Universidade de Aveiro Departamento de Física
2010

Pedro Miguel Ribeiro Cipriano

Simulações Numéricas de Jogos de Recursos Públicos

Numerical Simulations of Public Goods Games

Universidade de Aveiro Departamento de Física
2010

Pedro Miguel Ribeiro Cipriano

Simulações Numéricas de Jogos de Recursos Públicos

Numerical Simulations of Public Goods Games

Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Mestre em Física, realizada sob a orientação científica do Dr. António Luis Ferreira, Professor Associado do Departamento de Física da Universidade de Aveiro

A Claudino Domingues Ribeiro, o meu avó materno, que desde sempre pelo seu exemplo me estimulou a curiosidade e a vontade de saber sempre mais. Recordo com saudade as adivinhas e as histórias, que foram sem dúvida um dos factores mais importantes para manter os meus horizontes abertos.

O júri

Presidente

Prof. Dr. Armando Neves
Professor associado departamento de Física da Universidade de Aveiro

Prof. Dr. António Luís Ferreira
Professor associado no departamento de Física da Universidade de Aveiro

Prof. Dr. Francisco C. Santos
Investigador auxiliar no Centro de Inteligência Artificial CENTRIA da Universidade Nova de Lisboa

Agradecimentos

Em primeiro lugar o meu orientador, Professor Dr. António Luís Ferreira, por tornar este trabalho possível.
Aos meus colegas pela ajuda que me foram dando, nem só a nível prático, mas também de apoio moral. Citando alguns, mas não todos: Daniel Gouveia, Marinho Lopes, Lidia del Rio, Diogo Baptista, Bruno Coutinho, Nuno Fortunato, João Ferreira, Ricardo Albuquerque e Pedro Silva.
A Ana Piedade, um apoio sempre presente.
Ao Professor Dr. Manuel Barroso e a todos os que cuidam dos equipamentos informáticos, que sem eles as simulações teriam levado anos.
Ao Professor Dr. Sergey Dorogovtsev pelas suas aulas de Complementos de Física Estatística onde tomei o primeiro contacto com o mundo das redes. Novamente ao Professor Dr. António Luís Ferreira pelas suas aulas de Sistemas Complexos e Desordenados onde aprendi que Teoria de Jogos também pode ser estudada por físicos. Ao Professor Dr. Ricardo Dias por estar sempre pronto a ajudar.
A minha família em geral, pelo apoio incondicional.

Palavras-Chave

Teoria de jogos evolutiva, jogos de recursos publicos, redes scale-free, redes regulares, cadeias de Markov, simulações numericas, processo estocástico.

Resumo

Foram simulados numericamente jogos de recursos públicos em redes usando algoritmo de Monte Carlo. Foram usadas redes regulares unidimensionais em anel, redes regulares bidimensionais (rede quadrada) e redes scale-free. São apresentados os métodos seguidos, a teoria e os algoritmos usados. Estes jogos apresentam uma transição de fase entre uma fase dominada por oportunistas de uma fase dominada por cooperadores em função de um parâmetro de rendimento das contribuições. Foi encontrado um intervalo, dependente do número médio de vizinhos, para o qual a fracção de configurações sobreviventes tende para 1 quando o tamanho da rede aumenta. Foi também encontrada uma dependência no valor de parâmetro crítico de transição no número médio de vizinhos para as configurações sobreviventes. Esses efeitos foram observados em todos os tipos de rede estudados neste trabalho.

Keywords

Evolutionary game theory, public goods games, scale-free networks, regular networks, Markov chains, numerical simulation, stochastic process

Abstract

Public goods games were numerically simulated in networks using Monte Carlo Algorithm. Regular one-dimensional ring networks, regular two-dimensional lattice networks and scale-free networks had been used. The methods followed, the theory and the algorithms used are presented. This games have a phase transition between one phase dominated by defectors from one dominated by cooperators in function of the value of efficiency from the contributions. It was found an interval, dependent on the average number of neighbors, where the fraction of surviving configurations tens to 1 when the size of the network increases. It was found dependence in the critical value of transition value with the average number of neighbors. Both effects were observed in all types of networks studied in this work.

Contents

List of Figures **3**

List of Tables **5**

Introduction **6**

1 Evolutionary Game Theory **7**

2 Public Goods Games **10**
- 2.1 Networks . . . 10
- 2.2 Types of Games . . . 12
- 2.3 Evolution . . . 13

3 Markov Chains **14**
- 3.1 Algorithm . . . 17

4 Results for one-dimensional regular ring networks **19**
- 4.1 Asynchronous Update . . . 19
  - 4.1.1 One cooperator surrounded by defectors . . . 19
  - 4.1.2 A number $l \leq z/2+1$ cooperators surrounded by defectors . . . 20
  - 4.1.3 Between $z/2+1$ and $z+1$ cooperators surrounded by defectors . . . 21
  - 4.1.4 More than $z+1$ cooperators surrounded by defectors . . . 21
  - 4.1.5 One defector surrounded by cooperators . . . 21
  - 4.1.6 A number $l \leq z/2+1$ defectors surrounded by cooperators . . . 22
  - 4.1.7 Between $z/2+1$ and $z+1$ defectors surrounded by cooperators . . . 23
  - 4.1.8 More than $z+1$ defectors surrounded by cooperators . . . 23
  - 4.1.9 Initial random distribution of cooperators and defectors . . . 24
- 4.2 Synchronous Update . . . 29
  - 4.2.1 One cooperator surrounded by defectors . . . 29
  - 4.2.2 One defector surrounded by cooperators . . . 29
  - 4.2.3 Initial random distribution of cooperators and defectors . . . 30

5 Results for two-dimensional lattice regular networks **33**
- 5.1 One cooperator surrounded by defectors . . . 33
- 5.2 One defector surrounded by cooperators . . . 35
- 5.3 Random initial distribution of defectors and cooperators . . . 36

6 Results for scale-free networks **41**
- 6.1 Game A . . . 41
- 6.2 Game B . . . 44

7 Future Work **48**

Conclusion **49**

Bibliography **50**

List of Figures

2.1 Degree distribution for regular networks and scalefree networks 10
2.2 Schematics for different kinds of networks 11
2.3 Analisis of a generated scalefree network 12

3.1 Algorithm for the program with asyncronous update 17
3.2 Algorithm for the program with syncronous update 18

4.1 Fraction of configurations which fall in the state with only defectors for one-dimensional ring regular networks starting from configurations with only one colaborator with asynchronous update 20
4.2 Fraction of configurations for the syncronous update in the one-dimensional regular ring network with $z = 2$ 25
4.3 Fraction of configurations for the syncronous update in the one-dimensional regular ring network with $z = 4$ 25
4.4 Fraction of configurations for the syncronous update in the one-dimensional regular ring network with $z = 6$ 26
4.5 Fraction of configurations for the syncronous update in the one-dimensional regular ring network with $z = 16$ 26
4.6 Cooperator's density in the surviving configurations for the syncronous update in the one-dimensional regular ring network with $z = 2$ 27
4.7 Cooperator's density in the surviving configurations for the syncronous update in the one-dimensional regular ring network with $z = 4$ 27
4.8 Cooperator's density in the surviving configurations for the syncronous update in the one-dimensional regular ring network with $z = 6$ 28
4.9 Cooperator's density in the surviving configurations for the syncronous update in the one-dimensional regular ring network with $z = 16$ 28
4.10 Fraction of configurations for one-dimensional regular ring networks starting from configurations with only one defector 29
4.11 Fraction of configurations with only cooperatorsfor one-dimensional regular ring networks starting from configurations with only one defector 30
4.12 Total fraction of cooperators for one-dimensional regular ring networks starting from a random distribution of states 30
4.13 Fraction of cooperators in the surviving configurations for one-dimensional regular ring networks starting from a random distribution of states 31
4.14 Fraction of configurations for one-dimensional regular ring networks starting from a random distribution of states 32
4.15 Fraction of configurations for one-dimensional regular ring networks starting from a random distribution of states 32

