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Hierarquia de Leibniz

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Mestre em Matemática, realizada sob a orientação científica do Doutor Manuel António Gonçalves Martins, Professor Auxiliar do Departamento de Matemática da Universidade de Aveiro.

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Leibniz Hierarchy

Dissertation presented to the University of Aveiro to fulfil the necessary requirements for obtaining a Master´s Degree in Mathematics, written under the scientific orientation of Prof. Manuel António Gonçalves Martins, Professor in the Mathematics Department at the University of Aveiro.

Dedico este trabalho aos meus pais e ao meu marido.

I dedicate this work at my parents and husband.

O júri / the jury

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J. P.

Palavras-chave

Resumo

A Lógica Algébrica Abstracta estuda o processo pelo qual uma classe de álgebras pode ser associada a uma lógica. Nesta dissertação, analisamos este processo agrupando lógicas partilhando certas propriedades em classes. O conceito central neste estudo é a congruência de Leibniz que assume o papel desempenhado pela equivalência no processo tradicional de Lindenbaum-Tarski.

Apresentamos uma hierarquia entre essas classes que é designada por hierarquia de Leibniz, caracterizando as lógicas de cada classe por propriedades meta-lógicas, por exemplo propriedades do operador de Leibniz.

Estudamos também a recente abordagem comportamental que usa lógicas multigénero, lógica equacional comportamental e, consequentemente, uma versão comportamental do operador de Leibniz. Neste contexto, apresentamos alguns exemplos, aos quais aplicamos esta nova teoria, capturando alguns fenómenos de algebrização que não era possível formalizar com a abordagem standard.

Keywords

Abstract

Abstract Algebraic logic studies the process by which a class of algebras can be associated with a logic. In this dissertation, we analyse this process by grouping logics sharing certain properties into classes. The central concept in this study is the Leibniz Congruence that assumes the role developed by the equivalence in the traditional Lindenbaum-Tarski process.

We show a hierarchy between these classes, designated by Leibniz hierarchy, by characterizing logics in each class by meta-logical properties, for example properties of the Leibniz operator.

We also study a recent behavioral approach which uses many-sorted logics, behavioral equational logic and, consequently, a behavioral version of the Leibniz operator. In this context, we provide some examples, to which we apply this new theory, capturing some phenomena of algebraization that are not possible to formalize using the standard approach.

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Chapter 1

Introduction

The general theory of Abstract Algebraic Logic (AAL) studies the mechanism by which a class of algebras can be associated with a given logic. This theory provides a general context in which bridge theorems, relating metalogical properties of a logic to algebraic properties of its algebraic counterpart, can be formulated precisely. These contrast to the study of algebraic logic whose main setting is to examine the class of algebras that are canonically associated with a logic. The strong connection between a logic and its associated class of algebras can be very useful for metalogical investigation. The paradigm of the Lindenbaum-Tarski process is the way by which the class of Boolean Algebras (**BA**) appears from the Classical Propositional Logic (**CPL**). Actually, given a theory T, the Lindenbaum-Tarski algebra induced by T for **CPL**, is the quotient algebra $\mathrm{Fm}_{\mathcal{L}}/\equiv_T$, where $\equiv_T$ is the congruence on the formula algebra defined by $p \equiv_T q$ if and only if (iff) p and q are logically equivalent in T, that is, $p \leftrightarrow q \in T$ (the connective $\leftrightarrow$ denotes the usual classical propositional equivalence). This quotient algebra is a Boolean algebra. Conversely, every countable Boolean algebra is isomorphic to an algebra $\mathrm{Fm}_{\mathcal{L}}/\equiv_T$ for some theory T of **CPL**. In this way, the class **BA** is associated with **CPL**. A similar phenomena occurs in the Intuitionistic Propositional Logic (**IPL**) with the class of Heyting Algebras (**HA**). In order to generalize this process to other logics, the role played by the congruence $\equiv_T$ is substituted by the Leibniz congruence and the equivalence connective by a system of equivalence formulas.

Logics with some identical properties have been grouped by classes which can be characterized by Leibniz operator properties, parameterized system of equivalence formulas and closure properties of the class of reduced matrix models. In this dissertation, we study these classes for propositional logics and we generalize for many-sorted logics

using properties of the behavioral Leibniz operator. This generalization can capture logics that are not algebraizable in the standard approach but are behaviorally algebraizable. Nevertheless, this generalization does not trivialize the notion of algebraization because there are again logics which are not algebraizable in any way.

In Chapter 2, we introduce some important concepts and results around the central notions of logic and algebra. A logic is defined as a pair $\langle \mathcal{L}, \vdash \rangle$ where $\mathcal{L}$ is a language and $\vdash$ is a binary relation between sets of formulas and individual formulas satisfying reflexivity, cut, weakening and structurality conditions. We present other alternative definitions, in some concrete case for (finitary) logic (c.f. [BP89] and [Gon08]). We also introduce the notion of matrix and the notions of Leibniz and Suszko operators, which are central in a semantical approach. We conclude this chapter by defining equational logic, which is an important tool in the study of equivalent (algebraic) semantics for a logic.

In Chapter 3, we consider a wide class of logics which is the class of protoalgebraic logics. Blok and Pigozzi proved that this class of logics is exactly the class of non-pathological defined by Czelakowski. We give some characterizations of protoalgebraic logics using the Leibniz and the Suszko operators. We also show that a logic is protoalgebraic iff it has a parameterized system of equivalence formulas, or equivalently, if it has the parameterized local deduction-detachment theorem. We illustrate this latter result, with **BCK** logic. We also study the relationship between the structural properties of the class of reduced matrix models and metalogical properties of protoalgebraic logics. We prove that a logic is protoalgebraic iff the class of reduced matrix models is closed under subdirect products. As we work with logics for which the finitariness condition does not hold, we emphasize some results about finitary protoalgebraic logics.

In Chapter 4, we study the class of equivalential logics which are logics that have a (possibly infinite) system of equivalence formulas. These logics were introduced by Prucnal and Wroński in [PW74] and extensively studied by Czelakowski in [Cze81], [Cze01, Chapter 3] and [Cze04]. They form a proper subclass of the class of protoalgebraic logics. We prove a useful theorem, called Herrmann's Test, which provides some conditions that a set of formulas built up in two variables must satisfy in order for a protoalgebraic logic to become equivalential. We also study finitely equivalential logics which are equivalential logics that have a finite system of equivalence formulas. We give a characterization of (finitely) equivalential logics by means of the Leibniz operator properties. The class of equivalential logics is also characterized by closure properties

of the class of reduced matrix models which is closed under submatrices and direct products. Furthermore, as we did for protoalgebraic logics, we focus on the finitary (finitely) equivalential logics. We prove that a logic is finitary and finitely equivalential iff the class of reduced matrix models is a quasivariety. We conclude this chapter with the presentation of some examples of modal logics which show that the class of finitely equivalential logics is a proper subclass of equivalential logics (c.f. [Mal89]).

In Chapter 5, we study the algebraization phenomena in a broad sense. In literature there are several notions of algebraization. In this Chapter, we will present some of them. First we define the notion of algebraic semantics. Roughly speaking, a class $\mathbf{K}$ of algebras can be considered as an algebraic semantics of a logic S if the consequence relation $\vdash_S$ can be interpreted in the equational consequence relation $\models_{\mathbf{K}}$ in a natural way. We show that a logic can has (if any) many algebraic semantics. In addition, if there exists an inverse interpretation of $\models_{\mathbf{K}}$ in $\vdash_S$, then $\mathbf{K}$ is called an equivalent algebraic semantics for S. It is unique up to a quasivariety. If S is finitary then the equivalent algebraic semantics $\mathbf{K}$ is a quasivariety. We consider weakly algebraizable logics which are logics that have a pair of interpretations which commute with surjective substitutions and are mutually inverse. We also give a characterization of weakly algebraizable logics using the Leibniz operator. We define algebraizable logics as logics which have an equivalent algebraic semantics. The most of familiar deductive systems have equivalent algebraic semantics. The process of algebraization is related to the famous Lindenbaum-Tarski method. For instance, this establishes the relationship between $\mathbf{CPL}$ and the class $\mathbf{BA}$. The central idea is to look at the set of formulas as an algebra with operations induced by the logical connectives. Tarski observed that logical equivalence is a congruence on the formula algebra, and therefore a quotient algebra could be built. Many other logics are algebraizable, namely $\mathbf{IPL}$. We give some characterizations of the class of algebraizable logics. Among them, we have that $\mathbf{K}$ is an equivalent algebraic semantics for a logic S iff there exists an isomorphism between the theory lattice of S and the equational theory lattice of $\mathbf{K}$ that commutes with inverse substitution. We finalize this chapter by giving some examples of logics which show that the inclusion among the different classes of algebraizable logics are proper.

In Chapter 6, we go behind protoalgebraic logics studying the class of truth-equational logics, which includes the class of weakly algebraizable logics and has been recently studied by Raftery (c.f. [Raf06b]). We characterize this class of logics by

properties of the Suszko operator. A logic is truth equational iff the Suszko operator is injective on the lattice filters for every algebra. We give examples of logics which are non-protoalgebraic or truth equational, e.g. Intuitionistic Propositional Logic without implication **IPL***.

In Chapter 7, we study the generalization of the theory of standard AAL to many-sorted setting. This generalization is important since propositional logics are not enough expressive when we want to reason about complex systems. Thus we need logics over rich languages where elements can be distinguished by sorts. For instance, the First-Order Logic (FOL) is a logic with two sorts (a sort for terms and a sort for formulas). The predicates can be naturally seen as operations that transform terms in formulas and the connectives as operations over formulas. A many-sorted logic is introduced as logic whose language is obtained from a many-sorted signature with a distinguished sort ϕ of formulas, and satisfies structurality condition. The notion of hidden many-sorted signature is a many-sorted signature which is divided in a visible and in a hidden part. In a hidden algebra, the elements are naturally split into the visible and the hidden data. Since we cannot access immediately to the hidden data, it is not possible to reason directly about the equality of two hidden values. Hence, equational logic needs to be replaced by behavioral equational logic, also called hidden equational logic, based on the notion of behavioral equivalence. Two values are consider Γ-behaviorally equivalent if they cannot be distinguished by any experiments (visible output) that can be built with the operations in the subsignature Γ. The Γ-behavioral equivalence is the largest equivalence relation compatible with the operations in Γ whose visible part is the identity relation. Thus there is a natural connection with the Leibniz congruence. Actually, in the many-sorted AAL approach, the theory was developed by replacing the role of unsorted equational logic by many-sorted behavioral equational logic over the same signature and taking as unique visible sort, the sort ϕ of formulas. Since the sort ϕ is considered visible, we have equational reasoning about formulas, which compels every connective to be congruent. The standard notion of algebraization is a particular case of many-sorted algebraization (we have a unique sort ϕ of the signature Σ). And this former is a particular case of behaviorally algebraization by considering $\Gamma = \Sigma$. At the end of this chapter we give some examples of logics which are not algebraizable in the standard sense but are behaviorally algebraizable, e.g. the paraconsistent logic $\mathcal{C}_1$ of da Costa (c.f. [Gon08, Chapter 5]). However, there are many logics that are not algebraizable in any way.

Chapter 2

Preliminaries

In this chapter, we introduce some important concepts and results around the central notions of logic and algebra that will be necessary throughout this thesis. For the interested reader, we suggest [ANS01, Chapter 4], [Cze01] and [Wój88] for more on these subjects and for the proofs of the results presented herein. We assume that the reader is familiar with some notions of universal algebra, as the notions of homomorphism, equivalence relation, etc (c.f. [BS81]). We allow the reader to become acquainted with our notations, terminology, conventions and mathematical language. The expression "iff" is used as an abbreviation for "if and only if". We describe four alternative ways of defining a logic which are showing to be equivalent under some conditions (c.f. [BP89] and [Gon08]). We also define the notion of matrix and the notions of Leibniz and Suszko operators, which we need for a semantical approach of our subject. We conclude this chapter by defining equational logic, which is an important tool in the study of equivalent (algebraic) semantics for a logic.

2.1 Propositional Language and $\mathcal{L}$-algebra

A *propositional* (or *sentential*) *language* $\mathcal{L}$ is a set of propositional connectives (or *fundamental operations* in algebraic context), for each one it is associated a finite natural number called *rank* (or *arity*). We define $\mathcal{L}$-*constants* as usual, they are connectives (if any) of $\mathcal{L}$ that have rank 0. Given a propositional language $\mathcal{L}$, $\mathrm{Fm}_\mathcal{L}$ denotes the set of propositional $\mathcal{L}$-*formulas*, also called $\mathcal{L}$-*sentences* (or $\mathcal{L}$-*terms* in algebraic context) built in the usual recursive way from the countably infinite set $\mathrm{Var} = \{p_0, p_1, \dots\}$ of *propositional variables* (or *atomic formulas*) using the connectives in $\mathcal{L}$: all variables

and constants are formulas, and if $\varphi_0, \ldots, \varphi_{n-1}$ are formulas and ω is a connective of rank n, then $\omega(\varphi_0, \ldots, \varphi_{n-1})$ is a formula. Let φ be a formula, we write $\varphi(p_0, \ldots, p_{n-1})$ to indicate that the variables occurring in φ are all included in the list $p_0, \ldots, p_{n-1}$. We denote by $\mathrm{Var}(\varphi)$ the finite set of variables that actually occur in φ. If $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$, then $\mathrm{Var}(\Gamma) = \bigcup\{\mathrm{Var}(\varphi) : \varphi \in \Gamma\}$. In the sequel, the letters $p, q, \ldots$ denote variables and $X, Y, Z, \ldots$ represent arbitrary sets of variables. We represent formulas by lower case Greek letters and sets of formulas by upper case Greek letters.

By an algebra of type $\mathcal{L}$ (an $\mathcal{L}$-*algebra* for short) we mean a structure $\mathbf{A} = \langle A, \langle \omega^{\mathbf{A}} : \omega \in \mathcal{L} \rangle \rangle$ where A is a non-empty set called the *universe* of $\mathbf{A}$ and $\omega^{\mathbf{A}}$ is an operation on A of arity k for each connective ω of rank k, i.e., a mapping $\omega^{\mathbf{A}} : A^k \to A$. We represent $\mathcal{L}$-algebras by boldface roman letters and their universes by the corresponding lightface letters. We assume that the reader is familiar with notions of universal algebra, as the notions of subalgebra, direct product of family of algebras, etc. Let $\mathbf{A} = \langle A, \langle \omega^{\mathbf{A}} : \omega \in \mathcal{L} \rangle \rangle$ be an $\mathcal{L}$-algebra and $\mathcal{L}'$ a *sublanguage* of $\mathcal{L}$, i.e., $\mathcal{L}' \subseteq \mathcal{L}$. The $\mathcal{L}'$-algebra $\langle A, \langle \omega^{\mathbf{A}} : \omega \in \mathcal{L}' \rangle \rangle$ is called the $\mathcal{L}'$-*reduct* of $\mathbf{A}$. The *algebra of $\mathcal{L}$-formulas* is the absolutely free algebra $\mathbf{Fm}_{\mathcal{L}}$ (or $\mathbf{Te}_{\mathcal{L}}(\mathrm{Var})$) of type $\mathcal{L}$ over the set of generators Var. For any set X of variables, $\mathrm{Fm}_{\mathcal{L}}(X)$ denotes the set of formulas in which only variables from X occur, and we denote by $\mathbf{Fm}_{\mathcal{L}}(X)$ the corresponding subalgebra of $\mathbf{Fm}_{\mathcal{L}}$.

Let $\mathbf{A}$ be an $\mathcal{L}$-algebra and $\varphi \in \mathrm{Fm}_{\mathcal{L}}$. Depending on the values in $\mathbf{A}$ that variables of φ are assigned, the formula φ has a unique interpretation in $\mathbf{A}$. Since $\mathbf{Fm}_{\mathcal{L}}$ is absolutely freely generated by the set of variables, any mapping $h : \mathrm{Var} \to A$ can be uniquely extended to a homomorphism $\bar{h} : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, called *assignment* (also named *valuation* or *evaluation*). Conversely, if we have a homomorphism $g : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ then there exists a homomorphism $h : \mathrm{Var} \to A$ such that $\bar{h} = g$. Indeed, we can always consider $h := g_{|\mathrm{Var}}$. In the sequel, we also write h for $\bar{h}$. Let $\mathbf{A}$ be an algebra, $\varphi(p_0, \ldots, p_{n-1}) \in \mathrm{Fm}_{\mathcal{L}}$ and $a_0, \ldots, a_{n-1} \in A$. We write $\varphi^{\mathbf{A}}(a_0, \ldots, a_{n-1}) = h(\varphi)$, i.e., it is the interpretation of φ in $\mathbf{A}$ when $h(p_i) = a_i$ for $i = 0, \ldots, n-1$. An endomorphism $e : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$ is called a *substitution*. By a *substitution instance* of a formula φ we mean a formula of the form $e(\varphi)$ where e is any substitution.

A *congruence* on an algebra $\mathbf{A}$ is an equivalence relation that is compatible with the operations on $\mathbf{A}$ (c.f. [BS81, Definition 5.1 in Chapter II]). Let $\mathbf{A}$ be an algebra and $a_1, \ldots, a_n \in A$. We denote by $\theta(a_1, \ldots, a_n)$ the *congruence generated* by $\{(a_i, a_j) : 1 \leq i, j \leq n\}$, which is the smallest congruence such that $a_1, \ldots, a_n$ are in the same equivalence class. The congruence generated by the pair (a_1, a_2), $\theta(a_1, a_2)$, is called

principal congruence. The set of all congruences on an $\mathcal{L}$-algebra $\mathbf{A}$ is denoted by $\mathrm{Co}\mathbf{A}$. This set is always closed under arbitrary intersections and unions of directed sets. Thus, it forms a lattice with set-theoretical inclusion, where the meet operation is the intersection of congruences and the join operation is defined in the following way: $\theta_1 \vee \theta_2 = \theta_1 \cup (\theta_1 \circ \theta_2) \cup (\theta_1 \circ \theta_2 \circ \theta_1) \cup \ldots$ (c.f. [BS81, Theorem 4.6 in Chapter I]). Let $\mathbf{K}$ be a class of $\mathcal{L}$-algebras. We say that θ is a $\mathbf{K}$-*congruence* of $\mathbf{A}$ if $\theta \in \mathrm{Co}\mathbf{A}$ and $\mathbf{A}/\theta \in \mathbf{K}$. The set of all $\mathbf{K}$-congruences of $\mathbf{A}$ is denoted by $\mathrm{Co}_{\mathbf{K}}\mathbf{A}$. Given any $R \subseteq \mathrm{A}^2$, the intersection of all $\mathbf{K}$-congruences on $\mathbf{A}$ that includes R is denoted by $\theta_{\mathbf{K}}^{\mathbf{A}}R$ and is called the $\mathbf{K}$-*congruence generated* by R. If $R = \{(a, b)\}$, we simply write $\theta_{\mathbf{K}}^{\mathbf{A}}(a, b)$ for the smallest congruence θ of $\mathbf{A}$ such that $a \equiv b(\theta)$ and $\mathbf{A}/\theta \in \mathbf{K}$.

2.2   Logic and Deductive System

A *logic S* (or *logical system*) over a propositional language $\mathcal{L}$ is defined as a pair $S = \langle \mathcal{L}, \vdash_S \rangle$, where $\vdash_S$ is a relation between set of formulas and individual formulas, called the *consequence relation* of S, which satisfies the following conditions, for all $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$:

$$\varphi \in \Gamma \Rightarrow \Gamma \vdash_S \varphi \qquad \text{(Reflexivity)}$$

$$\Gamma \vdash_S \varphi \text{ and } \Gamma \subseteq \Delta \Rightarrow \Delta \vdash_S \varphi \qquad \text{(Cut)}$$

$$\Gamma \vdash_S \varphi \text{ and } \Delta \vdash_S \psi \text{ for every } \psi \in \Gamma \Rightarrow \Delta \vdash_S \varphi \qquad \text{(Weakening)}$$

$$\Gamma \vdash_S \varphi \Rightarrow e[\Gamma] \vdash_S e(\varphi) \text{ for every substitution } e \qquad \text{(Structurality)}$$

where $\Gamma \vdash_S \varphi$ abbreviates that $\langle \Gamma, \varphi \rangle \in S$ and reads Γ entails φ in S or φ is a consequence of Γ in S. Note that reflexivity and weakening conditions together imply cut condition. Let $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$, we write $\Gamma \vdash_S \Delta$ for $\Gamma \vdash_S \delta$ for all $\delta \in \Delta$, and we write $\Gamma \dashv\vdash_S \Delta$ when $\Gamma \vdash_S \Delta$ and $\Delta \vdash_S \Gamma$ hold. In the later case, we say that Γ and Δ are *interderivable.*

A very important property of a logic is the finitariness. Indeed, there are logicians (e.g. Blok and Pigozzi) that defined logic being finitary. We say that $\vdash_S$ is *finitary* if

$$\Gamma \vdash_S \varphi \Rightarrow \Gamma' \vdash_S \varphi \text{ for some finite } \Gamma' \subseteq \Gamma.$$

With this extra property, we obtain stronger results that we will emphasize throughout the text.

A formula is called a *theorem* of S (an *S-theorem* for short) if $\emptyset \vdash_S \varphi$ (we write $\vdash_S \varphi$ for short). The set of all theorems is denoted by $\mathrm{Thm}(S)$. By the structurality of S, $\mathrm{Thm}(S)$ is closed under substitutions. A set T of formulas is called a *theory* of S (an *S-theory* for short) if it is closed under the consequence relation $\vdash_S$, that is, if, for every $\varphi \in \mathrm{Fm}_\mathcal{L}$, $T \vdash_S \varphi$ implies $\varphi \in T$. We represent theories by uppercase Latin letters. The set of all S-theories is denoted by $\mathrm{Th}(S)$ and is closed under inverse substitutions. Indeed, let e be a substitution, $T \in \mathrm{Th}(S)$ and $\varphi \in \mathrm{Fm}_\mathcal{L}$. Suppose $e^{-1}[T] \vdash_S \varphi$. By structurality of S, $e[e^{-1}[T]] \vdash_S e(\varphi)$. Since we always have that $e[e^{-1}[T]] \subseteq T$, by cut condition, $T \vdash_S e(\varphi)$. As T is a theory, $e(\varphi) \in T$, i.e., $\varphi \in e^{-1}[T]$. Thus $e^{-1}[T] \in \mathrm{Th}(S)$. Observe that the theorems of S belong to every S-theory and it is not required that this set be nonempty. We say that a theory is *consistent* if it is not the set of all formulas. Otherwise, it is called *inconsistent*. The set $\mathrm{Th}(S)$ forms a complete lattice $\mathbf{Th}(S) = \langle \mathrm{Th}(S), \cap, \vee^S \rangle$, where the meet operation is the intersection of an arbitrary family of theories and the join operation is defined in the following way: for any $T, T' \in \mathrm{Th}(S)$, $T \vee^S T' = \bigcap \{R \in \mathrm{Th}(S) : T \cup T' \subseteq R\}$. The largest theory is the set $\mathrm{Fm}_\mathcal{L}$ and the smallest theory is the set $\mathrm{Thm}(S)$. For any $\Gamma \subseteq \mathrm{Fm}_\mathcal{L}$, we denote by $\mathrm{Cn}_S\Gamma$ the smallest S-theory including Γ, i.e., $\mathrm{Cn}_S\Gamma = \{\varphi \in \mathrm{Fm}_\mathcal{L} : \Gamma \vdash_S \varphi\}$ and we said that Γ *generates* $\mathrm{Cn}_S\Gamma$. It is not difficult to see that $T \vee^S T' = \mathrm{Cn}_S(T \cup T')$, i.e., $T \vee^S T'$ is the theory generated by $T \cup T'$. An S-theory T is *finitely axiomatized* or *finitely generated* if $T = \mathrm{Cn}_S\Gamma$ for some finite $\Gamma \subseteq \mathrm{Fm}_\mathcal{L}$.

Let $\Gamma \cup \{\varphi\} \in \mathrm{Fm}_\mathcal{L}$. We have that $\Gamma \vdash_S \varphi$ iff, for all substitutions e and $T \in \mathrm{Th}(S)$ such that $e[\Gamma] \subseteq T$ we have $e(\varphi) \in T$. Indeed, suppose that $\Gamma \vdash_S \varphi$. Let e be a substitution and $T \in \mathrm{Th}(S)$ such that $e[\Gamma] \subseteq T$. By structurality condition, $e[\Gamma] \vdash_S e(\varphi)$. And by cut condition, $T \vdash_S e(\varphi)$. Since T is a theory, $e(\varphi) \in T$. Conversely, let $\Gamma \cup \{\varphi\} \in \mathrm{Fm}_\mathcal{L}$ and consider the substitution $e = id_{\mathrm{Fm}_\mathcal{L}}$. Thus, for every $T \in \mathrm{Th}(S)$ such that $e[\Gamma] \subseteq T$, we have that $e(\varphi) \in T$. Let $T := \mathrm{Cn}_S(\Gamma)$. Since $e[\Gamma] = \Gamma \subseteq \mathrm{Cn}_S(\Gamma) = T$, we have that $\varphi = e(\varphi) \in T$. We conclude that $\Gamma \vdash_S \varphi$.

Let S be a logic. We can see Cn_S as a function on the power set of $\mathrm{Fm}_\mathcal{L}$ into itself, usually called the *consequence operator* of S. This operator satisfies the following conditions, for all $\Gamma, \Delta \subseteq \mathrm{Fm}_\mathcal{L}$:

$$\Gamma \subseteq \mathrm{Cn}_S\Gamma \qquad \text{(Reflexivity)}$$

$$\Gamma \subseteq \Delta \Rightarrow \mathrm{Cn}_S\Gamma \subseteq \mathrm{Cn}_S\Delta \qquad \text{(Monotonicity)}$$

$$\mathrm{Cn}_S\mathrm{Cn}_S\Gamma \subseteq \mathrm{Cn}_S\Gamma \qquad \text{(Idempotency)}$$

$$e[\mathrm{Cn}_S\Gamma] \subseteq \mathrm{Cn}_S(e[\Gamma]) \text{ for every substitution } e \qquad \text{(Structurality)}$$

In addition, if S is finitary then Cn_S is *finitary* in the sense

$$\mathrm{Cn}_S\Gamma \subseteq \bigcup\{\mathrm{Cn}_S\Gamma' \text{ for all finite set } \Gamma' \subseteq \Gamma\}.$$

Conversely, any function $\mathcal{C}$ from power set of $\mathrm{Fm}_\mathcal{L}$ into itself satisfying reflexivity, monotonicity, idempotency and structurality conditions, give rise to a logic S. Indeed, we defined the relation $\vdash_S$ in the following way: for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\mathcal{L}$, $\Gamma \vdash_S \varphi$ iff $\varphi \in \mathcal{C}(\Gamma)$. It is not difficult to see that the relation $\vdash_S$ satisfies reflexivity, cut, weakening and structurality conditions. Thus, we obtain a logic over $\mathcal{L}$, which can be proved finitary whenever $\mathcal{C}$ is finitary.

Theorem 2.2.1. *Let $\mathcal{C}$ be a set of subsets of $\mathrm{Fm}_\mathcal{L}$. $\mathcal{C}$ is the set of theories of some logic iff the following conditions hold:*

(i) $\mathcal{C}$ is closed under arbitrary intersection, i.e., $\bigcap X \in \mathcal{C}$ for every $X \subseteq \mathcal{C}$;

(ii) $\mathcal{C}$ is closed under inverse images of substitutions, i.e., if $T \in \mathcal{C}$ then $e^{-1}[T] \in \mathcal{C}$ for every substitution e.

Proof. Suppose that $\mathcal{C}$ is a set of theories of a logic S, i.e., $\mathcal{C} = \mathrm{Th}(S)$. Since $\mathrm{Th}(S)$ is always closed under arbitrary intersection and inverse images of substitutions, conditions (i) and (ii) hold.

Conversely, assume conditions (i) and (ii). We define a relation $\vdash_\mathcal{C}$ between set of formulas and individual formulas as follows, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\mathcal{L}$, $\Gamma \vdash_\mathcal{C} \varphi$ iff $\varphi \in \bigcap\{T \in \mathcal{C} : \Gamma \subseteq T\} := \mathrm{Cn}_\mathcal{C}(\Gamma)$. It is not difficult to see that reflexivity, cut and weakening conditions hold. Let $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\mathcal{L}$. Suppose $\Gamma \vdash_\mathcal{C} \varphi$, i.e., $\varphi \in \mathrm{Cn}_\mathcal{C}\Gamma$. Let e be a substitution. Thus $e(\varphi) \in e[\mathrm{Cn}_\mathcal{C}(\Gamma)]$. We always have $\Gamma \subseteq e^{-1}[e[\Gamma]]$. Since $e[\Gamma] \subseteq \mathrm{Cn}_\mathcal{C}(e[\Gamma])$, we have that $e^{-1}[e[\Gamma]] \subseteq e^{-1}[\mathrm{Cn}_\mathcal{C}(e[\Gamma])]$. Thus $\Gamma \subseteq e^{-1}[\mathrm{Cn}_\mathcal{C}(e[\Gamma])]$. By condition (ii), $e^{-1}[\mathrm{Cn}_\mathcal{C}(e[\Gamma])] \in \mathcal{C}$. Thus $\mathrm{Cn}_\mathcal{C}\Gamma \subseteq e^{-1}[\mathrm{Cn}_\mathcal{C}(e[\Gamma])]$, which implies that $e[\mathrm{Cn}_\mathcal{C}\Gamma] \subseteq e[e^{-1}[\mathrm{Cn}_\mathcal{C}(e[\Gamma])]]$. Since we always have $e[e^{-1}[\mathrm{Cn}_\mathcal{C}(e[\Gamma])]] \subseteq \mathrm{Cn}_\mathcal{C}(e[\Gamma])$, we deduce that $e[\mathrm{Cn}_\mathcal{C}\Gamma] \subseteq \mathrm{Cn}_\mathcal{C}(e[\Gamma])$. As $e(\varphi) \in e[\mathrm{Cn}_\mathcal{C}(\Gamma)]$, we have that $e(\varphi) \in \mathrm{Cn}_\mathcal{C}(e[\Gamma])$, i.e., $e[\Gamma] \vdash_\mathcal{C} e(\varphi)$. Thus structurality condition holds. We conclude that $\langle \mathcal{L}, \vdash_\mathcal{C} \rangle$ is a logic. It only remains to show that $\mathcal{C} = \mathrm{Th}(S)$. Suppose $T \in \mathrm{Th}(S)$, then $T \vdash_\mathcal{C} \varphi$ implies $\varphi \in T$, i.e., $\mathrm{Cn}_\mathcal{C}T \subseteq T$ and as $T \subseteq \mathrm{Cn}_\mathcal{C}T$ always holds, we have $\mathrm{Cn}_\mathcal{C}T = T$. Since $\mathcal{C}$ is closed under intersection, $T \in \mathcal{C}$. Conversely, assume $T \in \mathcal{C}$, i.e., $\mathrm{Cn}_\mathcal{C}T = T$. If $T \vdash_\mathcal{C} \varphi$ then $\varphi \in \mathrm{Cn}_\mathcal{C}T = T$, so $T \in \mathrm{Th}(S)$. $\qquad\square$

If S is a finitary logic then we have the following theorem.

Theorem 2.2.2. *Let $\mathcal{C}$ be a set of subsets of $\mathrm{Fm}_{\mathcal{L}}$. $\mathcal{C}$ is the set of theories of some finitary logic iff the following conditions hold:*

(i) $\mathcal{C}$ is closed under arbitrary intersection;

(ii) $\mathcal{C}$ is closed under inverse images of substitutions;

(iii) $\mathcal{C}$ is closed under directed unions, i.e., $\bigcup X \in \mathcal{C}$ for every $X \subseteq \mathcal{C}$ that is upward-directed in the sense that, for every pair $T, T' \in X$, there is an $R \in \mathcal{C}$ such that $T, T' \subseteq R$.

Moreover, condition (ii) can be replaced by

(ii') $\mathcal{C}$ is closed under inverse images of surjective substitutions.

Proof. Suppose that $\mathcal{C}$ is a set of theories of a finitary logic S, i.e., $\mathcal{C} = \mathrm{Th}(S)$. We see the proof of Theorem 2.2.1 for conditions (i) and (ii). Let $\{T_i : i \in I\}$ be an upward-directed subset of $\mathrm{Th}(S)$. Suppose $\bigcup_{i \in I} T_i \vdash_S \varphi$. Since S is finitary, there exists a finite $\Gamma' \subseteq \bigcup_{i \in I} T_i$ such that $\Gamma' \vdash_S \varphi$. As the set of T_i's is upward-directed, there is a $j \in I$ such that $\Gamma' \subseteq T_j$. By cut condition, $T_j \vdash_S \varphi$, i.e., $\varphi \in T_j \subseteq \bigcup_{i \in I} T_i$. Hence $\bigcup_{i \in I} T_i \in \mathrm{Th}(S)$.

Conversely, assume conditions (i), (ii') and (iii) hold. We define the relation $\vdash_{\mathcal{C}}$ as in the proof of Theorem 2.2.1. It is not difficult to see that the relation $\vdash_{\mathcal{C}}$ satisfies reflexivity, cut and weakening conditions. Let $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. Suppose $\Gamma \vdash_{\mathcal{C}} \varphi$. For each finite $\Gamma' \subseteq \Gamma$, we consider $\mathrm{Cn}_{\mathcal{C}}\Gamma' = \bigcap \{R \in \mathcal{C} : \Gamma' \subseteq R\}$. Since $\mathcal{C}$ is closed under intersection, $\mathrm{Cn}_{\mathcal{C}}\Gamma' \in \mathcal{C}$. The set $\{\mathrm{Cn}_{\mathcal{C}}\Gamma' : \Gamma' \text{ finite and } \Gamma' \subseteq \Gamma\}$ is obviously upward-directed. By condition (iii), $U = \bigcup\{\mathrm{Cn}_{\mathcal{C}}\Gamma' : \Gamma' \text{ finite and } \Gamma' \subseteq \Gamma\} \in \mathcal{C}$. Since $\Gamma = \bigcup\{\Gamma' : \Gamma' \text{ finite and } \Gamma' \subseteq \Gamma\} \subseteq U$, by cut condition, $U \vdash_{\mathcal{C}} \varphi$, i.e., $\varphi \in U$. Hence, $\varphi \in \mathrm{Cn}_{\mathcal{C}}\Gamma'$ for some finite $\Gamma' \subseteq \Gamma$, i.e., $\Gamma' \vdash_{\mathcal{C}} \varphi$. Thus finitary condition holds. Now, let $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. Suppose $\Gamma \vdash_{\mathcal{C}} \varphi$. By finitary condition, there exists a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_S \varphi$, i.e, $\varphi \in \mathrm{Cn}_{\mathcal{C}}\Gamma'$. Let e be substitution. Thus $e(\varphi) \in e[\mathrm{Cn}_{\mathcal{C}}(\Gamma')]$. Since there are only finitely many variables in $\Gamma' \cup \{\varphi\}$, there exists a surjective substitution e' such that $e'(\psi) = e(\psi)$ for every $\psi \in \Gamma' \cup \{\varphi\}$. Thus $e'(\varphi) \in e'[\mathrm{Cn}_{\mathcal{C}}(\Gamma')]$. We always have $\Gamma' \subseteq e'^{-1}[e'[\Gamma']]$. Since $e'[\Gamma'] \subseteq \mathrm{Cn}_{\mathcal{C}}(e'[\Gamma'])$, we have that $e'^{-1}[e'[\Gamma']] \subseteq e'^{-1}[\mathrm{Cn}_{\mathcal{C}}(e'[\Gamma'])]$. Thus $\Gamma' \subseteq e'^{-1}[\mathrm{Cn}_{\mathcal{C}}(e'[\Gamma'])]$. By condition (ii'), $e'^{-1}[\mathrm{Cn}_{\mathcal{C}}(e'[\Gamma'])] \in \mathcal{C}$. Thus $\mathrm{Cn}_{\mathcal{C}}\Gamma' \subseteq e'^{-1}[\mathrm{Cn}_{\mathcal{C}}(e'[\Gamma'])]$, which implies that $e'[\mathrm{Cn}_{\mathcal{C}}\Gamma'] \subseteq e'[e'^{-1}[\mathrm{Cn}_{\mathcal{C}}(e'[\Gamma'])]]$. By surjectivity of e', $e'[e'^{-1}[\mathrm{Cn}_{\mathcal{C}}(e'[\Gamma'])]] = \mathrm{Cn}_{\mathcal{C}}(e'[\Gamma'])$. Thus $e'[\mathrm{Cn}_{\mathcal{C}}\Gamma'] \subseteq \mathrm{Cn}_{\mathcal{C}}(e'[\Gamma'])$. Since $e'(\varphi) \in e'[\mathrm{Cn}_{\mathcal{C}}(\Gamma')]$, we have that $e'(\varphi) \in \mathrm{Cn}_{\mathcal{C}}(e'[\Gamma'])$, i.e., $e'[\Gamma'] \vdash_{\mathcal{C}} e'(\varphi)$. As $\Gamma' \subseteq \Gamma$,

we have that $e'[\Gamma'] \subseteq e'[\Gamma]$. By cut condition, $e'[\Gamma] \vdash_\mathcal{C} e'(\varphi)$, i.e., $e[\Gamma] \vdash_\mathcal{C} e(\varphi)$. Thus structurality condition holds. We conclude that $\langle \mathcal{L}, \vdash_\mathcal{C} \rangle$ is a finitary logic. To show that $\mathcal{C} = \mathrm{Th}(S)$, we see the end of the proof of Theorem 2.2.1. $\square$

These two last theorems show that the consequence operator Cn_S, and hence also the consequence relation $\vdash_S$, can be defined in terms of the lattice $\mathbf{Th}(S)$. Therefore a logic may be characterized by the properties of its set of theories, i.e, $S = \langle \mathcal{L}, \mathbf{Th}(S) \rangle$.

In the following lemma, we give some properties of $\mathrm{Th}(S)$ whenever S is a finitary logic.

Lemma 2.2.3. [BP89, Lemma 1.1] *Let S be a logic. The following conditions are equivalent:*

    (i) $\vdash_S$ is finitary;

    (ii) The compact elements of $\mathbf{Th}(S)$ coincide with the finitely generated S-theories;

    (iii) $\mathrm{Th}S$ is closed under directed unions;

    (iv) The lattice $\mathbf{Th}(S)$ is algebraic.

Another way of defining a finitary logic is by a set of axioms and a set of inference rules in the so called Hilbert style. By an *inference rule*, we mean any pair $\langle \Gamma, \varphi \rangle$ (also denoted by $\dfrac{\Gamma}{\varphi}$) where Γ is a finite set of formulas (the *premises* of the rule) and φ is a single formula (the *conclusion* of the rule). An *axiom*, is an inference rule with $\Gamma = \emptyset$, i.e., a pair $\langle \emptyset, \varphi \rangle$, usually just denoted by φ. The rules of this type are called *Hilbert-style* rules of inference, or *H-rules* for short.

Let AX be a set of axioms and IR a set of inference rules. We say that a formula φ is *directly derivable* from a set Γ of formulas by the inference rule $\langle \Delta, \psi \rangle$ if there is a substitution e such that $e(\psi) = \varphi$ and $e[\Delta] \subseteq \Gamma$. An *S-derivation* of φ from Γ is a finite sequence $\vartheta_0, \ldots, \vartheta_{n-1}$ of formulas such that $\vartheta_{n-1} = \varphi$ and, for each $i < n$, ϑ_i is either a member of Γ, a substitution instance of an axiom, or is directly derivable from $\{\vartheta_0, \ldots, \vartheta_{i-1}\}$. An *S*-derivation from $\emptyset$ is called an *S-proof.* We can defined a relation $\vdash_{\mathrm{AX,IR}}$ between set of formulas and individual formulas such that $\Gamma \vdash_{\mathrm{AX,IR}} \varphi$ iff φ is contained in the smallest set of formulas that includes Γ together with all substitution instances of the axioms of S and is closed under direct derivability by the inference rules of S. Clearly, $\Gamma \vdash_{\mathrm{AX,IR}} \varphi$ iff there is an S-derivation of φ from Γ, and $\vdash_{\mathrm{AX,IR}} \varphi$ iff there is an S-proof of φ. It is not difficult to see that the relation $\vdash_{\mathrm{AX,IR}}$ satisfies reflexivity, cut, weakening, structurality and finitary conditions. Thus $\langle \mathcal{L}, \vdash_{\mathrm{AX,IR}} \rangle$ is

a finitary logic, called the *deductive system* with the set of axioms AX and the set of inference rules IR. Conversely, let $S = \langle \mathcal{L}, \vdash_S \rangle$ be a finitary logic. By defining AX $:= \{\varphi : \emptyset \vdash_S \varphi\}$ and IR $:= \{\langle \Gamma, \varphi \rangle : \Gamma \vdash_S \varphi$ and Γ is finite$\}$, it is not difficult to see that $\vdash_{\text{AX,IR}}$ and $\vdash_S$ coincide. The set of axioms and inference rules, $\langle \text{AX, IR} \rangle$, is called an *axiomatization* (or a *presentation*) of S. Of course, a deductive system may have several axiomatizations. If both the set of axioms and the set of inference rules are finite then $\langle \text{AX, IR} \rangle$ is called a *finite axiomatization*.

A logic S is *trivial* if for all $\Gamma \neq \emptyset$, $\Gamma \vdash_S \varphi$ for every $\varphi \in \text{Fm}_{\mathcal{L}}$. There are exactly two trivial logics for each language $\mathcal{L}$: one has the empty set of theorems, called *almost inconsistent logic*, where $\emptyset$ and $\text{Fm}_{\mathcal{L}}$ are the only theories, and the other has every formulas as theorems, called *inconsistent logic*, where $\text{Fm}_{\mathcal{L}}$ is the only theory. To any logic we can associate several expansions, extensions, subsystems and fragments. By an *expansion* of a logic S we mean any system $S' = \langle \mathcal{L}', \vdash_{S'} \rangle$ such that $\mathcal{L} \subseteq \mathcal{L}'$ and $\Gamma \vdash_S \varphi \Rightarrow \Gamma \vdash_{S'} \varphi$ for all $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$. An expansion is called an *extension* if $\mathcal{L} = \mathcal{L}'$. In this case, S is called a *subsystem* of S'. A deductive system S' is an *axiomatic extension* of S if it is obtained by adjoining new axioms but leaving the inference rules invariant. Let $\mathcal{L}'$ be a sublanguage of $\mathcal{L}$, and $\vdash_{S'}$ the restriction of $\vdash_S$ to $\mathcal{L}'$ in the sense that, for all $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}'}$, $\Gamma \vdash_{S'} \varphi$ iff $\Gamma \vdash_S \varphi$. It is not difficult to see that $S' = \langle \mathcal{L}', \vdash_{S'} \rangle$ is a logic over $\mathcal{L}'$. S' is called the $\mathcal{L}'$-*fragment* of S and S is called a *conservative expansion* of S'.

2.3 Matrix with Leibniz and Suszko Operators

An $\mathcal{L}$-*matrix* (or a *logical matrix*) is a pair M $= \langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is an $\mathcal{L}$-algebra and D is an arbitrary subset of A. The elements of D are called *designated elements* (or *designated values*) of M. If $D = $ A or $\mathbf{A}$ is a trivial algebra (that is, A has only one element), then the matrix M is called *trivial*. If M is a matrix, $\models_{\text{M}}$ is the consequence relation defined between a (possibly infinite) set Γ of formulas and individual formulas φ, in the following way;

$\Gamma \models_{\text{M}} \varphi$ iff, for every homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, $h[\Gamma] \subseteq D$ implies $h(\varphi) \in D$.

Furthermore, the consequence relation $\models_{\text{M}}$ satisfies reflexivity, cut, weakening and structurality conditions. Thus $\langle \mathcal{L}, \models_{\text{M}} \rangle$ is a logic.

If $\mathcal{M}$ is a class of matrices, $\models_{\mathcal{M}}$ is the consequence relation between a (possibly

infinite) set Γ of formulas and a single formula φ defined as follows;

$$\Gamma \models_{\mathcal{M}} \varphi \text{ iff, for every } M \in \mathcal{M}, \ \Gamma \models_M \varphi.$$

A matrix M is called a *matrix model* of S (or an *S-matrix* for short) if, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_S \varphi$ implies $\Gamma \models_M \varphi$. The class of all matrix models of a logic S is denoted by $\mathbf{Mod}(S)$. A subset D of A is called a *deductive filter* or an *S-filter* (simply a *filter* when S is clear from context), if the matrix $\langle \mathbf{A}, D \rangle$ is an S-matrix. Usually we denote the S-filter of an S-matrix by the letters F, D and G. If S is a deductive system, F is an S-filter iff F contains all interpretations of the logical axioms and is closed under all inference rules of S. More precisely, for every homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, we have $h(\varphi) \in F$ for each axiom φ of S and $h[\Gamma] \subseteq F$ implies $h(\varphi) \in F$ for each inference rule $\langle \Gamma, \varphi \rangle$ of S. Given an $\mathcal{L}$-algebra $\mathbf{A}$, the set of all S-filters of $\mathbf{A}$, which is denoted by $\mathrm{Fi}_S(\mathbf{A})$, is closed under arbitrary intersection. Thus it is a complete lattice, denoted by $\mathbf{Fi}_S(\mathbf{A}) = \langle \mathrm{Fi}_S(\mathbf{A}), \bigcap, \bigvee \rangle$, where $\bigvee_{i \in I} F_i = \bigcap \{ G \in \mathrm{Fi}_S(\mathbf{A}) : \bigcup_{i \in I} F_i \subseteq G \}$. Therefore, given any subset X of A there is always the least S-filter of $\mathbf{A}$ that contains X; it is called the S-filter of $\mathbf{A}$ *generated* by X which we denote by $\mathrm{Fi}_S^{\mathbf{A}}(X)$. If X is the singleton $\{a\}$, we write $\mathrm{Fi}_S^{\mathbf{A}}(a)$ instead of $\mathrm{Fi}_S^{\mathbf{A}}(\{a\})$ and it is called *principal filter*. The S-filters on the formula algebra are exactly the S-theories and the corresponding matrix models $\langle \mathbf{Fm}_{\mathcal{L}}, T \rangle$ are called *formula matrix models* or *Lindenbaum matrix models* of S. The class of all Lindenbaum matrix models of a logic S is denoted by $\mathbf{L}(S)$. An *S-filter of* an S-matrix $M = \langle \mathbf{A}, D \rangle$ is an S-filter on the algebra $\mathbf{A}$ that includes D. We denote by $\mathrm{Fi}_S(M) = \{ E : E \in \mathrm{Fi}_S(\mathbf{A}) \text{ and } D \subseteq E \}$ the set of all S-filters of M which forms a complete sublattice of $\mathbf{Fi}_S(\mathbf{A})$.

