuous derivative by $\mathrm{C}^1_{\mathrm{rd}}$ or $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T})$.

**Example 12.** Consider $\mathbb{T} = \bigcup_{k=0}^{\infty}[2k, 2k+1]$. For this time scale,

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{k=0}^{\infty}[2k, 2k+1); \\ 1 & \text{if } t \in \bigcup_{k=0}^{\infty}\{2k+1\}. \end{cases}$$

Let us consider $t \in [0,1] \cap \mathbb{T}$. Then, we have

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}, \ t \in [0,1)\,,$$

provided this limit exists, and

$$f^{\Delta}(1) = \frac{f(2) - f(1)}{2 - 1},$$

provided $f$ is continuous at $t = 1$. Let $f$ be defined on $\mathbb{T}$ by

$$f(t) = \begin{cases} t & \text{if } t \in [0,1); \\ 2 & \text{if } t \geq 1. \end{cases}$$

We observe that at $t = 1$ $f$ is rd-continuous (since $\lim_{t \to 1^-} f(t) = 1$) but not continuous (since $f(1) = 2$).

**Definition 13.** A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T}^{\kappa} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$.

**Theorem 14.** *[24, Theorem 1.74] Every rd-continuous function has an antiderivative.*

Let $f : \mathbb{T}^{\kappa} :\to \mathbb{R}$ be a rd-continuous function and let $F : \mathbb{T} \to \mathbb{R}$ be an antiderivative of $f$. Then, the $\Delta$-*integral* is defined by $\int_s^r f(t)\Delta t = F(r) - F(s)$ for all $r, s \in \mathbb{T}$. It satisfies

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t), \quad t \in \mathbb{T}^{\kappa}. \tag{1.4}$$

**Theorem 15.** *[24, Theorem 1.77] Let $a, b, c \in \mathbb{T}$, $\xi \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T}^{\kappa})$. Then,*

1. $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$.
2. $\int_a^b (\xi f)(t)\Delta t = \xi \int_a^b f(t)\Delta t$.
3. $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$.
4. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$.
5. $\int_a^a f(t)\Delta t = 0$.
6. *if* $|f(t)| \leq g(t)$ *on* $[a,b]_{\mathbb{T}}^{\kappa}$, *then*

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t.$$

One can easily prove [24, Theorem 1.79] that, when $\mathbb{T} = \mathbb{R}$ then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, being the right-hand side of the equality the usual Riemann integral, and when $\mathbb{T} = \mathbb{Z}$ then $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$.

*Remark* 16. It is an immediate consequence of the last item in Theorem 15 that,

$$f(t) \leq g(t), \quad \forall t \in [a,b]_{\mathbb{T}}^{\kappa} \quad \text{implies} \quad \int_a^b f(t)\Delta t \leq \int_a^b g(t)\Delta t. \tag{1.5}$$

We now present the (often used in this text) integration by parts formulas for the $\Delta$-integral:

$$\int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^{\Delta}(t)g(t)\Delta t; \tag{1.6}$$

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t. \tag{1.7}$$

*Remark* 17. For analogous results on $\nabla$-integrals the reader can consult, e.g., [25].

Some more definitions and results must be presented since they will be used in the sequel. We start defining the polynomials on time scales. Throughout the text we put $\mathbb{N}_0 = \{0, 1, \ldots\}$.

**Definition 18.** The functions $g_k, h_k : \mathbb{T}^2 \to \mathbb{R}$, defined recursively by

$$g_0(t,s) = h_0(t,s) = 1, \quad s,t \in \mathbb{T},$$

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau), s)\Delta\tau, \quad h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\Delta\tau, \quad k \in \mathbb{N}_0, \quad s,t \in \mathbb{T},$$

are called the *polynomials on time scales*.

Now we define the *exponential function*. First we need the concept of *regressivity*.

**Definition 19.** A function $p : \mathbb{T}^{\kappa} \to \mathbb{R}$ is *regressive* provided

$$1 + \mu(t)p(t) \neq 0$$

holds for all $t \in \mathbb{T}^{\kappa}$. We denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions. The set of all *positively regressive* functions is defined by

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \text{ for all } t \in \mathbb{T}^{\kappa}\}.$$

**Theorem 20.** *[25, Theorem 1.37] Suppose $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then the initial value problem*

$$y^{\Delta} = p(t)y, \; y(t_0) = 1 \tag{1.8}$$

*has a unique solution on $\mathbb{T}$.*

**Definition 21.** Let $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$. The exponential function on time scales is defined by the solution of the IVP (1.8) and is denoted by $e_p(\cdot, t_0)$.

*Remark* 22. If $\mathbb{T} = \mathbb{R}$, the exponential function is given by

$$e_p(t, t_0) = e^{\int_{t_0}^{t} p(\tau) d\tau},$$

with $t, t_0 \in \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{R}$ a continuous function. For $\mathbb{T} = \mathbb{Z}$, the exponential function is

$$e_p(t, t_0) = \prod_{\tau = t_0}^{t} [1 + p(\tau)],$$

for $t, t_0 \in \mathbb{Z}$ with $t_0 < t$ and $p : \mathbb{Z} \to \mathbb{R}$ a sequence satisfying $p(t) \neq -1$ for all $t \in \mathbb{Z}$. Further examples of exponential functions can be found in [24, Sect. 2.3].

The reader can find several properties of the exponential function in [24]. Let us now present some results about linear dynamic equations.

For $n \in \mathbb{N}_0$ and rd-continuous functions $p_i : \mathbb{T} \to \mathbb{R}$, $1 \leq i \leq n$, let us consider the $n$th order linear dynamic equation

$$Ly = 0\,, \qquad \text{where } Ly = y^{\Delta^n} + \sum_{i=1}^{n} p_i y^{\Delta^{n-i}}\,. \tag{1.9}$$

A function $y : \mathbb{T} \to \mathbb{R}$ is said to be a solution of equation (1.9) on $\mathbb{T}$ provided $y$ is $n$ times delta differentiable on $\mathbb{T}^{\kappa^n}$ and satisfies $Ly(t) = 0$ for all $t \in \mathbb{T}^{\kappa^n}$.

**Lemma 23.** *[24, p. 239] If $z = (z_1, \ldots, z_n) : \mathbb{T} \to \mathbb{R}^n$ satisfies for all $t \in \mathbb{T}^\kappa$*

$$z^\Delta = A(t) z(t), \qquad \textit{where} \qquad A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ -p_n & \ldots & \ldots & -p_2 & -p_1 \end{pmatrix} \tag{1.10}$$

*then $y = z_1$ is a solution of equation* (1.9). *Conversely, if $y$ solves* (1.9) *on $\mathbb{T}$, then $z = \left(y, y^\Delta, \ldots, y^{\Delta^{n-1}}\right) : \mathbb{T} \to \mathbb{R}$ satisfies* (1.10) *for all $t \in \mathbb{T}^{\kappa^n}$.*

**Definition 24.** [24, p. 239] We say that equation (1.9) is *regressive* provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^\kappa$, where $A$ is the matrix in (1.10).

**Definition 25.** [24, p. 243] Let $y_k : \mathbb{T} \to \mathbb{R}$ be (m-1) times $\Delta$-differentiable functions for all $1 \leq k \leq m$. We then define the *Wronski determinant* $W = W(y_1, \ldots, y_m)$ of the set $\{y_1, \ldots, y_m\}$ by $W(y_1, \ldots, y_m) = \det V(y_1, \ldots, y_m)$, where

$$V(y_1, \ldots, y_m) = \begin{pmatrix} y_1 & y_2 & \ldots & y_m \\ y_1^\Delta & y_2^\Delta & \ldots & y_m^\Delta \\ \vdots & \vdots & & \vdots \\ y_1^{\Delta^{m-1}} & y_2^{\Delta^{m-1}} & \ldots & y_m^{\Delta^{m-1}} \end{pmatrix}. \tag{1.11}$$

**Definition 26.** A set of solutions $\{y_1, \ldots, y_n\}$ of the regressive equation (1.9) is called a *fundamental system* for (1.9) if there is $t_0 \in \mathbb{T}^{\kappa^{n-1}}$ such that $W(y_1, \ldots, y_n) \neq 0$.

**Definition 27.** [24, p. 250] We define the Cauchy function $y : \mathbb{T} \times \mathbb{T}^{\kappa^n} \to \mathbb{R}$ for the linear dynamic equation (1.9) to be, for each fixed $s \in \mathbb{T}^{\kappa^n}$, the solution of the initial value problem

$$Ly = 0, \quad y^{\Delta^i}(\sigma(s), s) = 0, \quad 0 \leq i \leq n-2, \quad y^{\Delta^{n-1}}(\sigma(s), s) = 1\,. \tag{1.12}$$

**Theorem 28.** *[24, p. 251] Suppose $\{y_1, \ldots, y_n\}$ is a fundamental system of the regressive equation (1.9). Let $f \in C_{rd}$. Then the solution of the initial value problem*

$$Ly = f(t), \quad y^{\Delta^i}(t_0) = 0, \quad 0 \leq i \leq n-1\,,$$

*is given by $y(t) = \int_{t_0}^t y(t,s) f(s) \Delta s$, where $y(t,s)$ is the Cauchy function for* (1.9).

We end this section enunciating three theorems, namely, the derivative under the integral sign, a chain rule and a mean value theorem on time scales.

**Theorem 29.** *[24, Theorem 1.117] Let $t_0 \in \mathbb{T}^\kappa$ and assume $k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is continuous at $(t,t)$, where $t \in \mathbb{T}^\kappa$ with $t > t_0$. In addition, assume that $k(t,\cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood $U$ of $t$, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$|k(\sigma(t), \tau) - k(s, \tau) - k^{\Delta_1}(t, \tau)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in U,$$

*where $k^{\Delta_1}$ denotes the delta derivative of $k$ with respect to the first variable. Then,*

$$g(t) := \int_{t_0}^t k(t,\tau) \Delta\tau \text{ implies } g^\Delta(t) = \int_{t_0}^t k^{\Delta_1}(t,\tau)\Delta\tau + k(\sigma(t), t).$$

**Theorem 30.** *[24, Theorem 1.90] Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is $\Delta$-differentiable on $\mathbb{T}^\kappa$. Then, $f \circ g$ is $\Delta$-differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t) g^\Delta(t)) dh \right\} g^\Delta(t), \quad t \in \mathbb{T}^\kappa$$

*holds.*

**Theorem 31.** *[25, Theorem 1.14] Let $f$ be a continuous function on $[a,b]_\mathbb{T}$ that is $\Delta$-differentiable on $[a,b)_\mathbb{T}$. Then there exist $\xi, \tau \in [a,b)_\mathbb{T}$ such that*

$$f^\Delta(\xi) \leq \frac{f(b) - f(a)}{b - a} \leq f^\Delta(\tau).$$

# Chapter 2

# Calculus of Variations and Optimal Control

The calculus of variations deals with finding extrema and, in this sense, it can be considered a branch of optimization. The problems and techniques in this branch, however, differ markedly from those involving the extrema of functions of several variables owing to the nature of the domain on the quantity to be optimized. The calculus of variations is concerned with finding extrema for functionals, i.e., for mappings from a set of functions to the real numbers. The candidates in the competition for an extremum are thus functions as opposed to vectors in $\mathbb{R}^n$, and this furnishes the subject a distinct character. The functionals are generally defined by definite integrals; the set of functions are often defined by boundary conditions and smoothness requirements, which arise in the formulation of the problem/model. Let us take a look at the classical (basic) problem of the calculus of variations: find a function $y \in C^1[a,b]$ such that

$$\mathcal{L}[y(\cdot)] = \int_a^b L(t, y(t), y'(t))dt \longrightarrow \min, \quad y(a) = y_a, \quad y(b) = y_b, \tag{2.1}$$

with $a, b, y_a, y_b \in \mathbb{R}$ and $L(t, u, v)$ satisfying some smoothness properties.

The enduring interest in the calculus of variations is in part due to its applications. We now present a historical example of this.

**Example 32** (Brachystochrones)**.** The history of the calculus of variations essentially begins with a problem posed by Johann Bernoulli (1696) as a challenge to the mathematical comunity and in particular to his brother Jacob. The problem is important in the history of the calculus of variations because the method developed by Johann's pupil, Euler, to solve this problem provided a sufficiently general framework to solve other variational problems [92].

The problem that Johann posed was to find the shape of a wire along which a bead initially at rest slides under gravity from one end to the other in minimal time. The endpoints of the wire are specified and the motion of the bead is assumed frictionless. The curve corresponding to the shape of the wire is called a *brachystochrone* or a curve of fastest descent.

The problem attracted the attention of various mathematicians throughout the time including Huygens, L'Hôpital, Leibniz, Newton, Euler and Lagrange (see [92] and references cited therein for more historical details).

To model Bernoulli's problem we use Cartesian coordinates with the positive $y$-axis oriented in the direction of the gravitational force. Let $(a, y_a)$ and $(b, y_b)$ denote the coordinates of the initial and final positions of the bead, respectively. Here, we require that $a < b$ and $y_a < y_b$. The problem consists of determining, among the curves that have $(a, y_a)$ and $(b, y_b)$ as endpoints, the curve on which the bead slides down from $(a, y_a)$ to $(b, y_b)$ in minimum time. The problem makes sense only for continuous curves. We make the additional simplifying (but reasonable) assumptions that the curve can be represented by a function $y : [a, b] \to \mathbb{R}$ and that $y$ is at least piecewise differentiable in the interval $[a, b]$. Now, the total time it takes the bead to slide down a curve is given by

$$T[y(\cdot)] = \int_0^l \frac{ds}{v(s)}, \tag{2.2}$$

where $l$ denotes the arclength of the curve, $s$ is the arclength parameter, and $v$ is the velocity of the bead $s$ units down the curve from $(a, b)$.

We now derive an expression for the velocity in terms of the function $y$. We use the law of conservation of energy to achieve this. At any position $(x, y(x))$ on the curve, the sum of the potential and kinetic energies of the bead is a constant. Hence

$$\frac{1}{2}mv^2(x) + mgy(x) = c, \tag{2.3}$$

where $m$ is the mass of the bead, $v$ is the velocity of the bead at $(x, y(x))$, and $c$ is a constant. Solving equation (2.3) for $v$ gives

$$v(x) = \sqrt{\frac{2c}{m} - 2gy(x)}.$$

Equality (2.2) becomes

$$T[y(\cdot)] = \int_a^b \frac{\sqrt{1 + y'^2(x)}}{\sqrt{\frac{2c}{m} - 2gy(x)}} dx.$$

We thus seek a function $y$ such that $T$ is minimum and $y(a) = y_a$, $y(b) = y_b$.

It can be shown that the extrema for $T$ is a portion of the curve called *cycloid* (cf. Example 2.3.4 in [92]).

Let us return to the problem given in (2.1) and write the first and second order necessary optimality conditions for it, i.e., the well-known Euler–Lagrange equation and Legendre's necessary condition, respectively.

**Theorem 33.** *(cf., e.g., [92]) Suppose that $L_{uu}$, $L_{uv}$ and $L_{vv}$ exist and are continuous. Let $\hat{y} \in C^1[a,b]$ be a solution to the problem given in* (2.1)*. Then, necessarily*

1. $\frac{d}{dt} L_v(t, \hat{y}(t), \hat{y}'(t)) = L_u(t, \hat{y}(t), \hat{y}'(t)), \quad t \in [a,b]$ (Euler–Lagrange equation).

2. $L_{vv}(t, \hat{y}(t), \hat{y}'(t)) \geq 0, \quad t \in [a,b]$ (Legendre's condition).

It can be constructed a discrete analogue of problem (2.1): Let $a, b \in \mathbb{Z}$. Find a function $y$ defined on the set $\{a, a+1, \dots, b-1, b\}$ such that

$$\mathcal{L}[y(\cdot)] = \sum_{t=a}^{b-1} L(t, y(t+1), \Delta y(t)) \longrightarrow \min, \quad y(a) = y_a, \quad y(b) = y_b, \tag{2.4}$$

with $y_a, y_b \in \mathbb{R}$. For this problem we have the following result.

**Theorem 34.** *(cf., e.g., [7]) Suppose that $L_{uu}$, $L_{uv}$ and $L_{vv}$ exist and are continuous. Let $\hat{y}$ be a solution to the problem given in* (2.4)*. Then, necessarily*

1. $\Delta L_v(t) = L_u(t), \quad t \in \{a, \dots, b-2\}$ (discrete Euler–Lagrange equation).

2. $L_{vv}(t) + 2L_{uv}(t) + L_{uu}(t) + L_{vv}(t+1) \geq 0$, $t \in \{a, \dots, b-2\}$, *(discrete Legendre's condition),*

*where* $(t) = (t, \hat{y}(t+1), \Delta\hat{y}(t))$.

In 2004, M. Bohner wrote a paper [21] in which it was introduced the calculus of variations on time scales. We now present some of the results obtained.

**Definition 35.** A function $f$ defined on $[a,b]_{\mathbb{T}} \times \mathbb{R}$ is called *continuous in the second variable, uniformly in the first variable*, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies $|f(t, x_1) - f(t, x_2)| < \varepsilon$ for all $t \in [a,b]_{\mathbb{T}}$.

**Lemma 36** (cf. Lemma 2.2 in [21])**.** *Suppose that $F(x) = \int_a^b f(t,x)\Delta t$ is well defined. If $f_x$ is continuous in $x$, uniformly in $t$, then $F'(x) = \int_a^b f_x(t,x)\Delta t$.*

We now introduce the basic problem of the calculus of variations on time scales: Let $a, b \in \mathbb{T}$ and $L(t, u, v) : [a,b]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^2 \to \mathbb{R}$ be a function. Find a function $y \in \mathrm{C}^1_{\mathrm{rd}}$ such that

$$\mathcal{L}[y(\cdot)] = \int_a^b L(t, y^{\sigma}(t), y^{\Delta}(t))\Delta t \longrightarrow \min, \quad y(a) = y_a, \quad y(b) = y_b, \tag{2.5}$$

with $y_a, y_b \in \mathbb{R}$.

**Definition 37.** For $f \in \mathrm{C}^1_{\mathrm{rd}}$ we define the norm

$$\|f\| = \sup_{t \in [a,b]_{\mathbb{T}}^{\kappa}} |f^{\sigma}(t)| + \sup_{t \in [a,b]_{\mathbb{T}}^{\kappa}} |f^{\Delta}(t)|.$$

A function $\hat{y} \in \mathrm{C}^1_{\mathrm{rd}}$ with $\hat{y}(a) = y_a$ and $\hat{y}(b) = y_b$ is called a (weak) local minimum for problem (2.5) provided there exists $\delta > 0$ such that $\mathcal{L}(\hat{y}) \leq \mathcal{L}(\hat{y})$ for all $y \in \mathrm{C}^1_{\mathrm{rd}}$ with $y(a) = y_a$ and $y(b) = y_b$ and $\|y - \hat{y}\| < \delta$.

**Definition 38.** A function $\eta \in \mathrm{C}^1_{\mathrm{rd}}$ is called an *admissible variation* provided $\eta \neq 0$ and $\eta(a) = \eta(b) = 0$.

**Lemma 39** (cf. Lemma 3.4 in [21])**.** *Let $y, \eta \in C^1_{rd}$ be arbitrary fixed functions. Put $f(t,\varepsilon) = L(t, y^\sigma(t) + \varepsilon\eta^\sigma(t), y^\Delta(t) + \varepsilon\eta^\Delta(t))$ and $\Phi(\varepsilon) = \mathcal{L}(y + \varepsilon\eta)$, $\varepsilon \in \mathbb{R}$. If $f_\varepsilon$ and $f_{\varepsilon\varepsilon}$ are continuous in $\varepsilon$, uniformly in $t$, then*

$$\Phi'(\varepsilon) = \int_a^b [L_u(t, y^\sigma(t), y^\Delta(t))\eta^\sigma(t) + L_v(t, y^\sigma(t), y^\Delta(t))\eta^\Delta(t)]\Delta t,$$

$$\Phi''(\varepsilon) = \int_a^b \{L_{uu}[y](t)(\eta^\sigma(t))^2 + 2\eta^\sigma(t)L_{uv}[y](t)\eta^\Delta(t) + L_{vv}[y](t)(\eta^\Delta(t))^2\}\Delta t,$$

*where $[y](t) = (t, y^\sigma(t), y^\Delta(t))$.*

The next lemma is the time scales version of the classical Dubois–Reymond lemma.

**Lemma 40** (cf. Lemma 4.1 in [21])**.** *Let $g \in C_{rd}([a,b]^\kappa_{\mathbb{T}})$. Then,*

$$\int_a^b g(t)\eta^\Delta(t)\Delta t = 0, \quad \textit{for all } \eta \in C^1_{rd}([a,b]_{\mathbb{T}}) \textit{ with } \eta(a) = \eta(b) = 0,$$

*holds if and only if*

$$g(t) = c, \quad \textit{on } [a,b]^\kappa_{\mathbb{T}} \textit{ for some } c \in \mathbb{R}.$$

Nest theorem contains first and second order necessary optimality conditions for problem defined by (2.5). Its proof can be seen in [21].

**Theorem 41.** *Suppose that $L$ satisfies the assumption of Lemma 39. If $\hat{y} \in C^1_{rd}$ is a (weak) local minimum for problem given by* (2.5)*, then necessarily*

1. $L^\Delta_v[\hat{y}](t) = L_u[\hat{y}](t), \quad t \in [a,b]^{\kappa^2}_{\mathbb{T}}$ (time scales Euler–Lagrange equation).
2. $L_{vv}[\hat{y}](t) + \mu(t)\{2L_{uv}[\hat{y}](t) + \mu(t)L_{uu}[\hat{y}](t) + (\mu^\sigma(t))^* L_{vv}[\hat{y}](\sigma(t))\} \geq 0, \quad t \in [a,b]^{\kappa^2}_{\mathbb{T}}$ (time scales Legendre's condition),

*where $[y](t) = (t, y^\sigma(t), y^\Delta(t))$ and $\alpha^* = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R}\backslash\{0\}$ and $0^* = 0$.*

**Example 42.** Consider the following problem

$$\mathcal{L}[y(\cdot)] = \int_a^b [y^\Delta(t)]^2 \Delta t \longrightarrow \min, \quad y(a) = y_a, \quad y(b) = y_b. \tag{2.6}$$

Its Euler–Lagrange equation is $y^{\Delta^2}(t) = 0$ for all $t \in [a,b]^{\kappa^2}_{\mathbb{T}}$. It follows that $y(t) = ct + d$, and the constants $c$ and $d$ are obtained using the boundary conditions $y(a) = y_a$ and $y(b) = y_b$. Note that Legendre's condition is always satisfied in this problem since $\mu(t) \geq 0$ and $L_{uu} = L_{uv} = 0$, $L_{vv} = 2$.

# Chapter 3

# Inequalities

Inequalities have proven to be one of the most important tools for the development of many branches of mathematics. Indeed, this importance seems to have increased considerably during the last century and the theory of inequalities may nowadays be regarded as an independent branch of mathematics. A particular feature that makes the study of this interesting topic so fascinating arises from the numerous fields of applications, such as fixed point theory and calculus of variations.

The integral inequalities of various types have been widely studied in most subjects involving mathematical analysis. In recent years, the application of integral inequalities has greatly expanded and they are now used not only in mathematics but also in the areas of physics, technology and biological sciences [85]. Moreover, many physical problems arising in a wide variety of applications are governed by both ordinary and partial difference equations [64] and, since the integral inequalities with explicit estimates are so important in the study of properties (including the existence itself) of solutions of differential and integral equations, their discrete analogues should also be useful in the study of properties of solutions of difference equations.

An early significant integral inequality and certainly a keystone in the development of the theory of differential equations can be stated as follows.

**Theorem 43.** *If $u \in C[a,b]$ is a nonnegative function and*

$$u(t) \leq c + \int_a^t du(s)ds,$$

*for all $t \in [a,b]$ where $c, d$ are nonnegative constants, then the function $u$ has the estimate*

$$u(t) \leq c\exp(d(t-a)), \quad t \in [a,b].$$

The above theorem was discovered by Gronwall [55] in 1919 and is now known as *Gronwall's inequality.* The discrete version of Theorem 43 seems to have appeared first in the work of Mikeladze [75] in 1935.

In 1956, Bihari [20] gave a nonlinear generalization of Gronwall's inequality by studying the inequality

$$u(t) \leq c + \int_a^t dw(u(s))ds,$$

where $w$ satisfies some prescribed conditions, which is of fundamental importance in the study of nonlinear problems and is known as *Bihari's inequality*.

Other fundamental inequalities as, the Hölder (in particular, Cauchy–Schwarz) inequality, the Minkowski inequality and the Jensen inequality caught the fancy of a number of research workers. Let us state and then provide a simple example of application (specifically to a calculus of variations problem) of the Jensen inequality.

**Theorem 44.** [78] *If $g \in C([a,b],(c,d))$ and $f \in C((c,d),\mathbb{R})$ is convex, then*

$$f\left(\frac{\int_a^b g(s)ds}{b-a}\right) \leq \frac{\int_a^b f(g(s))ds}{b-a}.$$

**Example 45.** Consider the following calculus of variations problem: Find the minimum of the functional $\mathcal{L}$ defined by

$$\mathcal{L}[y(\cdot)] = \int_0^1 [y'(t)]^4 dt,$$

with the boundary conditions $y(0) = 0$ and $y(1) = 1$. Since $f(x) = x^4$ is convex for all $x \in \mathbb{R}$ we have that

$$\mathcal{L}[y(\cdot)] \geq \left(\int_0^1 y'(t)dt\right)^4 = 1,$$

for all $y \in C^1[0,1]$. Now, the function $\hat{y}(t) = t$ satisfies the boundary conditions and is such that $\mathcal{L}[\hat{y}(t)] = 1$. Therefore the functional $\mathcal{L}$ achieves its minimum at $\hat{y}$.

At the time of the beginning of this work there was already work done in the development of inequalities of the above mentioned type, i.e., Jensen, Gronwall, Bihari, etc., within the time scales setting (cf. [2, 83, 84, 93, 94]). We now state two of them being the others presented when required. The first one is a comparison theorem.

**Theorem 46** (Theorem 5.4 of [2])**.** *Let $a \in \mathbb{T}$, $y \in C_{rd}(\mathbb{T})$ and $f \in C_{rd}(\mathbb{T}^\kappa)$ and $p \in \mathcal{R}^+$. Then,*

$$y^\Delta(t) \leq p(t)y(t) + f(t), \quad t \in \mathbb{T}^\kappa,$$

*implies*

$$y(t) \leq y(a)e_p(t,a) + \int_a^t e_p(t,\sigma(\tau))f(\tau)\Delta\tau, \quad t \in \mathbb{T}.$$

The next theorem presents Gronwall's inequality on time scales and can be found in [2, Theorem 5.6].

**Theorem 47** (Gronwall's inequality on time scales)**.** *Let $t_0 \in \mathbb{T}$. Suppose that $u, a, b \in C_{rd}(\mathbb{T})$ and $b \in \mathcal{R}^+$, $b \geq 0$. Then,*

$$u(t) \leq a(t) + \int_{t_0}^{t} b(\tau)u(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}$$

*implies*

$$u(t) \leq a(t) + \int_{t_0}^{t} a(\tau)b(\tau)e_b(t, \sigma(\tau))\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}.$$

If we consider the time scale $\mathbb{T} = h\mathbb{Z}$ we get, from Theorem 47, a discrete version of the Gronwall inequality (cf. [2, Example 5.1]).

**Corollary 48.** *If $c, d$ are two nonnegative constants, $a, b \in h\mathbb{Z}$ and $u$ is a function defined on $[a, b] \cap h\mathbb{Z}$, then the inequality*

$$u(t) \leq c + \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} du(kh)h, \quad t \in [a, b] \cap h\mathbb{Z},$$

*implies*

$$u(t) \leq c + \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} cd(1 + dh)^{\frac{t-h(k+1)}{h}}h, \quad t \in [a, b] \cap h\mathbb{Z}.$$

*Remark* 49. We note that many new inequalities were accomplished by the fact that the proofs are done in a general time scale. For example, to the best of our knowledge, no Gronwall's inequality was known for the time scale $\mathbb{T} = \bigcup_{k \in \mathbb{Z}}[k, k + \frac{1}{2}]$. For this time scale, if $a, b \in \mathbb{T}$ $(a < b)$, then the $\Delta$-integral is (see [58]):

$$\int_a^b f(t)\Delta t = \int_a^{[a]} f(t)dt + \sum_{k=[a]}^{[b]-1} \left[\int_k^{k+\frac{1}{2}} f(t)dt + \frac{1}{2}f(k + \frac{1}{2})\right] + \int_{[b]}^{b} f(t)dt,$$

where $[t]$ is the Gauss bracket.

# Part II

# Original Work

# Chapter 4

# Calculus of Variations on Time Scales

In this chapter we will present some of our achievements made in the calculus of variations within the time scales setting. In Section 4.1, we present the Euler–Lagrange equation for the problem of the calculus of variations depending on the $\Delta$-derivative of order $r \in \mathbb{N}$ of a function. In Section 4.2, we prove a necessary optimality condition for isoperimetric problems.

## 4.1 Higher order $\Delta$-derivatives

Working with functions that have more than the first $\Delta$-derivative on a general time scale can cause problems. Consider the simple example of $f(t) = t^2$. It is easy to see that $f^\Delta$ exists and $f^\Delta(t) = t + \sigma(t)$. However, if for example we take $\mathbb{T} = [0, 1] \cup [2, 3]$, then $f^{\Delta^2}$ doesn't exist at $t = 1$ because $t + \sigma(t)$ is not continuous at that point.

Here, we consider time scales such that:

**(H)** $\sigma(t) = a_1 t + a_0$ for some $a_1 \in \mathbb{R}^+$ and $a_0 \in \mathbb{R}$, $t \in [a, \rho(b)]_{\mathbb{T}}$.

Under hypothesis (H) we have, among others, the differential calculus ($\mathbb{T} = \mathbb{R}$, $a_1 = 1$, $a_0 = 0$), the difference calculus ($\mathbb{T} = \mathbb{Z}$, $a_1 = a_0 = 1$) and the quantum calculus ($\mathbb{T} = \{q^k : k \in \mathbb{N}_0\}$, with $q > 1$, $a_1 = q$, $a_0 = 0$).

*Remark* 50. From assumption (H) it follows by Lemma 9 that it is not possible to have points which are simultaneously left-dense and right-scattered. Also points that are simultaneously left-scattered and right-dense do not occur, since $\sigma$ is strictly increasing.

**Lemma 51** (cf. [45])**.** *Under hypothesis (H), if $f$ is a two times $\Delta$-differentiable function, then the next formula holds:*

$$f^{\sigma\Delta}(t) = a_1 f^{\Delta\sigma}(t), \quad t \in \mathbb{T}^{\kappa^2}. \tag{4.1}$$

*Proof.* We have $f^{\sigma\Delta}(t) = \left[f(t) + \mu(t)f^{\Delta}(t)\right]^{\Delta}$ by formula (1.2). By the hypothesis on $\sigma$, function $\mu$ is $\Delta$-differentiable, hence $\left[f(t) + \mu(t)f^{\Delta}(t)\right]^{\Delta} = f^{\Delta}(t) + \mu^{\Delta}(t)f^{\Delta\sigma}(t) + \mu(t)f^{\Delta^2}(t)$ and applying again formula (1.2) we obtain $f^{\Delta}(t) + \mu^{\Delta}(t)f^{\Delta\sigma}(t) + \mu(t)f^{\Delta^2}(t) = f^{\Delta\sigma}(t) + \mu^{\Delta}(t)f^{\Delta\sigma}(t) = (1 + \mu^{\Delta}(t))f^{\Delta\sigma}(t)$. Now we only need to observe that $\mu^{\Delta}(t) = \sigma^{\Delta}(t) - 1$ and the result follows. □

We consider the Calculus of Variations problem in which the Lagrangian depends on $\Delta$-derivatives up to order $r \in \mathbb{N}$. Our extension for problem given in (2.5) is now enunciated:

$$\begin{gathered}
\mathcal{L}[y(\cdot)] = \int_a^{\rho^{r-1}(b)} L(t, y^{\sigma^r}(t), y^{\sigma^{r-1}\Delta}(t), \ldots, y^{\sigma\Delta^{r-1}}(t), y^{\Delta^r}(t))\Delta t \longrightarrow \min, \\
y(a) = y_a, \quad y\left(\rho^{r-1}(b)\right) = y_b, \\
\vdots \\
y^{\Delta^{r-1}}(a) = y_a^{r-1}, \quad y^{\Delta^{r-1}}\left(\rho^{r-1}(b)\right) = y_b^{r-1},
\end{gathered} \tag{P}$$

Assume that the Lagrangian $L(t, u_0, u_1, \ldots, u_r)$ of problem (P) has (standard) partial derivatives with respect to $u_0, \ldots, u_r$, $r \geq 1$, and partial $\Delta$-derivative with respect to $t$ of order $r + 1$. Let $y \in \mathrm{C}^{2r}$, where

$$\mathrm{C}^{2r} = \left\{y : [a, b]_{\mathbb{T}} \to \mathbb{R} : y^{\Delta^{2r}} \text{ is continuous on } \mathbb{T}^{\kappa^{2r}}\right\}.$$

*Remark* 52. We will assume from now on and until the end of this section that the time scale $\mathbb{T}$ has at least $2r + 1$ points. Indeed, if the time scale has only $2r$ points, then it can be written as $\mathbb{T} = \{a, \sigma(a), \ldots, \sigma^{2r-1}(a)\}$. Noting that $\rho^{r-1}(b) = \rho^{r-1}(\sigma^{2r-1}(a)) = \sigma^r(a)$ and using formula (1.1) we conclude that $y(t)$ for $t \in \{a, \ldots, \sigma^{r-1}(a)\}$ can be determined using the boundary conditions $[y(a) = y_a, \ldots, y^{\Delta^{r-1}}(a) = y_a^{r-1}]$ and $y(t)$ for $t \in \{\sigma^r(a), \ldots, \sigma^{2r-1}(a)\}$ can be determined using the boundary conditions $[y(\rho^{r-1}(b)) = y_b, \ldots, y^{\Delta^{r-1}}(\rho^{r-1}(b)) = y_b^{r-1}]$. Therefore we would have nothing to minimize in problem (P).

**Definition 53.** We say that $y_* \in \mathrm{C}^{2r}$ is a *weak local minimum* for (P) provided there exists $\delta > 0$ such that $\mathcal{L}(y_*) \leq \mathcal{L}(y)$ for all $y \in \mathrm{C}^{2r}$ satisfying the constraints in (P) and $\|y - y_*\|_{r,\infty} < \delta$, where

$$||y||_{r,\infty} := \sum_{i=0}^{r} ||y^{(i)}||_{\infty},$$

with $y^{(i)} = y^{\sigma^i\Delta^{r-i}}$ and $||y||_{\infty} := \sup_{t\in\mathbb{T}^{\kappa^r}} |y(t)|$.

**Definition 54.** We say that $\eta \in C^{2r}$ is an admissible variation for problem (P) if

$$\begin{gathered}
\eta(a) = 0, \quad \eta\left(\rho^{r-1}(b)\right) = 0 \\
\vdots \\
\eta^{\Delta^{r-1}}(a) = 0, \quad \eta^{\Delta^{r-1}}\left(\rho^{r-1}(b)\right) = 0.
\end{gathered}$$

For simplicity of presentation, from now on we fix $r = 3$ (the reader can consult [74] for a presentation with an arbitrary $r$).

**Lemma 55** (cf. [45]). *Suppose that $f$ is defined on $[a, \rho^6(b)]_{\mathbb{T}}$ and is continuous. Then, under hypothesis (H), $\int_a^{\rho^5(b)} f(t)\eta^{\sigma^3}(t)\Delta t = 0$ for every admissible variation $\eta$ if and only if $f(t) = 0$ for all $t \in [a, \rho^6(b)]_{\mathbb{T}}$.*

*Proof.* If $f(t) = 0$, then the result is obvious.

Now suppose without loss of generality that there exists $t_0 \in [a, \rho^6(b)]_{\mathbb{T}}$ such that $f(t_0) > 0$. First we consider the case in which $t_0$ is right-dense, hence left-dense or $t_0 = a$ (see Remark 50). If $t_0 = a$, then by the continuity of $f$ at $t_0$ there exists a $\delta > 0$ such that for all $t \in [t_0, t_0 + \delta)_{\mathbb{T}}$ we have $f(t) > 0$. Let us define $\eta$ by

$$\eta(t) = \begin{cases} (t - t_0)^8(t - t_0 - \delta)^8 & \text{if } t \in [t_0, t_0 + \delta)_{\mathbb{T}}; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\eta$ is a $C^6$ function and satisfy the requirements of an admissible variation. But with this definition for $\eta$ we get the contradiction

$$\int_a^{\rho^5(b)} f(t)\eta^{\sigma^3}(t)\Delta t = \int_{t_0}^{t_0+\delta} f(t)\eta^{\sigma^3}(t)\Delta t > 0.$$

Now, consider the case where $t_0 \neq a$. Again, the continuity of $f$ ensures the existence of a $\delta > 0$ such that for all $t \in (t_0 - \delta, t_0 + \delta)_{\mathbb{T}}$ we have $f(t) > 0$. Defining $\eta$ by

$$\eta(t) = \begin{cases} (t - t_0 + \delta)^8(t - t_0 - \delta)^8 & \text{if } t \in (t_0 - \delta, t_0 + \delta)_{\mathbb{T}}; \\ 0 & \text{otherwise,} \end{cases}$$

and noting that it satisfy the properties of an admissible variation, we obtain

$$\int_a^{\rho^5(b)} f(t)\eta^{\sigma^3}(t)\Delta t = \int_{t_0-\delta}^{t_0+\delta} f(t)\eta^{\sigma^3}(t)\Delta t > 0,$$

which is again a contradiction.