5.1 Neighbors of a given site A in regular two-dimensional networks 33

5.2 Fraction of different configurations for the two-dimensional regular network starting from from the configuration one cooperator surrounded by defectors 34
5.3 Fraction of different configurations for the two-dimensional regular network starting from the configuration with one defector surrounded by cooperators 36
5.4 Classes of sites in regular two-dimensional networks 37
5.5 Total fraction of cooperators in regular two-dimensional lattice networks 38
5.6 Fraction of cooperators in the surviving configurations for regular two-dimensional lattice network using synchronous update 38
5.7 Fraction of cooperators in the surviving configurations for regular two-dimensional lattice network with $z = 4$ using synchronous update 39
5.8 Fraction of configurations for the regular two-dimensional lattice network with $z = 8$ using synchronous update 39
5.9 Fraction of ocnfigurations for the regular two-dimensional lattice network with $z = 12$ using synchronous update 40

6.1 Total fraction of cooperators in scale-free networks in the game A 41
6.2 Fraction of configurations for scale-free network with $N = 1000$ using the game A with synchronous update 42
6.3 Fraction of configurations for scale-free network with $z = 16$ using the game A with synchronous update 42
6.4 Fraction of configurations for scale-free network with $z = 8$ using the game A with synchronous update 43
6.5 Fraction of cooperators in the surviving configurations for scale-free network with $N = 1000$ using the game A with synchronous update 44
6.6 Total fraction of cooperators in scale-free networks in the game B 44
6.7 Fraction of configurations for scale-free network with $N = 1000$ using the game B with synchronous update 45
6.8 Fraction of surviving configurations for scale-free network with $z = 6$ using the game B with synchronous update 46
6.9 Number of configurations for scale-free network with $z = 8$ using the game B with synchronous update 46
6.10 Fraction of cooperators in the surviving configurations for scale-free network with $N = 1000$ using the game B with synchronous update 47

List of Tables

1.1 Payoff matrix for the Hawk-Dove game 7
1.2 Payoff matrix for Alice's game 8
1.3 Payoff matrix for the Prisoner's Dillema 9

4.1 One cooperator surrounded by defectors 19
4.2 Expected maximum values of η for which is possible to have configurations with only defectors 20
4.3 A number $l \leq z/2+1$ cooperators surrounded by defectors 20
4.4 Between $z/2+1 < l \leq z+1$ cooperators surrounded by defectors 21
4.5 More than $z+1$ cooperators surrounded by defectors 21
4.6 One defector surrounded by cooperators 22
4.7 Expected minimum values of η for which it is possible to have configurations with only cooperators 22
4.8 A number $l \leq z/2+1$ defectors surrounded by cooperators 22
4.9 Between $z/2+1 < l \leq z+1$ defectors surrounded by cooperators 23
4.10 More than $z+1$ defectors surrounded by cooperators 23
4.11 Interface between more than $z+1$ cooperators and more than $z+1$ defectors . . 24
4.12 Expected critical values of η for regular one-dimensional ring network with asynchronous update 24

5.1 Expected values of η for which one cooperator is able to win over the defectors for two-dimensional lattice networks 34
5.2 Expected values of η for which one defector is defeated by the cooperators for two-dimensional lattice networks 35

Introduction

Adam Smith published in 1776 the book *An Inquiry into the Nature and Causes of the Wealth of Nations*. That book was for economists as *Philosophiae Naturalis Principia Mathematica* was for physicists. Naturally, economics went beyond Smith's book as physics did with Newton's, so game theory is for economists as relativity is to physicists [1]. The importance of the book had gone beyond economics: the historian Silvan Schweber was the first to point out that Smith's book had influence on Darwin. Stephen Jay Gould wrote : " In fact, I would advance the even stronger claim that the theory of natural selection is, in essence, Adam Smith's economics transfered to nature..." [2]. It is not suprising that in nowdays evolutionary game theory is an object of study of both economists and biologists.

In the sixties, Stanley Milgram conducted a famous experiment [3], some people in Nebraska were instructed to send a parcel to someone they knew personally who in turn could forward it to another acquaintance with the eventual goal of reaching a Boston-area stockbroker. On average, it took less that six steps to reach the goal. Sugesting, that two people could be connected by less than "Six degrees of separation". That experiment introduced the ideia that human relations could be described as a network. In 1999 a milestone paper had been published in *Science* by R*é*ka Albert and Albert-L*á*szl*ó* Barab*á*si [4]. In that paper they noted the scale-free nature of many kinds of networks. In scale-free networks many lonely nodes will have almost no connections at all, some nodes will be moderately well connected, and few will be superconnected hubs. Those kind of networks obey a power law, being the World Wide Web one of the most studied networks. Following that article, many scientists began studying networks, in particular scale-free ones.

In the last three centuries, most of a physicist's notion of everything had been a bit limited to matter and the forces guiding its motion. In 1905 Einstein added cosmic time and space to the mix [5], that had simplified reality as he combined matter with energy and space with time [6]. At the end of the 20th century physicists realized that one ingredient was missing: information. Information is an indispensable element in codifying and quantifying the understanding of nature. It has opened physicists eyes to the rest of reality, and they started to use statistical mechanics for everything. It turned out that game theory and statistical mechanics could help describe everything from the stock market to quantum physics [1].

In this work, regular and scale-free networks will be combined with public goods games and simulated though a stochastic method: Markov chains and the Monte Carlo algorithm. Although there are several works on the subject [7] [8] [9] [10] [11] [12] [13] , some effects had not been studied. So the aim of this work is to add some information to the work already done.

Chapter 1

Evolutionary Game Theory

Games in game theory are generally formulated in terms of a payoff matrix, which define the payoff of a given choice when interacting with the choice of another player. In this work, fitness is the accumulated payoff of an individual in a game. One can easily conclude that the fitness of one player is dependent of the actions of the others. The players are often denoted as population [14].

A phenotype is any observable characteristic of an organism: such as its morphology, development, biochemistry or behavior [15]. In game theory the phenotype of an individual only interferes with its fitness. Therefore one can always define the fitness as a function of the phenotype. The most important feature in evolutionary game theory is the fitness dependence of the frequence of different phenotypes in the population. A good example is the Hawk-Dove game, whose payoff matrix can be found in table 1.1, being $C > G$.

Table 1.1: Payoff matrix for the Hawk-Dove game

	if it meets a hawk	if it meets a dove
A hawk recieves	$\frac{G-C}{2}$	G
A dove recieves	0	G/2

In a given population where the number of doves is much bigger than the number of hawks, hawks will have a bigger fitness as it is likely they will find doves and get a gain G, while the doves only get a gain of $G/2$. But in a population consisting mostly of hawks, doves will win as they will avoid the fight and have their fitness unchaged while the hawks will be figthing with an average loss of fitness of $\frac{G-C}{2}$. So one can conclude that none of the phenotypes is better than the other, their sucess depends on the frequence of that phenotype in the population [16].

The strategy is what will define the actions of a player (eg. the strategy of a player could be making all moves at random). The Nash equilibrium [17] [18] [19] [20] is a special case of a strategy, if two players adopt it, then none of them could improve their payoff by using another strategy. If an entire population adopts an evolutionarily stable strategy, which is a refinement of the Nash equilibrium, then no other strategy will be effective there.