A logic S over the language $\mathcal{L}$ is said to be *complete relative* to a class of S-matrices $\mathcal{M}$ if for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_S \varphi \Leftrightarrow \Gamma \models_{\mathcal{M}} \varphi$; when this holds, we say that $\mathcal{M}$ is a *matrix semantics* for S or that $\mathcal{M}$ is *strongly adequate* for S. The next theorem says that every logic has a matrix semantics.

Theorem 2.3.1 (Completeness Theorem)**.** *Let S be a logic. The class of all matrix models of S forms a matrix semantics for S. Furthermore, the class of all Lindenbaum matrix models of S is also a matrix semantics for S.*

Proof. Let $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $M \in \mathbf{Mod}(S)$. By definition of matrix model of S, $\Gamma \vdash_S \varphi \Rightarrow \Gamma \models_M \varphi$.

Conversely, assume $\Gamma \models_{\mathbf{Mod}(S)} \varphi$. Let $T = \mathrm{Cn}_S(\Gamma) \in \mathrm{Th}(S)$. Thus $\mathrm{M} = \langle \mathbf{Fm}_{\mathcal{L}}, T \rangle \in \mathbf{Mod}(S)$. Consider $h = id_{\mathrm{Fm}_{\mathcal{L}}}$ (the identity homomorphism on $\mathrm{Fm}_{\mathcal{L}}$), then $h[\Gamma] = \Gamma \subseteq \mathrm{Cn}_S(\Gamma) = T$. Hence $h(\varphi) = \varphi \in T$. Since $T = \mathrm{Cn}_S(\Gamma)$, $\Gamma \vdash_S \varphi$. $\qquad \square$

We may consider an $\mathcal{L}$-matrix $\langle \mathbf{A}, D \rangle$ as a structure over the first order language without equality containing the operation connectives of $\mathcal{L}$ and one unary predicate. If we consider the matrix model $\langle \mathbf{A}, D \rangle$ as a first order structure, the unary predicate is interpreted as D. It admits the intuitive interpretation "it is true that" and it is often called the *truth predicate* (c.f. [BP89]).

Let $\mathbf{A}$ be an algebra and $D \subseteq \mathrm{A}$. A (*matrix*) *congruence* on a matrix $\mathrm{M} = \langle \mathbf{A}, D \rangle$ (also called *strict congruence* of the matrix M) is a congruence on $\mathbf{A}$ that is *compatible with* D in the sense that, if, for all $a, b \in \mathrm{A}$, $a \in D$ and $\langle a, b \rangle \in \theta$ then $b \in D$. For any family $(\theta_i)_{i \in I}$ of congruences on $\mathbf{A}$ compatible with D, we have that $\bigvee_{i \in I} \theta_i$ is also a congruence on $\mathbf{A}$ compatible with D (c.f. [BP92, Lemma 5.2]).

Definition 2.3.2. *Let $\langle \mathbf{A}, D \rangle$ be a matrix. Then there is the largest matrix congruence (i.e., the largest congruence on $\mathbf{A}$ compatible with D) called the* Leibniz congruence *of D on $\mathbf{A}$ and denoted by $\Omega_{\mathbf{A}} D$.*

Observe that the definition of Leibniz congruence is completely independent of any logic; it is intrinsic to $\mathbf{A}$ and D. The Leibniz congruence is the largest congruence θ of $\mathbf{A}$ such that for all $a \in \mathrm{A}$ we have either $a/\theta \subseteq D$ or $a/\theta \cap D = \emptyset$. In other words, the Leibniz congruence does not identify elements inside D with elements outside D. In the following theorem, we give a characterization of the Leibniz congruence.

Theorem 2.3.3. *Let $\langle \mathbf{A}, D \rangle$ be a matrix. Then,*

$$\Omega_{\mathbf{A}} D = \{(a, b) \in \mathrm{A}^2 : \varphi^{\mathbf{A}}(a, c_0, \ldots, c_{k-1}) \in D \ iff \ \varphi^{\mathbf{A}}(b, c_0, \ldots, c_{k-1}) \in D,$$

$$for \ all \ \varphi(p, q_0, \ldots, q_{k-1}) \in \mathrm{Fm}_{\mathcal{L}}, k < \omega, \ and \ all \ c_0, \ldots, c_{k-1} \in \mathrm{A}\}.$$

This definition justifies the term "Leibniz congruence" for $\Omega_{\mathbf{A}} D$ since it formalizes the Leibniz second order criterion of equality according to which two objects of a domain are equal iff they share exactly the same properties expressed in the language of the discourse. For the Leibniz congruence on the formula algebra $\mathbf{Fm}_{\mathcal{L}}$ we simply write Ω instead of $\Omega_{\mathbf{Fm}_{\mathcal{L}}}$.

The *Leibniz operator* on $\mathbf{A}$ is a function $\Omega_{\mathbf{A}} : \mathcal{P}(\mathrm{A}) \to \mathrm{Co}(\mathbf{A})$, which for any $D \subseteq \mathrm{A}$ associates $\Omega_{\mathbf{A}} D$, the largest congruence of $\mathbf{A}$ compatible with D. In [Her96],

Herrmann considered only the Leibniz operator over the formula algebra $\mathbf{Fm}_{\mathcal{L}}$. For any $T \in \mathrm{Th}(S)$, he called ΩT the *relation of indiscernibility* with respect to T given by:

$$\alpha \equiv \beta(\Omega T) \text{ iff for all } \varphi \in \mathrm{Fm}_{\mathcal{L}}, p \in \mathrm{Var}, \varphi(p/\alpha) \in T \Leftrightarrow \varphi(p/\beta) \in T,$$

where $\varphi(p/\alpha)$ is the formula that results from φ replacing p by α.

There are some properties of the Leibniz operator that we need for our study.

Lemma 2.3.4. [BP92, Lemma 5.4] *Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathcal{L}$-algebras, and $h : \mathbf{A} \to \mathbf{B}$ a surjective homomorphism. Then, for every $F \in \mathrm{Fi}_S(\mathbf{B})$, $\Omega_{\mathbf{A}}(h^{-1}[F]) = h^{-1}[\Omega_{\mathbf{B}} F]$.*

Let $\mathbf{A}$ be an algebra. We say that the Leibniz operator is *monotone* on $\mathbf{Fi}_S(\mathbf{A})$ (also called *compatibility property* in [BP86, Definition 2.2] or *order-preserving*), if, for all $F, G \in \mathrm{Fi}_S(\mathbf{A})$ such that $F \subseteq G$ we have $\Omega_{\mathbf{A}} F \subseteq \Omega_{\mathbf{A}} G$. The Leibniz operator is said to be *commute with inverse substitutions* on $\mathbf{Fi}_S(\mathbf{A})$, if, for all $F \in \mathrm{Fi}_S(\mathbf{A})$ and all substitutions e, we have $e^{-1}[\Omega_{\mathbf{A}} F] = \Omega_{\mathbf{A}}(e^{-1}[F])$. If the Leibniz operator Ω is monotone and commutes with inverse substitutions on $\mathbf{Th}(S)$, then $e[\Omega T] \subseteq \Omega(\mathrm{Cn}_S(e[T]))$ for all substitutions e and $T \in \mathrm{Th}(S)$. Indeed, let $T \in \mathrm{Th}(S)$ and e a substitution. Since $e[T] \subseteq \mathrm{Cn}_S(e[T])$, we have $T \subseteq e^{-1}[\mathrm{Cn}_S(e[T])]$ and $e^{-1}[\mathrm{Cn}_S(e[T])] \in \mathrm{Th}(S)$. Hence, by monotonicity of the Leibniz operator, $\Omega T \subseteq \Omega(e^{-1}[\mathrm{Cn}_S(e[T])])$. Since Ω commutes with inverse substitution, $\Omega(e^{-1}[\mathrm{Cn}_S(e[T])]) = e^{-1}[\Omega(\mathrm{Cn}_S(e[T]))]$ which gives that $\Omega T \subseteq e^{-1}[\Omega(\mathrm{Cn}_S(e[T]))]$. Therefore, $e[\Omega T] \subseteq \Omega(\mathrm{Cn}_S(e[T]))$. We say that the Leibniz operator is *meet-continuous* on $\mathbf{Fi}_S(\mathbf{A})$, if, for every family $(F_i)_{i \in I}$ of $\mathrm{Fi}_S(\mathbf{A})$, $\Omega_{\mathbf{A}}\left(\bigcap\{F_i : i \in I\}\right) = \bigcap\{\Omega_{\mathbf{A}} F_i : i \in I\}$. And we say that it is *continuous* on $\mathbf{Fi}_S(\mathbf{A})$, if, for every directed family $(F_i)_{i \in I}$ of $\mathrm{Fi}_S(\mathbf{A})$ such that $\bigcup\{F_i : i \in I\} \in \mathrm{Fi}_S(\mathbf{A})$, $\Omega_{\mathbf{A}}\left(\bigcup\{F_i : i \in I\}\right) = \bigcup\{\Omega_{\mathbf{A}} F_i : i \in I\}$. The Leibniz operator is said to be *injective* on $\mathrm{Fi}_S(\mathbf{A})$, if, for all $F, G \in \mathrm{Fi}_S(\mathbf{A})$, $\Omega_{\mathbf{A}} F = \Omega_{\mathbf{A}} G$ implies $F = G$.

Lemma 2.3.5. [Raf06b, Lemma 5] *If a logic S has no theorem then its Leibniz operator is non-injective on $\mathrm{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$.*

There is another important operator, $\tilde{\Omega}_{\mathbf{A}}$, called the *Suszko operator* of S which maps each S-filter F of an algebra $\mathbf{A}$ to the intersection of the Leibniz congruences of all S-filters containing F. The Suszko operator of S on $\mathbf{A}$ is a function with domain $\mathrm{Fi}_S(\mathbf{A})$ that maps each S-filter F of $\mathbf{A}$ to the congruence on $\mathbf{A}$ compatible with D (not necessary the largest) such that $\tilde{\Omega}_{\mathbf{A}} F := \bigcap\{\Omega_{\mathbf{A}} G : F \subseteq G \in \mathrm{Fi}_S(\mathbf{A})\}$ which is called the *Suszko congruence*. Note that $\tilde{\Omega}_{\mathbf{A}} F \subseteq \Omega_{\mathbf{A}} F$, for every S-filter F of an

algebra $\mathbf{A}$ and it is not difficult to see that $\tilde{\Omega}_{\mathbf{A}}$ is always monotone on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. We can characterize the Suszko congruence by the following condition: for all $a, b \in A$, $\langle a, b \rangle \in \tilde{\Omega}_{\mathbf{A}} F$ iff for all $\varphi(p, q_0, \ldots, q_{k-1}) \in \mathrm{Fm}_{\mathcal{L}}$ and $c_0, \ldots, c_{k-1} \in A$, $\mathrm{Fi}_S^{\mathbf{A}}(F \cup \{\varphi^{\mathbf{A}}(a, c_0, \ldots, c_{k-1})\}) = \mathrm{Fi}_S^{\mathbf{A}}(F \cup \{\varphi^{\mathbf{A}}(b, c_0, \ldots, c_{k-1})\})$. Observe that the Suszko congruence does not only depend on $\mathbf{A}$ and F but also on S through the operator $\mathrm{Fi}_S^{\mathbf{A}}$ of S-filter generation on $\mathbf{A}$. In the case of theories, the Suszko operator is simply characterized in the following way: for all $\varphi, \psi, \alpha \in \mathrm{Fm}_{\mathcal{L}}$ and $p \in \mathrm{Var}(\alpha)$,

$$\varphi \equiv \psi(\tilde{\Omega} T) \text{ iff } T \cup \{\alpha(p/\varphi)\} \dashv\vdash_S T \cup \{\alpha(p/\psi)\}.$$

Given a matrix $\mathrm{M} = \langle \mathbf{A}, D \rangle$ and a matrix congruence θ of M, the quotient of M by θ is the matrix $\langle \mathbf{A}/\theta, D/\theta \rangle$, called the *quotient matrix of* M *by* θ, where $\mathbf{A}/\theta$ is the quotient algebra and D/θ is the set of equivalence classes of the elements in D. There is only one matrix congruence on the quotient of a matrix by its Leibniz congruence, which is the identity relation, denoted by $\triangle_{\mathbf{A}}$ (whereas $\nabla_{\mathbf{A}}$ denotes the totally relation). A matrix $\langle \mathbf{A}, D \rangle$ is said to be *reduced* (or *Leibniz-reduced* or *simple*) if $\Omega_{\mathbf{A}} D = \triangle_{\mathbf{A}}$. To each matrix $\mathrm{M} = \langle \mathbf{A}, D \rangle$ corresponds $\langle \mathbf{A}/\Omega_{\mathbf{A}} D, D/\Omega_{\mathbf{A}} D \rangle$ the *reduced matrix* (also called the *reduction* of M), denoted by $\mathrm{M}/\Omega \mathrm{M}$ or M^*. We denote by $\mathbf{Mod}^*(S)$ the class of all reduced matrix models of S and by $\mathbf{L}^*(S)$ the class of all reduced Lindenbaum matrix models of S.

The class of algebras that is traditionally associated with a logic S is the class of *algebraic reducts* (or *algebraic Leibniz-reducts*) of the reduced matrix models of S, denoted by $\mathbf{Alg}^*(S)$, i.e.,

$$\mathbf{Alg}^*(S) = \{\mathbf{A} : \text{ there exists } F \in \mathrm{Fi}_S(\mathbf{A}) \text{ such that } \langle \mathbf{A}, F \rangle \in \mathbf{Mod}^*(S)\}.$$

The class of algebraic reducts of the matrices in $\mathbf{L}^*$ is denoted by $\mathbf{LAlg}^*(S)$:

$$\mathbf{LAlg}^*(S) = \{\mathbf{Fm}_{\mathcal{L}} : \text{ there exists } T \in \mathrm{Th}(S) \text{ such that } \langle \mathbf{Fm}_{\mathcal{L}}, T \rangle \in \mathbf{L}^*(S)\}.$$

The elements of $\mathbf{LAlg}^*(S)$ are called the *Lindenbaum algebras* of S.

We have the corresponding notation for the Suszko operator. A matrix $\langle \mathbf{A}, D \rangle$ is said to be *Suszko-reduced* if $\tilde{\Omega}_{\mathbf{A}} D = \triangle_{\mathbf{A}}$. Thus to each matrix $\mathrm{M} = \langle \mathbf{A}, D \rangle$ corresponds the Suszko-reduced matrix $\langle \mathbf{A}/\tilde{\Omega}_{\mathbf{A}} D, D/\tilde{\Omega}_{\mathbf{A}} D \rangle$. Obviously, every Leibniz-reduced matrix of S is Suszko-reduced, but the converse is false. We denote by $\mathbf{Mod}^*_{Su}(S)$, $\mathbf{L}^*_{Su}(S)$, $\mathbf{Alg}^*_{Su}(S)$ and $\mathbf{LAlg}^*_{Su}(S)$ the class of Suszko-reduced matrix models of S, the class of Suszko-reduced Lindenbaum matrix models of S, the class of algebraic Suszko-reducts

of the Suszko-reduced matrix models of S and the class of algebraic Suszko-reducts of the Suszko-reduced Lindenbaum matrix models of S, respectively. Actually, the class of algebras that AAL canonically associates to a logic S is the class $\mathbf{Alg}^*_{Su}(S)$. However, this class coincides with $\mathbf{Alg}^*(S)$ for the protoalgebraic logics which are defined in the following chapter.

A *submatrix* of a matrix $\langle \mathbf{A}, D \rangle$ is a matrix of the form $\langle \mathbf{B}, E \rangle$, where $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $E = D \cap B$. The *direct product* of a family of matrices $(\langle \mathbf{A}_i, D_i \rangle)_{i \in I}$ is defined by $\langle \prod_{i \in I} \mathbf{A}_i, \prod_{i \in I} D_i \rangle$, where $\prod_{i \in I} \mathbf{A}_i$ is the usual direct product of the family of $\mathcal{L}$-algebras $(\mathbf{A}_i)_{i \in I}$ and $\prod_{i \in I} D_i$ is the cartesian product of the family of sets $(D_i)_{i \in I}$. A submatrix $\langle \mathbf{A}, D \rangle$ of this direct product is called a *subdirect product* of the family of matrices if $\mathbf{A}$ is a *subdirect product* of the family of algebras $(\mathbf{A}_i)_{i \in I}$, that is, if $\pi_i[A] = A_i$ for each projection function $\pi_i : \mathbf{A} \subseteq \prod_{i \in I} A_i \to A_i$.

Let F be a filter over I and $(\langle \mathbf{A}_i, D_i \rangle)_{i \in I}$ a family of matrices. We define on the direct product $\prod_{i \in I} \mathbf{A}_i$ the binary relation θ_F by the condition: for all $a, b \in \prod_{i \in I} A_i$,

$$a \equiv b(\theta_F) \text{ iff } \{i \in I : a(i) = b(i)\} \in F.$$

The relation θ_F is a congruence relation on the algebra $\prod_{i \in I} \mathbf{A}_i$. The quotient algebra $\prod_{i \in I} \mathbf{A}_i / F$ is called a *reduced product* of the family of algebras $(\mathbf{A}_i)_{i \in I}$. The set $\prod_{i \in I} D_i / F$ of designated elements is defined as follows: for all $a \in \prod_{i \in I} A_i$,

$$a/F \in D/F \text{ iff } \{i \in I : a(i) \in D_i\} \in F.$$

We denote by $\langle \_ \rangle_F$ the class of the congruence θ_F. The *reduced product* of a family of matrices $(\langle \mathbf{A}_i, D_i \rangle)_{i \in I}$ is the matrix $\langle \prod_{i \in I} \mathbf{A}_i / F, \prod_{i \in I} D_i / F \rangle$. If F consists on the set I only, the reduced product is isomorphic with the direct product of the family of matrices $(\langle \mathbf{A}_i, D_i \rangle)_{i \in I}$. We say that F is an *ultrafilter* over I, if F is a filter over I such that for all $X \in \mathcal{P}(I)$, $X \in F$ iff $I \setminus X \notin F$. If F is an ultrafilter over I, then $\langle \prod_{i \in I} \mathbf{A}_i / F, \prod_{i \in I} D_i / F \rangle$ is called an *ultraproduct* of the family of matrices $(\langle \mathbf{A}_i, D_i \rangle)_{i \in I}$.

2.4   Quasivariety

By an $\mathcal{L}$-*equation* (or simply an *equation*), we mean a formal expression $\varphi \approx \psi$, with $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$. Sometimes it is useful to see an equation as a pair of formulas $\langle \varphi, \psi \rangle$.

We denote by $\mathrm{Eq}_{\mathcal{L}}$ the set of all $\mathcal{L}$-equations. A *quasi-equation* is a formal expression $\xi_0 \approx \eta_0 \wedge \cdots \wedge \xi_{n-1} \approx \eta_{n-1} \rightarrow \varphi \approx \psi$ with $\xi_0, \ldots, \xi_{n-1}, \eta_0, \ldots, \eta_{n-1}, \varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$. Similarly, we can see quasi-equation as a pair $\langle \Gamma, \lambda \rangle$, where Γ is a finite set of equations and λ is an equation. Equations can be seen as special case of quasi-equations.

Let $\varphi \approx \psi$ be an equation, Γ a set of equations and $\mathbf{A}$ an algebra. We write $\Gamma \models_{\mathbf{A}} \varphi \approx \psi$ if, for every homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \rightarrow \mathbf{A}$,

$$h(\xi) = h(\eta) \text{ for every } \xi \approx \eta \in \Gamma \text{ implies } h(\varphi) = h(\psi).$$

If $\Gamma = \emptyset$, we write $\models_{\mathbf{A}} \varphi \approx \psi$ instead of $\emptyset \models_{\mathbf{A}} \varphi \approx \psi$. An equation $\varphi \approx \psi$ is an *identity* of $\mathbf{A}$ if $\models_{\mathbf{A}} \varphi \approx \psi$. Similarly, a quasi-equation $\xi_0 \approx \eta_0 \wedge \cdots \wedge \xi_{n-1} \approx \eta_{n-1} \rightarrow \varphi \approx \psi$ is a *quasi-identity* of $\mathbf{A}$ if $\{\xi_0 \approx \eta_0, \ldots, \xi_{n-1} \approx \eta_{n-1}\} \models_{\mathbf{A}} \varphi \approx \psi$.

Let $\mathbf{K}$ be a class of $\mathcal{L}$-algebras. The (*semantic*) *equational consequence relation* $\models_{\mathbf{K}}$ determined by $\mathbf{K}$ is the relation between a set Γ of equations and a single equation $\varphi \approx \psi$, denoted by $\Gamma \models_{\mathbf{K}} \varphi \approx \psi$ and defined in the following way:

$$\Gamma \models_{\mathbf{K}} \varphi \approx \psi \text{ iff, for every } \mathbf{A} \in \mathbf{K} \text{ we have } \Gamma \models_{\mathbf{A}} \varphi \approx \psi.$$

In this case we say that $\varphi \approx \psi$ is a $\mathbf{K}$-*consequence* of Γ. We write $\models_{\mathbf{K}} \varphi \approx \psi$ instead of $\emptyset \models_{\mathbf{K}} \varphi \approx \psi$. If Γ, Γ' are sets of equations, then we write $\Gamma \models_{\mathbf{K}} \Gamma'$ for $\Gamma \models_{\mathbf{K}} \varphi \approx \psi$ for all $\varphi \approx \psi \in \Gamma'$, and $\Gamma =\!\!\models_{\mathbf{K}} \Gamma'$ when $\Gamma \models_{\mathbf{K}} \Gamma'$ and $\Gamma' \models_{\mathbf{K}} \Gamma$ hold. If $\mathbf{K}$ is the class of the $\mathcal{L}$-algebras that satisfy a given set of equations then it is called a *variety* and if they satisfy a set of quasi-equations then it is called a *quasivariety*. A variety or a quasivariety is *trivial* if it contains, up to isomorphism, only the one-element algebra. Since the intersection of a class of varieties of type $\mathcal{L}$ is again a variety and the class of all $\mathcal{L}$-algebras forms a variety, we can conclude that for every class $\mathbf{K}$ of algebras of a same type there is a smallest variety containing $\mathbf{K}$, denoted by $V(\mathbf{K})$ and called the *variety generated* by $\mathbf{K}$. If $\mathbf{K}$ has a single member $\mathbf{A}$, we write simply $V(\mathbf{A})$. A variety V is *finitely generated* if $V = V(\mathbf{K})$ for some finite set $\mathbf{K}$ of algebras. We can defined the same notions for quasivariety. For example, there is a smallest quasivariety containing $\mathbf{K}$ and denoted by $Q(\mathbf{K})$.

We assume that the reader is familiar with notions of universal algebra, as homomorphism, isomorphism, direct product, etc. We introduce the following operators mapping classes of algebras to classes of algebras (all of the same type):

$\mathbf{A} \in I(\mathbf{K})$ iff $\mathbf{A}$ is isomorphic to some member of $\mathbf{K}$;

$\mathbf{A} \in S(\mathbf{K})$ iff $\mathbf{A}$ is an isomorphic copy of subalgebra of some member of $\mathbf{K}$;

$\mathbf{A} \in H(\mathbf{K})$ iff $\mathbf{A}$ is a homomorphic image of some member of $\mathbf{K}$;

$\mathbf{A} \in P(\mathbf{K})$ iff $\mathbf{A}$ is an isomorphic copy of direct product of a nonempty family of algebras in $\mathbf{K}$;

$\mathbf{A} \in P_S(\mathbf{K})$ iff $\mathbf{A}$ is an isomorphic copy of subdirect product of a nonempty family of algebras in $\mathbf{K}$;

$\mathbf{A} \in P_R(\mathbf{K})$ iff $\mathbf{A}$ is an isomorphic copy of reduced product of a nonempty family of algebras in $\mathbf{K}$;

$\mathbf{A} \in P_U(\mathbf{K})$ iff $\mathbf{A}$ is an isomorphic copy of ultraproduct of a nonempty family of algebras in $\mathbf{K}$.

A variety can be characterized as a nonempty class $\mathbf{K}$ of $\mathcal{L}$-algebras which is closed under homomorphic images, subalgebras and direct products.

Theorem 2.4.1. [BS81, Chatper II, Theorem 9.5] *Let $\mathbf{K}$ be a class of algebras. Then,* $V(\mathbf{K}) = \mathrm{HSP}(\mathbf{K})$

Theorem 2.4.2. [BS81, Chapter V, Theorem 2.25] *Let $\mathbf{K}$ be a class of algebras. Then the following are equivalent:*

(i) $\mathbf{K}$ can be axiomatized by quasi-identities;

(ii) $\mathbf{K}$ is a quasivariety;

(iii) $\mathbf{K}$ is closed under I,S,P *and* P_U *and contains a trivial algebra;*

(iv) $\mathbf{K}$ is closed under ISP_R *and contains a trivial algebra;*

(v) $\mathbf{K}$ is closed under ISPP_U *and contains a trivial algebra.*

2.5 Equational Logic

The equational consequence relation $\models_{\mathbf{K}}$ satisfies the following conditions:

$$\varphi \approx \psi \in \Gamma \Rightarrow \Gamma \models_{\mathbf{K}} \varphi \approx \psi \qquad \text{(Reflexivity)}$$

$$\Gamma \models_{\mathbf{K}} \varphi \approx \psi \text{ and } \Gamma \subseteq \Delta \Rightarrow \Delta \models_{\mathbf{K}} \varphi \approx \psi \qquad \text{(Cut)}$$

$$\Gamma \models_{\mathbf{K}} \varphi \approx \psi \text{ and } \Delta \models_{\mathbf{K}} \xi \approx \eta \text{ for all } \xi \approx \eta \in \Gamma \Rightarrow \Delta \models_{\mathbf{K}} \varphi \approx \psi \qquad \text{(Weakening)}$$

$$\Gamma \models_{\mathbf{K}} \varphi \approx \psi \Rightarrow e[\Gamma] \models_{\mathbf{K}} e(\varphi) \approx e(\psi) \text{ for all substitution } e \qquad \text{(Structurality)}$$

The relation $\models_{\mathbf{K}}$ is called *finitary* if

$$\Gamma \models_{\mathbf{K}} \varphi \approx \psi \text{ implies } \Gamma' \models_{\mathbf{K}} \varphi \approx \psi \text{ for some finite } \Gamma' \subseteq \Gamma.$$

The equational consequence relation $\models_{\mathbf{K}}$ associated with a quasivariety $\mathbf{K}$ is an example of a 2-deductive system. For more information about k-deductive systems (in a general case), we point out to [BP92, Chapter 1], [CP99, Definition 2.1] and [CP04a, Definition 1]. In this context, we deal with an equation $\varphi \approx \psi$ as a 2-formula $\langle \varphi, \psi \rangle$. We denote by $\langle \mathcal{L}, \models_{\mathbf{K}} \rangle$ the *equational logic* associated to $\mathbf{K}$. All notions applicable to deductive systems, which are 1-deductive systems, transfer naturally to 2-deductive systems, and in particular to equational logic.

A set of equations Γ is called an *equational theory* of $\mathbf{K}$ ($\models_{\mathbf{K}}$-*theory* or $\mathbf{K}$-*theory* for short) if $\Gamma \models_{\mathbf{K}} \varphi \approx \psi$ implies $\varphi \approx \psi \in \Gamma$, i.e., if Γ is closed under $\mathbf{K}$-consequence. The set of all $\mathbf{K}$-theories is denoted by $\mathrm{Th}(\mathbf{K})$. It is closed under arbitrary intersection. It forms a complete lattice $\mathbf{Th}(\mathbf{K}) = \langle \mathrm{Th}(\mathbf{K}), \cap, \vee \rangle$ where the largest theory is the set $\mathrm{Eq}_{\mathcal{L}}$ and the smallest is the set of identities of $\mathbf{K}$. Let Γ be a set of equations. We denote by $\mathrm{Cn}_{\mathbf{K}}\Gamma = \{\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}} : \Gamma \models_{\mathbf{K}} \varphi \approx \psi\}$ the smallest $\mathbf{K}$-theory containing Γ. The notion of *generators* of a $\mathbf{K}$-theory is defined in the obvious way. We note that the theories of an equational logic are exactly the $\mathbf{K}$-congruences on the formula algebra $\mathbf{Fm}_{\mathcal{L}}$.

Lemma 2.5.1. *Let $\mathbf{K}$ be a class of algebras. If $\mathbf{K}$ is closed under ultraproducts, then $\models_{\mathbf{K}}$ is finitary.*

Proof. Assume that $\models_{\mathbf{K}}$ is closed under ultraproducts. Suppose $\models_{\mathbf{K}}$ is not finitary. Let $\Gamma \cup \{\varphi \approx \psi\} \in \mathrm{Eq}_{\mathcal{L}}$. Consider $\Gamma \models_{\mathbf{K}} \varphi \approx \psi$. Thus for all finite set $\Gamma' \subseteq \Gamma$, $\Gamma' \not\models_{\mathbf{K}} \varphi \approx \psi$. Let I be the set of all indices i such that Γ_i is a finite subset of Γ, i.e., $I = \{i : \Gamma_i \text{ is a finite subset of } \Gamma\}$. Consider the set $i^* := \{j \in I : \Gamma_i \subseteq \Gamma_j\}$, for all $i \in I$. It is not difficult to see that the family $(i^*)_{i \in I}$ have a finite intersection property, in the sense that for all $I' \subseteq I$, $\bigcap_{i \in I'} i^* \neq \emptyset$ (because the propositional language have a countably infinite set of variables). Thus there exists a proper filter which contains the family $(i^*)_{i \in I}$. This proper filter can be extended to an ultrafilter, i.e., there exists an ultrafilter U such that $(i^*)_{i \in I} \subseteq U$. By hypothesis, we have that for each $i \in I$, there exists an $\mathcal{L}$-algebra $\mathbf{A}_i$ of $\mathbf{K}$ such that $\Gamma_i \not\models_{\mathbf{A}_i} \varphi \approx \psi$. Thus, for each $i \in I$, there exists a homomorphism $h_i : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}_i$ such that $h_i(\xi) = h_i(\eta)$, for all $\xi \approx \eta \in \Gamma_i$ and

$h_i(\varphi) \neq h_i(\psi)$. Let $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i/U$ be the ultraproducts of the family $(\mathbf{A}_i)_{i \in I}$. Since $\mathbf{K}$ is closed under ultraproducts, we have that $\mathbf{A} \in \mathbf{K}$. Let $h := \langle h_i : i \in I \rangle_F : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$. For all $\xi \approx \eta \in \Gamma$, if $h(\xi) = h(\eta)$ then $h_i(\xi) = h_i(\eta)$ for all $i \in I$. Which implies that $h_i(\varphi) \neq h_i(\psi)$ for all $i \in I$, i.e., $h(\varphi) \neq h(\psi)$. Thus, $\mathbf{A} \in \mathbf{K}$ and $\Gamma \nvDash_{\mathbf{A}} \varphi \approx \psi$. Hence we have a contradiction with the hypothesis. Therefore $\vDash_{\mathbf{K}}$ is finitary. $\qquad\square$

We can also show that if a logic S is finitary then the class $\mathbf{Mod}(S)$ is closed under ultraproducts.

In the following lemma, we give some properties of $\mathrm{Th}(\mathbf{K})$ whenever $\vDash_{\mathbf{K}}$ is a finitary equational logic.

Lemma 2.5.2. [BP89, Lemma 3.1] *Let $\mathbf{K}$ be a class of algebras. Then the following conditions are equivalent:*

(i) $\vDash_{\mathbf{K}}$ *is finitary;*

(ii) $\vDash_{\mathbf{K}}$ *coincides with* $\vDash_{Q(\mathbf{K})}$*;*

(iii) *The compact elements of* $\mathbf{Th}(\mathbf{K})$ *coincide with the finitely generated $\mathbf{K}$-theories;*

(iv) $\mathrm{Th}(\mathbf{K})$ *is closed under directed unions;*

(v) *The lattice* $\mathbf{Th}(\mathbf{K})$ *is algebraic.*

An equational logic $\vDash_{\mathbf{K}}$ can be viewed in several ways. Indeed, it can be also defined by the consequence operator $\mathrm{Cn}_{\mathbf{K}}$ or by the theory lattice $\mathbf{ThK}$.

If $\mathbf{K}$ is a quasivariety axiomatized by a set Γ of identities and quasi-identities, then $\vDash_{\mathbf{K}}$ can be viewed as an equational consequence relation over the set of $\mathcal{L}$-equations defined by axioms and inference rules as follows: for axioms, we have,

$p \approx p$

$\varphi \approx \psi$, for every equation $\varphi \approx \psi \in \Gamma$

And for inference rules,

$$\frac{p \approx q}{q \approx p}$$

$$\frac{p \approx q, \; q \approx r}{p \approx r}$$

$$\frac{\{p_i \approx q_i : i < m\}}{f(p_0, \ldots, p_{m-1}) \approx f(q_0, \ldots, q_{m-1})} \text{ for every } f \in \mathcal{L} \text{ with arity } m$$

$$\frac{\{\varphi_i \approx \psi_i : i < n\}}{\varphi \approx \psi}$$ for every quasi-equation $(\varphi_0 \approx \psi_0 \wedge \cdots \wedge \varphi_{n-1} \approx \psi_{n-1}) \to \varphi \approx \psi \in \Gamma.$

For each homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, the set of equations $\{\varphi \approx \psi : h(\varphi) = h(\psi)\}$ is a congruence called the *relation-kernel* of h. The relation-kernel of the natural mapping of $\mathbf{Fm}_{\mathcal{L}}$ onto $\mathbf{Fm}_{\mathcal{L}}/\theta$ is θ itself. For any homomorphism h of $\mathbf{Fm}_{\mathcal{L}}$ into a member of $\mathbf{K}$, the relation-kernel θ of h is a $\mathbf{K}$-theory. More generally, for any $\Gamma \subseteq \mathrm{Eq}_{\mathcal{L}}$, the $\mathbf{K}$-theory $\mathrm{Cn}_{\mathbf{K}}\Gamma$ generated by Γ can be characterized as the intersection of the relation-kernels θ of all homomorphisms of $\mathbf{Fm}_{\mathcal{L}}$ into members of $\mathbf{K}$ such that $h(\xi) = h(\eta)$ for all $\xi \approx \eta \in \Gamma$. If $\mathbf{K}$ is a quasivariety then a set of equations Γ is a $\mathbf{K}$-theory iff $\theta = \{(\varphi, \psi) : \varphi \approx \psi \in \Gamma\}$ is a $\mathbf{K}$-congruence on $\mathbf{Fm}_{\mathcal{L}}$, i.e., θ is a congruence on $\mathbf{Fm}_{\mathcal{L}}$ and $\mathbf{Fm}_{\mathcal{L}}/\theta \in \mathbf{K}$. A *matrix homomorphism* (a *strict homomorphism*) from $\mathrm{M} = \langle \mathbf{A}, D \rangle$ to $\mathrm{N} = \langle \mathbf{B}, E \rangle$ is an $h \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$ such that $D \subseteq h^{-1}[E]$ ($D = h^{-1}[E]$ respectively). We denote by $\mathrm{Hom}(\mathrm{M}, \mathrm{N})$ the set of all matrix homomorphisms from M to N, and by $\mathrm{Hom}_S(\mathrm{M}, \mathrm{N})$ the set of all strict homomorphisms from M to N. The kernel of a strict homomorphism from M to N is a matrix congruence on M, and every matrix congruence θ of a matrix $\langle \mathbf{A}, D \rangle$ can be obtained as the kernel of the projection of $\langle \mathbf{A}, D \rangle$ onto $\langle \mathbf{A}/\theta, D/\theta \rangle$.

The class $\mathbf{Mod}(S)$ is closed under strict homomorphic pre-image, strict homomorphic image, submatrices and direct products (c.f. [Cze01, Corollary 0.3.10]). Moreover, if S is finitary then $\mathbf{Mod}(S)$ is closed under reduced products (c.f. [Cze01, Corollary 0.3.10]). We say that a class of matrices is a *matrix-quasivariety* if it is closed under submatrices, direct products and ultraproducts.

Chapter 3

Protoalgebraic Logics

In this chapter, we consider a wide class of logics called protoalgebraic logics. We give some characterizations of this class using two operators, namely the Leibniz and the Suszko operators. We show that a logic S is protoalgebraic iff it has an k-parameterized system of equivalence formulas, or equivalently, if it has the parameterized local deduction-detachment theorem (PLDDT for short). We also study the relationship between the structural properties of the class of reduced matrix models and metalogical properties of protoalgebraic logics. As we have pointed out in the introduction, we emphasize some results about finitary protoalgebraic logics. We give examples to illustrate some results. For more details about protoalgebraic logics, we suggest [BP86] and [Cze01, Chapter 1 and 2].

3.1 Definitions

Let S be a logic and T an S-theory. Two formulas α and β are said to be *T-indiscernible* relative to S (or *T-equivalent* relative to S in [BP86]) if for every formula $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ and every variable p occurring in φ, $T \vdash_S \varphi(p/\alpha)$ iff $T \vdash_S \varphi(p/\beta)$, where $\varphi(p/\alpha)$ is the formula that results from φ replacing the variable p by α. Equivalently, α and β are T-indiscernible iff α and β are congruent modulo the Leibniz congruence ΩT on $\mathbf{Fm}_{\mathcal{L}}$.

We say that two formulas α and β are *T-interderivable* relative to S (or *inferentially equivalent* in [Mal89]) if $T, \alpha \vdash_S \beta$ and $T, \beta \vdash_S \alpha$ where $T, \alpha \vdash_S \beta$ means $T \cup \{\alpha\} \vdash_S \beta$. The notion of protoalgebraic logic was defined by Blok and Pigozzi in [BP86, Definition 2.1].

Definition 3.1.1. *A logic S is called* protoalgebraic *if, for every S-theory T, any two formulas which are T-indiscernible relative to S are T-interderivable relative to S, i.e., for all $T \in \mathrm{Th}(S)$ and $\alpha, \beta \in \mathrm{Fm}_\mathcal{L}$,*

$$\alpha \equiv \beta(\Omega T) \text{ implies } T, \alpha \vdash_S \beta \text{ and } T, \beta \vdash_S \alpha.$$

Moreover, if the reverse implication holds, the logic is called selfextensional.

Any conservative expansion of a protoalgebraic logic is also protoalgebraic ([BP86, Theorem 2.11]).

A logic S is called *non-pathological* by Czelakowski (c.f. [Her96]) if there is a set $\Delta(p, q)$ of formulas in two variables p and q such that

$$\vdash_S \Delta(p, p) \qquad\qquad \text{(Reflexivity)}$$

$$\{p\} \cup \Delta(p, q) \vdash_S q \qquad\qquad \text{(Modus Ponens)}$$

The set Δ is called a *system of implication formulas* or a *protoequivalence system* for

S.

The set $T_{pq} := \{\varphi(p, q, r_1, \ldots, r_n) : \ \vdash_S \varphi(p, p, r_1, \ldots, r_n)\}$ is a set of all formulas $\varphi(p, q, r_1, \ldots, r_n)$ which become theorems of S after the identification of the variables p and q in φ. We often use this set because if a logic is protoalgebraic then T_{pq} is a protoequivalence system and also an k-parameterized system of equivalence formulas.

Lemma 3.1.2. *Let S be a logic. Then,*

    (i) The set T_{pq} is closed under any substitution e such that $(e(p))(q/p) = (e(q))(q/p)$.

    (ii) $p \equiv q(\Omega T_{pq})$.

Proof. (i) Let $\varphi \in T_{pq}$ and e a substitution such that $(e(p))(q/p) = (e(q))(q/p)$. We have that $\vdash_S \varphi(q/p)$. By structurality of S, $\vdash_S e(\varphi(q/p))$. Note that $e(\varphi(q/p)) = (e(\varphi(q/p)))(q/p)$. Thus $\vdash_S e(\varphi(q/p))(q/p)$. This prove that $e(\varphi) \in T_{pq}$.

    (ii) Let $\varphi \in \mathrm{Fm}_\mathcal{L}$ and $r \in \mathrm{Var}$. We have that $(\varphi(r/p))(q/p) = (\varphi(r/q))(q/p)$. Thus $\vdash_S (\varphi(r/p))(q/p)$ iff $\vdash_S (\varphi(r/q))(q/p)$, i.e., $\varphi(r/p) \in T_{pq}$ iff $\varphi(r/q) \in T_{pq}$. Which means that $p \equiv q(\Omega T_{pq})$. $\qquad\square$

Blok and Pigozzi proved that the protoalgebraic logics are exactly the non-pathological ones.

Theorem 3.1.3. *Let S be a logic. The following conditions are equivalent:*

(i) *S is protoalgebraic;*

(ii) *S is non-pathological.*

Proof. Assume S is protoalgebraic. Let e be a substitution such that $e(p) = p$, $e(q) = q$ and $e(r_i) = p$ for all $i \in I$. Consider $\Delta(p, q) = e[T_{pq}]$. Since $(e(p))(q/p) = (e(q))(q/p)$, by Lemma 3.1.2, $\Delta(p, q) \subseteq T_{pq}$. Thus reflexivity condition holds. Again, by Lemma 3.1.2, $p \equiv q(\Omega T_{pq})$. Since S is protoalgebraic, we have that $T_{pq}, p \vdash_S q$ and $T_{pq}, q \vdash_S p$. Thus T_{pq} satisfies modus ponens condition. Obviously, $\Delta(p, q)$ also satisfies modus ponens condition. We conclude that $\Delta(p, q)$ is a protoequivalence system for S.

Conversely, assume S is non-pathological. There exists $\Delta(p, q)$ a protoequivalence system for S. Let $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}$ and $T \in \mathrm{Th}(S)$. Suppose $\alpha \equiv \beta(\Omega T)$. Then $\varphi(\alpha, \alpha) \equiv \varphi(\alpha, \beta)(\Omega T)$ for every formula $\varphi(p, q) \in \Delta(p, q)$. By compatibility, $\Delta(\alpha, \alpha) \subseteq T$ iff $\Delta(\alpha, \beta) \subseteq T$. Since $\Delta(p, q)$ is reflexive, $\vdash_S \Delta(\alpha, \alpha)$, which implies that $\Delta(\alpha, \alpha) \subseteq T$. Thus $\Delta(\alpha, \beta) \subseteq T$. Since $\Delta(\alpha, \beta) \cup \{\alpha\} \subseteq T \cup \{\alpha\}$, by modus ponens, $T, \alpha \vdash_S \beta$. In an analogous way we have $T, \beta \vdash_S \alpha$. $\square$

Whenever a logic S has a binary connective $\rightarrow$ for which $p \rightarrow p$ is a theorem and modus ponens is an inference rule, then S is protoalgebraic with $\Delta(p, q) = \{p \rightarrow q\}$ as a protoequivalence system. The set Δ may be empty and in this case the logic have the rule $p \vdash_S q$ for all $p, q \in \mathrm{Fm}_{\mathcal{L}}$, i.e., S is inconsistent or almost inconsistent.

Moreover, if S is a finitary and protoalgebraic logic, then the protoequivalence system Δ can be taken to be finite. Indeed, since $\{p\} \cup \Delta(p, q) \vdash_S q$, the finitariness of S implies $\{p\} \cup \Delta'(p, q) \vdash_S q$ for some finite $\Delta' \subseteq \Delta$. Trivially, $\vdash_S \Delta(p, p)$ implies $\vdash_S \Delta'(p, p)$. Thus Δ' is also a protoequivalence system for S.

Example 3.2 (Orthologic [Mal89])**.** An algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg \rangle$ is called an *ortholattice* if the reduct $\langle A, \wedge, \vee \rangle$ is a lattice, $0 := x \wedge \neg x$ and $1 := x \vee \neg x$ are distinguished constant terms in $\mathbf{A}$ interpreted as the least and the greatest element of the lattice $\langle A, \wedge, \vee \rangle$, and if $\mathbf{A}$ satisfies the identities $\neg(x \wedge y) \approx \neg x \vee \neg y$ and $\neg\neg x \approx x$. We denote by OL the class of all ortholattices. Each ortholattice $\mathbf{A}$ can be identified with the matrix $\langle \mathbf{A}, \{1\} \rangle$, where the unit element of the lattice is the only one designated element.

Let S_{OL} be the *minimal orthologic* defined in the language $\mathcal{L} = \{\wedge, \vee, \neg\}$ by the structural consequence relation $\vdash_{S_{OL}}$ determined by the class OL in the following way:

for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\Gamma \vdash_{S_{OL}} \varphi \text{ iff, for all } \mathbf{A} \in OL, \Gamma \models_{\langle \mathbf{A}, \{1\} \rangle} \varphi.$$

Let $p \to q := \neg p \vee q$. Since $p \to p = \neg p \vee p = 1$, we have that $\vdash_{S_{OL}} p \to p$. By lattices properties, we also have $p, p \to q \vdash_{S_{OL}} q$. Thus $\Delta(p, q) = \{p \to q\}$ is a protoequivalence system for S. By Theorem 3.1.3, S_{OL} is protoalgebraic. $\diamondsuit$

3.3 Leibniz Operator

In this section, we give theorems that characterized the class of protoalgebraic logics using the Leibniz operator defined in the Chapter 2 and other properties.

Theorem 3.3.1. *Let S be a logic. The following conditions are equivalent:*

(i) S is protoalgebraic;

(ii) The Leibniz operator $\Omega_{\mathbf{A}}$ is monotone on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$;

(iii) The Leibniz operator Ω is monotone on $\mathbf{Th}(S)$.