Assume now that $t_0$ is right-scattered. In view of Remark 50, all the points $t$ such that $t \geq t_0$ must be isolated. So, define $\eta$ such that $\eta^{\sigma^3}(t_0) = 1$ and is zero elsewhere. It is easy to see that $\eta$ satisfies all the requirements of an admissible variation. Further, using formula (1.4)

$$\int_a^{\rho^5(b)} f(t)\eta^{\sigma^3}(t)\Delta t = \int_{t_0}^{\sigma(t_0)} f(t)\eta^{\sigma^3}(t)\Delta t = \mu(t_0)f(t_0)\eta^{\sigma^3}(t_0) > 0,$$

which is a contradiction. □

**Theorem 56** (cf. [45]). *Let the Lagrangian $L(t, u_0, u_1, u_2, u_3)$ have standard partial derivatives with respect to $u_0, u_1, u_2, u_3$ and partial $\Delta$-derivative with respect to $t$ of order 4. On a time scale $\mathbb{T}$ satisfying (H), if $y_*$ is a weak local minimum for the problem of minimizing*

$$\int_a^{\rho^2(b)} L\left(t, y^{\sigma^3}(t), y^{\sigma^2\Delta}(t), y^{\sigma\Delta^2}(t), y^{\Delta^3}(t)\right) \Delta t$$

*subject to*

$$y(a) = y_a,\ y\left(\rho^2(b)\right) = y_b,$$
$$y^{\Delta}(a) = y_a^1,\ y^{\Delta}\left(\rho^2(b)\right) = y_b^1,$$
$$y^{\Delta^2}(a) = y_a^2,\ y^{\Delta^2}\left(\rho^2(b)\right) = y_b^2,$$

*then $y_*$ satisfies the Euler–Lagrange equation*

$$L_{u_0}(\cdot) - L_{u_1}^{\Delta}(\cdot) + \frac{1}{a_1} L_{u_2}^{\Delta^2}(\cdot) - \frac{1}{a_1^3} L_{u_3}^{\Delta^3}(\cdot) = 0, \quad t \in [a, \rho^6(b)]_{\mathbb{T}},$$

*where* $(\cdot) = (t, y_*^{\sigma^3}(t), y_*^{\sigma^2\Delta}(t), y_*^{\sigma\Delta^2}(t), y_*^{\Delta^3}(t))$.

*Proof.* Suppose that $y_*$ is a weak local minimum of $\mathcal{L}$. Let $\eta \in C^6$ be an admissible variation, i.e., $\eta$ is an arbitrary function such that $\eta$, $\eta^{\Delta}$ and $\eta^{\Delta^2}$ vanish at $t = a$ and $t = \rho^2(b)$. Define function $\Phi : \mathbb{R} \to \mathbb{R}$ by $\Phi(\varepsilon) = \mathcal{L}(y_* + \varepsilon\eta)$. This function has a minimum at $\varepsilon = 0$, so we must have

$$\Phi'(0) = 0.$$

Differentiating $\Phi$ with respect to $\varepsilon$ and setting $\varepsilon = 0$, we obtain

$$0 = \int_a^{\rho^2(b)} \left\{ L_{u_0}(\cdot)\eta^{\sigma^3}(t) + L_{u_1}(\cdot)\eta^{\sigma^2\Delta}(t) + L_{u_2}(\cdot)\eta^{\sigma\Delta^2}(t) + L_{u_3}(\cdot)\eta^{\Delta^3}(t) \right\} \Delta t. \tag{4.2}$$

Since we will $\Delta$-differentiate $L_{u_i}$, $i = 1, 2, 3$, we rewrite (4.2) in the following form:

$$\begin{aligned} 0 = \int_a^{\rho^3(b)} &\left\{ L_{u_0}(\cdot)\eta^{\sigma^3}(t) + L_{u_1}(\cdot)\eta^{\sigma^2\Delta}(t) + L_{u_2}(\cdot)\eta^{\sigma\Delta^2}(t) + L_{u_3}(\cdot)\eta^{\Delta^3}(t) \right\} \Delta t \\ &+ \mu(\rho^3(b)) \left\{ L_{u_0}\eta^{\sigma^3} + L_{u_1}\eta^{\sigma^2\Delta} + L_{u_2}\eta^{\sigma\Delta^2} + L_{u_3}\eta^{\Delta^3} \right\} (\rho^3(b)). \end{aligned} \tag{4.3}$$

Integrating (4.3) by parts gives

$$\begin{aligned} 0 = &\int_a^{\rho^3(b)} \left\{ L_{u_0}(\cdot)\eta^{\sigma^3}(t) - L_{u_1}^{\Delta}(\cdot)\eta^{\sigma^3}(t) - L_{u_2}^{\Delta}(\cdot)\eta^{\sigma\Delta\sigma}(t) - L_{u_3}^{\Delta}(\cdot)\eta^{\Delta^2\sigma}(t) \right\} \Delta t \\ &+ \left[ L_{u_1}(\cdot)\eta^{\sigma^2}(t) \right]_{t=a}^{t=\rho^3(b)} + \left[ L_{u_2}(\cdot)\eta^{\sigma\Delta}(t) \right]_{t=a}^{t=\rho^3(b)} + \left[ L_{u_3}(\cdot)\eta^{\Delta^2}(t) \right]_{t=a}^{t=\rho^3(b)} \\ &+ \mu(\rho^3(b)) \left\{ L_{u_0}\eta^{\sigma^3} + L_{u_1}\eta^{\sigma^2\Delta} + L_{u_2}\eta^{\sigma\Delta^2} + L_{u_3}\eta^{\Delta^3} \right\} (\rho^3(b)). \end{aligned} \tag{4.4}$$

Now we show how to simplify (4.4). We start by evaluating $\eta^{\sigma^2}(a)$:

$$\begin{aligned} \eta^{\sigma^2}(a) &= \eta^{\sigma}(a) + \mu(a)\eta^{\sigma\Delta}(a) \\ &= \eta(a) + \mu(a)\eta^{\Delta}(a) + \mu(a)a_1\eta^{\Delta\sigma}(a) \\ &= \mu(a)a_1 \left( \eta^{\Delta}(a) + \mu(a)\eta^{\Delta^2}(a) \right) \\ &= 0, \end{aligned} \tag{4.5}$$

where the last term of (4.5) follows from (4.1). Now, we calculate $\eta^{\sigma\Delta}(a)$. By (4.1) we have $\eta^{\sigma\Delta}(a) = a_1\eta^{\Delta\sigma}(a)$ and applying (1.2) we obtain

$$a_1\eta^{\Delta\sigma}(a) = a_1\left(\eta^{\Delta}(a) + \mu(a)\eta^{\Delta^2}(a)\right) = 0.$$

Now we turn to analyze what happens at $t = \rho^3(b)$. It is easy to see that if $b$ is left-dense, then the last terms of (4.4) vanish. So suppose that $b$ is left-scattered. Since $\sigma$ is $\Delta$-differentiable, by Lemma 9 we cannot have points that are simultaneously left-dense and right-scattered. Hence, $\rho(b)$, $\rho^2(b)$ and $\rho^3(b)$ are right-scattered points. Now, by hypothesis, $\eta^{\Delta}(\rho^2(b)) = 0$. Hence we obtain using (1.1) that

$$\frac{\eta(\rho(b)) - \eta(\rho^2(b))}{\mu(\rho^2(b))} = 0.$$

But $\eta(\rho^2(b)) = 0$, hence $\eta(\rho(b)) = 0$. Analogously, we have

$$\eta^{\Delta^2}(\rho^2(b)) = 0 \Leftrightarrow \frac{\eta^{\Delta}(\rho(b)) - \eta^{\Delta}(\rho^2(b))}{\mu(\rho^2(b))} = 0,$$

from what follows that $\eta^{\Delta}(\rho(b)) = 0$. This last equality implies $\eta(b) = 0$. Applying previous expressions to the last terms of (4.4), we obtain:

$$\begin{aligned}
\eta^{\sigma^2}(\rho^3(b)) &= \eta(\rho(b)) = 0,\\
\eta^{\sigma\Delta}(\rho^3(b)) &= \frac{\eta^{\sigma^2}(\rho^3(b)) - \eta^{\sigma}(\rho^3(b))}{\mu(\rho^3(b))} = 0,\\
\eta^{\sigma^3}(\rho^3(b)) &= \eta(b) = 0,\\
\eta^{\sigma^2\Delta}(\rho^3(b)) &= \frac{\eta^{\sigma^3}(\rho^3(b)) - \eta^{\sigma^2}(\rho^3(b))}{\mu(\rho^3(b))} = 0,\\
\eta^{\sigma\Delta^2}(\rho^3(b)) &= \frac{\eta^{\sigma\Delta}(\rho^2(b)) - \eta^{\sigma\Delta}(\rho^3(b))}{\mu(\rho^3(b))}\\
&= \frac{\frac{\eta^{\sigma}(\rho(b)) - \eta^{\sigma}(\rho^2(b))}{\mu(\rho^2(b))} - \frac{\eta^{\sigma}(\rho^2(b)) - \eta^{\sigma}(\rho^3(b))}{\mu(\rho^3(b))}}{\mu(\rho^3(b))}\\
&= 0.
\end{aligned}$$

In view of our previous calculations,

$$\begin{aligned}
&\left[L_{u_1}(\cdot)\eta^{\sigma^2}(t)\right]_{t=a}^{t=\rho^3(b)} + \left[L_{u_2}(\cdot)\eta^{\sigma\Delta}(t)\right]_{t=a}^{t=\rho^3(b)} + \left[L_{u_3}(\cdot)\eta^{\Delta^2}(t)\right]_{t=a}^{t=\rho^3(b)}\\
&\qquad + \mu(\rho^3(b))\left\{L_{u_0}\eta^{\sigma^3} + L_{u_1}\eta^{\sigma^2\Delta} + L_{u_2}\eta^{\sigma\Delta^2} + L_{u_3}\eta^{\Delta^3}\right\}(\rho^3(b))
\end{aligned}$$

is reduced to[1]

$$L_{u_3}(\rho^3(b))\eta^{\Delta^2}(\rho^3(b)) + \mu(\rho^3(b))L_{u_3}(\rho^3(b))\eta^{\Delta^3}(\rho^3(b)). \tag{4.6}$$

[1] There is some abuse of notation: $L_{u_3}(\rho^3(b))$ denotes $L_{u_3}(\cdot)|_{t=\rho^3(b)}$, that is, we substitute $t$ in $(\cdot) = (t, y_*^{\sigma^3}(t), y_*^{\sigma^2\Delta}(t), y_*^{\sigma\Delta^2}(t), y_*^{\Delta^3}(t))$ by $\rho^3(b)$.

Now note that

$$\eta^{\Delta^2\sigma}(\rho^3(b)) = \eta^{\Delta^2}(\rho^3(b)) + \mu(\rho^3(b))\eta^{\Delta^3}(\rho^3(b))$$

and by hypothesis $\eta^{\Delta^2\sigma}(\rho^3(b)) = \eta^{\Delta^2}(\rho^2(b)) = 0$. Therefore,

$$\mu(\rho^3(b))\eta^{\Delta^3}(\rho^3(b)) = -\eta^{\Delta^2}(\rho^3(b)),$$

from which follows that (4.6) must be zero. We have just simplified (4.4) to

$$0 = \int_a^{\rho^3(b)} \left\{L_{u_0}(\cdot)\eta^{\sigma^3}(t) - L_{u_1}^{\Delta}(\cdot)\eta^{\sigma^3}(t) - L_{u_2}^{\Delta}(\cdot)\eta^{\sigma\Delta\sigma}(t) - L_{u_3}^{\Delta}(\cdot)\eta^{\Delta^2\sigma}(t)\right\}\Delta t. \tag{4.7}$$

In order to apply again the integration by parts formula, we must first make some transformations in $\eta^{\sigma\Delta\sigma}$ and $\eta^{\Delta^2\sigma}$. By (4.1) we have

$$\eta^{\sigma\Delta\sigma}(t) = \frac{1}{a_1}\eta^{\sigma^2\Delta}(t) \tag{4.8}$$

and

$$\eta^{\Delta^2\sigma}(t) = \frac{1}{a_1^2}\eta^{\sigma\Delta^2}(t). \tag{4.9}$$

Hence, (4.7) becomes

$$0 = \int_a^{\rho^3(b)} \left\{L_{u_0}(\cdot)\eta^{\sigma^3}(t) - L_{u_1}^{\Delta}(\cdot)\eta^{\sigma^3}(t) - \frac{1}{a_1}L_{u_2}^{\Delta}(\cdot)\eta^{\sigma^2\Delta}(t) - \frac{1}{a_1^2}L_{u_3}^{\Delta}(\cdot)\eta^{\sigma\Delta^2}(t)\right\}\Delta t. \tag{4.10}$$

By the same reasoning as before, (4.10) is equivalent to

$$\begin{aligned} 0 = \int_a^{\rho^4(b)} &\left\{L_{u_0}(\cdot)\eta^{\sigma^3}(t) - L_{u_1}^{\Delta}(\cdot)\eta^{\sigma^3}(t) - \frac{1}{a_1}L_{u_2}^{\Delta}(\cdot)\eta^{\sigma^2\Delta}(t) - \frac{1}{a_1^2}L_{u_3}^{\Delta}(\cdot)\eta^{\sigma\Delta^2}(t)\right\}\Delta t \\ &+ \mu(\rho^4(b))\left\{L_{u_0}\eta^{\sigma^3} - L_{u_1}^{\Delta}\eta^{\sigma^3} - \frac{1}{a_1}L_{u_2}^{\Delta}\eta^{\sigma^2\Delta} - \frac{1}{a_1^2}L_{u_3}^{\Delta}\eta^{\sigma\Delta^2}\right\}(\rho^4(b)) \end{aligned}$$

and integrating by parts we obtain

$$\begin{aligned} 0 = &\int_a^{\rho^4(b)} \left\{L_{u_0}(\cdot)\eta^{\sigma^3}(t) - L_{u_1}^{\Delta}(\cdot)\eta^{\sigma^3}(t) + \frac{1}{a_1}L_{u_2}^{\Delta^2}(\cdot)\eta^{\sigma^3}(t) + \frac{1}{a_1^2}L_{u_3}^{\Delta^2}(\cdot)\eta^{\sigma\Delta\sigma}(t)\right\}\Delta t \\ &- \left[\frac{1}{a_1}L_{u_2}^{\Delta}(\cdot)\eta^{\sigma^2}(t)\right]_{t=a}^{t=\rho^4(b)} - \left[\frac{1}{a_1^2}L_{u_3}^{\Delta}(\cdot)\eta^{\sigma\Delta}(t)\right]_{t=a}^{t=\rho^4(b)} \\ &+ \mu(\rho^4(b))\left\{L_{u_0}\eta^{\sigma^3} - L_{u_1}^{\Delta}\eta^{\sigma^3} - \frac{1}{a_1}L_{u_2}^{\Delta}\eta^{\sigma^2\Delta} - \frac{1}{a_1^2}L_{u_3}^{\Delta}\eta^{\sigma\Delta^2}\right\}(\rho^4(b)). \end{aligned} \tag{4.11}$$

Using analogous arguments to those above, we simplify (4.11) to

$$\int_a^{\rho^4(b)} \left\{L_{u_0}(\cdot)\eta^{\sigma^3}(t) - L_{u_1}^{\Delta}(\cdot)\eta^{\sigma^3}(t) + \frac{1}{a_1}L_{u_2}^{\Delta}(\cdot)\eta^{\sigma^2\Delta}(t) + \frac{1}{a_1^3}L_{u_3}^{\Delta^2}(\cdot)\eta^{\sigma^2\Delta}(t)\right\}\Delta t = 0.$$

Calculations as done before lead us to the final expression

$$\int_a^{\rho^5(b)} \left\{ L_{u_0}(\cdot)\eta^{\sigma^3}(t) - L^{\Delta}_{u_1}(\cdot)\eta^{\sigma^3}(t) + \frac{1}{a_1} L^{\Delta^2}_{u_2}(\cdot)\eta^{\sigma^3}(t) - \frac{1}{a_1^3} L^{\Delta^3}_{u_3}(\cdot)\eta^{\sigma^3}(t) \right\} \Delta t = 0,$$

which is equivalent to

$$\int_a^{\rho^5(b)} \left\{ L_{u_0}(\cdot) - L^{\Delta}_{u_1}(\cdot) + \frac{1}{a_1} L^{\Delta^2}_{u_2}(\cdot) - \frac{1}{a_1^3} L^{\Delta^3}_{u_3}(\cdot) \right\} \eta^{\sigma^3}(t)\Delta t = 0. \tag{4.12}$$

Applying Lemma 55 to (4.12), we obtain the Euler–Lagrange equation

$$L_{u_0}(\cdot) - L^{\Delta}_{u_1}(\cdot) + \frac{1}{a_1} L^{\Delta^2}_{u_2}(\cdot) - \frac{1}{a_1^3} L^{\Delta^3}_{u_3}(\cdot) = 0, \quad t \in [a, \rho^6(b)]_{\mathbb{T}}.$$

The proof is complete. □

Following exactly the same steps of the proofs of Lemma 55 and Theorem 56 for an arbitrary $r \in \mathbb{N}$, one easily obtain the Euler–Lagrange equation for problem (P).

**Theorem 57** (cf. [45])**.** *(Necessary optimality condition for problems of the calculus of variations with higher-order Δ-derivatives) On a time scale $\mathbb{T}$ satisfying hypothesis (H), if $y_*$ is a weak local minimum for problem (P), then $y_*$ satisfies the Euler-Lagrange equation*

$$\sum_{i=0}^{r} (-1)^i \left(\frac{1}{a_1}\right)^{\frac{(i-1)i}{2}} L^{\Delta^i}_{u_i}\left(t, y_*^{\sigma^r}(t), y_*^{\sigma^{r-1}\Delta}(t), \ldots, y_*^{\sigma\Delta^{r-1}}(t), y_*^{\Delta^r}(t)\right) = 0, \tag{4.13}$$

$t \in [a, \rho^{2r}(b)]_{\mathbb{T}}$.

*Remark* 58. The factor $\left(\frac{1}{a_1}\right)^{\frac{(i-1)i}{2}}$ in (4.13) comes from the fact that, after each time we apply the integration by parts formula, we commute successively $\sigma$ with $\Delta$ using (4.1) [see formulas (4.8) and (4.9)], doing this $\sum_{j=1}^{i-1} j = \frac{(i-1)i}{2}$ times for each of the parcels within the integral.

**Example 59.** Let us consider the time scale $\mathbb{T} = \{q^k : k \in \mathbb{N}_0\}$, $a, b \in \mathbb{T}$ with $a < b$, and the problem

$$\begin{gathered} \mathcal{L}[y(\cdot)] = \int_a^{\rho(b)} (y^{\Delta^2}(t))^2 \Delta t \longrightarrow \min, \\ y(a) = 0, \quad y(\rho(b)) = 1, \\ y^{\Delta}(a) = 0, \quad y^{\Delta}(\rho(b)) = 0. \end{gathered} \tag{4.14}$$

We point out that for this time scale, $\sigma(t) = qt$, $\mu(t) = (q-1)t$, and

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}, \qquad \int_a^b f(t)\Delta t = \sum_{t \in [a,b)_{\mathbb{T}}} (q-1)tf(t).$$

By Theorem 57, the Euler–Lagrange equation for problem given in (4.14) is

$$\frac{1}{q} y^{\Delta^4}(t) = 0, \quad t \in [a, \rho^4(b)]_{\mathbb{T}}.$$

It follows that

$$y^{\Delta^2}(t) = ct + d,$$

for some constants $c, d \in \mathbb{R}$. From this we get, using $y^\Delta(a) = 0$ and $y^\Delta(\rho(b)) = 0$,

$$y^\Delta(t) = c\int_a^t s\Delta s - \frac{c\int_a^{\rho(b)} s\Delta s}{\rho(b) - a}(t-a).$$

Finally, using the boundary conditions $y(a) = 0$ and $y(\rho(b)) = 1$, and defining

$$\begin{aligned} A &= \frac{\int_a^{\rho(b)} s\Delta s}{\rho(b) - a}, \\ B &= \frac{1}{\int_a^{\rho(b)} \left(\int_a^s \tau\Delta\tau\right)\Delta s - A\int_a^{\rho(b)}(s-a)\Delta s}, \end{aligned}$$

we get

$$y(t) = B\left[\int_a^t \left(\int_a^s \tau\Delta\tau\right)\Delta s - A\int_a^t (s-a)\Delta s\right], \quad t \in [a,b]_\mathbb{T}, \tag{4.15}$$

provided the denominator of $B$ is nonzero. We end this example showing that this is indeed the case. We start by rewriting the denominator of $B$, which we denote by $D$, as

$$\begin{aligned} D = &\int_a^{\rho(b)} \left(\int_a^s (\tau - a)\Delta\tau\right)\Delta s \\ &+ a\int_a^{\rho(b)} (s-a)\Delta s - \frac{\int_a^{\rho(b)}(s-a)\Delta s + a(\rho(b)-a)}{\rho(b)-a}\int_a^{\rho(b)}(s-a)\Delta s. \end{aligned}$$

Now, note that

$$\begin{aligned} \int_a^{\rho(b)} \left(\int_a^s (\tau - a)\Delta\tau\right)\Delta s &= h_3(\rho(b), a) \\ \int_a^{\rho(b)} (s-a)\Delta s &= h_2(\rho(b), a), \end{aligned}$$

where $h_i$ are given by Definition 18. It is known (cf. [24, Example 1.104]) that

$$h_k(t,s) = \prod_{\nu=0}^{k-1} \frac{t - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}, \quad s, t \in \mathbb{T}.$$

Hence,

$$D = \prod_{\nu=0}^{2} \frac{\rho(b) - q^\nu a}{\sum_{\mu=0}^{\nu} q^\mu} + a\prod_{\nu=0}^{1} \frac{\rho(b) - q^\nu a}{\sum_{\mu=0}^{\nu} q^\mu} - \frac{\prod_{\nu=0}^{1} \frac{\rho(b) - q^\nu a}{\sum_{\mu=0}^{\nu} q^\mu} + a(\rho(b) - a)}{\rho(b) - a}\prod_{\nu=0}^{1} \frac{\rho(b) - q^\nu a}{\sum_{\mu=0}^{\nu} q^\mu}.$$

From this it is not difficult to achieve

$$D = \frac{q(-\rho(b) + a)(-\rho(b) + qa)(a - \rho(b)q)}{(1+q)^2(1+q+q^2)}.$$

Therefore, we conclude that $D \neq 0$. We also get another representation for (4.15), namely,

$$y(t) = \frac{(-t+a)(-t+qa)(a-tq)}{(-\rho(b)+a)(-\rho(b)+qa)(a-\rho(b)q)}, \quad t \in [a,b]_\mathbb{T}.$$

## 4.2 Isoperimetric problems

In this section we make a study of isoperimetric problems on a general time scale (see [92] and [6] for continuos and discrete versions, respectively), proving the corresponding necessary optimality condition. These problems add some constraints (in integral form) to the basic problem. In the end, we show that certain eigenvalue problems can be recast as an isoperimetric problem.

We start by giving a proof of a technical lemma.

**Lemma 60** (cf. [49])**.** *Suppose that a continuous function $f : \mathbb{T} \to \mathbb{R}$ is such that $f^\sigma(t) = 0$ for all $t \in \mathbb{T}^\kappa$. Then, $f(t) = 0$ for all $t \in \mathbb{T}$ except possibly at $t = a$ if $a$ is right-scattered.*

*Proof.* First note that, since $f^\sigma(t) = 0$, then $f^\sigma(t)$ is $\Delta$-differentiable, hence continuous for all $t \in \mathbb{T}^\kappa$. Now, if $t$ is right-dense, the result is obvious. Suppose that $t$ is right-scattered. We will analyze two cases: (i) if $t$ is left-scattered, then $t \neq a$ and by hypothesis $0 = f^\sigma(\rho(t)) = f(t)$; (ii) if $t$ is left-dense, then, by the continuity of $f^\sigma$ and $f$ at $t$, we can write

$$\forall \varepsilon > 0\, \exists \delta_1 > 0 : \forall s_1 \in (t - \delta_1, t]_\mathbb{T}, \text{ we have } |f^\sigma(s_1) - f^\sigma(t)| < \varepsilon\,, \tag{4.16}$$

$$\forall \varepsilon > 0\, \exists \delta_2 > 0 : \forall s_2 \in (t - \delta_2, t]_\mathbb{T}, \text{ we have } |f(s_2) - f(t)| < \varepsilon\,, \tag{4.17}$$

respectively. Let $\delta = \min\{\delta_1, \delta_2\}$ and take $s_1 \in (t - \delta, t)_\mathbb{T}$. As $\sigma(s_1) \in (t - \delta, t)_\mathbb{T}$, take $s_2 = \sigma(s_1)$. By (4.16) and (4.17), we have:

$$|-f^\sigma(t) + f(t)| = |f^\sigma(s_1) - f^\sigma(t) + f(t) - f(s_2)| \leq |f^\sigma(s_1) - f^\sigma(t)| + |f(s_2) - f(t)| < 2\varepsilon.$$

Since $\varepsilon$ is arbitrary, $|-f^\sigma(t) + f(t)| = 0$, which is equivalent to $f(t) = f^\sigma(t)$. □

We now define the isoperimetric problem on time scales. Let $J : \mathrm{C}^1_{\mathrm{rd}} \to \mathbb{R}$ be a functional of the form

$$J[y(\cdot)] = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t, \tag{4.18}$$

where $L(t, u, v) : [a, b]^\kappa_\mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the assumption of Lemma 39. The *isoperimetric problem* consists of finding functions $y \in \mathrm{C}^1_{\mathrm{rd}}$ satisfying given boundary conditions

$$y(a) = y_a,\ y(b) = y_b, \tag{4.19}$$

and a constraint of the form

$$I[y(\cdot)] = \int_a^b g(t, y^\sigma(t), y^\Delta(t))\Delta t = l, \tag{4.20}$$

where $g(t, u, v) : [a, b]^\kappa \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the assumption of Lemma 39, and $l$ is a specified real number, that takes (4.18) to a minimum.

**Definition 61.** We say that a function $y \in \mathrm{C}^1_{\mathrm{rd}}$ is *admissible* for the isoperimetric problem if it satisfies (4.19) and (4.20).

**Definition 62.** An admissible function $y_*$ is said to be an *extremal* for $I$ if it satisfies the following equation:

$$g_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t g_u(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau = c,$$

for all $t \in [a,b]^\kappa_{\mathbb{T}}$ and some constant $c$.

**Theorem 63** (cf. [49])**.** *Suppose that $J$ has a local minimum at $y_* \in C^1_{rd}$ subject to the boundary conditions (4.19) and the isoperimetric constraint (4.20), and that $y_*$ is not an extremal for the functional $I$. Then, there exists a Lagrange multiplier constant $\lambda$ such that $y_*$ satisfies the following equation:*

$$F_v^\Delta(t, y_*^\sigma(t), y_*^\Delta(t)) - F_u(t, y_*^\sigma(t), y_*^\Delta(t)) = 0, \quad \textit{for all } t \in [a,b]^{\kappa^2}_{\mathbb{T}}, \tag{4.21}$$

*where $F = L - \lambda g$.*

*Proof.* Let $y_*$ be a local minimum for $J$ and consider neighboring functions of the form

$$\hat{y} = y_* + \varepsilon_1\eta_1 + \varepsilon_2\eta_2, \tag{4.22}$$

where for each $i \in \{1,2\}$, $\varepsilon_i$ is a sufficiently small parameter ($\varepsilon_1$ and $\varepsilon_2$ must be such that $\|\hat{y} - y^*\| < \delta$, for some $\delta > 0$), $\eta_i(x) \in \mathrm{C}^1_{\mathrm{rd}}$ and $\eta_i(a) = \eta_i(b) = 0$. Here, $\eta_1$ is an arbitrary fixed function and $\eta_2$ is a fixed function that we will choose later.

First we show that (4.22) has a subset of admissible functions for the isoperimetric problem. Consider the quantity

$$I[\hat{y}(\cdot)] = \int_a^b g(t, y_*^\sigma(t) + \varepsilon_1\eta_1^\sigma(t) + \varepsilon_2\eta_2^\sigma(t), y_*^\Delta(t) + \varepsilon_1\eta_1^\Delta(t) + \varepsilon_2\eta_2^\Delta(t))\Delta t.$$

Then we can regard $I[\hat{y}(\cdot)]$ as a function of $\varepsilon_1$ and $\varepsilon_2$, say $I[\hat{y}(\cdot)] = \hat{Q}(\varepsilon_1, \varepsilon_2)$. Since $y_*$ is a local minimum for $J$ subject to the boundary conditions and the isoperimetric constraint, putting $Q(\varepsilon_1, \varepsilon_2) = \hat{Q}(\varepsilon_1, \varepsilon_2) - l$ we have that

$$Q(0,0) = 0. \tag{4.23}$$

By the conditions imposed on $g$, we have

$$\begin{aligned}\frac{\partial Q}{\partial \varepsilon_2}(0,0) &= \int_a^b \left[g_u(t, y_*^\sigma(t), y_*^\Delta(t))\eta_2^\sigma(t) + g_v(t, y_*^\sigma(t), y_*^\Delta(t))\eta_2^\Delta(t)\right]\Delta t \\ &= \int_a^b \left[g_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t g_u(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau\right]\eta_2^\Delta(t)\Delta t,\end{aligned} \tag{4.24}$$

where (4.24) follows from (1.6) and the fact that $\eta_2(a) = \eta_2(b) = 0$. Now, consider the function

$$E(t) = g_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t g_u(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau, \quad t \in [a,b]_{\mathbb{T}}^\kappa.$$

Then, we can apply Lemma 40 to show that there exists a function $\eta_2 \in \mathrm{C}_{\mathrm{rd}}^1$ such that

$$\int_a^b \left[ g_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t g_u(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau \right] \eta_2^\Delta(t)\Delta t \neq 0,$$

provided $y_*$ is not an extremal for $I$, which is indeed the case. We have just proved that

$$\frac{\partial Q}{\partial \varepsilon_2}(0,0) \neq 0. \tag{4.25}$$

Using (4.23) and (4.25), the implicit function theorem asserts that there exist neighborhoods $N_1$ and $N_2$ of 0, $N_1 \subseteq \{\varepsilon_1 \text{ from } (4.22)\} \cap \mathbb{R}$ and $N_2 \subseteq \{\varepsilon_2 \text{ from } (4.22)\} \cap \mathbb{R}$, and a function $\varepsilon_2 : N_1 \to \mathbb{R}$ such that for all $\varepsilon_1 \in N_1$ we have

$$Q(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = 0,$$

which is equivalent to $\hat{Q}(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = l$. Now we derive the necessary condition (4.21). Consider the quantity $J(\hat{y}) = K(\varepsilon_1, \varepsilon_2)$. By hypothesis, $K$ is minimum at $(0,0)$ subject to the constraint $Q(0,0) = 0$, and we have proved that $\nabla Q(0,0) \neq \mathbf{0}$. We can appeal to the Lagrange multiplier rule (see, e.g., [92, Theorem 4.1.1]) to assert that there exists a number $\lambda$ such that

$$\nabla(K(0,0) - \lambda Q(0,0)) = \mathbf{0}. \tag{4.26}$$

Having in mind that $\eta_1(a) = \eta_1(b) = 0$, we can write:

$$\begin{aligned} \frac{\partial K}{\partial \varepsilon_1}(0,0) &= \int_a^b \left[ L_u(t, y_*^\sigma(t), y_*^\Delta(t))\eta_1^\sigma(t) + L_v(t, y_*^\sigma(t), y_*^\Delta(t))\eta_1^\Delta(t) \right] \Delta t \\ &= \int_a^b \left[ L_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t L_u(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau \right] \eta_1^\Delta(t)\Delta t. \end{aligned} \tag{4.27}$$

Similarly, we have that

$$\frac{\partial Q}{\partial \varepsilon_1}(0,0) = \int_a^b \left[ g_v(t, y_*^\sigma(t), y_*^\Delta(t)) - \int_a^t g_u(\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))\Delta\tau \right] \eta_1^\Delta(t)\Delta t. \tag{4.28}$$

Combining (4.26), (4.27) and (4.28), we obtain

$$\int_a^b \left\{ L_v(\cdot) - \int_a^t L_u(\cdot\cdot)\Delta\tau - \lambda\left( g_v(\cdot) - \int_a^t g_u(\cdot\cdot)\Delta\tau \right) \right\} \eta_1^\Delta(t)\Delta t = 0,$$

where $(\cdot) = (t, y_*^\sigma(t), y_*^\Delta(t))$ and $(\cdot\cdot) = (\tau, y_*^\sigma(\tau), y_*^\Delta(\tau))$. Since $\eta_1$ is arbitrary, Lemma 40 implies that there exists a constant $d$ such that

$$L_v(\cdot) - \lambda g_v(\cdot) - \left( \int_a^t [L_u(\cdot\cdot) - \lambda g_u(\cdot\cdot)]\Delta\tau \right) = d, \quad t \in [a,b]_{\mathbb{T}}^\kappa,$$

or

$$F_v(\cdot) - \int_a^t F_x(\cdot\cdot)\Delta\tau = d, \tag{4.29}$$

with $F = L - \lambda g$. Since the integral and the constant in (4.29) are $\Delta$-differentiable, we obtain the desired necessary optimality condition (4.21). $\square$

*Remark* 64. Theorem 63 remains valid when $y_*$ is assumed to be a local maximizer of the isoperimetric problem (4.18)-(4.20).

*Remark* 65. In Theorem 63 we assume that $y_*$ is not an extremal for $I$. This corresponds to the normal case. A version of Theorem 63 that involves abnormal extremals can be found in [10]. A different hypothesis excluding abnormality is given in [69].

**Example 66.** Let $a, -a \in \mathbb{T}$ and suppose that we want to find functions defined on $[-a, a]_{\mathbb{T}}$ that take

$$J[y(\cdot)] = \int_{-a}^{a} y^\sigma(t)\Delta t$$

to its largest value (see Remark 64) and that satisfy the conditions

$$y(-a) = y(a) = 0, \quad I[y(\cdot)] = \int_{-a}^{a} \sqrt{1 + (y^\Delta(t))^2}\Delta t = l > 2a.$$

Note that if $y$ is an extremal for $I$, then $y$ is a line segment [21], and therefore $y(t) = 0$ for all $t \in [-a, a]_{\mathbb{T}}$. This implies that $I(y) = 2a > 2a$, which is a contradiction. Hence, $I$ has no extremals satisfying the boundary conditions and the isoperimetric constraint. Using Theorem 63, let

$$F = L - \lambda g = y^\sigma - \lambda\sqrt{1 + (y^\Delta)^2}\,.$$

Because

$$F_x = 1, \quad F_v = \lambda\frac{y^\Delta}{\sqrt{1 + (y^\Delta)^2}},$$

a necessary optimality condition is given by the following dynamic equation:

$$\lambda\left(\frac{y^\Delta}{\sqrt{1 + (y^\Delta)^2}}\right)^\Delta - 1 = 0, \quad t \in [-a, a]_{\mathbb{T}}^{\kappa^2}.$$

If we restrict ourselves to times scales $\mathbb{T}$ satisfying hypothesis (H) of Section 4.1 it follows that the same proof as in Theorem 63 can be used, *mutatis mutandis*, to obtain a necessary optimality condition for the higher-order isoperimetric problem (i.e., when $L$ and $g$ contain higher order $\Delta$-derivatives). In this case, the necessary optimality condition (4.21) is generalized to

$$\sum_{i=0}^{r}(-1)^i\left(\frac{1}{a_1}\right)^{\frac{(i-1)i}{2}} F_{u_i}^{\Delta^i}\left(t, y_*^{\sigma^r}(t), y_*^{\sigma^{r-1}\Delta}(t), \ldots, y_*^{\sigma\Delta^{r-1}}(t), y_*^{\Delta^r}(t)\right) = 0\,,$$

where again $F = L - \lambda g$.

### 4.2.1 Sturm–Liouville eigenvalue problems

Eigenvalue problems on time scales have been studied in [3]. Consider the following Sturm–Liouville eigenvalue problem: find nontrivial solutions to the $\Delta$-dynamic equation

$$y^{\Delta^2}(t) + q(t)y^\sigma(t) + \lambda y^\sigma(t) = 0, \quad t \in [a,b]_{\mathbb{T}}^{\kappa^2}, \tag{4.30}$$

for the unknown $y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ subject to the boundary conditions

$$y(a) = y(b) = 0. \tag{4.31}$$

Here $q : [a,b]_{\mathbb{T}}^{\kappa} \to \mathbb{R}$ is a continuous function.

Generically, the only solution to equation (4.30) that satisfies the boundary conditions (4.31) is the trivial solution, $y(t) = 0$ for all $t \in [a,b]_{\mathbb{T}}$. There are, however, certain values of $\lambda$ that lead to nontrivial solutions. These are called *eigenvalues* and the corresponding nontrivial solutions are called *eigenfunctions*. These eigenvalues may be arranged as $-\infty < \lambda_1 < \lambda_2 < \dots$ (cf. [3, Theorem 1]) and $\lambda_1$ is called the *first eigenvalue*.

Consider the functional defined by

$$J[y(\cdot)] = \int_a^b [(y^\Delta)^2(t) - q(t)(y^\sigma)^2(t)]\Delta t, \tag{4.32}$$

and suppose that $y_* \in \mathrm{C}^2_{\mathrm{rd}}$ (functions that are twice $\Delta$-differentiable with rd-continuous second $\Delta$-derivative) is a local minimum for $J$ subject to the boundary conditions (4.31) and the isoperimetric constraint

$$I[y(\cdot)] = \int_a^b (y^\sigma)^2(t)\Delta t = 1. \tag{4.33}$$

If $y_*$ is an extremal for $I$, then we would have $-2y^\sigma(t) = 0,\ t \in [a,b]_{\mathbb{T}}^{\kappa}$. Noting that $y(a) = 0$, using Lemma 60 we would conclude that $y(t) = 0$ for all $t \in [a,b]_{\mathbb{T}}$. No extremals for $I$ can therefore satisfy the isoperimetric condition (4.33). Hence, by Theorem 63 there is a constant $\lambda$ such that $y_*$ satisfies

$$F_v^\Delta(t, y_*^\sigma(t), y_*^\Delta(t)) - F_u(t, y_*^\sigma(t), y_*^\Delta(t)) = 0, \quad t \in [a,b]_{\mathbb{T}}^{\kappa^2}, \tag{4.34}$$

with $F = v^2 - qu^2 - \lambda u^2$. It is easily seen that (4.34) is equivalent to (4.30). The isoperimetric problem thus corresponds to the Sturm-Liouville problem augmented by the normalizing condition (4.33), which simply scales the eigenfunctions. Here, the Lagrange multiplier plays the role of the eigenvalue.