Nash equilibrium can be calculated for any game, if the payoff matrix is known. The "Alice's game" is presented in table 1.2. Bob owns Alice 10 euro. As alice wants to have some of her money back, she proposed to Bob a game: they go to the library every week, so based on the way they both use to reach the library Bob will pay to Alice a given amount of money as shown in table 1.2. Alice's payoff is always positive while the Bob's payoff is always negative, meaning that is a zero-sum game, what Bob loses Alice wins. This game is to be played in rounds. The

Table 1.2: Payoff matrix for Alice's game

	Bob goes on the Bus	Bob Walks
Alice goes on the Bus	3	6
Alice Walks	5	4

calculation of Nash equilibrium allows one to know which is the best mixed strategy. Alice chooses Bus with probability p and Walk with probability $1 - p$, while Bob chooses Bus with probability q and Walk with probability $1 - q$. Her expected payoff from choosing bus will be the sum of her payoff from Bus when Bob chooses Bus multiplied by the probability that Bob will choose Bus, plus her payoff from Bus when Bob chooses Walk times the probability that Bob plays Walk. Applying the same method for Alice expected payoff when she chooses Walk and for Bob, one obtains the following sets of payoffs:

Alice's expected payoff for Bus: $3q + 6(1 - q)$

Alice's expected payoff for Walk: $5q + 4(1 - q)$

Bob's expected payoff for Bus: $-3p - 5(1 - p)$

Bob's expected payoff for Walk: $-6p - 4(1 - p)$

Alice's total payoff will be her probability of choosing Bus times her Bus expected payoff plus her probability of choosing Walk times her Walk payoff. Bob's total payoff will be similar. To achieve the Nash equilibrium the expected payoff for both choices must be equal. Bob would not change his strategy if:

$$-3p - 5(1 - p) = -6p - 4(1 - p) \tag{1.1}$$

While Alice will not change her strategy if:

$$3q + 6(1 - q) = 5q + 4(1 - q) \tag{1.2}$$

Solving both equation 1.1 in order to p and equation 1.2 in order to q, one get $p = \frac{1}{4}$ and $q = \frac{1}{2}$. If Alice chooses Bus one in four times and Bob choose Bus half of the times, then both have achived the Nash equlibrium and none of them can improve their payoff by deviating themselves from those values [1].

There are many types of games, in which several characteristics may vary: number of iterated rounds, payoff matrix, number of players, etc (ie: The payoff matrix for the games stag hunt and chicken are different).

One could analyse many games, coming to many useful conclusions, but often scientists resume their study on the Prisoner's Dilemma. Two prisioners are suspected of having commited a joint crime, they are confined to different rooms and cannot talk to each other. The police do not have enough evidence to convince a judge and the attorney offers each of the suspects a deal: confess your crime and you will avoid a prison sentence. If one of them confesses and the other doesn't, then the first will go free immediately and the second will recieve a prison sentence of ten years. If both confess, they will each get seven years, and if neither of them confess, then both will recieve one year. The payoff matrix for the Prisoner's Dillema can be found in Table 1.3. Both players in the Prisioner's Dilemma would obtain the maximum fitness if they both cooperate. Then comes the relevant part, if one player looks at the game only by their own point of view, no matter what the other players does, the best choice is to defect [14].

The iterated Prisioner's Dilemma requires a more complex strategy, because if a player tries to cooperate and the oponent always defects the best approach is to not cooperate. Simple

Table 1.3: Payoff matrix for the Prisoner's Dillema

	Cooperate	Defect
Cooperate	-1	-10
Defect	0	-7

strategies, like always cooperating (ALLC) and always defecting (ALLD) could have very good scores in some games, but on average their score are smaller because they are higly sensitive to the frequence of the other strategies. One of the most sucessful strategies is Tit-for-tat (TFT), which cooperates in the first round and then copies the last choice of the oponent. TFT proves itself one of the most effective, because when competing with other strategies on average its fitness will usually be bigger. TFT is also a catalyst for cooperation, because it always cooperate with cooperators and always punish the defectors. There are some modifications to TFT, one of them is Generous Tit-for-tat (GTFT), which has a probability of not punishing an oponent's mistake in order to promote the cooperation, but it is weak when it finds an ALLD. Many other modifications, using the same concept, have been proposed and implemented, but TFT's lack of dependence of the frequence of strategies in the population still makes it the most efective strategy [14].

The N-person prisioner's dilemma is usally the chosen paradigm to study public goods games (PGGs). A network is defined as the space where the game is played. For each node of the network there is a player with a defined strategy, always cooperate or always defect. Usually cooperators (C) contribute with an amount c to the public good and defectors (D) do not contribute. The public good is then multiplied by a factor r and the result is equally distributed between all members of the group regardless of their strategy. The evolution of the strategies of each node is then calculated based in the gain of each node. Both in Prisioner's Dillema and in PGGs defecting looks a better choice if one analyses the game from it's own point of view, but cooperation is what gives better rewards for all the population. The results aren't only dependent on the game itself but also on the network where the game takes place. That network is nothing more than the relations between the nodes, meaning a node will not just interact directly with the nodes connected to it, but also with the next neighbor's and ultimatelly with all the network. Early experiments in the field, with regular networks, had shown that most of the poputation tends to defect, which is not what is observed in society where a large scale cooperation is known to take place. The use of scale-free networks in PGGs soon corrected that, showing an increase in the cooperation because it takes in account that in a society context everyone is different.

Chapter 2

Public Goods Games

One can see similarities between PGGs and human society. Different situations from family issues to global warming can be described as PGGs [7] [21]. Like PGGs, cooperation is the best option if we look from a population's point of view, and defection always looks better for a single human. In PGGs, cooperation can be achived even in absence of enforcing mechanisms [13].

PGGs on regular and scale-free networks have already been studied [7] [8] [9] [10] [11] [12] [13]. This chapter is a sumary of the methods used [13].

2.1 Networks

A network is a set of nodes connected by links.

The number of nodes in a given network will be denoted by the letter N. The total number of links of a node or connectivity is often called the node's degree and it is represented by the letter k. The average connectivity in this work will be represented by z.

In the examples presented, the networks do not include unitary loops, i.e. a given node cannot connect with itself. All these networks are undirected, meaning that the links are not oriented.

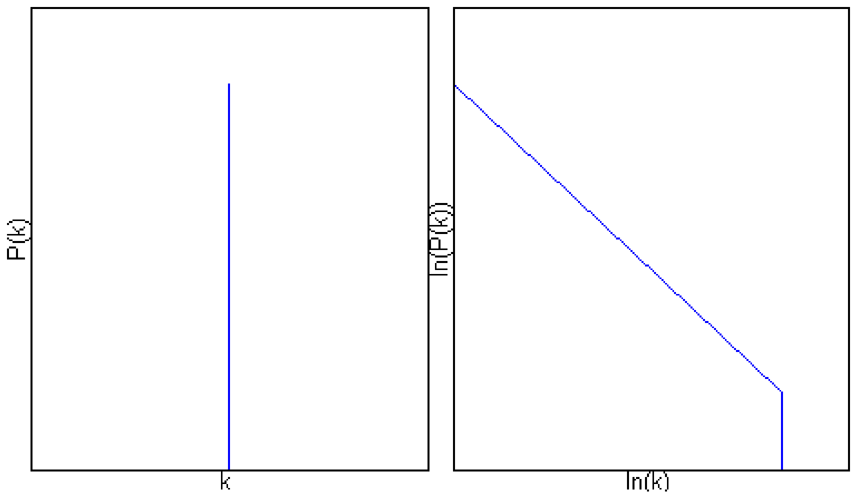

Figure 2.1: Degree distribution for (left) regular networks and (right) scale-free networks [23]

Three types of networks are considered and the schematics are presented in figure 2.2: one-dimensional regular ring networks, two-dimensional regular lattice networks and scale-free networks.