Proof. $(i) \Rightarrow (ii)$ Assume S is protoalgebraic. Let $\mathbf{A}$ be an $\mathcal{L}$-algebra and $E, F \in \mathrm{Fi}_S(\mathbf{A})$. Suppose $E \subseteq F$. To prove that $\Omega_{\mathbf{A}} E \subseteq \Omega_{\mathbf{A}} F$, it suffices to show that $\Omega_{\mathbf{A}} E$ is compatible with F. Let $a, b \in A$. Suppose $a \in F$ and $a \equiv b(\Omega_{\mathbf{A}} E)$. Since S is protoalgebraic, by Theorem 3.1.3, there exists a protoequivalence system $\Delta(p, q)$ for S. Let $\delta(p, q) \in \Delta$. Then $\delta(a, a) \equiv \delta(a, b)(\Omega_{\mathbf{A}} E)$. By compatibility with E, $\Delta^{\mathbf{A}}(a, a) \subseteq E$ iff $\Delta^{\mathbf{A}}(a, b) \subseteq E$. Since $\Delta(p, q)$ is reflexive, $\Delta^{\mathbf{A}}(a, a) \subseteq E$. Hence, $\Delta^{\mathbf{A}}(a, b) \subseteq E$. Since $a \in F$ and $E \subseteq F$, we have that $\{a\} \cup \Delta^{\mathbf{A}}(a, b) \subseteq F$. By Modus Ponens, we conclude that $b \in F$, i.e., $\Omega_{\mathbf{A}} E \subseteq \Omega_{\mathbf{A}} F$.

$(ii) \Rightarrow (iii)$ Assume the Leibniz operator $\Omega_{\mathbf{A}}$ is monotone on $\mathbf{Fi}_S(\mathbf{A})$, for every algebra $\mathbf{A}$. Since $\mathbf{Th}(S) \subseteq \mathbf{Fi}_S(\mathbf{A})$, Ω is monotone on $\mathbf{Th}(S)$.

$(iii) \Rightarrow (i)$ Assume the Leibniz operator Ω is monotone on $\mathbf{Th}(S)$. Let $T \in \mathrm{Th}(S)$ and $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}$. Suppose $\alpha \equiv \beta(\Omega T)$. Since $T \subseteq T \cup \{\alpha\}$, by monotonicity of Ω, we have $\Omega T \subseteq \Omega(Cn_S(T \cup \{\alpha\}))$. Thus $\alpha \equiv \beta(\Omega(Cn_S(T \cup \{\alpha\})))$. Since $\alpha \in Cn_S(T \cup \{\alpha\})$, by compatibility, $\beta \in Cn_S(T \cup \{\alpha\})$, i.e., $T, \alpha \vdash_S \beta$. In an analogous way, we obtain $T, \beta \vdash_S \alpha$. Therefore S is protoalgebraic. $\square$

In the next theorem, we give another characterization of protoalgebraic logics using again the Leibniz operator.

Theorem 3.3.2. *Let S be a logic. The following conditions are equivalent:*

(i) *S is protoalgebraic;*

(ii) *The Leibniz operator $\Omega_{\mathbf{A}}$ is meet-continuous on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$;*

(iii) *The Leibniz operator Ω is meet-continuous on $\mathbf{Th}(S)$.*

Proof. $(i) \Rightarrow (ii)$ Assume S is protoalgebraic. Let $\mathbf{A}$ be an $\mathcal{L}$-algebra. The inclusion $\bigcap\{\Omega_{\mathbf{A}}F_i : i \in I\} \subseteq \Omega_{\mathbf{A}}\big(\bigcap\{F_i : i \in I\}\big)$ always holds. Indeed, let $a \equiv b\big(\bigcap\{\Omega_{\mathbf{A}}F_i : i \in I\}\big)$ and $a \in \bigcap\{F_i : i \in I\}$. Thus, for all $i \in I$, $a \equiv b(\Omega_{\mathbf{A}}F_i)$ and $a \in F_i$. By compatibility, for all $i \in I$, $b \in F_i$, i.e., $b \in \bigcap\{F_i : i \in I\}$. Hence, $\bigcap\{\Omega_{\mathbf{A}}F_i : i \in I\} \subseteq \Omega_{\mathbf{A}}\big(\bigcap\{F_i : i \in I\}\big)$. For the reverse inclusion, we have $\bigcap\{F_i : i \in I\} \subseteq F_i$ for all $i \in I$. Since S is protoalgebraic, by Theorem 3.3.1, $\Omega_{\mathbf{A}}$ is monotone on $\mathbf{Fi}_S(\mathbf{A})$. Thus, $\Omega_{\mathbf{A}}\big(\bigcap\{F_i : i \in I\}\big) \subseteq \Omega_{\mathbf{A}}F_i$ for all $i \in I$, i.e., $\Omega_{\mathbf{A}}\big(\bigcap\{F_i : i \in I\}\big) \subseteq \bigcap\{\Omega_{\mathbf{A}}F_i : i \in I\}$. We conclude that $\Omega_{\mathbf{A}}$ is meet-continuous on $\mathbf{Fi}_S(\mathbf{A})$.

$(ii) \Rightarrow (iii)$ It is obvious.

$(iii) \Rightarrow (i)$ Let $T_1, T_2 \in \mathrm{Th}(S)$. Suppose $T_1 \subseteq T_2$. By assumption, $\Omega\big(T_1 \cap T_2\big) = \Omega T_1 \cap \Omega T_2$. Since $T_1 \cap T_2 = T_1$, we have $\Omega T_1 = \Omega\big(T_1 \cap T_2\big) = \Omega T_1 \cap \Omega T_2$. Thus $\Omega T_1 \subseteq \Omega T_2$. By Theorem 3.3.1, S is protoalgebraic. $\square$

A logic S has the *correspondence property* if, for every strict homomorphism $h : M \to N$ between matrix models of S and every filter $F \in \mathrm{Fi}_S(M)$, we have $F = h^{-1}[h[F]]$. We say that S has the *compatibility property* if, for every algebra $\mathbf{A}$, any $\theta \in \mathrm{Co}\mathbf{A}$ which is compatible with an S-filter F of $\mathbf{A}$, is also compatible with every S-filter that includes F. In the next theorem, we give a characterization of protoalgebraic logics using these properties.

Theorem 3.3.3. *Let S be a logic. The following conditions are equivalent:*

(i) *S is protoalgebraic;*

(ii) *S has the compatibility property;*

(iii) *S has the correspondence property.*

Proof. $(i) \Rightarrow (ii)$ Assume S is protoalgebraic. Let $\mathbf{A}$ be an S-algebra, $F \in \mathrm{Fi}_S(\mathbf{A})$ and $\theta \in \mathrm{Co}\mathbf{A}$ which is compatible with F. Let $G \in \mathrm{Fi}_S(\mathbf{A})$ such that $F \subseteq G$, and $a, b \in A$ such that $a \equiv b(\theta)$ and $a \in G$. Since S is protoalgebraic, by Theorem 3.3.1, $\theta \subseteq \Omega_{\mathbf{A}}F \subseteq \Omega_{\mathbf{A}}G$. Thus $a \equiv b(\Omega_{\mathbf{A}}G)$ and by compatibility, $b \in G$. Therefore θ is compatible with G.

$(ii) \Rightarrow (i)$ Let $F, G \in \mathrm{Fi}_S(\mathbf{A})$. Suppose $F \subseteq G$. Since $\Omega_{\mathbf{A}} F$ is compatible with F, by the compatibility property, $\Omega_{\mathbf{A}} F$ is also compatible with G. Thus $\Omega_{\mathbf{A}} F \subseteq \Omega_{\mathbf{A}} G$. By Theorem 3.3.1, S is protoalgebraic.

$(i) \Rightarrow (iii)$ Since S is protoalgebraic, by Theorem 3.1.3, there exists a protoequivalence system $\Delta(p, q)$ for S. Let $\mathrm{M} = \langle \mathbf{A}, D \rangle$ and $\mathrm{N} = \langle \mathbf{B}, E \rangle$ be matrix models of S, h a strict homomorphism of M into N and $F \in \mathrm{Fi}_S(M)$. We always have that $F \subseteq h^{-1}[h[F]]$. For the reverse inclusion, assume $a \in h^{-1}h[F]$, i.e, $h(a) \in h[F]$. Then, there exists $c \in F$ such that $h(a) = h(c)$. Since $E \in \mathrm{Fi}_S(\mathbf{B})$, $\Delta^{\mathbf{B}}\big(h(c), h(a)\big) \subseteq E$. As h is a homomorphism, $h\big(\Delta^{\mathbf{A}}(c, a)\big) = \Delta^{\mathbf{B}}\big(h(c), h(a)\big)$. Thus $h\big(\Delta^{\mathbf{A}}(c, a)\big) \subseteq E$. Since h is strict, $\Delta^{\mathbf{A}}(c, a) \subseteq h^{-1}[E] = D \subseteq F$. So $\{c\} \cup \Delta^{\mathbf{A}}(c, a) \subseteq F$ and by modus ponens $a \in F$. We conclude that $F = h^{-1}h[F]$.

$(iii) \Rightarrow (i)$ Assume S has the correspondence property. Let $T_1, T_2 \in \mathrm{Th}(S)$. Suppose $T_1 \subseteq T_2$. The canonical mapping $h : \mathrm{Fm}_{\mathcal{L}} \to \mathrm{Fm}_{\mathcal{L}}/\Omega T_1$ is a strict homomorphism from $\mathrm{M} := \langle \mathrm{Fm}_{\mathcal{L}}, T_1 \rangle$ onto $\mathrm{N} := \langle \mathrm{Fm}_{\mathcal{L}}/\Omega T_1, T_1/\Omega T_1 \rangle$. Since $T_2 \in \mathrm{Fi}_S(M)$, by assumption, $T_2 = h^{-1}[h[T_2]]$. Thus h is a strict homomorphism from $\langle \mathrm{Fm}_{\mathcal{L}}, T_2 \rangle$ onto $\langle \mathrm{Fm}_{\mathcal{L}}/\Omega T_1, h[T_2] \rangle = \langle \mathrm{Fm}_{\mathcal{L}}/\Omega T_1, T_2/\Omega T_1 \rangle$ which implies that ΩT_1 is compatible with T_2. Thus $\Omega T_1 \subseteq \Omega T_2$. By Theorem 3.3.1, S is protoalgebraic. $\square$

3.4 Parameterized System of Equivalence Formulas

We say that $E(p, q, \underline{r}) = \{\epsilon_i(p, q, \underline{r}) : i \in I\}$ is an k-*parameterized system of formulas* if it is a set of formulas of S built up from the variables p, q and possibly other variables $\underline{r} = r_1, r_2, \ldots$ called *parameters* with k the length of the string $\underline{r}$. Note that $E(p, q, \underline{r})$ may be infinite; hence the length of the string $\underline{r}$ could be ω. In order to define the notion of parameterized system of equivalence formulas, we need to introduce some notations. Let $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$, we denote by $E(\langle \varphi, \psi \rangle)$ the set of all substitution instances $e(\epsilon_i(p, q, \underline{r}))$ where i ranges over I and e over all substitutions such that $e(p) = \varphi$ and $e(q) = \psi$, i.e.,

$$E(\langle \varphi, \psi \rangle) := \{\epsilon_i(p/\varphi, q/\psi, \underline{r}/\underline{\gamma}) : i \in I, \underline{\gamma} \in (\mathrm{Fm}_{\mathcal{L}})^k\}.$$

We extend this notation to $\mathcal{L}$-algebras. If $\mathbf{A}$ is an $\mathcal{L}$-algebra and $a, b \in \mathrm{A}$, we denote by $E^{\mathbf{A}}(\langle a, b \rangle)$ the set of all elements of A of the form $h(\epsilon_i(p, q, \underline{r}))$ where i ranges over I and h over all homomorphisms $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ such that $h(p) = a$ and $h(q) = b$, i.e.,

$$E^{\mathbf{A}}(\langle a, b \rangle) := \{\epsilon_i^{\mathbf{A}}(p/a, q/b, \underline{r}/\underline{c}) : i \in I, \underline{c} \in \mathrm{A}^k\}$$

where $\epsilon_i^{\mathbf{A}} := h(\epsilon_i)$. If $\langle \mathbf{A}, D \rangle$ is an S-matrix, then $E_{\mathbf{A}}(D)$ is a binary relation on $\mathbf{A}$, called the (*universally parameterized*) *analytical relation in* $\langle \mathbf{A}, D \rangle$ *determined by* $E(p, q, \underline{r})$, and is defined in the following way:

$$a \equiv b(E_{\mathbf{A}}(D)) \text{ iff } E^{\mathbf{A}}(\langle a, b \rangle) \subseteq D.$$

In general, $E_{\mathbf{A}}(D)$ need not be an equivalence relation on $\mathbf{A}$.

Definition 3.4.1. *Let S be a logic. A set $E(p, q, \underline{r})$ is called an k-parameterized system of equivalence formulas for S (an k-parameterized equivalence for S for short) if the following conditions hold:*

p-(R) $\vdash_S E(\langle p, p \rangle)$ (Reflexivity)

p-(S) $E(\langle p, q \rangle) \vdash_S E(\langle q, p \rangle)$ (Symmetry)

p-(T) $E(\langle p, q \rangle) \cup E(\langle q, t \rangle) \vdash_S E(\langle p, t \rangle)$ (Transitivity)

p-(MP) $E(\langle p, q \rangle) \cup \{p\} \vdash_S q$ (Modus Ponens)

p-($\mathbf{RP}_{sim}$) *for each connective f of rank $n \geq 0$* (Simple Replacement)

$$E(\langle p_1, q_1 \rangle) \cup \cdots \cup E(\langle p_n, q_n \rangle) \vdash_S E\big(\langle f(p_1, \ldots, p_n), f(q_1, \ldots, q_n) \rangle\big)$$

Symmetry and transitivity conditions are derivable from the remaining ones and thus they are redundant (c.f. [Cze01, Corollary 1.2.5]).

A parameterized system of equivalence formulas for S may be empty. In this case the logic have the rule $p \vdash_S q$ for all $p, q \in \mathrm{Fm}_{\mathcal{L}}$, i.e., S is inconsistent or almost inconsistent.

Simple replacement condition can be substituted by *single replacement* condition, that is, for each $\varphi \in \mathrm{Fm}_{\mathcal{L}}$,

$$E(\langle p, q \rangle) \vdash_S E\big(\langle \varphi(p), \varphi(q) \rangle\big).$$

Indeed, suppose that simple replacement condition holds. We prove by induction on formulas. If φ is a constant, then single replacement condition holds. Let $\varphi = f\big(\varphi_1(p), \ldots, \varphi_n(p)\big) \in \mathrm{Fm}_{\mathcal{L}}$, where f is a connective of rank n. By hypothesis of induction, for every $i = 1 \ldots n$ we have that $E(\langle p, q \rangle) \vdash_S E\big(\langle \varphi_i(p), \varphi_i(q) \rangle\big)$. Since simple replacement condition holds, by structurality condition, we have that $E(\langle \varphi_1(p), \varphi_1(q) \rangle) \bigcup \cdots \bigcup$

$E(\langle \varphi_n(p), \varphi_n(q)\rangle) \vdash_S E(\langle f(\varphi_1(p), \ldots, \varphi_n(p)), f(\varphi_1(q), \ldots, \varphi_n(q))\rangle)$. Thus by considering $\varphi = f(\varphi_1(p), \ldots, \varphi_n(p))$, we have that $E(\langle p, q\rangle) \vdash_S E(\langle \varphi(p), \varphi(q)\rangle)$, i.e., single replacement condition holds. Conversely, suppose that single replacement condition holds. Let f be a connective of rank n and $p_1, \ldots, p_n, q_1, \ldots, q_n \in \mathrm{Var}$. By hypothesi $E(\langle p_1, q_1\rangle) \vdash_S E(\langle f(p_1, \ldots, p_n), f(q_1, p_2, \ldots, p_n)\rangle)$ and $E(\langle p_2, q_2\rangle) \vdash_S E(\langle f(q_1, p_2, \ldots, p_n), f(q_1, q_2, p_3, \ldots, p_n)\rangle)$. Since $E(p, q, \underline{r})$ is transitive, we have that $E(\langle f(p_1, \ldots, p_n), f(q_1, p_2, \ldots, p_n)\rangle) \cup E(\langle f(q_1, p_2, \ldots, p_n), f(q_1, q_2, p_3, \ldots, p_n)\rangle) \vdash_S E(\langle f(p_1, \ldots, p_n), f(q_1, q_2, p_3, \ldots, p_n)\rangle)$. And by cut condition, $E(\langle p_1, q_1\rangle) \cup E(\langle p_2, q_2\rangle) \vdash_S E(\langle f(p_1, \ldots, p_n), f(q_1, q_2, p_3, \ldots, p_n)\rangle)$. In a similar way, we can substitute the variables $p_3, \ldots, p_n$ by $q_3, \ldots, q_n$. Therefore we obtain simple replacement condition.

Theorem 3.4.2. *Let S be a logic and $E(p, q, \underline{r})$ an k-parameterized system of formulas. The following conditions are equivalent:*

(i) $E(p, q, \underline{r})$ is an k-parameterized system of equivalence formulas for S;

(ii) $E_{\mathbf{A}}(D) = \Omega_{\mathbf{A}} D$ for all $\langle \mathbf{A}, D\rangle \in \mathbf{Mod}(S)$;

(iii) $E_{\mathbf{A}}(D) = \triangle_{\mathbf{A}}$ for all $\langle \mathbf{A}, D\rangle \in \mathbf{Mod}^*(S)$.

Proof. $(i) \Rightarrow (ii)$. Suppose $E(p, q, \underline{r})$ is an k-parameterized system of equivalence formulas for S. Let $\mathrm{M} = \langle \mathbf{A}, D\rangle$ be an S-matrix. Reflexivity, symmetry, transitivity and simple replacement conditions guarantee that the relation $E_{\mathbf{A}}(D)$ is a congruence relation on $\mathbf{A}$ and modus ponens condition guarantees that $E_{\mathbf{A}}(D)$ is compatible with D. Thus $E_{\mathbf{A}}(D) \subseteq \Omega_{\mathbf{A}} D$. For the reverse inclusion, let $a, b \in A$. Assume $a \equiv b(\Omega_{\mathbf{A}} D)$. Let $\epsilon(p, q, \underline{r}) \in E(p, q, \underline{r})$ and $\underline{c} \in A^k$. We have that $\epsilon^{\mathbf{A}}(a, a, \underline{c}) \equiv \epsilon^{\mathbf{A}}(a, b, \underline{c})(\Omega_{\mathbf{A}} D)$. By reflexivity condition, $\epsilon^{\mathbf{A}}(a, a, \underline{c}) \in D$. Since $\Omega_{\mathbf{A}} D$ is compatible with D, $\epsilon^{\mathbf{A}}(a, b, \underline{c}) \in D$, i.e., $E^{\mathbf{A}}(\langle a, b\rangle) \subseteq D$. Thus $a \equiv b(E_{\mathbf{A}}(D))$.

$(ii) \Rightarrow (i)$. Suppose that $E_{\mathbf{A}}(D) = \Omega_{\mathbf{A}} D$ for all $\langle \mathbf{A}, D\rangle \in \mathbf{Mod}(S)$. Let $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$ and $T \in \mathrm{Th}(S)$. Since $\varphi \equiv \varphi(\Omega T)$, we have that $E(\langle \varphi, \varphi\rangle) \subseteq T$, i.e., reflexivity condition holds. Now suppose $E(\langle \varphi, \psi\rangle) \subseteq T$. Then $\varphi \equiv \psi(\Omega T)$. Hence $\psi \equiv \varphi(\Omega T)$ and consequently $E(\langle \psi, \varphi\rangle) \subseteq T$. Since this holds for every $T \in \mathrm{Th}(S)$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$, symmetry condition holds. The transitivity and simple replacement conditions can be shown in a similar way. Now, assume $\{\varphi\} \cup E(\langle \varphi, \psi\rangle) \subseteq T$. Then $\varphi \equiv \psi(\Omega T)$ and $\varphi \in T$. Since ΩT is compatible with T, $\psi \in T$. Thus modus ponens condition holds.

Therefore, $E(p, q, \underline{r})$ is an k-parameterized system of equivalence formulas for S.

$(ii) \Rightarrow (iii)$. This is obvious.

$(iii) \Rightarrow (ii)$. Let M $= \langle \mathbf{A}, D \rangle$ be an S-matrix and $a, b \in$ A. We denote by $[a]$ the equivalence class of a relative to $\Omega_\mathbf{A} D$. We have $a \equiv b(\Omega_\mathbf{A} D)$ iff $[a] = [b]$ iff (by assumption) $E^{\mathbf{A}/D}(\langle [a], [b] \rangle) \subseteq D/\Omega_\mathbf{A} D$ iff $E^\mathbf{A}(\langle a, b \rangle) \subseteq D$ iff $a \equiv b(E_\mathbf{A}(D))$. Thus $E_\mathbf{A}(D) = \Omega_\mathbf{A} D$ for all $\langle \mathbf{A}, D \rangle \in \mathbf{Mod}(S)$. $\qquad\square$

We say that a non-empty k-parameterized system $E(p, q, \underline{r})$ *defines the leibniz congruences* in a logic S if, for every $\langle \mathbf{A}, D \rangle \in \mathbf{Mod}(S)$, $\Omega_\mathbf{A} F = \{(a, b) \in$ A$^2 : E^\mathbf{A}(\langle a, b \rangle) \subseteq F\}$. We can reformulate Theorem 3.4.2 saying that $E(p, q, \underline{r})$ is an k-parameterized system of equivalence formulas for S iff $E(p, q, \underline{r})$ defines the Leibniz congruences in S.

In the following theorem, we give a characterization of protoalgebraic logics as the ones which have an k-parameterized system of equivalence formulas.

Theorem 3.4.3. *Let S be a logic. The following conditions are equivalent:*

    (i) S is protoalgebraic;

    (ii) S has an k-parameterized system of equivalence formulas.

Proof. Let S be a protoalgebraic logic. We can represent T_{pq} as the k-parameterized system $T(p, q, \underline{r})$ with $\underline{r}$ the string of all variables distinct from p, q. Let $\delta(p, q, r_1, \ldots, r_k) \in T_{pq}$ and $\gamma_1, \ldots, \gamma_k$ a string of formulas. Consider the substitution e such that $e(p) = p$, $e(q) = q$ and $e(r_1) = \gamma_1, \ldots, e(r_k) = \gamma_k$. Since e satisfies the condition $(e(p))(q/p) = (e(q))(q/p)$, by Lemma 3.1.2, we have $\delta(p, q, \gamma_1, \ldots, \gamma_k) = e(\delta(p, q, r_1, \ldots, r_k)) \in T_{pq}$. Hence, $T(p, q, \underline{\gamma}) \subseteq T(p, q, \underline{r})$ for every string $\underline{\gamma}$ of length k of formulas of S. It is not difficult to see that $T(p, q, \underline{r})$ satisfies reflexivity condition. Now we verify single replacement condition. Let $\delta(p, q, r_1, \ldots, r_k) \in T(p, q, \underline{r})$ and e a substitution such that $e(p) = \varphi(p)$, $e(q) = \varphi(q)$, where $\varphi \in \mathrm{Fm}_\mathcal{L}$, and $e(r_1) = \gamma_1, \ldots, e(r_k) = \gamma_k$. Since $(e(p))(q/p) = (e(q))(q/p)$, by Lemma 3.1.2, we obtain that $\delta(\varphi(p), \varphi(q), \gamma_1, \ldots, \gamma_k) = e(\delta(p, q, r_1, \ldots, r_k)) \in T_{pq}$, i.e., $T(\varphi(p), \varphi(q), \underline{\gamma}) \subseteq T(p, q, \underline{r})$ for any formula φ and any string $\underline{\gamma}$ of formulas. Thus single replacement condition holds. Moreover, by Lemma 3.1.2, $p \equiv q(\Omega T(p, q, \underline{r}))$. We always have that $\mathrm{Cn}_S(T(p, q, \underline{r})) \subseteq \mathrm{Cn}_S(T(p, q, \underline{r}) \cup \{p\})$. Since S is protoalgebraic, by Theorem 3.3.1, the Leibniz operator is monotone on $\mathbf{Th}(S)$. Thus, $\Omega(\mathrm{Cn}_S(T(p, q, \underline{r}))) \subseteq \Omega(\mathrm{Cn}_S(T(p, q, \underline{r}) \cup \{p\}))$. We deduce that $p \equiv q(\Omega(\mathrm{Cn}_S(T(p, q, \underline{r}) \cup \{p\})))$. By compatibility, $q \in \mathrm{Cn}_S(T(p, q, \underline{r}) \cup \{p\})$, i.e., modus ponens condition holds. We conclude that the set $T(p, q, \underline{r})$ is an k-parameterized system of equivalence formulas for S.

Conversely, suppose that $E(p, q, \underline{r})$ is an k-parameterized system of equivalence formulas for S. It is not difficult to see that $E(p, q, p, p, p, p, \ldots)$ (the set of formulas

obtained by replacing every parameter by p) is a protoequivalence system for S. By Theorem 3.1.3, S is protoalgebraic. □

This proof is constructive in the sense that it produces an k-parameterized system of equivalence formulas for any protoalgebraic logic, namely the set T_{pq}.

Suppose that $E(p, q, \underline{r})$ and $E'(p, q, \underline{s})$ are respectively k and l-parameterized systems of equivalence formulas for a logic S. Then, $E(p, q, \underline{r})$ and $E'(p, q, \underline{s})$ are interderivable. Conversely, if $E(p, q, \underline{r})$ and $E'(p, q, \underline{s})$ are respectively k and l-parameterized systems of formulas, which are interderivable, then $E(p, q, \underline{r})$ is an k-parameterized system of equivalence formulas for S iff $E'(p, q, \underline{s})$ is also an l-parameterized system of equivalence formulas for S.

In Chapter 2, we have defined the Suszko operator and seen that $\tilde{\Omega}T \subseteq \Omega T$ for every $T \in \mathrm{Th}(S)$. When the Suszko operator coincides with the Leibniz operator in a logic S then S is protoalgebraic.

Theorem 3.4.4. *Let S be a logic. The following conditions are equivalent:*

(i) S is protoalgebraic;

(ii) The Leibniz operator coincide with the Suszko operator on $\mathbf{Th}(S)$, i.e., $\tilde{\Omega}T = \Omega T$ for every $T \in \mathrm{Th}(S)$.

Proof. Assume S is protoalgebraic. By Theorem 3.4.3, there exists an k-parameterized system of equivalence formulas $E(p, q, \underline{r})$ for S. Let $T \in \mathrm{Th}(S)$ and $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}$. Suppose $\alpha \equiv \beta(\Omega T)$. Then $\varphi(p/\alpha) \equiv \varphi(p/\beta)(\Omega T)$ for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ and $p \in \mathrm{Var}(\varphi)$. Thus $E(\langle\varphi(p/\alpha), \varphi(p/\beta)\rangle) \subseteq T$. By modus ponens condition, we have $E(\langle\varphi(p/\alpha), \varphi(p/\beta)\rangle) \cup \{\varphi(p/\alpha)\} \vdash_S \varphi(p/\beta)$. And by cut condition, $T \cup \{\varphi(p/\alpha)\} \vdash_S \varphi(p/\beta)$. Furthermore, by symmetry condition, $E(\langle\varphi(p/\beta), \varphi(p/\alpha)\rangle) \subseteq T$. And again by modus ponens, we obtain $T \cup \{\varphi(p/\beta)\} \vdash_S \varphi(p/\alpha)$. Hence $\Omega T \subseteq \tilde{\Omega}T$. Since the reverse inclusion always holds, we conclude that $\tilde{\Omega}T = \Omega T$ for every $T \in \mathrm{Th}(S)$.

Conversely, assume $\tilde{\Omega}T = \Omega T$ for every $T \in \mathrm{Th}(S)$. Let $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}$ and $T_1, T_2 \in \mathrm{Th}(S)$. Suppose that $T_1 \subseteq T_2$ and $\alpha \equiv \beta(\tilde{\Omega}T_1)$. By definition of Suszko congruence we have $T_1 \cup \{\varphi(p/\alpha)\} \dashv\vdash_S T_1 \cup \{\varphi(p/\beta)\}$ for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ and $p \in \mathrm{Var}(\varphi)$. Since $T_1 \subseteq T_2$, we have $T_2 \cup \{\varphi(p/\alpha)\} \vdash_S \varphi(p/\beta)$ and $T_2 \cup \{\varphi(p/\beta)\} \vdash_S \varphi(p/\alpha)$ for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ and $p \in \mathrm{Var}(\varphi)$. Thus $\alpha \equiv \beta(\tilde{\Omega}T_2)$. Therefore the Suszko operator is monotone on $\mathbf{Th}(S)$. Since $\tilde{\Omega}T = \Omega T$ for every $T \in \mathrm{Th}(S)$, the Leibniz operator is also monotone on $\mathbf{Th}(S)$. By Theorem 3.3.1, S is protoalgebraic. □

3.5 Reduced Matrix Models

Herein, we study the relationship between the structural properties of the class of reduced matrix models and metalogical properties of protoalgebraic logics.

Theorem 3.5.1. *Let S be a logic. The following conditions are equivalent:*

(i) S is protoalgebraic;

(ii) The class $\mathbf{Mod}^*(S)$ is closed under subdirect products.

Proof. Assume S is protoalgebraic. By Theorem 3.4.3, there exists an k-parameterized system $E(p, q, \underline{r})$ of equivalence formulas for S. Let $M = \langle \mathbf{A}, D \rangle$ be a subdirect products of the family $(M_i = \langle \mathbf{A}_i, D_i \rangle)_{i \in I}$ of reduced matrix models of S and $a, b \in A$. Since the class $\mathbf{Mod}(S)$ is closed under submatrices and direct products, it is also closed under subdirect products. Thus $M \in \mathbf{Mod}(S)$. Suppose $a \equiv b(\Omega_{\mathbf{A}} D)$. By Theorem 3.4.2, $a \equiv b(E_{\mathbf{A}}(D))$, i.e., $E^{\mathbf{A}}(\langle a, b \rangle) \subseteq D$. Then,

$$E^{\mathbf{A}_i}(a(i), b(i), \underline{c}(i)) \subseteq D_i, \text{ for all } i \in I \text{ and all } \underline{c} \in A^k.$$

Since the projection $\pi_i : \mathbf{A} \to \mathbf{A}_i$ is surjective, we have

$$E^{\mathbf{A}_i}(a(i), b(i), \underline{d}) \subseteq D_i, \text{ for all } i \in I \text{ and all string } \underline{d} \in A_i^k.$$

That is, $a(i) \equiv b(i)(E_{\mathbf{A}_i}(D_i))$, for all $i \in I$. Since each M_i is reduced, by Theorem 3.4.2, $E_{\mathbf{A}_i}(D_i) = \Delta_{\mathbf{A}_i}$. Thus $a(i) = b(i)$ for all $i \in I$. Hence $a = b$, which implies that M is reduced. Therefore the class $\mathbf{Mod}^*(S)$ is closed under subdirect products.

Conversely, let $\mathbf{A}$ be an $\mathcal{L}$-algebra and $F, G \in \mathrm{Fi}_S(\mathbf{A})$. Suppose $F \subseteq G$. Let $f : \mathbf{A} \to \mathbf{A}/F$ and $g : \mathbf{A} \to \mathbf{A}/G$ be natural homomorphisms and $\Theta := \Omega_{\mathbf{A}} F \cap \Omega_{\mathbf{A}} G$. Consider $M := \langle \mathbf{A}/\Theta, F/\Theta \rangle$, $M_1 := \langle \mathbf{A}/\Omega_{\mathbf{A}} F, F/\Omega_{\mathbf{A}} F \rangle$ and $M_2 := \langle \mathbf{A}/\Omega_{\mathbf{A}} G, G/\Omega_{\mathbf{A}} G \rangle$. It is not difficult to see that M is isomorphic to a subdirect product of M_1 and M_2 (via the mapping $h([a]_\Theta) := \langle f(a), g(a) \rangle$, for every $a \in A$). The matrices M_1 and M_2 are reduced and hence members of $\mathbf{Mod}^*(S)$. Since by hypothesis $\mathbf{Mod}^*(S)$ is closed under subdirect products, M is reduced as well. This means that Θ is the largest congruence of $\mathbf{A}$ compatible with F, i.e., $\Theta = \Omega_{\mathbf{A}} F$. Therefore $\Omega_{\mathbf{A}} F \subseteq \Omega_{\mathbf{A}} G$. By Theorem 3.3.1, S is protoalgebraic. $\square$

It follows from the above theorem that the class of reduced matrix models of a protoalgebraic logic is closed under direct products.

3.6 Parameterized Local Deduction - Detachment Theorem

For the Classical Propositional Logic **CPL**, the Deduction-Detachment Theorem (DDT for short) has been studied by many logicians. They have proved that for all $\Gamma \cup \{\alpha, \beta\} \in \mathrm{Fm}_{\mathcal{L}}$,

$$\Gamma \cup \{\alpha\} \vdash_{\mathbf{CPL}} \beta \text{ iff } \Gamma \vdash_{\mathbf{CPL}} \alpha \to \beta$$

where the binary connective $\to$ is the usual propositional implication. We can generalize this notion for logics that does not have an implication connective. It still is possible to find a family of sets of formulas that play the same role of the binary connective $\to$. This generalization of DDT has been called Parameterized Local Deduction-Detachment Theorem (PLDDT for short). It was shown that a logic S has this property iff it is protoalgebraic.

We illustrate this result giving an example, namely the **BCK** logic, that has the Local Deduction-Detachment Theorem (LDDT for short), and consequently it is protoalgebraic. We point out to [Cze01, Chapter 2], [CP04a] and [CP04b] for more details about the relation between the various kinds of DDT and properties of protoalgebraic logics.

Definition 3.6.1. *A logic S has the* Parameterized Local Deduction-Detachment Theorem, PLDDT, *with respect to a family of sets of formulas* Φ*, if, for all* $\Gamma \cup \{\alpha, \beta\} \subseteq \mathrm{Fm}_{\mathcal{L}}$*,*

$\Gamma \cup \{\alpha\} \vdash_S \beta$ iff there exists $V(p, q, \underline{r}) \in \Phi$, and exists $\underline{\gamma} \in (\mathrm{Fm}_{\mathcal{L}})^k$, $\Gamma \vdash_S V(\alpha, \beta, \underline{\gamma})$.

The implication from right to left in the above equivalence is called *detachment property*, and the implication in the opposite direction is called the *deduction property*. Moreover, if S is finitary then for each $V \in \Phi$, we can choose a finite subset $V_f \subseteq V$ such that the family $\Phi_f = \{V_f : V_f \subseteq V \text{ and } V \in \Phi\}$ also determines PLDDT for S. The only logics that have the PLDDT with respect to the family $\{\emptyset\}$ are the trivial logics.

We say that a logic has the *Local Deduction-Detachment Theorem* if it has the PLDDT with an empty set of parameters. More precisely, if there is a family of sets of formulas Φ in two variables such that for all $\Gamma \cup \{\alpha, \beta\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \cup \{\alpha\} \vdash_S \beta$ iff there exists $V(p, q) \in \Phi$, $\Gamma \vdash_S V(\alpha, \beta)$. And we say that a logic has the *Deduction-Detachment Theorem* if it has the LDDT such that Φ is the family of a single set of

finite formulas, i.e., if there is a finite set of formulas $\Phi := V(p, q)$ in two variables such that for all $\Gamma \cup \{\alpha, \beta\} \subseteq \mathrm{Fm}_\mathcal{L}$, $\Gamma \cup \{\alpha\} \vdash_S \beta$ iff $\Gamma \vdash_S V(\alpha, \beta)$. If the set Φ is unitary then we say that the logic has the *Uniterm Deduction-Detachment Theorem* (UDDT for short). The general case is simply referred as DDT or *Multiterm Deduction-Detachment Theorem* (MDDT for short). For more information about these kinds of DDT, the reader can see [CP04a] and [CP04b], where Czelakowski and Pigozzi described the MDDT for k-deductive systems in general, and [Cze01, Chapter 2], where Czelakowski examined in detail the properties of protoalgebraic logics for which the DDT holds.

Theorem 3.6.2. *Let S be a logic. The following conditions are equivalent:*

   (i) S is protoalgebraic;

   (ii) S has the PLDDT *with respect to the family Φ.*

Proof. Assume S is protoalgebraic. If S is a logic without theorems then we consider $\Phi := \{\emptyset\}$. Let S be a logic with $\mathrm{Thm}(S) \neq \emptyset$ and $p, q \in \mathrm{Var}$. Consider $\Phi = \{T \in \mathrm{Th}(S) : T \cup \{p\} \vdash_S q\}$. The family Φ is non-empty because the set $\{\varphi : q \vdash_S \varphi\} \in \Phi$. Since $\mathrm{Thm}(S)$ is non-empty, $p, q \in \mathrm{Var}(T)$ for every $T \in \mathrm{Th}(S)$. We show that Φ determines PLDDT for S. Let $\Gamma \cup \{\alpha, \beta\} \subseteq \mathrm{Fm}_\mathcal{L}$. Suppose $\Gamma \cup \{\alpha\} \vdash_S \beta$. There exists a surjective substitution e such that $e(p) = \alpha$ and $e(q) = \beta$. Let $T := e^{-1}[Cn_S(\Gamma)]$, $\mathrm{M} := \langle \mathbf{Fm}_\mathcal{L}, T \rangle$ and $\mathrm{N} := \langle \mathbf{Fm}_\mathcal{L}, Cn_S(\Gamma) \rangle$. Since $\mathrm{Th}(S)$ is closed under inverse substitution, $T \in \mathrm{Th}(S)$. It is not difficult to see that e is a strict homomorphism from M onto N and $\mathrm{Cn}_S(T \cup \{p\}) \in \mathrm{Fi}_S(\mathrm{M})$. As S is protoalgebraic, by Theorem 3.3.3, S has the correspondence property and hence $\mathrm{Cn}_S(T \cup \{p\}) = e^{-1}[e[\mathrm{Cn}_S(T \cup \{p\})]]$. By surjectivity of e, $e[T] = e[e^{-1}[\mathrm{Cn}_S(\Gamma)]] = \mathrm{Cn}_S(\Gamma)$. We have that $\mathrm{Cn}_S(\Gamma \cup \{\alpha\}) = \mathrm{Cn}_S(\mathrm{Cn}_S(\Gamma) \cup \{\alpha\}) = \mathrm{Cn}_S(e[T] \cup \{e(p)\}) = \mathrm{Cn}_S(e[T \cup \{p\}])$ and it is not difficult to prove that $\mathrm{Cn}_S(e[\Gamma \cup \{p\}]) = e[\mathrm{Cn}_S(T \cup \{p\})]$. Since $\mathrm{Cn}_S(\Gamma \cup \{\alpha\}) \in \mathrm{Fi}_S(\mathrm{N})$, $e[\mathrm{Cn}_S(T \cup \{p\})] \in \mathrm{Fi}_S(\mathrm{N})$. Thus e is a strict homomorphism from $\langle \mathbf{Fm}_\mathcal{L}, \mathrm{Cn}_S(T \cup \{p\}) \rangle$ onto $\langle \mathbf{Fm}_\mathcal{L}, \mathrm{Cn}_S(\Gamma \cup \{\alpha\}) \rangle$. As $e(q) = \beta \in \mathrm{Cn}_S(\Gamma \cup \{\alpha\})$, we have $q \in \mathrm{Cn}_S(T \cup \{p\})$. So $T = T(p, q, \underline{r}) \in \Phi$. Furthermore $T(\alpha, \beta, e(\underline{r})) = e[T] = \mathrm{Cn}_S(\Gamma)$. Consider $V := T$, we have that $V \cup \{\alpha\} \vdash_S \beta$ implies $\Gamma \vdash_S V(\alpha, \beta, \underline{\gamma})$ for some string of formulas $\underline{\gamma} \in (\mathrm{Fm}_\mathcal{L})^k$. The reverse implication is obvious by the definition of the family Φ. Therefore S has the PLDDT with respect to the family Φ.

   Conversely, assume that some family Φ determines PLDDT for a logic S. Thus the sets in Φ have the detachment property, i.e., $V \cup \{p\} \vdash_S q$ for all $V \in \Phi$. Since $p \vdash_S p$, by PLDDT, there exists a set $V(p, q, \underline{r}) \in \Phi$ and a string $\underline{\gamma}$ of formulas of S such that

$\emptyset \vdash_S V(p, p, \underline{\gamma})$. Let e be a substitution such that $e(p) = p$, $e(q) = q$, $e(\underline{r}) = p$ and $e(\underline{\gamma}) = p$. Consider $\Delta(p, q) := e[V(p, q, \underline{r})]$. Since $V(p, q, \underline{r}) \cup \{p\} \vdash_S q$, by structurality of S, we have $V(p, q, \underline{p}) \cup \{p\} \vdash_S q$, i.e., $\Delta(p, q) \cup \{p\} \vdash_S q$ modus ponens condition holds. Moreover, since $\vdash_S V(p, p, \underline{\gamma})$, we have $\vdash_S \Delta(p, p)$, i.e., reflexivity condition holds. Hence the set $\Delta(p, q)$ is a protoequivalence system for S. By Theorem 3.1.3, S is protoalgebraic. $\qquad \square$

In the following example, we show that the logic **BCK** is protoalgebraic since the *LDDT* holds.

Example 3.7 (**BCK** Logic [Cze01])**.** Let **BCK** the deductive system defined in the language $\mathcal{L} = \{\rightarrow\}$, where $\rightarrow$ is a binary connective, by the following axioms:

$$(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)) \tag{B}$$

$$(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r)) \tag{C}$$

$$p \rightarrow (q \rightarrow p) \tag{K}$$

and the only inference rule,

$$\frac{p, p \rightarrow q}{q} \tag{Modus Ponens}$$

Let $\Phi := (\{p \rightarrow_n q\})_{n \in \mathbb{N}}$, where $p \rightarrow_0 q := q$, and $p \rightarrow_{n+1} q := p \rightarrow (p \rightarrow_n q)$ for all $n \in \mathbb{N}$. The (one-element) sets of Φ do not involve parametric variables. We have that **BCK** has the LDDT with respect to the family Φ, i.e., for any $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\Gamma \cup \{\varphi\} \vdash_{\mathbf{BCK}} \psi \text{ iff, } \Gamma \vdash_{\mathbf{BCK}} \varphi \rightarrow_n \psi \text{ for some } n \in \mathbb{N}.$$

Indeed, suppose that $\Gamma \cup \{\varphi\} \vdash_{\mathbf{BCK}} \psi$. We show by induction on the length of the proof of ψ from $\Gamma \cup \{\varphi\}$. If ψ is an axiom or $\psi = \varphi$ then $\Gamma \vdash_{\mathbf{BCK}} \psi \rightarrow (\varphi \rightarrow \psi)$ since axiom (K) holds. By modus ponens, we have $\Gamma \cup \{\psi\} \vdash_{\mathbf{BCK}} \varphi \rightarrow \psi$, i.e., detachment property holds for $n = 1$. If ψ belongs to $\Gamma \cup \{\varphi\}$ then $\Gamma \vdash_{\mathbf{BCK}} \psi$, i.e., detachment property holds for $n = 0$. If ψ is obtained by modus ponens then there exists a formula ξ such that applying modus ponens to ξ and $\xi \rightarrow \psi$ we have ψ. By inductive hypothesis, there exist $i \in \mathbb{N}$ such that $\Gamma \vdash_{\mathbf{BCK}} \varphi \rightarrow_i \xi$ and $j \in \mathbb{N}$ such that $\Gamma \vdash_{\mathbf{BCK}} \varphi \rightarrow_j (\xi \rightarrow \psi)$. By induction on $i + j$, we can show that

$$\vdash_{\mathbf{BCK}} (p \rightarrow_i q) \rightarrow ((p \rightarrow_j (q \rightarrow r)) \rightarrow (p \rightarrow_{i+j} r))$$

Thus, $\Gamma \vdash_{\mathbf{BCK}} (\varphi \rightarrow_i \xi) \rightarrow ((\varphi \rightarrow_j (\xi \rightarrow \psi)) \rightarrow (\varphi \rightarrow_{i+j} \psi))$. Since $\Gamma \vdash_{\mathbf{BCK}} \varphi \rightarrow_i \xi$, by modus ponens, we obtain $\Gamma \vdash_{\mathbf{BCK}} (\varphi \rightarrow_j (\xi \rightarrow \psi)) \rightarrow (\varphi \rightarrow_{i+j} \psi)$. And since $\Gamma \vdash_{\mathbf{BCK}} \varphi \rightarrow_j (\xi \rightarrow \psi)$, by modus ponens, $\Gamma \vdash_{\mathbf{BCK}} \varphi \rightarrow_{i+j} \psi$. Thus, detachment property holds for $n = i + j$.

Conversely, if $\Gamma \vdash_{\mathbf{BCK}} \varphi \rightarrow_n \psi$ for some $n \in \mathbb{N}$, applying modus ponens n times we have that $\Gamma \cup \{\varphi\} \vdash_{\mathbf{BCK}} \psi$.

Therefore $\mathbf{BCK}$ has the LDDT with respect to the family Φ. By Theorem 3.6.2, $\mathbf{BCK}$ is protoalgebraic. $\diamondsuit$

3.8 Finitary and Protoalgebraic Logics

In this section we only present some results of finitary protoalgebraic logics without their proofs.

Theorem 3.8.1. [Cze01, Theorem 1.4.1] *Let S be a finitary and protoalgebraic logic. Then the following conditions are equivalent:*

(i) The class $\mathbf{Mod}^*(S)$ is closed under ultraproducts;

(ii) Every k-parameterized system of equivalence formulas for S contains a finite k-parameterized system of equivalence formulas for S;

(iii) There exists a finite k-parameterized system of equivalence formulas for S.

The class $\mathbf{Mod}^*(S)$ need not be closed under ultraproducts for a finitary and protoalgebraic logic S. Indeed, let S be a deductive system defined as the expansion of the Intuitionistic Propositional Logic $\mathbf{IPL}$, by adjoining the unary connective $\square$ and two axioms $\square \top$ and $\square(p \rightarrow q) \rightarrow (\square p \rightarrow \square q)$. The only inference rule is modus ponens. It is not difficult to see that the set $\Delta(p, q) = \{p \rightarrow q\}$ is a protoequivalence system for S. By Theorem 3.1.3, S is protoalgebraic. However, the class $\mathbf{Mod}^*(S)$ is not closed under ultraproducts (c.f. [BP92]).

In this chapter, we have seen some theorems in which properties of the Leibniz operator defined on $\mathbf{Th}(S)$ can be transfer on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. Indeed, in Theorem 3.3.1, the fact that the Leibniz operator is monotone on $\mathbf{Th}(S)$ transfer on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. The same happens for the property of being meet-continuous in Theorem 3.3.2 . Furthermore, in Chapters 4 and 5, we will see others properties on $\mathbf{Th}(S)$ of the Leibniz operator that can be transfer on $\mathbf{Fi}_S(\mathbf{A})$ for every

algebra **A**. This phenomena can be formalized as a so called transfer principle which is stated in the following theorem.