The first eigenvalue has the notable property that the corresponding eigenfunction produces the minimum value for the functional $J$.

**Theorem 67.** *Let $\lambda_1$ be the first eigenvalue for the Sturm-Liouville problem (4.30) with boundary conditions (4.31), and let $y_1$ be the corresponding eigenfunction normalized to satisfy the isoperimetric constraint (4.33). Then, among functions in $C^2_{rd}$ that satisfy the boundary conditions (4.31) and the isoperimetric condition (4.33), the functional $J$ defined by (4.32) has a minimum at $y_1$. Moreover, $J[y_1(\cdot)] = \lambda_1$.*

*Proof.* Suppose that $J$ has a minimum at $y$ satisfying conditions (4.31) and (4.33). Then $y$ satisfies equation (4.30) and multiplying this equation by $y^\sigma$ and $\Delta$-integrating from $a$ to $\rho(b)$ (note that $y^{\Delta^2}$ is defined only on $[a,b]_{\mathbb{T}}^{\kappa^2}$), we obtain

$$\int_a^{\rho(b)} y^\sigma(t) y^{\Delta^2}(t)\Delta t + \int_a^{\rho(b)} q(t)(y^\sigma)^2(t)\Delta t + \lambda \int_a^{\rho(b)} (y^\sigma)^2(t)\Delta t = 0. \tag{4.35}$$

Since $y(a) = y(b) = 0$, we get

$$\begin{aligned}\int_a^{\rho(b)} y^\sigma(t) y^{\Delta^2}(t)\Delta t &= \left[y(t)y^\Delta(t)\right]_{t=a}^{t=\rho(b)} - \int_a^{\rho(b)} (y^\Delta)^2(t)\Delta t\\ &= y(\rho(b))y^\Delta(\rho(b)) - \int_a^{\rho(b)} (y^\Delta)^2(t)\Delta t.\end{aligned} \tag{4.36}$$

If $b$ is left-dense it is clear that (4.36) is equal to $-\int_a^b (y^\Delta)^2(t)\Delta t$. If $b$ is left-scattered (hence $\rho(b)$ is right-scattered) we obtain for (4.36)

$$\begin{aligned} y(\rho(b))y^\Delta(\rho(b)) - \int_a^{\rho(b)} (y^\Delta)^2(t)\Delta t &= y(\rho(b))\frac{y(b) - y(\rho(b))}{\mu(\rho(b))} - \int_a^{\rho(b)} (y^\Delta)^2(t)\Delta t\\ &= -\frac{y(\rho(b))^2}{\mu(\rho(b))} - \int_a^{\rho(b)} (y^\Delta)^2(t)\Delta t\\ &= -\int_a^b (y^\Delta)^2(t)\Delta t,\end{aligned}$$

where the last equality follows by (1.4). With the help of (1.4) it is easy to prove that $\int_a^{\rho(b)} q(t)(y^\sigma)^2(t)\Delta t = \int_a^b q(t)(y^\sigma)^2(t)\Delta t$ and $\int_a^{\rho(b)} (y^\sigma)^2(t)\Delta t = \int_a^b (y^\sigma)^2(t)\Delta t$. By (4.33), (4.35) reduces to

$$\int_a^b [(y^\Delta)^2(t) - q(t)(y^\sigma)^2(t)]\Delta t = \lambda,$$

that is, $J[y(\cdot)] = \lambda$. Due to the isoperimetric condition, $y$ must be a nontrivial solution to (4.30) and therefore $\lambda$ must be an eigenvalue. Since there exists a least element within the eigenvalues, $\lambda_1$, and a corresponding eigenfunction $y_1$ normalized to meet the isoperimetric condition, the minimum value for $J$ is $\lambda_1$ and $J[y_1(\cdot)] = \lambda_1$. □

## 4.3 State of the Art

The results of this chapter are already published in the following international journals and chapters in books: [45, 49]. In particular, the original results of the papers [45, 49] were presented in the International Workshop on Mathematical Control Theory and Finance, Lisbon, 10-14 April, 2007, and in the Conference on Nonlinear Analysis and Optimization, June 18-24, 2008, Technion, Haifa, Israel, respectively. It is worth mentioning that nowadays other researchers are dedicating their time to the development of the theory of the calculus of variations on time scales (see [10, 23, 62, 71, 72, 74, 96] and references therein).

# Chapter 5

# Inequalities applied to some Calculus of Variations and Optimal Control problems

In this chapter we prove a result that complements Jensen's inequality on time scales (cf. Proposition 69) and state some useful consequences of it. These are then applied in Section 5.2 to solve some classes of variational problems on time scales. A simple illustrative example is given in Section 5.2.1.

The method here proposed is direct, in the sense that permits to find directly the optimal solution instead of using variational arguments and go through the usual procedure of solving the associated $\Delta$ Euler–Lagrange equation. This is particularly useful since even simple classes of problems of the calculus of variations on time scales lead to dynamic Euler–Lagrange equations for which methods to compute explicit solutions are not known. A second advantage of the method here promoted is that it provides directly an optimal solution, while the variational method on time scales is based on necessary optimality conditions, being necessary further analysis in order to conclude if the candidate is a local minimizer, a local maximizer, or just a saddle. Finally, while all the previous methods of the calculus of variations on time scales only establish local optimality, here we provide global solutions.

The use of inequalities to solve certain classes of optimal control problems is an old idea with a rich history [30, 38, 56, 57, 89]. We trust that the present study will be the beginning of a class of direct methods for optimal control problems on time scales, to be investigated with the help of dynamic inequalities.

## 5.1 Some integral inequalities

The next theorem is a generalization of the Jensen inequality on time scales.

**Theorem 68** (Generalized Jensen's inequality [94]). *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose $f : [a, b]_{\mathbb{T}}^{\kappa} \to (c, d)$ is rd-continuous and $F : (c, d) \to \mathbb{R}$ is convex. Moreover, let $h : [a, b]_{\mathbb{T}}^{\kappa} \to \mathbb{R}$ be rd-continuous with*

$$\int_a^b |h(t)|\Delta t > 0.$$

*Then,*

$$\frac{\int_a^b |h(t)|F(f(t))\Delta t}{\int_a^b |h(t)|\Delta t} \geq F\left(\frac{\int_a^b |h(t)|f(t)\Delta t}{\int_a^b |h(t)|\Delta t}\right). \tag{5.1}$$

The next observation is crucial to solve variational problems. Follows the statement and a proof.

**Proposition 69** (cf. [22]). *If in Theorem 68 $F$ is strictly convex and $h(t) \neq 0$ for all $t \in [a, b]_{\mathbb{T}}^{\kappa}$, then the equality in (5.1) holds if and only if $f$ is constant.*

*Proof.* Consider $x_0 \in (c, d)$ defined by

$$x_0 = \frac{\int_a^b |h(t)|f(t)\Delta t}{\int_a^b |h(t)|\Delta t}.$$

From the definition of strictly convexity, there exists $m \in \mathbb{R}$ such that

$$F(x) - F(x_0) > m(x - x_0),$$

for all $x \in (c, d)\backslash\{x_0\}$. Assume $f$ is not constant. Then, $f(t_0) \neq x_0$ for some $t_0 \in [a, b]_{\mathbb{T}}^{\kappa}$. We split the proof in two cases. (i) Assume that $t_0$ is right-dense. Then, since $f$ is rd-continuous, we have that $f(t) \neq x_0$ on $[t_0, t_0 + \delta)_{\mathbb{T}}$ for some $\delta > 0$. Hence,

$$\begin{aligned}
\int_a^b |h(t)|F(f(t))\Delta t - \int_a^b |h(t)|\Delta t F(x_0) &= \int_a^b |h(t)|[F(f(t)) - F(x_0)]\Delta t \\
&> m \int_a^b |h(t)|[f(t) - x_0]\Delta t \\
&= 0\,.
\end{aligned}$$

(ii) Assume now that $t_0$ is right-scattered. Then [note that $\int_{t_0}^{\sigma(t_0)} f(t)\Delta t = \mu(t_0)f(t_0)$],

$$\begin{aligned}
&\int_a^b |h(t)|F(f(t))\Delta t - \int_a^b |h(t)|\Delta t F(x_0) \\
&\quad= \int_a^b |h(t)|[F(f(t)) - F(x_0)]\Delta t \\
&\quad= \int_a^{t_0} |h(t)|[F(f(t)) - F(x_0)]\Delta t + \int_{t_0}^{\sigma(t_0)} |h(t)|[F(f(t)) - F(x_0)]\Delta t \\
&\qquad+ \int_{\sigma(t_0)}^b |h(t)|[F(f(t)) - F(x_0)]\Delta t \\
&\quad> \int_a^{t_0} |h(t)|[F(f(t)) - F(x_0)]\Delta t + m\int_{t_0}^{\sigma(t_0)} |h(t)|[f(t) - x_0]\Delta t \\
&\qquad+ \int_{\sigma(t_0)}^b |h(t)|[F(f(t)) - F(x_0)]\Delta t \\
&\quad\geq m\left\{\int_a^{t_0} |h(t)|[f(t) - x_0]\Delta t + \int_{t_0}^{\sigma(t_0)} |h(t)|[f(t) - x_0]\Delta t\right. \\
&\qquad\left.+ \int_{\sigma(t_0)}^b |h(t)|[f(t) - x_0]\Delta t\right\} \\
&\quad= m\int_a^b |h(t)|[f(t) - x_0]\Delta t = 0\,.
\end{aligned}$$

Finally, if $f$ is constant, it is obvious that the equality in (5.1) holds. □

*Remark* 70. If in Theorem 68 $F$ is a concave function, then the inequality sign in (5.1) must be reversed. Obviously, Proposition 69 remains true if we let $F$ to be strictly concave.

Before proceeding, we state a particular case of Theorem 68.

**Corollary 71.** *Let $a = q^n$ and $b = q^m$ for some $n, m \in \mathbb{N}_0$ with $n < m$. Define $f$ and $h$ on $[q^n, q^{m-1}]_{q^{\mathbb{N}_0}}$ and assume $F : (c, d) \to \mathbb{R}$ is convex, where $(c, d) \supset [f(q^n), f(q^{m-1})]_{q^{\mathbb{N}_0}}$. If*

$$\sum_{k=n}^{m-1} (q-1)q^k|h(q^k)| > 0,$$

*then*

$$\frac{\sum_{k=n}^{m-1} q^k|h(q^k)|F(f(q^k))}{\sum_{k=n}^{m-1} q^k|h(q^k)|} \geq F\left(\frac{\sum_{k=n}^{m-1} q^k|h(q^k)|f(q^k)}{\sum_{k=n}^{m-1} q^k|h(q^k)|}\right).$$

*Proof.* Choose $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$, $q > 1$, in Theorem 68. □

We now present Jensen's inequality on time scales, complemented by Proposition 69.

**Theorem 72** (cf. [22])**.** *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose $f : [a, b]_{\mathbb{T}}^{\kappa} \to (c, d)$ is rd-continuous and $F : (c, d) \to \mathbb{R}$ is convex (resp., concave). Then,*

$$\frac{\int_a^b F(f(t))\Delta t}{b-a} \geq F\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \tag{5.2}$$

*(resp., the reverse inequality). Moreover, if $F$ is strictly convex or strictly concave, then equality in (5.2) holds if and only if $f$ is constant.*

*Proof.* Particular case of Theorem 68 and Proposition 69 with $h(t) = 1$ for all $t \in [a, b]_{\mathbb{T}}^{\kappa}$. ☐

We now state and prove some consequences of Theorem 72.

**Corollary 73** (cf. [22])**.** *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose $f : [a, b]_{\mathbb{T}}^{\kappa} \to (c, d)$ is rd-continuous and $F : (c, d) \to \mathbb{R}$ is such that $F'' \geq 0$ (resp., $F'' \leq 0$). Then,*

$$\frac{\int_a^b F(f(t))\Delta t}{b-a} \geq F\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \tag{5.3}$$

*(resp., the reverse inequality). Furthermore, if $F'' > 0$ or $F'' < 0$, equality in (5.3) holds if and only if $f$ is constant.*

*Proof.* This follows immediately from Theorem 72 and the facts that a function $F$ with $F'' \geq 0$ (resp., $F'' \leq 0$) is convex (resp., concave) and with $F'' > 0$ (resp., $F'' \leq 0$) is strictly convex (resp., strictly concave). ☐

**Corollary 74** (cf. [22])**.** *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose $f : [a, b]_{\mathbb{T}}^{\kappa} \to (c, d)$ is rd-continuous and $\varphi, \psi : (c, d) \to \mathbb{R}$ are continuous functions such that $\varphi^{-1}$ exists, $\psi$ is strictly increasing, and $\psi \circ \varphi^{-1}$ is convex (resp., concave) on $\operatorname{Im}(\varphi)$. Then,*

$$\psi^{-1}\left(\frac{\int_a^b \psi(f(t))\Delta t}{b-a}\right) \geq \varphi^{-1}\left(\frac{\int_a^b \varphi(f(t))\Delta t}{b-a}\right)$$

*(resp., the reverse inequality). Furthermore, if $\psi \circ \varphi^{-1}$ is strictly convex or strictly concave, the equality holds if and only if $f$ is constant.*

*Proof.* Since $\varphi$ is continuous and $\varphi \circ f$ is rd-continuous, it follows from Theorem 72 with $f = \varphi \circ f$ and $F = \psi \circ \varphi^{-1}$ that

$$\frac{\int_a^b (\psi \circ \varphi^{-1})((\varphi \circ f)(t))\Delta t}{b-a} \geq (\psi \circ \varphi^{-1})\left(\frac{\int_a^b (\varphi \circ f)(t)\Delta t}{b-a}\right).$$

Since $\psi$ is strictly increasing, we obtain

$$\psi^{-1}\left(\frac{\int_a^b \psi(f(t))\Delta t}{b-a}\right) \geq \varphi^{-1}\left(\frac{\int_a^b \varphi(f(t))\Delta t}{b-a}\right).$$

Finally, when $\psi \circ \varphi^{-1}$ is strictly convex, the equality holds if and only if $\varphi \circ f$ is constant, or equivalently (since $\varphi$ is invertible), $f$ is constant. The case when $\psi \circ \varphi^{-1}$ is concave is treated analogously. ☐

**Corollary 75** (cf. [22]). *Assume $f : [a,b]_{\mathbb{T}}^{\kappa} \to \mathbb{R}$ is rd-continuous and positive. If $\alpha < 0$ or $\alpha > 1$, then*

$$\int_a^b (f(t))^\alpha \Delta t \geq (b-a)^{1-\alpha} \left( \int_a^b f(t)\Delta t \right)^\alpha .$$

*If $0 < \alpha < 1$, then*

$$\int_a^b (f(t))^\alpha \Delta t \leq (b-a)^{1-\alpha} \left( \int_a^b f(t)\Delta t \right)^\alpha .$$

*Furthermore, in both cases equality holds if and only if $f$ is constant.*

*Proof.* Define $F(x) = x^\alpha,\ x > 0$. Then

$$F''(x) = \alpha(\alpha-1)x^{\alpha-2}, \quad x > 0.$$

Hence, when $\alpha < 0$ or $\alpha > 1$, $F'' > 0$, i.e., $F$ is strictly convex. When $0 < \alpha < 1$, $F'' < 0$, i.e., $F$ is strictly concave. Applying Corollary 73 with this function $F$, we obtain the above inequalities with equality if and only if $f$ is constant. □

**Corollary 76** (cf. [22]). *Assume $f : [a,b]_{\mathbb{T}}^{\kappa} \to \mathbb{R}$ is rd-continuous and positive. If $\alpha < -1$ or $\alpha > 0$, then*

$$\left( \int_a^b \frac{1}{f(t)} \Delta t \right)^\alpha \int_a^b (f(t))^\alpha \Delta t \geq (b-a)^{1+\alpha} .$$

*If $-1 < \alpha < 0$, then*

$$\left( \int_a^b \frac{1}{f(t)} \Delta t \right)^\alpha \int_a^b (f(t))^\alpha \Delta t \leq (b-a)^{1+\alpha}.$$

*Furthermore, in both cases the equality holds if and only if $f$ is constant.*

*Proof.* This follows from Corollary 75 by replacing $f$ by $1/f$ and $\alpha$ by $-\alpha$. □

**Corollary 77** (cf. [22]). *If $f : [a,b]_{\mathbb{T}}^{\kappa} \to \mathbb{R}$ is rd-continuous, then*

$$\int_a^b e^{f(t)} \Delta t \geq (b-a) e^{\frac{1}{b-a}\int_a^b f(t)\Delta t} . \tag{5.4}$$

*Moreover, the equality in (5.4) holds if and only if $f$ is constant.*

*Proof.* Choose $F(x) = e^x$, $x \in \mathbb{R}$, in Corollary 73. □

**Corollary 78** (cf. [22]). *If $f : [a,b]_{\mathbb{T}}^{\kappa} \to \mathbb{R}$ is rd-continuous and positive, then*

$$\int_a^b \ln(f(t))\Delta t \leq (b-a) \ln \left( \frac{1}{b-a} \int_a^b f(t)\Delta t \right) . \tag{5.5}$$

*Moreover, the equality in (5.5) holds if and only if $f$ is constant.*

*Proof.* Let $F(x) = \ln(x)$, $x > 0$, in Corollary 73. □

**Corollary 79** (cf. [22]). *If $f : [a,b]_{\mathbb{T}}^{\kappa} \to \mathbb{R}$ is rd-continuous and positive, then*

$$\int_a^b f(t)\ln(f(t))\Delta t \geq \int_a^b f(t)\Delta t \ln\left(\frac{1}{b-a}\int_a^b f(t)\Delta t\right). \tag{5.6}$$

*Moreover, the equality in* (5.6) *holds if and only if $f$ is constant.*

*Proof.* Let $F(x) = x\ln(x)$, $x > 0$. Then, $F''(x) = 1/x$, i.e., $F''(x) > 0$ for all $x > 0$. By Corollary 73, we get

$$\frac{1}{b-a}\int_a^b f(t)\ln(f(t))\Delta t \geq \frac{1}{b-a}\int_a^b f(t)\Delta t \ln\left(\frac{1}{b-a}\int_a^b f(t)\Delta t\right),$$

and the result follows. □

## 5.2 Applications to the Calculus of Variations

We now show how the results obtained in Section 5.1 can be applied to determine the minimum or maximum of certain problems of calculus of variations and optimal control on time scales.

**Theorem 80** (cf. [22]). *Let $\mathbb{T}$ be a time scale, $a$, $b \in \mathbb{T}$ with $a < b$, and $\varphi : \mathbb{R} \to \mathbb{R}$ be a positive and continuous function. Consider the functional*

$$F[y(\cdot)] = \int_a^b \left[\left\{\int_0^1 \varphi(y(t) + h\mu(t)y^{\Delta}(t))dh\right\} y^{\Delta}(t)\right]^{\alpha} \Delta t, \quad \alpha \in \mathbb{R}\backslash\{0,1\},$$

*defined on all $C_{\mathrm{rd}}^1$ functions $y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ satisfying $y^{\Delta}(t) > 0$ on $[a,b]_{\mathbb{T}}^{\kappa}$, $y(a) = 0$, and $y(b) = B$. Define a function $G(x) = \int_0^x \varphi(s)ds$, $x \geq 0$, and let $G^{-1}$ denote its inverse. Let*

$$C = \frac{\int_0^B \varphi(s)ds}{b-a}. \tag{5.7}$$

**(i)** *If $\alpha < 0$ or $\alpha > 1$, then the minimum of $F$ occurs when*

$$y(t) = G^{-1}(C(t-a)), \quad t \in [a,b]_{\mathbb{T}},$$

*and $F_{\min} = (b-a)C^{\alpha}$.*

**(ii)** *If $0 < \alpha < 1$, then the maximum of $F$ occurs when*

$$y(t) = G^{-1}(C(t-a)), \quad t \in [a,b]_{\mathbb{T}},$$

*and $F_{\max} = (b-a)C^{\alpha}$.*

*Remark* 81. Since $\varphi$ is continuous and positive, $G$ and $G^{-1}$ are well defined.

*Remark* 82. In cases $\alpha = 0$ or $\alpha = 1$ there is nothing to minimize or maximize, i.e., the problem of extremizing $F[y(\cdot)]$ is trivial. Indeed, if $\alpha = 0$, then $F[y(\cdot)] = b - a$; if $\alpha = 1$, then it follows from Theorem 30 that

$$\begin{aligned} F[y(\cdot)] &= \int_a^b \left\{ \int_0^1 \varphi(y(t) + h\mu(t)y^\Delta(t))dh \right\} y^\Delta(t)\Delta t \\ &= \int_a^b (G \circ y)^\Delta(t)\Delta t \\ &= G(B)\,. \end{aligned}$$

In both cases $F$ is a constant and does not depend on function $y$.

*Proof of Theorem 80.* Suppose that $\alpha < 0$ or $\alpha > 1$. Using Corollary 75 we can write

$$\begin{aligned} F[y(\cdot)] \geq (b-a)^{1-\alpha} &\left[ \int_a^b \left\{ \int_0^1 \varphi(y(t) + h\mu(t)y^\Delta(t))dh \right\} y^\Delta(t)\Delta t \right]^\alpha \\ &= (b-a)^{1-\alpha}(G(y(b)) - G(y(a)))^\alpha\,, \end{aligned}$$

where the equality holds if and only if

$$\left\{ \int_0^1 \varphi(y(t) + h\mu(t)y^\Delta(t))dh \right\} y^\Delta(t) = c, \quad \text{for some } c \in \mathbb{R}, \quad t \in [a,b]^\kappa_\mathbb{T}.$$

Using Theorem 30 we arrive at

$$(G \circ y)^\Delta(t) = c.$$

$\Delta$-integrating from $a$ to $t$ yields (note that $y(a) = 0$ and $G(0) = 0$)

$$G(y(t)) = c(t-a),$$

from which we get

$$y(t) = G^{-1}(c(t-a)).$$

The value of $c$ is obtained using the boundary condition $y(b) = B$:

$$c = \frac{G(B)}{b-a} = \frac{\int_0^B \varphi(s)ds}{b-a} = C,$$

with $C$ as in (5.7). Finally, in this case

$$F_{\min} = \int_a^b C^\alpha \Delta t = (b-a)C^\alpha.$$

The proof of the second part of the theorem is done analogously using the second part of Corollary 75. □

*Remark* 83. We note that the optimal solution found in the proof of the previous theorem satisfies $y^\Delta > 0$. Indeed,

$$\begin{aligned} y^\Delta(t) &= \left(G^{-1}(C(t-a))\right)^\Delta \\ &= \int_0^1 (G^{-1})'[C(t-a) + h\mu(t)C]dh\ C \\ &> 0, \end{aligned}$$

because $C > 0$ and $(G^{-1})'(x) = \frac{1}{\varphi(G^{-1}(x))} > 0$ for all $x \geq 0$.

**Theorem 84** (cf. [22])**.** *Let $\varphi : [a,b]_{\mathbb{T}}^\kappa \to \mathbb{R}$ be a positive and rd-continuous function. Then, among all $C_{\mathrm{rd}}^1$ functions $y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ with $y(a) = 0$ and $y(b) = B$, the functional*

$$F[y(\cdot)] = \int_a^b \varphi(t) e^{y^\Delta(t)} \Delta t$$

*has minimum value $F_{\min} = (b-a)e^C$ attained when*

$$y(t) = -\int_a^t \ln(\varphi(s))\Delta s + C(t-a), \quad t \in [a,b]_{\mathbb{T}},$$

*where*

$$C = \frac{\int_a^b \ln(\varphi(t))\Delta t + B}{b-a}. \tag{5.8}$$

*Proof.* By Corollary 77,

$$\begin{aligned} F[y(\cdot)] &= \int_a^b e^{\ln(\varphi(t)) + y^\Delta(t)} \Delta t \\ &\geq (b-a) e^{\frac{1}{b-a}\int_a^b [\ln(\varphi(t)) + y^\Delta(t)]\Delta t} = (b-a) e^{\frac{1}{b-a}\left[\int_a^b \ln(\varphi(t))\Delta t + B\right]}, \end{aligned}$$

with $F(y(\cdot)) = (b-a)e^{\frac{1}{b-a}\left[\int_a^b \ln(\varphi(t))\Delta t + B\right]}$ if and only if

$$\ln(\varphi(t)) + y^\Delta(t) = c, \quad \text{for some } c \in \mathbb{R}, \quad t \in [a,b]_{\mathbb{T}}^\kappa. \tag{5.9}$$

Integrating (5.9) from $a$ to $t$ (note that $y(a) = 0$) gives

$$y(t) = -\int_a^t \ln(\varphi(s))\Delta s + c(t-a), \quad t \in [a,b]_{\mathbb{T}}.$$

Using the boundary condition $y(b) = B$ we have

$$c = \frac{\int_a^b \ln(\varphi(t))\Delta t + B}{b-a} = C,$$

with $C$ as in (5.8). A simple calculation shows that $F_{\min} = (b-a)e^C$. □

*Remark* 85. If we let $\mathbb{T} = \mathbb{R}$ in the previous theorem we get [30, Theorem 3.4].

**Theorem 86** (cf. [22]). *Let $\varphi : [a,b]_{\mathbb{T}}^{\kappa} \to \mathbb{R}$ be a positive and rd-continuous function. Then, among all $C_{\mathrm{rd}}^1$-functions $y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ satisfying $y^{\Delta} > 0$, $y(a) = 0$, and $y(b) = B$, with*

$$\frac{B + \int_a^b \varphi(s)\Delta s}{b-a} > \varphi(t), \quad t \in [a,b]_{\mathbb{T}}^{\kappa}, \tag{5.10}$$

*the functional*

$$F[y(\cdot)] = \int_a^b [\varphi(t) + y^{\Delta}(t)] \ln[\varphi(t) + y^{\Delta}(t)]\Delta t$$

*has minimum value $F_{\min} = (b-a)C\ln(C)$ attained when*

$$y(t) = C(t-a) - \int_a^t \varphi(s)\Delta s, \quad t \in [a,b]_{\mathbb{T}},$$

*where*

$$C = \frac{B + \int_a^b \varphi(s)\Delta s}{b-a}. \tag{5.11}$$

*Proof.* By Corollary 79,

$$\begin{aligned} F[y(\cdot)] \geq \int_a^b [\varphi(t) + y^{\Delta}(t)]\Delta t \ln\left(\frac{1}{b-a}\int_a^b [\varphi(t) + y^{\Delta}(t)]\Delta t\right) \\ = \left(\int_a^b \varphi(t)\Delta t + B\right) \ln\left(\frac{\int_a^b \varphi(t)\Delta t + B}{b-a}\right) \end{aligned}$$

with $F[y(\cdot)] = \left(\int_a^b \varphi(t)\Delta t + B\right) \ln\left(\frac{\int_a^b \varphi(t)\Delta t+B}{b-a}\right)$ if and only if

$$\varphi(t) + y^{\Delta}(t) = c, \quad \text{for some } c \in \mathbb{R}, \quad t \in [a,b]_{\mathbb{T}}^{\kappa}.$$

Upon integration from $a$ to $t$ (note that $y(a) = 0$),

$$y(t) = c(t-a) - \int_a^t \varphi(s)\Delta s, \ t \in [a,b]_{\mathbb{T}}.$$

Using the boundary condition $y(b) = B$, we have

$$c = \frac{B + \int_a^b \varphi(s)\Delta s}{b-a} = C,$$

where $C$ is as in (5.11). Note that with this choice of $y$ we have, using (5.10), that $y^{\Delta}(t) = C - \varphi(t) > 0$, $t \in [a,b]_{\mathbb{T}}^{\kappa}$. It follows that $F_{\min} = (b-a)C\ln(C)$. $\square$

In order to close this subject we would like to point out that Theorem 3.6 in [30] is not true. This is due to the fact that the bound on the functional $I$ considered in the proof is not constant. Let us quote the "theorem":

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a positive and continuous function and $a > 0$. Then, among all $C^1$ functions $y : [0, a] \to \mathbb{R}$ satisfying $y' > 0$, $y(0) = 0$, and $y(a) = A$, the functional

$$I = \int_0^a \ln(\varphi(x)y'(x))dx$$

attains its maximum when

$$y = \frac{1}{C}\int_0^x \frac{1}{\varphi(s)}ds,$$

where

$$C = \frac{1}{A}\int_0^a \frac{1}{\varphi(s)}ds, \tag{5.12}$$

and

$$I_{\max} = -a\ln(C). \tag{5.13}$$

Now we give a counterexample to [30, Theorem 3.6]. Let $a = A = 1$, $\varphi(x) = x + 1$, and $\tilde{y}(x) = x$ for all $x \in [0, 1]$. Then, the hypotheses of the "theorem" are satisfied. Moreover,

$$I[\tilde{y}(x)] = \int_0^1 \ln(\varphi(x)\tilde{y}'(x))dx = [(x + 1)(\ln(x + 1) - 1)]_{x=0}^{x=1} = 2\ln(2) - 1 \approx 0.386.$$

According with (5.12) and (5.13) the maximum of the functional $I$ is given by $I_{\max} = -\ln(C)$, where

$$C = \int_0^1 \frac{1}{\varphi(s)}ds.$$

A simple calculation shows that $C = \ln(2)$, hence $I_{\max} = -\ln(\ln(2)) \approx 0.367$. Therefore, $I[\tilde{y}(x)] > I_{\max}$, which proves our claim.

### 5.2.1 An example

Let us consider $\mathbb{T} = \mathbb{Z}$, $a = 0$, $b = 5$, $B = 25$ and $\varphi(t) = 2t + 1$ in Theorem 86:

**Corollary 87** (cf. [22])**.** *The functional*

$$F[y(\cdot)] = \sum_{t=0}^{4}[(2t + 1) + (y(t + 1) - y(t))]\ln[(2t + 1) + (y(t + 1) - y(t))],$$

*defined for all $y : [0, 5] \cap \mathbb{Z} \to \mathbb{R}$ such that $y(t + 1) > y(t)$ for all $t \in [0, 4] \cap \mathbb{T}$, attains its minimum when*

$$y(t) = 10t - t^2, \quad t \in [0, 5] \cap \mathbb{Z},$$

*and $F_{\min} = 50\ln(10)$.*

*Proof.* First we note that $\max\{\varphi(t) : t \in [0, 4] \cap \mathbb{Z}\} = 9$, hence

$$\frac{B + \sum_{k=0}^{4}\varphi(k)}{b - a} = \frac{25 + 25}{5} = 10 > 9 \geq \varphi(t).$$

Observing that, when $\mathbb{T} = \mathbb{Z}$, $(t^2)^\Delta = 2t + 1$, we just have to invoke Theorem 86 to get the desired result. □

## 5.3 State of the Art

The results of this chapter are already published [22]. Direct methods are an important subject to the calculus of variations theory and further research is in progress, extending Leitmann's direct method to time scales [73].

# Chapter 6

# Inequalities on Time Scales

This chapter is devoted to the development of some integral inequalities as well to some of its applications. These inequalities will be used to prove existence of solution(s) to some dynamic equations and to estimate them, and this is shown in Sections 6.1 and 6.2. In Section 6.3 we prove Hölder, Cauchy–Schwarz, Minkowski and Jensen's type inequalities in the more general setting of $\Diamond_\alpha$-integrals. Finally, in Section 6.4, we obtain some Gronwall–Bellman–Bihari type inequalities for functions depending on two time scales variables.

Throughout we use the notations $\mathbb{R}_0^+ = [0, \infty)$ and $\mathbb{R}^+ = (0, \infty)$.

## 6.1 Gronwall's type inequalities

We start by proving a lemma which is essential in the proofs of the next theorems.

**Lemma 88** (cf. [46]). *Let $a, b \in \mathbb{T}$, and consider a function $r \in C_{rd}^1([a, b]_\mathbb{T}, \mathbb{R}^+)$ with $r^\Delta(t) \geq 0$ on $[a, b]_\mathbb{T}^\kappa$. Suppose that a function $g \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ is positive and nondecreasing on $\mathbb{R}^+$ and define,*

$$G(x) = \int_{x_0}^{x} \frac{ds}{g(s)},$$

*where $x \geq 0$, $x_0 \geq 0$ if $\int_0^x \frac{ds}{g(s)} < \infty$ and $x > 0$, $x_0 > 0$ if $\int_0^x \frac{ds}{g(s)} = \infty$. Then, for each $t \in [a, b]_\mathbb{T}$, we have*

$$G(r(t)) \leq G(r(a)) + \int_a^t \frac{r^\Delta(\tau)}{g(r(\tau))} \Delta\tau. \tag{6.1}$$

*Proof.* Since $g$ is positive and nondecreasing on $(0,\infty)$, we have, successively, that

$$\begin{gathered} r(t) \le r(t) + h\mu(t)r^{\Delta}(t)\,,\\ g(r(t)) \le g(r(t) + h\mu(t)r^{\Delta}(t))\,,\\ \frac{1}{g(r(t)+h\mu(t)r^{\Delta}(t))} \le \frac{1}{g(r(t))}\,,\\ \int_0^1 \frac{1}{g(r(t)+h\mu(t)r^{\Delta}(t))}dh \le \int_0^1 \frac{1}{g(r(t))}dh = \frac{1}{g(r(t))}\,,\\ \left\{\int_0^1 \frac{1}{g(r(t)+h\mu(t)r^{\Delta}(t))}dh\right\} r^{\Delta}(t) \le \frac{r^{\Delta}(t)}{g(r(t))}\,, \end{gathered} \tag{6.2}$$

for all $t \in [a,b]_{\mathbb{T}}^{\kappa}$ and $h \in [0,1]$. By $\Delta$-integrating the last inequality in (6.2) from $a$ to $t$ and having in mind that Theorem 30 guarantees that

$$\begin{aligned} (G\circ r)^{\Delta}(t) &= \left\{\int_0^1 G'(r(t)+h\mu(t)r^{\Delta}(t))dh\right\} r^{\Delta}(t)\\ &= \left\{\int_0^1 \frac{1}{g(r(t)+h\mu(t)r^{\Delta}(t))}dh\right\} r^{\Delta}(t), \end{aligned}$$

we obtain the desired result [note that the case $t = b$ if $\rho(b) < b$ is also proved because of (1.5)]. □

**Theorem 89** (cf. [46]). *Let $u(t)$ and $f(t)$ be nonnegative rd-continuous functions in the time scales interval $\mathbb{T}_* := [a,b]_{\mathbb{T}}$ and $\mathbb{T}_*^{\kappa}$, respectively. Let $k(t,s)$ be defined as in Theorem 29 in such a way that $k(t,s)$ and $k^{\Delta_1}(t,s)$ are nonnegative for every $t,s \in \mathbb{T}_*$ with $s \le t$ for which they are defined (it is assumed that $k$ is not identically zero on $\mathbb{T}_*^{\kappa}\times\mathbb{T}_*^{\kappa^2}$). Let $\Phi \in C(\mathbb{R}_0^+,\mathbb{R}_0^+)$ be a nondecreasing, subadditive and submultiplicative function, such that $\Phi(u) > 0$ for $u > 0$ and let $W \in C(\mathbb{R}_0^+,\mathbb{R}_0^+)$ be a nondecreasing function such that for $u > 0$ we have $W(u) > 0$. Assume that $a(t)$ is a positive rd-continuous function and nondecreasing for $t \in \mathbb{T}_*$. If*

$$u(t) \le a(t) + \int_a^t f(s)u(s)\Delta s + \int_a^t f(s)W\left(\int_a^s k(s,\tau)\Phi(u(\tau))\Delta\tau\right)\Delta s, \tag{6.3}$$

*for $a \le \tau \le s \le t \le b$, $\tau, s, t \in \mathbb{T}_*$, then for all $t \in \mathbb{T}_*$ satisfying*

$$\Psi(\zeta) + \int_a^{\rho(t)} k(\rho(t),s)\Phi(p(s))\Phi(\int_a^s f(\tau)\Delta\tau)\Delta s \in \mathrm{Dom}(\Psi^{-1})$$

*we have*

$$\begin{aligned} u(t) &\le p(t)a(t)\\ &\quad + p(t)\int_a^t f(s)W\left[\Psi^{-1}\left(\Psi(\zeta) + \int_a^s k(s,\tau)\Phi(p(\tau))\Phi\left(\int_a^{\tau} f(\theta)\Delta\theta\right)\Delta\tau\right)\right]\Delta s\,, \end{aligned} \tag{6.4}$$

*where*

$$p(t) = 1 + \int_a^t f(s)e_f(t,\sigma(s))\Delta s\,, \tag{6.5}$$

$$\zeta = \int_a^{\rho(b)} k(\rho(b),s)\Phi(p(s)a(s))\Delta s\,,$$

$$\Psi(x) = \int_{x_0}^{x} \frac{1}{\Phi(W(s))}ds,\ \ x>0,\ \ x_0>0\,, \tag{6.6}$$

*and, as usual, $\Psi^{-1}$ denotes the inverse of $\Psi$.*

*Remark* 90. We are interested to study the situation when $k$ is not identically zero on $\mathbb{T}_*^{\kappa}\times\mathbb{T}_*^{\kappa^2}$. That comprise the new cases, not considered previously in the literature. The case $k(t,s)\equiv 0$ was studied in [8, Theorem 3.1] and is not discussed here.