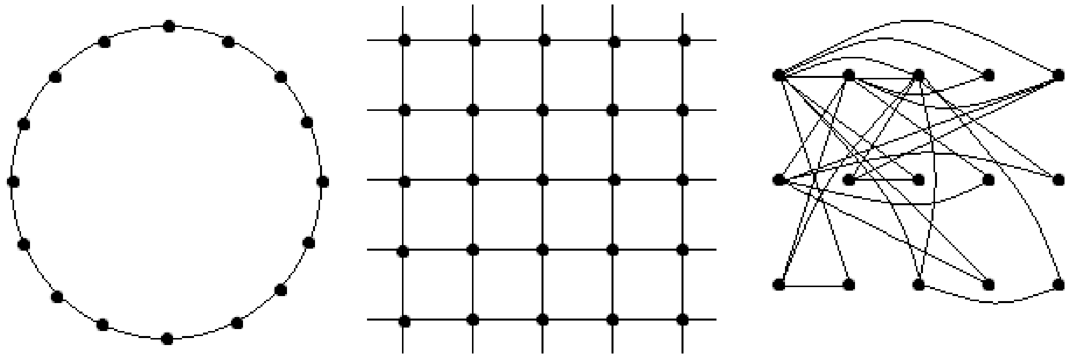

Figure 2.2: Schematics for different kinds of networks
(left) One-dimensional regular network with $N = 16$ and $z = 2$
(center) Two-dimensional regular network with $N = 25$ and $z = 4$
(right) Scale-free network with $N = 15$ and $z = 4$

In the regular network type, all nodes have exactly the same number of connections $z = 2m$ as show in figure 2.1 (left). An easy algorithm to build a network with $m = 1$ is:

- node a links to node *a + 1*;
- node N links to node *1*.

This produces a ring-shaped network as presented in figure 2.2 (left). With $m = 2$, the logic is similar, but instead of one ring, two concentric rings are obtained. In this situation the algorithm is:

- node a links to node *a + 1* and *a + 2*;
- node *N - 1* links to node N and *1*;
- node N links to node *1* and *2*.

In general, for one-dimensional regular ring networks, node a will connect to nodes $a-m, a-m+1, ..., a-1, a+1, ..., a+m-1$ and $a+m$.

An algorithm to build two-dimensional regular networks could be the following:

- place all nodes on a square latice;
- each node connect with its k nearest neighbors, making sure there are periodic boundary conditions, i.e. for $k = 4$ each node will connect will with all nodes that are a-distant; for $k = 8$ each node will connect to all nodes that are $\sqrt{2}a$-distant or less.

In the scale-free network type, node's degree follow a power law distribution $P(k) \propto k^{-\gamma}, k \neq 0$ [23], where γ is the exponent of the distribution as show in figure 2.1 (right). A way of building scale-free networks was proposed by Albert & Barabási (1999) [4]. The schematic for scale-free networks is presented in figure 2.2 (right). The method to build such networks with $z = 2m$ is:

- connect m nodes with each other;

- keep a list of the nodes that have been linked to the new nodes. However, only the old nodes (nodes which are linked to new ones) must be added to the list;

- build the network by adding nodes. A new node connects with m already existing nodes. Each time a node is added, there is a probability p that it will link to a node, chosen with probability proportional to $(k-m)$. Otherwise, the new node will be linked to an existing node at random. The best way to choose a node with a probability proportional to $(k-m)$ is to select a node at random from the list mentioned above.

The exponent γ obtained with this algorithm is given by $\gamma = 1 + 1/p$, and it can vary from 2 to infinity.

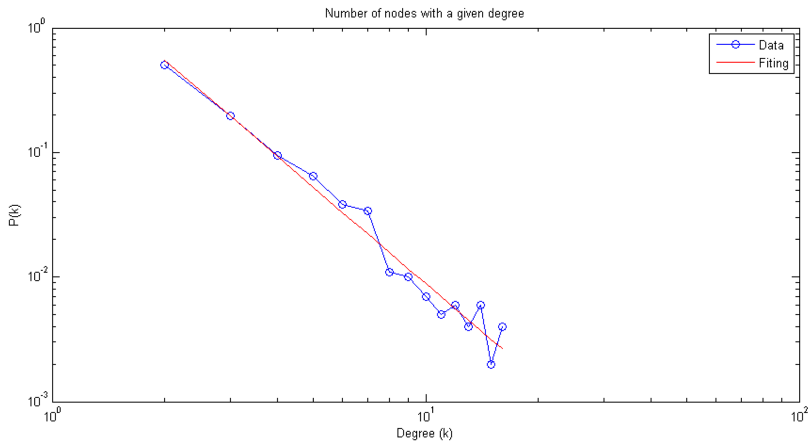

Figure 2.3: Number of nodes with a given degree k for our scalefree network with $N = 1000$, $m = 2$ and $p = 0.5$. The modulus of the slope obtained give us the γ and it is equal to 2.566.

For the scale-free case with p = 0.5, the expected exponent is $\gamma = 3$. In this case, the obtained exponent was 2.566 as presented in figure 2.1. Such difference is acceptable given that the theoretical value of γ was calculated considering infinite-size networks, whereas in this finite-size network there are cut-offs and $N = 1000$ is a rather small network.

In this work we will use the following networks:

- regular one-dimensional ring networks with $N = 1000$ and 500; $z = 2, 4, 6, 8$ and 16;

- regular two-dimensional lattice networks with $N = 1024$ (32×32) and 529 (23×23); $z = 4, 8$ and 12;

- scale-free networks with $N = 1000$ and 500; $p = 0.5$; $z = 2, 4, 6, 8$ and 16;

2.2 Types of Games

A group is a set of nodes connected to a center node, that is, one node (the center) and all nodes connected to it. For each node there is a game centered in that node and all nodes from its group take part.

Two types of games will be considered. The games only differ in the quantity payed by the cooperators. In the first type, from now on called game A, each cooperator will pay in each interaction a quantity c. In the other, from now on called game B, each cooperator contributes with a total c for all interactions in the same round, meaning it will pay $c/(k+1)$ for each interaction where k is the number of neighbors. After that, the accumulated gains from all interactions are divided by all nodes, regardless of being cooperators or not [13].

The strategy from node i is denoted by s_i which takes the value of 1 in case of a cooperator and 0 if defector.

For the game A, in a single game, the gains are:

$$\begin{cases} G(D) = \frac{crn_c}{k+1} \\ G(C) = G(D) - c \end{cases} \tag{2.1}$$

$G(D)$ and $G(C)$ are the respective gains of a defector and a cooperator, $k+1$ is the number of members of the group and n_c is the number of cooperators in the neighborhood of the defector [13]. For game B, for a single round, the gains of node y in the game centered in x follow this form:

$$G(s_y) = \sum_x G_x(s_y) = \frac{r}{k_x+1} \sum_{i=0}^{k_x} \frac{c}{k_i+1} s_i - \frac{c}{k_y+1} s_y \tag{2.2}$$

$G(s_y)$ is the gain of the node y from the group centered in x. s_y and s_i are the respective strategies from the nodes y and i which belong in the group centered in x, and take the value 0 if defector and 1 if cooperator. k_x and k_i are the respective number of neighbors of the nodes x and i [13].

The two types were simulated separately, and only for the scale-free networks as one can easily show that for regular networks both games give the same results.

Instead of r, in this work $\eta = r/(z+1)$ will be used.

2.3 Evolution

Each node x will compare his gain with one of his neighbors, chosen at random. If the gain of x is smaller than the gain og the neighbor y then x will adopt the strategy of y with a probability given by:

$$p = \frac{G(s_y) - G(s_x)}{M} \tag{2.3}$$

Where M is the normalization constant, $G(s_x)$ and $G(s_y)$ are the respective sum of the gains of each game in which they had participated in the round. If the gain of x is bigger or equal than the gain of y the strategy of x will remain unchanged [13].

Into this work two different M have been used. A fixed M, equal to the maximum possible value of the probability when $\eta = 3.5$, is used when the time dependence is important. A $M = max(G(s_y) - G(s_x))$ is used when the time dependence isn't important and a better computacional performance is required.

Two types of actualization had been used, the synchronous and asynchronous. The synchronous one only compares the gains after all nodes in all groups from the network interact, calculating all the changes of strategies at same time. The asynchronous one calculates those changes for a given node x after calculating all its gains from interation with its neighbors, updating only one node each time.