Theorem 3.8.2. [Cze01, Theorem 1.7.1] *Let S be a finitary and protoalgebraic logic. A property expressible by a universal formula of elementary lattice theory holds in* **Th**S *iff it holds in* **Fi**$_S$**A** *for every algebra* **A**.

A class $\mathcal{M}$ of matrix models of a logic S is said to have the *S-filter extension property* (*FEP* for short) if for all M $= \langle \mathbf{A}, D \rangle \in \mathcal{M}$, every S-filter F on an arbitrary submatrix N $= \langle \mathbf{B}, G \rangle$ of M can be extend to an S-filter on M, i.e., if $F \in \mathrm{Fi}_S(\mathrm{N})$ then there exists an S-filter $E \in \mathrm{Fi}_S(\mathrm{M})$ such that $E \cap \mathrm{B} = F$. For a finitary and protoalgebraic logic S, the class $\mathbf{Mod}(S)$ has the *FEP* iff the class $\mathbf{Mod}^*(S)$ also has the *FEP* iff S has the LDDT (c.f. [Cze01, Theorem 2.3.5]).

Chapter 4

Equivalential Logics

Equivalential logics have been introduced by Prucnal and Wroński in [PW74] and extensively studied by Czelakowski in [Cze81], [Cze01, Chapter 3] and [Cze04]. In this chapter we define equivalential and finitely equivalential logics, and we give some characterizations using the Leibniz operator. We also study the relationship between the structural properties of the class of reduced matrix models and metalogical properties of equivalential logics. Moreover, as we did for protoalgebraic logics we focus on the finitary logics. We conclude, this chapter, by discussing some examples of logics which show that the class of finitely equivalential logic is a proper subclass of equivalential logics and the latter is a proper subclass of protoalgebraic logics.

4.1   Definitions and Characterizations

Equivalential logics are logics which have a system of equivalence formulas without parameters. Thus, they constitute a subclass of protoalgebraic logics.

Definition 4.1.1. *Let S be a logic. A set $E(p,q)$ of formulas of S built-up in two variables p and q is called a* system of equivalence formulas for S (an *equivalence for S for short) if the following conditions are satisfied:*

(R) $\vdash_S E(p,p)$ (Reflexivity)

(S) $E(p,q) \vdash_S E(q,p)$ (Symmetry)

(T) $E(p,q) \cup E(q,r) \vdash_S E(p,r)$ (Transitivity)

(MP) $E(p,q) \cup \{p\} \vdash_S q$ (Modus Ponens)

(RP$_{sim}$) *for each connective f of rank $n \geq 0$* (Simple Replacement)

$$E(p_1, q_1) \cup \cdots \cup E(p_n, q_n) \vdash_S E(f(p_1, \ldots, p_n), f(q_1, \ldots, q_n))$$

This definition of system of equivalence formulas is the original definition due to Prucnal and Wroński (c.f. [PW74]). However, Wójcicki has pointed out that symmetry and transitivity conditions are derivable from the remaining ones and thus redundant (c.f. [Wój88, Lemma 3.4.3] and [Cze01, Corollary 3.1.4]). Therefore, when we need to prove that a set $E(p, q)$ is a system of equivalence formulas for S, it is enough to verify reflexivity, modus ponens and simple replacement conditions. Moreover, we can substitute simple replacement condition by single replacement condition that is, for each $\varphi \in \mathrm{Fm}_{\mathcal{L}}$,

$$E(p, q) \vdash_S E\big(\varphi(p), \varphi(q)\big).$$

Definition 4.1.2. *A logic S is called* equivalential (finitely equivalential) *if it has a system (a finite system, respectively) of equivalence formulas.*

Every equivalential logic S is protoalgebraic, since any system of equivalence formulas for S is a free parameterized system of equivalence formulas for S. Furthermore, the system of equivalence formulas $E(p, q)$ may be empty. Indeed, it is not difficult to see that a logic S has the empty system of equivalence formulas iff S is trivial, i.e., S is inconsistent or almost inconsistent.

Any extension of (finitely) equivalential logic is also (finitely) equivalential with the same system of equivalence formulas.

If a logic S is equivalential and $E(p, q)$ is a system of equivalence formulas for S, then, by Theorem 3.4.2, the Leibniz congruence ΩT for any theory $T \in \mathrm{Th}(S)$ has a simple characterization in terms of $E(p, q)$ that is, for any $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$,

$$\varphi \equiv \psi (\Omega T) \text{ iff } E(\varphi, \psi) \subseteq T.$$

If we have two systems of equivalence formulas $E(p, q)$ and $E'(p, q)$ for an equivalential logic S, then $E(p, q)$ and $E'(p, q)$ are interderivable relative to S; moreover, if two sets $E(p, q)$ and $E'(p, q)$ are interderivable relative to S then, $E(p, q)$ is a system of equivalence formulas for S iff $E'(p, q)$ is a system of equivalence formulas for S. Indeed, since $E(p, q) \subseteq Cn_S(E(p, q))$, we have $p \equiv q\big(\Omega(Cn_S(E(p, q)))\big)$. Let $\varphi \in E'(p, q)$. Thus $\varphi(p, p) \equiv \varphi(p, q)\big(\Omega(Cn_S(E(p, q)))\big)$. By compatibility, $\varphi(p, p) \in Cn_S(E(p, q))$ iff $\varphi(p, q) \in Cn_S(E(p, q))$. Since $\vdash_S E(p, p)$, we have that $\varphi(p, q) \in Cn_S(E(p, q))$.

Thus $E(p,q) \vdash_S E'(p,q)$. Analogously, we can show that $E'(p,q) \vdash_S E(p,q)$. Conversely, suppose that $E(p,q)$ and $E'(p,q)$ are sets which are interderivable relative to a logic S and $E(p,q)$ is a system of equivalence formulas for S. Since $E'(p,p) \subseteq Cn_S(E'(p,p)) = Cn_S(E(p,p))$, we have $\vdash_S E'(p,p)$, i.e., reflexivity condition holds. As $E'(p,q) \subseteq Cn_S(E'(p,q)) = Cn_S(E(p,q))$ and $E(p,q) \subseteq Cn_S(E(q,p)) = Cn_S(E'(q,p))$, we have $E'(p,q) \subseteq Cn_S(E'(q,p))$, i.e., symmetry condition is satisfied. In an analogous way, we show for the remaining conditions.

In the following theorem, we give some conditions that a set $E(p,q)$ must satisfied in order for a protoalgebraic logic to become equivalential.

Theorem 4.1.3 (Herrmann's Test)**.** *Let S be a protoalgebraic logic. The following conditions are equivalent:*

(i) S is equivalential;

(ii) There exists some set $E(p,q)$ that satisfies the following conditions:

$$\vdash_S E(p,p) \text{ and } p \equiv q\big(\Omega\big(Cn_S(E(p,q))\big)\big)$$

Proof. Assume S is equivalential. Then there exists a system of equivalence formulas $E(p,q)$ for S. By reflexivity, we have $\vdash_S E(p,p)$. Since $E(p,q) \subseteq \mathrm{Cn}_S(E(p,q))$. By the characterization of the Leibniz congruence in terms of $E(p,q)$, we have $p \equiv q\big(\Omega\big(Cn_S(E(p,q))\big)\big)$.

Conversely, assume that there is some set $E(p,q)$ such that $\vdash_S E(p,p)$ and $p \equiv q\big(\Omega\big(Cn_S\big(E(p,q)\big)\big)\big)$. Since $\vdash_S E(p,p)$, reflexivity condition holds. We always have $E(p,q) \subseteq E(p,q) \cup \{p\}$. As S is protoalgebraic, by Theorem 3.3.1, $\Omega\big(Cn_S\big(E(p,q)\big)\big) \subseteq \Omega\big(Cn_S\big(E(p,q) \cup \{p\}\big)\big)$. Thus $p \equiv q\big(\Omega(Cn_S(E(p,q) \cup \{p\}))\big)$, i.e., modus ponens condition holds. In order to show that single replacement condition is satisfied, we claim that $\mathrm{Cn}_S(T_{pq} \cap \mathrm{Fm}_{\mathcal{L}}(\{p,q\})) = \mathrm{Cn}_S(E(p,q))$. By reflexivity of $E(p,q)$, we have $E(p,q) \subseteq T_{pq} \cap \mathrm{Fm}_{\mathcal{L}}(\{p,q\})$. Thus $\mathrm{Cn}_S(E(p,q)) \subseteq \mathrm{Cn}_S(T_{pq} \cap \mathrm{Fm}_{\mathcal{L}}(\{p,q\}))$. For the reverse inclusion, let $\varphi \in T_{pq} \cap \mathrm{Fm}_{\mathcal{L}}(\{p,q\})$. Since $p \equiv q\big(\Omega\big(Cn_S(E(p,q))\big)\big)$, we have $\varphi(p,p) \equiv \varphi(p,q)\big(\Omega\big(Cn_S(E(p,q))\big)\big)$. By compatibility, $\varphi(p,p) \in \mathrm{Cn}_S(E(p,q))$ iff $\varphi(p,q) \in \mathrm{Cn}_S(E(p,q))$. By reflexivity, $\varphi(p,p) \in \mathrm{Cn}_S(E(p,q))$. Thus $\varphi(p,q) \in \mathrm{Cn}_S(E(p,q))$. Now, let $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ and e a substitution such that $e(p) = \varphi(p)$ and $e(q) = \varphi(q)$. Since $(e(p))(q/p) = (e(q))(q/p)$, by Lemma 3.1.2, $E(\varphi(p),\varphi(q)) = E(e(p),e(q)) = e(E(p,q)) \subseteq T_{pq} \cap \mathrm{Fm}_{\mathcal{L}}(\{p,q\})$. Thus, $E(\varphi(p),\varphi(q)) \subseteq \mathrm{Cn}_S(T_{pq} \cap \mathrm{Fm}_{\mathcal{L}}(\{p,q\})) = \mathrm{Cn}_S(E(p,q))$, i.e., $E(p,q) \vdash_S E(\varphi(p),\varphi(q))$. We conclude that $E(p,q)$ is a system of equivalence formulas for S. Therefore S is equivalential. $\square$

With this proof, it is not difficult to show the following result.

Corollary 4.1.4. *Let S be a logic. The following conditions are equivalent:*

  (i) S is equivalential;

  (ii) $T_{pq} \cap \mathrm{Fm}_{\mathcal{L}}(\{p, q\})$ is a system of equivalence formulas for S.

Proof. Suppose that S is equivalential. Thus S is protoalgebraic. By Theorem 4.1.3, there exists a set $E(p, q)$ such that $\vdash_S E(p, p)$ and $p \equiv q\big(\Omega\big(Cn_S(E(p, q))\big)\big)$. In the second part of the proof of Theorem 4.1.3, we show that $\mathrm{Cn}_S(E(p, q)) = \mathrm{Cn}_S(T_{pq} \cap \mathrm{Fm}_{\mathcal{L}}(\{p, q\}))$ and that $E(p, q)$ is a system of equivalence formulas for S. Therefore $T_{pq} \cap \mathrm{Fm}_{\mathcal{L}}(\{p, q\})$ is a system of equivalence formulas for S.

The converse is obvious. □

The following theorems give characterizations of equivalential and finitely equivalential logics using the Leibniz operator.

Theorem 4.1.5. *Let S be a logic. The following conditions are equivalent:*

  (i) S is equivalential;

  (ii) The Leibniz operator Ω is monotone and commutes with inverse substitutions on **Th**(S).

Proof. Assume S is equivalential. Thus S is protoalgebraic and by Theorem 3.3.1, the Leibniz operator Ω is monotone on **Th**(S). On the other hand, since S is equivalential, there exists a system of equivalence formulas $E(p, q)$ for S. Let $T \in \mathrm{Th}(S)$ and e a substitution. As $\mathrm{Th}(S)$ is closed under inverse substitutions, $e^{-1}[T] \in \mathrm{Th}(S)$. Let $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$. Suppose that $\varphi \equiv \psi(e^{-1}[\Omega T])$. Thus $e(\varphi) \equiv e(\psi)(\Omega T)$, i.e., $E(e(\varphi), e(\psi)) \subseteq T$. Since $E(e(\varphi), e(\psi)) = e[E(\varphi, \psi)]$, we have that $e[E(\varphi, \psi)] \subseteq T$, i.e., $E(\varphi, \psi) \subseteq e^{-1}[T]$. Therefore $\varphi \equiv \psi(\Omega e^{-1}[T])$.

Conversely, assume the Leibniz operator is monotone and commutes with inverse substitutions on **Th**(S). Let e be a substitution such that $e(p) = p$, $e(q) = q$ and $e(r) = p$ for the remaining variables r. Consider $E(p, q) = e[T_{pq}]$. By Lemma 3.1.2, T_{pq} is closed with respect to e, i.e, $E(p, q) \subseteq T_{pq}$ which means that $\vdash_S E(p, p)$. Again by Lemma 3.1.2, $p \equiv q(\Omega(T_{pq}))$. Thus $e(p) \equiv e(q)(e[\Omega(T_{pq})])$, i.e., $p \equiv q(e[\Omega(T_{pq})])$. By hypothesis, $e[\Omega(T_{pq})] \subseteq \Omega(\mathrm{Cn}_S(e[T_{pq}]))$. Thus $p \equiv q(\Omega(\mathrm{Cn}_S(e[T_{pq}])))$, i.e., $p \equiv q\big(\Omega(\mathrm{Cn}_S(E(p, q)))\big)$. By Theorem 4.1.3, S is equivalential. □

Theorem 4.1.6. *Let S be a logic. The following conditions are equivalent:*

(i) *S is finitely equivalential;*

(ii) $\Omega_{\mathbf{A}}$ *is continuous on* $\mathbf{Fi}_S(\mathbf{A})$ *for every algebra* $\mathbf{A}$*;*

(iii) Ω *is continuous on* $\mathbf{Th}(S)$*.*

Proof. $(i) \Rightarrow (ii)$. Assume S is finitely equivalential. Let $E(p, q)$ be a finite system of equivalence formulas for S, $\mathbf{A}$ an $\mathcal{L}$-algebra and $(D_i)_{i \in I}$ an upward directed family of $\mathrm{Fi}_S(\mathbf{A})$ such that $\bigcup\{D_i : i \in I\} \in \mathrm{Fi}_S(\mathbf{A})$. We have, $D_i \subseteq \bigcup\{D_i : i \in I\}$ for all $i \in I$. Since S is also protoalgebraic, by Theorem 3.3.1, $\Omega_{\mathbf{A}}(D_i) \subseteq \Omega_{\mathbf{A}}\left(\bigcup\{D_i : i \in I\}\right)$, for all $i \in I$. Hence, $\bigcup\{\Omega_{\mathbf{A}}(D_i) : i \in I\} \subseteq \Omega_{\mathbf{A}}\left(\bigcup\{D_i : i \in I\}\right)$. For the reverse inclusion, suppose that $a \equiv b\left(\Omega_{\mathbf{A}}\left(\bigcup\{D_i : i \in I\}\right)\right)$, i.e., $E^{\mathbf{A}}(a, b) \subseteq \bigcup\{D_i : i \in I\}$. Since $(D_i)_{i \in I}$ is upward directed and $E(p, q)$ is finite, there exists an $i \in I$ such that $E^{\mathbf{A}}(a, b) \subseteq D_i$. Thus, $a \equiv b(\Omega_{\mathbf{A}}(D_i))$ and therefore $\Omega_{\mathbf{A}}\left(\bigcup\{D_i : i \in I\}\right) \subseteq \bigcup\{\Omega_{\mathbf{A}}(D_i) : i \in I\}$.

$(ii) \Rightarrow (iii)$. This is obvious.

$(iii) \Rightarrow (i)$. Assume Ω is continuous on $\mathbf{Th}(S)$. First, we show that Ω is monotone on $\mathbf{Th}(S)$ and the theory T_{pq} is finitely axiomatizable. Let $T_1, T_2 \in \mathrm{Th}(S)$. Suppose that $T_1 \subseteq T_2$. The family (T_1, T_2) is upward directed and $T_1 \cup T_2 = T_2 \in \mathrm{Th}(S)$. By hypothesis, $\Omega T_2 = \Omega(T_1 \cup T_2) = \Omega T_1 \cup \Omega T_2$. Thus $\Omega T_1 \subseteq \Omega T_2$. By Theorem 3.3.1, S is protoalgebraic. Now, let $(T_i)_{i \in I}$ be the family of all finitely axiomatizable closed subtheories of T_{pq}. Thus each T_i is of the form $T_i = \mathrm{Cn}_S(X_i)$ with X_i a finite subset of T_{pq}. The family $(T_i)_{i \in I}$ is upward directed and $T_{pq} = \bigcup\{T_i : i \in I\}$. By Lemma 3.1.2, $p \equiv q(\Omega(T_{pq}))$. Since the Leibniz operator is continuous, $p \equiv q(\Omega(T_i))$ for some $i \in I$. We have seen that T_{pq} is an k-parameterized system of equivalence formulas for S. Thus, $T_{pq}(p, q, \underline{\delta}) \subseteq T_i$ for every string $\underline{\delta}$ of formulas. In particular, $T_{pq} \subseteq T_i$. Hence, $T_{pq} = T_i = \mathrm{Cn}_S(X_i)$ with X_i finite, i.e., T_{pq} is finitely axiomatizable.

In order to proved that S is finitely equivalential, we show that $T_{pq} = \mathrm{Cn}_S(E(p, q))$ for some finite set $E(p, q)$ of formulas. We fix an infinite set $V = \{u_1, u_2, \dots\}$ of variables disjoint from the variables in $Var(X_i) \cup \{p, q\}$ and we consider a substitution e such that $e(p) = p$, $e(q) = q$, $e(r) = p$ for every variable $r \in Var(X_i) \setminus \{p, q\}$ and $e(u) = u$ for every $u \in V$. Let $E(p, q) = e(X_i)$. By Lemma 3.1.2, we have $e(T_{pq}) = \mathrm{Cn}_S(e(X_i)) = \mathrm{Cn}_S(E(p, q)) \subseteq T_{pq}$. For the reverse inclusion, let $\varphi(p, q, r_1, \dots, r_n) \in T_{pq}$ where all the variables of φ are displayed. If we substitute in φ all the variables $r_1, \dots, r_n$ by arbitrary formulas, we obtain a formula which belongs to T_{pq}. In particular, $\varphi(p, q, u_1, \dots, u_n) \in T_{pq}$, i.e., $\varphi(p, q, u_1, \dots, u_n) \in \mathrm{Cn}_S(X_i)$. Hence by structurality, $\varphi(p, q, u_1, \dots, u_n) = \varphi(e(p), e(q), e(u_1), \dots, e(u_n)) = e(\varphi(p, q, u_1, \dots, u_n)) \in$

$\text{Cn}_S(e(X_i)) = \text{Cn}_S(E(p,q))$. Replacing the variables $u_1, \ldots, u_n$ by $r_1, \ldots, r_n$, we have $\varphi(p,q,r_1,\ldots,r_n) \in \text{Cn}_S(E(p,q))$. We conclude that $T_{pq} = \text{Cn}_S(E(p,q))$. Now, we prove that $E(p,q)$ is a finite system of equivalence formulas for S. Since $E(p,q) \subseteq T_{pq}$, reflexivity condition holds. As T_{pq} is an k-parameterized system of equivalence formulas for S, modus ponens condition holds. Thus, $E(p,q)$ satisfies modus ponens condition. To prove single replacement condition, we can use the same argument as in proof of Theorem 4.1.3. Hence $E(p,q)$ is a finite system of equivalence formulas for S. Therefore S is finitely equivalential. $\qquad\square$

In the following theorem, we characterize equivalential logics by closure properties of the class $\mathbf{Mod}^*(S)$.

Theorem 4.1.7. *Let S be a logic. The following conditions are equivalent:*

(i) S is equivalential;

(ii) The class $\mathbf{Mod}^*(S)$ is closed under submatrices and direct products.

Proof. Let S be an equivalential logic. Thus S is protoalgebraic and, by Theorem 3.5.1, $\mathbf{Mod}^*(S)$ is closed under subdirect products. Then it is also closed under direct products. Let $\text{N} = \langle \mathbf{B}, E \rangle$ be a submatrix of a matrix $\text{M} = \langle \mathbf{A}, D \rangle \in \mathbf{Mod}^*(S)$. Since $\mathbf{Mod}(S)$ is closed under submatrix, $\text{N} \in \mathbf{Mod}(S)$. As S is equivalential, there exists $E(p,q)$ a system of equivalence formulas for S. Let $a, b \in \text{B}$. We prove by contraposition that if $a \equiv b(\Omega_{\mathbf{B}} E)$ then $a \equiv b(\Omega_{\mathbf{A}} D)$. Suppose that $(a,b) \notin \Omega_{\mathbf{A}} D$, i.e., $E^{\mathbf{A}}(a,b) \nsubseteq D$. Since $E^{\mathbf{B}}(a,b) \subseteq E^{\mathbf{A}}(a,b)$, $E^{\mathbf{B}}(a,b) \nsubseteq D$. Thus $E^{\mathbf{B}}(a,b) \nsubseteq D \cap \text{B}$, i.e., $(a,b) \notin \Omega_{\mathbf{B}}(D \cap \text{B})$. As $E = D \cap \text{B}$, $\Omega_{\mathbf{B}} E = \Omega_{\mathbf{B}}(\text{B} \cap D)$. So $(a,b) \notin \Omega_{\mathbf{B}}(E)$. Now suppose that $a \equiv b(\Omega_{\mathbf{B}} E)$. Then $a \equiv b(\Omega_{\mathbf{A}} D)$. Since $\Omega_{\mathbf{A}} D = \triangle_{\mathbf{A}}$, $a = b$. Thus $\Omega_{\mathbf{B}} E = \triangle_{\mathbf{B}}$.

Conversely, assume that $\mathbf{Mod}^*(S)$ is closed under submatrices and direct products. Thus $\mathbf{Mod}^*(S)$ is closed under subdirect products and by Theorem 3.5.1, S is protoalgebraic. By Theorem 3.4.3, there exists an k-parameterized system of equivalence formulas $E(p,q,\underline{r}) = \{\epsilon_i(p,q,\underline{r}) : i \in I\}$ for S. Let $E'(p,q)$ be the set of all formulas of the form $\epsilon_i(p,q,\varphi_1(p,q),\ldots,\varphi_{k_i}(p,q))$, where $i \in I$ and $\varphi_1(p,q),\ldots,\varphi_{k_i}(p,q)$ range over all formulas that contain only the variables p,q. Let $\langle \mathbf{A}, F \rangle \in \mathbf{Mod}(S)$ and $a, b \in \text{A}$. Consider $\mathbf{B}$ the subalgebra of $\mathbf{A}$ generated by a and b. Thus, each elements of $\mathbf{B}$ is of the form $\varphi^{\mathbf{A}}(a,b)$ for some $\varphi(p,q) \in \text{Fm}_{\mathcal{L}}(\{p,q\})$. The matrix $\langle \mathbf{A}/\Omega_{\mathbf{A}} F, F/\Omega_{\mathbf{A}} F \rangle \in \mathbf{Mod}^*(S)$, since it is the reduction of the matrix $\langle \mathbf{A}, F \rangle$. As the

matrix $\langle \mathbf{B}/(\Omega_{\mathbf{A}}F \cap B^2), (F \cap B)/(\Omega_{\mathbf{A}}F \cap B^2)\rangle$ is isomorphic to a submatrix of $\langle \mathbf{A}, F\rangle$, by hypothesis, it is reduced. Hence, $\Omega_{\mathbf{A}}F \cap B^2$ is the largest congruence on $\mathbf{B}$ compatible with $F \cap B$, i.e., $\Omega_{\mathbf{B}}(F \cap B) = \Omega_{\mathbf{A}}F \cap B^2$. We have that:

$a \equiv b(\Omega_{\mathbf{A}}F)$ iff $a \equiv b(\Omega_{\mathbf{A}}F \cap B^2)$ iff $a \equiv b(\Omega_{\mathbf{B}}(F \cap B))$

$\quad$ iff $\epsilon_i^{\mathbf{B}}(a, b, \underline{c}) \in F \cap B$ for all $i \in I$, and $\underline{c} \in B^{k_i}$

$\quad$ iff $\epsilon_i^{\mathbf{B}}(a, b, \varphi_1^{\mathbf{B}}(p, q), \ldots, \varphi_{k_i}^{\mathbf{B}}(p, q)) \in F \cap B$ for all $i \in I$, and $\varphi_1, \ldots, \varphi_{k_i} \in \mathrm{Fm}_{\mathcal{L}}(\{p, q\})$

$\quad$ iff $\epsilon_i^{\mathbf{A}}(a, b, \varphi_1^{\mathbf{A}}(p, q), \ldots, \varphi_{k_i}^{\mathbf{A}}(p, q)) \in F$ for all $i \in I$, and $\varphi_1, \ldots, \varphi_{k_i} \in \mathrm{Fm}_{\mathcal{L}}(\{p, q\})$

$\quad$ iff $E'^{\mathbf{A}}(a, b) \subseteq F$

By Theorem 3.4.2, $E'(p, q)$ is a system of equivalence formulas (without parameters) for S. Therefore, S is equivalential. $\qquad\square$

As expected, the class of equivalential logics is a proper subclass of the class of protoalgebraic logics. The next example illustrate this result.

Example 4.2 (Orthologic [Mal89])**.** In Example 3.2, we have seen that the minimal orthologic S_{OL} is protoalgebraic. Now we prove that this logic is not equivalential using the fact that the class $\mathbf{Mod}^*(S_{OL})$ is not closed under submatrices. It is not difficult to prove that if the matrix $\langle \mathbf{A}, D\rangle \in \mathbf{Mod}^*(S_{OL})$ then $\mathbf{A}$ is an ortholattice and $D = \{1\}$, where 1 is the unit element of A. An *orthomodular* lattice is an ortholattice which satisfies the orthomodularity law: $y \approx (x \wedge y) \vee (y \wedge (\neg(x \wedge y)))$. The Benzene Ring $\mathbf{B}_6$ is an ortholattice which is not orthomodular because the orthomodularity law does not hold. Indeed, $a < b$ but $a \vee (b \wedge \neg a) = a \vee 0 = a \neq b$.
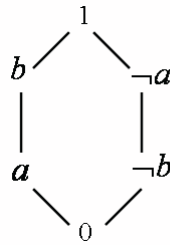
Figure 4.1: The Benzene Ring $\mathbf{B}_6$

As, we have seen that any class of ortholattices $\mathbf{K}$ can be identified with the class of matrices $\{\langle \mathbf{A}, \{1\}\rangle : \mathbf{A} \in \mathbf{K}\}$, we have that $\mathbf{B}_6$ can be identified with the matrix $\langle \mathbf{B}_6, \{1\}\rangle$. As $\Omega_{\mathbf{B}_6}(\{1\}) = \triangle_{\mathbf{B}_6} \cup \{(a, b), (b, a), (\neg a, \neg b), (\neg b, \neg a)\}$, the matrix $\langle \mathbf{B}_6, \{1\}\rangle$ is not reduced. Since S_{OL} is an orthologic which is not orthomodular, there exists a

matrix $\langle \mathbf{A}, \{1\} \rangle \in \mathbf{Mod}^*(S_{OL})$ such that the ortholattice $\mathbf{A}$ is not orthomodular. Malinowski proved in [Mal89, Theorem 3.3.1] that if an ortholattice $\mathbf{A}$ is not orthomodular then it contains the Benzene Ring $\mathbf{B}_6$ as a subortholattice. Thus $\mathbf{B}_6$ is a subalgebra of $\mathbf{A}$ and consequently $\langle \mathbf{B}_6, \{1\} \rangle$ is a submatrix of $\langle \mathbf{A}, \{1\} \rangle$. Since $\langle \mathbf{B}_6, \{1\} \rangle$ is not reduced, $\mathbf{Mod}^*(S_{OL})$ is not closed under submatrices. By Theorem 4.1.7, S_{OL} is not equivalential. The reader can see [Mal89, Section 3.3], [Mal90] and [CJ00, Chapter 6] for more information about orthologic and orthomodular logic; for instance, in the class of orthomodular logics, the DDT fails. $\diamond$

4.3 Finitary and (Finitely) Equivalential Logics

In this section, we focus on some results related to finitary logics.

Theorem 4.3.1. *Let S be a logic. The following conditions are equivalent:*

(i) S is finitary and finitely equivalential;

(ii) $\mathbf{Mod}^*(S)$ is a matrix-quasivariety.

Proof. Assume S is finitary and finitely equivalential. There exists a finite system of equivalence formulas $E(p, q)$ for S. Since S is finitary and also protoalgebraic, by Theorem 3.8.1, $\mathbf{Mod}^*(S)$ is closed under ultraproducts. Moreover S is also equivalential, by Theorem 4.1.7, $\mathbf{Mod}^*(S)$ is closed under submatrices and direct products. Therefore, $\mathbf{Mod}^*(S)$ is a matrix-quasivariety.

Conversely, assume $\mathbf{Mod}^*(S)$ is a matrix-quasivariety, i.e., $\mathbf{Mod}^*(S)$ is closed under submatrices, direct products and ultraproducts. Since $\mathbf{Mod}^*(S)$ is closed under ultraproducts, S is a finitary logic. By Theorem 4.1.7, S is equivalential. Thus, there exists a (possibly infinite) system of equivalence formulas $E(p, q)$ for S. As S is a finitary protoalgebraic logic and $\mathbf{Mod}^*(S)$ is closed under ultraproducts, by Theorem 3.8.1, $E(p, q)$ contains a finite subsystem of equivalence formulas for S. Therefore S is finitely equivalential. $\square$

4.4 Examples

Herein, we study some examples of modal logics with respect to the existence of system of equivalence formulas. We also discuss the finitariness of the system of equivalence formulas.

We begin by defining a large class of logics which are called implicative logics and which are finitely equivalential. This class of implicative logics have been extensively studied by Rasiowa (c.f. [Ras74]) and Sikorski.

Definition 4.4.1. *A logic S is called* implicative *if the language only contains a finite number of connectives of rank at most 2 and if there exists a formula $\varphi(p, q)$, called* implication *of S, such that:*

(i) $\vdash_S \varphi(p, p)$

(ii) $q \vdash_S \varphi(p, q)$

(iii) $\{\varphi(p, q)\} \cup \{\varphi(q, r)\} \vdash_S \varphi(p, r)$

(iv) $\{\varphi(p, q)\} \cup \{p\} \vdash_S q$

(v) *for each connective f of rank $n \geq 0$,*

$$\{\varphi(p_1, q_1), \varphi(q_1, p_1)\} \cup \cdots \cup \{\varphi(p_n, q_n), \varphi(q_n, p_n)\} \vdash_S \varphi(f(p_1, \ldots, p_n), f(q_1, \ldots, q_n))$$

It is not difficult to see that every implicative logic is finitely equivalential. Moreover, if $\varphi(p, q)$ is an implication for a logic S then S is finitely equivalential with $\{\varphi(p, q), \varphi(q, p)\}$ its system of equivalence formulas.

Among modal logics, we can find a variety of logics which show that the class of finitely equivalential logics is a proper subclass of equivalential logics. We present examples without proof and we give references where the reader can find detailed discussions.

Example 4.5 (Modal Logic [Mal89]). Let $\mathcal{L} = \{\wedge, \vee, \neg, \square\}$ be the language of Modal Logics, where $\wedge, \vee, \neg$ are the familiar connectives of conjunction, disjunction and negation, and $\square$ is a unary connective representing the logical necessity ($\square\varphi$ reads: "It is necessary that φ"). The notation $\square^n p$ is defined recursively, for all $n \in \mathbb{N}$, by: $\square^0 p = p$ and $\square^{n+1} p = \square(\square^n p)$. We adopt the usual notation: the formula $\varphi \rightarrow \psi$ is an abbreviation for $\neg\varphi \vee \psi$, and $\varphi \leftrightarrow \psi$ is an abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, for any $\varphi, \psi \in \text{Fm}_\mathcal{L}$. We denote by $Sb(X, r_1, \ldots, r_n)$ the least invariant set of modal formulas that includes the set $X \subseteq \text{Fm}_\mathcal{L}$ and is closed under the inference rules $r_1, \ldots, r_n$. We list some axioms and inference rules that we need to define some modal systems.

(MP) $\dfrac{p, p \rightarrow q}{q}$ (Modus Ponens)

(RE) $\dfrac{p \leftrightarrow q}{\square p \leftrightarrow \square q}$ (Extensionality)

$(NR)\ \dfrac{p}{\Box p}$ (Necessitation)

$(Kr)\ \Box(p \to q) \to (\Box p \to \Box q)$

$(T)\ \Box p \to p$

$(S4_n)\ \Box^n p \to \Box^{n+1} p,\ \text{for all } n \in \mathbb{N}$

We also defined CL as the least invariant set in $\mathrm{Fm}_{\mathcal{L}}$ containing all classical tautologies. By a *Modal System* we mean an invariant set of $\mathrm{Fm}_{\mathcal{L}}$ that contains all classical tautologies and is closed under modus ponens. In the sequel, we define some interesting modal systems:

$E = Sb(CL, (MP), (RE))$ is the least classical modal system;

$Kr = Sb(E, (Kr), (MP), (NR))$ is the least normal classical modal system, called Kripke system;

$T = Sb(K, (T), (MP), (NR))$;

$S4_n = Sb(T, (S4_n), (MP), (NR))$ for $n \in \mathbb{N}$.

If L is a modal system, we denote by $\overrightarrow{L}$ the *Modal Logic* defined in the language $\mathrm{Fm}_{\mathcal{L}}$ by the set of axioms L and the inference rule modus ponens. All modal logics are protoalgebraic because the DDT holds: for any $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\Gamma \cup \{\varphi\} \vdash_{\overrightarrow{L}} \psi \text{ iff } \Gamma \vdash_{\overrightarrow{L}} \varphi \to \psi$$

Malinowski has shown, in [Mal89, Corollary 2.1.3] and in [Mal86, Corollary II.3], that the logic $\overrightarrow{E}$ is not equivalential. But the Kripke logic $\overrightarrow{Kr}$ is finitary and equivalential with the set of equivalence formulas $E(p, q) := \{\Box^n(p \leftrightarrow q) : n \in \mathbb{N}\}$. Since the modal logic $\overrightarrow{T}$ is an axiomatic extension of $\overrightarrow{Kr}$, it is also equivalential. Malinowski has also shown, in [Mal89, Theorem 2.2.1] and in [Mal86, Theorem III.1], that $\overrightarrow{T}$ is not finitely equivalential. We deduced that $\overrightarrow{Kr}$ is not finitely equivalential. Furthermore, the modal logic $\overrightarrow{S4_n}$ is finitely equivalential, for all $n \in \mathbb{N}$ with $E(p, q) = \{\Box^n(p \leftrightarrow q)\}$ as the finite system of equivalence formulas. $\Diamond$

Chapter 5

Algebraizable Logics

The general theory of AAL studies the mechanism by which a class of algebras can be associated with a given logic. This contrasts to the study of algebraic logic where the main setting is to examine the class of algebras that are canonically associated with a logic. Boole could be considered as the first logician who studied the relationship between **CPL** and the class **BA**. The paradigm of the Lindenbaum-Tarski process is the way by which the class **BA** appears from **CPL**. In [BP89], Blok and Pigozzi give a precise meaning of the notion of finitary finitely algebraizable logics which are logics that have equivalent algebraic semantics with a finite set of equivalence formulas and a finite set of defining equations.

In this chapter, we study the algebraization phenomena in a wide sense. First we define the notion of algebraic semantics. Roughly speaking, a class $\mathbf{K}$ of algebras can be considered as an algebraic semantics of a logic S if the consequence relation $\vdash_S$ can be interpreted in the equational consequence relation $\models_{\mathbf{K}}$ in a natural way. In addition, if there exists an inverse interpretation of $\models_{\mathbf{K}}$ in $\vdash_S$, then $\mathbf{K}$ is called an equivalent algebraic semantics for S (it is unique up to a quasivariety). We study the class of weakly algebraizable logics which are logics that have a pair of interpretations that commute with surjective substitutions and are mutually inverse. We characterized this class using Leibniz operator properties. We also define algebraizable logics as logics which have an equivalent algebraic semantics and give some characterizations. Among them, we have that $\mathbf{K}$ is an equivalent algebraic semantics for S iff there exists an isomorphism between the theory lattice of S and the equational theory lattice of $\mathbf{K}$ that commutes with inverse substitution. We finalize by giving examples of logics which show that the inclusion among different classes of algebraizable logics are proper.

5.1 Algebraic Semantics

The reader should not confuse the notion of algebraic semantics presented in this chapter with the notion of matrix semantics defined in Chapter 2. Although every logic has a matrix semantics, it can does not have an algebraic semantics. We give an example to illustrate this result. Furthermore, if the logic is finitary, then it has (if any) a quasivariety semantics which can be axiomatized by a set of axioms and a set of inference rules. We point out to [BR03], where Blok and Rebagliato have studied sufficient conditions for a logic to have an algebraic semantics and presented some examples that illustrate their results.

Definition 5.1.1. *Let S be a logic and* **K** *a class of $\mathcal{L}$-algebras. We say that $\vdash_S$ is interpretable in* $\models_{\mathbf{K}}$ *if there exists a mapping $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ such that for all $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,*

$$\Gamma \vdash_S \alpha \ \textit{iff} \ \tau[\Gamma] \models_{\mathbf{K}} \tau(\alpha)$$

The mapping τ is called an interpretation *of $\vdash_S$ in* $\models_{\mathbf{K}}$.

We say that $\Delta(p, \underline{v}) = \{\delta_i(p, \underline{v}) \approx \epsilon_i(p, \underline{v}) : i \in I\}$ is an *l-parameterized system of equations* if it is a set of equations in a single variable p and possibly other variables $\underline{v} = v_1, v_2, \ldots$ called *parameters* with l the length of the string $\underline{v}$. Note that $\Delta(p, \underline{v})$ may be infinite; hence the length of the string $\underline{v}$, could be ω. Let $\alpha \in \mathrm{Fm}_{\mathcal{L}}$, we denote by $\Delta(\langle \alpha \rangle)$ the set of all substitution instances $e(\delta_i(p, \underline{v})) \approx e(\epsilon_i(p, \underline{v}))$ where i ranges over I and e over all substitutions such that $e(p) = \alpha$, i.e.,

$$\Delta(\langle \alpha \rangle) := \{\delta_i(p/\alpha, \underline{v}/\underline{\xi}) \approx \epsilon_i(p/\alpha, \underline{v}/\underline{\xi}) : i \in I, \underline{\xi} \in (\mathrm{Fm}_{\mathcal{L}})^l\}.$$

We extend this notation to $\mathcal{L}$-algebras. If **A** is an $\mathcal{L}$-algebra and $a, b \in A$, we denote by $\Delta^{\mathbf{A}}(\langle a \rangle)$ the set of all elements of A of the form $h(\delta_i(p, \underline{v})) \approx h(\epsilon_i(p, \underline{v}))$ where i ranges over I and h over all homomorphisms $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ such that $h(p) = a$, i.e.,

$$\Delta^{\mathbf{A}}(\langle a \rangle) := \{\delta_i^{\mathbf{A}}(p/a, \underline{v}/\underline{c}) \approx \epsilon_i^{\mathbf{A}}(p/a, \underline{v}/\underline{c}) : i \in I, \underline{c} \in A^l\},$$

where $\epsilon_i^{\mathbf{A}} := h(\epsilon_i)$ and $\delta_i^{\mathbf{A}} := h(\delta_i)$.

In the following propositions, the mapping τ can be represented, with some assumptions, by a particular set of equations.

Proposition 5.1.2. *Let $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ be an arbitrary mapping. The following conditions are equivalent:*

(i) τ commutes with surjective substitutions;

(ii) There exists an l-parameterized system of equations $\Delta(p,\underline{v})$ such that for all $\alpha \in \mathrm{Fm}_{\mathcal{L}}$, $\tau(\alpha) = \Delta(\langle\alpha\rangle)$.

Proof. Assume τ commutes with surjective substitutions. We fix a variable p and define $\Delta(p,\underline{v}) := \tau(p)$. Let l be the length of $\underline{v}$. Clearly, $\tau(p) \subseteq \{\delta(p,\underline{\xi}) \approx \epsilon(p,\underline{\xi}) : \delta(p,\underline{v}) \approx \epsilon(p,\underline{v}) \in \Delta(p,\underline{v}), \underline{\xi} \in (\mathrm{Fm}_{\mathcal{L}})^l\}$. For the reverse inclusion, let $\underline{\xi} \in (\mathrm{Fm}_{\mathcal{L}})^l$ and e a surjective substitution such that $e(p) = p$, $e(\underline{v}) = \underline{\xi}$. For all $\delta(p,\underline{v}) \approx \epsilon(p,\underline{v}) \in \Delta(p,\underline{v})$, we have that $\delta\big(e(p),e(\underline{v})\big) \approx \epsilon\big(e(p),e(\underline{v})\big) \in e[\tau(p)]$. By hypothesis, $e[\tau(p)] = \tau(e(p)) = \tau(p)$. Thus for all $\delta(p,\underline{v}) \approx \epsilon(p,\underline{v}) \in \Delta(p,\underline{v})$, $\delta(p,\underline{\xi}) \approx \epsilon(p,\underline{\xi}) \in \tau(p)$, i.e., $\{\delta(p,\underline{\xi}) \approx \epsilon(p,\underline{\xi}) : \delta(p,\underline{v}) \approx \epsilon(p,\underline{v}) \in \Delta(p,\underline{v}), \underline{\xi} \in (\mathrm{Fm}_{\mathcal{L}})^l\} \subseteq \tau(p)$. We conclude that $\tau(p) = \{\delta(p,\underline{\xi}) \approx \epsilon(p,\underline{\xi}) : \delta(p,\underline{v}) \approx \epsilon(p,\underline{v}) \in \Delta(p,\underline{v}), \underline{\xi} \in (\mathrm{Fm}_{\mathcal{L}})^l\}$. Now, let $\alpha \in \mathrm{Fm}_{\mathcal{L}}$ and e a surjective substitution such that $e(p) = \alpha$. We have that $\tau(\alpha) = \tau(e(p)) = e[\tau(p)] = e[\{\delta(p,\underline{\xi}) \approx \epsilon(p,\underline{\xi}) : \delta(p,\underline{v}) \approx \epsilon(p,\underline{v}) \in \Delta(p,\underline{v}), \underline{\xi} \in (\mathrm{Fm}_{\mathcal{L}})^l\}] = \{\delta\big(e(p),e(\underline{\xi})\big) \approx \epsilon\big(e(p),e(\underline{\xi})\big) : \delta(p,\underline{v}) \approx \epsilon(p,\underline{v}) \in \Delta(p,\underline{v}), \underline{\xi} \in (\mathrm{Fm}_{\mathcal{L}})^l\} = \{\delta\big(\alpha,e(\underline{\xi})\big) \approx \epsilon\big(\alpha,e(\underline{\xi})\big) : \delta(p,\underline{v}) \approx \epsilon(p,\underline{v}) \in \Delta(p,\underline{v}), \underline{\xi} \in (\mathrm{Fm}_{\mathcal{L}})^l\}$. Since e is surjective, $\tau(\alpha) = \{\delta(\alpha,\underline{\eta}) \approx \epsilon(\alpha,\underline{\eta}) : \delta(p,\underline{v}) \approx \epsilon(p,\underline{v}) \in \Delta(p,\underline{v}), \underline{\eta} \in (\mathrm{Fm}_{\mathcal{L}})^l\} = \Delta(\langle\alpha\rangle)$.

Conversely, let e be a surjective substitution and $\alpha \in \mathrm{Fm}_{\mathcal{L}}$. We have that $e[\tau(\alpha)] = e[\Delta(\langle\alpha\rangle)] = \Delta(\langle e(\alpha)\rangle) = \{\delta_i(e(\alpha),e(\underline{\xi})) \approx \epsilon_i(e(\alpha),e(\underline{\xi})) : i \in I, \underline{\xi} \in (\mathrm{Fm}_{\mathcal{L}})^l\}$. By surjectivity of e, $e[\tau(\alpha)] = \{\delta_i(e(\alpha),\underline{\eta}) \approx \epsilon_i(e(\alpha),\underline{\eta}) : i \in I, \underline{\eta} \in (\mathrm{Fm}_{\mathcal{L}})^l\} = \tau(e(\alpha))$. $\square$

Proposition 5.1.3. *Let $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ be an arbitrary mapping. The following conditions are equivalent:*

(i) τ commutes with arbitrary substitutions;

(ii) There exists a set $\Delta(p) = \{\delta_i(p) \approx \epsilon_i(p) : i \in I\}$ of equations in a single variable p such that for all $\alpha \in \mathrm{Fm}_{\mathcal{L}}$, $\tau(\alpha) = \Delta(\alpha)$.

Proof. Assume τ commutes with arbitrary substitutions. We fix a variable p and define $\Delta(p) := \tau(p)$. Suppose $\mathrm{Var}(\Delta(p)) \subseteq \{p, r_1, r_2, \dots\}$. Let e be a substitution such that for every $q \in \mathrm{Var}(\Delta(p))$, $e(q) = p$. Then $\{\delta(p,p,p,\dots) \approx \epsilon(p,p,p,\dots) : \delta(p,\underline{v}) \approx \epsilon(p,\underline{v}) \in \Delta(p)\} = e[\{\delta(p) \approx \epsilon(p) : \delta(p) \approx \epsilon(p) \in \Delta(p)\}] = e[\tau(p)] = \tau\big(e(p)\big) = \tau(p) = \{\delta(p,r_1,r_2,\dots) \approx \epsilon(p,r_1,r_2,\dots) : \delta(p,\underline{v}) \approx \epsilon(p,\underline{v}) \in \Delta(p)\}$. Hence, $\{r_1,r_2,\dots\} \subseteq \{p\}$. Thus $\mathrm{Var}(\Delta(p)) \subseteq \{p\}$. Now, let $\alpha \in \mathrm{Fm}_{\mathcal{L}}$ and e a substitution such that $e(p) = \alpha$. Then $\tau(\alpha) = \tau\big(e(p)\big) = e[\tau(p)] = e[\Delta(p)] = \Delta\big(e(p)\big) = \Delta(\alpha)$.