*Proof.* Define the function $z(t)$ in $\mathbb{T}_*$ by

$$z(t) = a(t) + \int_a^t f(s)W\left(\int_a^s k(s,\tau)\Phi(u(\tau))\Delta\tau\right)\Delta s\,. \tag{6.7}$$

Then, (6.3) can be rewritten as

$$u(t) \le z(t) + \int_a^t f(s)u(s)\Delta s.$$

Clearly, $z(t)$ is rd-continuous in $t\in\mathbb{T}_*$. Using Theorem 47, we get

$$u(t) \le z(t) + \int_a^t f(s)z(s)e_f(t,\sigma(s))\Delta s\,.$$

Moreover, it is easy to see that $z(t)$ is nondecreasing in $t\in\mathbb{T}_*$. We get

$$u(t) \le z(t)p(t), \tag{6.8}$$

where $p(t)$ is defined by (6.5). Define

$$v(t) = \int_a^t k(t,s)\Phi(u(s))\Delta s,\ \ t\in\mathbb{T}_*^{\kappa}.$$

From (6.8), and taking into account the properties of $\Phi$, we get

$$\begin{aligned}
v(t) &\le \int_a^t k(t,s)\Phi\left[p(s)\left(a(s)+\int_a^s f(\tau)W(v(\tau))\Delta\tau\right)\right]\Delta s\\
&\le \int_a^t k(t,s)\Phi(p(s)a(s))\Delta s + \int_a^t k(t,s)\Phi\left(p(s)\int_a^s f(\tau)W(v(\tau))\Delta\tau\right)\Delta s\\
&\le \int_a^{\rho(b)} k(\rho(b),s)\Phi(p(s)a(s))\Delta s\\
&\qquad + \int_a^t k(t,s)\Phi\left(p(s)\int_a^s f(\tau)\Delta\tau\right)\Phi(W(v(s))\Delta s\\
&= \zeta + \int_a^t k(t,s)\Phi\left(p(s)\int_a^s f(\tau)\Delta\tau\right)\Phi(W(v(s))\Delta s\,.
\end{aligned}$$

Define function $r(t)$ on $\mathbb{T}_*^\kappa$ by

$$r(t) = \zeta + \int_a^t k(t,s)\Phi\left(p(s)\int_a^s f(\tau)\Delta\tau\right)\Phi(W(v(s))\Delta s\,.$$

Since $p$ and $a$ are positive functions, we have that $\Phi(a(s)p(s)) > 0$ for all $s \in \mathbb{T}_*$. Since $k^{\Delta_1} \geq 0$, we must have $\zeta > 0$, hence $r(t)$ is a positive function on $\mathbb{T}_*^\kappa$. In addition, $r(t)$ is $\Delta$-differentiable on $\mathbb{T}_*^{\kappa^2}$ with

$$\begin{aligned} r^\Delta(t) &= k(\sigma(t),t)\Phi\left(p(t)\int_a^t f(\tau)\Delta\tau\right)\Phi(W(v(t)) \\ &\quad + \int_a^t k^{\Delta_1}(t,s)\Phi\left(p(s)\int_a^s f(\tau)\Delta\tau\right)\Phi(W(v(s))\Delta s \\ &\leq \Phi(W(r(t)))\left[k(\sigma(t),t)\Phi\left(p(t)\int_a^t f(\tau)\Delta\tau\right)\right. \\ &\quad \left. + \int_a^t k^{\Delta_1}(t,s)\Phi\left(p(s)\int_a^s f(\tau)\Delta\tau\right)\Delta s\right]. \end{aligned} \tag{6.9}$$

Dividing both sides of inequality (6.9) by $\Phi(W(r(t)))$, we obtain

$$\frac{r^\Delta(t)}{\Phi(W(r(t)))} \leq \left[\int_a^t k(t,s)\Phi\left(p(s)\int_a^s f(\tau)\Delta\tau\right)\Delta s\right]^\Delta.$$

Let us consider the function $\Psi$ defined by (6.6). $\Delta$-integrating this last inequality from $a$ to $t$ and using Lemma 88, we obtain

$$\Psi(r(t)) \leq \Psi(r(a)) + \int_a^t k(t,s)\Phi\left(p(s)\int_a^s f(\tau)\Delta\tau\right)\Delta s,$$

from which it follows that

$$r(t) \leq \Psi^{-1}\left(\Psi(\zeta) + \int_a^t k(t,s)\Phi(p(s))\Phi(\int_a^s f(\tau)\Delta\tau)\Delta s\right),\ t \in \mathbb{T}_*^\kappa. \tag{6.10}$$

Combining (6.10), (6.8) and (6.7), we obtain the desired inequality (6.4). $\square$

If we let $\mathbb{T} = \mathbb{R}$ in Theorem 89, we get [35, Theorem 2.1]. If in turn we consider $\mathbb{T} = \mathbb{Z}$, then we obtain the following result:

**Corollary 91** (cf. [46])**.** *Let $u(t)$ and $f(t)$ be nonnegative functions in the time scales interval $\mathbb{T}_* := [a,b]_\mathbb{Z}$ and $[a,b-1]_\mathbb{Z}$, respectively. Let $k(t,s)$ be defined as in Theorem 29 in such a way that $k(t,s)$ and $k^{\Delta_1}(t,s) = k(\sigma(t),s) - k(t,s)$ are nonnegative for every $t,s \in \mathbb{T}_*$ with $s \leq t$ for which they are defined (it is assumed that $k$ is not identically zero on $[a,b-1]_{\mathbb{T}_*} \times [a,b-2]_{\mathbb{T}_*}$). Let $\Phi \in C(\mathbb{R}_0^+,\mathbb{R}_0^+)$ be a nondecreasing, subadditive and submultiplicative function such that*

*$\Phi(u) > 0$ for $u > 0$ and let $W \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ be a nondecreasing function such that for $u > 0$ we have $W(u) > 0$. Assume that $a(t)$ is a positive and nondecreasing function for $t \in \mathbb{T}_*$. If*

$$u(t) \leq a(t) + \sum_{s=a}^{t-1} f(s)u(s) + \sum_{s=a}^{t-1} f(s)W\left(\sum_{\tau=a}^{s-1} k(s,\tau)\Phi(u(\tau))\right),$$

*for $a \leq \tau \leq s \leq t \leq b$, $\tau, s, t \in \mathbb{T}_*$, then for all $t \in \mathbb{T}_*$ satisfying*

$$\Psi(\zeta) + \sum_{s=a}^{t-2} k(t-1,s)\Phi(p(s))\Phi(\sum_{\tau=a}^{s-1} f(\tau)) \in \mathrm{Dom}(\Psi^{-1})$$

*we have*

$$u(t) \leq p(t)\left\{a(t) + \sum_{s=a}^{t-1} f(s)W\left[\Psi^{-1}\left(\Psi(\zeta) + \sum_{\tau=a}^{s-1} k(s,\tau)\Phi(p(\tau))\Phi(\sum_{\theta=a}^{\tau-1} f(\theta))\right)\right]\right\},$$

*where*

$$p(t) = 1 + \sum_{s=a}^{t-1} f(s)e_f(t, s+1),$$

$$\zeta = \sum_{s=a}^{b-1} k(b-1,s)\Phi(p(s)a(s)),$$

$$\Psi(x) = \int_{x_0}^{x} \frac{1}{\Phi(W(s))}ds, \quad x > 0, \quad x_0 > 0,$$

*and $\Psi^{-1}$ is the inverse of $\Psi$.*

For the particular case $\mathbb{T} = \mathbb{R}$, Theorem 92 generalizes the result obtained by Oguntuase in [80, Theorems 2.3 and 2.9].

**Theorem 92** (cf. [46])**.** *Suppose that $u(t)$ is a nonnegative rd-continuous function in the time scales interval $\mathbb{T}_* = [a,b]_{\mathbb{T}}$ and that $h(t)$, $f(t)$ are nonnegative rd-continuous functions in the time scales interval $\mathbb{T}_*^{\kappa}$. Assume that $b(t)$ is a nonnegative rd-continuous function and not identically zero on $\mathbb{T}_*^{\kappa^2}$. Let $\Phi(u)$, $W(u)$, and $a(t)$ be as defined in Theorem 89. If*

$$u(t) \leq a(t) + \int_a^t f(s)u(s)\Delta s + \int_a^t f(s)h(s)W\left(\int_a^s b(\tau)\Phi(u(\tau))\Delta\tau\right)\Delta s,$$

*for $a \leq \tau \leq s \leq t \leq b$, $\tau, s, t \in \mathbb{T}_*$, then for all $t \in \mathbb{T}_*$ satisfying*

$$\Psi(\xi) + \int_a^{\rho(t)} b(\tau)\Phi(p(\tau))\Phi(\int_a^{\tau} f(\theta)h(\theta)\Delta\theta)\Delta\tau \in \mathrm{Dom}(\Psi^{-1})$$

*we have*

$$\begin{aligned} u(t) &\leq p(t)a(t) \\ &\quad + p(t)\int_a^t f(s)h(s)W\left[\Psi^{-1}\left(\Psi(\xi) + \int_a^s b(\tau)\Phi(p(\tau))\Phi(\int_a^{\tau} f(\theta)h(\theta)\Delta\theta)\Delta\tau\right)\right]\Delta s, \end{aligned}$$

*where $p(t)$ is defined by (6.5), $\Psi$ is defined by (6.6), and*

$$\xi = \int_a^{\rho(b)} b(s)\Phi(p(s)a(s))\Delta s.$$

*Proof.* similar to the proof of Theorem 89. $\square$

For the remaining of this section, we use the following class of $S$ functions.

**Definition 93** ($S$ function)**.** A nondecreasing continuous function $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $S$ if it satisfies the following conditions:

1. $g(x)$ is positive for $x > 0$;

2. $(1/z)g(x) \leq g(x/z)$ for $x \geq 0$ and $z \geq 1$.

*Remark* 94. For a brief discussion about this class of $S$ functions, the reader is invited to consult [19, Section 4].

**Theorem 95** (cf. [46])**.** *Let $u(t)$, $f(t)$, $k(t,s)$, $\Phi$ and $W$ be as defined in Theorem 89 and assume that $g \in S$. Suppose that $a(t)$ is a positive, rd-continuous and nondecreasing function. If*

$$u(t) \leq a(t) + \int_a^t f(s)g(u(s))\Delta s + \int_a^t f(s)W\left(\int_a^s k(s,\tau)\Phi(u(\tau))\Delta\tau\right)\Delta s, \tag{6.11}$$

*for $a \leq \tau \leq s \leq t \leq b$, $\tau, s, t \in \mathbb{T}_*$, then for all $t \in \mathbb{T}_*$ satisfying*

$$G(1) + \int_a^t f(\tau)\Delta\tau \in \mathrm{Dom}(G^{-1})$$

*and*

$$\Psi(\bar{\zeta}) + \int_a^{\rho(t)} k(\rho(t),\tau)\Phi(q(\tau))\Phi(\int_a^{\tau} f(\theta)\Delta\theta)\Delta\tau \in \mathrm{Dom}(\Psi^{-1}),$$

*we have*

$$\begin{aligned} u(t) &\leq q(t)\max\{a(t),1\} \\ &\quad + q(t)\int_a^t f(s)W\left[\Psi^{-1}\left(\Psi(\bar{\zeta}) + \int_a^s k(s,\tau)\Phi(q(\tau))\Phi(\int_a^{\tau} f(\theta)\Delta\theta)\Delta\tau\right)\right]\Delta s\,, \end{aligned}$$

*where $\Psi$ is defined by (6.6),*

$$\begin{aligned} G(x) &= \int_{\delta}^{x} \frac{ds}{g(s)},\ x > 0,\ \delta > 0, \\ q(t) &= G^{-1}\left(G(1) + \int_a^t f(\tau)\Delta\tau\right), \qquad (6.12) \\ \bar{\zeta} &= \int_a^{\rho(b)} k(\rho(b),s)\Phi(q(s)\max\{a(s),1\})\Delta s, \end{aligned}$$

*and $G^{-1}$ is the inverse function of $G$.*

*Proof.* Define the function

$$z(t) = \max\{a(t), 1\} + \int_a^t f(s) W\left(\int_a^s k(s,\tau)\Phi(u(\tau))\Delta\tau\right)\Delta s.$$

Then, from (6.11) we have that

$$u(t) \leq z(t) + \int_a^t f(s) g(u(s))\Delta s.$$

Clearly, $z(t) \geq 1$ is rd-continuous and nondecreasing. Since $g \in S$, we have

$$\frac{u(t)}{z(t)} \leq 1 + \int_a^t f(s) g\left(\frac{u(s)}{z(s)}\right)\Delta s,$$

or

$$x(t) \leq 1 + \int_a^t f(s) g(x(s))\Delta s, \tag{6.13}$$

with $x(t) = u(t)/z(t)$. If we define $v(t)$ as the right-hand side of inequality (6.13), we have that $v(a) = 1$,

$$v^\Delta(t) = f(t) g(x(t)),$$

and since $g$ is nondecreasing,

$$v^\Delta(t) \leq f(t) g(v(t)),$$

or

$$\frac{v^\Delta(t)}{g(v(t))} \leq f(t). \tag{6.14}$$

Being the case that $v^\Delta(t) \geq 0$, $\Delta$-integrating (6.14) from $a$ to $t$ and applying Lemma 88, we obtain

$$G(v(t)) \leq G(1) + \int_a^t f(\tau)\Delta\tau,$$

which implies that

$$v(t) \leq G^{-1}\left(G(1) + \int_a^t f(\tau)\Delta\tau\right).$$

We have just proved that $x(t) \leq q(t)$, which is equivalent to

$$u(t) \leq q(t) z(t).$$

Following the same arguments as in the proof of Theorem 89, we obtain the desired inequality. □

If we consider the time scale $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then we obtain the following result.

**Corollary 96** (cf. [46]). *Let $a, b \in h\mathbb{Z}$, $h > 0$. Suppose that $u(t)$, $f(t)$, $k(t,s)$, $\Phi$ and $W$ are as defined in Theorem 89 and assume that $g \in S$. Suppose that $a(t)$ is a positive and nondecreasing function. If*

$$u(t) \leq a(t) + \sum_{s\in[a,t)_{\mathbb{T}_*}} f(s)g(u(s))h + \sum_{s\in[a,t)_{\mathbb{T}_*}} f(s)W\left(\sum_{\tau\in[a,s)_{\mathbb{T}_*}} k(s,\tau)\Phi(u(\tau))h\right)h,$$

*for $a \leq \tau \leq s \leq t \leq b$, $\tau, s, t \in \mathbb{T}_* = [a,b]_{h\mathbb{Z}}$, then for all $t \in \mathbb{T}_*$ satisfying*

$$G(1) + \sum_{\tau\in[a,t)_{\mathbb{T}_*}} f(\tau)h \in \mathrm{Dom}(G^{-1})$$

*and*

$$\Psi(\bar{\zeta}) + \sum_{\tau\in[a,t-h)_{\mathbb{T}_*}} k(t-h,\tau)\Phi(q(\tau))\Phi\left(\sum_{\theta\in[a,\tau)_{\mathbb{T}_*}} f(\theta)h\right)h \in \mathrm{Dom}(\Psi^{-1}),$$

*we have*

$$\begin{aligned} u(t) \leq q(t)\Bigg\{&\max\{a(t),1\} \\ &+ \sum_{s\in[a,t)_{\mathbb{T}_*}} f(s)W\left[\Psi^{-1}\left(\Psi(\bar{\zeta}) + \sum_{\tau\in[a,s)_{\mathbb{T}_*}} k(s,\tau)\Phi(q(\tau))\Phi\left(\sum_{\theta\in[a,\tau)_{\mathbb{T}_*}} f(\theta)h\right)h\right)\right]h\Bigg\},\end{aligned}$$

*where $\Psi$ is defined by (6.6),*

$$G(x) = \int_\delta^x \frac{ds}{g(s)},\ x > 0,\ \delta > 0,$$

$$q(t) = G^{-1}\left(G(1) + \sum_{\tau\in[a,t)_{\mathbb{T}_*}} f(\tau)h\right),$$

$$\bar{\zeta} = \sum_{s\in[a,b-h)\mathbb{T}_*} k(b-h,s)\Phi(q(s)\max\{a(s),1\})h,$$

*and $G^{-1}$ is the inverse function of $G$.*

**Theorem 97** (cf. [46]). *Let $u(t)$, $f(t)$, $b(t)$, $h(t)$, $\Phi$ and $W$ be as defined in Theorem 92 and assume that $g \in S$. Suppose that $a(t)$ is a positive, rd-continuous and nondecreasing function. If*

$$u(t) \leq a(t) + \int_a^t f(s)g(u(s))\Delta s + \int_a^t f(s)h(s)W\left(\int_a^s b(\tau)\Phi(u(\tau))\Delta\tau\right)\Delta s,$$

*for $a \leq \tau \leq s \leq t \leq b$, $\tau, s, t \in \mathbb{T}_*$, then for all $t \in \mathbb{T}_*$ satisfying*

$$\Psi(\bar{\xi}) + \int_a^{\rho(t)} b(\tau)\Phi(q(\tau))\Phi(\int_a^\tau f(\theta)h(\theta)\Delta\theta)\Delta\tau \in \mathrm{Dom}(\Psi^{-1}),$$

*we have*

$$u(t) \le q(t)\max\{a(t),1\} + q(t)\int_a^t f(s)h(s)W\left[\Psi^{-1}\left(\Psi(\bar{\xi}) + \int_a^s b(\tau)\Phi(q(\tau))\Phi(\int_a^\tau f(\theta)h(\theta)\Delta\theta)\Delta\tau\right)\right]\Delta s,$$

*where* $\Psi$ *is defined by (6.6),* $q(t)$ *is defined by (6.12) and*

$$\bar{\xi} = \int_a^{\rho(b)} b(s)\Phi(q(s)\max\{a(s),1\})\Delta s,$$

*Proof.* similar to the proof of Theorem 95. □

We now make use of Theorem 92 to the qualitative analysis of a nonlinear dynamic equation. Let $a, b \in \mathbb{T}$ and consider the initial value problem

$$u^\Delta(t) = F\left(t, u(t), \int_a^t K(t, u(s))\Delta s\right), \quad t \in \mathbb{T}_*^\kappa, \quad u(a) = u_a, \tag{6.15}$$

where $\mathbb{T}_* = [a,b]_\mathbb{T}$, $u \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}_*)$, $F \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_* \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $K \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_* \times \mathbb{R}, \mathbb{R})$.

In what follows, we shall assume that the IVP (6.15) has a unique solution, which we denote by $u_*(t)$.

**Theorem 98** (cf. [46]). *Assume that the functions* $F$ *and* $K$ *in (6.15) satisfy the conditions*

$$|K(t,u)| \le h(t)\Phi(|u|), \tag{6.16}$$

$$|F(t,u,v)| \le |u| + |v|, \tag{6.17}$$

*where* $h$ *and* $\Phi$ *are as defined in Theorem 92. Then, for* $t \in \mathbb{T}_*$ *such that*

$$\Psi(\xi) + \int_a^{\rho(t)} \Phi(p(\tau))\Phi(\int_a^\tau h(\theta)\Delta\theta)\Delta\tau \in \mathrm{Dom}(\Psi^{-1}),$$

*we have the estimate*

$$|u_*(t)| \le p(t)\left\{|u_a| + \int_a^t h(s)\Psi^{-1}\left(\Psi(\xi) + \int_a^s \Phi(p(\tau))\Phi(\int_a^\tau h(\theta)\Delta\theta)\Delta\tau\right)\Delta s\right\}, \tag{6.18}$$

*where*

$$p(t) = 1 + \int_a^t e_1(t,\sigma(s))\Delta s,$$

$$\xi = \int_a^{\rho(b)} \Phi(p(s)|u_a|)\Delta s,$$

$$\Psi(x) = \int_{x_0}^x \frac{1}{\Phi(s)}ds, \ x > 0, \ x_0 > 0\,.$$

*Proof.* Let $u_*(t)$ be the solution of the IVP (6.15). Then, we have

$$u_*(t) = u_a + \int_a^t F\left(s, u_*(s), \int_a^s K(s, u_*(\tau))\Delta\tau\right)\Delta s. \tag{6.19}$$

Using (6.16) and (6.17) in (6.19), we have

$$\begin{aligned} |u_*| &\le |u_a| + \int_a^t \left(|u_*(s)| + \int_a^s |K(s, u_*(\tau))|\Delta\tau\right)\Delta s \\ &\le |u_a| + \int_a^t \left(|u_*(s)| + h(s)\int_a^s \Phi(|u_*(\tau))|)\Delta\tau\right)\Delta s. \end{aligned} \tag{6.20}$$

A suitable application of Theorem 92 to (6.20), with $a(t) = |u_a|$, $f(t) = b(t) = 1$ and $W(u) = u$, yields (6.18). ☐

## 6.2 Some more integral inequalities and applications

In this section we shall be concern with the existence of solutions of the following integrodynamic equation

$$x^\Delta(t) = F\left(t, x(t), \int_a^t K[t, \tau, x(\tau)]\Delta\tau\right), \quad x(a) = A, \quad t \in [a, b]_{\mathbb{T}}^\kappa, \tag{6.21}$$

where $a, b \in \mathbb{T}$, $K : [a, b]_{\mathbb{T}}^\kappa \times [a, b]_{\mathbb{T}}^{\kappa^2} \times \mathbb{R} \to \mathbb{R}$ and $F : [a, b]_{\mathbb{T}}^\kappa \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

As is well known integrodifferential equations and their discrete analogues find many applications in various mathematical problems. Moreover, it appears to be advantageous to model certain processes by employing a suitable combination of both differential equations and difference equations at different stages in the process under consideration (see [67] and references therein for more details).

Some results concerning existence of a solution to some particular cases of the integrodynamic equation in (6.21) were obtained in [63, 95]. Here we will make use of the well-known in the literature *Topological Transversality Theorem* to prove the existence of a solution to the above mentioned equation.

We first introduce some basic definitions and results on fixed point theory. Let $\mathcal{B}$ be a Banach space and $C \subset \mathcal{B}$ be convex. By a *pair* $(X, A)$ in $C$ is meant an arbitrary subset $X$ of $C$ and an $A \subset X$ closed in $X$. We call a homotopy $H : X \times [0, 1] \to Y$ *compact* if it is a compact map. If $X \subset Y$, the homotopy $H$ is called *fixed point free* on $A \subset X$ if for each $\lambda \in [0, 1]$, the map $H|A \times \{\lambda\} : A \to Y$ has no fixed point. We denote by $\mathcal{C}_A(X, C)$ the set of all compact maps $F : X \to C$ such that the restriction $F|A : A \to C$ is fixed point free.

Two maps $F, G \in \mathcal{C}_A(X, C)$ are called *homotopic*, written $F \simeq G$ in $\mathcal{C}_A(X, C)$, provided there is a compact homotopy $H_\lambda : X \to C$ $(\lambda \in [0, 1])$ that is fixed point free on $A$ and such that $H_0 = F$ and $H_1 = G$.

**Definition 99.** Let $(X, A)$ be a pair in a convex $C \subset \mathcal{B}$. A map $F \in \mathcal{C}_A(X, C)$ is called *essential* provided every $G \in \mathcal{C}_A(X, C)$ such that $F|A = G|A$ has a fixed point.

**Theorem 100** (Topological Transversality [54])**.** *Let $(X, A)$ be a pair in a convex $C \subset \mathcal{B}$, and let $F, G$ be maps in $\mathcal{C}_A(X, C)$ such that $F \simeq G$ in $\mathcal{C}_A(X, C)$. Then, $F$ is essential if and only if $G$ is essential.*

The next theorem is very useful to the application of the Topological Transversality Theorem. Its proof can be found in [54].

**Theorem 101.** *Let $U$ be an open subset of a convex set $C \subset \mathcal{B}$, and let $(\bar{U}, \partial U)$ be the pair consisting of the closure of $U$ in $C$ and the boundary of $U$ in $C$. Then, for any $u_0 \in U$, the constant map $F|\bar{U} = u_0$ is essential in $\mathcal{C}_{\partial U}(\bar{U}, C)$.*

If more details are needed about fixed point theory we recommend the books [28, 54].

Here, as usual, $C([a, b]_{\mathbb{T}}, \mathbb{R})$ [sometimes we will only write $C[a, b]_{\mathbb{T}}$] denotes the set of all real valued continuous functions on $[a, b]_{\mathbb{T}}$ and $C^1([a, b]_{\mathbb{T}}, \mathbb{R})$ the set of all $\Delta$-differentiable functions whose derivative is continuous on $[a, b]_{\mathbb{T}}^{\kappa}$, and we equip the spaces $C([a, b]_{\mathbb{T}}, \mathbb{R})$, $C^1([a, b]_{\mathbb{T}}, \mathbb{R})$ with the norms $\|u\|_0 = \sup_{t\in[a,b]_{\mathbb{T}}} |u(t)|$, $\|u\|_1 = \sup_{t\in[a,b]_{\mathbb{T}}} |u(t)| + \sup_{t\in[a,b]_{\mathbb{T}}^{\kappa}} |u^{\Delta}(t)|$, respectively.

To make use of the Topological Transversality Theorem it is essential to obtain a priori bounds on the possible solutions of the equation in study. To achieve that, the next lemma is indispensable.

**Lemma 102.** *Let $f, g \in C([a, b]_{\mathbb{T}}^{\kappa}, \mathbb{R}_0^+)$ and $u, p \in C([a, b]_{\mathbb{T}}, \mathbb{R}_0^+)$, being $p(t)$ positive and nondecreasing on $[a, b]_{\mathbb{T}}$, $k \in C([a, b]_{\mathbb{T}}^{\kappa} \times [a, b]_{\mathbb{T}}^{\kappa^2}, \mathbb{R}_0^+)$, and $c \in \mathbb{R}_0^+$. Moreover, let $w, \tilde{w} \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ be nondecreasing functions with $\{w, \tilde{w}\}(x) > 0$ for $x > 0$. Let $B, D > 0$ and $M = (b - a)D$. Define a function $G$ by,*

$$G(x) = \int_1^x \frac{1}{w(\max\{s, M\tilde{w}(s)\})} ds, \quad x > 0,$$

*and assume that $\lim_{x\to\infty} G(x) = \infty$. If, for all $t \in [a, b]_{\mathbb{T}}$, the following inequality holds*

$$u(t) \le p(t) + B \int_a^t w\left(\max\left\{u(s) + c, D\int_a^s \tilde{w}(u(\tau) + c)\Delta\tau\right\}\right)\Delta s,$$

*then,*

$$u(t) \le G^{-1}\left[G(p(t) + c) + B(t - a)\right] - c, \quad t \in [a, b]_{\mathbb{T}}.$$

*Proof.* We start by noting that the result trivially holds for $t = a$. Consider an arbitrary number $t_0 \in (a, b]_{\mathbb{T}}$ and define a positive and nondecreasing function $z(t)$ on $[a, t_0]_{\mathbb{T}}$ by

$$z(t) = p(t_0) + B \int_a^t w\left(\max\left\{u(s) + c, D\int_a^s \tilde{w}(u(\tau) + c)\Delta\tau\right\}\right)\Delta s.$$

Then, $z(a) = p(t_0)$, for all $t \in [a, t_0]_{\mathbb{T}}$ we have $u(t) \leq z(t)$, and

$$\begin{aligned} z^{\Delta}(t) &= Bw\left(\max\left\{u(t) + c, D\int_a^t \tilde{w}(u(\tau) + c)\Delta\tau\right\}\right) \\ &\leq Bw\left(\max\left\{u(t) + c, D(b-a)\tilde{w}(z(t) + c)\right\}\right) \\ &= Bw\left(\max\left\{u(t) + c, M\tilde{w}(z(t) + c)\right\}\right), \end{aligned}$$

for all $t \in [a, t_0]_{\mathbb{T}}^{\kappa}$. Hence,

$$\frac{z^{\Delta}(t)}{w\left(\max\left\{u(t) + c, M\tilde{w}(z(t) + c)\right\}\right)} \leq B,$$

and, after integrating from $a$ to $t$ and with the help of Lemma 88, we get

$$G(z(t) + c) \leq G(z(a) + c) + B(t - a), \quad t \in [a, t_0]_{\mathbb{T}}.$$

In view of the hypothesis on function $G$, we may write

$$z(t) \leq G^{-1}\left[G(z(a) + c) + B(t - a)\right] - c.$$

Therefore,

$$u(t) \leq G^{-1}\left[G(p(t_0) + c) + B(t - a)\right] - c,$$

for all $t \in [a, t_0]_{\mathbb{T}}$. Setting $t = t_0$ in the above inequality and having in mind that $t_0$ is arbitrary, we conclude the proof. □

*Remark* 103. We note that by item 1 of Theorem 7 there is an inclusion of $C^1[a, b]_{\mathbb{T}}$ into $C[a, b]_{\mathbb{T}}$.

**Theorem 104.** *The embedding $j : C^1[a, b]_{\mathbb{T}} \to C[a, b]_{\mathbb{T}}$ is completely continuous.*

*Proof.* Let $B$ be a bounded set in $C^1[a, b]_{\mathbb{T}}$. Then, there exists $M > 0$ such that $\|x\|_1 \leq M$ for all $x \in B$. By the definition of the norm $\|\cdot\|_1$, we have that $\sup_{t\in[a,b]_{\mathbb{T}}} |x(t)| \leq M$, hence $\|x\|_0 \leq M$, i.e., $B$ is bounded in $C[a, b]_{\mathbb{T}}$. Let now $\varepsilon > 0$ be given and put $\delta = \frac{\varepsilon}{M}$. Then, it is easily seen that, for $x \in B$ we have $|x^{\Delta}(t)| \leq M$ for all $t \in [a, b]_{\mathbb{T}}^{\kappa}$. For arbitrary $t_1$, $t_2 \in [a, b]_{\mathbb{T}}$ with $t_1 \neq t_2$, we use Theorem 31 to get

$$\frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|} \leq M,$$

i.e., for $|t_1 - t_2| < \delta$ we have $|x(t_1) - x(t_2)| < \varepsilon$ and this proves that $B$ is equicontinuous on $C[a, b]_{\mathbb{T}}$ (the case $t_1 = t_2$ is obvious). By the Arzela–Ascoli Theorem, $B$ is relatively compact and therefore $j$ is completely continuous. □

**Theorem 105.** *Assume that the functions $F$ and $K$ introduced in* (6.21) *satisfy*

$$|F(t,x,y)| \le Bw(\max\{|x|,|y|\}) + C, \tag{6.22}$$

$$|K(t,s,x)| \le D\tilde{w}(|x|), \tag{6.23}$$

*where $B$, $C$ and $D$ are positive constants and $w, \tilde{w}$ are defined as in Lemma 102.*

*Then, the integrodynamic equation in* (6.21) *has a solution $x \in C^1([a,b]_{\mathbb{T}}, \mathbb{R})$.*

*Proof.* We start considering the following auxiliary problem,

$$y^{\Delta}(t) = F\left(t, y(t) + A, \int_a^t K[t,s,y(s)+A]\Delta s\right), \quad y(a) = 0, \quad t \in [a,b]_{\mathbb{T}}^{\kappa}, \tag{6.24}$$

and showing that it has a solution. To do this, we first establish a priori bounds to the (possible) solutions of the family of problems

$$y^{\Delta}(t) = \lambda F\left(t, y(t) + A, \int_a^t K[t,s,y(s)+A]\Delta s\right), \quad y(a) = 0, \quad t \in [a,b]_{\mathbb{T}}^{\kappa}, \tag{6.25}$$

independently of $\lambda \in [0,1]$.

Integrating both sides of the equation in (6.25) on $[a,t]_{\mathbb{T}}$, we obtain,

$$y(t) = \lambda \int_a^t F\left(s, y(s) + A, \int_a^s K[s,\tau,y(\tau)+A]\Delta\tau\right)\Delta s,$$

for all $t \in [a,b]_{\mathbb{T}}$. Applying the modulus function to both sides of the last equality we obtain, using the triangle inequality and inequalities (6.22) and (6.23),

$$|y(t)| \le p(t) + B\int_a^t w\left(\max\left\{|y(s)+A|, D\int_a^s \tilde{w}(|y(\tau)+c|)\Delta\tau\right\}\right)\Delta s,$$

with $p(t) = C(t-a)$. By Lemma 102 (with $u(t) = |y(t)|$, $c = |A|$), we deduce that (note also that $|y(\cdot) + A| \le |y(\cdot)| + |A|$)

$$|y(t)| \le G^{-1}\left[G(p(t) + |A|) + B(t-a)\right] - |A|, \tag{6.26}$$

for all $t \in [a,b]_{\mathbb{T}}$. Denote the right-hand side of (6.26) by $R(t)$.

Let $\mathcal{B} = C^1[a,b]_{\mathbb{T}}$ be the Banach space equipped with the norm $\|\cdot\|_1$ and define a set $C = \{u \in C^1[a,b]_{\mathbb{T}} : u(a) = 0\}$. Observe that $C$ is a convex subset of $\mathcal{B}$.

Define the linear operator $L : C \to C[a,b]_{\mathbb{T}}$ by $Lu = u^{\Delta}$. It is clearly bijective with a continuous inverse $L^{-1} : C[a,b]_{\mathbb{T}} \to C$. Let $M_0$ and $M_1$ be defined by

$$M_0 = R(b),$$

$$M_1 = \sup_{t\in[a,b]_{\mathbb{T}}, |y|\le M_0} \left|F\left(t, y, \int_a^t K[t,s,y]\Delta s\right)\right|.$$

Consider the family of maps $\mathcal{F}_\lambda : C[a,b]_{\mathbb{T}} \to C[a,b]_{\mathbb{T}}$, $0 \leq \lambda \leq 1$, defined by

$$(\mathcal{F}_\lambda y)(t) = \lambda F\left(t, y(t)+A, \int_a^t K[t,s,y(s)+A]\Delta s\right),$$

and the completely continuous embedding $j : C \to C[a,b]_{\mathbb{T}}$ (it is easily seen that the restriction here used of $j$ of Theorem 104 is also completely continuous). Let us consider the set $Y = \{y \in C : \|y\|_1 \leq r\}$ with $r = 1 + M_0 + M_1$. Then we can define a homotopy $H_\lambda : Y \to C$ by $H_\lambda = L^{-1}\mathcal{F}_\lambda j$. Since $L^{-1}$ and $\mathcal{F}_\lambda$ are continuous and $j$ is completely continuous, $H$ is a compact homotopy. Moreover, it is fixed point free on the boundary of $Y$. Since $H_0$ is the zero map, it is essential. Because $H_0 \simeq H_1$, Theorem 100 implies that $H_1$ is also essential. In particular, $H_1$ has a fixed point which is a solution of (6.24). To finish the proof, let $y \in C^1[a,b]_{\mathbb{T}}$ be a solution of (6.24) and define the function $x(t) = y(t) + A$, $t \in [a,b]_{\mathbb{T}}$. Then, it is easily seen that $x(a) = A$ and

$$x^\Delta(t) = F\left(t, x(t), \int_a^t K[t,s,x(s)]\Delta s\right), \quad t \in [a,b]_{\mathbb{T}}^\kappa,$$

i.e., $x \in C^1[a,b]_{\mathbb{T}}$ is a solution of (6.21). □

Theorem 105 was inspired by the work of Adrian Constantin in [32]. It provides a nontrivial generalization to the one presented there, and as a consequence, Theorem 105 seems to be new even if we let the time scale to be $\mathbb{T} = \mathbb{R}$. Indeed, let

$$F(t,x,y) = |y| + \begin{cases} (x+1)\ln(x+1) & \text{if } x \geq 0; \\ 0 & \text{if } x < 0. \end{cases}$$

We will show that, for all $f, h \in C([a,b]_{\mathbb{T}}^\kappa, \mathbb{R}_0^+)$, $|F(t,x,y)| > f(t)|x| + |y| + h(t)$ for some $(t,x,y) \in [a,b]_{\mathbb{T}}^\kappa \times \mathbb{R}^2$ [in [32] the assumption on $F$ was that $|F(t,x,y)| \leq f(t)|x| + |y| + h(t)$]. For that, suppose the contrary, i.e., assume that there exist $f, h \in C([a,b]_{\mathbb{T}}^\kappa, \mathbb{R}_0^+)$ such that $|F(t,x,y)| \leq f(t)|x| + |y| + h(t)$ for all $(t,x,y) \in [a,b]_{\mathbb{T}}^\kappa \times \mathbb{R}^2$. In particular, for arbitrary $x > 0$, we have that,

$$|y| + (x+1)\ln(x+1) = |F(t,x,y)| \leq f(t)x + |y| + h(t),$$

which is equivalent to

$$(x+1)\ln(x+1) \leq f(t)x + h(t),$$

or

$$x[\ln(x+1) - f(t)] + \ln(x+1) \leq h(t). \tag{6.27}$$

Now, we fix $t \in [a,b]_{\mathbb{T}}^\kappa$ and let $x \to \infty$ in both sides of (6.27) getting a contradiction.

Note that the function $\bar{w}(x) = (x+1)\ln(x+1)$, $x \geq 0$, is nondecreasing and is such that $\int_1^\infty \frac{1}{\bar{w}(s)} = \infty$. Moreover, for $F$ as above, we have

$$\begin{aligned} |F(t,x,y)| &\leq |y| + \bar{w}(|x|) \\ &\leq \max\{|x|,|y|\} + \bar{w}(\max\{|x|,|y|\}). \end{aligned}$$

If we define $w(x) = \max\{x, \bar{w}(x)\}$, $x \geq 0$, then $w(x) \leq x + \bar{w}(x)$ and (see [31]) $\int_1^\infty \frac{1}{w(s)} ds = \infty$ which shows that we can apply our result to such a function $F$.

**Corollary 106.** *Assume that $p \in C([a,b]_{\mathbb{T}}^{\kappa}, \mathbb{R})$. Then, the initial value problem*

$$x^{\Delta}(t) = p(t)x(t), \quad x(a) = A, \tag{6.28}$$

*has a unique solution $x \in C^1([a,b]_{\mathbb{T}}, \mathbb{R})$.*

*Proof.* Define $F(t,x,y) = p(t)x$, for $(t,x) \in [a,b]_{\mathbb{T}}^{\kappa} \times \mathbb{R}$. Then, $|F(t,x,y)| = |p(t)||x|$ and the existence part follows by Theorem 105.

Suppose now that $x, y \in C^1([a,b]_{\mathbb{T}}, \mathbb{R})$ both satisfy (6.28). An integration on $[a,t]_{\mathbb{T}}$ yields,

$$\begin{aligned} x(t) &= A + \int_a^t p(s)x(s)\Delta s, \\ y(t) &= A + \int_a^t p(s)y(s)\Delta s. \end{aligned}$$

Hence,

$$|x(t) - y(t)| \leq \int_a^t |p(s)||x(s) - y(s)|\Delta s.$$

Using Gronwall's inequality (cf. Theorem 47) [note that $|p(s)| \in \mathcal{R}^+$] it follows that $|x(t) - y(t)| \leq 0$ and finally that $x(t) = y(t)$ for all $t \in [a,b]_{\mathbb{T}}$. Hence, the solution is unique. □

*Remark* 107. S. Hilger in his seminal work [58] proved (not only but also) the existence of a unique solution to equation (6.28) with $p \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}^{\kappa}, \mathbb{R})$ being regressive. In this paper we are requiring $p(t)$ to be continuous but not a regressive function.