Chapter 3

Markov Chains

Games in finite populations are defined stochastically corresponding to Markov Chains.

A Markov chain [24] is a discrete random process where the evolution of the system only depends on the present state, and does not depend on the previous states. As the process is random it is impossible to predict the future states, but statistical properties can be studied.

In an unidimensional system with z neighbours with length N there are 2^N possible states. The quantity $p(..., s_{i-1},\ s_i,\ s_{i+1}, ...; t)$ is the probability of finding one state of the system $..., s_{i-1}, s_i, s_{i+1}, ...$ at the time t. $p(s_i; t)$ is the probability of the node i to be in the state s_i at the time t. $G(i)$ is the total gain of the node i (ie: for a regular network is the sum of the gain from $z+1$ games, one centered in the node and z in each of the z neighbours):

$$G(i) = \frac{rc}{z+1} \sum_{k=-z/2}^{z/2} \sum_{j=-z/2}^{z/2} s_{i+j-k} - (z+1)cs_i \tag{3.1}$$

The gain of the node i depends on the state of the system in s_{i-z}, s_{i-z+1}, $\cdots$, s_i, $\ldots$, s_{i+z-1}, s_{i+z}, a total of $2z+1$ variables. w_i represents the probability of the node i changing from the state s_i to the state $1-s_i$ and can be writen as:

$$w_i(S_i) = \frac{1}{zM} \sum_{k=-z/2}^{z/2} max((G(i-k) - G(i))|s_{i-k} - s_i|, 0) \tag{3.2}$$

where M is the chosen normalization.

If the nodes i and $i-k$ are in the same state, $|s_{i-k} - s_i| = 0$, then there is no contribution from that term. If both nodes are in different state then $|s_{i-k} - s_i| = 1$. w_i depends on the states of the nodes $\ldots$, $s_{i-z-z/2}$, $s_{i-z-z/2+1}$, $\cdots$, $s_{i+z+z/2-1}$, $s_{i+z+z/2}$, making a total of $3z+1$ variables.

Each one of these states can be transformed in another N states by changing one variable s_k with $1 \leq k \leq N$.

The time dependence of the probability of one state of the system is described by the Master Equation:

$$\begin{aligned} \frac{d}{dt} p(..., s_{i-1}, s_i, s_{i+1}, ...; t) = & \sum_{k=-N/2}^{N/2} p(..., 1-s_k, ...; t) w_k(1-s_k) \\ & -p(..., s_{i-1}, s_i, s_{i+1}, ...; t) \sum_{k=-N/2}^{N/2} w_k(s_k) \end{aligned} \tag{3.3}$$

The state $..., 1 - s_k, ...$ is equal to the state $..., s_{i-1}, s_i, s_{i+1}, ...$, except in the node k where state is $1 - s_k$ instead of s_k. The first term has all the contributions from the different states that are different from the original state $..., s_{i-1}, s_i, s_{i+1}, ...$ in only one node and put the system back in the original state. The second term subtracts all the transitions of only one node that change the system from the original state. This dynamic is only valid for the asynchronous update, as one assume that only one node could change at each step.

In a network with N nodes, one can label all possible i (states) by numbers, starting from 1 (only defectors) to 2^N (only cooperators).

A $\Pi_{i,j}$ matrix can be defined to represent the rate of transition between the state j and the state i. The equation (3.3) can be writen as a matrix:

$$p(i, t + \delta t) = \sum_j \Pi_{i,j} p(j, t) \tag{3.4}$$

Where $\Pi_{i,j} = dtw_k(1 - s_k)$ if $i \neq j$ and the state i is different from the state j in only a value of the variable s_k that takes the value $1 - s_k$ in the state i. As the states $i = 1$ and $i = 2^N$ are absorvent states, one get the following relations:

$$\Pi_{j,j} = 1 - \delta t \sum_{k=-N/2}^{N/2} w_k(s_k) \tag{3.5a}$$

$$\Pi_{i,2^N} = 0 \, for \, i \neq 2^N \tag{3.5b}$$

$$\Pi_{2^N,2^N} = 1 \tag{3.5c}$$

$$\Pi_{i,1} = 0 \, for \, i \neq 1 \tag{3.5d}$$

$$\Pi_{1,1} = 1 \tag{3.5e}$$

One defines the matrix $Q_{i,j}$ by eliminating the lines 1 and N and the collums 1 and N, leaving all the other elements unchanged. So $Q_{i,j} = \Pi_{i+1,j+1}$ with $i = 1, 2, ..., 2^L - 2$ and $j = 1, 2, ..., 2^L - 2$.

One can obtain the probability distribution for $t = ndt$ from $p(i, t = n\delta t) = \sum_j \Pi_{ij}^n p(j, 0)$ and so Π_{ij}^n represents the probability that in n steps the system goes from the state j to the state i. Also $p(i, t = n\delta t) = \sum_j Q_{ij}^n p(j, 0)$ if i isn't an absorbing state. One defines $p(i, t = n\delta t | j, 0; s)$ as the probability of the system to be in the state i starting from the state j without hiting any absorvent states. The probability of not falling in any absorbing state in $t = n\delta t$ starting from $t = 0$ in the state j is given by:

$$p_s(t = n\delta t | j, 0) = \sum_i Q_{ij}^n \tag{3.6}$$

Then:

$$p(i, t = n\delta t | j, 0; s) = \frac{Q_{ij}^n}{\sum_i Q_{ij}^n} \tag{3.7}$$

The average number of steps which the system spends in the state i starting from the state j, U_{ij} can be writen as $Uij = \delta_{ij} + Q_{ij} + Q_{ij}^2 + Q_{ij}^3 + ...$ so the matrix U is given by $U = (I - Q)^{-1}$.

The average number of steps before falling into an absorbing state when the system starts in the state j is $t_s = \sum_i U_{ij}$.

Being $R_{i,j}$ the matrix obtained from $\Pi_{i,j}$ which contains the transition rates from any state j to an absorbing state i, then the probability for the system to hit the absorbing state i starting from the state j, the fixation probabilities, can be writen as:

$$A_{i,j} = \sum_{k\, not\, absorvent} (\delta_{k,j} + Q_{kj} + Q_{kj}^2 + ...)R_{ik} = \sum_k R_{ik} U_{kj} \tag{3.8}$$

Taking into account that the system visits the state k an arbirtrary number of times before it falls in the absorbing state in the next step and defining $\rho(i)$ as the particle density in the state i, one can calculate the average density of cooperators in configurations that haven't fallen into an absorbing state, assuming that the system starts with equal probability from any of the starting states j.

$$\rho(t) = \frac{1}{2^N - 2} \sum_{i,j} \rho(i) p(i, t = n\delta t | j, 0; s) \tag{3.9}$$

With:

$$p(i, t = n\delta t | j, 0; s) = \frac{Q_{ij}^n}{\sum_i Q_{ij}^n} \tag{3.10}$$

One can also calculate $v(t)$ which represents the number of neighbors of a node (in the neighborhood of z) that are in a state different than the state of the central node. Also the number of active nodes $\Delta(t)$ which are the number of nodes with at least one neighbor in a different state can be calculated. And finally the probabilitiy for an infinite time for the system to hit a given absorvent state $\bar{A}_i$ can also be defined. The formulae to do the calculations are:

$$\rho(t) = \frac{1}{N} \langle \sum_i s_i \rangle \tag{3.11a}$$

$$v(t) = \frac{1}{N} \langle \sum_i \sum_{k=-z/2}^{z/2} |s_i - s_{i-k}| \rangle \tag{3.11b}$$

$$\Delta(t) = \frac{1}{N} \langle \sum_i \left(1 - \prod_{k=-z/2, k\neq 0}^{z/2} (1 - |s_i - s_{i-k}|) \right) \rangle \tag{3.11c}$$

$$\bar{A}_i = \frac{1}{2^N - 2} \sum_j A_{ij} \tag{3.11d}$$

In the results section $\bar{A}_1$ will be denoted by $p(0)$, while $\bar{A}_{2^L}$ will be denoted by $p(1)$. The probability to survive, $p(2)$ is then equal to $1 - \bar{A}_1 - \bar{A}_{2^L}$.