The converse is obvious. $\square$

In [Cze01, Definition 4.2.1], Czelakowski defined the notion of interpretation in more general sense (between two arbitrary logics) and used the term "transformer". When an interpretation commutes with arbitrary substitutions and is defined by a finite set of equations then it coincides with the notion of "translation" introduced by Blok and Pigozzi in [BP01, Definition 4.1]. There are others logicians that study mappings between logic (Feitosa and Ottaviano have described in [FD01, Definition 1.10] a more general notion of translation. They have studied the conservative translation and established some logical properties that may be preserved via translations).

The following definition of algebraic semantics is due to Blok and Pigozzi in [BP89, definition 2.2] but with the requirement that the set of equations is finite and the logic is finitary.

Definition 5.1.4. *Let S be a logic and $\mathbf{K}$ a class of $\mathcal{L}$-algebras. We say that $\mathbf{K}$ is an* algebraic semantics *of S if $\vdash_S$ is interpretable in $\models_{\mathbf{K}}$ in the following sense: there exists $\Delta(p)$ a set of equations such that for all $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,*

$$\Gamma \vdash_S \alpha \ \textit{iff} \ \Delta(\Gamma) \models_{\mathbf{K}} \Delta(\alpha)$$

where $\Delta(\Gamma) = \{\delta_i(p/\gamma) \approx \epsilon_i(p/\gamma) : i \in I, \gamma \in \Gamma\}$.

The equations in $\Delta(p)$ are called the defining equations *for $\vdash_S$ and $\models_{\mathbf{K}}$.*

In other words, $\mathbf{K}$ is an algebraic semantics of S iff there exists an interpretation $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ of $\vdash_S$ in $\models_{\mathbf{K}}$ that commutes with arbitrary substitutions. The algebraic semantics $\mathbf{K}$ is also called τ-*algebraic semantics* of S by Raftery in [Raf06b, Definition 1].

If τ is an interpretation of $\vdash_S$ in $\models_{\mathbf{K}}$, that commutes with arbitrary substitutions, then for every algebra $\mathbf{A} \in \mathbf{K}$, we write $F_{\mathbf{A}}^{\tau} := \{a \in A : \delta_i^{\mathbf{A}}(a) = \epsilon_i^{\mathbf{A}}(a), i \in I\}$.

Theorem 5.1.5. *Let S be a logic, $\mathbf{K}$ a class of $\mathcal{L}$-algebras and $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ an arbitrary mapping that commutes with arbitrary substitutions. Then the following conditions are equivalent:*

(i) $\mathbf{K}$ is an algebraic semantics of S with the set of defining equations $\tau(p)$;

(ii) The class $\mathcal{M} = \{\langle \mathbf{A}, F_{\mathbf{A}}^{\tau} \rangle : \mathbf{A} \in \mathbf{K}\}$ is a matrix semantics of S.

Proof. Let $\mathcal{M} = \{\langle \mathbf{A}, F_{\mathbf{A}}^{\tau} \rangle : \mathbf{A} \in \mathbf{K}\}$ be a class of matrices and $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. We have that

$\Gamma \models_{\mathcal{M}} \alpha$ iff, for all M $= \langle \mathbf{A}, F_{\mathbf{A}}^{\tau} \rangle \in \mathcal{M}$, $\Gamma \models_{\mathrm{M}} \alpha$ iff, for all $\mathbf{A} \in$
$\mathbf{K}$ and all homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, $h[\Gamma] \subseteq F_{\mathbf{A}}^{\tau}$ implies $h(\alpha) \in F_{\mathbf{A}}^{\tau}$
iff, for all $\mathbf{A} \in \mathbf{K}$, all homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ and all $i \in I$; $\delta_i^{\mathbf{A}}(\gamma) =$
$\epsilon_i^{\mathbf{A}}(\gamma)$ for all $\gamma \in \Gamma$ implies $\delta_i^{\mathbf{A}}(\alpha) = \epsilon_i^{\mathbf{A}}(\alpha)$ iff, for all $\mathbf{A} \in \mathbf{K}$, all homomorphism $h :$
$\mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ and all $i \in I$; $h(\delta_i(\gamma)) = h(\epsilon_i(\gamma))$ for all $\gamma \in \Gamma$ implies $h(\delta_i(\alpha)) = h(\epsilon_i(\alpha))$
iff, for all $\mathbf{A} \in \mathbf{K}$, $\Delta(\Gamma) \models_{\mathbf{A}} \Delta(\alpha)$ iff $\Delta(\Gamma) \models_{\mathbf{K}} \Delta(\alpha)$.

Now, assume that the class $\mathbf{K}$ is an algebraic semantics of S. Let $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$.
We have that $\Gamma \models_{\mathcal{M}} \alpha$ iff $\Delta[\Gamma] \models_{\mathbf{K}} \Delta(\alpha)$. By assumption, $\Delta[\Gamma] \models_{\mathbf{K}} \Delta(\alpha)$ iff $\Gamma \vdash_S \alpha$.
Thus $\mathcal{M}$ is a matrix semantics of S.

Conversely, assume that the class $\mathcal{M}$ is a matrix semantics of S. Let $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$.
By assumption, $\Gamma \vdash_S \alpha$ iff $\Gamma \models_{\mathcal{M}} \alpha$. Since, $\Gamma \models_{\mathcal{M}} \alpha$ iff $\Delta[\Gamma] \models_{\mathbf{K}} \Delta(\alpha)$, we have that
$\mathbf{K}$ is an algebraic semantics of S. □

We give an example of logic that has an algebraic semantics.

Example 5.2 (Classical Propositional Logic [BP01]). Let **CPL** be the *Classical Propositional Logic* defined in the language $\mathcal{L} = \{\to, \wedge, \vee, \neg, \bot, \top\}$ by the following axioms:

$A_1 \ p \to (q \to p)$

$A_2 \ (p \to (q \to r)) \to ((p \to q) \to (p \to r))$

$A_3 \ (\neg q \to \neg p) \to (p \to q)$

$A_4 \ (p \wedge q) \to p$

$A_5 \ (p \wedge q) \to q$

$A_6 \ (r \to p) \to ((r \to q) \to (r \to (p \wedge q)))$

$A_7 \ p \to (p \vee q)$

$A_8 \ q \to (p \vee q)$

$A_9 \ (p \to r) \to ((q \to r) \to ((p \vee q) \to r))$

$A_{10} \ \bot \to p$

$A_{11} \ p \to \top$

and the inference rule:

$$\frac{p, p \to q}{q} \qquad \text{(Modus Ponens)}$$

We define *Boolean algebra* as an $\mathcal{L}$-algebra $\mathbf{A} = \langle \mathrm{A}, \to^{\mathbf{A}}, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \neg^{\mathbf{A}}, \bot^{\mathbf{A}}, \top^{\mathbf{A}} \rangle$ such that $\langle \mathrm{A}, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \neg^{\mathbf{A}}, \bot^{\mathbf{A}}, \top^{\mathbf{A}} \rangle$ is a bounded, complemented, distributive lattice with smallest element $\bot^{\mathbf{A}}$, largest element $\top^{\mathbf{A}}$ and complementation $\neg^{\mathbf{A}}$, while $\to^{\mathbf{A}}$ is relative complementation ($a \to^{\mathbf{A}} b = \neg^{\mathbf{A}} a \vee^{\mathbf{A}} b$). We denote by **BA** the class of all

Boolean algebras. Since $\{\langle \mathbf{A}, \{\top^{\mathbf{A}}\}\rangle : \mathbf{A} \in \mathbf{BA}\}$ is a matrix semantics of $\mathbf{CPL}$, by Theorem 5.1.5, the class $\mathbf{BA}$ is an algebraic semantics of $\mathbf{CPL}$ with the set of defining equation $\{p \approx \top\}$. Therefore, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathbf{CPL}} \varphi$ iff $\Gamma \approx \top \models_{\mathbf{BA}} \varphi \approx \top$. However, the logic $\mathbf{CPL}$ has another algebraic semantics which is the class of Heyting algebras ($\mathbf{HA}$) with the defining equation $\{\neg\neg p \approx \top\}$ (c.f. [BR03, Proposition 2.6]).$\diamondsuit$

We say that an algebra $\mathbf{A}$ is a τ-*model* of a logic S if for all $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\Gamma \vdash_S \alpha \text{ implies } \tau[\Gamma] \models_{\mathbf{A}} \tau(\alpha).$$

We denote by $\mathbf{K}(S, \tau)$ the class of all τ-models of S.

With the above example, we can note that if a logic has an algebraic semantics then it is not unique. Nevertheless, in the following proposition, we prove that if a logic has an algebraic semantics then it has the largest one.

Proposition 5.2.1. *Let S be a logic and $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ an arbitrary mapping that commutes with arbitrary substitutions. If S has an algebraic semantics with the set of defining equations $\tau(p)$, then $\mathbf{K}(S, \tau)$ is the largest algebraic semantics.*

Proof. Let $\mathbf{K}$ be an algebraic semantics of S with the set of defining equations $\tau(p)$ and $\Gamma \cup \{\alpha\} \in \mathrm{Fm}_{\mathcal{L}}$. By the definition of $\mathbf{K}(S, \tau)$, we have that $\Gamma \vdash_S \alpha$ implies $\tau[\Gamma] \models_{\mathbf{K}(S,\tau)} \tau(\alpha)$. Since $\mathbf{K}$ is an algebraic semantics, all the algebras $\mathbf{A} \in \mathbf{K}$ are τ-models. Thus $\mathbf{K} \subseteq \mathbf{K}(S, \tau)$. Hence $\tau[\Gamma] \models_{\mathbf{K}(S,\tau)} \tau(\alpha)$ implies $\tau[\Gamma] \models_{\mathbf{K}} \tau(\alpha)$. Since $\mathbf{K}$ is an algebraic semantics of S, $\Gamma \vdash_S \alpha$. We conclude that $\mathbf{K}(S, \tau)$ is an algebraic semantics of S. It is not difficult to see that for all $\mathbf{K}$ algebraic semantics of S, $\mathbf{K} \subseteq \mathbf{K}(S, \tau)$. Thus $\mathbf{K}(S, \tau)$ is the largest algebraic semantics of S. $\square$

Moreover, if S is a deductive system, i.e, is axiomatized by a set of axioms and a set of inference rules, then the largest algebraic semantics is also axiomatized.

Proposition 5.2.2. [BR03, Proposition 2.9] *Let S be a deductive system axiomatized by a set AX of axioms and a set IR of inference rules, and $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ an arbitrary mapping that commutes with arbitrary substitutions which is defined by a finite set of defining equations. Then, the algebraic 2-deductive system $\models_{\mathbf{K}(S,\tau)}$ is axiomatized by the axioms,*

(i) $p \approx p$

(ii) $\tau(\alpha)$ i.e., $\delta_i(\alpha) \approx \epsilon_i(\alpha)$, for all $i \in I$, and all $\alpha \in \mathrm{AX}$

and the inference rules,

(iii) $\dfrac{p \approx q}{q \approx p}$

(iv) $\dfrac{p \approx q, q \approx r}{p \approx r}$

(v) $\dfrac{\{p_i \approx q_i : i < m\}}{f(p_0, \ldots, p_{m-1}) \approx f(q_0, \ldots, q_{m-1})}$, for all connectives f of rank m

(vi) $\dfrac{\bigcup\{\tau(\alpha_j) : j < n\}}{\tau(\beta)}$ i.e., $\dfrac{\{\delta_i(\alpha_j) \approx \epsilon_i(\alpha_j) : i \in I, j < n\}}{\delta_i(\beta) \approx \epsilon_i(\beta)}$, for all $i \in I$, all $\langle\{\alpha_j : j < n\}, \beta\rangle \in \mathrm{IR}$ and $n \in \mathbb{N}$.

If a finitary logic S has an algebraic semantics $\mathbf{K}$ with the set of defining equations $\Delta(p)$ then it has a *quasivariety semantics* $Q(\mathbf{K})$ with the same set of defining equations. We give an example of logic that has a quasivariety semantics.

Example 5.3 (Classical Propositional Logic [BP01])**.** Let $\mathbf{2} = \langle\{0,1\}, \to^{\mathbf{2}}, \wedge^{\mathbf{2}}, \vee^{\mathbf{2}}, \neg^{\mathbf{2}},$ $\perp^{\mathbf{2}}, \top^{\mathbf{2}}\rangle$ be the two-element Boolean algebra where $\perp^{\mathbf{2}} = 0$ and $\top^{\mathbf{2}} = 1$ denote respectively "false" and "true", and $\to^{\mathbf{2}}, \wedge^{\mathbf{2}}, \vee^{\mathbf{2}} : \{0,1\}^2 \to \{0,1\}$ and $\neg^{\mathbf{2}} : \{0,1\} \to \{0,1\}$ are given by the usual truth tables. We have that, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathbf{CPL}} \varphi$ iff $\Gamma \approx$ $\top \models_{\mathbf{2}} \varphi \approx \top$. Thus the class of two-element Boolean algebras is an algebraic semantics for $\mathbf{CPL}$ with the set of defining equations $\Delta(p) = \{p \approx \top\}$. The variety of Boolean algebras is generated by the two-element Boolean algebra, i.e., $\mathbf{BA} = HSP(\mathbf{2})$. We also have that the variety of Boolean algebras is generated by $\mathbf{2}$ as a quasivariety, i.e., $\mathbf{BA} = SP(\mathbf{2})$. $\diamondsuit$

If a logic S has an algebraic semantics, then any fragment of S, whose language includes the connectives occurring in the set of defining equations, has an algebraic semantics; and any extension of S, also has an algebraic semantics with the same set of defining equations. Blok and Pigozzi proved this result, in [BP89, Corollary 2.5], for finitary logic and finite set of defining equations.

Theorem 5.3.1. *Let S be a logic, $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ an arbitrary mapping that commutes with arbitrary substitutions, $\mathbf{K}$ an algebraic semantics of S with the set of defining equations $\tau(p)$ and $\mathcal{L}'$ a sublanguage of $\mathcal{L}$ that contains all the primitive connectives occurring in $\tau(p)$. Then the class $\mathbf{K}'$ of all $\mathcal{L}'$-reducts of members of $\mathbf{K}$ is an algebraic semantics of any $\mathcal{L}'$-fragment S' of S. Furthermore, if $\mathbf{K}(S, \tau)$ is a quasivariety, then $\mathbf{K}'(S', \tau)$ is a quasivariety semantics for S'.*

In order to prove that any extension of logic, which has an algebraic semantics, also has an algebraic semantics, we need the following lemmas.

Let $\tau_{S,\mathbf{K}} : \mathrm{Th}S \to \mathrm{Th}\mathbf{K}$ be the mapping defined by $\tau_{S,\mathbf{K}}[T] = \mathrm{Cn}_{\mathbf{K}}(\tau[T])$, for all $T \in \mathrm{Th}S$.

Lemma 5.3.2. *Let S be a logic, $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ an arbitrary mapping that commutes with arbitrary substitutions and $\mathbf{K} \subseteq \mathbf{K}(S, \tau)$ a class of τ-models of S. Then the following conditions are equivalent:*

(i) $\mathbf{K}$ is an algebraic semantics of S with the set of defining equations $\tau(p)$.

(ii) $\tau_{S,\mathbf{K}}$ is injective.

Proof. Let $T_1, T_2 \in \mathrm{Th}S$. Suppose $\tau_{S,\mathbf{K}}[T_1] = \tau_{S,\mathbf{K}}[T_2]$. Let $\alpha \in T_1$. We have that $\tau(\alpha) \subseteq \tau[T_1] \subseteq \tau_{S,\mathbf{K}}[T_1] = \tau_{S,\mathbf{K}}[T_2]$. Hence, $\tau[T_2] \models_{\mathbf{K}} \tau(\alpha)$. Since $\mathbf{K}$ is an algebraic semantics of S, we have $T_2 \vdash_S \alpha$, i.e., $\alpha \in T_2$. In an analogous way, we can prove that $T_2 \subseteq T_1$. Therefore $\tau_{S,\mathbf{K}}$ is injective.

Conversely, let $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. Since $\mathbf{K}$ is a class of τ-models of S, we have that $\Gamma \vdash_S \alpha$ implies $\tau[\Gamma] \models_{\mathbf{K}} \tau(\alpha)$. Now, suppose $\tau[\Gamma] \models_{\mathbf{K}} \tau(\alpha)$. Thus, $\mathrm{Cn}_{\mathbf{K}}(\tau[\Gamma]) = \mathrm{Cn}_{\mathbf{K}}(\tau[\Gamma \cup \{\alpha\}])$. Since $\Gamma \subseteq \mathrm{Cn}_S(\Gamma)$, we have that $\tau[\Gamma] \subseteq \tau[\mathrm{Cn}_S(\Gamma)]$. So, $\mathrm{Cn}_{\mathbf{K}}(\tau[\Gamma]) \subseteq \mathrm{Cn}_{\mathbf{K}}(\tau[\mathrm{Cn}_S(\Gamma)]) = \tau_{S,\mathbf{K}}[\mathrm{Cn}_S(\Gamma)]$. For the reverse inclusion, let $\alpha \approx \beta \in \tau_{S,\mathbf{K}}[\mathrm{Cn}_S(\Gamma)]$, i.e., $\tau[\mathrm{Cn}_S(\Gamma)] \models_{\mathbf{K}} \alpha \approx \beta$. Thus $\{\tau(\xi) : \Gamma \vdash_S \xi\} \models_{\mathbf{K}} \alpha \approx \beta$. As $\mathbf{K}$ is a class of τ-models of S, for all $\xi \in \mathrm{Fm}_{\mathcal{L}}$ we have that $\Gamma \vdash_S \xi$ implies $\tau[\Gamma] \models_{\mathbf{K}} \tau(\xi)$. Therefore, $\tau[\Gamma] \models_{\mathbf{K}} \alpha \approx \beta$, i.e., $\alpha \approx \beta \in \mathrm{Cn}_K(\tau[\Gamma])$. We conclude that for all $\Gamma \in \mathrm{Fm}_{\mathcal{L}}$, $\tau_{S,\mathbf{K}}[\mathrm{Cn}_S(\Gamma)] = \mathrm{Cn}_{\mathbf{K}}(\tau[\Gamma])$. By these results, we have that $\tau_{S,\mathbf{K}}[\mathrm{Cn}_S(\Gamma)] = \tau_{S,\mathbf{K}}[\mathrm{Cn}_S(\Gamma \cup \{\alpha\})]$. Since $\tau_{S,\mathbf{K}}$ is injective, $\mathrm{Cn}_S(\Gamma) = \mathrm{Cn}_S(\Gamma \cup \{\alpha\})$ and so, $\Gamma \vdash_S \alpha$. $\square$

Lemma 5.3.3. *Let S be a deductive system and $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ an arbitrary mapping that commutes with arbitrary substitutions. Suppose that $\mathbf{K} = \mathbf{K}(S, \tau)$ is an algebraic semantics of S with the finite set of defining equations $\tau(p)$. If S' is an extension of S and $\mathbf{K}' = \mathbf{K}'(S', \tau)$ then $\tau_{S',\mathbf{K}'}$ equals $\tau_{S,\mathbf{K}}$ restricted to $\mathrm{Th}S'$.*

Proof. Let $T \in \mathrm{Th}S'$. Since S' is an extension of S, $\mathbf{K}' \subseteq \mathbf{K}$, so $\models_{\mathbf{K}'}$ is an extension of $\models_{\mathbf{K}}$. Hence $\mathrm{Cn}_{\mathbf{K}}(\tau[T]) \subseteq \mathrm{Cn}_{\mathbf{K}'}(\tau[T])$, i.e., $\tau_{S,\mathbf{K}}[T] \subseteq \tau_{S',\mathbf{K}'}[T]$. For the reverse inclusion, by Proposition 5.2.2, $\mathbf{K}'$ can be axiomatized by a set of axioms and a set of inference rules. It is not difficult to see that $\mathrm{Cn}_{\mathbf{K}}(\tau[T])$ contains all substitution instances of the axioms of $\models_{\mathbf{K}'}$ and is closed under the inference rules of $\models_{\mathbf{K}'}$. Indeed, it is obvious that $\mathrm{Cn}_{\mathbf{K}}(\tau[T])$ contains all substitution instances of the axiom *(i)* and is closed under the inference rules *(iii)* and *(iv)* of Proposition 5.2.2 . Let $\alpha \in \mathrm{Thm}(S')$ and e a substitution. Since $\mathrm{Thm}(S')$ is closed under substitutions, $\vdash_{S'} e(\alpha)$. Thus for

all $T \in \text{Th}S'$, we have that $e(\alpha) \in T$. As τ commutes with arbitrary substitutions, $e[\tau(\alpha)] = \tau[e(\alpha)] \subseteq \tau[T] \subseteq \text{Cn}_{\mathbf{K}}(\tau[T])$. Thus $\text{Cn}_{\mathbf{K}}(\tau[T])$ contains all substitution instances of axioms of $\mathbf{K}'$. Now, let $\{\alpha_i : i < n\} \vdash_{S'} \beta$ be an inference rule of S' and e a substitution such that $\{e[\tau(\alpha_i)] : i < n\} \subseteq \text{Cn}_{\mathbf{K}}(\tau[T])$, i.e., $\tau[T] \models_{\mathbf{K}} e[\tau(\alpha_i)]$ for all $i < n$. Since τ commutes with arbitrary substitutions, $e[\tau(\alpha_i)] = \tau[e(\alpha_i)]$ for all $i < n$, i.e., $\tau[T] \models_{\mathbf{K}} \tau[e(\alpha_i)]$ for all $i < n$. As $\mathbf{K}$ is an algebraic semantics of S, it follows that $T \vdash_S e(\alpha_i)$ for all $i < n$, i.e., $\{e(\alpha_i) : i < n\} \subseteq T$. By structurality of S', $\{e(\alpha_i) : i < n\} \vdash_{S'} e(\beta)$. Since $T \in \text{Th}S'$, we have that $e(\beta) \in T$. Thus $e[\tau(\beta)] = \tau[e(\beta)] \subseteq \tau[T] \subseteq \text{Cn}_{\mathbf{K}}(\tau[T])$. Therefore $\text{Cn}_{\mathbf{K}}(\tau[T])$ is closed under the inference rules of $\mathbf{K}'$. By the characterization of a theory in a deductive system, we have proved that $\text{Cn}_{\mathbf{K}}(\tau[T]) \in \text{Th}(\mathbf{K}')$. Since $\tau[T] \subseteq \text{Cn}_{\mathbf{K}}(\tau[T])$ and $\text{Cn}_{\mathbf{K}'}(\tau[T])$ is the least S'-theory that contains $\tau[T]$, we have that $\text{Cn}_{\mathbf{K}'}(\tau[T]) \subseteq \text{Cn}_{\mathbf{K}}(\tau[T])$, i.e., $\tau_{S',\mathbf{K}'}[T] \subseteq \tau_{S,\mathbf{K}}[T]$. $\qquad\square$

Theorem 5.3.4. *Let S be a deductive system. If S has an algebraic semantics, then any extension of S also has an algebraic semantics with the same set of defining equations.*

Proof. Assume that S has an algebraic semantics $\mathbf{K}$ with the set of defining equations $\tau(p)$. Let S' be an extension of S and $\mathbf{K}' = \mathbf{K}(S', \tau)$. By Proposition 5.2.1, the class $\mathbf{K}(S, \tau)$ is an algebraic semantics of S with the set of defining equations $\tau(p)$. Since, by Lemma 5.3.2, the mapping $\tau_{S,\mathbf{K}}$ is injective, we have, by Lemma 5.3.3, that the mapping $\tau_{S',\mathbf{K}'}$ is also injective. Again by Lemma 5.3.2, we have that $\mathbf{K}'$ is an algebraic semantics of S' with the set of defining equations $\tau(p)$. $\qquad\square$

In Example 5.2, we see that a logic can have several algebraic semantics. Now, we give an example of logic which does not have any algebraic semantics.

Example 5.4. [BR03, Theorem 2.19] Let $\mathcal{L} = \{\rightarrow\}$ be the language with just one binary connective and S the deductive system over $\mathcal{L}$ with the single axiom,

$$p \rightarrow p$$

and the single inference rule, Modus Ponens,

$$\frac{p, p \rightarrow q}{q}$$

The set $\Delta(p, q) = \{p \rightarrow q\}$ is a protoequivalence system for S. By Theorem 3.1.3, S is protoalgebraic. In order to prove that S does not have an algebraic semantics, we argue by contradiction and we need the following Lemmas.

Lemma 5.4.1. [BR03, Theorem 2.16] *Let S be a deductive system which has an algebraic semantics with the set of defining equations $\Delta(p) = \{\delta_i(p) \approx \epsilon_i(p) : i < n\}$. Then*

$$\{p, \gamma(\delta_i(p), \psi_0, \ldots, \psi_{k-1})\} \vdash_S \gamma(\epsilon_i(p), \psi_0, \ldots, \psi_{k-1}) \text{ for all } i < n$$

and

$$\{p, \gamma(\epsilon_i(p), \psi_0, \ldots, \psi_{k-1})\} \vdash_S \gamma(\delta_i(p), \psi_0, \ldots, \psi_{k-1}) \text{ for all } i < n$$

for every $\psi_0, \ldots, \psi_{k-1}, \gamma(p, q_0, \ldots, q_{k-1}) \in \mathrm{Fm}_{\mathcal{L}}$, where $k < \omega$.

Lemma 5.4.2. [BR03, Lemma 2.18] *Let p, q be distinct variables and $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$. Then $\{p, q \rightarrow \varphi\} \vdash_S \psi$ iff $\psi \in \{p, q \rightarrow \varphi\} \cup \{\gamma \rightarrow \gamma : \gamma \in \mathrm{Fm}_{\mathcal{L}}\}$.*

Suppose that S has an algebraic semantics, denoted by $\mathbf{K}$, with the set of defining equations $\Delta(p) = \{\delta_i(p) \approx \epsilon_i(p) : i < n\}$. Let $q \in \mathrm{Var}$ such that $q \neq p$. By Lemma 5.4.1, $\{p, q \rightarrow \delta_i(p)\} \vdash_S q \rightarrow \epsilon_i(p)$ for all $i < n$. By Lemma 5.4.2, $q \rightarrow \epsilon_i(p) \in \{p, q \rightarrow \delta_i(p)\} \cup \{\gamma \rightarrow \gamma : \gamma \in \mathrm{Fm}_{\mathcal{L}}\}$. Since $\Delta(p)$ is a set in only one variable p, $q \neq \epsilon_i(p)$ and since $q \neq p$, $q \rightarrow \epsilon_i(p) \neq p$. Thus, $q \rightarrow \epsilon_i(p) = q \rightarrow \delta_i(p)$. Hence $\delta_i(p)$ is equal to $\epsilon_i(p)$ for all $i < n$, i.e., $\models_{\mathbf{K}} \Delta(\varphi)$ for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}$. As $\mathbf{K}$ is an algebraic semantics of S, $\vdash_S \varphi$ for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}$. Therefore S is a trivial logic which is a contradiction. $\diamondsuit$

This example prove that not every logic has an algebraic semantics. There are many other examples of logics which have no algebraic semantics, for instance the deducibility relation of the formal system from relevance logic, denoted by $\mathbf{P} - \mathbf{W}$ (c.f. [Raf06b, Proposition 38]).

5.5 Equivalent Algebraic Semantics

Herein, we define the notion of equivalent algebraic semantics of a logic which is a useful tool for the study of the class of algebraizable logics.

Definition 5.5.1. *Let S be a logic and $\mathbf{K}$ a class of $\mathcal{L}$-algebras. We say that $\vdash_S$ is equivalent to $\models_{\mathbf{K}}$ iff*

1. *$\vdash_S$ is interpretable in $\models_{\mathbf{K}}$, i.e., there exists an interpretation $\tau : \mathrm{Fm}_{\mathcal{L}} \rightarrow \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ such that for all $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,*

   (i) $\Gamma \vdash_S \alpha$ iff $\tau[\Gamma] \models_{\mathbf{K}} \tau(\alpha)$

2. $\models_{\mathbf{K}}$ *is interpretable in* $\vdash_S$, *that is, there exists a mapping* $\rho : \mathrm{Eq}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$, *called* interpretation of $\models_{\mathbf{K}}$ *in* $\vdash_S$, *such that for all* $\Sigma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$,

   (ii) $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ *iff* $\rho[\Sigma] \vdash_S \rho(\varphi \approx \psi)$

3. *and, the interpretations* τ *and* ρ *are mutually inverse, that is,*

   (iii) $\varphi \approx \psi = \models\models_{\mathbf{K}} \tau[\rho(\varphi \approx \psi)]$

   (iv) $\alpha \dashv\vdash_S \rho[\tau(\alpha)]$

Let S be a logic, $\mathbf{K}$ a class of $\mathcal{L}$-algebras and $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ and $\rho : \mathrm{Eq}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$ arbitrary mappings. Conditions (i) and (iii) are equivalent to conditions (ii) and (iv). Indeed, assume conditions (i) and (iii) hold. Let $\Sigma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$. We have that $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ iff (by (iii)) $\tau[\rho[\Sigma]] \models_{\mathbf{K}} \tau[\rho(\varphi \approx \psi)]$ iff (by (i)) $\rho[\Sigma] \vdash_S \rho(\varphi \approx \psi)$. Thus (ii) holds. Now let $\alpha \in \mathrm{Fm}_{\mathcal{L}}$. Applying (iii) to $\tau(\alpha) \subseteq \mathrm{Eq}_{\mathcal{L}}$, we have $\tau(\alpha) =\models\models_{\mathbf{K}} \tau[\rho[\tau(\alpha)]]$. Since $\tau[\rho[\tau(\alpha)]] \models_{\mathbf{K}} \tau(\alpha)$, by (i), $\rho[\tau(\alpha)] \vdash_S \alpha$. And since $\tau(\alpha) \models_{\mathbf{K}} \tau[\rho[\tau(\alpha)]]$, by (i), $\alpha \vdash_S \rho[\tau(\alpha)]$. Thus (iv) holds. In an analogous way we can prove the converse.

Like for the mapping τ, we can represent the mapping ρ, with some assumptions, by a particular set of formulas.

Proposition 5.5.2. *Let* $\rho : \mathrm{Eq}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$ *be an arbitrary mapping. The following conditions are equivalent:*

   (i) ρ *commutes with surjective substitutions;*

   (ii) *There exists an k-parameterized system of formulas* $E(p, q, \underline{r})$ *such that for all* $\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}$, $\rho(\varphi \approx \psi) = E(\langle \varphi, \psi \rangle)$.

Proof. Assume ρ commutes with surjective substitutions. We fix two variables p, q and define $E(p, q, \underline{r}) := \rho(p \approx q)$. Let k be the length of $\underline{r}$. Clearly, $\rho(p \approx q) \subseteq \{\epsilon(p, q, \underline{\gamma}) : \epsilon(p, q, \underline{r}) \in E(p, q, \underline{r}), \underline{\gamma} \in (\mathrm{Fm}_{\mathcal{L}})^k\}$. For the reverse inclusion, let $\underline{\gamma} \in (\mathrm{Fm}_{\mathcal{L}})^k$ and e a surjective substitution such that $e(p) = p$, $e(q) = q$, $e(\underline{r}) = \underline{\gamma}$. We have that for all $\epsilon(p, q, \underline{r}) \in E(p, q, \underline{r})$, $\epsilon\big(e(p), e(q), e(\underline{r})\big) \in e[\rho(p \approx q)]$. By assumption, $e[\rho(p \approx q)] = \rho(e(p) \approx e(q)) = \rho(p \approx q)$. Thus, for all $\epsilon(p, q, \underline{r}) \in E(p, q, \underline{r})$, $\epsilon(p, q, \underline{\gamma}) \in \rho(p \approx q)$, i.e., $\{\epsilon(p, q, \underline{\gamma}) : \epsilon(p, q, \underline{r}) \in E(p, q, \underline{r}), \underline{\gamma} \in (\mathrm{Fm}_{\mathcal{L}})^k\} \subseteq \rho(p \approx q)$. Hence, $\rho(p \approx q) = \{\epsilon(p, q, \underline{\gamma}) : \epsilon(p, q, \underline{r}) \in E(p, q, \underline{r}), \underline{\gamma} \in (\mathrm{Fm}_{\mathcal{L}})^k\}$. Now, let $\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}$ and e a surjective substitution such that $e(p) = \varphi$ and $e(q) = \psi$. We have that $\rho(\varphi \approx \psi) = \rho(e(p) \approx e(q)) = e(\rho(p \approx q)) = e\big(\{\epsilon(p, q, \underline{\gamma}) : \epsilon(p, q, \underline{r}) \in E(p, q, \underline{r}), \underline{\gamma} \in (\mathrm{Fm}_{\mathcal{L}})^k\}\big) =$

$\{\epsilon\big(e(p), e(q), e(\underline{\gamma})\big) : \epsilon(p, q, \underline{r}) \in E(p, q, \underline{r}), \underline{\gamma} \in (\mathrm{Fm}_{\mathcal{L}})^k\} = \{\epsilon\big(\varphi, \psi, e(\underline{\gamma})\big) : \epsilon(p, q, \underline{r}) \in E(p, q, \underline{r}), \underline{\gamma} \in (\mathrm{Fm}_{\mathcal{L}})^k\}$. Since e is surjective, $\rho(\varphi \approx \psi) = \{\epsilon(\varphi, \psi, \underline{\xi}) : \epsilon(p, q, \underline{r}) \in E(p, q, \underline{r}), \underline{\xi} \in (\mathrm{Fm}_{\mathcal{L}})^k\} = E(\langle \varphi, \psi \rangle)$.

Conversely, let e be a surjective substitution and $\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}$. We have that $e[\rho(\varphi \approx \psi)] = e[E(\langle \varphi, \psi \rangle)] = E\big(\langle e(\varphi), e(\psi) \rangle\big)$. By surjectivity of e, $e[\rho(\varphi \approx \psi)] = \{\epsilon_i\big(e(\varphi), e(\psi), \underline{\xi}\big) : i \in I, \underline{\xi} \in (\mathrm{Fm}_{\mathcal{L}})^k\} = \rho(e(\varphi) \approx e(\psi))$. Thus, ρ commutes with surjective substitutions. $\square$

If the consequence relation of a logic is equivalent to an equational consequence relation of a class of algebras then, with some conditions on the interpretations, the logic is protoalgebraic.

Proposition 5.5.3. *Let S be a logic and $\mathbf{K}$ a class of $\mathcal{L}$-algebras. Suppose that $\models_{\mathbf{K}}$ is equivalent to $\vdash_S$ by means of interpretations τ and ρ which commute with surjective substitutions and are determined, respectively, by an l-parameterized system of equations $\Delta(p, \underline{v})$ and by an k-parameterized system of formulas $E(p, q, \underline{r})$. Then S is protoalgebraic and $E(p, q, \underline{r})$ is an k-parameterized system of equivalence formulas for S.*

Proof. We show that $E(p, q, \underline{r})$ satisfies reflexivity, modus ponens, and simple replacement conditions. Since $\models_{\mathbf{K}} p \approx p$, by condition (ii) of Definition 5.5.1, $\vdash_S \rho(p \approx p)$, i.e., $\vdash_S E(\langle p, p \rangle)$. Thus reflexivity condition holds.

Since $\Delta(p, \underline{v}) \cup \{p \approx q\} \models_{\mathbf{K}} \Delta(q, \underline{v})$ is satisfied by any equational consequence relation, by condition (ii) of Definition 5.5.1, $\rho[\Delta(p, \underline{v})] \cup \rho(p \approx q) \vdash_S \rho[\Delta(q, \underline{v})]$, i.e, $E\big(\langle \Delta(p, \underline{v}) \rangle\big) \cup E(\langle p, q \rangle) \vdash_S E\big(\langle \Delta(q, \underline{v}) \rangle\big)$. And by condition (iv) of Definition 5.5.1, $\{p\} \cup E(\langle p, q \rangle) \vdash_S q$. Thus modus ponens condition holds.

Let f be a connective of S of rank n. Since $\{p_1 \approx q_1\} \cup \cdots \cup \{p_n \approx q_n\} \models_{\mathbf{K}} f(p_1, \ldots, p_n) \approx f(q_1, \ldots, q_n)$, by condition (ii) of Definition 5.5.1, $\rho(p_1 \approx q_1) \cup \cdots \cup \rho(p_n \approx q_n) \vdash_S \rho\big(f(p_1, \ldots, p_n) \approx f(q_1, \ldots, q_n)\big)$, i.e., $E(\langle p_1, q_1 \rangle) \cup \cdots \cup E(\langle p_n, q_n \rangle) \vdash_S E(\langle f(p_1, \ldots, p_n), f(q_1, \ldots, q_n) \rangle)$. Thus simple replacement condition holds.

We conclude that $E(p, q, \underline{r})$ is an k-parameterized system of equivalence formulas for S. By Theorem 3.4.3, S is protoalgebraic. $\square$

Proposition 5.5.4. *Let $\rho : \mathrm{Eq}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$ be an arbitrary mapping. The following conditions are equivalent:*

(i) ρ commutes with arbitrary substitutions;

(ii) There exists a set $E(p,q)$ of formulas in two variables p, q such that for all $\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}$, $\rho(\varphi \approx \psi) = E(\varphi, \psi)$.

Proof. Assume ρ commutes with arbitrary substitutions. We fix p, q distinct variables and define $E(p,q) := \rho(p \approx q)$. Suppose $\mathrm{Var}(E(p,q)) \subseteq \{p, q, r_1, r_2, \dots\}$. Let e be a substitution such that $e(p) = p$, $e(q) = q$ and $e(r_i) = p$ for all $i \in I$. By assumption, $E(p, q, p, \dots) = e(E(p,q)) = e(\rho(p \approx q)) = \rho(e(p) \approx e(q)) = \rho(p \approx q) = E(p, q, r_1, r_2, \dots)$. Hence, $\{r_1, r_2, \dots\} \subseteq \{p, q\}$. Thus $\mathrm{Var}(E(p,q)) \subseteq \{p, q\}$. Now, let $\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}$ and e a substitution such that $e(p) = \varphi$ and $e(q) = \psi$. We have that $\rho(\varphi \approx \psi) = \rho(e(p) \approx e(q)) = e(\rho(p \approx q)) = e(E(p,q)) = E(e(p), e(q)) = E(\varphi, \psi)$.

The converse is obvious. $\square$

If the consequence relation of a logic is equivalent to an equational consequence relation of a class of algebras then, with some conditions on the interpretations, the logic is equivalential.

Proposition 5.5.5. *Let S be a logic and $\mathbf{K}$ a class of $\mathcal{L}$-algebras. Suppose that $\models_{\mathbf{K}}$ is equivalent to $\vdash_S$ by means of interpretations τ and ρ which commute with arbitrary substitutions and are determined, respectively, by set of equations $\Delta(p)$ and a set of formulas $E(p,q)$. Then S is equivalential and $E(p,q)$ is a system of equivalence formulas for S.*

Proof. We show that $E(p,q)$ satisfies reflexivity, modus ponens, and simple replacement conditions. Since $\models_{\mathbf{K}} p \approx p$, by condition *(ii)* of Definition 5.5.1, $\vdash_S \rho(p \approx p)$, i.e., $\vdash_S E(p,p)$. Thus reflexivity condition holds.

Since $\Delta(p) \cup \{p \approx q\} \models_{\mathbf{K}} \Delta(q)$ is satisfied by any equational consequence relation, by condition *(ii)* of Definition 5.5.1, $\rho[\Delta(p)] \cup \rho(p \approx q) \vdash_S \rho[\Delta(q)]$, i.e, $E(\Delta(p)) \cup E(p,q) \vdash_S E(\Delta(q))$. And by condition *(iv)* of Definition 5.5.1, $\{p\} \cup E(p,q) \vdash_S q$. Thus modus ponens condition holds.

Let f be a connective of S of rank n. Since $\{p_1 \approx q_1\} \cup \dots \cup \{p_n \approx q_n\} \models_{\mathbf{K}} f(p_1, \dots, p_n) \approx f(q_1, \dots, q_n)$, by condition *(ii)* of Definition 5.5.1, $\rho(p_1 \approx q_1) \cup \dots \cup \rho(p_n \approx q_n) \vdash_S \rho(f(p_1, \dots, p_n) \approx f(q_1, \dots, q_n))$, i.e., $E(p_1, q_1) \cup \dots \cup E(p_n, q_n) \vdash_S E(f(p_1, \dots, p_n), f(q_1, \dots, q_n))$. Thus simple replacement condition holds.

We conclude that $E(p,q)$ is a system of equivalence formulas for S. By Theorem 4.1.1, S is equivalential. $\square$

Now, we define the notion of equivalent algebraic semantics.

Definition 5.5.6. *Let S be a logic and* **K** *a class of $\mathcal{L}$-algebras. We say that* **K** *is an equivalent algebraic semantics for S if $\vdash_S$ is equivalent to $\models_{\mathbf{K}}$ in the following sense: there exist $\Delta(p)$ a set of equation in a single variable and $E(p,q)$ a set of formulas in two variables such that, for every $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\Sigma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$,*

> *(i) $\Gamma \vdash_S \alpha$ iff $\Delta(\Gamma) \models_{\mathbf{K}} \Delta(\alpha)$*

> *(ii) $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ iff $E(\Sigma) \vdash_S E(\varphi, \psi)$*

> *(iii) $\varphi \approx \psi =\!\!\models_{\mathbf{K}} \Delta(E(\varphi, \psi))$*

> *(iv) $\alpha \dashv\vdash_S E(\Delta(\alpha))$*

The set $E(p,q)$ is called a set of equivalence formulas *and $\Delta(p)$ a set of* defining equations *for S and* **K***.*

In other words, **K** is an equivalent algebraic semantics for S iff there exists a pair of interpretations $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ and $\rho : \mathrm{Eq}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$ that commute with arbitrary substitutions and are mutually inverse. It is not difficult to see that conditions (i) and (iii) are equivalent to conditions (ii) and (iv).

We have seen, in Example 5.2, that the class **BA** is an algebraic semantics of **CPL**. Now, we show that it is an equivalent algebraic semantics.

Example 5.6 (Classical Propositional Logic [BP01])**.** The interpretation τ of $\vdash_{\mathbf{CPL}}$ in $\models_{\mathbf{BA}}$ commutes with arbitrary substitutions and is defined in the following way: for all $\alpha \in \mathrm{Fm}_{\mathcal{L}}$, $\tau(\alpha) = \{\alpha \approx \top\}$. There exists an interpretation ρ of $\models_{\mathbf{BA}}$ in $\vdash_{\mathbf{CPL}}$ that commutes with arbitrary substitutions and is defined in the following way: for all $\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}$, $\rho(\varphi \approx \psi) = \{\varphi \to \psi, \psi \to \varphi\}$. We can prove that conditions (i) and (iii) hold. Therefore, the class of **BA** forms an equivalent algebraic semantics for **CPL**, $\Delta(p) = \{p \approx \top\}$ is the set of defining equations and $E(p,q) = \{p \to q, q \to p\}$ the set of equivalence formulas for **CPL** and **BA**. $\diamond$

5.7 Weakly Algebraizable Logics

In this section, we define the class of weakly algebraizable logics which is a proper subclass of the class of protoalgebraic logics. We also characterized weakly algebraizable logics using the Leibniz operator. We point out to [CJ00] for more details about these logics.

Definition 5.7.1. *A logic S is called* weakly algebraizable *if there exist a class* **K** *of $\mathcal{L}$-algebras, an l-parameterized system of equations $\Delta(p, \underline{v})$ and an k-parameterized system of formulas $E(p, q, \underline{r})$ such that the following conditions hold, for every $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\Sigma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$:*

> *(i) $\Gamma \vdash_S \alpha$ iff $\Delta(\langle \Gamma \rangle) \models_{\mathbf{K}} \Delta(\langle \alpha \rangle)$;*

> *(ii) $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ iff $E(\langle \Sigma \rangle) \vdash_S E(\langle \varphi, \psi \rangle)$;*

> *(iii) $\varphi \approx \psi =\!\!\models\!\models_{\mathbf{K}} \Delta(\langle E(\langle \varphi, \psi \rangle) \rangle)$;*

> *(iv) $\alpha \dashv\vdash_S E(\langle \Delta(\langle \alpha \rangle) \rangle)$.*

In other words, a logic S is weakly algebraizable iff there exists an equational consequence relation $\models_{\mathbf{K}}$ on $\mathrm{Eq}_{\mathcal{L}}$ equivalent to $\vdash_S$, and the equivalence between $\vdash_S$ and $\models_{\mathbf{K}}$ is established by means of interpretations τ and ρ that commute with surjective substitutions and are mutually inverse.

In order to prove the theorem that gives a characterization of weakly algebraizable logics using the Leibniz operator, we need the following lemma.

Lemma 5.7.2. [Cze01, Lemma 1.6.2] *Let S be a protoalgebraic logic and $E(p, q, \underline{r})$ an k-parameterized system of equivalence formulas for S. Then the following conditions are equivalent:*

> *(i) Ω is injective on $\mathrm{Th}(S)$;*

> *(ii) $p \dashv\vdash_S \bigcup \{ E(\langle \varphi, \psi \rangle) : \varphi \equiv \psi(\Omega(\mathrm{Cn}_S(p))) \}$;*

> *(iii) There exists a set $\Delta(p)$ of equations in a single variable p such that $p \dashv\vdash_S \bigcup \{ E(\langle \varphi, \psi \rangle) : \varphi \approx \psi \in \Delta(p) \}$.*

Theorem 5.7.3. *Let S be a logic. The following conditions are equivalent:*

> *(i) S is weakly algebraizable;*

> *(ii) The Leibniz operator Ω is monotone and injective on $\mathbf{Th}(S)$;*

> *(iii) The Leibniz operator $\Omega_{\mathbf{A}}$ is monotone and injective on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$.*

Proof. $(i) \Rightarrow (ii)$ Assume S is weakly algebraizable. There exists a class **K** of $\mathcal{L}$-algebras, an l-parameterized system of equations $\Delta(p, \underline{v})$ and an k-parameterized system of formulas $E(p, q, \underline{r})$ such that they satisfy conditions in Definition 5.7.1. By Proposition 5.5.3, S is protoalgebraic and $E(p, q, \underline{r})$ is an k-parameterized system of

equivalence formulas for S. By Theorem 3.3.1, the Leibniz operator Ω is monotone on $\mathbf{Th}(S)$. Now, let $T_1, T_2 \in \mathrm{Th}(S)$. Suppose $\Omega T_1 = \Omega T_2$. Let $\alpha \in T_1$, i.e, $T_1 \vdash_S \alpha$. By condition (iv) of Definition 5.7.1, $\alpha \dashv\vdash_S E(\langle \Delta(\langle\alpha\rangle)\rangle)$. Hence $T_1 \vdash_S E(\langle\Delta(\langle\alpha\rangle)\rangle)$, i.e., $E(\langle\Delta(\langle\alpha\rangle)\rangle) \subseteq \mathrm{Cn}_S(E(\langle\Delta(\langle\alpha\rangle)\rangle)) \subseteq T_1$. Thus, for all $\varphi \approx \psi \in \Delta(\langle\alpha\rangle)$, $\varphi \equiv \psi(\Omega T_1)$. Since $\Omega(T_1) = \Omega(T_2)$; for all $\varphi \approx \psi \in \Delta(\langle\alpha\rangle)$, $\varphi \equiv \psi(\Omega T_2)$, i.e., $E(\langle\Delta(\langle\alpha\rangle)\rangle) \subseteq T_2$. So, $T_2 \vdash_S E(\langle\Delta(\langle\alpha\rangle)\rangle)$. By condition (iv) of Definition 5.7.1, $T_2 \vdash_S \alpha$, i.e., $\alpha \in T_2$. In an analogous way, we can prove that $T_2 \subseteq T_1$. Therefore $T_1 = T_2$.