In view of the previous remark it is interesting to think of what happens when the function $p$ is not regressive. This is shown in the following result.

**Proposition 108.** *Suppose that $p \in C([a,b]_{\mathbb{T}}^{\kappa}, \mathbb{R})$ is not regressive. Then, there exists $t_0 \in [a,b]_{\mathbb{T}}^{\kappa}$ such that the solution of (6.28) satisfies $x(t) = 0$ for all $t \in [\sigma(t_0), b]_{\mathbb{T}}$.*

*Proof.* Since $p$ is not regressive, there exists a point $t_0 \in [a,b]_{\mathbb{T}}^{\kappa}$ such that $1 + \mu(t_0)p(t_0) = 0$. This immediately implies that $\mu(t_0) \neq 0$ and $p(t_0) \neq 0$. Let $x \in C^1([a,b]_{\mathbb{T}}, \mathbb{R})$ be the unique solution of (6.28). Using formula (1.2) it is easy to derive from (6.28) that $x^{\Delta}(t)[1+\mu(t)p(t)] = p(t)x(\sigma(t))$ and in particular that $x^{\Delta}(t_0)[1 + \mu(t_0)p(t_0)] = p(t_0)x(\sigma(t_0))$. It follows that $x(\sigma(t_0)) = 0$. Now note that, since $t_0$ is right-scattered, we must have $\sigma(t_0) \neq a$. Moreover, $\tilde{x}(t) = 0$ satisfies $x^{\Delta}(t) = p(t)x(t)$ for all $t \in [\sigma(t_0), b]_{\mathbb{T}}$. The uniqueness of the solution completes the proof. □

### 6.2.1 Some particular integrodynamic equations

We now use Theorem 105 to prove existence of solution to an integral equation describing some physical phenomena. We emphasize that, for each time scale we get an equation, i.e., we can construct various models in order to (hopefully) better describe the phenomena in question.

**Theorem 109.** *Let $0, b \in \mathbb{T}$ with $b > 0$ and $L \in C^1([0,b]_{\mathbb{T}}, \mathbb{R}_0^+)$. Moreover, assume that a function $M(t,s) \in C([0,b]_{\mathbb{T}} \times [0,b]_{\mathbb{T}}^{\kappa}, \mathbb{R}_0^+)$ has continuous partial $\Delta$-derivative with respect to its first variable [denoted by $M^{\Delta_t}(t,s)$] and $M(\sigma(t),t)$ is continuous for all $t \in [0,b]_{\mathbb{T}}^{\kappa}$. Suppose that the function $\bar{w} \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ is nondecreasing, such that $\bar{w}(x) > 0$ for all $x > 0$ and $\int_1^{\infty} \frac{1}{\bar{w}(s)} ds = \infty$. Then, the integral equation,*

$$x(t) = L(t) + \int_0^t M(t,s)\bar{w}(x(s))\Delta s, \quad t \in [0,b]_{\mathbb{T}}, \tag{6.29}$$

*has a solution $x \in C^1([0,b]_{\mathbb{T}}, \mathbb{R}_0^+)$.*

*Proof.* Let us define

$$\begin{aligned} F(t,x,y) &= L^{\Delta}(t) + M(\sigma(t),t)\bar{w}(|x|) + y, \quad (t,x,y) \in [0,b]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^2, \\ K(t,s,x) &= M^{\Delta_t}(t,s)\bar{w}(|x|), \quad (t,s,x) \in [0,b]_{\mathbb{T}}^{\kappa} \times [0,b]_{\mathbb{T}}^{\kappa^2} \times \mathbb{R}. \end{aligned}$$

Then, we have

$$\begin{aligned} |F(t,x,y)| &\leq |L^{\Delta}(t)| + M(\sigma(t),t)\bar{w}(|x|) + |y|, \\ |K(t,s,x)| &\leq |M^{\Delta_t}(t,s)|\bar{w}(|x|). \end{aligned}$$

Using Theorem 105 with

$$\begin{aligned} B &> \max_{t \in [0,b]_{\mathbb{T}}^{\kappa}} M(\sigma(t),t), \\ C &> \max_{t \in [0,b]_{\mathbb{T}}^{\kappa}} |L^{\Delta}(t)|, \\ D &> \max_{(t,s) \in [0,b]_{\mathbb{T}}^{\kappa} \times [0,b]_{\mathbb{T}}^{\kappa^2}} |M^{\Delta_t}(t,s)|, \end{aligned}$$

$\tilde{w}(x) = \bar{w}(x)$ and $w(x) = \max\{x, \bar{w}(x)\}$ we conclude that the equation

$$x^{\Delta}(t) = L^{\Delta}(t) + M(\sigma(t),t)\bar{w}(|x(t)|) + \int_0^t M^{\Delta_t}(t,s)\bar{w}(|x(s)|)\Delta s,$$

with initial value $x(0) = L(0)$ has a solution $x \in C^1([0,b]_{\mathbb{T}}, \mathbb{R})$. Now, an integration on $[0,t]_{\mathbb{T}}$ gives, using Lemma 29,

$$x(t) = L(t) + \int_0^t M(t,s)\bar{w}(|x(s)|)\Delta s, \quad t \in [0,b]_{\mathbb{T}}.$$

By the assumptions on functions $L$, $M$ and $\bar{w}$ we conclude that $x(t) \geq 0$ for all $t \in [0,b]_{\mathbb{T}}$, hence $x$ is a nonnegative solution of (6.29). □

**Corollary 110.** *If in equation (6.29), $L(t) > 0$ on $[0,b]_{\mathbb{T}}$ and $w = \bar{w}$ is continuously differentiable on $(0,\infty)$, then the solution obtained by Theorem 109 is unique.*

*Proof.* Assume that there exist two positive solutions $x, y$ on $[0,b]_{\mathbb{T}}$ to (6.29). Then,

$$x(t) - y(t) = \int_0^t M(t,s)[w(x(s)) - w(y(s))]\Delta s, \quad t \in [0,b]_{\mathbb{T}},$$

hence,

$$|x(t) - y(t)| \leq \int_0^t M(t,s)|w(x(s)) - w(y(s))|\Delta s, \quad t \in [0,b]_{\mathbb{T}}.$$

Let us now define the following numbers:

$$\begin{aligned}
\gamma_1 &= \min_{s\in[0,b]_{\mathbb{T}}} \{x(s), y(s)\},\\
\gamma_2 &= \max_{s\in[0,b]_{\mathbb{T}}} \{x(s), y(s)\},\\
\nu &= \max_{x\in[\gamma_1,\gamma_2]} w'(x),\\
\mu &= \max_{(t,s)\in[0,b]_{\mathbb{T}}\times[0,b]_{\mathbb{T}}^{\kappa}} M(t,s).
\end{aligned}$$

The mean value theorem guarantees that

$$|w(x(t)) - w(y(t))| \leq \nu|(x(t) - y(t))|, \quad t \in [0,b]_{\mathbb{T}},$$

hence,

$$|x(t) - y(t)| \leq \int_0^t \mu\nu|x(s) - y(s)|\Delta s, \quad t \in [0,b]_{\mathbb{T}}.$$

Using Theorem 47, we conclude that $|x(t)-y(t)| \leq 0$ on $[0,b]_{\mathbb{T}}$ and this implies that $x(t) = y(t)$ on $[0,b]_{\mathbb{T}}$. □

Now we want to mention some particular cases of equation (6.29). Define on $\mathbb{R}_0^+$ the function $\bar{w}(x) = x^r$, $0 \leq r \leq 1$. Then, the equation

$$x(t) = L(t) + \int_0^t M(t,s)[x(s)]^r \Delta s, \quad t \in [0,b]_{\mathbb{T}}, \tag{6.30}$$

has a unique positive solution if $L(t) > 0$ for all $t \in [0,b]_{\mathbb{T}}$. This type of equation appears in many applications such as nonlinear diffusion, cellular mass change dynamics, or studies concerning the shape of liquid drops [79]. Some results concerning existence and uniqueness of solutions to (6.30) [the time scale being $\mathbb{T} = \mathbb{R}$] were previously obtained (consult [79] and references therein).

If we define $u(t) = [x(t)]^r$, it follows from (6.30) that,

$$[u(t)]^{\frac{1}{r}} = L(t) + \int_0^t M(t,s)u(s)\Delta s, \quad t \in [0,b]_{\mathbb{T}}. \tag{6.31}$$

It is easily seen that $u(t)$ is the unique solution of (6.31) and, if we let $r = \frac{1}{2}$ and $M(t,s) = K(t-s)$ for $K \in C^1(\mathbb{R}_0^+, \mathbb{R}_0^+)$, it follows that

$$[u(t)]^2 = L(t) + \int_0^t K(t-s)u(s)\Delta s, \quad t \in [0,b]_{\mathbb{T}}. \tag{6.32}$$

This equation appears in the mathematical theory of the infiltration of a fluid from a cylindrical reservoir into an isotropic homogeneous porous medium [81, 82]. Some existence and uniqueness theorems regarding solutions of (6.32) were obtained in [32, 33, 34, 81, 82].

*Remark* 111. Corollary 110 proves uniqueness of a solution to the integral equation (6.32) under different assumptions on the function $L$ than those in [32, 33, 34, 81, 82] (obviously considering $\mathbb{T} = \mathbb{R}$). For example, (to prove uniqueness) in [33], the author considered $L \in C^1(\mathbb{R}_0^+, \mathbb{R}_0^+)$ such that $L'(0) \neq 0$. Therefore, if we, e.g., let $L(t) = t^2 + 1$, $t \in [0,b]$, we see that $L'(0) = 0$, hence we cannot use the results obtained in [33]. If we let $L(t) = t$, Corollary 110 cannot be applied.

We close this section with an example of what the integrodynamic equation in (6.21) could be on a particular time scale different from $\mathbb{R}$, namely, in $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ for some $h > 0$. Let $a = A = 0$ and $b = hm$ for some $m \in \mathbb{N}$. Then, on this time scale, (6.21) becomes

$$\frac{x(t+h) - x(t)}{h} = F\left(t, x(t), \sum_{k=0}^{m-1} hK[t, kh, x(kh)]\right), \quad x(0) = 0, \quad t \in [a,b]_{h\mathbb{Z}}^{\kappa}.$$

## 6.3 $\Diamond_\alpha$-integral inequalities

In this section we propose to prove inequalities of Jensen's, Minkowski's and Holder's type using the $\Diamond_\alpha$– integral (cf. definition in (6.34) below).

Based on the $\Delta$ and $\nabla$ dynamic derivatives, a combined dynamic derivative, so called $\Diamond_\alpha$–derivative, was introduced as a linear combination of $\Delta$ and $\nabla$ dynamic derivatives on time scales [90], i.e, for $0 \leq \alpha \leq 1$,

$$f^{\Diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \quad t \in \mathbb{T}_\kappa^\kappa, \tag{6.33}$$

provided $f^\Delta$ and $f^\nabla$ exist. The $\Diamond_\alpha$-derivative reduces to the $\Delta$-derivative for $\alpha = 1$ and to the $\nabla$-derivative for $\alpha = 0$. On the other hand, it represents a "weighted dynamic derivative" on any uniformly discrete time scale when $\alpha = \frac{1}{2}$. A motivation for study such derivatives is for designing more balanced adaptive algorithms on nonuniform grids with reduced spuriosity [90].

Let $a, t \in \mathbb{T}$, and $h : \mathbb{T} \to \mathbb{R}$. Then, the $\Diamond_\alpha$-integral of $h$ from $a$ to $t$ is defined by

$$\int_a^t h(\tau)\Diamond_\alpha \tau = \alpha \int_a^t h(\tau)\Delta\tau + (1-\alpha)\int_a^t h(\tau)\nabla\tau, \quad 0 \leq \alpha \leq 1, \tag{6.34}$$

provided that there exist $\Delta$ and $\nabla$ integrals of $h$ on $\mathbb{T}$. It is clear that the diamond-$\alpha$ integral of $h$ exists when $h$ is a continuous function. We refer the reader to [70, 87, 90] for an account of the calculus associated with the $\Diamond_\alpha$-derivatives and integrals.

The next lemma provides a straightforward but useful result for what follows.

**Lemma 112** (cf. [11]). *Assume that $f$ and $g$ are continuous functions on $[a, b]_\mathbb{T}$. If $f(t) \leq g(t)$ for all $t \in [a, b]_\mathbb{T}$, then $\int_a^b f(t)\Diamond_\alpha t \leq \int_a^b g(t)\Diamond_\alpha t$.*

*Proof.* Let $f(t)$ and $g(t)$ be continuous functions on $[a, b]_\mathbb{T}$. Using (1.5) and the analogous inequality for the $\nabla$-integral we conclude that $\int_a^b f(t)\Delta t \leq \int_a^b g(t)\Delta t$ and $\int_a^b f(t)\nabla t \leq \int_a^b g(t)\nabla t$. Since $\alpha \in [0, 1]$ the result follows. □

We now obtain Jensen's type $\Diamond_\alpha$-integral inequalities.

**Theorem 113** (cf. [11]). *Let $\mathbb{T}$ be a time scale, $a$, $b \in \mathbb{T}$ with $a < b$, and $c$, $d \in \mathbb{R}$. If $g \in C([a, b]_\mathbb{T}, (c, d))$ and $f \in C((c, d), \mathbb{R})$ is convex, then*

$$f\left(\frac{\int_a^b g(s)\Diamond_\alpha s}{b-a}\right) \leq \frac{\int_a^b f(g(s))\Diamond_\alpha s}{b-a}. \tag{6.35}$$

*Remark* 114. In the particular case $\alpha = 1$, inequality (6.35) reduces to that of Theorem 72. If $\mathbb{T} = \mathbb{R}$, then Theorem 113 gives the classical Jensen inequality, i.e., Theorem 44. However, if $\mathbb{T} = \mathbb{Z}$ and $f(x) = -\ln(x)$, then one gets the well-known arithmetic-mean geometric-mean inequality (6.39).

*Proof.* Since $f$ is convex we have

$$\begin{aligned} f\left(\frac{\int_a^b g(s)\Diamond_\alpha s}{b-a}\right) &= f\left(\frac{\alpha}{b-a}\int_a^b g(s)\Delta s + \frac{1-\alpha}{b-a}\int_a^b g(s)\nabla s\right) \\ &\leq \alpha f\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right) + (1-\alpha) f\left(\frac{1}{b-a}\int_a^b g(s)\nabla s\right). \end{aligned}$$

Using now Jensen's inequality on time scales (cf. Theorem 72), we get

$$\begin{aligned} f\left(\frac{\int_a^b g(s)\Diamond_\alpha s}{b-a}\right) &\leq \frac{\alpha}{b-a}\int_a^b f(g(s))\Delta s + \frac{1-\alpha}{b-a}\int_a^b f(g(s))\nabla s \\ &= \frac{1}{b-a}\left(\alpha\int_a^b f(g(s))\Delta s + (1-\alpha)\int_a^b f(g(s))\nabla s\right) \\ &= \frac{1}{b-a}\int_a^b f(g(s))\Diamond_\alpha s. \end{aligned}$$

□

Now we give an extended Jensen's inequality (cf. [11]).

**Theorem 115** (Generalized Jensen's inequality)**.** *Let $\mathbb{T}$ be a time scale, $a$, $b \in \mathbb{T}$ with $a < b$, $c$, $d \in \mathbb{R}$, $g \in C([a,b]_{\mathbb{T}}, (c,d))$ and $h \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ with*

$$\int_a^b |h(s)| \Diamond_\alpha s > 0\,.$$

*If $f \in C((c,d), \mathbb{R})$ is convex, then*

$$f\left(\frac{\int_a^b |h(s)| g(s) \Diamond_\alpha s}{\int_a^b |h(s)| \Diamond_\alpha s}\right) \leq \frac{\int_a^b |h(s)| f(g(s)) \Diamond_\alpha s}{\int_a^b |h(s)| \Diamond_\alpha s}. \tag{6.36}$$

*Remark* 116. In the particular case $h = 1$, Theorem 115 reduces to Theorem 113.

*Remark* 117. Similar result to Theorem 115 holds if one changes the condition "$f$ is convex" to "$f$ is concave", by replacing the inequality sign "$\leq$" in (6.36) by "$\geq$".

*Proof.* Since $f$ is convex, it follows, for example from [51, Exercise 3.42C], that for $t \in (c,d)$ there exists $a_t \in \mathbb{R}$ such that

$$a_t(x - t) \leq f(x) - f(t) \text{ for all } x \in (c,d). \tag{6.37}$$

Setting

$$t = \frac{\int_a^b |h(s)| g(s) \Diamond_\alpha s}{\int_a^b |h(s)| \Diamond_\alpha s}\,,$$

then using (6.37) and Lemma 112, we get

$$\begin{aligned}
&\int_a^b |h(s)| f(g(s)) \Diamond_\alpha s - \left(\int_a^b |h(s)| \Diamond_\alpha s\right) f\left(\frac{\int_a^b |h(s)| g(s) \Diamond_\alpha s}{\int_a^b |h(s)| \Diamond_\alpha s}\right)\\
&= \int_a^b |h(s)| f(g(s)) \Diamond_\alpha s - \left(\int_a^b |h(s)| \Diamond_\alpha s\right) f(t) = \int_a^b |h(s)| \left(f(g(s)) - f(t)\right) \Diamond_\alpha s\\
&\geq a_t \left(\int_a^b |h(s)| (g(s) - t)\right) \Diamond_\alpha s = a_t \left(\int_a^b |h(s)| g(s) \Diamond_\alpha s - t \int_a^b |h(s)| \Diamond_\alpha s\right)\\
&= a_t \left(\int_a^b |h(s)| g(s) \Diamond_\alpha s - \int_a^b |h(s)| g(s) \Diamond_\alpha s\right) = 0\,.
\end{aligned}$$

This leads to the desired inequality. □

Now we sate two (well-known in the literature) corollaries of the previous theorem.

**Corollary 118.** ($\mathbb{T} = \mathbb{R}$) *Let $g, h : [a,b] \to \mathbb{R}$ be continuous functions with $g([a,b]) \subseteq (c,d)$ and $\int_a^b |h(x)| dx > 0$. If $f \in C((c,d), \mathbb{R})$ is convex, then*

$$f\left(\frac{\int_a^b |h(x)| g(x) dx}{\int_a^b |h(x)| dx}\right) \leq \frac{\int_a^b |h(x)| f(g(x)) dx}{\int_a^b |h(x)| dx}.$$

**Corollary 119.** ($\mathbb{T} = \mathbb{Z}$) *Given a convex function $f$, we have for any $x_1, \ldots, x_n \in \mathbb{R}$ and $c_1, \ldots, c_n \in \mathbb{R}$ with $\sum_{k=1}^n |c_k| > 0$:*

$$f\left(\frac{\sum_{k=1}^n |c_k| x_k}{\sum_{k=1}^n |c_k|}\right) \leq \frac{\sum_{k=1}^n |c_k| f(x_k)}{\sum_{k=1}^n |c_k|}. \tag{6.38}$$

## Particular Cases

(i) Let $g(t) > 0$ on $[a,b]_{\mathbb{T}}$ and $f(t) = t^\beta$ on $(0,+\infty)$. One can see that $f$ is convex on $(0,+\infty)$ for $\beta < 0$ or $\beta > 1$, and $f$ is concave on $(0,+\infty)$ for $\beta \in (0,1)$. Therefore,

$$\left(\frac{\int_a^b |h(s)|g(s)\Diamond_\alpha s}{\int_a^b |h(s)|\Diamond_\alpha s}\right)^\beta \leq \frac{\int_a^b |h(s)|g^\beta(s)\Diamond_\alpha s}{\int_a^b |h(s)|\Diamond_\alpha s}, \text{ if } \beta < 0 \text{ or } \beta > 1;$$

$$\left(\frac{\int_a^b |h(s)|g(s)\Diamond_\alpha s}{\int_a^b |h(s)|\Diamond_\alpha s}\right)^\beta \geq \frac{\int_a^b |h(s)|g^\beta(s)\Diamond_\alpha s}{\int_a^b |h(s)|\Diamond_\alpha s}, \text{ if } \beta \in (0,1).$$

(ii) Let $g(t) > 0$ on $[a,b]_{\mathbb{T}}$ and $f(t) = \ln(t)$ on $(0,+\infty)$. One can also see that $f$ is concave on $(0,+\infty)$. It follows that

$$\ln\left(\frac{\int_a^b |h(s)|g(s)\Diamond_\alpha s}{\int_a^b |h(s)|\Diamond_\alpha s}\right) \geq \frac{\int_a^b |h(s)|\ln(g(s))\Diamond_\alpha s}{\int_a^b |h(s)|\Diamond_\alpha s}.$$

(iii) Let $h = 1$, then

$$\ln\left(\frac{\int_a^b g(s)\Diamond_\alpha s}{b-a}\right) \geq \frac{\int_a^b \ln(g(s))\Diamond_\alpha s}{b-a}.$$

(iv) Let $\mathbb{T} = \mathbb{R}$, $g : [0,1] \to (0,\infty)$ and $h(t) = 1$. Applying Theorem 115 with the convex and continuous function $f = -\ln$ on $(0,\infty)$, $a = 0$ and $b = 1$, we get:

$$\ln \int_0^1 g(s)ds \geq \int_0^1 \ln(g(s))ds.$$

Therefore,

$$\int_0^1 g(s)ds \geq \exp\left(\int_0^1 \ln(g(s))ds\right).$$

(v) Let $\mathbb{T} = \mathbb{Z}$ and $n \in \mathbb{N}$. Fix $a = 1$, $b = n+1$ and consider a function $g : \{1, \ldots, n+1\} \to (0,\infty)$. Obviously, $f = -\ln$ is convex and continuous on $(0,\infty)$, so we may apply Jensen's inequality to obtain

$$\begin{aligned}
\ln\left[\frac{1}{n}\left(\alpha\sum_{t=1}^{n}g(t)+(1-\alpha)\sum_{t=2}^{n+1}g(t)\right)\right] &= \ln\left[\frac{1}{n}\int_{1}^{n+1}g(t)\Diamond_{\alpha}t\right]\\
&\geq \frac{1}{n}\int_{1}^{n+1}\ln(g(t))\Diamond_{\alpha}t\\
&= \frac{1}{n}\left[\alpha\sum_{t=1}^{n}\ln(g(t))+(1-\alpha)\sum_{t=2}^{n+1}\ln(g(t))\right]\\
&= \ln\left\{\prod_{t=1}^{n}g(t)\right\}^{\frac{\alpha}{n}}+\ln\left\{\prod_{t=2}^{n+1}g(t)\right\}^{\frac{1-\alpha}{n}},
\end{aligned}$$

and hence

$$\frac{1}{n}\left(\alpha\sum_{t=1}^{n}g(t)+(1-\alpha)\sum_{t=2}^{n+1}g(t)\right)\geq\left\{\prod_{t=1}^{n}g(t)\right\}^{\frac{\alpha}{n}}\left\{\prod_{t=2}^{n+1}g(t)\right\}^{\frac{1-\alpha}{n}}.$$

When $\alpha=1$, we obtain the well-known arithmetic-mean geometric-mean inequality:

$$\frac{1}{n}\sum_{t=1}^{n}g(t)\geq\left\{\prod_{t=1}^{n}g(t)\right\}^{\frac{1}{n}}. \tag{6.39}$$

When $\alpha=0$, we also have

$$\frac{1}{n}\sum_{t=2}^{n+1}g(t)\geq\left\{\prod_{t=2}^{n+1}g(t)\right\}^{\frac{1}{n}}.$$

(vi) Let $\mathbb{T}=2^{\mathbb{N}_0}$ and $N\in\mathbb{N}$. We can apply Theorem 115 with $a=1, b=2^N$ and $g:\{2^k : 0\leq k\leq N\}\to(0,\infty)$. Then, we get:

$$\begin{aligned}
\ln\left\{\frac{\int_1^{2^N}g(t)\Diamond_\alpha t}{2^N-1}\right\} &= \ln\left\{\alpha\frac{\int_1^{2^N}g(t)\Delta t}{2^N-1}+(1-\alpha)\frac{\int_1^{2^N}g(t)\nabla t}{2^N-1}\right\}\\
&= \ln\left\{\frac{\alpha\sum_{k=0}^{N-1}2^k g(2^k)}{2^N-1}+\frac{(1-\alpha)\sum_{k=1}^{N}2^k g(2^k)}{2^N-1}\right\}\\
&\geq \frac{\int_1^{2^N}\ln(g(t))\Diamond_\alpha t}{2^N-1}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\frac{\int_1^{2^N}\ln(g(t))\Delta t}{2^N-1} + (1-\alpha)\frac{\int_1^{2^N}\ln(g(t))\nabla t}{2^N-1}\\
&= \frac{\alpha\sum_{k=0}^{N-1}2^k\ln(g(2^k))}{2^N-1} + \frac{(1-\alpha)\sum_{k=1}^{N}2^k\ln(g(2^k))}{2^N-1}\\
&= \frac{\sum_{k=0}^{N-1}\ln(g(2^k))^{\alpha 2^k}}{2^N-1} + \frac{\sum_{k=1}^{N}\ln(g(2^k))^{(1-\alpha)2^k}}{2^N-1}\\
&= \frac{\ln\prod_{k=0}^{N-1}(g(2^k))^{\alpha 2^k}}{2^N-1} + \frac{\ln(\prod_{k=1}^{N}g(2^k))^{(1-\alpha)2^k}}{2^N-1}\\
&= \ln\left\{\prod_{k=0}^{N-1}(g(2^k))^{\alpha 2^k}\right\}^{\frac{1}{2^N-1}} + \ln\left\{\prod_{k=1}^{N}(g(2^k))^{(1-\alpha)2^k}\right\}^{\frac{1}{2^N-1}}\\
&= \ln\left(\left\{\prod_{k=0}^{N-1}(g(2^k))^{\alpha 2^k}\right\}^{\frac{1}{2^N-1}}\left\{\prod_{k=1}^{N}(g(2^k))^{(1-\alpha)2^k}\right\}^{\frac{1}{2^N-1}}\right).
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&\ln\left\{\frac{\alpha\sum_{k=0}^{N-1}2^k g(2^k)+(1-\alpha)\sum_{k=1}^{N}2^k g(2^k)}{2^N-1}\right\}\\
&\qquad\qquad \geq \ln\left(\left\{\prod_{k=0}^{N-1}(g(2^k))^{\alpha 2^k}\right\}^{\frac{1}{2^N-1}}\left\{\prod_{k=1}^{N}(g(2^k))^{(1-\alpha)2^k}\right\}^{\frac{1}{2^N-1}}\right).
\end{aligned}
$$

On the other hand,

$$
\alpha\sum_{k=0}^{N-1}2^k g(2^k)+(1-\alpha)\sum_{k=1}^{N}2^k g(2^k)=\sum_{k=1}^{N-1}2^k g(2^k)+\alpha g(1)+(1-\alpha)2^N g(2^N).
$$

It follows that

$$
\begin{aligned}
&\frac{\sum_{k=1}^{N-1}2^k g(2^k)+\alpha g(1)+(1-\alpha)2^N g(2^N)}{2^N-1}\\
&\qquad\qquad \geq \left\{\prod_{k=0}^{N-1}(g(2^k))^{\alpha 2^k}\right\}^{\frac{1}{2^N-1}}\left\{\prod_{k=1}^{N}(g(2^k))^{(1-\alpha)2^k}\right\}^{\frac{1}{2^N-1}}.
\end{aligned}
$$

In the particular case when $\alpha = 1$ we have

$$
\frac{\sum_{k=0}^{N-1}2^k g(2^k)}{2^N-1} \geq \left\{\prod_{k=0}^{N-1}(g(2^k))^{2^k}\right\}^{\frac{1}{2^N-1}},
$$

and when $\alpha = 0$ we get the inequality

$$
\frac{\sum_{k=1}^{N}2^k g(2^k)}{2^N-1} \geq \left\{\prod_{k=1}^{N}(g(2^k))^{2^k}\right\}^{\frac{1}{2^N-1}}.
$$

### 6.3.1 Related $\Diamond_\alpha$-integral inequalities

The usual proof of Hölder's inequality use the basic Young inequality $x^{\frac{1}{p}}y^{\frac{1}{q}} \le \frac{x}{p} + \frac{y}{q}$ for nonnegative $x$ and $y$. Here we present a proof based on the application of Jensen's inequality (cf. Theorem 115).

**Theorem 120** (Hölder's inequality)**.** *Let $\mathbb{T}$ be a time scale, $a$, $b \in \mathbb{T}$ with $a < b$, and $f$, $g$, $h \in C([a,b]_{\mathbb{T}}, [0,\infty))$ with $\int_a^b h(x)g^q(x)\Diamond_\alpha x > 0$, where $q$ is the Hölder conjugate number of $p$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p$. Then, we have:*

$$\int_a^b h(x)f(x)g(x)\Diamond_\alpha x \le \left(\int_a^b h(x)f^p(x)\Diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b h(x)g^q(x)\Diamond_\alpha x\right)^{\frac{1}{q}}. \tag{6.40}$$

*Proof.* Choosing $f(x) = x^p$ in Theorem 115, which for $p > 1$ is obviously a convex function on $[0,\infty)$, we have

$$\left(\frac{\int_a^b |h(s)|g(s)\Diamond_\alpha s}{\int_a^b |h(s)|\Diamond_\alpha s}\right)^p \le \frac{\int_a^b |h(s)|(g(s))^p\Diamond_\alpha s}{\int_a^b |h(s)|\Diamond_\alpha s}. \tag{6.41}$$

Inequality (6.40) is trivially true in the case when $g$ is identically zero. We consider two cases: (i) $g(x) > 0$ for all $x \in [a,b]_{\mathbb{T}}$; (ii) there exists at least one $t \in [a,b]_{\mathbb{T}}$ such that $g(x) = 0$. We begin with situation (i). Replacing $g$ by $fg^{\frac{-q}{p}}$ and $|h(x)|$ by $hg^q$ in inequality (6.41), we get:

$$\left(\frac{\int_a^b h(x)g^q(x)f(x)g^{\frac{-q}{p}}(x)\Diamond_\alpha x}{\int_a^b h(x)g^q(x)\Diamond_\alpha x}\right)^p \le \frac{\int_a^b h(x)g^q(x)(f(x)g^{\frac{-q}{p}}(x))^p\Diamond_\alpha x}{\int_a^b h(x)g^q(x)\Diamond_\alpha x}.$$

Using the fact that $\frac{1}{p} + \frac{1}{q} = 1$, we obtain that

$$\int_a^b h(x)f(x)g(x)\Diamond_\alpha x \le \left(\int_a^b h(x)f^p(x)\Diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b h(x)g^q(x)\Diamond_\alpha x\right)^{\frac{1}{q}}. \tag{6.42}$$

We now consider situation (ii). Let $G = \{x \in [a,b]_{\mathbb{T}} \mid g(x) = 0\}$. Then,

$$\begin{aligned}\int_a^b h(x)f(x)g(x)\Diamond_\alpha x &= \int_{[a,b]_{\mathbb{T}}-G} h(x)f(x)g(x)\Diamond_\alpha x + \int_G h(x)f(x)g(x)\Diamond_\alpha x\\ &= \int_{[a,b]_{\mathbb{T}}-G} h(x)f(x)g(x)\Diamond_\alpha x\end{aligned}$$

because $\int_G h(x)f(x)g(x)\Diamond_\alpha x = 0$. For the set $[a,b]_{\mathbb{T}} - G$ we are in case (i), i.e. $g(x) > 0$, and it follows from (6.42) that

$$\begin{aligned}\int_a^b h(x)f(x)g(x)\Diamond_\alpha x &= \int_{[a,b]_{\mathbb{T}}-G} h(x)f(x)g(x)\Diamond_\alpha x\\ &\le \left(\int_{[a,b]_{\mathbb{T}}-G} h(x)f^p(x)\Diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_{[a,b]_{\mathbb{T}}-G} h(x)g^q(x)\Diamond_\alpha x\right)^{\frac{1}{q}}\\ &\le \left(\int_a^b h(x)f^p(x)\Diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b h(x)g^q(x)\Diamond_\alpha x\right)^{\frac{1}{q}}.\end{aligned}$$

□

*Remark* 121. In the particular case $h = 1$, Theorem 120 gives the $\Diamond_\alpha$ version of classical Hölder's inequality:

$$\int_a^b |f(x)g(x)|\Diamond_\alpha x \leq \left(\int_a^b |f|^p(x)\Diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |g|^q(x)\Diamond_\alpha x\right)^{\frac{1}{q}},$$

where $p > 1$ and $q = \frac{p}{p-1}$.

*Remark* 122. In the special case $p = q = 2$, (6.40) reduces to the following $\Diamond_\alpha$ Cauchy–Schwarz integral inequality on time scales:

$$\int_a^b |f(x)g(x)|\Diamond_\alpha x \leq \sqrt{\left(\int_a^b f^2(x)\Diamond_\alpha x\right)\left(\int_a^b g^2(x)\Diamond_\alpha x\right)}.$$

We are now in position to prove a Minkowski inequality using Hölder's inequality (6.40).

**Theorem 123** (Minkowski's inequality)**.** *Let $\mathbb{T}$ be a time scale, $a$, $b \in \mathbb{T}$ with $a < b$, and $p > 1$. For continuous functions $f, g : [a, b]_\mathbb{T} \to \mathbb{R}$ we have*

$$\left(\int_a^b |(f+g)(x)|^p\Diamond_\alpha x\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p\Diamond_\alpha x\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p\Diamond_\alpha x\right)^{\frac{1}{p}}.$$

*Proof.* We have, by the triangle inequality, that

$$\begin{aligned}\int_a^b |f(x)+g(x)|^p\Diamond_\alpha x &= \int_a^b |f(x)+g(x)|^{p-1}|f(x)+g(x)|\Diamond_\alpha x\\ &\leq \int_a^b |f(x)||f(x)+g(x)|^{p-1}\Diamond_\alpha x + \int_a^b |g(x)||f(x)+g(x)|^{p-1}\Diamond_\alpha x. \quad (6.43)\end{aligned}$$

Applying now Hölder's inequality with $q = p/(p-1)$ to (6.43), we obtain:

$$\begin{aligned}\int_a^b |f(x)+g(x)|^p\Diamond_\alpha x &\leq \left[\int_a^b |f(x)|^p\Diamond_\alpha x\right]^{\frac{1}{p}}\left[\int_a^b |f(x)+g(x)|^{(p-1)q}\Diamond_\alpha x\right]^{\frac{1}{q}}\\ &\quad + \left[\int_a^b |g(x)|^p\Diamond_\alpha x\right]^{\frac{1}{p}}\left[\int_a^b |f(x)+g(x)|^{(p-1)q}\Diamond_\alpha x\right]^{\frac{1}{q}}\\ &= \left\{\left[\int_a^b |f(x)|^p\Diamond_\alpha x\right]^{\frac{1}{p}} + \left[\int_a^b |g(x)|^p\Diamond_\alpha x\right]^{\frac{1}{p}}\right\}\left[\int_a^b |f(x)+g(x)|^p\Diamond_\alpha x\right]^{\frac{1}{q}}.\end{aligned}$$

Dividing both sides of the last inequality by

$$\left[\int_a^b |f(x)+g(x)|^p\Diamond_\alpha x\right]^{\frac{1}{q}},$$

we get the desired conclusion. □

## 6.4 Some integral inequalities with two independent variables

We establish some linear and nonlinear integral inequalities of Gronwall–Bellman–Bihari type for functions with two independent variables. Some particular time scales are considered as examples and we use one of our results to estimate solutions of a partial delta dynamic equation (cf. (6.62) below).

As was mentioned earlier, several integral inequalities of Gronwall-Bellman-Bihari type on time scales but for functions of a single variable were obtained in the papers [2], [8], [83] and [93]. However, to the best of the author's knowledge, no such a paper exists on the literature when functions of two independent variables are considered. Therefore we aim to give a first insight on such a type of inequalities.

### 6.4.1 Linear inequalities

Throughout we let $\tilde{\mathbb{T}}_1 = [a_1, \infty)_{\mathbb{T}_1}$ and $\tilde{\mathbb{T}}_2 = [a_2, \infty)_{\mathbb{T}_2}$, for $a_1 \in \mathbb{T}_1$, $a_2 \in \mathbb{T}_2$ being $\mathbb{T}_1$ and $\mathbb{T}_2$ time scales.