Solving 3.10 requires Q_{ij}^n. Q_{ij} is a big sparse matrix, and therefore lots of memory can be saved, but Q_{ij}^n isn't sparse anymore. So for a big N one can't calculate the exact result for a given network (ie: Matlab suports matrices up to 40000×40000 elements so it can't be used to solve any system with $N > 15$). The solution used in this work is to use the Monte Carlo Method to generate some initial states, apply the algoritm explained in the next section and then derive the statistical properties of the system.

3.1 Algorithm

In this section the algorithm will be explained, as the basic formulae and dynamical rules have already been discussed, this section will only describe the numerical method used in our computer program. The algorithm only contemplates one realization of the system.

As it is impossible to apply the algorithm to all the possible initial states, three types of initial states have been selected: one cooperator surrounded by defectors, one defector surrounded by cooperators and a random distribution of cooperators and defectors. In one cooperator surrounded by defectors all the nodes are set as defector, then one node at random is chosen and changed to cooperator. In one defector surrounded by cooperators all nodes are set as cooperator and one of them chosen at random is set as defector. In the random distribution for each node the state defector or cooperator is chosen at random. In the algoritm the initial state is set in the function $set\_initial\_state()$.

The algorithm for the asyncronous update program can be found in figure 3.1, while the syncronous one can be found in figure 3.2.

The function $select\_active\_nodes(state)$ will make a list of all nodes which are active, meaning they have at least one neighbor in a different state. The function $length(active)$ returns the number of active nodes. The function $random\_select\_active\_node(active)$ selects one node at random from the active nodes. The function $gain(node)$ will calculate the gain of a node, using the formula (2.1) for the game A and (2.2) for the game B, suming the calculation coming from the game at that node and each of its neighbors. The function $select\_neighbors(node)$ will create a list of the neighbors from a given node. The function $random\_choose\_neighbor(node)$ will choose one neighbor of given node at random. Finally, the function $calculate\_probability(gain(node), gain(neig))$ will output one number between 0 and 1 which is the probability that the node *node* will change its state to the state of the node *neig* or not, using (2.3).

```
state = set_initial_state();
for r = 1:rounds
    active = select_active_nodes(state);
    active_nodes = length(active);
    for a = 1:active_nodes
        node = random_select_active_node(active);
        gain(node) = calculate_gain(node);
        neighbors = select_neighbors(node);
        gain(neighbors) = calculate_gain(neighbors);
        neig = random_choose_neighbor(node);
        if gain(node) < gain(neig)
            prob = calculate_probability(gain(node),gain(neig));
            if random_number < prob
                state(node) = state(neig);
            end
        end
        active = select_active_nodes(state);
    end
end
```

Figure 3.1: Algorithm for the program with asyncronous update

```
state = set_initial_state();
for r = 1:rounds
    state_temp = state;
    for a = 1:number_of_nodes
        gain(node) = calculate_gain(node);
    end
    for a = 1:number_of_nodes
         neig = random_choose_neighbor(node);
         if gain(node) < gain(neig)
             prob = calculate_probability(gain(node),gain(neig));
             if random_number < prob
                 state(node) = state(neig);
             end
         end
    end
    state = state_tem;
end
```

Figure 3.2: Algorithm for the program with syncronous update

Chapter 4

Results for one-dimensional regular ring networks

In this chapter results for one-dimensional regular ring networks are presented.

4.1 Asynchronous Update

4.1.1 One cooperator surrounded by defectors

In order to understand more complex configurations, one must first study the simpler ones. A configuration with one cooperator surrounded by defectors is presented in table 4.1. In table 4.1 the first line corresponds to the index i of the position of the node in the lattice.

Table 4.1: One cooperator surrounded by defectors

i	$-z/2$	...	-2	-1	0	1	2	...	$z/2$
state	D	D	D	D	C	D	D	D	D

One wants to compare the defectors with smaller and bigger gain that are in the range to be converted by the cooperator. Nodes that satisfy that condition are the defectors with smaller gain present in $i = z/2$ or $i = -z/2$ and the defectors with bigger gain present in $i = 1$ or $i = -1$, the cooperator is present in $i = 0$. The respective gains for them are:

$$\begin{cases} G(0) = \eta(z+1) - (z+1) \\ G(1) = G(-1) = \eta z \\ G(z/2) = G(-z/2) = \eta(z/2+1) \end{cases} \tag{4.1}$$

Solving $G(0) \leq G(1)$ and $G(0) \geq G(z/2)$ one gets $\eta \leq z+1$ and $\eta \geq \frac{2(z+1)}{z}$. Figure 4.1 shows the fraction of configurations that fall in the absorbing abstate with only defectors after 10^5 rounds, starting from the configuration with one cooperator surrounded by defectors, as a function of η for asynchronous update. It's not possible to have a configuration with only defectors for $\eta > z+1$ as one can see in figure 4.1 for different values of z, and it's not possible to have a configuration with only cooperators for $\eta < \frac{2(z+1)}{z}$. The expected transition values are present in table 4.2. Figure 4.1 clearly displays the transition's dependence on z, at $\eta = z+1$, all realizations with different z show the same behavior, and the value was exact (neglecting the inherent small errors of a numeric calculation).

Table 4.2: Expected maximum values of η for which is possible to have configurations with only defectors

z	2	4	6	8	16
$\eta = z+1$	3	5	7	9	17

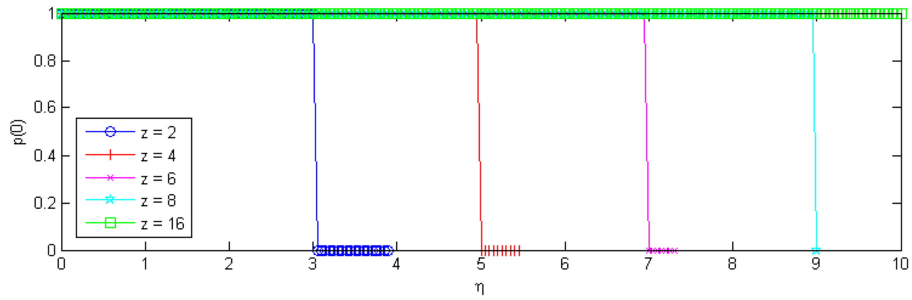

Figure 4.1: Fraction of configurations which have fall in the state with only defectors $p(0)$, for one-dimensional regular ring networks with $n = 1000$ and $z = 2, 4, 6, 8$ and 16, using 10^4 samples and 10^5 rounds. The initial state was one cooperator surrounded by defectors and the update was asynchronous

4.1.2 A number $l \leq z/2 + 1$ cooperators surrounded by defectors

The next configuration to be analised is the group which have $l \leq z/2 + 1$ cooperators surrounded by defectors, as can be seen in table 4.3. For lower values of η it will eventually fall in one configuration with one cooperator surrounded by defectors. For $\eta > \frac{2(z+1)}{z}$ the configurations with one cooperator surrounded by defectors have a probability to fall in one configuration with more than one cooperator.

Table 4.3: A number $l \leq z/2 + 1$ cooperators surrounded by defectors

i	$\frac{-z}{2}$	...	-2	-1	0	1	2	...	$l-1$	...	$\frac{z}{2}+1$	$\frac{z}{2}+2$
state	D	D	D	D	C	C	C	C	C	D	D	D

One wants to compare the gains from the weakest cooperator $G(0)$ with the gain from the strongest defector $G(-1)$. The gains for them are:

$$\begin{cases} G(-z \leq i \leq l-z-1) = \frac{\eta}{2}(2+i+z)(i+z+1) \\ G(l-z-1 \leq i < 0) = \frac{\eta}{2}l(l+1) + \eta l(i-l+z+1) \\ G(0) = G(-1) + \eta l - (z+1) \end{cases} \tag{4.2}$$

Solving $G(0) \leq G(-1)$ one gets $\eta \leq \frac{z+1}{l}$, meaning that the cooperator at $i = 0$ can only be converted into a defector when this condition is verified. For $\eta > \frac{z+1}{l}$ the number of cooperators increases.