$(ii) \Rightarrow (iii)$ Since the Leibniz operator is monotone on $\mathbf{Th}(S)$, by Theorem 3.3.1, it is also monotone on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. Now let $\mathbf{A}$ be an algebra and $F_1, F_2 \in \mathrm{Fi}_S(\mathbf{A})$. Suppose that $\Omega_{\mathbf{A}} F_1 = \Omega_{\mathbf{A}} F_2$. Let $a \in F_1$. Then $\mathrm{Fi}_S^{\mathbf{A}}(a) \subseteq F_1$. Since the Leibniz operator is monotone on $\mathbf{Th}(S)$, by Theorem 3.3.1, S is protoalgebraic. By Theorem 3.4.3, S has an k-parameterized system of equivalence formulas $E(p, q, \underline{r})$. As Ω is injective on $\mathbf{Th}(S)$, by Lemma 5.7.2, there exists a set $\Delta(p)$ of equations in a single variable p such that $p \dashv\vdash_S \bigcup\{E(\langle\varphi, \psi\rangle) : \varphi \approx \psi \in \Delta(p)\}$. We fixe $\varphi(p) \approx \psi(p) \in \Delta(p)$. We have that $E^{\mathbf{A}}(\langle\varphi^{\mathbf{A}}(a), \psi^{\mathbf{A}}(a)\rangle) \subseteq \mathrm{Fi}_S^{\mathbf{A}}(a)$. Hence, $\mathrm{Fi}_S^{\mathbf{A}}(\bigcup\{E^{\mathbf{A}}(\langle\varphi^{\mathbf{A}}(a), \psi^{\mathbf{A}}(a)\rangle) : \varphi \approx \psi \in \Delta(p)\}) \subseteq \mathrm{Fi}_S^{\mathbf{A}}(a)$. We also have that for every homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ such that $h(p) = a$, $a \in \mathrm{Fi}_S^{\mathbf{A}}(\bigcup\{E^{\mathbf{A}}(\langle h(\varphi), h(\psi)\rangle) : \varphi \approx \psi \in \Delta(p)\})$, i.e., $a \in \mathrm{Fi}_S^{\mathbf{A}}(\bigcup\{E^{\mathbf{A}}(\langle\varphi^{\mathbf{A}}(a), \psi^{\mathbf{A}}(a)\rangle) : \varphi \approx \psi \in \Delta(p)\})$. Thus $\mathrm{Fi}_S^{\mathbf{A}}(a) \subseteq \mathrm{Fi}_S^{\mathbf{A}}(\bigcup\{E^{\mathbf{A}}(\langle\varphi^{\mathbf{A}}(a), \psi^{\mathbf{A}}(a)\rangle) : \varphi \approx \psi \in \Delta(p)\})$. Therefore $\mathrm{Fi}_S^{\mathbf{A}}(a) = \mathrm{Fi}_S^{\mathbf{A}}(\bigcup\{E^{\mathbf{A}}(\langle\varphi^{\mathbf{A}}(a), \psi^{\mathbf{A}}(a)\rangle) : \varphi \approx \psi \in \Delta(p)\})$.

$a \in F_1$ iff $\mathrm{Fi}_S^{\mathbf{A}}(a) \subseteq F_1$ iff $E^{\mathbf{A}}(\langle\varphi^{\mathbf{A}}(a), \psi^{\mathbf{A}}(a)\rangle) \subseteq F_1$ for all $\varphi \approx \psi \in \Delta(p)$

iff $\varphi^{\mathbf{A}}(a) \equiv \psi^{\mathbf{A}}(a)(\Omega_{\mathbf{A}} F_1)$ for all $\varphi \approx \psi \in \Delta(p)$

iff $\varphi^{\mathbf{A}}(a) \equiv \psi^{\mathbf{A}}(a)(\Omega_{\mathbf{A}} F_2)$ for all $\varphi \approx \psi \in \Delta(p)$

iff $E^{\mathbf{A}}(\langle\varphi^{\mathbf{A}}(a), \psi^{\mathbf{A}}(a)\rangle) \subseteq F_2$ for all $\varphi \approx \psi \in \Delta(p)$

iff $\mathrm{Fi}_S^{\mathbf{A}}(\bigcup\{E^{\mathbf{A}}(\langle\varphi^{\mathbf{A}}(a), \psi^{\mathbf{A}}(a)\rangle) : \varphi \approx \psi \in \Delta(p)\}) \subseteq F_2$ iff $\mathrm{Fi}_S^{\mathbf{A}}(a) \subseteq F_2$ iff $a \in F_2$.

Hence $F_1 = F_2$.

$(iii) \Rightarrow (i)$ Assume that the Leibniz operator is monotone and injective on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. By Theorem 3.3.1, S is protoalgebraic. And by Theorem 3.4.3, there exists $E(p, q, \underline{r})$ an k-parameterized system of equivalence formulas for S. On the other hand, since the Leibniz operator is injective on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$, it is also injective on $\mathbf{Th}(S)$. By Lemma 5.7.2 and structurality of S, there exists a set $\Delta(p)$ of equations such that for all $\alpha \in \mathrm{Fm}_{\mathcal{L}}$, $\alpha \dashv\vdash_S \bigcup\{E(\langle\varphi, \psi\rangle) : \varphi \approx \psi \in \Delta(\alpha)\}$, i.e., $\alpha \dashv\vdash_S E(\langle\Delta(\alpha)\rangle)$. Let $\mathbf{K}$ be the class of $\mathcal{L}$-algebras. We define the relation $\models_{\mathbf{K}}$

on $\mathcal{P}(\mathrm{Eq}_{\mathcal{L}})$ as follows: for all $\Sigma \subseteq \mathrm{Eq}_{\mathcal{L}}$, $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ iff $E(\langle \Sigma \rangle) \vdash_S E(\langle \varphi, \psi \rangle)$. It is not difficult to see that $\models_{\mathbf{K}}$ is an equational consequence relation on $\mathrm{Eq}_{\mathcal{L}}$. Therefore S is weakly algebraizable. $\qquad\square$

A class $\mathcal{M}$ of matrices has its filters *equationally definable* by a set of equations $\Delta(p) = \{\delta_i(p) \approx \epsilon_i(p) : i \in I\}$ in a single variable p, if for any matrix $\langle \mathbf{A}, F \rangle \in \mathcal{M}$ and any $a \in \mathrm{A}$; $a \in F$ iff $\delta_i^{\mathbf{A}}(a) = \epsilon_i^{\mathbf{A}}(a)$. We say that a class of matrices $\mathcal{M}$ has its filters *implicitly definable* if; for any algebra $\mathbf{A}$ if $\langle \mathbf{A}, F \rangle, \langle \mathbf{A}, G \rangle \in \mathcal{M}$ then $F = G$, i.e., the matrices in $\mathcal{M}$ are uniquely determined by their algebraic reducts. Obviously, if the filters are equationally definable in $\mathcal{M}$ by a set of equations $\Delta(p)$ then they are also implicitly definable.

Theorem 5.7.4. *Let S be a logic. The following conditions are equivalent:*

    (i) S is weakly algebraizable;

    (ii) the Leibniz operator is monotone on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$ and the class $\mathbf{Mod}^*(S)$ has its filters equationally definable;

    (iii) the Leibniz operator is monotone on $\mathbf{Th}(S)$ and the class $\mathbf{L}^*(S)$ has its filters equationally definable;

    (iv) the Leibniz operator is monotone on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$ and the class $\mathbf{Mod}^*(S)$ has its filters implicitly definable;

    (v) the Leibniz operator is monotone on $\mathbf{Th}(S)$ and the class $\mathbf{L}^*(S)$ has its filters implicitly definable.

Proof. $(i) \Rightarrow (ii)$ Assume S is weakly algebraizable. By Theorem 5.7.3, the Leibniz operator is monotone on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. And by Theorem 3.3.1, S is protoalgebraic. Thus by Theorem 3.4.3, there exists $E(p, q, \underline{r})$ an k-parameterized system of equivalence formulas for S. Since S is weakly algebraizable, by Theorem 5.7.3, the Leibniz operator is injective on $\mathbf{Th}(S)$. And so, by Lemma 5.7.2, there exists a set of equations $\Delta(p)$ in a single variable such that $p \dashv\vdash_S \bigcup\{E(\langle \varphi, \psi \rangle) : \varphi \approx \psi \in \Delta(p)\}$, i.e., $p \dashv\vdash_S E(\langle \Delta(p) \rangle)$. Let $\langle \mathbf{A}, F \rangle \in \mathbf{Mod}^*(S)$ and $a \in F$. Since, $p \dashv\vdash_S E(\langle \Delta(p) \rangle)$, we have that $E^{\mathbf{A}}(\langle \Delta(a) \rangle) \subseteq F$. Thus $\bigcup\{E^{\mathbf{A}}(\langle \varphi^{\mathbf{A}}(a), \psi^{\mathbf{A}}(a) \rangle) : \varphi \approx \psi \in \Delta(p)\}) \subseteq F$, i.e., $\varphi^{\mathbf{A}}(a) \equiv \psi^{\mathbf{A}}(a)(\Omega_{\mathbf{A}} F)$ for all $\varphi \approx \psi \in \Delta(p)$. Since $\langle \mathbf{A}, F \rangle \in \mathbf{Mod}^*(S)$, $\Omega_{\mathbf{A}} F = \triangle_{\mathbf{A}}$. Hence for all $\varphi \approx \psi \in \Delta(p)$, $\varphi^{\mathbf{A}}(a) = \psi^{\mathbf{A}}(a)$.

    $(ii) \Rightarrow (iii)$ It is obvious, since $\mathbf{L}^*(S) \subseteq \mathbf{Mod}^*(S)$.

$(ii) \Rightarrow (iv)$ It is obvious, since the property equationally definable implies the property implicitly definable.

$(iii) \Rightarrow (v)$ It is obvious.

$(iv) \Rightarrow (v)$ It is obvious.

$(v) \Rightarrow (i)$ Let $T_1, T_2 \in \text{Th}(S)$. Suppose that $\Omega T_1 = \Omega T_2$. The respective reduction matrices $\langle \mathbf{Fm}_{\mathcal{L}}/\Omega T_1, T_1/\Omega T_1 \rangle$ and $\langle \mathbf{Fm}_{\mathcal{L}}/\Omega T_2, T_2/\Omega T_2 \rangle$ are in $\mathbf{L}^*(S)$. Since $\Omega T_1 = \Omega T_2$, we have that $\mathbf{Fm}_{\mathcal{L}}/\Omega T_1 = \mathbf{Fm}_{\mathcal{L}}/\Omega T_2$. By assumption, $T_1/\Omega T_1 = T_2/\Omega T_2$, i.e., $T_1/\Omega T_1 = T_2/\Omega T_1$. Thus, by compatibility, $T_1 = T_2$. Hence the Leibniz operator is injective on $\mathbf{Th}(S)$. Since the Leibniz operator is monotone on $\mathbf{Th}(S)$, by Theorem 5.7.3, S is weakly algebraizable. □

In the Examples 3.2 and 4.2, we have seen that the minimal orthologic S_{OL} is protoalgebraic but not equivalential. Now, we prove that S_{OL} is weakly algebraizable using Theorem 5.7.4 .

Example 5.8 (Orthologic [Mal89])**.** Since S_{OL} is protoalgebraic, by Theorem 3.3.1, the Leibniz operator is monotone on $\mathbf{Th}(S_{OL})$. Consider a set of equation $\Delta(p) = \{p \approx 1\}$, where 1 denotes an arbitrary but fixed theorem of S_{OL} (e.g., $1 := p \vee \neg p$). If we prove that for any $T \in \text{Th}(S_{OL})$ and any $\alpha \in \text{Fm}_{\mathcal{L}}$, $\alpha \in T$ iff $\alpha \equiv 1(\Omega T)$; then we have shown that the class $\mathbf{L}^*(S)$ has its filters equationally definable by the set $\Delta(p)$.

Let $T \in \text{Th}(S)$ and $\alpha \in \text{Fm}_{\mathcal{L}}$. Suppose that $\alpha \in T$. Let $\varphi \in \text{Fm}_{\mathcal{L}}$. Since $1 \in \text{Thm}(S)$, $1 \in T$. As $\alpha, \varphi(\alpha) \vdash_{S_{OL}} \varphi(1)$ (because it is an inference rule of S_{OL}), we have that $\varphi(\alpha) \in T$ implies $\varphi(1) \in T$. Since $\alpha, \varphi(1) \vdash_{S_{OL}} \varphi(\alpha)$ (because it is also an inference rules of S_{OL}), we have that $\varphi(1) \in T$ implies $\varphi(\alpha) \in T$. Therefore for all $\varphi \in \text{Fm}_{\mathcal{L}}$, $\varphi(\alpha) \in T$ iff $\varphi(1) \in T$. Thus $\alpha \equiv 1(\Omega T)$. Conversely, suppose that $\alpha \equiv 1(\Omega T)$. Since $1 \in \text{Thm}(S)$, $1 \in T$. By compatibility, $\alpha \in T$.

Since the class $\mathbf{L}^*(S)$ has its filters equationally definable by the set $\Delta(p)$ and the Leibniz operator is monotone on $\mathbf{Th}(S_{OL})$, by Theorem 5.7.4, the minimal orthologic S_{OL} is weakly algebraizable. ◇

5.9 Algebraizable Logics

In [BP89], Blok and Pigozzi have defined the notion of "algebraizable logic" for finitary logic. They are logics that have an equivalent algebraic semantics. The idea underlying the definition is the following: a logic is algebraizable if there exists a class of algebras

that can be associated to the logic in the same way as the class of **BA** has been associated to **CPL**.

Definition 5.9.1. *A logic S is called* algebraizable *(also called* possibly infinitely algebraizable *in* [Her96]*) if there exist a class* **K** *of $\mathcal{L}$-algebras, a set of equations $\Delta(p)$ and a set of formulas $E(p, q)$ such that the following conditions hold, for every $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\Sigma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$:*

(i) $\Gamma \vdash_S \alpha$ iff $\Delta(\Gamma) \models_{\mathbf{K}} \Delta(\alpha)$;

(ii) $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ iff $E(\Sigma) \vdash_S E(\varphi, \psi)$;

(iii) $\varphi \approx \psi =\!\!\models\!\!\models_{\mathbf{K}} \Delta(E(\varphi, \psi))$;

(iv) $\alpha \dashv\vdash_S E(\Delta(\alpha))$.

The class **K** *is called an* equivalent algebraic semantics; *and $E(p, q)$ the set of* equivalence formulas *and $\Delta(p)$ the set of* defining equations *for S and* **K**.

In other words, a logic S is algebraizable iff there exists an equational consequence relation $\models_{\mathbf{K}}$ on $\mathrm{Eq}_{\mathcal{L}}$ equivalent to $\vdash_S$, and the equivalence between $\vdash_S$ and $\models_{\mathbf{K}}$ is established by means of interpretations τ and ρ that commute with arbitrary substitutions and are mutually inverse. That is, if there exists a class **K** of $\mathcal{L}$-algebras that is an equivalent algebraic semantics for S with $\tau(p)$ the set of defining equations and $\rho(p \approx q)$ the set of equivalence formulas for S and **K**.

We note that the difference between weakly algebraizable logic and algebraizable logic is that in the latter, the set of equations and the set of formulas do not have parameters. Thus, if a logic S is algebraizable, then it is weakly algebraizable.

Theorem 5.9.2. *Let S be a logic. The following conditions are equivalent:*

(i) S is algebraizable;

(ii) The Leibniz operator Ω is monotone, injective and commutes with inverse substitutions on **Th**(S).

Proof. Assume S is algebraizable. Hence S is weakly algebraizable and by Theorem 5.7.3, the Leibniz operator Ω is monotone and injective on **Th**(S). Let **K** be a class of $\mathcal{L}$-algebras, $\Delta(p)$ a set of equations and $E(p, q)$ a set of formulas such that conditions of Definition 5.7.1 hold. By Proposition 5.5.5, S is equivalential and $E(p, q)$ is a system of equivalence formulas for S. And by Theorem 4.1.5, the Leibniz operator Ω commutes with inverse substitutions on **Th**(S).

Conversely, suppose Ω is monotone, injective and commutes with inverse substitutions on $\mathbf{Th}(S)$. By Theorem 4.1.5, S is equivalential. There exists a system of equivalence formulas $E(p,q)$ for S. Consider the class of algebras $\mathbf{K} = \{\mathbf{Fm}_{\mathcal{L}}/\Omega T : T \in \mathrm{Th}(S)\}$. It is not difficult to see that for every $\Sigma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$, $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ iff $E(\Sigma) \vdash_S E(\varphi, \psi)$. Let $T = \mathrm{Cn}_S(p)$ and e a substitution such that $e(r) = p$ for every $r \in \mathrm{Var}$. Consider $\Delta(p) = e[\Omega T]$. As we can see pair of formulas as equations, $\Delta(p)$ is a set of equations. Let $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$. We have that $\varphi \equiv \psi\big(\Omega(\mathrm{Cn}_S(E(\Omega T)))\big)$ iff $E(\varphi, \psi) \subseteq \mathrm{Cn}_S(E(\Omega T))$ iff $E(\Omega T) \vdash_S E(\varphi, \psi)$ iff $\Omega T \models_{\mathbf{K}} \varphi \approx \psi$ iff $\varphi \equiv \psi(\Omega T)$. Thus $\Omega(\mathrm{Cn}_S(E(\Omega T))) = \Omega T$. Since the Leibniz operator is injective, $\mathrm{Cn}_S(E(\Omega T)) = T$. As $T = \mathrm{Cn}_S(p)$, $p \dashv\vdash_S T$. Thus $p \dashv\vdash_S E(\Omega T)$. By structurality of S, $e(p) \dashv\vdash_S e[E(\Omega T)]$. Since $e[E(\Omega T)] = E(e[\Omega T]) = E(\Delta(p))$ and $e(p) = p$, we have that $p \dashv\vdash_S E(\Delta(p))$. Thus S is algebraizable. $\qquad\square$

By the above characterization of algebraizable logics, it is not difficult to prove the following Corollary.

Corollary 5.9.3. *The class of algebraizable logics is the intersection of the class of equivalential logics and the class of weakly algebraizable logics.*

If a deductive system is algebraizable then any fragment, whose language includes the connectives occurring in the set of defining equations and the set of equivalence formulas; and any extension are algebraizable with the same set of defining equations and set of equivalence formulas.

Corollary 5.9.4. *Let S be a deductive system. If S is algebraizable, then any extension of S is itself algebraizable with the same set of equivalence formulas and set of defining equations.*

Thus, any axiomatic extension of an algebraizable deductive system is itself algebraizable.

Corollary 5.9.5. *Let S be an algebraizable deductive system. Then any $\mathcal{L}'$-fragment of S, where $\mathcal{L}'$ contains all primitive connectives that occur in the set of equivalence formulas and set of defining equations, is algebraizable with the same set of equivalence formulas and set of defining equation.*

A logic may have (if any) many algebraic semantics, but the equivalent algebraic semantics associated with an algebraizable logic is unique in a following sense.

Theorem 5.9.6. *Let S be an algebraizable logic, $\mathbf{K}$ and $\mathbf{K}'$ two equivalent algebraic semantics for S such that $E(p, q)$ is the set of equivalence formulas and $\Delta(p)$ the set of defining equations for S and $\mathbf{K}$, and similarly $E'(p', q')$ and $\Delta'(p')$ for S and $\mathbf{K}'$. Then $\models_{\mathbf{K}} = \models_{\mathbf{K}'}$, $E(p, q) \dashv\vdash_S E'(p', q')$ and $\Delta(p) =\!\!\models\!\!\models_{\mathbf{K}} \Delta'(p')$.*

Therefore, there is (if any) one equivalent algebraic semantics that can be canonically associated with the logic. We write *the equivalent algebraic semantics* when we want to refer to the largest equivalent algebraic semantics.

The duality of this result fails to hold. There are distinct logics with the same equivalent algebraic semantics. In [BP89, Theorem 5.12], Blok and Pigozzi have given an example of two distinct finitely algebraizable deductive systems with the same equivalent quasivariety semantics.

Theorem 5.9.7. *Let S be a logic. A sufficient condition for S to be algebraizable is that is equivalential with a set of equivalence formulas $E(p, q)$ that satisfies:*

$$p, q \vdash_S E(p, q) \tag{G-rule}$$

In this case $E(p, q)$ and $\Delta(p) = \{p \approx E(p, p)\}$ are, respectively, the set of equivalence formulas and the set of defining equations. If the sufficient condition of the above theorem is satisfied the logic is said to be *regularly algebraizable* (or 1-*equivalential* in [Her96]). That is, a logic is regularly algebraizable if there exists a set $E(p, q)$ of formulas such that conditions of reflexivity, symmetry, transitivity, simple replacement, modus ponens and G-rule hold (c.f. [CP04b, Definition 2.3]). Moreover, if the set of equivalence formulas is finite then we say that the logic is *finitely regularly algebraizable*. Thus, every (finitely) regularly algebraizable logic is (finitely) algebraizable. We point out to [Cze01, Chapter 5] and [CP04a] for a detailed study of this class of logics.

5.10   Finitely Algebraizable Logics

In this section, we study the class of finitely algebraizable logics that is a proper subclass of algebraizable logics.

Definition 5.10.1. *A logic S is said to be* finitely algebraizable *if there exist a class $\mathbf{K}$ of $\mathcal{L}$-algebras, a finite set of equations $\Delta(p)$ and a finite set of formulas $E(p, q)$ such that the following conditions hold, for every $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\Sigma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$:*

(i) $\Gamma \vdash_S \alpha$ iff $\Delta(\Gamma) \models_{\mathbf{K}} \Delta(\alpha)$;

  (ii) $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ iff $E(\Sigma) \vdash_S E(\varphi, \psi)$;

  (iii) $\varphi \approx \psi =\!\models_{\mathbf{K}} \Delta(E(\varphi, \psi))$;

  (iv) $\alpha \dashv\vdash_S E(\Delta(\alpha))$.

The class $\mathbf{K}$ *is called an* equivalent algebraic semantics*; and* $E(p, q)$ *the set of* equivalence formulas *and* $\Delta(p)$ *the set of* defining equations *for S and* $\mathbf{K}$.

In other words, a finitely algebraizable logic is an algebraizable logic where the set of defining equations and the set of equivalence formulas are finite.

In the following theorem, we give a characterization of this class of logics using the Leibniz operator.

Theorem 5.10.2. *Let S be a logic. The following conditions are equivalent:*

  (i) S is finitely algebraizable;

  (ii) The Leibniz operator $\Omega_{\mathbf{A}}$ is injective and continuous on $\mathbf{Fi}_S(\mathbf{A})$ for every
   algebra $\mathbf{A}$.

  (iii) The Leibniz operator Ω is injective and continuous on $\mathbf{Th}(S)$.

Proof. $(i) \Rightarrow (ii)$ Assume S is finitely algebraizable. There exist a class $\mathbf{K}$ of $\mathcal{L}$-algebras, a finite set of equations $\Delta(p)$ and a finite set of formulas $E(p, q)$ that satisfy conditions of Definition 5.10.1. Thus S is weakly algebraizable. By Theorem 5.7.3, the Leibniz operator is injective on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. And by Theorem 5.5.5, S is equivalential and $E(p, q)$ is a system of equivalence formulas for S. Since $E(p, q)$ is finite, S is finitely equivalential. By Theorem 4.1.6, the Leibniz operator Ω is continuous on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$.

  $(ii) \Rightarrow (iii)$ It is obvious.

  $(iii) \Rightarrow (i)$ Since the Leibniz operator is continuous on $\mathbf{Th}(S)$, by Theorem 4.1.6, S is finitely equivalential. Thus S is equivalential and by Theorem 4.1.5, the Leibniz operator is monotone and commutes with inverse substitutions on $\mathbf{Th}(S)$. By assumption, the Leibniz operator is also injective on $\mathbf{Th}(S)$. Thus by Theorem 5.9.2, S is algebraizable, i.e, there exists a class $\mathbf{K}$ of $\mathcal{L}$-algebras which is an equivalent algebraic semantics for S with $\Delta(p)$ the set of defining equations and $E(p, q)$ the set of equivalence formulas. By Theorem 5.5.5, S is equivalential with $E(p, q)$ the system of equivalence formulas for S. Since S is finitely equivalential, the set $E(p, q)$ must be finite. Thus the set $\Delta(p)$ must be also finite. Therefore S is finitely algebraizable. $\square$

We prove that if a finitely algebraizable logic S is finitary then the equational consequence relation which is equivalent to $\vdash_S$ is also finitary.

Proposition 5.10.3. *Let S be a finitely algebraizable logic with the correspondent equivalent algebraic semantics* $\mathbf{K}$. *If S is finitary, then* $\models_{\mathbf{K}}$ *is also finitary.*

Proof. Assume S is a finitary finitely algebraizable logic. Let $\mathbf{K}$ be the equivalent algebraic semantics for S; and $\Delta(p)$ the finite set of defining equations and $E(p,q)$ the finite set of equivalence formulas for S and $\mathbf{K}$. Let $\Sigma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$. By condition (ii) of Definition 5.5.1, $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ iff $E(\Sigma) \vdash_S E(\varphi, \psi)$. Since S is finitary and $E(p,q)$ is finite, there exists a finite $\Sigma_f \subseteq \Sigma$ such that $E(\Sigma) \vdash_S E(\varphi, \psi)$ iff $E(\Sigma_f) \vdash_S E(\varphi, \psi)$. And by condition (ii) of Definition 5.5.1, $E(\Sigma_f) \vdash_S E(\varphi, \psi)$ iff $\Sigma_f \models_{\mathbf{K}} \varphi \approx \psi$. Thus $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ iff $\Sigma_f \models_{\mathbf{K}} \varphi \approx \psi$ for some finite $\Sigma_f \subseteq \Sigma$. Therefore $\models_{\mathbf{K}}$ is finitary. $\qquad\square$

Now, we define the notion of algebraization as Blok and Pigozzi have studied in [BP89].

Definition 5.10.4. *A logic S is said to be* algebraizable in the sense of Blok and Pigozzi *if it is finitary and has an equivalent algebraic semantics* $\mathbf{K}$ *with $\Delta(p)$ the finite set of defining equations and $E(p,q)$ the finite set of equivalence formulas for S and* $\mathbf{K}$.

I.e., S is algebraizable in the sense of Blok and Pigozzi iff S is finitary and finitely algebraizable.

We show that there is (if any) an equivalent algebraic semantics which is a quasivariety for algebraizable in the sense of Blok and Pigozzi logic.

Theorem 5.10.5. *Let S be an algebraizable in the sense of Blok and Pigozzi logic; and $\mathbf{K}$ and $\mathbf{K}'$ two equivalent algebraic semantics for S. Then $Q(\mathbf{K}) = Q(\mathbf{K}')$, i.e., $\mathbf{K}$ and $\mathbf{K}'$ generate the same quasivariety, and $Q(\mathbf{K})$ is also an equivalent algebraic semantics called* equivalent quasivariety semantics *(it is unique and is the largest equivalent algebraic semantics for S).*

If S is an algebraizable logic whose algebraic counterpart is a variety, we say that S is *strongly algebraizable*.

In the following theorem, we give a characterization for logics which are algebraizable in the sense of Blok and Pigozzi.

Theorem 5.10.6. [Cze01, Theorem 4.6.5] *Let S be a finitary logic and $\mathbf{K}$ a quasivariety of $\mathcal{L}$-algebras. The following conditions are equivalent:*

    (i) S is finitely algebraizable with equivalent quasivariety semantics $\mathbf{K}$;

    (ii) For every $\mathcal{L}$-algebra $\mathbf{A}$, not necessarily in $\mathbf{K}$, the Leibniz operator $\Omega_{\mathbf{A}}$ established an isomorphism between the lattices $\mathbf{Fi}_S(\mathbf{A})$ of S-filters and $\mathrm{Co}_{\mathbf{K}}(\mathbf{A})$ of $\mathbf{K}$-congruences of $\mathbf{A}$.

Corollary 5.10.7. *Let S be an algebraizable in the sense of Blok and Pigozzi logic and $\mathbf{K}$ the equivalent quasivariety semantics. Then $\mathbf{K}$ is the class of all algebraic reducts of $\mathbf{Mod}^*(S)$, i.e., $\mathbf{K} = \boldsymbol{Alg}^*(S)$.*

Proof. Let $\mathrm{M} = \langle \mathbf{A}, D \rangle \in \mathbf{Mod}^*(S)$. By assumption and Theorem 5.10.6, $\Omega_{\mathbf{A}} D \in \mathrm{Co}_{\mathbf{K}}(\mathbf{A})$. Thus $\mathbf{A}/\Omega_{\mathbf{A}} D \in \mathbf{K}$. Since M is reduced, $\Omega_{\mathbf{A}} D = \triangle_{\mathbf{A}}$. Therefore $\mathbf{A} \in \mathbf{K}$.

For the reverse inclusion, suppose that $\mathbf{A} \in \mathbf{K}$. Thus $\triangle_{\mathbf{A}} \in \mathrm{Co}_{\mathbf{K}}(\mathbf{A})$. By Theorem 5.10.6, there exists a unique $D \in \mathrm{Fi}_S(\mathbf{A})$ such that $\triangle_{\mathbf{A}} = \Omega_{\mathbf{A}} D$. Therefore the matrix $\langle \mathbf{A}, D \rangle \in \mathbf{Mod}^*(S)$. $\qquad\square$

An expansion S' of an algebraizable in the sense of Blok and Pigozzi logic S is not necessarily finitely algebraizable. In [Cai04, Example 1], Caicedo has given an example of an expansion of the **IPL** that is not finitely algebraizable. Nevertheless, if an expansion of S is finitely algebraizable, then it is not necessarily with the same set of equivalence formulas and set of defining equations as S. Indeed, in [Cai04, Example 2], we can find an expansion of the **CPL** that is finitely algebraizable with the set of equivalence formulas $E(p, q) = \{\Box(p \to q), \Box(q \to p)\}$ and the set of defining equations $\Delta(p) = \{\neg\neg p \approx \top\}$ which are different that the ones for **CPL**. However, if the axioms and inference rules of S' define implicitly the new connectives then S' is finitely algebraizable with the same set of equivalence formulas and set of defining equations as S and its equivalent quasivariety semantics is the class of algebras that is increased algebras of $\mathbf{Alg}^*(S)$ (c.f. [Cai04, Theorem 1]).

We have seen that we can characterized all the classes of logics until studied by properties of the Leibniz operator. Thus, in the following figure, we present the relation between these different classes which is called the Leibniz hierarchy.

Finitely Algebraizable

monotone

injective

continuous

Finitely Equivalential

monotone

continuous

Algebraizable

monotone

injective

commutes with inverse substitution

Equivalential

monotone

commutes with inverse substitution

Weakly Algebraizable

monotone

injective

Protoalgebraic

monotone
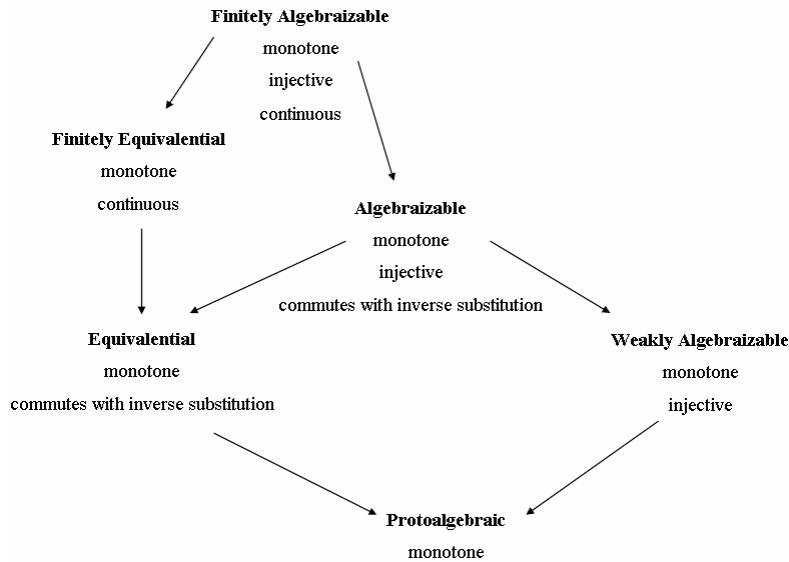
Figure 5.1: A view of the Leibniz Hierarchy

5.10.1   Bridge Theorems

The relationship between the class of algebras that we associate to algebraizable logic and a logic itself is very strong. We can find in the literature, several relations between properties of finitary finitely algebraizable logic S and its equivalent quasivariety semantics $\mathbf{Alg}^*(S)$. These relations are important because sometimes it is easier to understand a problem concerning logic S by translating it to correspondent property of the algebra in $\mathbf{Alg}^*(S)$. In this section, we only enunciate some bridge theorems without their proofs.

A class $\mathbf{K}$ of algebras has the *amalgamation property* if for any pair of embedding mappings $f : \mathbf{C} \to \mathbf{A}$ and $g : \mathbf{C} \to \mathbf{B}$ with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{K}$, there exists $\mathbf{D} \in \mathbf{K}$ and embedding mappings $h : \mathbf{A} \to \mathbf{D}$ and $k : \mathbf{B} \to \mathbf{D}$ such that $h \circ f = k \circ g$. A logic S has the *Craig´s interpolation property* if for any two formulas φ and ψ such that $\varphi \vdash_S \psi$, there exists a third formula α, called an *interpolant*, such that every variable in α occurs both in φ and ψ, and we have that $\varphi \vdash_S \alpha$ and $\alpha \vdash_S \psi$. A finitary finitely algebraizable logic S has the Craig´s interpolation property iff $\mathbf{Alg}^*(S)$ has the amalgamation property (c.f. [ANS01, Theorem 6.15] and [CP99, Theorem 3.5]).

A logic S has the Beth´s definability property if implicit definability equals explicit definability (c.f. [BH06, Definition 3.3]). Let $\mathbf{K}$ be a class of algebras and $\mathbf{A}, \mathbf{B} \in \mathbf{K}$. A morphism $h : \mathbf{A} \to \mathbf{B}$ is called an *epimorphism* of $\mathbf{K}$ if; for every $\mathbf{C} \in \mathbf{K}$ and

every $f, g \in \text{Hom}(\mathbf{B}, \mathbf{C})$, we have that $f \circ h = g \circ h$ implies $f = g$. A finitary finitely algebraizable logic S has the Beth´s definability property iff all the epimorphisms of $\mathbf{Alg}^*(S)$ are surjective (c.f. [ANS01, Theorem 6.11] and [BH06, Theorem 3.17]).

We can state that $\mathbf{CPL}$ has the Craig´s interpolation property and the Beth´s definability property. Thus the class $\mathbf{BA}$ has the amalgamation property and all epimorphisms in $\mathbf{BA}$ are surjective.

We say that a class of algebras $\mathbf{K}$ has *equationally definable principal relative congruence* (EDPRC for short) if there is a finite set of equations in at most four variables $\{\varepsilon_i(x_0, x_1, y_0, y_1) \approx \delta_i(x_0, x_1, y_0, y_1) : i \leq n\}$ such that for every algebra $\mathbf{A} \in \mathbf{K}$ and all $a, b, c, d \in \mathrm{A}$, $c \equiv d\big(\theta_{\mathbf{K}}^{\mathbf{A}}(a, b)\big)$ iff $\varepsilon_i^{\mathbf{A}}(a, b, c, d) = \delta_i^{\mathbf{A}}(a, b, c, d)$ for all $i \leq n$. A finitary finitely algebraizable logic S has the DDT iff its equivalent quasivariety semantics $\mathbf{Alg}^*(S)$ has the EDPRC property (c.f. [BP01, Theorem 5.5]). We have a generalization of this result in [CP04b, Theorem 3.3], where Czelakowski and Pigozzi described the relation between the MDDT and the EDPRC property.

5.11   Examples

Example 5.12 (Classical Propositional Logic [BP01])**.** In Example 5.6, we have seen that $\mathbf{CPL}$ has an equivalent algebraic semantics, which is the class $\mathbf{BA}$, with $\Delta(p) = \{p \approx \top\}$ the set of defining equation and $E(p, q) = \{p \to q, q \to p\}$ the set of equivalence formulas for $\mathbf{CPL}$ and $\mathbf{BA}$. Thus $\mathbf{CPL}$ is finitary and finitely algebraizable.

Furthermore, $\mathbf{CPL}$ has the DDT, that is, for every $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, we have that

$$\Gamma \cup \{\varphi\} \vdash_{\mathbf{CPL}} \psi \text{ iff } \Gamma \vdash_{\mathbf{CPL}} \varphi \to \psi.$$

Thus the class $\mathbf{BA}$ has the EDPRC property (c.f. [BP01, Theorem 5.6]). Indeed, for every $\mathbf{A} \in \mathbf{BA}$ and $a, b, c, d \in \mathrm{A}$, we have that

$$c \equiv d(\theta_{\mathbf{BA}}^{\mathbf{A}}(a, b)) \text{ iff } (a \leftrightarrow b) \wedge c = (a \leftrightarrow b) \wedge d,$$

where $a \leftrightarrow b$ abbreviates $(a \to b) \wedge (b \to a)$. $\diamondsuit$

Example 5.13 (Intuitionistic Propositional Logic [BP01])**.** The *Intuitionistic Propositional Logic*, $\mathbf{IPL}$, has the same language as $\mathbf{CPL}$. It is defined by the following axioms: $(A_1), (A_2), (A_4) - (A_{11})$, together with

(A_{12}) $\neg p \to (p \to \bot)$

(A_{13}) $(p \to \bot) \to \neg p$

and by the only inference rule, modus ponens.

A *Heyting algebra* is an algebra $\mathbf{A} = \langle \mathrm{A}, \to^{\mathbf{A}}, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \neg^{\mathbf{A}}, \bot^{\mathbf{A}}, \top^{\mathbf{A}} \rangle$ such that $\langle \mathrm{A}, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \bot^{\mathbf{A}}, \top^{\mathbf{A}} \rangle$ is a bounded (where $\bot$ is the minimum and $\top$ the maximum), distributive lattice and for $a, b \in \mathrm{A}$, $a \to^{\mathbf{A}} b$ is the largest element c (with respect to the lattice order) such that $a \wedge^{\mathbf{A}} c \leq b$ and $\neg^{\mathbf{A}} a = a \to^{\mathbf{A}} \top^{\mathbf{A}}$. Thus $\to^{\mathbf{A}}$ is a binary operation with the property, for all $a, b, c \in \mathrm{A}$; $c \leq a \to^{\mathbf{A}} b$ iff $a \wedge^{\mathbf{A}} c \leq b$. The operation $\to^{\mathbf{A}}$ is called *relative pseudo-complementation*. Each finite distributive lattice admits a unique relative pseudo-complementation operation. Hence every finite distributive lattice is the reduct of a unique Heyting algebra. In contrast with the class of Boolean algebras, the variety of Heyting algebras, **HA**, is not generated by a finite algebra.

The class **HA** forms an equivalent algebraic semantics for **IPL** with the same set of defining equations and set of equivalence formulas as for **CPL**, i.e., $\Delta(p) = \{p \approx \top\}$ and $E(p, q) = \{p \to q, q \to p\}$. Thus **IPL** is finitary and finitely algebraizable.

Furthermore, **IPL** has the DDT. Indeed, for every $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, we have that

$$\Gamma \cup \{\varphi\} \vdash_{\mathbf{IPL}} \psi \text{ iff } \Gamma \vdash_{\mathbf{IPL}} \varphi \to \psi.$$

Thus the class **HA** has the EDPRC property (c.f. [BP01, Example 5.2.2]). Actually, for every $\mathbf{A} \in \mathbf{HA}$ and $a, b, c, d \in \mathrm{A}$, we have that

$$c \equiv d(\theta^{\mathbf{A}}_{\mathbf{HA}}(a, b)) \text{ iff } (a \leftrightarrow b) \wedge c = (a \leftrightarrow b) \wedge d,$$

where $a \leftrightarrow b$ abbreviates $(a \to b) \wedge (b \to a)$. $\diamondsuit$

The various implication fragments of **CPL** and **IPL** are all finitely algebraizable. Since the equivalent algebraic semantics of **CPL** and **IPL** are the varieties **BA** and **HA**, respectively, the equivalent algebraic semantics of each fragment of **CPL** or **IPL** that contains either $\to$ or $\leftrightarrow$ is the class of all subalgebras of the appropriate reducts of Boolean or Heyting algebras, respectively. In particular, the equivalent algebraic semantics of the $\{\wedge, \to\}$, the $\{\wedge, \leftrightarrow\}$, the $\{\to\}$, and the $\{\leftrightarrow\}$ fragments are called the varieties of Brouwerian semilattices, Skolem semilattices, Hilbert algebras, and equivalential algebras.

Example 5.14 (Logic of Andréka and Németi [CJ00])**.** The finitary logic of Andréka and Németi, denoted by $S_{\mathbf{AN}}$, is defined in the language $\mathcal{L} = \{\leftrightarrow, *\}$ (where $\leftrightarrow$ is a

binary connective and $*$ is unary), by the axioms:

$$*p \text{ and } p \leftrightarrow p$$

and the infinite family of inference rules:

(i) $p, p \leftrightarrow q \vdash_{\mathbf{AN}} q$;

(ii) $p \vdash_{\mathbf{AN}} \varphi \leftrightarrow \varphi(p/ * p)$, for each $\varphi \in \mathrm{Fm}_{\mathcal{L}}$;

(iii) $p \vdash_{\mathbf{AN}} \varphi(p/ * p) \leftrightarrow \varphi$, for each $\varphi \in \mathrm{Fm}_{\mathcal{L}}$.

Theorem 5.14.1. $S_{\mathbf{AN}}$ *is weakly algebraizable but not algebraizable.*

Proof. We show that the Leibniz operator is monotone on $\mathbf{Th}(S_{\mathbf{AN}})$ and the class $\mathbf{L}^*(S_{\mathbf{AN}})$ has its filters equationally definable. Let $\Delta(p, q) = \{p \leftrightarrow q\}$. Since $p \leftrightarrow p$ is an axiom and modus ponens can be deduce by the inference rules, $\Delta(p, q)$ is a protoequivalence system for $S_{\mathbf{AN}}$. By Theorem 3.1.3, $S_{\mathbf{AN}}$ is protoalgebraic. And by Theorem 3.3.1, the Leibniz operator is monotone on $\mathbf{Th}(S_{\mathbf{AN}})$. Now, let $\{p \approx *p\}$ be a set of equation in a single variable p. We have that for every $T \in \mathrm{Th}(S_{\mathbf{AN}})$, $\alpha \in T$ iff $\alpha \equiv *\alpha(\Omega T)$. Indeed, let $\alpha \in T$. By the inference rules (ii) and (iii), we have that $\varphi(p/\alpha) \leftrightarrow \varphi(p/ * \alpha) \in T$ and $\varphi(p/ * \alpha) \leftrightarrow \varphi(p/\alpha) \in T$, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ and $p \in \mathrm{Var}$. And, by the inference rule (i), $\varphi(p/\alpha) \in T$ iff $\varphi(p/ * \alpha) \in T$, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ and $p \in \mathrm{Var}$. Thus $\alpha \equiv *\alpha(\Omega T)$. Conversely, suppose that $\alpha \equiv *\alpha(\Omega T)$. Since ΩT is a congruence, $\alpha \leftrightarrow \alpha \equiv *\alpha \leftrightarrow \alpha(\Omega T)$. By the second axiom, $\alpha \leftrightarrow \alpha \in T$. And by compatibility, $*\alpha \leftrightarrow \alpha \in T$. Moreover, by the first axiom, $*\alpha \in T$. And by the inference rule (i), we have that $\alpha \in T$. Therefore $\mathbf{L}^*(S_{\mathbf{AN}})$ has its filters equationally definable by $\{p \approx *p\}$. By Theorem 5.7.4, $S_{\mathbf{AN}}$ is weakly algebraizable.

Finally, we show that the class $\mathbf{Mod}^*(S_{AN})$ is not closed under submatrices. Let $M = \langle \mathbf{A}, F \rangle$ be the matrix where $F = \{1, 2\}$ and $\mathbf{A}$ is a four-element algebra, $A = \{0, 1, 2, 3\}$ with $\leftrightarrow^{\mathbf{A}}$ given by the table

$\leftrightarrow^{\mathbf{A}}$	0	1	2	3
0	1	0	0	0
1	0	1	1	0
2	0	1	2	3
3	0	0	3	2

The operation $*^{\mathbf{A}}$ is defined, for every $a \in A$, by $*^{\mathbf{A}}a := a \leftrightarrow^{\mathbf{A}} a$. It is not difficult to prove that the matrix $M \in \mathbf{Mod}^*(S_{\mathbf{AN}})$. Let $N = \langle \mathbf{B}, B \cap F \rangle$ be a submatrix of M where

B = $\{0, 1, 2\}$. Since $\mathbf{Mod}(S_{\mathbf{AN}})$ is closed under submatrices, $N \in \mathbf{Mod}(S_{\mathbf{AN}})$. And as $1 \equiv 2(\Omega_{\mathbf{B}}(B \cap F))$, N is not reduced. By Theorem 4.1.7, $S_{\mathbf{AN}}$ is not equivalential. Therefore $S_{\mathbf{AN}}$ is not algebraizable. $\qquad \blacksquare$

$\diamondsuit$

Example 5.15 (Last Judgement [Her96])**.** The finitary logic *Last Judgement*, denoted by **LJ**, is defined in the modal language $\mathcal{L} = \{\rightarrow, \neg, \wedge, \vee, \square\}$, where $\rightarrow, \wedge, \vee$ are binary connectives and $\neg, \square$ are unary, by the following axioms:

(i) *all classical tautologies;*

(ii) *$\square^n \phi$ for all intuitionistic tautologies ϕ and $n \geq 0$;*

(iii) *$\square^n(\square(p \rightarrow q) \rightarrow (\square p \rightarrow \square q))$, for all $n \geq 0$;*

(iv) *$(p \rightarrow q) \rightarrow \square^n(\neg q \rightarrow \neg p)$, for all $n \geq 0$.*

and the inference rule modus ponens.