**Theorem 124** (cf. [48]). *Let $u(t_1,t_2), a(t_1,t_2), f(t_1,t_2) \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$ with $a(t_1,t_2)$ nondecreasing in each of the variables. If*

$$u(t_1,t_2) \le a(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2, \tag{6.44}$$

*for $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, then*

$$u(t_1,t_2) \le a(t_1,t_2) e_{\int_{a_2}^{t_2} f(t_1,s_2)\Delta_2 s_2}(t_1,a_1), \quad (t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2. \tag{6.45}$$

*Proof.* Since $a(t_1,t_2)$ is nondecreasing on $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, inequality (6.44) implies that, for an arbitrary $\varepsilon > 0$,

$$r(t_1,t_2) \le 1 + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2) r(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2,$$

where $r(t_1,t_2) = \frac{u(t_1,t_2)}{a(t_1,t_2)+\varepsilon}$. Define $v(t_1,t_2)$ by the right-hand side of the last inequality. Then

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right) = f(t_1,t_2) r(t_1,t_2) \le f(t_1,t_2) v(t_1,t_2), \quad (t_1,t_2) \in \tilde{\mathbb{T}}_1^k \times \tilde{\mathbb{T}}_2^k. \tag{6.46}$$

From (6.46) and taking into account that $v(t_1,t_2)$ is positive and nondecreasing, we obtain

$$\frac{v(t_1,t_2)\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)}{v(t_1,t_2)v(t_1,\sigma_2(t_2))} \le f(t_1,t_2),$$

from which it follows that

$$\frac{v(t_1,t_2)\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)}{v(t_1,t_2)v(t_1,\sigma_2(t_2))} \le f(t_1,t_2) + \frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\frac{\partial v(t_1,t_2)}{\Delta_2 t_2}}{v(t_1,t_2)v(t_1,\sigma_2(t_2))}.$$

The previous inequality can be rewritten as

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}}{v(t_1,t_2)}\right) \le f(t_1,t_2).$$

Integrating with respect to the second variable from $a_2$ to $t_2$ and noting that $\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\mid_{(t_1,a_2)}= 0$, we have

$$\frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}}{v(t_1,t_2)} \le \int_{a_2}^{t_2} f(t_1,s_2)\Delta_2 s_2,$$

that is,

$$\frac{\partial v(t_1,t_2)}{\Delta_1 t_1} \le \int_{a_2}^{t_2} f(t_1,s_2)\Delta_2 s_2 v(t_1,t_2).$$

Fixing $t_2 \in \tilde{\mathbb{T}}_2$ arbitrarily, we have that $p(t_1) := \int_{a_2}^{t_2} f(t_1,s_2)\Delta_2 s_2 \in \mathcal{R}^+$ and by Theorem 46

$$v(t_1,t_2) \le e_p(t_1,a_1),$$

since $v(a_1,t_2) = 1$. Now, inequality (6.45) follows from the inequality

$$u(t_1,t_2) \le [a(t_1,t_2)+\varepsilon]v(t_1,t_2),$$

and the arbitrariness of $\varepsilon$. □

**Corollary 125** (cf. Lemma 2.1 of [65]). *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ and assume that the functions $u(x,y), a(x,y), f(x,y) \in C([x_0,\infty)\times[y_0,\infty),\mathbb{R}_0^+)$ with $a(x,y)$ nondecreasing in its variables. If*

$$u(x,y) \le a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} f(t,s)u(t,s)dtds, \tag{6.47}$$

*for $(x,y) \in [x_0,\infty)\times[y_0,\infty)$, then*

$$u(x,y) \le a(x,y)\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} f(t,s)dtds\right), \tag{6.48}$$

*for $(x,y) \in [x_0,\infty)\times[y_0,\infty)$.*

**Corollary 126** (cf. Theorem 2.1 of [88]). *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ and assume that the functions $u(m,n), a(m,n), f(m,n)$ are nonnegative and that $a(m,n)$ is nondecreasing for $m \in [m_0,\infty)\cap\mathbb{Z}$ and $n \in [n_0,\infty)\cap\mathbb{Z}$, $m_0, n_0 \in \mathbb{Z}$. If*

$$u(m,n) \le a(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f(s,t)u(s,t), \tag{6.49}$$

*for all $(m,n) \in [m_0,\infty)\cap\mathbb{Z}\times[n_0,\infty)\cap\mathbb{Z}$, then*

$$u(m,n) \le a(m,n)\prod_{s=m_0}^{m-1}\left[1+\sum_{t=n_0}^{n-1} f(s,t)\right], \tag{6.50}$$

*for all $(m,n) \in [m_0,\infty)\cap\mathbb{Z}\times[n_0,\infty)\cap\mathbb{Z}$.*

*Remark* 127. We note that, following the same steps of the proof of Theorem 124, it can be obtained other bound on the function $u$, namely

$$u(t_1,t_2) \leq a(t_1,t_2) e_{\int_{a_1}^{t_1} f(s_1,t_2)\Delta_1 s_1}(t_2,a_2). \tag{6.51}$$

When $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then the bounds in (6.51) and (6.45) coincide (see Corollary 125). But if, for example, we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, the bounds obtained can be different; moreover, at different points one can be sharper than the other and vice-versa. Note up the following example:

Let $f(t_1,t_2)$ be a function defined by $f(0,0) = 1/4$, $f(1,0) = 1/5$, $f(2,0) = 1$, $f(0,1) = 1/2$, $f(1,1) = 0$ and $f(2,1) = 5$. Set $a_1 = a_2 = 0$.

Then, from (6.45) we get

$$\begin{aligned} u(2,1) &\leq a(2,1)\frac{3}{2}, \\ u(3,2) &\leq a(3,2)\frac{147}{10}, \end{aligned}$$

while from (6.51) we get

$$\begin{aligned} u(2,1) &\leq a(2,1)\frac{29}{20}, \\ u(3,2) &\leq a(3,2)\frac{637}{40}. \end{aligned}$$

We now present the particular case of Theorem 124 when $\mathbb{T}_1 = \mathbb{Z}$ and $\mathbb{T}_2 = \mathbb{R}$.

**Corollary 128** (cf. [48]). *Let $\mathbb{T}_1 = \mathbb{Z}$ and $\mathbb{T}_2 = \mathbb{R}$. Assume that the functions $u(t,x)$, $a(t,x)$ and $f(t,x)$ satisfy the hypothesis of Theorem 124 for all $(t,x) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, with $a_1 = a_2 = 0$. If*

$$u(t,x) \leq a(t,x) + \sum_{k=0}^{t-1} \int_0^x f(k,\tau)u(k,\tau)d\tau, \tag{6.52}$$

*for all $(t,x) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, then*

$$u(t,x) \leq a(t,x) \prod_{k=0}^{t-1} \left[1 + \int_0^x f(k,\tau)d\tau\right], \tag{6.53}$$

*for all $(t,x) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$.*

We now generalize Theorem 124. If in Theorem 129 $f$ does not depend on the first two variables, then we obtain Theorem 124.

**Theorem 129** (cf. [48]). *Let $u(t_1,t_2), a(t_1,t_2) \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$, with $a(t_1,t_2)$ nondecreasing in each of the variables and $f(t_1,t_2,s_1,s_2) \in C(S,\mathbb{R}_0^+)$ be nondecreasing in $t_1$ and $t_2$, where $S = \{(t_1,t_2,s_1,s_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \times \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 : a_1 \leq s_1 \leq t_1, a_2 \leq s_2 \leq t_2\}$. If*

$$u(t_1,t_2) \leq a(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(t_1,t_2,s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2, \tag{6.54}$$

*for* $(t_1, t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, *then*

$$u(t_1, t_2) \leq a(t_1, t_2) e_{\int_{a_2}^{t_2} f(t_1,t_2,t_1,s_2)\Delta_2 s_2}(t_1, a_1), \; (t_1, t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2. \tag{6.55}$$

*Proof.* We start by fixing arbitrary numbers $t_1^* \in \tilde{\mathbb{T}}_1$ and $t_2^* \in \tilde{\mathbb{T}}_2$ and consider the function defined on $[a_1, t_1^*] \cap \tilde{\mathbb{T}}_1 \times [a_2, t_2^*] \cap \tilde{\mathbb{T}}_2$ by

$$v(t_1, t_2) = a(t_1^*, t_2^*) + \varepsilon + \int_{a_1}^{t_1} \int_{a_2}^{t_2} f(t_1^*, t_2^*, s_1, s_2) u(s_1, s_2) \Delta_1 s_1 \Delta_2 s_2, \tag{6.56}$$

for an arbitrary $\varepsilon > 0$. From our hypothesis we see that

$$u(t_1, t_2) \leq v(t_1, t_2), \text{ for all } (t_1, t_2) \in [a_1, t_1^*] \cap \tilde{\mathbb{T}}_1 \times [a_2, t_2^*] \cap \tilde{\mathbb{T}}_2. \tag{6.57}$$

Moreover, $\Delta$-differentiating with respect to the first variable and then with respect to the second, we obtain

$$\begin{aligned} \frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1, t_2)}{\Delta_1 t_1}\right) &= f(t_1^*, t_2^*, t_1, t_2) u(t_1, t_2) \\ &\leq f(t_1^*, t_2^*, t_1, t_2) v(t_1, t_2), \end{aligned}$$

for all $(t_1, t_2) \in [a_1, t_1^*]^\kappa \cap \tilde{\mathbb{T}}_1 \times [a_2, t_2^*]^\kappa \cap \tilde{\mathbb{T}}_2$. From this last inequality, we can write

$$\frac{v(t_1, t_2) \frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)}{v(t_1, t_2) v(t_1, \sigma_2(t_2))} \leq f(t_1^*, t_2^*, t_1, t_2),$$

hence,

$$\frac{v(t_1, t_2) \frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)}{v(t_1, t_2) v(t_1, \sigma_2(t_2))} \leq f(t_1^*, t_2^*, t_1, t_2) + \frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1} \frac{\partial v(t_1,t_2)}{\Delta_2 t_2}}{v(t_1, t_2) v(t_1, \sigma_2(t_2))}.$$

The previous inequality can be rewritten as

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}}{v(t_1, t_2)}\right) \leq f(t_1^*, t_2^*, t_1, t_2).$$

$\Delta$-integrating with respect to the second variable from $a_2$ to $t_2$ and noting that $\frac{\partial v(t_1,t_2)}{\Delta_1 t_1} |_{(t_1,a_2)}= 0$, we have

$$\frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}}{v(t_1, t_2)} \leq \int_{a_2}^{t_2} f(t_1^*, t_2^*, t_1, s_2) \Delta_2 s_2,$$

that is,

$$\frac{\partial v(t_1, t_2)}{\Delta_1 t_1} \leq \int_{a_2}^{t_2} f(t_1^*, t_2^*, t_1, s_2) \Delta_2 s_2 v(t_1, t_2).$$

Fix $t_2 = t_2^*$ and put $p(t_1) := \int_{a_2}^{t_2^*} f(t_1^*, t_2^*, t_1, s_2) \Delta_2 s_2 \in \mathcal{R}^+$. By Theorem 46

$$v(t_1, t_2^*) \leq (a(t_1^*, t_2^*) + \varepsilon) e_p(t_1, a_1).$$

Letting $t_1 = t_1^*$ in the above inequality, and remembering that $t_1^*$, $t_2^*$ and $\varepsilon$ are arbitrary, inequality (6.55) follows. $\square$

### 6.4.2 Nonlinear inequalities

**Theorem 130** (cf. [48]). *Let $u(t_1,t_2), f(t_1,t_2) \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$. Moreover, let $a(t_1,t_2) \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R}^+)$ and be a nondecreasing function in each of the variables. If $p$ and $q$ are two positive real numbers such that $p \geq q$ and if*

$$u^p(t_1,t_2) \leq a(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)u^q(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2, \tag{6.58}$$

*for $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, then*

$$u(t_1,t_2) \leq a^{\frac{1}{p}}(t_1,t_2)\left[e_{\int_{a_2}^{t_2} f(t_1,s_2)a^{\frac{q}{p}-1}(t_1,s_2)\Delta_2 s_2}(t_1,a_1)\right]^{\frac{1}{p}}, \quad (t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2. \tag{6.59}$$

*Proof.* Since $a(t_1,t_2)$ is positive and nondecreasing on $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, inequality (6.58) implies that,

$$u^p(t_1,t_2) \leq a(t_1,t_2)\left(1 + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)\frac{u^q(s_1,s_2)}{a(s_1,s_2)}\Delta_1 s_1 \Delta_2 s_2\right).$$

Define $v(t_1,t_2)$ on $\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$ by

$$v(t_1,t_2) = 1 + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)\frac{u^q(s_1,s_2)}{a(s_1,s_2)}\Delta_1 s_1 \Delta_2 s_2.$$

Then,

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right) = f(t_1,t_2)\frac{u^q(t_1,t_2)}{a(t_1,t_2)} \leq f(t_1,t_2)a^{\frac{q}{p}-1}(t_1,t_2)v^{\frac{q}{p}}(t_1,t_2).$$

Now, note that $v^{\frac{q}{p}}(t_1,t_2) \leq v(t_1,t_2)$ and therefore

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right) \leq f(t_1,t_2)a^{\frac{q}{p}-1}(t_1,t_2)v(t_1,t_2).$$

From here, we can follow the same procedure as in the proof of Theorem 124 to obtain

$$v(t_1,t_2) \leq e_p(t_1,a_1),$$

where $p(t_1) = \int_{a_2}^{t_2} f(t_1,s_2)a^{\frac{q}{p}-1}(t_1,s_2)\Delta_2 s_2$. Noting that

$$u(t_1,t_2) \leq a^{\frac{1}{p}}(t_1,t_2)v^{\frac{1}{p}}(t_1,t_2),$$

we obtain the desired inequality (6.59). $\square$

**Theorem 131** (cf. [48]). *Let $u(t_1,t_2), a(t_1,t_2) \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$, with $a(t_1,t_2)$ nondecreasing in each of the variables and $f(t_1,t_2,s_1,s_2) \in C(S, \mathbb{R}_0^+)$, where $S = \{(t_1,t_2,s_1,s_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \times \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 : a_1 \leq s_1 \leq t_1, a_2 \leq s_2 \leq t_2\}$. If $p$ and $q$ are two positive real numbers such that $p \geq q$ and if*

$$u^p(t_1,t_2) \leq a(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(t_1,t_2,s_1,s_2)u^q(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2, \tag{6.60}$$

*for all* $(t_1, t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$*, then*

$$u(t_1, t_2) \le a^{\frac{1}{p}}(t_1, t_2) \left[ e_{\int_{a_2}^{t_2} a^{\frac{q}{p}-1}(t_1, s_2) f(t_1, t_2, t_1, s_2) \Delta_2 s_2}(t_1, a_1) \right]^{\frac{1}{p}},$$

*for all* $(t_1, t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$.

*Proof.* Since $a(t_1, t_2)$ is positive and nondecreasing on $(t_1, t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, inequality (6.60) implies that,

$$u^p(t_1, t_2) \le a(t_1, t_2) \left( 1 + \int_{a_1}^{t_1} \int_{a_2}^{t_2} f(t_1, t_2, s_1, s_2) \frac{u^q(s_1, s_2)}{a(s_1, s_2)} \Delta_1 s_1 \Delta_2 s_2 \right).$$

Fix $t_1^* \in \tilde{\mathbb{T}}_1$ and $t_2^* \in \tilde{\mathbb{T}}_2$ arbitrarily and define a function $v(t_1, t_2)$ on $[a_1, t_1^*] \cap \tilde{\mathbb{T}}_1 \times [a_2, t_2^*] \cap \tilde{\mathbb{T}}_2$ by

$$v(t_1, t_2) = 1 + \int_{a_1}^{t_1} \int_{a_2}^{t_2} f(t_1^*, t_2^*, s_1, s_2) \frac{u^q(s_1, s_2)}{a(s_1, s_2)} \Delta_1 s_1 \Delta_2 s_2.$$

Then,

$$\begin{aligned} \frac{\partial}{\Delta_2 t_2} \left( \frac{\partial v(t_1, t_2)}{\Delta_1 t_1} \right) &= f(t_1^*, t_2^*, t_1, t_2) \frac{u^q(t_1, t_2)}{a(t_1, t_2)} \\ &\le f(t_1^*, t_2^*, t_1, t_2) a^{\frac{q}{p}-1}(t_1, t_2) v^{\frac{q}{p}}(t_1, t_2). \end{aligned}$$

Since $v^{\frac{q}{p}}(t_1, t_2) \le v(t_1, t_2)$, we have that

$$\frac{\partial}{\Delta_2 t_2} \left( \frac{\partial v(t_1, t_2)}{\Delta_1 t_1} \right) \le f(t_1^*, t_2^*, t_1, t_2) a^{\frac{q}{p}-1}(t_1, t_2) v(t_1, t_2).$$

We can follow the same steps as before to reach the inequality

$$\frac{\partial v(t_1, t_2)}{\Delta_1 t_1} \le \int_{a_2}^{t_2} f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, s_2) a^{\frac{q}{p}-1}(t_1, s_2) \Delta_2 s_2 v(t_1, t_2).$$

Fix $t_2 = t_2^*$ and put $p(t_1) := \int_{a_2}^{t_2^*} f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, s_2) a^{\frac{q}{p}-1}(t_1, s_2) \Delta_2 s_2 \in \mathcal{R}^+$. Again, an application of Theorem 46 gives

$$v(t_1, t_2^*) \le e_p(t_1, a_1),$$

and putting $t_1 = t_1^*$ we obtain the desired inequality. $\square$

We consider now the following time scale: let $\{\alpha_k\}_{k \in \mathbb{N}}$ be a sequence of positive numbers and let $t_0^\alpha \in \mathbb{R}$. Let

$$t_k^\alpha = t_0^\alpha + \sum_{n=1}^{k} \alpha_n, \; k \in \mathbb{N},$$

and assume that $\lim_{k \to \infty} t_k^\alpha = \infty$. Then we can define the following time scale $\mathbb{T}^\alpha = \{t_k^\alpha : k \in \mathbb{N}_0\}$. Further, for $p \in \mathcal{R}$ (cf. [1, Example 4.6]),

$$e_p(t_k^\alpha, t_0^\alpha) = \prod_{n=1}^{k} (1 + \alpha_n p(t_{n-1})), \quad \text{for all} \quad k \in \mathbb{N}_0. \tag{6.61}$$

Now, for two sequences $\{\alpha_k, \beta_k\}_{k\in\mathbb{N}}$ and two numbers $t_0^\alpha, t_0^\beta \in \mathbb{R}$ as above, we define two time scales $\mathbb{T}^\alpha = \{t_k^\alpha : k \in \mathbb{N}_0\}$ and $\mathbb{T}^\beta = \{t_k^\beta : k \in \mathbb{N}_0\}$. We can therefore state the following result:

**Corollary 132** (cf. [48]). *Let $u(t,s), a(t,s)$ be defined on $\mathbb{T}^\alpha \times \mathbb{T}^\beta$, be nonnegative with a nondecreasing in its variables. Further, let $f(t,s,\tau,\xi)$, where $(t,s,\tau,\xi) \in \mathbb{T}^\alpha \times \mathbb{T}^\beta \times \mathbb{T}^\alpha \times \mathbb{T}^\beta$ with $\tau \le t$ and $\xi \le s$, be a nonnegative function and nondecreasing in its first two variables. If $p$ and $q$ are two positive real numbers such that $p \ge q$ and if*

$$u^p(t,s) \le a(t,s) + \sum_{\tau\in[t_0^\alpha,t)_{\mathbb{T}^\alpha}} \sum_{\xi\in[t_0^\beta,s)_{\mathbb{T}^\beta}} \mu^\alpha(\tau)\mu^\beta(\xi) f(t,s,\tau,\xi) u^q(\tau,\xi),$$

*for all $(t,s) \in \mathbb{T}^\alpha \times \mathbb{T}^\beta$, where $\mu^\alpha$ and $\mu^\beta$ are the graininess functions of $\mathbb{T}^\alpha$ and $\mathbb{T}^\beta$, respectively, then*

$$u(t,s) \le a^{\frac{1}{p}}(t,s) \left[ e_{\int_{t_0^\beta}^s f(t,s) a^{\frac{q}{p}-1}(t,\xi) g(t,s,t,\xi)\Delta^\beta \xi}(t, t_0^\alpha) \right]^{\frac{1}{p}},$$

*for all $(t,s) \in \mathbb{T}^\alpha \times \mathbb{T}^\beta$, where $e$ is given by (6.61).*

We end this section showing how Theorem 130 can be applied to estimate the solutions of the following partial delta dynamic equation:

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial u^2(t_1,t_2)}{\Delta_1 t_1}\right) = F(t_1,t_2,u(t_1,t_2)), \tag{6.62}$$

with the given initial boundary conditions (we assume that $a_1 = a_2 = 0$)

$$u^2(t_1,0) = g(t_1), \quad u^2(0,t_2) = h(t_2), \quad g(0) = 0, \quad h(0) = 0, \tag{6.63}$$

where $F \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \times \mathbb{R}_0^+, \mathbb{R}_0^+)$, $g \in C(\tilde{\mathbb{T}}_1, \mathbb{R}_0^+)$, $h \in C(\tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$, with $g$ and $h$ nondecreasing functions and positive on their domains except at $a_1$ and $a_2$, respectively.

**Theorem 133.** *Assume that on its domain, $F$ satisfies*

$$F(t_1,t_2,u) \le t_2 u.$$

*If $u(t_1,t_2)$ is a solution of the IBVP above for $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, then*

$$u(t_1,t_2) \le \sqrt{(g(t_1)+h(t_2))}\left[ e_{\int_0^{t_2} s_2 (g(t_1)+h(s_2))^{-\frac{1}{2}}\Delta_2 s_2}(t_1,0)\right]^{\frac{1}{2}}, \tag{6.64}$$

*for $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, except at the point $(0,0)$.*

*Proof.* Let $u(t_1,t_2)$ be a solution of the given IBVP (6.62)–(6.63). Then it satisfies the following delta integral equation

$$u^2(t_1,t_2) = g(t_1) + h(t_2) + \int_0^{t_1}\int_0^{t_2} F(s_1,s_2,u(s_1,s_2))\Delta_1 s_1 \Delta_2 s_2.$$

The hypothesis on $F$ imply that

$$u^2(t_1,t_2) \leq g(t_1) + h(t_2) + \int_0^{t_1}\int_0^{t_2} s_2 u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2.$$

An application of Theorem 130 with $a(t_1,t_2) = g(t_1)+h(t_2)$ and $f(t_1,t_2) = t_2$ gives (6.64). $\square$

## 6.5 State of the Art

The results of this chapter are published or accepted for publication in [11, 43, 46, 48]. Moreover the three papers [41, 42, 47] are already available in the literature and contain related results. Some partial results of [41] were presented in the *Time Scales seminar* at the Missouri University of Science and Technology.

The study of integral inequalities on time scales (with both $\Delta$-integrals or $\Diamond_\alpha$-integrals) and its applications is under strong current research. We provide here some more (recent) references within this subject: [12, 13, 26, 36, 37, 68].

# Chapter 7

# Discrete Fractional Calculus

We introduce a discrete-time fractional calculus of variations on the time scale $h\mathbb{Z}$, $h > 0$. First and second order necessary optimality conditions are established. Examples illustrating the use of the new Euler–Lagrange and Legendre type conditions are given. They show that solutions to the considered fractional problems become the classical discrete-time solutions when the fractional order of the discrete-derivatives are integer values, and that they converge to the fractional continuous-time solutions when $h$ tends to zero. Our Legendre type condition is useful to eliminate false candidates identified via the Euler–Lagrange fractional equation.

## 7.1 Introduction

The *Fractional Calculus* is an important research field in several different areas: physics (including classical and quantum mechanics as well as thermodynamics), chemistry, biology, economics, and control theory [77, 91]. It has its origin more than 300 years ago when L'Hopital asked Leibniz what should be the meaning of a derivative of non-integer order. After that episode several more famous mathematicians contributed to the development of Fractional Calculus: Abel, Fourier, Liouville, Riemann, Riesz, just to mention a few names.

In [76] Miller and Ross define a fractional sum of order $\nu > 0$ *via* the solution of a linear difference equation. They introduce it as

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{(\nu-1)} f(s). \tag{7.1}$$

Definition (7.1) is the discrete analogue of the Riemann-Liouville fractional integral

$${}_a\mathbf{D}_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x - s)^{\nu-1} f(s) ds$$

of order $\nu > 0$, which can be obtained *via* the solution of a linear differential equation [76, 77]. Basic properties of the operator $\Delta^{-\nu}$ in (7.1) were obtained in [76]. More recently, Atici and Eloe introduced the fractional difference of order $\alpha > 0$ by $\Delta^\alpha f(t) = \Delta(\Delta^{-(1-\alpha)} f(t))$, and developed some of its properties that allow to obtain solutions of certain fractional difference equations [15, 16].

The fractional differential calculus has been widely developed in the past few decades due mainly to its demonstrated applications in various fields of science and engineering [66, 77, 86]. The study of necessary optimality conditions for fractional problems of the calculus of variations and optimal control is a fairly recent issue attracting an increasing attention – see [4, 9, 39, 40, 52, 53] and references therein – but available results address only the continuous-time case. Therefore, it is pertinent to start a fractional discrete-time theory of the calculus of variations, namely, using the time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Computer simulations show that this time scale is particularly interesting because when $h$ tends to zero one recovers previous fractional continuous-time results.

Our objective is two-fold. On one hand we proceed to develop the theory of *fractional difference calculus*, e.g., introducing the concept of left and right fractional sum/difference (cf. Definitions 138 and 144). On the other hand, we believe that the results herein presented will potentiate research not only in the fractional calculus of variations but also in solving fractional difference equations, specifically, fractional equations in which left and right fractional differences appear [see the Euler–Lagrange Equation (7.15)].

Main results so far obtained by us contain a fractional formula of $h$-summation by parts (Theorem 146), and necessary optimality conditions of first and second order (Theorems 151 and 154, respectively) for the proposed $h$-fractional problem of the calculus of variations (7.14). We also provide some illustrative examples.

We remark that it remains a challenge how to generalize the present results to an arbitrary time scale $\mathbb{T}$. This is a difficult open problem since an explicit formula for the Cauchy function (cf. Definition 27) of the linear dynamic equation $y^{\Delta^n} = 0$ on an arbitrary time scale is not known.

## 7.2 Preliminaries

It is known that $y(t, s) := H_{n-1}(t, \sigma(s))$ is the Cauchy function (cf. Definition 27) for $y^{\Delta^n} = 0$ [24, Example 5.115], where $H_{n-1}$ is the time scale generalized polynomial $h_{n-1}$ of Definition 18 (we use here $H$ instead of $h$ in order to avoid confuse with the parameter $h$ of the time scale $\mathbb{T} = h\mathbb{Z}$).

From now on we restrict ourselves to the time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$.

If we have a function $f$ of two variables, $f(t, s)$, its partial forward $h$-differences will be denoted by $\Delta_{t,h}$ and $\Delta_{s,h}$, respectively. We will make use of the standard conventions $\sum_{t=c}^{c-1} f(t) = 0$, $c \in \mathbb{Z}$, and $\prod_{i=0}^{-1} f(i) = 1$.

Before giving an explicit formula for the generalized polynomials $H_k$ on $\mathbb{T} = h\mathbb{Z}$ we introduce the following definition:

**Definition 134** (cf. [18])**.** For arbitrary $x, y \in \mathbb{R}$ the $h$-factorial function is defined by

$$x_h^{(y)} := h^y \frac{\Gamma(\frac{x}{h}+1)}{\Gamma(\frac{x}{h}+1-y)},$$

where $\Gamma$ is the well-known Euler gamma function, and we use the convention that division at a pole yields zero.

*Remark* 135. For $h = 1$, and in accordance with the previous literature [see formula (7.1)], we write $x^{(y)}$ to denote $x_h^{(y)}$.

**Proposition 136.** *For the time-scale $\mathbb{T}$ one has*

$$H_k(t,s) := \frac{(t-s)_h^{(k)}}{k!} \quad \text{for all} \quad s,t \in h\mathbb{Z} \text{ and } k \in \mathbb{N}_0\,. \tag{7.2}$$

To prove (7.2) we use the following technical lemma. We remind the reader the basic property $\Gamma(x+1) = x\Gamma(x)$ of the gamma function.

**Lemma 137.** *Let $s \in \mathbb{T}$. Then, for all $t \in \mathbb{T}^\kappa$ one has*

$$\Delta_{t,h}\left\{\frac{(t-s)_h^{(k+1)}}{(k+1)!}\right\} = \frac{(t-s)_h^{(k)}}{k!}\,.$$

*Proof.* The equality follows by direct computations:

$$\begin{aligned}
\Delta_{t,h}\left\{\frac{(t-s)_h^{(k+1)}}{(k+1)!}\right\} &= \frac{1}{h}\left\{\frac{(\sigma(t)-s)_h^{(k+1)}}{(k+1)!} - \frac{(t-s)_h^{(k+1)}}{(k+1)!}\right\}\\
&= \frac{h^{k+1}}{h(k+1)!}\left\{\frac{\Gamma((t+h-s)/h+1)}{\Gamma((t+h-s)/h+1-(k+1))} - \frac{\Gamma((t-s)/h+1)}{\Gamma((t-s)/h+1-(k+1))}\right\}\\
&= \frac{h^k}{(k+1)!}\left\{\frac{\Gamma((t-s)/h+2)}{\Gamma((t-s)/h+1-k)} - \frac{\Gamma((t-s)/h+1)}{\Gamma((t-s)/h-k)}\right\}\\
&= \frac{h^k}{(k+1)!}\left\{\frac{((t-s)/h+1)\Gamma((t-s)/h+1)}{((t-s)/h-k)\Gamma((t-s)/h-k)} - \frac{\Gamma((t-s)/h+1)}{\Gamma((t-s)/h-k)}\right\}\\
&= \frac{h^k}{(k+1)!}\left\{\frac{(k+1)\Gamma((t-s)/h+1)}{((t-s)/h-k)\Gamma((t-s)/h-k)}\right\} = \frac{h^k}{k!}\left\{\frac{\Gamma((t-s)/h+1)}{\Gamma((t-s)/h+1-k)}\right\}\\
&= \frac{(t-s)_h^{(k)}}{k!}\,.
\end{aligned}$$

□

*Proof.* (of Proposition 136) We proceed by mathematical induction. For $k = 0$

$$H_0(t,s) = \frac{1}{0!}h^0\frac{\Gamma(\frac{t-s}{h}+1)}{\Gamma(\frac{t-s}{h}+1-0)} = \frac{\Gamma(\frac{t-s}{h}+1)}{\Gamma(\frac{t-s}{h}+1)} = 1\,.$$

Assume that (7.2) holds for $k$ replaced by $m$. Then, by Lemma 137

$$H_{m+1}(t,s) = \int_s^t H_m(\tau,s)\Delta\tau = \int_s^t \frac{(\tau-s)_h^{(m)}}{m!}\Delta\tau = \frac{(t-s)_h^{(m+1)}}{(m+1)!},$$

which is (7.2) with $k$ replaced by $m+1$. □

Let $y_1(t), \ldots, y_n(t)$ be $n$ linearly independent solutions of the linear homogeneous dynamic equation $y^{\Delta^n} = 0$. From Theorem 28 we know that the solution of (1.12) (with $L = \Delta^n$ and $t_0 = a$) is

$$y(t) = \Delta^{-n} f(t) = \int_a^t \frac{(t - \sigma(s))_h^{(n-1)}}{\Gamma(n)} f(s)\Delta s = \frac{1}{\Gamma(n)} \sum_{k=a/h}^{t/h-1} (t - \sigma(kh))_h^{(n-1)} f(kh)h \, .$$

Since $y^{\Delta_i}(a) = 0$, $i = 0, \ldots, n-1$, then we can write

$$\begin{aligned} \Delta^{-n} f(t) &= \frac{1}{\Gamma(n)} \sum_{k=a/h}^{t/h-n} (t - \sigma(kh))_h^{(n-1)} f(kh)h \\ &= \frac{1}{\Gamma(n)} \int_a^{\sigma(t-nh)} (t - \sigma(s))_h^{(n-1)} f(s)\Delta s \, . \end{aligned} \tag{7.3}$$

Note that function $t \to (\Delta^{-n} f)(t)$ is defined for $t = a + nh \bmod(h)$ while function $t \to f(t)$ is defined for $t = a \bmod(h)$. Extending (7.3) to any positive real value $\nu$, and having as an analogy the continuous left and right fractional derivatives [77], we define the left fractional $h$-sum and the right fractional $h$-sum as follows. We denote by $\mathcal{F}_{\mathbb{T}}$ the set of all real valued functions defined on the time scale $\mathbb{T}$.

**Definition 138.** Let $f \in \mathcal{F}_{\mathbb{T}}$. The left and right fractional $h$-sum of order $\nu > 0$ are, respectively, the operators ${}_a\Delta_h^{-\nu} : \mathcal{F}_{\mathbb{T}} \to \mathcal{F}_{\tilde{\mathbb{T}}_\nu^+}$ and ${}_h\Delta_b^{-\nu} : \mathcal{F}_{\mathbb{T}} \to \mathcal{F}_{\tilde{\mathbb{T}}_\nu^-}$, $\tilde{\mathbb{T}}_\nu^\pm = \{a \pm \nu h : a \in \mathbb{T}\}$, defined by

$$\begin{aligned} {}_a\Delta_h^{-\nu} f(t) &= \frac{1}{\Gamma(\nu)} \int_a^{\sigma(t-\nu h)} (t - \sigma(s))_h^{(\nu-1)} f(s)\Delta s = \frac{1}{\Gamma(\nu)} \sum_{k=\frac{a}{h}}^{\frac{t}{h}-\nu} (t - \sigma(kh))_h^{(\nu-1)} f(kh)h \\ {}_h\Delta_b^{-\nu} f(t) &= \frac{1}{\Gamma(\nu)} \int_t^{\sigma(b)} (s - \sigma(t))_h^{(\nu-1)} f(s)\Delta s = \frac{1}{\Gamma(\nu)} \sum_{k=\frac{t}{h}+\nu}^{\frac{b}{h}} (kh - \sigma(t))_h^{(\nu-1)} f(kh)h. \end{aligned}$$

**Lemma 139.** *Let $\nu > 0$ be an arbitrary positive real number. For any $t \in \mathbb{T}$ we have: (i) $\lim_{\nu \to 0} {}_a\Delta_h^{-\nu} f(t + \nu h) = f(t)$; (ii) $\lim_{\nu \to 0} {}_h\Delta_b^{-\nu} f(t - \nu h) = f(t)$.*

*Proof.* Since

$$\begin{aligned} {}_a\Delta_h^{-\nu} f(t + \nu h) &= \frac{1}{\Gamma(\nu)} \int_a^{\sigma(t)} (t + \nu h - \sigma(s))_h^{(\nu-1)} f(s)\Delta s \\ &= \frac{1}{\Gamma(\nu)} \sum_{k=\frac{a}{h}}^{\frac{t}{h}} (t + \nu h - \sigma(kh))_h^{(\nu-1)} f(kh)h \\ &= h^\nu f(t) + \frac{\nu}{\Gamma(\nu+1)} \sum_{k=\frac{a}{h}}^{\frac{\rho(t)}{h}} (t + \nu h - \sigma(kh))_h^{(\nu-1)} f(kh)h \, , \end{aligned}$$

it follows that $\lim_{\nu\to 0}{}_a\Delta_h^{-\nu} f(t+\nu h) = f(t)$. To prove (ii) we use a similar method:

$$\begin{aligned}
{}_h\Delta_b^{-\nu} f(t-\nu h) &= \frac{1}{\Gamma(\nu)}\int_t^{\sigma(b)} (s+\nu h-\sigma(t))_h^{(\nu-1)} f(s)\Delta s \\
&= h^\nu f(t) + \frac{\nu}{\Gamma(\nu+1)}\sum_{k=\frac{\sigma(t)}{h}}^{\frac{b}{h}} (kh+\nu h-\sigma(t))_h^{(\nu-1)} f(kh)h\,,
\end{aligned}$$

and therefore $\lim_{\nu\to 0}{}_h\Delta_b^{-\nu} f(t-\nu h) = f(t)$. □

For any $t\in\mathbb{T}$ and for any $\nu\geq 0$ we define ${}_a\Delta_h^0 f(t) := {}_h\Delta_b^0 f(t) := f(t)$ and write

$$\begin{aligned}
{}_a\Delta_h^{-\nu} f(t+\nu h) &= h^\nu f(t) + \frac{\nu}{\Gamma(\nu+1)}\int_a^t (t+\nu h-\sigma(s))_h^{(\nu-1)} f(s)\Delta s\,, \\
{}_h\Delta_b^{-\nu} f(t) &= h^\nu f(t-\nu h) + \frac{\nu}{\Gamma(\nu+1)}\int_{\sigma(t)}^{\sigma(b)} (s+\nu h-\sigma(t))_h^{(\nu-1)} f(s)\Delta s\,.
\end{aligned} \tag{7.4}$$

**Theorem 140.** *Let $f\in\mathcal{F}_\mathbb{T}$ and $\nu\geq 0$. For all $t\in\mathbb{T}^\kappa$ we have*

$$ {}_a\Delta_h^{-\nu} f^\Delta(t+\nu h) = ({}_a\Delta_h^{-\nu} f(t+\nu h))^\Delta - \frac{\nu}{\Gamma(\nu+1)}(t+\nu h-a)_h^{(\nu-1)} f(a)\,. \tag{7.5}$$

To prove Theorem 140 we make use of the following

**Lemma 141.** *Let $t\in\mathbb{T}^\kappa$. The following equality holds for all $s\in\mathbb{T}^\kappa$:*

$$\begin{aligned}
&\Delta_{s,h}\left((t+\nu h-s)_h^{(\nu-1)} f(s)\right) \\
&\qquad = (t+\nu h-\sigma(s))_h^{(\nu-1)} f^\Delta(s) - (v-1)(t+\nu h-\sigma(s))_h^{(\nu-2)} f(s)\,.
\end{aligned} \tag{7.6}$$

*Proof.* Direct calculations give the intended result:

$$\begin{aligned}
&\Delta_{s,h}\left((t+\nu h-s)_h^{(\nu-1)} f(s)\right) \\
&= \Delta_{s,h}\left((t+\nu h-s)_h^{(\nu-1)}\right) f(s) + (t+\nu h-\sigma(s))_h^{(\nu-1)} f^\Delta(s) \\
&= \frac{(t+\nu h-\sigma(s))_h^{(\nu-1)} - (t+\nu h-s)_h^{(\nu-1)}}{h} f(s) + (t+\nu h-\sigma(s))_h^{(\nu-1)} \\
&= \frac{f(s)}{h}\left[h^{\nu-1}\frac{\Gamma\left(\frac{t+\nu h-\sigma(s)}{h}+1\right)}{\Gamma\left(\frac{t+\nu h-\sigma(s)}{h}+1-(\nu-1)\right)} - h^{\nu-1}\frac{\Gamma\left(\frac{t+\nu h-s}{h}+1\right)}{\Gamma\left(\frac{t+\nu h-s}{h}+1-(\nu-1)\right)}\right] \\
&\quad + (t+\nu h-\sigma(s))_h^{(\nu-1)} f^\Delta(s) \\
&= f(s)\left[h^{\nu-2}\left[\frac{\Gamma(\frac{t+\nu h-s}{h})}{\Gamma(\frac{t-s}{h}+1)} - \frac{\Gamma(\frac{t+\nu h-s}{h}+1)}{\Gamma(\frac{t-s}{h}+2)}\right]\right] + (t+\nu h-\sigma(s))_h^{(\nu-1)} f^\Delta(s) \\
&= f(s)h^{\nu-2}\left[\frac{\Gamma(\frac{t+\nu h-s}{h})}{\Gamma(\frac{t-s}{h}+2)}\left(\frac{t-s}{h}+1-\frac{t+\nu h-s}{h}\right)\right] + (t+\nu h-\sigma(s))_h^{(\nu-1)} f^\Delta(s) \\
&= f(s)h^{\nu-2}\frac{\Gamma(\frac{t+\nu h-s-h}{h}+1)}{\Gamma(\frac{t-s+\nu h-h}{h}+1-(\nu-2))}(-(\nu-1)) + (t+\nu h-\sigma(s))_h^{(\nu-1)} f^\Delta(s) \\
&= -(\nu-1)(t+\nu h-\sigma(s))_h^{(\nu-2)} f(s) + (t+\nu h-\sigma(s))_h^{(\nu-1)} f^\Delta(s)\,,
\end{aligned}$$

where the first equality follows directly from (1.3). □

*Remark* 142. Given an arbitrary $t \in \mathbb{T}^\kappa$ it is easy to prove, in a similar way as in the proof of Lemma 141, the following equality analogous to (7.6): for all $s \in \mathbb{T}^\kappa$

$$\begin{aligned}\Delta_{s,h}&\left((s+\nu h-\sigma(t))_h^{(\nu-1)} f(s)\right)\\ &= (\nu-1)(s+\nu h-\sigma(t))_h^{(\nu-2)} f^\sigma(s) + (s+\nu h-\sigma(t))_h^{(\nu-1)} f^\Delta(s)\,. \end{aligned} \tag{7.7}$$