4.1.3 Between $z/2+1$ and $z+1$ cooperators surrounded by defectors

The configuration in table 4.4 represents the group of configurations with a number of cooperators between $z/2+1$ and $z+1$ surrounded by defectors.

Table 4.4: Between $z/2+1 < l \leq z+1$ cooperators surrounded by defectors

i	$-z/2$	...	-2	-1	0	1	2	...	$l-1$	l	...	$z+1$
state	D	D	D	D	C	C	C	C	C	D	D	D

Like the previous case one wants to compare the weakest cooperator ($i=0$) with the strongest defector ($i=-1$). The respective gains, assuming $l=\frac{z}{2}+1$ are:

$$\begin{cases} G(-1) = \frac{\eta}{2}(\frac{z}{2}+2)(\frac{z}{2}+1) + \eta(\frac{z}{2}-1)(\frac{z}{2}+1) \\ G(0) = G(-1) + \eta l - (z+1) \end{cases} \tag{4.3}$$

The same result as the previous case is obtained when solving $G(0) \leq G(-1)$. So, the cooperator can only be converted into a defector if $\eta \leq \frac{z+1}{l}$ otherwise the number of connected cooperators will increase.

4.1.4 More than $z+1$ cooperators surrounded by defectors

Finally, the configurations with more than $z+1$ cooperators surrounded by defectors, which are represented in table 4.5 will be studied.

Table 4.5: More than $z+1$ cooperators surrounded by defectors

i	$-z/2$	...	-2	-1	0	1	2	...	$l-1$	l
state	D	D	D	D	C	C	C	C	C	D

Yet again, one wants to compare the gain of the strongest defector $G(-1)$ with the gain of the weakest cooperator $G(0)$ and the gain from the weakest defector $G(-z/2)$ with the gain from the weakest cooperator. The gains for them are:

$$\begin{cases} G(-z/2) = \eta(\frac{z^2}{8} + \frac{3z}{4} + 1) \\ G(-1) = \frac{\eta z^2 + \eta z}{2} \\ G(0) = G(-1) + \eta(z+1) - (z+1) \end{cases} \tag{4.4}$$

Solving $G(-1) < G(0)$ one gets $\eta > 1$, so a group with more than $z+1$ cooperators can resist the invasion of defectors and grow if that condition is verified. Solving $G(-1) < G(-z/2)$ the result $\eta < \frac{z+1}{\frac{3z^2}{8}+\frac{z}{4}+1}$ is obtained, which is the interval where the defectors will win over the cooperators.

4.1.5 One defector surrounded by cooperators

So far, only configurations formed by groups of cooperators had been discussed, but now groups of defectors will be considered. The schematic for the configuration with one defector surrounded by cooperators is presented in table 4.6.

Table 4.6: One defector surrounded by cooperators

i	$-z/2$	...	-2	-1	0	1	2	...	$z/2$
state	C	C	C	C	D	C	C	C	C

One wants to compare the cooperators with smaller and bigger gain that are in range to be converted by the defector. The nodes that satisfy that condition are the cooperators present in $i = z/2$ or $i = -z/2$ and the cooperators present in $i = 1$ or $i = -1$. The respective gains for them are:

$$\begin{cases} G(0) = \eta(z+1)z \\ G(1) = G(-1) = \eta(z+1+z^2) - (z+1) \\ G(z/2) = G(-z/2) = \eta(z/2(z+1) + z(z/2)) - (z+1) \end{cases} \tag{4.5}$$

Solving $G(0) \leq G(1)$ and $G(0) \geq G(z/2)$ one gets $\eta \leq z+1$ and $\eta \geq \frac{2(z+1)}{z}$, that are the same values for one cooperator in the middle of defectors, so the same conclusions can be derived. The expected values of η are presented in table 4.7.

Table 4.7: Expected mininum values of η for which it is possible to have configurations with only cooperators

z	2	4	6	8	16
$\eta = \frac{2(z+1)}{z}$	3	$\frac{5}{2} = 2.5$	$\frac{7}{3} = 2.33(3)$	$\frac{9}{4} = 2.25$	$\frac{17}{8} = 2.125$

4.1.6 A number $l \leq z/2 + 1$ defectors surrounded by cooperators

A configuration with a maximum number of $z/2 + 1$ defectors surrounded by cooperators is presented in table 4.8.

Table 4.8: A number $l \leq z/2 + 1$ defectors surrounded by cooperators

i	$\frac{-z}{2}$	...	-2	-1	0	1	2	...	$l-1$	...	$\frac{z}{2}+1$	$\frac{z}{2}+2$
state	C	C	C	C	D	D	D	D	D	C	C	C

One wants to compare the strongest cooperator $i = -z/2$ with the defector in the border $i = 0$ to know what is the minimum value of η for which it is possible for the cooperators to take over the defectors. It is also important to compare the weakest cooperator $i = -1$ with the defector present in $i = 0$. The gains for them are:

$$\begin{cases} G(0) = \frac{\eta}{2}\left(z^2 + 2z + \frac{l^2}{2} + 1 - lz - \frac{3l}{2}\right) \\ G(-1) = G(0) + \eta l - (z+1) \\ G(-z/2) = \eta\left(z^2 + 2z + \frac{l^2}{2} + 1 - \frac{l(z+3)}{2}\right) - (z+1) \end{cases} \tag{4.6}$$

Solving $G(0) > G(-1)$ the result $\eta < \frac{z+1}{l}$ is obtained, which means the number of defectors will increase for values of η less than $\frac{z+1}{l}$. By solving $G(-z/2) > G(0)$ one get the solution

$\eta > \frac{2(z+1)}{\left(z^2+2z+\frac{l^2}{2}+1-lz-\frac{3l}{2}\right)}$, which means that the number of defectors can not decrease for smaller values of η.

4.1.7 Between $z/2+1$ and $z+1$ defectors surrounded by cooperators

Following the same steps for the groups of cooperators, a schematic for the group of configurations with a number of defectors between $z/2+1$ and $z+1$ surrounded by cooperators is presented in table 4.9.

Table 4.9: Between $z/2+1 < l \leq z+1$ defectors surrounded by cooperatos

i	$-z/2$	...	-2	-1	0	1	2	...	$l-1$	l	...	$z+1$
state	C	C	C	C	D	D	D	D	D	C	C	C

Like in previous examples, the nodes in which the gain must be compared are $i=0$ with $i=-1$ and $i=0$ with $i=-z/2$. The gains for these nodes are:

$$\begin{cases} G(0) = \frac{\eta}{2}\left(5l(z+1) - z^2 - 3z - 2 - 3l^2\right) \\ G(-1) = G(0) + \eta l - (z+1) \\ G(-z/2) = \frac{\eta z(7z+10)}{8} - (z+1) \end{cases} \tag{4.7}$$

When one solves $G(0) > G(-1)$ the result is $\eta < \frac{z+1}{l}$ which is the interval where the number of defectors can increase. Solving $G(0) < G(-z/2)$ the interval where the number of defectors can not decrease is found to be $\eta < \frac{8(z+1)}{20l(z+1)-23z^2-22z-8-12l}$.

4.1.8 More than $z+1$ defectors surrounded by cooperators

Finally, a configuration with more than $z+1$ defectors surrounded by cooperators is presented in table 4.10.