Theorem 5.15.1. LJ *is algebraizable but not finitely algebraizable.*

Proof. We show that $E(p, q) = \{\square^n(p \rightarrow q) : n \geq 0\} \cup \{\square^n(q \rightarrow p) : n \geq 0\}$ is a system of equivalence formulas for **LJ**. We prove by induction on $n \geq 0$ that $\square^n(p \rightarrow q) \vdash_{\mathbf{LJ}} \square^n p \rightarrow \square^n q$. If $n = 0$, then $p \rightarrow q \vdash_{\mathbf{LJ}} p \rightarrow q$, which is always true. Now assume that it is true for n and we show for $n+1$. Let e be a substitution such that $e(p) = \square(p \rightarrow q)$ and $e(q) = \square p \rightarrow \square q$. By structurality of **LJ** and induction hypothesi, $e(\square^n(p \rightarrow q)) \vdash_{\mathbf{LJ}} e(\square^n p \rightarrow \square^n q)$, i.e., $\square^n(e(p) \rightarrow e(q)) \vdash_{\mathbf{LJ}} \square^n e(p) \rightarrow \square^n e(q)$. Thus, $\square^n(\square(p \rightarrow q) \rightarrow (\square p \rightarrow \square q)) \vdash_{\mathbf{LJ}} \square^n(\square(p \rightarrow q)) \rightarrow \square^n(\square p \rightarrow \square q)$. By axiom (iii), we have that $\vdash_{\mathbf{LJ}} \square^n(\square(p \rightarrow q)) \rightarrow \square^n(\square p \rightarrow \square q)$. And by modus ponens, $\square^{n+1}(p \rightarrow q) \vdash_{\mathbf{LJ}} \square^n(\square p \rightarrow \square q)$. Let e' be another substitution such that $e'(p) = \square p$ and $e'(q) = \square q$. By structurality of **LJ** and induction hypothesi, $e'(\square^n(p \rightarrow q)) \vdash_{\mathbf{LJ}} e'(\square^n p \rightarrow \square^n q)$, i.e., $\square^n(e'(p) \rightarrow e'(q)) \vdash_{\mathbf{LJ}} \square^n e'(p) \rightarrow \square^n e'(q)$. Thus, $\square^n(\square p \rightarrow \square q) \vdash_{\mathbf{LJ}} \square^n \square p \rightarrow \square^n \square q$. Since we have proved that $\square^{n+1}(p \rightarrow q) \vdash_{\mathbf{LJ}} \square^n(\square p \rightarrow \square q)$, we obtain that $\square^{n+1}(p \rightarrow q) \vdash_{\mathbf{LJ}} \square^n \square p \rightarrow \square^n \square q$, i.e., the hypothesi is true for $n + 1$. Now, we proved that the set $E(p, q)$ satisfies the conditions of Definition 4.1.1. From axiom (ii), we have that reflexivity condition holds and by the inference rule, modus ponens condition is satisfied. We verify simple replacement condition for all connectives in the language $\mathcal{L}$.

For $\square$: We proved above that $\square^{n+1}(p \rightarrow q) \vdash_{\mathbf{LJ}} \square^n(\square p \rightarrow \square q)$. Thus we have $E(p, q) \vdash_{\mathbf{LJ}} E(\square p, \square q)$.

For $\neg$: We have by axiom (iv) that $E(p,q) \vdash_{\mathbf{LJ}} E(\neg p, \neg q)$.

For $\rightarrow, \wedge, \vee$: It suffices to show that $\Box^n(p_1 \rightarrow q_1), \Box^n(q_1 \rightarrow p_1), \Box^n(p_2 \rightarrow q_2), \Box^n(q_2 \rightarrow p_2) \vdash_{\mathbf{LJ}} \Box^n(p_1 \cdot p_2 \rightarrow q_1 \cdot q_2)$ for each $n \geq 0$ where $\cdot \in \{\rightarrow, \wedge, \vee\}$. Since $p_1 \rightarrow q_1, q_1 \rightarrow p_1, p_2 \rightarrow q_2, q_2 \rightarrow p_2 \vdash_{\mathbf{IPL}} p_1 \cdot p_2 \rightarrow q_1 \cdot q_2$, we have that $E(p_1, q_1), E(p_2, q_2) \vdash_{\mathbf{LJ}} E(p_1 \cdot p_2, q_1 \cdot q_2)$.

Therefore $E(p,q)$ is a system of equivalence formulas for $\mathbf{LJ}$. Then $\mathbf{LJ}$ is equivalential.

We show that $\mathbf{L}^*(\mathbf{LJ})$ has its filters equationally definable. Let $\Delta(p) = \{\neg p \approx 0\}$, where $0 := \neg(p \rightarrow p)$. Since $p \rightarrow (q \rightarrow p)$ is an intuitionistic tautology, by axiom (ii), $\vdash_{\mathbf{LJ}} \Box^n(p \rightarrow (p \rightarrow p))$ for all $n \geq 0$. And by axiom (iv), $(p \rightarrow p) \rightarrow p \vdash_{\mathbf{LJ}} \Box^n(\neg p \rightarrow (\neg(p \rightarrow p)))$ for all $n \geq 0$. Thus $p \vdash_{\mathbf{LJ}} \Box^n(\neg p \rightarrow (\neg(p \rightarrow p)))$ for all $n \geq 0$, i.e., $p \vdash_{\mathbf{LJ}} \Box^n(\neg p \rightarrow 0)$ for all $n \geq 0$. Since $\neg(p \rightarrow p) \rightarrow \neg p$ is an intuitionistic tautology, by axiom (ii), $\vdash_{\mathbf{LJ}} \Box^n(\neg(p \rightarrow p) \rightarrow \neg p)$ for all $n \geq 0$. So $p \vdash_{\mathbf{LJ}} \Box^n(0 \rightarrow \neg p)$ for all $n \geq 0$. We have proved that $p \vdash_{\mathbf{LJ}} E(\neg p, 0)$. As $(\neg p \rightarrow \neg(p \rightarrow p)) \rightarrow p$ is a classical tautology, by axiom (i), $\vdash_{\mathbf{LJ}} (\neg p \rightarrow \neg(p \rightarrow p)) \rightarrow p$, i.e., $\vdash_{\mathbf{LJ}} (\neg p \rightarrow 0) \rightarrow p$. By modus ponens, we have that $\neg p \rightarrow 0 \vdash_{\mathbf{LJ}} p$, i.e., $E(\neg p, 0) \vdash_{\mathbf{LJ}} p$. Therefore $p \dashv\vdash_{\mathbf{LJ}} E(\neg p, 0)$. Thus $\mathbf{L}^*(\mathbf{LJ})$ has its filters equationally definable. Since we have proved that $\mathbf{LJ}$ is equivalential, by Theorem 4.1.5, the Leibniz operator is monotone on $\mathbf{Th}(\mathbf{LJ})$. And by Theorem 5.7.4, $\mathbf{LJ}$ is weakly algebraizable. As the intersection of the classes of equivalential and weakly algebraizable logics is the class of algebraizable logics, we have that $\mathbf{LJ}$ is algebraizable.

Now, we show by contradiction that $\mathbf{LJ}$ is not finitely equivalential. Suppose that $\mathbf{LJ}$ is finitely equivalential, i.e., there exists a finite set of equivalence formulas $E_f(p,q)$. Since two systems of equivalence formulas are interderivable, we have that $E_f(p,q) \vdash_{\mathbf{LJ}} E(p,q)$. Let $1 := p \rightarrow p$ and e a substitution such that $e(p) = 1$ and $e(q) = p$. By structurality, $e[E_f(p,q)] \vdash_{\mathbf{LJ}} e[E(p,q)]$, i.e., $E_f(e(p), e(q)) \vdash_{\mathbf{LJ}} E(e(p), e(q))$. Thus $E_f(1, p) \vdash_{\mathbf{LJ}} E(1, p)$, i.e., $\{p, \Box p, \ldots, \Box^{n-1}p\} \vdash_{\mathbf{LJ}} \Box^n p$ for some n. Let $\mathbf{A} = \langle\{0, \ldots, n+1\}, \rightarrow, \wedge, \vee, \neg, \Box\rangle$ be the $(n+2)$-element linearly ordered Heyting algebra, where 0 is the smallest element and $n+1$ the largest element. The operation $\Box$ satisfies the following conditions: $\Box 0 = 0$, $\Box(n+1) = n+1$ and $\Box k = k - 1$ for $1 \leq k \leq n$. Let $D := \{1, \ldots, n+1\}$. It is not difficult to see that $\langle\mathbf{A}, D\rangle \in \mathbf{Mod}^*(\mathbf{LJ})$. Let $h : \mathbf{Fm}_{\mathcal{L}} \rightarrow \mathbf{A}$ be a homomorphism such that $h(p) = n$. Then $h[\{p, \Box p, \ldots, \Box^{n-1}p\}] \subseteq D$ while $h(\Box^n p) = 0 \notin D$. Hence, $p, \Box p, \ldots, \Box^{n-1}p \nvdash_{\mathbf{LJ}} \Box^n p$ and we have a contradiction. Therefore $\mathbf{LJ}$ is not finitely equivalential. $\qquad\qquad\qquad\square$

$\diamondsuit$

Example 5.16 (Relevance [FR94])**.** The *Relevance* logic **R** is defined in the language $\mathcal{L} = \{\neg, \rightarrow, \wedge\}$, where $\rightarrow, \wedge$ are binary connectives and $\neg$ is unary. We admit that the formula $\varphi \vee \psi$ is an abbreviation for $\neg(\neg\varphi \wedge \neg\psi)$, $\varphi \leftrightarrow \psi$ an abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, and $\varphi * \psi$ an abbreviation for $\neg(\varphi \rightarrow \neg\psi)$, for any $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$. The binary connective $*$ is commonly called *intensional conjunction* or *fusion*. The Relevance logic is defined by the following axioms,

R_1 $\varphi \rightarrow \varphi$

R_2 $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \eta) \rightarrow (\varphi \rightarrow \eta))$

R_3 $\varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi)$

R_4 $(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$

R_5 $(\varphi \wedge \psi) \rightarrow \varphi$

R_6 $(\varphi \wedge \psi) \rightarrow \psi$

R_7 $((\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \eta)) \rightarrow (\varphi \rightarrow (\psi \wedge \eta))$

R_8 $\varphi \rightarrow (\varphi \vee \psi)$

R_9 $\psi \rightarrow (\varphi \vee \psi)$

R_{10} $((\varphi \rightarrow \eta) \wedge (\psi \rightarrow \eta)) \rightarrow ((\varphi \vee \psi) \rightarrow \eta)$

R_{11} $(\varphi \wedge (\psi \vee \eta)) \rightarrow ((\varphi \wedge \psi) \vee \eta)$

R_{12} $(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$

R_{13} $\neg\neg\varphi \rightarrow \varphi$

and inference rules,

$$\varphi, \psi \vdash_{\mathbf{R}} \varphi \wedge \psi \qquad\qquad\qquad \text{(Adjunction)}$$

$$\varphi, \varphi \rightarrow \psi \vdash_{\mathbf{R}} \psi \qquad\qquad\qquad \text{(Modus Ponens)}$$

An algebra $\mathbf{A} = \langle A, \neg, \rightarrow, \wedge \rangle$ is a *De Morgan semigroup* when the following conditions hold, for any $a, b, c \in A$:

(i) $\langle A, \wedge, \neg \rangle$ *is a De Morgan lattice, whose ordering relation is denoted by $\leq$ and the supremum operation is $a \vee b = \neg(\neg a \wedge \neg b)$;*

(ii) $a \rightarrow (b \rightarrow c) \leq b \rightarrow (a \rightarrow c)$;

(iii) $a \leq ((a \rightarrow b) \wedge c) \rightarrow b$;

(iv) $a \rightarrow \neg a \leq \neg a$;

(v) $a \rightarrow b \leq \neg b \rightarrow \neg a$.

We say that an algebra $\mathbf{A}$ is an $\mathbf{R}$-*algebra* if $\mathbf{A} = \langle A, \neg, \rightarrow, \wedge \rangle$ is a De Morgan semigroup such that, for any $a, b, c \in A$, $\big((a \rightarrow a) \wedge (b \rightarrow b)\big) \rightarrow c \leq c$. The class of all $\mathbf{R}$-algebras constitute a variety denoted by R.

Theorem 5.16.1. *The Relevance logic* $\mathbf{R}$ *is finitely algebraizable with the class* R *as the equivalent algebraic semantics,* $E(p, q) = \{p \rightarrow q, p \rightarrow q\}$ *the set of equivalence formulas and* $\Delta(p) = \{p \wedge (p \rightarrow p) \approx (p \rightarrow p)\}$ *the set of defining equation for* R *and* $\mathbf{R}$.

Proof. By the axioms R_1 and R_2, the inference rule of modus ponens and the definition of $E(p, q)$, conditions of reflexivity, transitivity, modus ponens and symmetry are satisfied, respectively. It is not difficult to see that simple replacement condition holds. Thus $E(p, q)$ is a system of equivalence formulas for $\mathbf{R}$.

Now, we show that $p \dashv\vdash_{\mathbf{R}} E(\Delta(p))$. By axiom R_3, $\vdash_{\mathbf{R}} p \rightarrow ((p \rightarrow p) \rightarrow p)$. And by modus ponens, we have that $p \vdash_{\mathbf{R}} (p \rightarrow p) \rightarrow p$. On the other hand, by axiom R_1, $\vdash_{\mathbf{R}} (p \rightarrow p) \rightarrow (p \rightarrow p)$. And by the inference rule adjunction, we have that $p \vdash_{\mathbf{R}} ((p \rightarrow p) \rightarrow p) \wedge ((p \rightarrow p) \rightarrow (p \rightarrow p))$. Which gives, by axiom R_7 and modus ponens, $p \vdash_{\mathbf{R}} (p \rightarrow p) \rightarrow (p \wedge (p \rightarrow p))$. Thus we obtain that $p \vdash_{\mathbf{R}} E(\Delta(p))$. For the inference in the other direction, by axiom R_1 and modus ponens, we have that $(p \rightarrow p) \rightarrow (p \wedge (p \rightarrow p)) \vdash_{\mathbf{R}} p \wedge (p \rightarrow p)$. And by axiom R_5 and modus ponens, $p \wedge (p \rightarrow p) \vdash_{\mathbf{R}} p$. Thus $E(\Delta(p)) \vdash_{\mathbf{R}} p$. We can conclude that the logic $\mathbf{R}$ is finitely algebraizable. □

Furthermore, since the logic $\mathbf{RM}$ is an axiomatic extension of $\mathbf{R}$ (by the mingle axiom $p \rightarrow (p \rightarrow p)$), it is also algebraizable with the same set of equivalence formulas and defining equation (c.f. [BP89, Theorem 5.8]). ◇

Example 5.17 (*BCK* Logic [BP01]). In Example 3.7, we have defined the logic $\mathbf{BCK}$. We say that $\mathbf{A} = \langle A, *, 1, \leq \rangle$ is a *partially ordered monoid* if $\mathbf{A} = \langle A, *, 1 \rangle$ is a monoid, $\leq$ is a partially order on A and for all $x, y, z \in A$; if $x \leq y$ then $x * z \leq y * z$ and $z * x \leq z * y$. A structure $\mathbf{A}$ is called *integral* if $x \leq 1$, for all $x \in A$, and is called *residuated* if; for all $x, y \in A$, the set $\{z : x * z \leq y\}$ contains a largest element called the *residual* of x relative to y and denoted by $x \rightarrow y$. Since the partial order $\leq$ can be recovered via $x \leq y$ iff $x \rightarrow y = 1$, a partially ordered commutative, residuated and integral monoid $\langle A, *, 1, \leq \rangle$ can be treated as an algebra $\langle A, *, \rightarrow, 1 \rangle$, called *pocrim*. The class PO of all procrims is a quasivariety definable by:

(PO1) $x * 1 \approx x$ (PO5) $(z \to x) \to ((x \to y) \to (z \to y)) \approx 1$

(PO2) $x * y \approx y * x$ (PO6) $x \to (y \to z) \approx (x * y) \to z$

(PO3) $x \to 1 \approx 1$ (PO7) $x \to y \approx 1$ and $y \to x \approx 1 \Rightarrow x \approx y$

(PO4) $1 \to x \approx x$

A BCK algebra is defined as an algebras $\mathbf{A} = \langle A, \to, 1 \rangle$ satisfying $PO3$, $PO4$, $PO5$ and $PO7$ together with

(PO8) $x \to x \approx 1$

(PO9) $x \to (y \to z) \approx y \to (x \to z)$

We denoted by BCK, the class of all BCK algebras.

Theorem 5.17.1. *The logic* **BCK** *is finitely algebraizable with* BCK *the equivalent algebraic semantics,* $E(p, q) = \{p \to q, q \to p\}$ *the set of equivalence formulas and* $\Delta(p) = \{p \approx p \to p\}$ *the set of defining equations for* BCK *and* **BCK**.

Proof. By axiom (B), modus ponens and the definition of $E(p, q)$; reflexivity, transitivity, modus ponens and symmetry conditions are satisfied, respectively. We verify simple replacement condition for the connective $\to$. By axiom (B), we have that $\vdash_{\mathbf{BCK}} (p_1 \to q_1) \to ((q_1 \to p_2) \to (p_1 \to p_2))$ and $\vdash_{\mathbf{BCK}} (q_1 \to p_1) \to ((p_1 \to p_2) \to (q_1 \to p_2))$. And by modus ponens, $E(p_1, q_1) \vdash_{\mathbf{BCK}} E(p_1 \to p_2, q_1 \to p_2)$. Thus $E(p_1, q_1), E(p_2, q_2) \vdash_{\mathbf{BCK}} E(p_1 \to p_2, q_1 \to q_2)$. Therefore $E(p, q)$ is a system of equivalence formulas for **BCK**. Now, we show that $p \dashv\vdash_{\mathbf{BCK}} E(\Delta(p))$. By axiom (K), $p \vdash_{\mathbf{BCK}} p \to (p \to p)$. Since $\vdash_{\mathbf{BCK}} p \to p$, by modus ponens and axiom (B), we have that $\vdash_{\mathbf{BCK}} (p \to p) \to (p \to p)$. Using axiom (C), $\vdash_{\mathbf{BCK}} p \to ((p \to p) \to p)$ and by modus ponens, we have that $p \vdash_{\mathbf{BCK}} (p \to p) \to p$. Thus, $p \vdash_{\mathbf{BCK}} E(\Delta(p))$. For the inference in the other direction, we have that $\vdash_{\mathbf{BCK}} ((p \to p) \to p) \to ((p \to p) \to p)$. By axiom (C) and modus ponens, we have $p \to p \vdash_{\mathbf{BCK}} ((p \to p) \to p) \to p$. Again by modus ponens, $(p \to p) \to p \vdash_{\mathbf{BCK}} p$. Thus $E(\Delta(p)) \vdash_{\mathbf{BCK}} p$. We can conclude that the logic **BCK** is finitely algebraizable. $\qquad\square$

$\diamondsuit$

Chapter 6

Non-Protoalgebraic Logics

Protoalgebraic logics have been considered by logicians as the largest class of logics for which an interesting algebraic theory can be evolved. We have seen, in previous chapters, that some meta-properties of a logic, can be characterized intrinsically by properties of the Leibniz operator, obtaining, by this way, a hierarchy of classes of logics called "Leibniz hierarchy". When S is not protoalgebraic, its Suszko operator still monotone on $\mathbf{Th}(S)$ and the class $\mathbf{Mod}^*_{Su}(S)$ play the role of $\mathbf{Mod}^*(S)$. Some non-protoalgebraic logics have been investigated individually. In this chapter, we will provide some well known examples of non-protoalgebraic logics. We study the class of truth-equational logics which contains the class of weakly algebraizable logics and some non-protoalgebraic logics by discussing some examples and presenting its introductory theory. This class has been investigated by Raftery (c.f. [Raf06b]).

Definition 6.0.2. *A logic S is* truth-equational *if the class $\mathbf{L}^*_{Su}(S)$ has its filters equationally definable.*

By Theorem 5.7.4, we note that the class of truth-equational logics encompass the class of weakly algebraizable logics. Indeed, the latter is the intersection of the class of truth-equational logics and the class of protoalgebraic logics.

Theorem 6.0.3. [Raf06b, Theorem 28] *Let S be a logic. The following conditions are equivalent:*

> *(i) S is truth-equational;*

> *(ii) The class $\mathbf{Mod}^*_{Su}(S)$ has its filters equationally definable;*

> *(iii) The Suszko operator is injective on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$.*

It is not difficult to see that if the Suszko operator is injective on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$ then the Leibniz operator is also injective, but the converse is false [Raf06b, example 2]. Thus truth-equational logics do not encompass all logics which Leibniz operator is injective on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. The Leibniz operator´s injectivity by itself was investigated in [DM05].

In this thesis, when the logic is finitary, we use the Hilbert systems (defined in Chapter 2). However, a logic can be also defined in a Gentzen style, which informally consist on an axiomatization of the consequence relation. These two systems tend to serve different purposes. In spite of this difference, there exists a relation between Gentzen and Hilbert systems that we do not explain here. Nevertheless, the reader can see [Raf06a], where Raftery proved that a logic described in Hilbert system, can be always seen as Gentzen system (c.f. [Raf06a]). But if a logic is defined by Gentzen system, it may not have a Hilbert system (c.f. [Raf06a]). He also defined analogous classes of logics studied in this thesis using Gentzen system. Obviously, these new classes are different from ours, but the main idea is the same. Furthermore, he proved that if a logic is algebraizable with Hilbert system (i.e., algebraizable in our sense) then it is also algebraizable as a Gentzen system (called *Gentzen algebraizable*), but the converse is false. We also refer to [GR06], where Gil and Rebagliato give more details about finitely equivalential Gentzen system, and [Pig97].

Example 6.1 (Intuitionistic Propositional Logic without Implication [BP89])**.** *Intuitionistic Propositional Logic without Implication*, denoted by $\mathbf{IPL}^*$, is the $\{\vee, \wedge, \neg, \top, \perp\}$-fragment of $\mathbf{IPL}$.

In example 5.13, we have defined a Heyting algebra $\mathbf{A} = \langle A, \to^{\mathbf{A}}, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \neg^{\mathbf{A}}, \perp^{\mathbf{A}}, \top^{\mathbf{A}} \rangle$. The binary operation, which for all pair of elements x, y correspond the element $x \to y$, is called *implication*. A *pseudocomplementation* is an operation which for all x associates $x^* = x \to 0$; the element x is called *pseudocomplemented*. Let $\mathbf{A} = \langle \{\top, a, b, \perp\}, \vee, \wedge, \neg, \top, \perp \rangle$ be the four-element chain pseudocomplemented lattice: $\perp < b < a < \top$, $\neg\top = \neg a = \neg b = \perp$, and $\neg\perp = \top$. Let $F_1 = \{\top\}$ and $F_2 = \{\top, a\}$. The algebra $\mathbf{A}$ is the reduct of the four-element chain Heyting algebra, and F_1 and F_2 are filters of the Heyting algebra. Since the class $\mathbf{HA}$ is an equivalent algebraic semantics for $\mathbf{IPL}$, we have that F_1 and F_2 are $\mathbf{IPL}$-filters. Thus, they are also $\mathbf{IPL}^*$-filters of $\mathbf{A}$. It is not difficult to see that $\Omega_{\mathbf{A}} F_1 = \triangle_{\mathbf{A}} \cup \{(a, b), (b, a)\}$ and $\Omega_{\mathbf{A}} F_2 = \triangle_{\mathbf{A}} \cup \{(a, \top), (\top, a)\}$. Although $F_1 \subseteq F_2$, we have that $\Omega_{\mathbf{A}} F_1 \not\subseteq \Omega_{\mathbf{A}} F_2$. By Theorem 3.3.1, $\mathbf{IPL}^*$ is not protoalgebraic.

Furthermore, it is not difficult to prove that **IPL*** is truth-equational (c.f. [Raf06b, Example 7]). Since **IPL*** is not protoalgebraic, it is also not algebraizable. However, Font, Jansana and Pigozzi have proved in [FJP03] that **IPL*** is Gentzen algebraizable.

$\diamondsuit$

Example 6.2 ($\{\wedge, \vee\}$-fragment of Classical Propositional Logic [FV91])**.** The $\{\wedge, \vee\}$-*fragment of Classical Propositional Logic*, denoted by $\{\wedge, \vee\}$-**CPL**, is defined in the usual Hilbert-style with no axioms and with the following inference rules:

R_1 $\varphi \wedge \psi \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} \varphi$

R_7 $\varphi \vee (\psi \vee \xi) \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} (\varphi \vee \psi) \vee \xi$

R_2 $\varphi \wedge \psi \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} \psi \wedge \varphi$

R_8 $(\varphi \vee \psi) \vee \xi \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} \varphi \vee (\psi \vee \xi)$

R_3 $\{\varphi, \psi\} \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} \varphi \wedge \psi$

R_9 $\varphi \vee (\psi \wedge \xi) \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} (\varphi \vee \psi) \wedge (\varphi \vee \xi)$

R_4 $\varphi \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} \varphi \vee \psi$

R_{10} $(\varphi \vee \psi) \wedge (\varphi \vee \xi) \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} \varphi \vee (\psi \wedge \xi)$

R_5 $\varphi \vee \psi \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} \psi \vee \varphi$

R_{11} $\varphi \wedge (\psi \vee \xi) \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} (\varphi \wedge \psi) \vee (\varphi \wedge \xi)$

R_6 $\varphi \vee (\varphi \vee \psi) \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} \varphi \vee \psi$

R_{12} $\varphi \vee \varphi \vdash_{\{\wedge,\vee\}-\mathbf{CPL}} \varphi$

We note that rules (R_6), (R_8) and (R_{11}) are derivable from the remaining ones (c.f. [FGV91]). We denote by $\mathbf{D}_2$ the two-element distributive lattice on the set $D_2 = \{0, 1\}$. It is well-know that $\langle \mathbf{D}_2, \{1\}\rangle$ is a matrix model for $\{\wedge, \vee\}$-**CPL**. Consider the algebra $\mathbf{D}_2$, there are three $\{\wedge, \vee\}$-filters on it, namely $\emptyset$, $\{1\}$ and $\{0, 1\}$. It is not difficult to see that $\Omega_{\mathbf{D}_2}(\emptyset) = \nabla_{\mathbf{D}_2}$ and $\Omega_{\mathbf{D}_2}(\{1\}) = \triangle_{D_2}$. Thus we have $\emptyset \subseteq \{1\}$ while $\Omega_{\mathbf{D}_2}(\emptyset) \not\subseteq \Omega_{\mathbf{D}_2}(\{1\})$. By Theorem 3.3.1, $\{\wedge, \vee\}$-**CPL** is not protoalgebraic.

Moreover, the Leibniz operator is non-injective on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. Indeed, it is not difficult to see that $\Omega_{\mathbf{D}_2}(\{0, 1\}) = \nabla_{\mathbf{D}_2}$. And we have $\Omega_{\mathbf{D}_2}(\emptyset) = \Omega_{\mathbf{D}_2}(\{0, 1\})$, while $\emptyset \neq \{0, 1\}$. Thus the Suszko operator is non-injective on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. By Theorem 6.0.3, the logic $\{\wedge, \vee\}$-**CPL** is not truth-equational.

Since $\{\wedge, \vee\}$-**CPL** is not protoalgebraic, it is not algebraizable. However, in [FJP03], Font, Jansana and Pigozzi have shown that $\{\wedge, \vee\}$-**CPL** is Gentzen algebraizable. $\diamondsuit$

Example 6.3 (Minimum System of Positive Modal Logic [Jan02])**.** The *minimum system of Positive Modal logic*, denoted by $\mathcal{PML}$, is the restriction of the minimum normal modal logic **K** with the local consequence relation to the positive, or negation-free, modal language with connectives $\wedge, \vee, \top, \bot, \Box$ and $\diamond$. The reader can find a representation of the Gentzen system of this logic in [Jan02, Chapter 3].

A *positive modal algebra* $\mathbf{A} = \langle A, \wedge, \vee, \Box, \diamond, 0, 1 \rangle$ is an algebra such that $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice and for any $a, b \in A$,

$$\Box(a \wedge b) = \Box a \wedge \Box b \qquad\qquad \Box(a \vee b) \leq \Box a \vee \diamond b$$

$$\diamond(a \vee b) = \diamond a \vee \diamond b \qquad\qquad \Box 1 = 1$$

$$\Box a \wedge \diamond b \leq \diamond(a \wedge b) \qquad\qquad \diamond 0 = 0$$

The class of positive modal algebras forms a variety. Consider the four element chain distributive lattice with universe $A = \{0, b, a, 1\}$ ordered by $0 < b < a < 1$. We define the operations $\Box$ and $\diamond$ by:

$$\Box p = p \ \ if \ p \in \{0, 1\} \qquad \diamond p = p \ \ if \ p \in \{0, 1\}$$

$$\Box p = b \ \ if \ p \in \{a, b\} \qquad \diamond p = a \ \ if \ p \in \{a, b\}$$

The sets $\{1\}$ and $\{1, a\}$ are $\mathcal{PML}$-filters. It is not difficult to see that $\Omega_{\mathbf{A}}(\{1\}) = Id_{\mathbf{A}} \cup \{\langle a, b \rangle, \langle b, a \rangle, \langle a, 0 \rangle, \langle 0, a \rangle, \langle 0, b \rangle, \langle b, 0 \rangle\}$ and $\Omega_{\mathbf{A}}(\{1, a\}) = Id_{\mathbf{A}} \cup \{\langle 1, a \rangle, \langle a, 1 \rangle\}$. Although $\{1\} \subseteq \{1, a\}$, we have $\Omega_{\mathbf{A}}(\{1\}) \not\subseteq \Omega_{\mathbf{A}}(\{1, a\})$. By Theorem 3.3.1, $\mathcal{PML}$ is not protoalgebraic.

Thus it is not algebraizable in our sense. However, Jansana proved in [Jan02, Theorem 26] that the logic $\mathcal{PML}$ is Gentzen algebraizable and its equivalent algebraic semantics is the variety of positive modal algebras. Moreover, he proved in [Jan02, Theorem 10] that the variety of positive modal algebras is not the equivalent algebraic semantics, in our sense, of any algebraizable logic. $\diamond$

Example 6.4 (Belnap's four valued logic [Fon97]). *Belnap's four valued logic*, denoted by $\mathcal{B}$ is a finitary logic $\mathcal{B} = \langle \mathbf{Fm}_{\mathcal{L}}, \vdash_{\mathcal{B}} \rangle$ which has no axioms and is defined by the following inference rules:

R_1 $\varphi \wedge \psi \vdash_{\mathcal{B}} \varphi$ $\qquad\qquad\qquad$ R_6 $\varphi \vee \varphi \vdash_{\mathcal{B}} \varphi$

R_2 $\varphi \wedge \psi \vdash_{\mathcal{B}} \psi$ $\qquad\qquad\qquad$ R_7 $\varphi \vee (\psi \vee \eta) \vdash_{\mathcal{B}} (\varphi \vee \psi) \vee \eta$

R_3 $\varphi, \psi \vdash_{\mathcal{B}} \varphi \wedge \psi$ $\qquad\qquad\qquad$ R_8 $\varphi \vee \psi \vdash_{\mathcal{B}} \neg\neg\varphi \vee \psi$

R_4 $\varphi \vdash_{\mathcal{B}} \varphi \vee \psi$ $\qquad\qquad\qquad$ R_9 $\varphi \vee (\psi \wedge \eta) \vdash_{\mathcal{B}} (\varphi \vee \psi) \wedge (\varphi \vee \eta)$

R_5 $\varphi \vee \psi \vdash_{\mathcal{B}} \psi \vee \varphi$ $\qquad\qquad\qquad$ R_{10} $(\varphi \vee \psi) \wedge (\varphi \vee \eta) \vdash_{\mathcal{B}} \varphi \vee (\psi \wedge \eta)$

R_{11} $\neg(\varphi \vee \psi) \vee \eta \vdash_{\mathcal{B}} (\neg\varphi \wedge \neg\psi) \vee \eta$ R_{14} $(\neg\varphi \wedge \neg\psi) \vee \eta \vdash_{\mathcal{B}} \neg(\varphi \vee \psi) \vee \eta$

R_{12} $\neg(\varphi \wedge \psi) \vee \eta \vdash_{\mathcal{B}} (\neg\varphi \vee \neg\psi) \vee \eta$ R_{15} $(\neg\varphi \vee \neg\psi) \vee \eta \vdash_{\mathcal{B}} \neg(\varphi \wedge \psi) \vee \eta$

R_{13} $\neg\neg\varphi \vee \psi \vdash_{\mathcal{B}} \varphi \vee \psi$

A *De Morgan lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg \rangle$ such that:

(DM1) the reduct $\langle A, \wedge, \vee \rangle$ is a distributive lattice;

(DM2) The unary operation $\neg$ satisfies the following equations:

$$p \approx \neg\neg p \ , \ \neg(p \vee q) \approx (\neg p \wedge \neg q) \ , \ \neg(p \wedge q) \approx (\neg p \vee \neg q)$$

The variety of De Morgan lattices, denote by $\mathbf{DM}$, is generated, as a variety, by the four-element De Morgan lattice $\mathcal{M}_4$ with the universe $M_4 = \{f, n, b, t\}$ and with the algebraic structure specified by the Hasse diagram and negation table as follows:
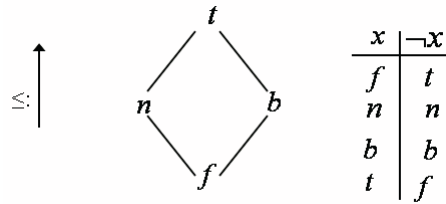
Figure 6.1: Hasse Diagram and Negation Table

It can be proved that the sets $\emptyset$ and $F_n = \{t, n\}$ are $\mathcal{B}$-filters (c.f. [Fon97, Theorem 2.11]). It is not difficult to see that $\Omega_{\mathcal{M}_4}\emptyset = \nabla_{\mathcal{M}_4}$ and $\Omega_{\mathcal{M}_4}F_n = \Delta_{\mathcal{M}_4}$. Although $\emptyset \subseteq F_n$, we have that $\Omega_{\mathcal{M}_4}\emptyset \nsubseteq \Omega_{\mathcal{M}_4}F_n$. By Theorem 3.3.1, $\mathcal{B}$ is not protoalgebraic.

Thus it is not algebraizable in our sense. However, Font proved in [Fon97, Theorem 4.11] that $\mathcal{B}$ is Gentzen algebraizable and its equivalent algebraic semantics is the variety $\mathbf{DM}$ of De Morgan lattices. The reader can see [FJ96, Chapter 5] for more information about the relation between the logic $\mathcal{B}$ and the four-element De Morgan lattice $\mathcal{M}_4$. $\Diamond$

Example 6.5 (Weaker Relevance Logic [FR94])**.** The *Weaker Relevance Logic*, denoted by $\mathbf{WR}$, is defined in the same language as the logic $\mathbf{R}$ (c.f. Example 5.16), i.e, $\mathcal{L} = \{\wedge, \rightarrow, \neg\}$. The consequence relation, $\vdash_{\mathbf{WR}}$, is defined in the following way: for

any $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathbf{WR}} \varphi \text{ iff there are } \varphi_1, \ldots, \varphi_n \in \Gamma \text{ such that } \vdash_{\mathbf{R}} (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \varphi.$$

Thus $\mathbf{WR}$ is finitary, has no theorems and for every $\varphi, \varphi_1, \ldots, \varphi_n \in \mathrm{Fm}_{\mathcal{L}}$, $\{\varphi_1, \ldots, \varphi_n\}$ $\vdash_{\mathbf{WR}} \varphi$ iff $\vdash_{\mathbf{R}} (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \varphi$. By Lemma 2.3.5, the Leibniz operator is non-injective on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. Thus the Suszko operator is non-injective on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. And by Theorem 6.0.3, $\mathbf{WR}$ is not truth-equational.

Since the only protoalgebraic logic without theorems are the trivial logic, we have that $\mathbf{WR}$ is not protoalgebraic (c.f. [FR94, Proposition 3.7]). Indeed, the set of theorems of $\mathbf{R}$ is nonempty and constitutes a proper theory of $\mathbf{WR}$. Thus $\mathbf{WR}$ is neither inconsistent nor almost inconsistent, i.e., it is not a trivial logic. For more information about the logic $\mathbf{WR}$, the reader can see [FJ96, Chapter 5] and [FR94, Chapter 3]. $\diamondsuit$

Chapter 7

Generalizations with Many-Sorted Logic

In this chapter, we study the generalization of the theory of standard AAL to many-sorted setting. This generalization is important since propositional logics are not enough expressive when we want to reason about complex systems. Thus we need logics over rich languages where elements can be distinguished by sorts. Herein, we study the theory in which logics that lack an algebraic counterpart become algebraizable in a behavior context. Actually, in the many-sorted AAL approach, the theory was developed by replacing the role of unsorted equational logic by many-sorted behavioral equational logic over the same signature and taking as unique visible sort, the sort ϕ of formulas. The paradigmatic examples are the Paraconsistent logic C_1 of da Costa and the Carnap-style presentation of modal logic S_5 which are not algebraizable in the standard sense but they are behaviorally algebraizable. However, there are many logics that are not algebraizable in any way. All the notations, notions, proofs and examples can be found in [Gon08], [CGM09] or in [CG07].

7.1 Basic Notions

Given a set A, we denote by A^*, the set of all finite strings with elements in A. If S is a set, $A = \{A_s\}_{s \in S}$ is an S-sorted or S-indexed set. Given $w = s_1 s_2 \ldots s_n \in S^*$, we denote by A_w the product $A_{s_1} \times A_{s_2} \times \cdots \times A_{s_n}$. A many-sorted signature is a pair $\Sigma = \langle S, F \rangle$ where S is a set of sorts and $F = \{F_{ws}\}_{w \in S^*, s \in S}$ is an indexed family of sets of operations. We say that a many-sorted signature $\Sigma = \langle S, F \rangle$ is n-sorted if

$n = |S|$. We write $f : s_1 \ldots s_n \to s \in F$ for an element f of $F_{s_1 \ldots s_n s}$. We denote by $T_\Sigma(X) = \{T_{\Sigma,s}(X)\}_{s \in S}$ the S-indexed family of carrier sets of the free Σ-algebra $\mathbf{T}_\Sigma(\mathbf{X})$ with generators taken from a sorted family $X = \{X_s\}_{s \in S}$ of variable sets. We write $x : s$ for $x \in X_s$. An element of $T_{\Sigma,s}(X)$ is called a Σ-*term of sort s* (an *s-term* for short). A term without variables is called a *closed term*. A many-sorted signature $\Sigma = \langle S, F \rangle$ is called *standard* if, for every $s \in S$, there exists a closed s-term. If the set of variables of a terms is finite then we often write $t \in T_\Sigma(x_1 : s_1, \ldots, x_n : s_n)$ (or simply $t(x_1 : s_1, \ldots, x_n : s_n)$). Moreover, if T is a set whose elements are all terms of this form, we write $T(x_1 : s_1, \ldots, x_n : s_n)$.

Given a many-sorted signature $\Sigma = \langle S, F \rangle$, a *substitution* over Σ is an S-indexed family of functions $e = \{e_s : X_s \to T_{\Sigma,s}(X)\}_{s \in S}$. As usual, $e(t)$ denotes the term obtained by uniformly applying e to each variable in t. Given a term $t(x_1 : s_1, \ldots, x_n : s_n)$ and terms $t_1 \in T_{\Sigma,s_1}(X), \ldots, t_n \in T_{\Sigma,s_n}(X)$, if e is a substitution such that $e_{s_1}(x_1) = t_1, \ldots, e_{s_n}(x_n) = t_n$ then we write $e(t) = t(t_1, \ldots, t_n)$. We extend this notion to any set, given $T(x_1 : s_1, \ldots, x_n : s_n)$ and $U \in T_{\Sigma,s_1}(X) \times \cdots \times T_{\Sigma,s_n}(X)$, we write $T[U] = \bigcup_{\langle t_1, \ldots, t_n \rangle \in U} T(t_1, \ldots, t_n)$, where $T(t_1, \ldots, t_n) := \{t(t_1, \ldots, t_n) : t \in T\}$.

A derived *operation of type* $s_1, \ldots, s_n \to s$ over Σ is a term in $T_{\Sigma,s}(\{x_1 : s_1, \ldots, x_n : s_n\})$ for some n. We denote by $Der_{\Sigma,s_1 \ldots s_n s}$ the set of all derived operations of type $s_1, \ldots, s_n \to s$. A *general many-sorted subsignature* of $\Sigma = \langle S, F \rangle$ is a many-sorted signature $\Gamma = \langle S, F' \rangle$ such that, for each $w \in S^*$, $F'_w \subseteq Der_{\Sigma,w}$. Let Γ be a subsignature of Σ, we say that Σ is Γ-*standard* if, for every $s \in S$ there exists a closed Γ-term of sort s, that is, a Γ-term of sort s without variables.

We assume fixed a signature $\Sigma = \langle S, F \rangle$ with a distinguished sort ϕ for formulas and a set of variables X. We define the induced set of formulas $L_\Sigma(X)$ to be the carrier set of sort ϕ of the free algebra $\mathbf{T}_\Sigma(\mathbf{X})$ with generators taken from X. We define a *structural many-sorted logic* as a pair $\mathcal{L} = \langle \Sigma, \vdash \rangle$ where Σ is a many-sorted signature and $\vdash$ is a consequence relation, such that $\langle L_\Sigma(X), \vdash \rangle$ is a logic (i.e., $\vdash$ satisfies reflexivity, cut and weakening conditions) that satisfies, for every $\Gamma \cup \{\varphi\} \subseteq L_\Sigma(X)$ and every substitution e: if $\Gamma \vdash \varphi$ then $e[\Gamma] \vdash e(\varphi)$ (c.f. [Gon08, Examples 2.1.1.5]). Propositional logics are a particular case of many-sorted logics which are called *single-sorted logics*, that is, logics over one sorted signature (i.e., $\Sigma = \langle S, F \rangle$ with $S = \{\phi\}$).

The notion of standard universal algebra is the same for many sorted signatures. However, we describe some definitions to remind the reader. Given a many-sorted signature $\Sigma = \langle S, F \rangle$, a Σ-*algebra* is a pair $\mathbf{A} = \langle \{A_s\}_{s \in S}, \_\_\mathbf{A} \rangle$ where each A_s is a

non-empty set, the *carrier of sorts s*, and $\_\mathbf{A}$ assigns to each symbol $f : s_1 \ldots s_n \to s$ of Σ a function $\underline{f}_{\mathbf{A}} : A_{s_1} \times \cdots \times A_{s_n} \to A_s$. The set of all Σ-algebras is denoted by Alg_Σ. When the signature is clear from the context we just write algebra instead of Σ-algebra. A Σ-algebra is *trivial* if each of its carriers contains exactly one element. A Σ-algebra $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$, in symbols $\mathbf{B} \subseteq \mathbf{A}$, if B is non-empty subuniverse of $\mathbf{A}$ and for each operation $f : s_1 \ldots s_n \to s$ and every $b_1 \in B_{s_1}, \ldots, b_n \in B_{s_n}$ we have that $\underline{f}_{\mathbf{B}}(b_1, \ldots, b_n) = \underline{f}_{\mathbf{A}}(b_1, \ldots, b_n)$. Given a subsignature Γ of Σ and a Σ-algebra $\mathbf{A} = \langle (A_s)_{s \in S}, \_\mathbf{A} \rangle$, the *reduct* of $\mathbf{A}$ to Γ, denoted by $\mathbf{A}_\Gamma$, is a Γ-algebra $\mathbf{A}_\Gamma = \langle (A_s)_{s \in S}, \_\mathbf{A}_\Gamma \rangle$ where $\_\mathbf{A}_\Gamma$ is the restriction of $\_\mathbf{A}$ to the operations in Γ. We can defined as usual direct product, subdirect product, reduced product, ultraproduct, etc (c.f. [MT92]).

A *homomorphism* $h : \mathbf{A} \to \mathbf{B}$ from the Σ-algebra $\mathbf{A}$ to the Σ-algebra $\mathbf{B}$ is an S-sorted set $\{h_s : A_s \to B_s\}_{s \in S}$, such that for all $f \in F_{s_1 \ldots s_n s}$, we have that $h_s(\underline{f}_{\mathbf{A}}(a_1, \ldots, a_n)) = \underline{f}_{\mathbf{B}}(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))$. We can defined as usual epimorphism, embedding, isomorphism, etc. An *assignment* of X over a Σ-algebra $\mathbf{A}$ is a family $h = \{h_s\}_{s \in S}$ such that, for every $s \in S$, $h_s : X_s \to A_s$. The *interpretation* of terms (or *denotation* of terms) is the free extension of h to $\mathbf{T}_\Sigma(\mathbf{X})$, that we also denote by h. Given a term $t(x_1 : s_1, \ldots, x_n : s_n)$ and $\langle a_1, \ldots, a_n \rangle \in A_{s_1} \times \cdots \times A_{s_n}$, we denote by $t_{\mathbf{A}}(a_1, \ldots, a_n)$ the denotation of t in $\mathbf{A}$ when the variables $x_1, .., x_n$ are interpreted by $a_1, \ldots, a_n$, respectively. Algebraically, $t_{\mathbf{A}}(a_1, \ldots, a_n) = h(t)$, where h is any assignment such that $h(x_i) = a_i$ for all $i \leq n$.