*Proof.* (of Theorem 140) From Lemma 141 we obtain that

$$\begin{aligned}{}_a\Delta_h^{-\nu} f^\Delta(t+\nu h) &= h^\nu f^\Delta(t) + \frac{\nu}{\Gamma(\nu+1)}\int_a^t (t+\nu h-\sigma(s))_h^{(\nu-1)} f^\Delta(s)\Delta s\\ &= h^\nu f^\Delta(t) + \frac{\nu}{\Gamma(\nu+1)}\left[(t+\nu h-s)_h^{(\nu-1)} f(s)\right]_{s=a}^{s=t}\\ &\quad+\frac{\nu}{\Gamma(\nu+1)}\int_a^{\sigma(t)} (\nu-1)(t+\nu h-\sigma(s))_h^{(\nu-2)} f(s)\Delta s\\ &= -\frac{\nu(t+\nu h-a)_h^{(\nu-1)}}{\Gamma(\nu+1)} f(a) + h^\nu f^\Delta(t) + \nu h^{\nu-1} f(t)\\ &\quad+\frac{\nu}{\Gamma(\nu+1)}\int_a^{t} (\nu-1)(t+\nu h-\sigma(s))_h^{(\nu-2)} f(s)\Delta s.\end{aligned} \tag{7.8}$$

We now show that $({}_a\Delta_h^{-\nu} f(t+\nu h))^\Delta$ equals (7.8):

$$\begin{aligned}({}_a\Delta_h^{-\nu} f(t+\nu h))^\Delta &= \frac{1}{h}\left[h^\nu f(\sigma(t)) + \frac{\nu}{\Gamma(\nu+1)}\int_a^{\sigma(t)} (\sigma(t)+\nu h-\sigma(s))_h^{(\nu-1)} f(s)\Delta s\right.\\ &\qquad\left. -h^\nu f(t) - \frac{\nu}{\Gamma(\nu+1)}\int_a^t (t+\nu h-\sigma(s))_h^{(\nu-1)} f(s)\Delta s\right]\\ &= h^\nu f^\Delta(t) + \frac{\nu}{h\Gamma(\nu+1)}\left[\int_a^t (\sigma(t)+\nu h-\sigma(s))_h^{(\nu-1)} f(s)\Delta s\right.\\ &\qquad\left.-\int_a^t (t+\nu h-\sigma(s))_h^{(\nu-1)} f(s)\Delta s\right] + h^{\nu-1}\nu f(t)\\ &= h^\nu f^\Delta(t) + \frac{\nu}{\Gamma(\nu+1)}\int_a^t \Delta_{t,h}\left((t+\nu h-\sigma(s))_h^{(\nu-1)}\right) f(s)\Delta s + h^{\nu-1}\nu f(t)\\ &= h^\nu f^\Delta(t) + \frac{\nu}{\Gamma(\nu+1)}\int_a^t (\nu-1)(t+\nu h-\sigma(s))_h^{(\nu-2)} f(s)\Delta s + \nu h^{\nu-1} f(t)\,.\end{aligned}$$

□

Follows the counterpart of Theorem 140 for the right fractional $h$-sum:

**Theorem 143.** *Let $f \in \mathcal{F}_\mathbb{T}$ and $\nu \geq 0$. For all $t \in \mathbb{T}^\kappa$ we have*

$${}_h\Delta_{\rho(b)}^{-\nu} f^\Delta(t-\nu h) = \frac{\nu}{\Gamma(\nu+1)}(b+\nu h-\sigma(t))_h^{(\nu-1)} f(b) + ({}_h\Delta_b^{-\nu} f(t-\nu h))^\Delta\,. \tag{7.9}$$

*Proof.* From (7.7) we obtain that

$$
\begin{aligned}
{}_h\Delta_{\rho(b)}^{-\nu} f^{\Delta}(t-\nu h) &= h^{\nu} f^{\Delta}(t) + \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^{b} (s+\nu h-\sigma(t))_h^{(\nu-1)} f^{\Delta}(s)\Delta s \\
&= h^{\nu} f^{\Delta}(t) + \left[\frac{\nu(s+\nu h-\sigma(t))_h^{(\nu-1)}}{\Gamma(\nu+1)} f(s)\right]_{s=\sigma(t)}^{s=b} \\
&\quad - \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^{b} (\nu-1)(s+\nu h-\sigma(t))_h^{(\nu-2)} f^{\sigma}(s)\Delta s \qquad (7.10)\\
&= \frac{\nu(b+\nu h-\sigma(t))_h^{(\nu-1)}}{\Gamma(\nu+1)} f(b) + h^{\nu} f^{\Delta}(t) - \nu h^{\nu-1} f(\sigma(t)) \\
&\quad - \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^{b} (\nu-1)(s+\nu h-\sigma(t))_h^{(\nu-2)} f^{\sigma}(s)\Delta s.
\end{aligned}
$$

We show that $({}_h\Delta_b^{-\nu} f(t-\nu h))^{\Delta}$ equals (7.10):

$$
\begin{aligned}
&({}_h\Delta_b^{-\nu} f(t-\nu h))^{\Delta} \\
&\quad = \frac{1}{h}\Bigg[h^{\nu} f(\sigma(t)) - h^{\nu} f(t) + \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma^2(t)}^{\sigma(b)} (s+\nu h-\sigma^2(t)))_h^{(\nu-1)} f(s)\Delta s \\
&\qquad - \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^{\sigma(b)} (s+\nu h-\sigma(t))_h^{(\nu-1)} f(s)\Delta s\Bigg] \\
&= h^{\nu} f^{\Delta}(t) + \frac{\nu}{h\Gamma(\nu+1)} \Bigg[\int_{\sigma^2(t)}^{\sigma(b)} (s+\nu h-\sigma^2(t)))_h^{(\nu-1)} f(s)\Delta s \\
&\quad - \int_{\sigma^2(t)}^{\sigma(b)} (s+\nu h-\sigma(t))_h^{(\nu-1)} f(s)\Delta s\Bigg] - \nu h^{\nu-1} f(\sigma(t)) \\
&= h^{\nu} f^{\Delta}(t) + \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma^2(t)}^{\sigma(b)} \Delta_{t,h}\left((s+\nu h-\sigma(t))_h^{(\nu-1)}\right) f(s)\Delta s - \nu h^{\nu-1} f(\sigma(t)) \\
&= h^{\nu} f^{\Delta}(t) - \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma^2(t)}^{\sigma(b)} (\nu-1)(s+\nu h-\sigma^2(t))_h^{(\nu-2)} f(s)\Delta s - \nu h^{\nu-1} f(\sigma(t)) \\
&= h^{\nu} f^{\Delta}(t) - \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^{b} (\nu-1)(s+\nu h-\sigma(t))_h^{(\nu-2)} f(s)\Delta s - \nu h^{\nu-1} f(\sigma(t)).
\end{aligned}
$$

□

**Definition 144.** Let $0 < \alpha \leq 1$ and set $\gamma := 1-\alpha$. The *left fractional difference* ${}_a\Delta_h^{\alpha} f(t)$ and the *right fractional difference* ${}_h\Delta_b^{\alpha} f(t)$ of order $\alpha$ of a function $f \in \mathcal{F}_{\mathbb{T}}$ are defined as

$$
{}_a\Delta_h^{\alpha} f(t) := ({}_a\Delta_h^{-\gamma} f(t+\gamma h))^{\Delta} \quad \text{and} \quad {}_h\Delta_b^{\alpha} f(t) := -({}_h\Delta_b^{-\gamma} f(t-\gamma h))^{\Delta},
$$

for all $t \in \mathbb{T}^{\kappa}$.

## 7.3 Main results

Our aim is to introduce the $h$-fractional problem of the calculus of variations and to prove corresponding necessary optimality conditions. In order to obtain an Euler–Lagrange type equation (cf. Theorem 151) we first prove a fractional formula of $h$-summation by parts.

### 7.3.1 Fractional $h$-summation by parts

A big challenge was to discover a fractional $h$-summation by parts formula within this time scale setting. Indeed, there is no clue of what such a formula should be. We found it eventually, making use of the following lemma.

**Lemma 145.** *Let $f$ and $k$ be two functions defined on $\mathbb{T}^\kappa$ and $\mathbb{T}^{\kappa^2}$, respectively, and $g$ a function defined on $\mathbb{T}^\kappa \times \mathbb{T}^{\kappa^2}$. Then, the following equality holds:*

$$\int_a^b f(t)\left[\int_a^t g(t,s)k(s)\Delta s\right]\Delta t = \int_a^{\rho(b)} k(t)\left[\int_{\sigma(t)}^b g(s,t)f(s)\Delta s\right]\Delta t\,.$$

*Proof.* Consider the matrices $R = [f(a+h), f(a+2h), \cdots, f(b-h)]$,

$$C_1 = \begin{bmatrix} g(a+h,a)k(a) \\ g(a+2h,a)k(a) + g(a+2h,a+h)k(a+h) \\ \vdots \\ g(b-h,a)k(a) + g(b-h,a+h)k(a+h) + \cdots + g(b-h,b-2h)k(b-2h) \end{bmatrix}$$

$$C_2 = \begin{bmatrix} g(a+h,a) \\ g(a+2h,a) \\ \vdots \\ g(b-h,a) \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 \\ g(a+2h,a+h) \\ \vdots \\ g(b-h,a+h) \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(b-h,b-2h) \end{bmatrix}.$$

Direct calculations show that

$$\begin{aligned}
\int_a^b f(t)\left[\int_a^t g(t,s)k(s)\Delta s\right]\Delta t &= h^2 \sum_{i=a/h}^{b/h-1} f(ih) \sum_{j=a/h}^{i-1} g(ih,jh)k(jh) = h^2 R\cdot C_1 \\
&= h^2 R\cdot[k(a)C_2 + k(a+h)C_3 + \cdots + k(b-2h)C_4] \\
&= h^2\left[k(a)\sum_{j=a/h+1}^{b/h-1} g(jh,a)f(jh) + k(a+h)\sum_{j=a/h+2}^{b/h-1} g(jh,a+h)f(jh)\right. \\
&\qquad\left. + \cdots + k(b-2h)\sum_{j=b/h-1}^{b/h-1} g(jh,b-2h)f(jh)\right] \\
&= \sum_{i=a/h}^{b/h-2} k(ih)h \sum_{j=\sigma(ih)/h}^{b/h-1} g(jh,ih)f(jh)h = \int_a^{\rho(b)} k(t)\left[\int_{\sigma(t)}^b g(s,t)f(s)\Delta s\right]\Delta t.
\end{aligned}$$

□

**Theorem 146** (fractional $h$-summation by parts)**.** *Let $f$ and $g$ be real valued functions defined on $\mathbb{T}^\kappa$ and $\mathbb{T}$, respectively. Fix $0 < \alpha \leq 1$ and put $\gamma := 1 - \alpha$. Then,*

$$\begin{aligned}
\int_a^b f(t)_a\Delta_h^\alpha g(t)\Delta t &= h^\gamma f(\rho(b))g(b) - h^\gamma f(a)g(a) + \int_a^{\rho(b)} {}_h\Delta_{\rho(b)}^\alpha f(t) g^\sigma(t)\Delta t\\
&+ \frac{\gamma}{\Gamma(\gamma+1)} g(a)\left(\int_a^b (t+\gamma h - a)_h^{(\gamma-1)} f(t)\Delta t - \int_{\sigma(a)}^b (t+\gamma h-\sigma(a))_h^{(\gamma-1)} f(t)\Delta t\right). \qquad (7.11)
\end{aligned}$$

*Proof.* By (7.5) we can write

$$\begin{aligned}
\int_a^b f(t)_a\Delta_h^\alpha g(t)\Delta t &= \int_a^b f(t)({}_a\Delta_h^{-\gamma} g(t+\gamma h))^\Delta \Delta t\\
&= \int_a^b f(t)\left[{}_a\Delta_h^{-\gamma} g^\Delta(t+\gamma h) + \frac{\gamma}{\Gamma(\gamma+1)}(t+\gamma h - a)_h^{(\gamma-1)} g(a)\right]\Delta t \qquad (7.12)\\
&= \int_a^b f(t)_a\Delta_h^{-\gamma} g^\Delta(t+\gamma h)\Delta t + \int_a^b \frac{\gamma}{\Gamma(\gamma+1)}(t+\gamma h - a)_h^{(\gamma-1)} f(t) g(a)\Delta t.
\end{aligned}$$

Using (7.4) we get

$$\begin{aligned}
&\int_a^b f(t)_a\Delta_h^{-\gamma} g^\Delta(t+\gamma h)\Delta t\\
&= \int_a^b f(t)\left[h^\gamma g^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h - \sigma(s))_h^{(\gamma-1)} g^\Delta(s)\Delta s\right]\Delta t\\
&= h^\gamma \int_a^b f(t) g^\Delta(t)\Delta t + \frac{\gamma}{\Gamma(\gamma+1)}\int_a^b f(t)\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)} g^\Delta(s)\Delta s\Delta t\\
&= h^\gamma \int_a^b f(t) g^\Delta(t)\Delta t + \frac{\gamma}{\Gamma(\gamma+1)}\int_a^{\rho(b)} g^\Delta(t)\int_{\sigma(t)}^b (s+\gamma h-\sigma(t))_h^{(\gamma-1)} f(s)\Delta s\Delta t\\
&= h^\gamma f(\rho(b))[g(b)-g(\rho(b))] + \int_a^{\rho(b)} g^\Delta(t)_h\Delta_{\rho(b)}^{-\gamma} f(t-\gamma h)\Delta t,
\end{aligned}$$

where the third equality follows by Lemma 145. We proceed to develop the right-hand side of the last equality as follows:

$$\begin{aligned}
&h^\gamma f(\rho(b))[g(b)-g(\rho(b))] + \int_a^{\rho(b)} g^\Delta(t)_h\Delta_{\rho(b)}^{-\gamma} f(t-\gamma h)\Delta t\\
&= h^\gamma f(\rho(b))[g(b)-g(\rho(b))] + \left[g(t)_h\Delta_{\rho(b)}^{-\gamma} f(t-\gamma h)\right]_{t=a}^{t=\rho(b)}\\
&\quad - \int_a^{\rho(b)} g^\sigma(t)({}_h\Delta_{\rho(b)}^{-\gamma} f(t-\gamma h))^\Delta \Delta t\\
&= h^\gamma f(\rho(b))g(b) - h^\gamma f(a)g(a)\\
&\quad - \frac{\gamma}{\Gamma(\gamma+1)} g(a)\int_{\sigma(a)}^b (s+\gamma h-\sigma(a))_h^{(\gamma-1)} f(s)\Delta s + \int_a^{\rho(b)}\left({}_h\Delta_{\rho(b)}^\alpha f(t)\right) g^\sigma(t)\Delta t,
\end{aligned}$$

where the first equality follows from (1.7). Putting this into (7.12) we get (7.11). □

Let us prove that Theorem 146 indeed generalizes the usual summation by parts formula

**Corollary 147.** *Suppose that $h = \alpha = 1$ in Theorem 146. Then, the next formula holds:*

$$\int_a^b f(t)\Delta g(t)\Delta t = f(\rho(b))g(b) - f(a)g(a) - \int_a^{\rho(b)} \Delta f(t) g^\sigma(t)\Delta t. \tag{7.13}$$

*Remark* 148. Since $h = 1$, then (7.13) becomes

$$\sum_{t=a}^{b-1} f(t)[g(t+1) - g(t)] = f(b-1)g(b) - f(a)g(a) - \sum_{t=a}^{b-2}[f(t+1) - f(t)]g(t+1).$$

If we allow $f$ to be defined at $t = b$, then we have

$$\begin{aligned}\sum_{t=a}^{b-1} f(t)[g(t+1) - g(t)] &= f(b-1)g(b) - f(a)g(a) - \sum_{t=a}^{b-2}[f(t+1) - f(t)]g(t+1)\\ &= f(b)g(b) - f(a)g(a) - [f(b) - f(b-1)]g(b)\\ &\qquad - \sum_{t=a}^{b-2}[f(t+1) - f(t)]g(t+1)\\ &= f(b)g(b) - f(a)g(a) - \sum_{t=a}^{b-1}[f(t+1) - f(t)]g(t+1).\end{aligned}$$

This is the usual summation by parts formula.

### 7.3.2 Necessary optimality conditions

We begin to fix two arbitrary real numbers $\alpha$ and $\beta$ such that $\alpha, \beta \in (0, 1]$. Further, we put $\gamma := 1 - \alpha$ and $\nu := 1 - \beta$.

Let a function $L(t, u, v, w) : \mathbb{T}^\kappa \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given. We consider the problem of minimizing (or maximizing) a functional $\mathcal{L} : \mathcal{F}_\mathbb{T} \to \mathbb{R}$ subject to given boundary conditions:

$$\mathcal{L}[y(\cdot)] = \int_a^b L(t, y^\sigma(t), {}_a\Delta_h^\alpha y(t), {}_h\Delta_b^\beta y(t))\Delta t \longrightarrow \min,\ y(a) = A,\ y(b) = B\,. \tag{7.14}$$

Our main aim is to derive necessary optimality conditions for problem (7.14).

**Definition 149.** For $f \in \mathcal{F}_\mathbb{T}$ we define the norm

$$\|f\| = \max_{t\in\mathbb{T}^\kappa} |f^\sigma(t)| + \max_{t\in\mathbb{T}^\kappa} |{}_a\Delta_h^\alpha f(t)| + \max_{t\in\mathbb{T}^\kappa} |{}_h\Delta_b^\beta f(t)|.$$

A function $\hat{y} \in \mathcal{F}_\mathbb{T}$ with $\hat{y}(a) = A$ and $\hat{y}(b) = B$ is called a local minimum for problem (7.14) provided there exists $\delta > 0$ such that $\mathcal{L}(\hat{y}) \le \mathcal{L}(y)$ for all $y \in \mathcal{F}_\mathbb{T}$ with $y(a) = A$ and $y(b) = B$ and $\|y - \hat{y}\| < \delta$.

**Definition 150.** A function $\eta \in \mathcal{F}_\mathbb{T}$ is called an admissible variation provided $\eta \neq 0$ and $\eta(a) = \eta(b) = 0$.

From now on we assume that the second-order partial derivatives $L_{uu}$, $L_{uv}$, $L_{uw}$, $L_{vw}$, $L_{vv}$, and $L_{ww}$ exist and are continuous.

**First order optimality condition**

The next theorem gives a first order necessary condition for problem (7.14), i.e., an Euler–Lagrange type equation for the fractional $h$-difference setting.

**Theorem 151** (The $h$-fractional Euler-Lagrange equation for problem (7.14))**.** *If $\hat{y} \in \mathcal{F}_{\mathbb{T}}$ is a local minimum for problem* (7.14)*, then the equality*

$$L_u[\hat{y}](t) + {}_h\Delta^{\alpha}_{\rho(b)} L_v[\hat{y}](t) + {}_a\Delta^{\beta}_{h} L_w[\hat{y}](t) = 0 \tag{7.15}$$

*holds for all $t \in \mathbb{T}^{\kappa^2}$ with operator $[\cdot]$ defined by $[y](s) = (s, y^{\sigma}(s), {}_a\Delta^{\alpha}_{s} y(s), {}_s\Delta^{\beta}_{b} y(s))$.*

*Proof.* Suppose that $\hat{y}(\cdot)$ is a local minimum of $\mathcal{L}[\cdot]$. Let $\eta(\cdot)$ be an arbitrarily fixed admissible variation and define a function $\Phi : \left(-\frac{\delta}{\|\eta(\cdot)\|}, \frac{\delta}{\|\eta(\cdot)\|}\right) \rightarrow \mathbb{R}$ by

$$\Phi(\varepsilon) = \mathcal{L}[\hat{y}(\cdot) + \varepsilon\eta(\cdot)]. \tag{7.16}$$

This function has a minimum at $\varepsilon = 0$, so we must have $\Phi'(0) = 0$, i.e.,

$$\int_a^b \left[L_u[\hat{y}](t)\eta^{\sigma}(t) + L_v[\hat{y}](t){}_a\Delta^{\alpha}_{h}\eta(t) + L_w[\hat{y}](t){}_h\Delta^{\beta}_{b}\eta(t)\right] \Delta t = 0,$$

which we may write equivalently as

$$\begin{aligned} hL_u[\hat{y}](t)\eta^{\sigma}(t)|_{t=\rho(b)} + \int_a^{\rho(b)} L_u[\hat{y}](t)\eta^{\sigma}(t)\Delta t + \int_a^b L_v[\hat{y}](t){}_a\Delta^{\alpha}_{h}\eta(t)\Delta t \\ + \int_a^b L_w[\hat{y}](t){}_h\Delta^{\beta}_{b}\eta(t)\Delta t = 0. \end{aligned} \tag{7.17}$$

Using Theorem 146 and the fact that $\eta(a) = \eta(b) = 0$, we get

$$\int_a^b L_v[\hat{y}](t){}_a\Delta^{\alpha}_{h}\eta(t)\Delta t = \int_a^{\rho(b)} \left({}_h\Delta^{\alpha}_{\rho(b)}\left(L_v[\hat{y}]\right)(t)\right)\eta^{\sigma}(t)\Delta t \tag{7.18}$$

for the third term in (7.17). Using (7.9) it follows that

$$\begin{aligned} &\int_a^b L_w[\hat{y}](t){}_h\Delta^{\beta}_{b}\eta(t)\Delta t \\ &= -\int_a^b L_w[\hat{y}](t)({}_h\Delta^{-\nu}_{b}\eta(t-\nu h))^{\Delta}\Delta t \\ &= -\int_a^b L_w[\hat{y}](t)\left[{}_h\Delta^{-\nu}_{\rho(b)}\eta^{\Delta}(t-\nu h) - \frac{\nu}{\Gamma(\nu+1)}(b+\nu h-\sigma(t))_h^{(\nu-1)}\eta(b)\right]\Delta t \\ &= -\int_a^b L_w[\hat{y}](t){}_h\Delta^{-\nu}_{\rho(b)}\eta^{\Delta}(t-\nu h)\Delta t + \frac{\nu\eta(b)}{\Gamma(\nu+1)}\int_a^b (b+\nu h-\sigma(t))_h^{(\nu-1)}L_w[\hat{y}](t)\Delta t. \end{aligned} \tag{7.19}$$

We now use Lemma 145 to get

$$\begin{aligned}
&\int_a^b L_w[\hat{y}](t)_h\Delta_{\rho(b)}^{-\nu}\eta^\Delta(t-\nu h)\Delta t\\
&\quad=\int_a^b L_w[\hat{y}](t)\left[h^\nu\eta^\Delta(t)+\frac{\nu}{\Gamma(\nu+1)}\int_{\sigma(t)}^b(s+\nu h-\sigma(t))_h^{(\nu-1)}\eta^\Delta(s)\Delta s\right]\Delta t\\
&\quad=\int_a^b h^\nu L_w[\hat{y}](t)\eta^\Delta(t)\Delta t\\
&\qquad+\frac{\nu}{\Gamma(\nu+1)}\int_a^{\rho(b)}\left[L_w[\hat{y}](t)\int_{\sigma(t)}^b(s+\nu h-\sigma(t))_h^{(\nu-1)}\eta^\Delta(s)\Delta s\right]\Delta t \qquad (7.20)\\
&\quad=\int_a^b h^\nu L_w[\hat{y}](t)\eta^\Delta(t)\Delta t\\
&\qquad+\frac{\nu}{\Gamma(\nu+1)}\int_a^b\left[\eta^\Delta(t)\int_a^t(t+\nu h-\sigma(s))_h^{(\nu-1)}L_w[\hat{y}](s)\Delta s\right]\Delta t\\
&\quad=\int_a^b\eta^\Delta(t)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)\Delta t.
\end{aligned}$$

We apply again the time scale integration by parts formula (1.7), this time to (7.20), to obtain,

$$\begin{aligned}
&\int_a^b\eta^\Delta(t)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)\Delta t\\
&\quad=\int_a^{\rho(b)}\eta^\Delta(t)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)\Delta t\\
&\qquad+(\eta(b)-\eta(\rho(b)))_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)|_{t=\rho(b)}\\
&\quad=\left[\eta(t)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)\right]_{t=a}^{t=\rho(b)}-\int_a^{\rho(b)}\eta^\sigma(t)({}_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h))^\Delta\Delta t \qquad (7.21)\\
&\qquad+\eta(b)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)|_{t=\rho(b)}-\eta(\rho(b))_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)|_{t=\rho(b)}\\
&\quad=\eta(b)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)|_{t=\rho(b)}-\eta(a)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)|_{t=a}\\
&\qquad-\int_a^{\rho(b)}\eta^\sigma(t)_a\Delta_h^\beta\left(L_w[\hat{y}]\right)(t)\Delta t.
\end{aligned}$$

Since $\eta(a)=\eta(b)=0$ we obtain, from (7.20) and (7.21), that

$$\int_a^b L_w[\hat{y}](t)_h\Delta_{\rho(b)}^{-\nu}\eta^\Delta(t)\Delta t=-\int_a^{\rho(b)}\eta^\sigma(t)_a\Delta_h^\beta\left(L_w[\hat{y}]\right)(t)\Delta t\,,$$

and after inserting in (7.19), that

$$\int_a^b L_w[\hat{y}](t)_h\Delta_b^\beta\eta(t)\Delta t=\int_a^{\rho(b)}\eta^\sigma(t)_a\Delta_h^\beta\left(L_w[\hat{y}]\right)(t)\Delta t. \qquad (7.22)$$

By (7.18) and (7.22) we may write (7.17) as

$$\int_a^{\rho(b)}\left[L_u[\hat{y}](t)+{}_h\Delta_{\rho(b)}^\alpha\left(L_v[\hat{y}]\right)(t)+{}_a\Delta_h^\beta\left(L_w[\hat{y}]\right)(t)\right]\eta^\sigma(t)\Delta t=0\,.$$

Since the values of $\eta^\sigma(t)$ are arbitrary for $t \in \mathbb{T}^{\kappa^2}$, the Euler-Lagrange equation (7.15) holds along $\hat{y}$. □

The next result is a direct consequence of Theorem 151.

**Corollary 152** (The $h$-Euler–Lagrange equation – cf., e.g., [21])**.** *Let $\mathbb{T}$ be the time scale $h\mathbb{Z}$, $h > 0$, with the forward jump operator $\sigma$ and the delta derivative $\Delta$. Assume $a, b \in \mathbb{T}$, $a < b$. If $\hat{y}$ is a solution to the problem*

$$\mathcal{L}[y(\cdot)] = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t \longrightarrow \min, \ y(a) = A, \ y(b) = B\,,$$

*then the equality $L_u(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) - \left(L_v(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))\right)^\Delta = 0$ holds for all $t \in \mathbb{T}^{\kappa^2}$.*

*Proof.* Choose $\alpha = 1$ and a $L$ that does not depend on $w$ in Theorem 151. □

*Remark* 153. If we take $h = 1$ in Corollary 152 we have that

$$L_u(t, \hat{y}^\sigma(t), \Delta\hat{y}(t)) - \Delta L_v(t, \hat{y}^\sigma(t), \Delta\hat{y}(t)) = 0,$$

holds for all $t \in \mathbb{T}^{\kappa^2}$ (see also Theorem 34).

### Natural boundary conditions

If the initial condition $y(a) = A$ is not present in problem (7.14) (i.e., $y(a)$ is free), besides the $h$-fractional Euler–Lagrange equation (7.15) the following supplementary condition must be fulfilled:

$$\begin{aligned} - h^\gamma L_v[\hat{y}](a) + \frac{\gamma}{\Gamma(\gamma+1)} &\left(\int_a^b (t+\gamma h - a)_h^{(\gamma-1)} L_v[\hat{y}](t)\Delta t \right. \\ &\left. - \int_{\sigma(a)}^b (t+\gamma h - \sigma(a))_h^{(\gamma-1)} L_v[\hat{y}](t)\Delta t \right) + L_w[\hat{y}](a) = 0. \end{aligned} \tag{7.23}$$

Similarly, if $y(b) = B$ is not present in (7.14) ($y(b)$ is free), the extra condition

$$\begin{aligned} hL_u[\hat{y}](\rho(b)) &+ h^\gamma L_v[\hat{y}](\rho(b)) - h^\nu L_w[\hat{y}](\rho(b)) \\ &+ \frac{\nu}{\Gamma(\nu+1)} \left(\int_a^b (b+\nu h - \sigma(t))_h^{(\nu-1)} L_w[\hat{y}](t)\Delta t \right. \\ &\left. - \int_a^{\rho(b)} (\rho(b)+\nu h - \sigma(t))_h^{(\nu-1)} L_w[\hat{y}](t)\Delta t \right) = 0 \end{aligned} \tag{7.24}$$

is added to Theorem 151. The proofs of these facts are immediate and we don't write them here. We just note that the first term in (7.24) arises from the first term of the left-hand side of (7.17). Equalities (7.23) and (7.24) are called *natural boundary conditions*.

### Second order optimality condition

We now obtain a second order necessary condition for problem (7.14), i.e., we prove a Legendre optimality type condition for the fractional $h$-difference setting.

**Theorem 154** (The $h$-fractional Legendre necessary condition). *If $\hat{y} \in \mathcal{F}_{\mathbb{T}}$ is a local minimum for problem* (7.14)*, then the inequality*

$$\begin{aligned}
&h^2 L_{uu}[\hat{y}](t) + 2h^{\gamma+1} L_{uv}[\hat{y}](t) + 2h^{\nu+1}(\nu-1) L_{uw}[\hat{y}](t) + h^{2\gamma}(\gamma-1)^2 L_{vv}[\hat{y}](\sigma(t)) \\
&\quad + 2h^{\nu+\gamma}(\gamma-1) L_{vw}[\hat{y}](\sigma(t)) + 2h^{\nu+\gamma}(\nu-1) L_{vw}[\hat{y}](t) + h^{2\nu}(\nu-1)^2 L_{ww}[\hat{y}](t) \\
&\quad + h^{2\nu} L_{ww}[\hat{y}](\sigma(t)) + \int_a^t h^3 L_{ww}[\hat{y}](s) \left(\frac{\nu(1-\nu)}{\Gamma(\nu+1)}(t+\nu h-\sigma(s))_h^{(\nu-2)}\right)^2 \Delta s \\
&\quad + h^{\gamma} L_{vv}[\hat{y}](t) + \int_{\sigma(\sigma(t))}^b h^3 L_{vv}[\hat{y}](s) \left(\frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)}(s+\gamma h-\sigma(\sigma(t)))_h^{(\gamma-2)}\right)^2 \Delta s \geq 0
\end{aligned} \tag{7.25}$$

*holds for all $t \in \mathbb{T}^{\kappa^2}$, where $[\hat{y}](t) = (t, \hat{y}^\sigma(t), {}_a\Delta_t^\alpha \hat{y}(t), {}_t\Delta_b^\beta \hat{y}(t))$.*

*Proof.* By the hypothesis of the theorem, and letting $\Phi$ be as in (7.16), we have as necessary optimality condition that $\Phi''(0) \geq 0$ for an arbitrary admissible variation $\eta(\cdot)$. Inequality $\Phi''(0) \geq 0$ is equivalent to

$$\begin{aligned}
&\int_a^b \Big[L_{uu}[\hat{y}](t)(\eta^\sigma(t))^2 + 2L_{uv}[\hat{y}](t)\eta^\sigma(t)_a\Delta_h^\alpha \eta(t) + 2L_{uw}[\hat{y}](t)\eta^\sigma(t)_h\Delta_b^\beta \eta(t) \\
&\quad + L_{vv}[\hat{y}](t)({}_a\Delta_h^\alpha \eta(t))^2 + 2L_{vw}[\hat{y}](t)_a\Delta_h^\alpha \eta(t)_h\Delta_b^\beta \eta(t) + L_{ww}(t)({}_h\Delta_b^\beta \eta(t))^2\Big] \Delta t \geq 0.
\end{aligned} \tag{7.26}$$

Let $\tau \in \mathbb{T}^{\kappa^2}$ be arbitrary, and choose $\eta : \mathbb{T} \to \mathbb{R}$ given by

$$\eta(t) = \begin{cases} h & \text{if } t = \sigma(\tau); \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\eta(a) = \eta(b) = 0$, i.e., $\eta$ is an admissible variation. Using (7.5) we get

$$\begin{aligned}
&\int_a^b \left[L_{uu}[\hat{y}](t)(\eta^\sigma(t))^2 + 2L_{uv}[\hat{y}](t)\eta^\sigma(t)_a\Delta_h^\alpha \eta(t) + L_{vv}[\hat{y}](t)({}_a\Delta_h^\alpha \eta(t))^2\right] \Delta t \\
&\quad = \int_a^b \Bigg[L_{uu}[\hat{y}](t)(\eta^\sigma(t))^2 \\
&\qquad\qquad + 2L_{uv}[\hat{y}](t)\eta^\sigma(t)\left(h^\gamma \eta^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)} \eta^\Delta(s)\Delta s\right) \\
&\qquad\qquad + L_{vv}[\hat{y}](t)\left(h^\gamma \eta^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)} \eta^\Delta(s)\Delta s\right)^2\Bigg] \Delta t \\
&\quad = h^3 L_{uu}[\hat{y}](\tau) + 2h^{\gamma+2} L_{uv}[\hat{y}](\tau) + h^{\gamma+1} L_{vv}[\hat{y}](\tau) \\
&\qquad + \int_{\sigma(\tau)}^b L_{vv}[\hat{y}](t)\left(h^\gamma \eta^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)} \eta^\Delta(s)\Delta s\right)^2 \Delta t.
\end{aligned}$$

Observe that

$$
\begin{aligned}
&h^{2\gamma+1}(\gamma-1)^2 L_{vv}[\hat{y}](\sigma(\tau)) \\
&\qquad + \int_{\sigma^2(\tau)}^{b} L_{vv}[\hat{y}](t)\left(\frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s\right)^2 \Delta t \\
&\qquad = \int_{\sigma(\tau)}^{b} L_{vv}[\hat{y}](t)\left(h^{\gamma}\eta^{\Delta}(t)+\frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s\right)^2 \Delta t.
\end{aligned}
$$

Let $t \in [\sigma^2(\tau), \rho(b)] \cap h\mathbb{Z}$. Since

$$
\begin{aligned}
&\frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s \\
&\quad = \frac{\gamma}{\Gamma(\gamma+1)}\left[\int_a^{\sigma(\tau)} (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s \right.\\
&\qquad\qquad \left. + \int_{\sigma(\tau)}^{t} (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s\right] \\
&\quad = h\frac{\gamma}{\Gamma(\gamma+1)}\left[(t+\gamma h-\sigma(\tau))_h^{(\gamma-1)} - (t+\gamma h-\sigma(\sigma(\tau)))_h^{(\gamma-1)}\right] \\
&\quad = \frac{\gamma}{\Gamma(\gamma+1)}h\left[h^{\gamma-1}\frac{\Gamma\left(\frac{t-\tau}{h}+\gamma\right)}{\Gamma\left(\frac{t-\tau}{h}+1\right)} - h^{\gamma-1}\frac{\Gamma\left(\frac{t-\tau}{h}+\gamma-1\right)}{\Gamma\left(\frac{t-\tau}{h}\right)}\right] \\
&\quad = \frac{\gamma h^{\gamma}}{\Gamma(\gamma+1)}\left[\frac{\left(\frac{t-\tau}{h}+\gamma-1\right)\Gamma\left(\frac{t-\tau}{h}+\gamma-1\right) - \left(\frac{t-\tau}{h}\right)\Gamma\left(\frac{t-\tau}{h}+\gamma-1\right)}{\left(\frac{t-\tau}{h}\right)\Gamma\left(\frac{t-\tau}{h}\right)}\right] \\
&\quad = \frac{\gamma h^{\gamma}}{\Gamma(\gamma+1)}\left[\frac{(\gamma-1)\Gamma\left(\frac{t-\tau}{h}+\gamma-1\right)}{\Gamma\left(\frac{t-\tau}{h}+1\right)}\right] \\
&\quad = h^2\frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)}(t+\gamma h-\sigma(\sigma(\tau)))_h^{(\gamma-2)},
\end{aligned}
\tag{7.27}
$$

we conclude that

$$
\begin{aligned}
&\int_{\sigma^2(\tau)}^{b} L_{vv}[\hat{y}](t)\left(\frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s\right)^2 \Delta t \\
&\qquad\qquad = \int_{\sigma^2(\tau)}^{b} L_{vv}[\hat{y}](t)\left(h^2\frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)}(t+\gamma h-\sigma^2(\tau))_h^{(\gamma-2)}\right)^2 \Delta t.
\end{aligned}
$$

Note that we can write ${}_t\Delta_b^{\beta}\eta(t) = -{}_h\Delta_{\rho(b)}^{-\nu}\eta^{\Delta}(t-\nu h)$ because $\eta(b)=0$. It is not difficult to see that the following equality holds:

$$
\begin{aligned}
\int_a^b 2L_{uw}[\hat{y}](t)\eta^{\sigma}(t)_h\Delta_b^{\beta}\eta(t)\Delta t &= -\int_a^b 2L_{uw}[\hat{y}](t)\eta^{\sigma}(t)_h\Delta_{\rho(b)}^{-\nu}\eta^{\Delta}(t-\nu h)\Delta t \\
&= 2h^{2+\nu}L_{uw}[\hat{y}](\tau)(\nu-1)\,.
\end{aligned}
$$

Moreover,

$$\begin{aligned}
&\int_a^b 2L_{vw}[\hat{y}](t)_a\Delta_h^\alpha\eta(t)_h\Delta_b^\beta\eta(t)\Delta t\\
&\quad= -2\int_a^b L_{vw}[\hat{y}](t)\left\{\left(h^\gamma\eta^\Delta(t)+\frac{\gamma}{\Gamma(\gamma+1)}\cdot\int_a^t(t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^\Delta(s)\Delta s\right)\right.\\
&\qquad\qquad\left.\cdot\left[h^\nu\eta^\Delta(t)+\frac{\nu}{\Gamma(\nu+1)}\int_{\sigma(t)}^b(s+\nu h-\sigma(t))_h^{(\nu-1)}\eta^\Delta(s)\Delta s\right]\right\}\Delta t\\
&\quad= 2h^{\gamma+\nu+1}(\nu-1)L_{vw}[\hat{y}](\tau)+2h^{\gamma+\nu+1}(\gamma-1)L_{vw}[\hat{y}](\sigma(\tau)).
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
&\int_a^b L_{ww}[\hat{y}](t)({}_h\Delta_b^\beta\eta(t))^2\Delta t\\
&= \int_a^b L_{ww}[\hat{y}](t)\left[h^\nu\eta^\Delta(t)+\frac{\nu}{\Gamma(\nu+1)}\int_{\sigma(t)}^b(s+\nu h-\sigma(t))_h^{(\nu-1)}\eta^\Delta(s)\Delta s\right]^2\Delta t\\
&= \int_a^{\sigma(\sigma(\tau))} L_{ww}[\hat{y}](t)\left[h^\nu\eta^\Delta(t)+\frac{\nu}{\Gamma(\nu+1)}\int_{\sigma(t)}^b(s+\nu h-\sigma(t))_h^{(\nu-1)}\eta^\Delta(s)\Delta s\right]^2\Delta t\\
&= \int_a^\tau L_{ww}[\hat{y}](t)\left[\frac{\nu}{\Gamma(\nu+1)}\int_{\sigma(t)}^b(s+\nu h-\sigma(t))_h^{(\nu-1)}\eta^\Delta(s)\Delta s\right]^2\Delta t\\
&\qquad + hL_{ww}[\hat{y}](\tau)(h^\nu-\nu h^\nu)^2+h^{2\nu+1}L_{ww}[\hat{y}](\sigma(\tau))\\
&= \int_a^\tau L_{ww}[\hat{y}](t)\left[h\frac{\nu}{\Gamma(\nu+1)}\left\{(\tau+\nu h-\sigma(t))_h^{(\nu-1)}-(\sigma(\tau)+\nu h-\sigma(t))_h^{(\nu-1)}\right\}\right]^2\\
&\qquad + hL_{ww}[\hat{y}](\tau)(h^\nu-\nu h^\nu)^2+h^{2\nu+1}L_{ww}[\hat{y}](\sigma(\tau)).
\end{aligned}$$

Similarly as we did in (7.27), we can prove that

$$\begin{aligned}
&h\frac{\nu}{\Gamma(\nu+1)}\left\{(\tau+\nu h-\sigma(t))_h^{(\nu-1)}-(\sigma(\tau)+\nu h-\sigma(t))_h^{(\nu-1)}\right\}\\
&\qquad\qquad\qquad\qquad = h^2\frac{\nu(1-\nu)}{\Gamma(\nu+1)}(\tau+\nu h-\sigma(t))_h^{(\nu-2)}.
\end{aligned}$$

Thus, we have that inequality (7.26) is equivalent to

$$\begin{aligned}
h\Bigg\{&h^2L_{uu}[\hat{y}](t)+2h^{\gamma+1}L_{uv}[\hat{y}](t)+h^\gamma L_{vv}[\hat{y}](t)+L_{vv}(\sigma(t))(\gamma h^\gamma-h^\gamma)^2\\
&+\int_{\sigma(\sigma(t))}^b h^3L_{vv}(s)\left(\frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)}(s+\gamma h-\sigma(\sigma(t)))_h^{(\gamma-2)}\right)^2\Delta s\\
&+2h^{\nu+1}L_{uw}[\hat{y}](t)(\nu-1)+2h^{\gamma+\nu}(\nu-1)L_{vw}[\hat{y}](t)\\
&+2h^{\gamma+\nu}(\gamma-1)L_{vw}(\sigma(t))+h^{2\nu}L_{ww}[\hat{y}](t)(1-\nu)^2+h^{2\nu}L_{ww}[\hat{y}](\sigma(t))\\
&+\int_a^t h^3L_{ww}[\hat{y}](s)\left(\frac{\nu(1-\nu)}{\Gamma(\nu+1)}(t+\nu h-\sigma(s))^{\nu-2}\right)^2\Delta s\Bigg\}\geq 0. \qquad (7.28)
\end{aligned}$$

Because $h > 0$, (7.28) is equivalent to (7.25). The theorem is proved. □

The next result is a simple corollary of Theorem 154.