Now, we define all the notions that we need to describe many-sorted equational logic. Given a many-sorted signature Σ, we denote an *equation* over Σ by $t_1 \approx t_2$ where $t_1, t_2 \in T_{\Sigma,s}(X)$ for some $s \in S$. The sets $Eq_{\Sigma,s}(X) = \{t_1 \approx t_2 : t_1, t_2 \in T_{\Sigma,s}(X)\}$, $Eq_\Sigma(X) = \{t_1 \approx t_2 : t_1, t_2 \in T_{\Sigma,s}(X) \text{ and } s \in S\}$ and $QEq_\Sigma(X)$ denote, respectively, the set of all equations over Σ of sort s, the set of all equations over Σ and the set of all *quasi-equations* (or *conditional equations*) over Σ. Given a Σ-algebra $\mathbf{A}$, an assignment h over $\mathbf{A}$ and $t_1 \approx t_2 \in Eq_\Sigma(X)$, we write $\mathbf{A}, h \Vdash t_1 \approx t_2$ if $h(t_1) = h(t_2)$. We say that $\mathbf{A}$ *satisfies* (or, is a *model* of) $t_1 \approx t_2$ if $\mathbf{A} \Vdash t_1 \approx t_2$, that is, if $\mathbf{A}, h \Vdash t_1 \approx t_2$ for every assignment h over $\mathbf{A}$. We have similar notions for quasi-equations. Given a class $\mathbf{K}$ of Σ-algebras, the *equational consequence relation* associated with $\mathbf{K}$, denoted by $\models_\Sigma^{\mathbf{K}} \subseteq 2^{Eq_\Sigma(X)} \times Eq_\Sigma(X)$ is defined by $\Psi \models_\Sigma^{\mathbf{K}} t_1 \approx t_2$ if, for every $\mathbf{A} \in \mathbf{K}$ and every assignment h over $\mathbf{A}$, we have that $\mathbf{A}, h \Vdash r_1 \approx r_2$ for every $r_1 \approx r_2 \in \Psi$ implies $\mathbf{A}, h \Vdash t_1 \approx t_2$.

Contrary to many-sorted equational logic, in many-sorted behavioral logic, the

set of sorts is explicitly divided in the visible sorts and the hidden sorts. A *hidden many-sorted signature* is a tuple $\langle \Sigma, V \rangle$ where $\Sigma = \langle S, F \rangle$ is a many-sorted signature and $V \subseteq S$, called the set of *visible sorts*. The subset of *hidden sorts* is denoted by $H = S \setminus V$. A *hidden subsignature* of a hidden many-sorted signature $\langle \Sigma, V \rangle$ is a hidden signature $\langle \Gamma, V \rangle$ such that Γ is a many-sorted subsignature of Σ. Given a hidden subsignature Γ of Σ, a Γ-*context* for sorts s is a term $t(x : s, x_1 : s_1, \ldots, x_m : s_m) \in T_\Gamma(X)$, with a distinguished variable x of sort s and parametric variables $x_1, \ldots, x_m$ of sorts $s_1, \ldots, s_m$ respectively. We denote by $\mathcal{C}_\Sigma^\Gamma[x : s]$, the set of all Γ-contexts for sort s, and by $\mathcal{E}_\Sigma^\Gamma[x : s]$, the set of all Γ-*experiments*, which are the Γ-contexts of visible sort. When Γ is clear from the context we just write context and experiment instead of Γ-context and Γ-experiment. Given $c \in \mathcal{C}_{\Sigma, s'}^\Gamma[x : s]$ (this set denotes the set of Γ-contexts of sort s' for sort s) and $t \in T_{\Sigma, s}(X)$, we denote by $c[t]$ the term obtained from c by substituting x by t.

Let $\mathbf{A}$ be a Σ-algebra, Γ a hidden subsignature of Σ and $s \in S$. We say that $a, b \in A_s$ are Γ-*behaviorally equivalent*, in symbols $a \equiv_\Gamma b$, if for every $\epsilon(x : s, x_1 : s_1, \ldots, x_n : s_n) \in \mathcal{E}_\Sigma^\Gamma[x : s]$ and for all $\langle a_1, \ldots, a_n \rangle \in A_{s_1} \times \cdots \times A_{s_n}$, we have that $\epsilon_\mathbf{A}(a, a_1, \ldots, a_n) = \epsilon_\mathbf{A}(b, a_1, \ldots, a_n)$. Equivalently, two objects are behaviorally equivalent if they cannot be distinguished by any experiment. Given a Σ-algebra $\mathbf{A}$, an assignment h over $\mathbf{A}$ and an equation $t_1 \approx t_2$ of sort $s \in S$, we say that $\mathbf{A}$ and h Γ-*behaviorally satisfy* the equation $t_1 \approx t_2$, in symbols $\mathbf{A}, h \Vdash_\Gamma t_1 \approx t_2$ if $h(t_1) \equiv_\Gamma h(t_2)$. And we say that $\mathbf{A}$ *behaviorally satisfies* $t_1 \approx t_2$, in symbols $\mathbf{A} \Vdash_\Gamma t_1 \approx t_2$ if $\mathbf{A}, h \Vdash_\Gamma t_1 \approx t_2$ for every assignment h over $\mathbf{A}$. We define similar notions for quasi-equations. Given a class $\mathbf{K}$ of Σ-algebras, the *behavioral consequence relation* over Σ associated with $\mathbf{K}$ and Γ, $\models_\Sigma^{\mathbf{K}, \Gamma} \subseteq 2^{Eq_\Sigma(X)} \times Eq_\Sigma(X)$ is defined by $\Psi \models_\Sigma^{\mathbf{K}, \Gamma} t_1 \approx t_2$ if, for every $\mathbf{A} \in \mathbf{K}$ and every assignment h over $\mathbf{A}$, $\mathbf{A}, h \Vdash_\Gamma u_1 \approx u_2$ for every $u_1 \approx u_2 \in \Psi$ implies $\mathbf{A}, h \Vdash_\Gamma t_1 \approx t_2$. Let $\Psi(x : s)$ be a set of equations with a distinguished variable x of sort s, then we say that Ψ is a *compatible set of equations* for $\models_\Sigma^{\mathbf{K}, \Gamma}$, if $\{x_1 \approx x_2\} \cup \Psi(x_1) \models_\Sigma^{\mathbf{K}, \Gamma} \Psi(x_2)$. We denote by $Comp_\Sigma^{\mathbf{K}, \Gamma}(Y)$ the set of all compatible sets of equations for the consequence relation $\models_\Sigma^{\mathbf{K}, \Gamma}$ whose variables are contained in $Y \subseteq X$.

Given a hidden signature $\langle \Sigma, V \rangle$, a class $\mathbf{K}$ of Σ-algebras is a *hidden variety* if there exists a set $E \subseteq Eq_\Sigma(X)$ of equations such that $\mathbf{K}$ contains exactly the Σ-algebras that behaviorally satisfy all equations in E. Given a class $\mathbf{K}$ of Σ-algebras, the *hidden variety generated* by $\mathbf{K}$ is the smallest hidden variety containing $\mathbf{K}$ and it is denoted by $HV(\mathbf{K})$. We can similarly define the notion for hidden quasivariety by

considering quasi-equations instead of equations.

Given a many-sorted signature $\Sigma = \langle S, F \rangle$, we defined an *extended signature* $\Sigma^o = \langle S^o, F^o \rangle$ such that $S^o = S \uplus \{v\}$, where v is to be considered the *sort of observations* of formulas. The indexed set of operations $F^o = \{F^o_{ws}\}_{w \in (S^o)^*, s \in S^o}$ is such that:

- $F^o_{ws} = F_{ws}$ if $ws \in S^*$;

- $F^o_{\phi v} = \{o\}$;

- $F^o_{ws} = \emptyset$ otherwise.

Roughly speaking, we extend the signature with a new sort v for the observations that we can perform on formulas using operation o. The *extended hidden signature* obtained from Σ, that we also denote by Σ^o, is the pair $\langle \Sigma^o, \{v\} \rangle$, where v is intended to represent the only visible sort of the extended hidden signature. Given a signature $\Sigma = \langle S, F \rangle$, a subsignature Γ of Σ and a class $\mathbf{K}$ of Σ^o-algebras, $Bhv_\Sigma^{\mathbf{K}, \Gamma}$ designates the logic $\langle Eq_{\Sigma^o}, \models_\Sigma^{\mathbf{K}, \Gamma} \rangle$, where $\models_\Sigma^{\mathbf{K}, \Gamma}$ is the behavioral consequence relation over Σ^o associated with $\mathbf{K}$ and Γ. We can define the logic $BEqn_\Sigma^{\mathbf{K}, \Gamma} = \langle Eq_\Sigma, \models_{\Sigma, bhv}^{\mathbf{K}, \Gamma} \rangle$, where $\models_{\Sigma, bhv}^{\mathbf{K}, \Gamma}$ is just the restriction of $\models_\Sigma^{\mathbf{K}, \Gamma}$ to Σ. The set of theories of $BEqn_\Sigma^{\mathbf{K}, \Gamma}$ is denoted by $Th_\Sigma^{\mathbf{K}, \Gamma}$.

Now we define some notions of matrix semantics to the behavioral setting. Thus, a *many-sorted logical matrix* over a many-sorted signature Σ is a tuple $\mathrm{M} = \langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is a Σ-algebra and $D \subseteq A_\phi$ is the set of *designated values*. We define a *consequence relation* over Σ, denoted by $\vdash_{\mathrm{M}}$, in the following way, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$; $T \vdash_{\mathrm{M}} \varphi$ iff for every *assignment* h over $\mathbf{A}$, we have that $h[T] \subseteq D$ implies $h(\varphi) \in D$. We say that a matrix M is a *model* of a logic $\mathcal{L}$ if, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$, $T \vdash_{\mathcal{L}} \varphi$ implies $\Gamma \vdash_{\mathrm{M}} \varphi$. In this case, D is called an $\mathcal{L}$-*filter* of $\mathbf{A}$. The set of all $\mathcal{L}$-filters of $\mathbf{A}$ is denoted by $\mathrm{Fi}_{\mathcal{L}}(\mathbf{A})$ and the class of all matrix models of $\mathcal{L}$ is denoted by $\mathbf{Mod}(\mathcal{L})$.

A *congruence* on a Σ-algebra $\mathbf{A}$ is an S-sorted set $\{\theta_s\}_{s \in S}$ such that, for every $s \in S$ and every θ_s is an equivalence relation on A_s and, for each $f \in F_{s_1 \ldots s_n s}$ and each $\langle a_1, \ldots, a_n \rangle, \langle b_1, \ldots, b_n \rangle \in A_{s_1} \times \cdots \times A_{s_n}$, we have $a_1 \equiv b_1(\theta_{s_1}), \ldots, a_n \equiv b_n(\theta_{s_n})$ implies $\underline{f}_{\mathbf{A}}(a_1, \ldots, a_n) \equiv \underline{f}_{\mathbf{A}}(b_1, \ldots, b_n)(\theta_s)$. Let Γ a subsignature of the signature $\Sigma = \langle S, F \rangle$ then a Γ-*congruence* over a Σ-algebra $\mathbf{A}$ is an equivalence relation θ over $\mathbf{A}$ such that, for every $a_1 \equiv b_1(\theta_{s_1}), \ldots, a_n \equiv b_n(\theta_{s_n})$ and $f : s_1 \ldots s_n \to s \in \Gamma$, we have that $f_{\mathbf{A}}(a_1, \ldots, a_n) \equiv f_{\mathbf{A}}(b_1, \ldots, b_n)(\theta_s)$. We denote by $Con_\Gamma^\Sigma(\mathbf{A})$ the set of all Γ-congruences over a Σ-algebra $\mathbf{A}$. It is a complete sublattice of the complete lattice of

equivalence relations on $\mathbf{A}$, $Eqv^\Sigma(\mathbf{A})$. We note that the difference between this notion and the standard notion of congruence over $\mathbf{A}$ is that a Γ-congruence is assumed to satisfy the congruence property just for contexts generated from the subsignature Γ. A Γ-congruence θ over a Σ-algebra $\mathbf{A}$ is *compatible with* a set $\Phi \subseteq A_\phi$ if for every $a_1, a_2 \in A_\phi$, $a_1 \equiv a_2(\theta_\phi)$ and $a_1 \in \Phi$ implies $a_2 \in \Phi$. A *matrix Γ-congruence* over a matrix M is a Γ-congruence θ over $\mathbf{A}$ compatible with D, i.e., θ is a Γ-congruence over $\mathbf{A}$ and for every $a, b \in A_\phi$, if $a \in D$ and $a \equiv b(\theta_\phi)$ then $b \in D$. A *Γ_ϕ-congruence* over a Σ-algebra $\mathbf{A}$ is a ϕ-reduct of a Γ-congruence over $\mathbf{A}$. It is always an equivalence relation θ over A_ϕ which satisfies the condition: if $a_1 \equiv b_1(\theta), \ldots, a_n \equiv b_n(\theta)$ and $f : \phi^n \to \phi \in Der_{\Gamma,\phi^n,\phi}$ then $f_\mathbf{A}(a_1, \ldots, a_n) \equiv f_\mathbf{A}(b_1, \ldots, b_n)(\theta)$. We denote by $Con^\Sigma_{\Gamma,\phi}(\mathbf{A})$ the set of all Γ_ϕ-congruences of $\mathbf{A}$. A *matrix Γ_ϕ-congruence* over a matrix M is the ϕ-restriction of a matrix Γ-congruence.

For each $T \in \mathrm{Th}(\mathcal{L})$, there is the largest Γ-congruence compatible with T denoted by $\Omega^{bhv}_\Gamma(T)$ which we call *behavioral Leibniz congruence*. Obviously, we can defined the *behavioral Leibniz operator* on the term algebra has a function Ω^{bhv}_Γ whose domain is the set $\mathrm{Th}(\mathcal{L})$ and it associates to each $T \in \mathrm{Th}(\mathcal{L})$, the largest Γ-congruence over $\mathbf{T}_\Sigma(\mathbf{X})$ compatible with T. We also define a *Γ-behavioral Leibniz congruence* of M as the largest matrix Γ-congruence over M, which we denote by $\Omega^{bhv}_{\Gamma,\mathbf{A}}(D)$. Given a matrix M $= \langle \mathbf{A}, D \rangle$ over Σ, we have that, for every $s \in S$, $a \equiv b(\Omega^{bhv}_{\Gamma,\mathbf{A}}(D))_s$ iff for every $c(x : s, x_1 : s_1, \ldots, x_n : s_n) \in \mathcal{C}^\Gamma_{\Sigma,\phi}[x : s]$ and every $\langle a_1, \ldots, a_n \rangle \in A_{s_1} \times \cdots \times A_{s_n}$ we have that $c_\mathbf{A}(a, a_1, \ldots, a_n) \in D$ iff $c_\mathbf{A}(b, a_1, \ldots, a_n) \in D$. We denote by $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)$ the restriction of $\Omega^{bhv}_{\Gamma,\mathbf{A}}(D)$ to the sort ϕ. As we cannot perform quotients, since the behavioral Leibniz congruence is not a congruence (in general), we extend the signature and the algebras over the extended signature. Given a matrix Γ_ϕ-congruence θ over a matrix M, consider the Σ-algebra $\mathbf{A}^o_\theta$ such that $\mathbf{A}^o_{\theta|\Sigma} = \mathbf{A}$, $(A^o_\theta)_v = A_\phi/\theta = \{[a]_\theta : a \in A_\phi\}$ and $o_{\mathbf{A}^o_\theta}(a) = [a]_\phi$. Thus, we use the visible part $(A^o_\theta)_v$ to simulate the quotient. A Γ_ϕ-congruence θ over $\mathbf{A}$ is said to be a $\mathbf{K}$-Γ_ϕ-*congruence* if $\mathbf{A}^o_\theta \in \mathbf{K}$ and we denote by $Con^\mathbf{K}_\Gamma(\mathbf{A})$ the set of all $\mathbf{K}$-Γ_ϕ-congruence over $\mathbf{A}$.

7.2   The Behavioral Leibniz Hierarchy

We do not describe the Leibniz hierarchy for many-sorted logics because it is a particular case of the behavioral Leibniz hierarchy taking the subsignature Γ as the signature Σ, i.e., $\Gamma = \Sigma$. We follow the same order as we have done for the standard case,

i.e., first we study the class of behaviorally protoalgebraic logics, then the behaviorally equivalential logics and we finish with the class of behaviorally weakly algebraizable and behaviorally (finitely) algebraizable logics.

Definition 7.2.1. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. We say that $\mathcal{L}$ is Γ-behaviorally* protoalgebraic *if, for every $T \in \text{Th}(\mathcal{L})$ and $\varphi, \psi \in L_\Sigma(X)$, we have*

$$\varphi \equiv \psi\big(\Omega_\Gamma^{bhv}(T)\big) \ \text{implies} \ T \cup \{\varphi\} \vdash \psi \ \text{and} \ T \cup \{\psi\} \vdash \varphi.$$

We say that a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is behaviorally protoalgebraic *if there exists a subsignature Γ of Σ such that $\mathcal{L}$ is Γ-behaviorally protoalgebraic.*

A set $\Delta(\xi_1, \xi_2, \underline{z}) \subseteq L_\Gamma(\{\xi_1, \xi_2, \underline{z}\})$, where $\underline{z} = \langle z_1 : s_1, z_2 : s_2, \dots \rangle$ is a set of parametric variables with sort different from ϕ and at most one variable of each sort, is said a Γ-*protoequivalence system* for a many-sorted logic $\mathcal{L}$ if it satisfies the following conditions:

$$\vdash \Delta(\xi, \xi, \underline{z}) \tag{Reflexivity}$$

$$\{\xi_1\} \cup \Delta(\langle \xi_1, \xi_2 \rangle) \vdash \xi_2 \tag{Modus Ponens}$$

where $\Delta(\langle \xi_1, \xi_2 \rangle) := \{\delta_i(\xi_1, \xi_2, \underline{z}) : i \in I, \underline{z} \in L_\Gamma(X)\}$. Note that there are no parametric variables in protoequivalence system for a single-sorted logic (defined at the beginning of the Chapter 3). Here we cannot remove parametric variables because substitutions must respect each sort. However, if a behaviorally protoalgebraic logic is standard then we obtain a Γ-protoequivalence system without parameters.

A set $\Delta(\xi_1, \xi_2, \underline{z}) \subseteq L_\Gamma(\{\xi_1, \xi_2, \underline{z}\})$, where $\underline{z} = \langle z_1 : s_1, z_2 : s_2, \dots \rangle$ is a set of parametric variables, is said a *parameterized Γ-equivalence system* for a many-sorted logic $\mathcal{L}$ if it satisfies the following conditions:

$$(i) \ \vdash \Delta(\xi, \xi, \underline{z}) \tag{Reflexivity}$$

$$(ii) \ \Delta(\langle \xi_1, \xi_2 \rangle) \vdash \Delta(\langle \xi_2, \xi_1 \rangle) \tag{Symmetry}$$

$$(iii) \ \Delta(\langle \xi_1, \xi_2 \rangle) \cup \Delta(\langle \xi_2, \xi_3 \rangle) \vdash \Delta(\langle \xi_1, \xi_3 \rangle) \tag{Transitivity}$$

$$(iv) \ \{\xi_1\} \cup \Delta(\langle \xi_1, \xi_2 \rangle) \vdash \xi_2 \tag{Modus Ponens}$$

$$(v) \ \Delta(\langle \xi_1, \xi_2 \rangle) \vdash \Delta(\langle c[\xi_1], c[\xi_2] \rangle) \ \text{for every} \ c \in \mathcal{C}_{\Sigma,\phi}^\Gamma[\xi : \phi] \tag{Single Replacement}$$

In the following theorem, we give a characterization of behaviorally protoalgebraic logics as we have seen in the case of single-sorted logic, in Chapter 3.

Theorem 7.2.2. [Gon08, Theorem 3.2.9] *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. The following conditions are equivalent:*

(i) $\mathcal{L}$ is Γ-behaviorally protoalgebraic;

(ii) The behavioral Leibniz operator $\Omega^{bhv}_{\Gamma,\phi}$ is monotone on $\mathbf{Th}(\mathcal{L})$;

(iii) There exists a Γ-protoequivalence system for $\mathcal{L}$;

(iv) There exists a parameterized Γ-equivalence system for $\mathcal{L}$.

The proof of the above theorem is similar as in the case of single-sorted logic.

Now, we define behaviorally equivalential logics.

Definition 7.2.3. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. We say that $\mathcal{L}$ is Γ-behaviorally equivalential if there exists a Γ-behavioral equivalence set of formulas, that is, a set $\Delta(\xi_1, \xi_2) \subseteq L_\Gamma(\{\xi_1, \xi_2\})$ of formulas such that for every $\varphi, \psi, \delta, \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in L_\Sigma(X)$:*

(i) $\vdash \Delta(\varphi, \varphi)$ *(Reflexivity)*

(ii) $\Delta(\varphi, \psi) \vdash \Delta(\psi, \varphi)$ *(Symmetry)*

(iii) $\Delta(\varphi, \psi) \cup \Delta(\psi, \delta) \vdash \Delta(\varphi, \delta)$ *(Transitivity)*

(iv) $\Delta(\varphi, \psi) \cup \{\varphi\} \vdash \psi$ *(Modus Ponens)*

(v) $\Delta(\varphi_1, \psi_1) \cup \cdots \cup \Delta(\varphi_n, \psi_n) \vdash \Delta(c[\varphi_1, \ldots, \varphi_n], c[\psi_1, \ldots, \psi_n])$ for every $c : \phi^n \to$
$\phi \in Der_{\Gamma,\phi}$ *(Replacement)*

We say that a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is behaviorally equivalential *if there exists a subsignature Γ of Σ such that $\mathcal{L}$ is Γ-behaviorally equivalential.*

Note that the main difference between this behavioral version of equivalential logic and the standard notion is that in the former the set Δ is not assumed to define a congruence, i.e, an equivalence relation compatible with all operations.

We define the set $T^\Gamma_{\xi_1,\xi_2} := \{\varphi(\xi_1, \xi_2, \underline{z}) : \emptyset \vdash \varphi(\xi_1, \xi_1, \underline{z})\}$ as the set of formulas that becomes theorems of $\mathcal{L}$ after the identification of the variables ξ_1 and ξ_2 in φ. In the following proposition and theorem, we give a characterization of class of logics as we have seen for single-sorted logic (c.f. Chapter 4).

Proposition 7.2.4. [Gon08, Proposition 3.2.16] *Let* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *be a many-sorted logic,* Γ *a subsignature of* Σ *and* $\Delta(\xi_1, \xi_2) \subseteq L_\Gamma(\{\xi_1, \xi_2\})$ *a set of formulas. Then,*

(i) *if* $\Delta(\xi_1, \xi_2)$ *is a* Γ-*behavioral equivalence set for* $\mathcal{L}$ *then, for every* $T \in \text{Th}(\mathcal{L})$ *and* $\varphi, \psi \in L_\Sigma(X)$, *we have that*

$$\varphi \equiv \psi\left(\Omega^{bhv}_{\Gamma,\phi}(T)\right) \text{ iff } \Delta(\varphi, \psi) \subseteq T.$$

(ii) *Herrmann's Test: suppose* $\mathcal{L}$ *is* Γ-*behaviorally protoalgebraic. Then,* $\Delta(\xi_1, \xi_2) \subseteq L_\Gamma(\{\xi_1, \xi_2\})$ *is a* Γ-*behavioral equivalence set for* $\mathcal{L}$ *iff it satisfies,*

$$\Delta(\xi_1, \xi_2) \subseteq T^\Gamma_{\xi_1, \xi_2} \text{ and } \xi_1 \equiv \xi_2\left(\Omega^{bhv}_{\Gamma,\phi}\left(Cn_\mathcal{L}(\Delta(\xi_1, \xi_2))\right)\right).$$

Theorem 7.2.5. [Gon08, Theorem 3.2.17] *Let* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *be a many-sorted logic and* Γ *a subsignature of* Σ. *If* $\mathcal{L}$ *is* Γ-*standard then the following conditions are equivalent:*

(i) $\mathcal{L}$ *is* Γ-*behaviorally equivalential;*

(ii) *The behavioral Leibniz operator* $\Omega^{bhv}_{\Gamma,\phi}$ *is monotone and commutes with inverse substitutions on* $\textbf{Th}(\mathcal{L})$.

Herein, we study the class of behaviorally weakly algebraizable logics and the class of behaviorally (finitely) algebraizable logics.

Definition 7.2.6. *Let* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *be a many-sorted logic and* Γ *a subsignature of* Σ. *We say that* $\mathcal{L}$ *is* Γ-*behaviorally weakly algebraizable if there exist a class* $\textbf{K}$ *of* Σ^o-*algebras, a set* $\Theta(\xi, \underline{z}) \subseteq Comp^{\textbf{K}, \Gamma}_\Sigma(X)$ *of* ϕ-*equations and a set* $\Delta(\xi_1, \xi_2, \underline{w}) \subseteq L_\Gamma(\{\xi_1, \xi_2, \underline{w}\})$ *of formulas such that, for every* $T \cup \{\varphi\} \subseteq L_\Sigma(X)$ *and for every set* $\Phi \cup \{\varphi_1 \approx \varphi_2\}$ *of* ϕ-*equations,*

(i) $T \vdash \varphi$ *iff* $\Theta[\langle T \rangle] \models^{\textbf{K},\Gamma}_{\Sigma,bhv} \Theta(\langle \varphi \rangle)$;

(ii) $\Phi \models^{\textbf{K},\Gamma}_{\Sigma,bhv} \varphi_1 \approx \varphi_2$ *iff* $\Delta[\langle \Phi \rangle] \vdash \Delta(\langle \varphi_1, \varphi_2 \rangle)$;

(iii) $\varphi \dashv\vdash \Delta[\langle \Theta(\langle \varphi \rangle) \rangle]$;

(iv) $\varphi_1 \approx \varphi_2 \models\!\!\!\models^{\textbf{K},\Gamma}_{\Sigma,bhv} \Theta[\langle \Delta(\langle \varphi_1, \varphi_2 \rangle) \rangle]$.

We say that a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *is* behaviorally weakly algebraizable *if there exists a subsignature* Γ *of* Σ *such that* $\mathcal{L}$ *is* Γ-*behaviorally weakly algebraizable.*

As in standard AAL, conditions (*i*) and (*iv*) jointly imply (*ii*) and (*iii*), and vice-versa.

Theorem 7.2.7. [Gon08, Theorem 3.2.15] *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. The following conditions are equivalent:*

(i) *$\mathcal{L}$ is Γ-behaviorally weakly algebraizable;*

(ii) *The behavioral Leibniz operator $\Omega_{\Gamma,\phi}^{bhv}$ is monotone and injective on $\mathbf{Th}(\mathcal{L})$.*

Definition 7.2.8. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. We say that $\mathcal{L}$ is Γ-behaviorally algebraizable if there exist a class $\mathbf{K}$ of Σ^o-algebras, a set $\Theta(\xi) \subseteq Comp_{\Sigma}^{\mathbf{K},\Gamma}(\{\xi\})$ of ϕ-equations and a set $\Delta(\xi_1, \xi_2) \subseteq L_\Gamma(\{\xi_1, \xi_2\})$ of formulas such that, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$ and every set $\Phi \cup \{\varphi_1 \approx \varphi_2\}$ of ϕ-equations,*

(i) $T \vdash \varphi$ *iff* $\Theta[T] \models_{\Sigma,bhv}^{\mathbf{K},\Gamma} \Theta(\varphi)$;

(ii) $\Phi \models_{\Sigma,bhv}^{\mathbf{K},\Gamma} \varphi_1 \approx \varphi_2$ *iff* $\Delta[\Phi] \vdash \Delta(\varphi_1, \varphi_2)$;

(iii) $\varphi \dashv\vdash \Delta[\Theta(\varphi)]$;

(iv) $\varphi_1 \approx \varphi_2 \mathrel{\dashv\models}_{\Sigma,bhv}^{\mathbf{K},\Gamma} \Theta[\Delta(\varphi_1, \varphi_2)]$.

We say that a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is behaviorally algebraizable if there exists a subsignature Γ of Σ such that $\mathcal{L}$ is Γ-behaviorally algebraizable.

As in standard AAL, Θ is called the set of *defining equations*, Δ the set of *equivalence formulas* and $\mathbf{K}$ a *behaviorally equivalent algebraic semantics* for $\mathcal{L}$. If the set of defining equations and the set of equivalence formulas are finite and without parameters, we say that $\mathcal{L}$ is *finitely Γ-behaviorally algebraizable*. The difference between behaviorally weakly algebraizable logic and behaviorally algebraizable logic is that in the former both the set of equivalence formulas and the set of defining equations may have parametric variables.

Theorem 7.2.9. [Gon08, Theorem 3.2.18] *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. If $\mathcal{L}$ is Γ-standard, then the following conditions are equivalent:*

(i) *$\mathcal{L}$ is Γ-behaviorally algebraizable;*

(ii) *The behavioral Leibniz operator $\Omega_{\Gamma,\phi}^{bhv}$ is monotone, injective and commutes with inverse substitutions on $\mathbf{Th}(\mathcal{L})$.*

As in the Chapter 5, we give a sufficient condition on the behavioral equivalence set for a behaviorally equivalential logic becomes behaviorally algebraizable.

Corollary 7.2.10. [Gon08, Corollary 3.3.2] *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. A sufficient condition for $\mathcal{L}$ to be Γ-behaviorally algebraizable is that it is Γ-behaviorally equivalential with Γ-behavioral equivalence set $\Delta(\xi_1, \xi_2)$ that also satisfies:*

$$\{\xi_1, \xi_2\} \vdash \Delta(\xi_1, \xi_2) \qquad\qquad\qquad \text{(G-rule)}$$

In this case, $\Delta(\xi_1, \xi_2)$ is the set of equivalence formulas and $\Theta(\xi) = \{\xi \approx \delta(\xi, \xi) : \delta \in \Delta\}$ the set of defining equations for $\mathcal{L}$.

In the behavioral algebraization process we cannot have a uniqueness result because it is parameterized by the choice of the subsignature Γ. Nevertheless, it is interesting to note that, once the subsignature Γ is fixed, we can prove the uniqueness result as Blok and Pigozzi have proved in [BP89, Theorem 2.15].

Theorem 7.2.11. [Gon08, Theorem 4.1.2] *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a Γ-behaviorally algebraizable many-sorted logic, where Γ is a subsignature of Σ, and let $\mathbf{K}$ and $\mathbf{K}'$ be two Γ-behaviorally equivalent algebraic semantics for $\mathcal{L}$ such that $\Delta(\xi_1, \xi_2)$ and $\Theta(\xi)$ are the equivalence formulas and defining equations for $\mathbf{K}$, and similarly $\Delta'(\xi_1, \xi_2)$ and $\Theta'(\xi)$ for $\mathbf{K}'$. Then, $\models_{\Sigma, bhv}^{\mathbf{K}, \Gamma} = \models_{\Sigma, bhv}^{\mathbf{K}', \Gamma}$, $\Delta(\xi_1, \xi_2) \dashv\vdash \Delta'(\xi_1, \xi_2)$ and $\Theta(\xi) \mathrel{=\!\!|\!\!|\!\!=}_{\Sigma, bhv}^{\mathbf{K}, \Gamma} \Theta'(\xi)$.*

Thus we can consider *the largest Γ-behaviorally equivalent algebraic semantics*, denoted by $\mathbf{K}_{\mathcal{L}}^{\Gamma}$. But contrarily to the case of standard AAL, in this approach $\mathbf{K}_{\mathcal{L}}^{\Gamma}$ is not the class of algebras that should be canonically associated with $\mathcal{L}$. Indeed, it is a subclass of $\mathbf{K}_{\mathcal{L}}^{\Gamma}$ that will allow us to generalize the standard results of AAL (c.f. [Gon08]).

Moreover, if $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is a many-sorted finitary and finitely Γ-behaviorally algebraizable logic for some subsignature Γ of Σ and if $\mathbf{K}$ and $\mathbf{K}'$ are two Γ-behaviorally equivalent algebraic semantics for $\mathcal{L}$ then $\mathbf{K}$ and $\mathbf{K}'$ generate the same Γ-hidden quasivariety, i.e., $\mathbf{K}$ and $\mathbf{K}'$ Γ-behaviorally satisfy the same quasi-equations. Therefore, this Γ-hidden quasivariety is also a Γ-behaviorally equivalent algebraic semantics for $\mathcal{L}$ and we can construct a basis for the quasi-equations of this unique equivalent Γ-hidden quasivariety semantics given an axiomatization of $\mathcal{L}$, in the following theorem.

Theorem 7.2.12. [Gon08, Theorem 4.1.3]

Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a finitary many-sorted logic defined by a set of axioms AX *and*

a set of inference rules IR *and consider* Γ *a subsignature of* Σ. *If* $\mathcal{L}$ *is finitely* Γ-*behaviorally algebraizable with the set of defining equations* $\Theta(\xi)$ *and the set of equivalence formulas* $\Delta(\xi_1, \xi_2)$, *then the unique equivalent* Γ-*hidden quasivariety semantics for* $\mathcal{L}$ *is axiomatized by the following equations and quasi-equations:*

(i) $\Theta(\varphi)$, *for every theorem* φ *of* $\mathcal{L}$;

(ii) $\Theta[\Delta(\xi, \xi)]$;

(iii) $\Theta(\psi_1) \wedge \cdots \wedge \Theta(\psi_n) \to \Theta(\varphi)$ *for every* $\langle \psi_1, \ldots, \psi_n, \varphi \rangle \in$ IR;

(iv) $\Theta[\Delta(\xi_1, \xi_2)] \to \xi_1 \approx \xi_2$.

The following theorem is a semantic characterization of behaviorally algebraizable logics.

Theorem 7.2.13. [Gon08, Theorem 4.2.8]

Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *be a many-sorted logic,* Γ *a subsignature of* Σ *and* **K** *a class of* Σ^o-*algebras.*

1. *The following conditions are equivalent:*

   (i) $\mathcal{L}$ *is* Γ-*behaviorally algebraizable and* **K** *is the* Γ-*behaviorally equivalent algebraic semantics;*

   (ii) *For every* Σ-*algebra* **A** *we have that* $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}$ *is an isomorphism between the lattices of* $\mathcal{L}$-*filters and* **K**-Γ_ϕ-*congruences of* **A**, *that commutes with inverse substitutions.*

2. *Assume* $\mathcal{L}$ *is* Γ-*behaviorally algebraizable with* **K** *the* Γ-*behaviorally equivalent algebraic semantics. Let* $\Theta(\xi)$ *be the set of defining equations for* **K**. *For each* Σ-*algebra* **A** *and* Γ_ϕ-*congruence* θ *of* **A** *define:*

$$H_{\mathbf{A}}(\theta) = \{a \in A_\phi : \gamma_{\mathbf{A}}(a) \equiv \delta_{\mathbf{A}}(a)(\theta), \text{ for every } \gamma \approx \delta \in \theta\}.$$

Then $H_{\mathbf{A}}$ *restricted to the* **K**-Γ_ϕ-*congruences of* **A** *is the inverse of* $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}$.

We have seen that all the classes of logics can be characterized by properties of the Leibniz operator. Thus, we present, in the following figure (c.f. [Gon08, figure 3.2]), the behavioral Leibniz hierarchy.
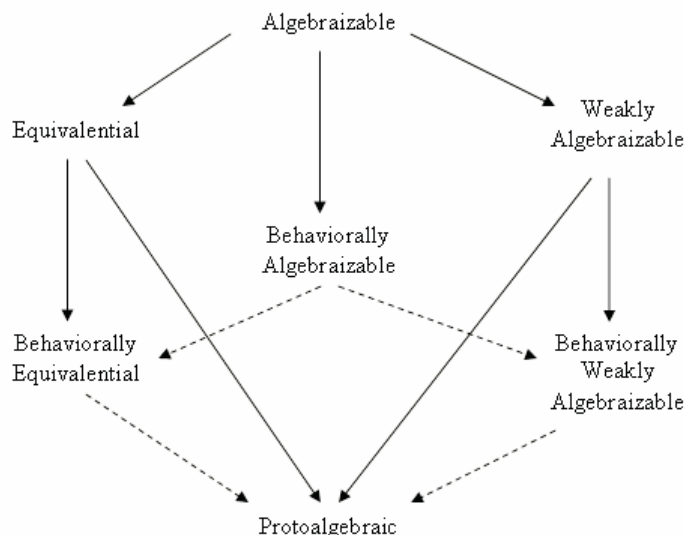
Figure 7.1: A view of the Behavioral Leibniz Hierarchy

7.3 Examples

Herein, we only present some examples of behaviorally algebraizable logics which appear in [Gon08, Chapter 5].

Example 7.4 (First Order Classical Logic)**.** The problem of algebraizing predicate logic is of a different character than the problem for propositional logics because the standard deductive systems for predicate logic are not structural. Indeed, the individual variables may be free or bound variables and it is due to the process of substituting terms for the free occurrences of an individual variables in a formula. Thus, there are different approaches to the algebraization of First Order Logic (FOL). One give rise to the variety of cylindric algebras (c.f [BP86, Appendix C] and [CG07]) and the other to polyadic algebras. In the cylindric approach, FOL is view as a single-sorted logic (i.e., as a structural propositional logic), called PR, where atomic formulas of FOL have to be represented as propositional variables in PR and PR is algebraizable with the variety of cylindric algebras has an equivalent algebraic semantics. But we can see FOL as a two-sorted logic, with a sort for terms and a sort for formulas. Thus, FOL can be shown many-sorted algebraizable with the variety of two-sorted version of cylindric algebras as equivalent algebraic semantics. $\diamondsuit$

Example 7.5 (Paraconsistent Logic $\mathcal{C}_1$ of da Costa)**.** The logic $\mathcal{C}_1 = \langle \Sigma_{\mathcal{C}_1}, \vdash_{\mathcal{C}_1} \rangle$, where the single-sorted signature $\Sigma_{\mathcal{C}_1} = \langle \{\phi\}, F \rangle$ is such that $F_\phi = \{t, f\}$, $F_{\phi\phi} = \{\neg\}$,

$F_{\phi\phi\phi} = \{\wedge, \vee, \rightarrow\}$ and $F_{ws} = \emptyset$ otherwise; is the paraconsistent logic of da Costa. A logic is said to be *paraconsistent* if its consequence relation is not *explosive* with respect to a negation connective $\neg$ (that is, if for all formulas φ and ψ, $\{\varphi, \neg\varphi\} \vdash \psi$). Moreover, $\mathcal{C}_1$ is a non-truth-functional logic, namely it lacks congruence for paraconsistent negation connective with respect to the equivalence $\leftrightarrow$ that algebraizes the Classical Propositional logic fragment. In general, it may happen that $\vdash_{\mathcal{C}_1} (\varphi \leftrightarrow \psi)$ but $\nvdash_{\mathcal{C}_1} (\neg\varphi \leftrightarrow \neg\psi)$. It was proved in [LMS91] that this logic is not algebraizable in the standard sense (it lack congruence for paraconsistent negation) but it is behaviorally algebraizable by taking as subsignature Γ all connectives without paraconsistent negation (c.f. [Gon08, Example 2.3.26] and [CGM09, Theorem 19]). The algebraic counterpart is the class of the so-called da Costa algebras. $\diamond$

Example 7.6 (Carnap-style presentation of modal logic S_5)**.** The logic $S_5 = \langle \Sigma_{S_5}, \vdash_{S_5} \rangle$ where the single-sorted signature $\Sigma_{S_5} = \langle \{\phi\}, F \rangle$ is such that $F_{\phi\phi} = \{\neg, \Box\}$, $F_{\phi\phi\phi} = \{\wedge, \vee, \rightarrow\}$ and $F_{ws} = \emptyset$ otherwise; is the modal logic S_5. This logic can be seen as an extension of **CPL**, with the modality $\Box$. Although S_5 is non-algebraizable in the standard sense (it lack congruence of its modal operator $\Box$), we can identify an algebraizable fragment of it, namely **CPL**. The logic S_5 is Γ-behaviorally algebraizable with a subsignature Γ that does not contain the modal operator $\Box$ and the behaviorally equivalent algebraic semantics is a class of algebras based on Boolean algebras which is the algebraic counterpart of **CPL**. $\diamond$

Since the class of behaviorally protoalgebraic logics coincides with the class of protoalgebraic logics, we have that the $\{\wedge, \vee\}$-fragment of **CPL** (c.f. Example 6.2) is not behaviorally protoalgebraic. Consequently, it is not behaviorally algebraizable. Thus, this generalization does not trivialize the notion of algebraization.

Chapter 8

Conclusion

In this dissertation, we studied several classes of logics. We initiated with the wider class of protoalgebraic logics which coincides with the class of non-pathological logics defined by Czelakowski. This class admits various characterizations, namely by Leibniz operator properties, by the existence of a parameterized system of equivalence formulas, by properties of the class of reduced matrices models and by the existence of a parameterized local deduction-detachment theorem. Another characterization of this class is when the Suszko operator coincides with the Leibniz operator. Since, for us, a logic $\langle \mathrm{Fm}_{\mathcal{L}}, \vdash_S \rangle$ is not necessarily finitary, we examine, throughout this thesis, some properties concerning finitary logics.

Afterwards, we examined the class of equivalential logics defined as the logics having a system of equivalence formulas. Obviously, it is a proper subclass of protoalgebraic logics. Herrmann´s test gives conditions for a protoalgebraic logic becomes equivalential. When the system of equivalence formulas can be taken finite, we have finitely equivalential logics. The class of finitely equivalential logics is a proper subclass of equivalential logics. For both classes, we also have characterizations using properties of the Leibniz operator and properties of the class of reduced matrix models.

We also analyzed the class of weakly algebraizable logics which contains the class of algebraizable logics and the class of finitely algebraizable logics. As in the other classes, we can give a characterization using properties of the Leibniz operator. A paradigmatic example of finitely algebraizable logic is the **CPL** for which the class **BA** can be associated with a meaningful relationship.

Although, most of the classes investigated in the literature are protoalgebraic logics, there are many others logics which are non-protoalgebraic. Among them, we have some

logics which belong to the class of truth-equational logics. This latter class is constituted by logics for which the class $\mathbf{L}^*_{Su}(S)$ has its filters equationally definable. Thus it contains some non-protoalgebraic logics and the class of weakly algebraizable logics. Besides, we have seen logics which are neither protoalgebraic nor truth-equational.

All classes of logics that we have studied, can be characterized by properties of the Leibniz Operator. This common point, enables the elaboration of a Leibniz Hierarchy.
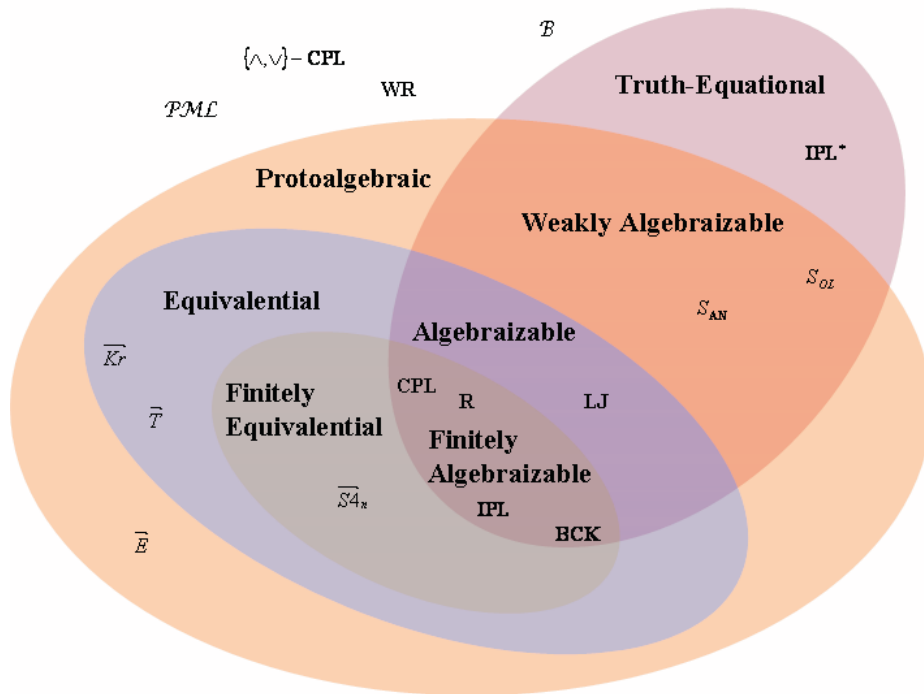
Figure 8.1: Leibniz Hierarchy and some Examples

We have examined each class for single-sorted logics. However, we have seen that this study of classes can be generalized for many-sorted logics. In order to capture some phenomena of behaviorally algebraization, we use the behavioral Leibniz operator for characterizing each new class. For instance, the Paraconsistent logic of da Costa is not algebraizable in a standard sense but it is behaviorally algebraizable.

8.0.1 Future work

There are still many open problems in this area: some classic ones and many others that have emerged through the behavioral approach. For example, we have seen that a logic S is finitary and finitely equivalential iff the class $\mathbf{Mod}^*(S)$ of reduced matrix

models is a matrix-quasivariety (Theorem 4.3.1). Consequently, the class $\mathbf{Alg}^*(S)$ of algebraic reducts of the reduced matrix models is also a quasivariety. But the fact that the class $\mathbf{Alg}^*(S)$ is a quasivariety does not imply that the logic S is finitary and finitely equivalential. As a counter-example, we have that $\overrightarrow{Kr}$ is a finitary equivalential logic which is not finitely equivalential (c.f. Example 4.5); however, the class $\mathbf{Alg}^*(\overrightarrow{Kr})$ is a variety, namely the class of modal algebras (c.f. [Wój88, Theorem 3.6.5]). We may investigate conditions, weaker than finitely equivalential, which $\mathbf{Alg}^*(S)$ must be a quasivariety. Obviously, this new class includes the class of finitary and finitely equivalential logics.

In the behavioral approach, the class of behaviorally finitely equivalential logics has not been considered yet. However, we have already obtained similar results to the ones on the standard case, namely the class of behaviorally finitely algebraizable logics can be characterized by the property of Behavioral Leibniz operator being continuous on $\mathbf{Fi}_S(\mathbf{A})$ for every algebra $\mathbf{A}$. This class can capture some phenomena occurred within modal logics (c.f. [Mal89] and [BM]).

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