**Corollary 155** (The $h$-Legendre necessary condition – cf. Result 1.3 of [21]). *Let $\mathbb{T}$ be the time scale $h\mathbb{Z}$, $h > 0$, with the forward jump operator $\sigma$ and the delta derivative $\Delta$. Assume $a, b \in \mathbb{T}$, $a < b$. If $\hat{y}$ is a solution to the problem*

$$\mathcal{L}[y(\cdot)] = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t \longrightarrow \min, \;\; y(a) = A, \;\; y(b) = B\,,$$

*then the inequality*

$$h^2 L_{uu}[\hat{y}](t) + 2hL_{uv}[\hat{y}](t) + L_{vv}[\hat{y}](t) + L_{vv}[\hat{y}](\sigma(t)) \geq 0 \tag{7.29}$$

*holds for all $t \in \mathbb{T}^{\kappa^2}$, where $[\hat{y}](t) = (t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))$.*

*Proof.* Choose $\alpha = 1$ and a Lagrangian $L$ that does not depend on $w$. Then, $\gamma = 0$ and the result follows immediately from Theorem 154. □

*Remark* 156. When $h \to 0$ we get $\sigma(t) = t$ and inequality (7.29) coincides with Legendre's classical necessary optimality condition $L_{vv}[\hat{y}](t) \geq 0$ (cf. Theorem 33).

## 7.4 Examples

In this section we present some illustrative examples. Calculations were done using Maxima Software.

**Example 157.** Let us consider the following problem:

$$\mathcal{L}[y(\cdot)] = \frac{1}{2}\int_0^1 \left({}_0\Delta_h^{\frac{3}{4}} y(t)\right)^2 \Delta t \longrightarrow \min\,, \quad y(0) = 0\,, \quad y(1) = 1\,. \tag{7.30}$$

We consider (7.30) with different values of $h$. Numerical results show that when $h$ tends to zero the $h$-fractional Euler–Lagrange extremal tends to the fractional continuous extremal: when $h \to 0$ (7.30) tends to the fractional continuous variational problem in the Riemann–Liouville sense studied in [4, Example 1], with solution given by

$$y(t) = \frac{1}{2}\int_0^t \frac{dx}{[(1-x)(t-x)]^{\frac{1}{4}}}\,. \tag{7.31}$$

This is illustrated in Figure 7.1. In this example for each value of $h$ there is a unique $h$-fractional Euler–Lagrange extremal, solution of (7.15), which always verifies the $h$-fractional Legendre necessary condition (7.25).

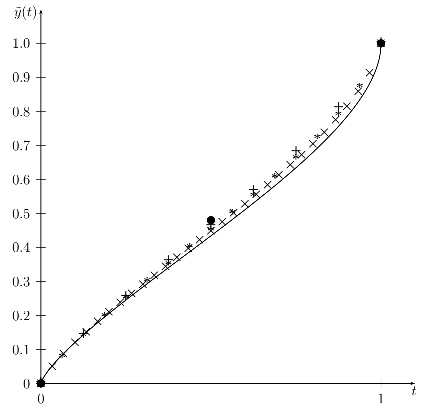


Figure 7.1: Extremal $\tilde{y}(t)$ for problem of Example 157 with different values of $h$: $h = 0.50$ ($\bullet$); $h = 0.125$ ($+$); $h = 0.0625$ ($*$); $h = 1/30$ ($\times$). The continuous line represent function (7.31).

**Example 158.** Let us consider the following problem:

$$\mathcal{L}[y(\cdot)] = \int_0^1 \left[\frac{1}{2}\left({}_0\Delta_h^\alpha y(t)\right)^2 - y^\sigma(t)\right]\Delta t \longrightarrow \min\,, \quad y(0) = 0\,, \quad y(1) = 0\,. \tag{7.32}$$

We begin by considering problem (7.32) with a fixed value for $\alpha$ and different values of $h$. The extremals $\tilde{y}$ are obtained using our Euler–Lagrange equation (7.15). As in Example 157 the numerical results show that when $h$ tends to zero the extremal of the problem tends to the extremal of the corresponding continuous fractional problem of the calculus of variations in the Riemann–Liouville sense. More precisely, when $h$ approximates zero problem (7.32) tends to the fractional continuous problem studied in [5, Example 2]. For $\alpha = 1$ and $h \to 0$ the extremal of (7.32) is given by $y(t) = \frac{1}{2}t(1-t)$, which coincides with the extremal of the classical problem of the calculus of variations

$$\mathcal{L}[y(\cdot)] = \int_0^1 \left(\frac{1}{2}y'(t)^2 - y(t)\right) dt \longrightarrow \min\,, \quad y(0) = 0\,, \quad y(1) = 0\,.$$

This is illustrated in Figure 7.2 for $h = \frac{1}{2^i}$, $i = 1, 2, 3, 4$. In this example, for each value of $\alpha$ and $h$, we only have one extremal (we only have one solution to (7.15) for each $\alpha$ and $h$). Our Legendre condition (7.25) is always verified along the extremals. Figure 7.3 shows the extremals of problem (7.32) for a fixed value of $h$ ($h = 1/20$) and different values of $\alpha$. The numerical results show that when $\alpha$ tends to one the extremal tends to the solution of the classical (integer order) discrete-time problem.

Our last example shows that the $h$-fractional Legendre necessary optimality condition can be a very useful tool. In Example 159 we consider a problem for which the $h$-fractional Euler–Lagrange equation gives several candidates but just a few of them verify the Legendre condition (7.25).

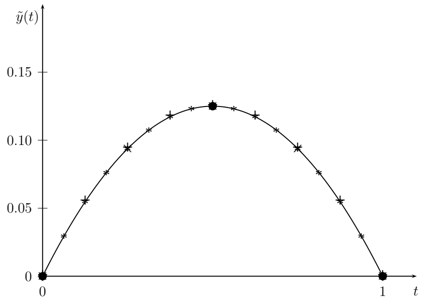


Figure 7.2: Extremal $\tilde{y}(t)$ for problem (7.32) with $\alpha = 1$ and different values of $h$: $h = 0.5$ ($\bullet$); $h = 0.25$ ($\times$); $h = 0.125$ ($+$); $h = 0.0625$ ($*$).

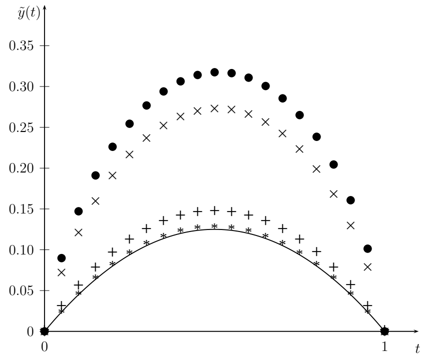


Figure 7.3: Extremal $\tilde{y}(t)$ for (7.32) with $h = 0.05$ and different values of $\alpha$: $\alpha = 0.70$ ($\bullet$); $\alpha = 0.75$ ($\times$); $\alpha = 0.95$ ($+$); $\alpha = 0.99$ ($*$). The continuous line is $y(t) = \frac{1}{2}t(1-t)$.

| # | $\tilde{y}\left(\frac{1}{4}\right)$ | $\tilde{y}\left(\frac{1}{2}\right)$ | $\tilde{y}\left(\frac{3}{4}\right)$ | $\mathcal{L}(\tilde{y})$ | Legendre condition (7.25) |
|---|---|---|---|---|---|
| 1 | -0.5511786 | 0.0515282 | 0.5133134 | 9.3035911 | Not verified |
| 2 | 0.2669091 | 0.4878808 | 0.7151924 | 2.0084203 | Verified |
| 3 | -2.6745703 | 0.5599360 | -2.6730125 | 698.4443232 | Not verified |
| 4 | 0.5789976 | 1.0701515 | 0.1840377 | 12.5174960 | Not verified |
| 5 | 1.0306820 | 1.8920322 | 2.7429222 | -32.7189756 | Verified |
| 6 | 0.5087946 | -0.1861431 | 0.4489196 | 10.6730959 | Not verified |
| 7 | 4.0583690 | -1.0299054 | -5.0030989 | 2451.7637948 | Not verified |
| 8 | -1.7436106 | -3.1898449 | -0.8850511 | 238.6120299 | Not verified |

Table 7.1: There exist 8 Euler-Lagrange extremals for problem (7.33) with $\alpha = 0.8$, $\beta = 0.5$, $h = 0.25$, $a = 0$, $b = 1$, and $\theta = 1$, but only 2 of them satisfy the fractional Legendre condition (7.25).

**Example 159.** Let us consider the following problem:

$$\mathcal{L}[y(\cdot)] = \int_a^b \left({}_a\Delta_h^\alpha y(t)\right)^3 + \theta\left({}_h\Delta_b^\alpha y(t)\right)^2 \Delta t \longrightarrow \min\,, \quad y(a) = 0\,, \quad y(b) = 1\,. \tag{7.33}$$

For $\alpha = 0.8$, $\beta = 0.5$, $h = 0.25$, $a = 0$, $b = 1$, and $\theta = 1$, problem (7.33) has eight different Euler–Lagrange extremals. As we can see on Table 7.1 only two of the candidates verify the Legendre condition. To determine the best candidate we compare the values of the functional $\mathcal{L}$ along the two good candidates. The extremal we are looking for is given by the candidate number five on Table 7.1.

For problem (7.33) with $\alpha = 0.3$, $h = 0.1$, $a = 0$, $b = 0.5$, and $\theta = 0$, we obtain the results of Table 7.2: there exist sixteen Euler–Lagrange extremals but only one satisfy the fractional Legendre condition. The extremal we are looking for is given by the candidate number six on Table 7.2.

| # | $\tilde{y}(0.1)$ | $\tilde{y}(0.2)$ | $\tilde{y}(0.3)$ | $\tilde{y}(0.4)$ | $\mathcal{L}(\tilde{y})$ | (7.25) |
|---|---|---|---|---|---|---|
| 1 | -0.305570704 | -0.428093486 | 0.223708338 | 0.480549114 | 12.25396166 | No |
| 2 | -0.427934654 | -0.599520948 | 0.313290997 | -0.661831134 | 156.2317667 | No |
| 3 | 0.284152257 | -0.227595659 | 0.318847274 | 0.531827387 | 8.669645848 | No |
| 4 | -0.277642565 | 0.222381632 | 0.386666793 | 0.555841555 | 6.993518478 | No |
| 5 | 0.387074742 | -0.310032839 | 0.434336603 | -0.482903047 | 110.7912605 | No |
| 6 | 0.259846344 | 0.364035314 | 0.463222456 | 0.597907505 | 5.104389191 | Yes |
| 7 | -0.375094681 | 0.300437245 | 0.522386246 | -0.419053781 | 93.95316858 | No |
| 8 | 0.343327771 | 0.480989769 | 0.61204299 | -0.280908953 | 69.23497954 | No |
| 9 | 0.297792192 | 0.417196073 | -0.218013689 | 0.460556635 | 14.12227593 | No |
| 10 | 0.41283304 | 0.578364133 | -0.302235104 | -0.649232892 | 157.8272685 | No |
| 11 | -0.321401682 | 0.257431098 | -0.360644857 | 0.400971272 | 19.87468886 | No |
| 12 | 0.330157414 | -0.264444122 | -0.459803086 | 0.368850105 | 24.84475504 | No |
| 13 | -0.459640837 | 0.368155651 | -0.515763025 | -0.860276767 | 224.9964788 | No |
| 14 | -0.359429958 | -0.50354835 | -0.640748011 | 0.294083676 | 34.43515839 | No |
| 15 | 0.477760586 | -0.382668914 | -0.66536683 | -0.956478654 | 263.3075289 | No |
| 16 | -0.541587541 | -0.758744525 | -0.965476394 | -1.246195157 | 392.9592508 | No |

Table 7.2: There exist 16 Euler-Lagrange extremals for problem (7.33) with $\alpha = 0.3$, $h = 0.1$, $a = 0$, $b = 0.5$, and $\theta = 0$, but only 1 (candidate #6) satisfy the fractional Legendre condition (7.25).

## 7.5 State of the Art

Discrete fractional calculus is a theory that is in its infancy. To the best of the author's knowledge there is no research paper providing any kind of result for the time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Nevertheless, the three papers [15, 16, 76] are already published and contain some basic results for the time scale $\mathbb{T} = \mathbb{Z}$ and also some methods to solve initial value problems with discrete left-fractional derivatives. None of these papers contain results involving any subject related with the calculus of variations. We have already two research papers [17, 18] within this topic and further investigation is being undertaken by Nuno R. O. Bastos in his PhD thesis under the supervision of Delfim F. M. Torres.

The results of this chapter were presented at the ICDEA 2009, International Conference on Difference Equations and Applications, Lisbon, Portugal, October 19–23, 2009, in a contributed talk entitled *Calculus of Variations with Discrete Fractional Derivatives*.

# Chapter 8

# Conclusions and future work

This thesis had two major objectives: the development of the calculus of variations on a general time scale and the development of a discrete fractional calculus of variations theory.

For the calculus of variations on time scales we contributed with a necessary optimality condition for the problem depending on more than one $\Delta$-derivative (cf. Theorem 57) and also for the isoperimetric problem (cf. Theorem 63). Some Sturm–Liouville eigenvalue problems and their relation with isoperimetric problems were also considered (cf. Section 4.2.1). In a first step towards the development of direct methods for the calculus of variations on a general time scale we derived some useful integral inequalities[1] and solved some variational problems (see Section 5.2). As an "outsider" consequence of the herein obtained integral inequalities we were able to prove existence of solution to an integrodynamic equation (cf. Theorem 105). In fact, it is our purpose to study the existence of solutions to boundary value problems of the form

$$\begin{aligned} y^{\Delta^2}(t) &= f(t, y^\sigma(t), y^\Delta(t)), \quad t \in [a,b]_{\mathbb{T}}^{\kappa^2}, \\ y(a) &= A, \quad y(b) = B, \end{aligned} \tag{8.1}$$

and apply the possible obtained results to prove existence of solution to some classes of Euler–Lagrange equations (remember that the Euler–Lagrange equation is a second order dynamic equation). To the best of our knowledge no such a study exists for a general time scale (we don't even are aware of such a study for discrete Euler–Lagrange equations).

*Remark* 160. While studying integral inequalities and its applications we were able to derive some results which are not within the scope of the calculus of variations and optimal control. This work is published in the three research papers [41, 42, 47].

We would also like to give answer to a deeper question: find the Euler–Lagrange equation for the higher-order calculus of variations problem without restriction (H) on the forward jump operator. We have some work in progress towards this end, namely, we obtain in [50],

[1] We remark that the Gronwall–Bellman–Bihari type inequalities presented in Section 6.4 for functions of two variables can be very useful in studying partial delta dynamic equations.

using the weak maximum principle on time scales [62], the desired Euler–Lagrange equation but in integral form.

With respect to the discrete fractional calculus theory we proved some properties for the fractional sum and difference operators in Section 7.2 and then, in Section 7.3, we proved the Euler–Lagrange equation as well as Legendre's necessary condition. We also indicate, in Section 7.4, that as the parameter $h$ tends to zero, the $h$-fractional Euler–Lagrange extremal seems to converge to the continuous fractional one. This indicates that the discrete extremal can be used to approximate the continuous one. One of the subjects that we are willing to study within the discrete fractional calculus is the existence of solutions to the Euler–Lagrange equation (7.15). Note that this equation contains both the left and the right discrete fractional derivatives. Since we obtained so far two main results, there is plenty to do in the development of the theory of discrete fractional calculus of variations. For example, we want to discover what one gets with a general variable endpoint variational problem. One can also think in obtaining some criteria in order to establish a sufficient condition. Moreover, we want to find if it is possible to make an analogue study to that done in Chapter 5 in order to solve directly some discrete fractional variational problems. We think that this is a fruitful area with much to be done, in both of theoretical and practical directions, and therefore we would like to make part of the efforts to accomplish it.

To close this manuscript it follows a list of the author publications during his PhD thesis: [11, 41, 42, 43, 44, 45, 46, 47, 48, 49].

# References


[1] R. Agarwal, M. Bohner, D. O'Regan, A. Peterson. *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. **141** (2002), no. 1-2, 1–26. {81}

[2] R. Agarwal, M. Bohner and A. Peterson, Inequalities on time scales: a survey, Math. Inequal. Appl. **4** (2001), no. 4, 535–557. {20,21,76}

[3] R. Agarwal, M. Bohner and P. J. Y. Wong, Sturm–Liouville eigenvalue problems on time scales, Appl. Math. Comput. **99** (1999), no. 2-3, 153–166. {37}

[4] O. P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, J. Math. Anal. Appl. **272** (2002), no. 1, 368–379. {86,101}

[5] O. P. Agrawal, A general finite element formulation for fractional variational problems, J. Math. Anal. Appl. **337** (2008), no. 1, 1–12. {102}

[6] C. D. Ahlbrandt and B. J. Harmsen, Discrete versions of continuous isoperimetric problems, J. Differ. Equations Appl. **3** (1998), no. 5-6, 449–462. {33}

[7] C. D. Ahlbrandt and A. C. Peterson. Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations, volume 16 of Kluwer Texts in the Mathematical Sciences. Kluwer Academic Publishers, Boston, 1996. {17}

[8] E. Akin-Bohner, M. Bohner and F. Akin, Pachpatte inequalities on time scales, JIPAM. J. Inequal. Pure Appl. Math. **6** (2005), no. 1, Article 6, 23 pp. (electronic). {53,76}

[9] R. Almeida and D. F. M. Torres, Calculus of variations with fractional derivatives and fractional integrals, Appl. Math. Lett. **22** (2009), no. 12, 1816–1820. {86}

[10] R. Almeida and D. F. M. Torres, Isoperimetric problems on time scales with nabla derivatives, Journal of Vibration and Control, Vol. 15 (2009), no. 6, 951–958. {36,38}

[11] M. R. Sidi Ammi, Rui A. C. Ferreira and D. F. M. Torres, Diamond-$\alpha$ Jensen's Inequality on Time Scales, Journal of Inequalities and Applications, vol. 2008, Article ID 576876, 13 pages, 2008. {69,83,106}

[12] M. R. Sidi Ammi and D. F. M. Torres, Combined dynamic Grüss inequalities on time scales, Journal of Mathematical Sciences, Vol. 161, No. 6, 2009, 792–802. {83}

[13] M. R. Sidi Ammi and D. F. M. Torres, Hölder's and Hardy's two Dimensional Diamond-alpha Inequalities on Time Scales. An. Univ. Craiova Ser. Mat. Inform. 37 (2010), no. 1, 1–11. {83}

[14] F. M. Atici, D. C. Biles and A. Lebedinsky, An application of time scales to economics, Math. Comput. Modelling **43** (2006), no. 7-8, 718–726. {1,2}

[15] F. M. Atici and P. W. Eloe, A transform method in discrete fractional calculus, Int. J. Difference Equ. **2** (2007), no. 2, 165–176. {2,86,104}

[16] F. M. Atici and P. W. Eloe, Initial value problems in discrete fractional calculus, Proc. Amer. Math. Soc. **137** (2009), no. 3, 981–989. {2,86,104}

[17] N. R. O. Bastos, R. A. C. Ferreira and D. F. M. Torres, The Fractional Difference Calculus of Variations, Symposium on Fractional Signals and Systems Lisbon09, M. Ortigueira et. al. (eds.), Lisbon, Portugal, November 4-6, 2009, 15pp. {104}

[18] N. R. O. Bastos, R. A. C. Ferreira and D. F. M. Torres, *Discrete-time fractional variational problems*, Signal Processing, 2010, doi: 10.1016/j.sigpro.2010.05.001. {87,104}

[19] P. R. Beesack, On some Gronwall-type integral inequalities in $n$ independent variables, J. Math. Anal. Appl. **100** (1984), no. 2, 393–408. {56}

[20] I. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, Acta Math. Acad. Sci. Hungar. **7** (1956), 81–94. {20}

[21] M. Bohner. *Calculus of variations on time scales*. Dynam. Systems Appl. **13** (2004), no. 3-4, pp. 339–349. {1,17,18,36,97,101}

[22] M. Bohner, R. A. C. Ferreira and D. F. M. Torres, Integral Inequalities and their Applications to the Calculus of Variations on Time Scales, Math. Inequal. Appl. 13 (2010), no. 3, 511–522. {40,42,43,44,46,47,48,49}

[23] M. Bohner and G. Sh. Guseinov, Double integral calculus of variations on time scales, Comput. Math. Appl. **54** (2007), no. 1, 45–57. {38}

[24] M. Bohner and A. Peterson, *Dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2001. {1,8,9,10,11,12,13,32,86}

[25] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2003. {1,11,13}

[26] M. Bohner and O. Duman. Opial type inequalities for diamond-alpha derivatives and integrals on time scales. Differ. Equ. Dyn. Syst., 2009, in press. {83}

[27] A. M. C. Brito da Cruz, H. S. Rodrigues and D. F. M. Torres, Escalas Temporais e Mathematica, Bol. Soc. Port. Mat. No. 62 (2010), 1–18. {8}

[28] R. F. Brown, *A topological introduction to nonlinear analysis*, Birkhäuser Boston, Boston, MA, 1993. {61}

[29] M. C. Caputo, Time scales: from nabla calculus to delta calculus and viceversa via duality, `arXiv:0910.0085` {8}

[30] W.-S. Cheung, Integral inequalities and applications to the calculus of variations, Int. J. Appl. Math. **9** (2002), no. 1, 85–108. {39,46,47,48}

[31] A. Constantin, Solutions globales d'equations differentielles perturbées, C. R. Acad. Sci. Paris, **320** (1995), 1319-1322. {65}

[32] A. Constantin, Topological transversality: Application to an integro-differential equation, *J. Math. Anal. Appl.*, **197**(1) (1996), 855-863. {64,68}

[33] A. Constantin, Monotone iterative technique for a nonlinear integral equation, J. Math. Anal. Appl. **205** (1997), no. 1, 280–283. {68}

[34] A. Constantin, Nonlinear alternative: application to an integral equation, J. Appl. Anal. **5** (1999), no. 1, 119–123. {68}

[35] M. Denche and H. Khellaf, Integral inequalities similar to Gronwall inequality, Electron. J. Differential Equations **2007**, No. 176, 14 pp. (electronic). {54}

[36] C. Dinu, A weighted Hermite Hadamard inequality for Steffensen-Popoviciu and Hermite-Hadamard weights on time scales, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **17** (2009), no. 1, 77–90. {83}

[37] C. Dinu, Hermite-Hadamard inequality on time scales, J. Inequal. Appl. **2008**, Art. ID 287947, 24 pp. {83}

[38] O. Dunkel, Integral Inequalities With Applications to the Calculus of Variations, Amer. Math. Monthly **31** (1924), no. 7, 326–337. {39}

[39] R. A. El–Nabulsi and D. F. M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order $(\alpha, \beta)$, Math. Methods Appl. Sci. **30** (2007), no. 15, 1931–1939. {86}

[40] R. A. El–Nabulsi and D. F. M. Torres, Fractional actionlike variational problems, J. Math. Phys. **49** (2008), no. 5, 053521, 7 pp. {86}

[41] R. A. C. Ferreira, Constantin integral inequalities on time scales, Arch. Math. (Basel) **93** (2009), no. 2, 153–163. {83,105,106}

[42] R. A. C. Ferreira, Existence of solutions to some retarded integrodifferential equations via the topological transversality theorem, Nonlinear Analysis **72** (2010), pp. 3062–3068. doi: 10.1016/j.na.2009.11.045 {83,105,106}

[43] R. A. C. Ferreira. *Existence of solution to an integrodynamic equation*, Dynam. Systems Appl., to appear. {83,106}

[44] R. A. C. Ferreira and D. F. M. Torres, Remarks on the Calculus of Variations on time scales, Int. J. Ecol. Econ. Stat., Vol. 9, Nr. F07, 2007, pp. 65–73. {106}

[45] R. A. C. Ferreira and D. F. M. Torres, Higher-order calculus of variations on time scales, in *Mathematical control theory and finance*, 149–159, Springer, Berlin. {25,27,31,38,106}

[46] R. A. C. Ferreira and D. F. M. Torres, Generalizations of Gronwall-Bihari inequalities on time scales, J. Difference Equ. Appl. **15** (2009), no. 6, 529–539. {51,52,54,55,56,58,59,83,106}

[47] R. A. C. Ferreira and D. F. M. Torres, Generalized retarded integral inequalities, Appl. Math. Lett. **22** (2009), no. 6, 876–881. {83,105,106}

[48] R. A. C. Ferreira and D. F. M. Torres, Some linear and nonlinear integral inequalities on time scales in two independent variables, Nonlinear Dyn. Syst. Theory **9** (2009), no. 2, 161–169. {76,78,80,82,83,106}

[49] R. A. C. Ferreira and D. F. M. Torres, Isoperimetric problems of the calculus of variations on time scales, Nonlinear Analysis and Optimization II. Optimization, pp. 123–131, Contemp. Math., Amer. Math. Soc., Providence, RI, 2010. {33,34,38,106}

[50] R. A. C. Ferreira, A. B. Malinowska and D. F. M. Torres, Optimality conditions for the calculus of variations with higher-order delta derivatives, submitted. {105}

[51] G. B. Folland, *Real analysis*, Second edition, Wiley, New York, 1999. {70}

[52] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether's theorem for fractional problems of the calculus of variations, J. Math. Anal. Appl. **334** (2007), no. 2, 834–846. {86}

[53] G. S. F. Frederico and D. F. M. Torres, Fractional conservation laws in optimal control theory, Nonlinear Dynam. **53** (2008), no. 3, 215–222. {86}

[54] A. Granas and J. Dugundji, *Fixed point theory*, Springer, New York, 2003. {61}

[55] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann. of Math. (2) **20** (1919), no. 4, 292–296. {19}

[56] G. H. Hardy, J. E. Littlewood, Some integral inequalities connected with the calculus of variations, Q. J. Math., Oxf. Ser. **3** (1932), 241–252. {39}

[57] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, 1952. {39}

[58] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, Results Math. **18** (1990), no. 1-2, 18–56. {8,21,65}

[59] S. Hilger, Differential and difference calculus—unified!, Nonlinear Anal. **30** (1997), no. 5, 2683–2694. {8}

[60] S. Hilger. *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*. PhD thesis, Universität Würzburg, 1988. {1}

[61] R. Hilscher, V. Zeidan. *Calculus of variations on time scales: weak local piecewise $C^1_{\mathrm{rd}}$ solutions with variable endpoints*. J. Math. Anal. Appl. **289** (2004), no. 1, pp. 143–166. {1}

[62] R. Hilscher, V. Zeidan. R. Hilscher and V. Zeidan, Weak maximum principle and accessory problem for control problems on time scales, Nonlinear Anal. **70** (2009), no. 9, 3209–3226. {38,106}

[63] T. Jankowski, The generalized quasilinearization for integro-differential equations of Volterra type on time scales, Rocky Mountain J. Math. **37** (2007), no. 3, 851–864. {60}

[64] W. G. Kelley and A. C. Peterson, *Difference equations*, Academic Press, Boston, MA, 1991. {19}

[65] H. Khellaf, On integral inequalities for functions of several independent variables, Electron. J. Differential Equations **2003**, No. 123, 12 pp. (electronic). {77}

[66] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006. {86}

[67] T. Kulik and C. C. Tisdell, Volterra integral equations on time scales: Basic qualitative and quantitative results with applications to initial value problems on unbounded domains. Int. J. Difference Equ. 3 (2008), no. 1, 103–133. {60}

[68] W. Liu, Quốc Anh Ngô and W. Chen, A perturbed Ostrowski-type inequality on time scales for $k$ points for functions whose second derivatives are bounded, J. Inequal. Appl. **2008**, Art. ID 597241, 12 pp. {83}

[69] A. B. Malinowska and Delfim F. M. Torres, Necessary and sufficient conditions for local Pareto optimality on time scales, Journal of Mathematical Sciences, Vol. 161, No. 6, 2009, 803–810. {36}

[70] A. B. Malinowska and D. F. M. Torres, On the diamond-alpha Riemann integral and mean value theorems on time scales, Dynamic Systems and Applications, Volume 18 (2009), no. 3-4, pp. 469–482. {69}

[71] A. B. Malinowska and D. F. M. Torres, Strong minimizers of the calculus of variations on time scales and the Weierstrass condition, Proceedings of the Estonian Academy of Sciences, Vol. 58, no. 4, 2009, 205–212. {38}

[72] A. B. Malinowska and D. F. M. Torres, The delta-nabla calculus of variations, Fasc. Math. 44 (2010), 75–83. {38}

[73] A. B. Malinowska and Delfim F. M. Torres, Leitmann's direct method of optimization for absolute extrema of certain problems of the calculus of variations on time scales, Applied Mathematics and Computation, 2010, in press. doi: 10.1016/j.amc.2010.01.015 {49}

[74] N. Martins, D. F. M. Torres, Calculus of variations on time scales with nabla derivatives, Nonlinear Analysis: Theory, Methods & Applications (2009), no.12, pp. e763–e773. {27,38}

[75] Mikeladze, Sh.E., De la résolution numérique des équations intégrales, Bull.Acad.Sci.URSS VII (1935), 255–257 (in Russian). {19}

[76] K. S. Miller and B. Ross, Fractional difference calculus, in *Univalent functions, fractional calculus, and their applications (Kōriyama, 1988)*, 139–152, Horwood, Chichester, 1989. {2,85,104}

[77] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993. {2,85,86,88}

[78] D. S. Mitrinović, *Analytic inequalities*, Springer, New York, 1970. {20}

[79] W. Mydlarczyk and W. Okrasiński, Nonlinear Volterra integral equations with convolution kernels, Bull. London Math. Soc. **35** (2003), no. 4, 484–490. {67}

[80] J. A. Oguntuase, On an inequality of Gronwall, JIPAM. J. Inequal. Pure Appl. Math. **2** (2001), no. 1, Article 9, 6 pp. (electronic). {55}

[81] W. Okrasiński, On the existence and uniqueness of nonnegative solutions of a certain nonlinear convolution equation, Ann. Polon. Math. **36** (1979), no. 1, 61–72. {68}

[82] W. Okrasiński, On a nonlinear convolution equation occurring in the theory of water percolation, Ann. Polon. Math. **37** (1980), no. 3, 223–229. {68}

[83] S. A. Özgün, A. Zafer and B. Kaymakçalan, Gronwall and Bihari type inequalities on time scales, in *Advances in difference equations (Veszprém, 1995)*, 481–490, Gordon and Breach, Amsterdam, 1997. {20,76}

[84] U. M. Özkan, M. Z. Sarikaya and H. Yildirim, Extensions of certain integral inequalities on time scales, Appl. Math. Lett. **21** (2008), no. 10, 993–1000. {20}

[85] B. G. Pachpatte, *Integral and finite difference inequalities and applications*, Mathematics studies 205, Elsevier, 2006. {19}

[86] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, CA, 1999. {86}

[87] J. W. Rogers Jr. and Q. Sheng, Notes on the diamond-$\alpha$ dynamic derivative on time scales, J. Math. Anal. Appl., **326** (2007), no. 1, 228-241. {69}

[88] Sh. Salem and K. R. Raslan, Some new discrete inequalities and their applications, JIPAM. J. Inequal. Pure Appl. Math. **5** (2004), no. 1, Article 2, 9 pp. (electronic). {77}

[89] C. Sbordone, On some integral inequalities and their applications to the calculus of variations, Boll. Un. Mat. Ital. C (6) **5** (1986), no. 1, 73–94. {39}

[90] Q. Sheng, M. Fadag, J. Henderson and J. M. Davis, An exploration of combined dynamic derivatives on time scales and their applications, Nonlinear Anal. Real World Appl. **7** (2006), no. 3, 395-413. {68,69}

[91] M. F. Silva, J. A. Tenreiro Machado and R. S. Barbosa, Using fractional derivatives in joint control of hexapod robots, J. Vib. Control **14** (2008), no. 9-10, 1473–1485. {85}

[92] B. van Brunt, *The calculus of variations*, Springer, New York, 2004. {15,16,17,33,35}

[93] F.-H. Wong, C.-C. Yeh and C.-H. Hong, Gronwall inequalities on time scales, Math. Inequal. Appl. **9** (2006), no. 1, 75–86. {20,76}

[94] F.-H. Wong, C.-C. Yeh and W.-C. Lian, An extension of Jensen's inequality on time scales, Adv. Dyn. Syst. Appl. **1** (2006), no. 1, 113–120. {20,40}

[95] Y. Xing, M. Han and G. Zheng, Initial value problem for first-order integro-differential equation of Volterra type on time scales, Nonlinear Anal. **60** (2005), no. 3, 429–442. {60}

[96] Z. Zhan, W. Wei and H. Xu, Hamilton-Jacobi-Bellman equations on time scales, Math. Comput. Modelling **49** (2009), no. 9-10, 2019–2028. {38}

# Index