Table 4.10: More than $z+1$ defectors surrounded by cooperators

i	$-z/2$	...	-2	-1	0	1		...	$l-1$	l
state	C	C	C	C	D	D	D	D	D	C

The gains of the nodes $i=0$ with $i=-1$ and $i=0$ with $i=-z/2$ will be compared:

$$\begin{cases} G(0) = \frac{\eta z(z+1)}{2} \\ G(-1) = G(0) + \eta(z+1) - (z+1) \\ G(-z/2) = \frac{\eta z(7z+10)}{8} - (z+1) \end{cases} \tag{4.8}$$

Solving $G(0) > G(-1)$ one sees that the number of defectors can only increase for $\eta < 1$. The number of defectors will not decrease for $\eta < \frac{8(z+1)}{3z(z+2)}$, as it can be obtained also from the condition $G(-z/2) > G(0)$.

4.1.9 Initial random distribution of cooperators and defectors

Two groups, one with a large number of cooperators and the other with a large number of defectors will be considered, so conclusions can be obtained. The schematic for such configuration is presented in table 4.11.

Table 4.11: Interface between more than $z+1$ cooperators and more than $z+1$ defectors

i	...	$-z/2$	...	-1	0	...	$z/2-1$	...
state	C	C	C	C	D	D	D	D

One want to compare the rates of change between the nodes $i=0$ and $i=-z/2$ with the rates for the nodes $i=-1$ and $i=0$, when these respective nodes have the following gains:

$$\begin{cases} G(0) = \frac{\eta z(z+1)}{2} \\ G(-1) = G(0) + \eta(z+1) - (z+1) \\ G(-z/2) = \frac{\eta z(7z+10)}{8} - (z+1) \end{cases} \tag{4.9}$$

First one defines the rate R_c as $G(0) - G(-z/2)$ which is equal to $(z+1) + \frac{3\eta z(z+2)}{8}$ and the rate R_d as $G(-1) - G(0)$ which is equal to $-(z+1) + \eta(z+1)$. The rate R_c, which compares the gains from the strongest cooperator with the strongest defector, points out that the cooperators can win over the defectors if $\eta > \frac{8(z+1)}{3z(z+2)}$. The rate R_d, which compares the gains from the weakest cooperator with the strongest defector, shows that the defectors can win over the cooperators if $\eta < 1$. If one equates both rates, a critical value of η will be obtained which is equal to $\eta_c = \frac{2(z+1)}{\frac{3z(z+2)}{8}+z+1}$. At η_c both rates are equal and neither defectors or cooperators win over. If the fraction of cooperators in surviving configurations are analysed, that value should correspond to a discontinuity between one phase with only defectors, for lower values of η to a phase dominated by cooperators for higher values of η. The critical values of η_c can be found in table 4.12, and the fraction of cooperators in the surviving configurations $\rho(2)$ is present in figure 4.6 ($z=2$), figure 4.7 ($z=4$), figure 4.8 ($z=6$) and figure 4.9 ($z=16$).

Table 4.12: Expected critical values of η for regular one-dimensional ring network with asynchronous update

z	2	4	6	8	16
$\eta_c = \frac{2(z+1)}{\frac{3z(z+2)}{8}+z+1}$	1	$\frac{5}{7} = 0.714$	$\frac{14}{25} = 0.56$	$\frac{6}{13} = 0.462$	$\frac{34}{125} = 0.272$

A dependence on the number of surviving configurations on the size of the network, N, had been found, as shown in figure 4.2 ($z=2$), figure 4.3 ($z=4$), figure 4.4 ($z=6$) and figure 4.5 ($z=16$). Those figures show that when the size of the system grows to infinity, the probability of getting a surviving configuration goes to 1, and the probability to get an absorbing state goes to 0, so, information for the behaviour of infinite systems should be studied considering only surviving configurations.

In figures 4.6, 4.7, 4.8 and 4.9 one can see that the calculated values agree with the computational results. For different z, the η_c agree with the values from table 4.12, and a transition between a state dominated by defectors for values of $\eta < \eta_c$ and a state dominated by cooperators for values of $\eta > \eta_c$ is observed. For different N regardeless of the value of z, the obtained

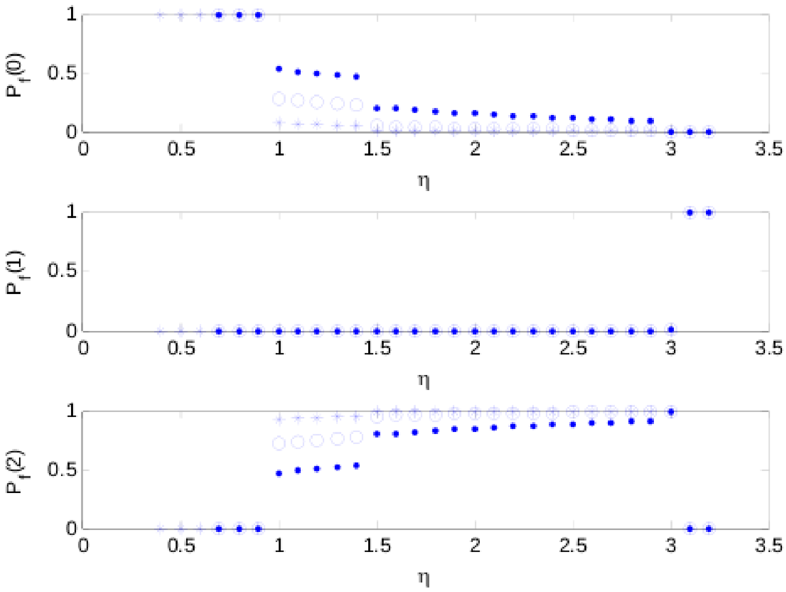

Figure 4.2: Fraction of surviving configurations $p(2)$, fraction of configurations with only defectors $p(0)$ and fraction of configurations with only cooperators $p(1)$, for infinite time using an asynchronous update in the regular one-dimensional ring network, with $z = 2$ and $M = 7.5$; $N = 10$ (.), $N = 20$ (o) and $N = 40$ (*). The vertical line is the critical value predicted, $\eta_c = 1$

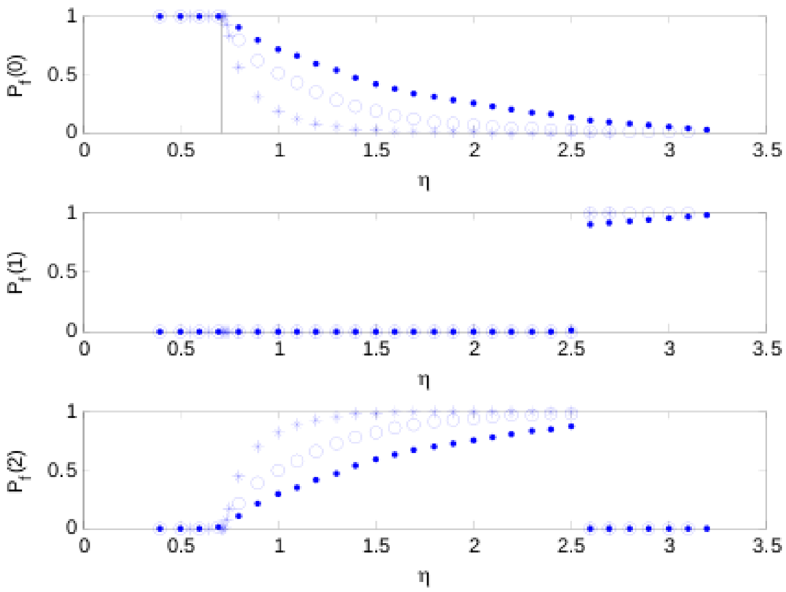

Figure 4.3: Fraction of surviving configurations $p(2)$, fraction of configurations with only defectors $p(0)$ and fraction of configurations with only cooperators $p(1)$, for infinite time using an asynchronous update in the regular one-dimensional ring network, with $z = 4$ and $M = 40$; $N = 20$ (.), $N = 40$ (o) and $N = 100$ (*). The vertical line is the critical value predicted, $\eta_c = \frac{5}{7}$.

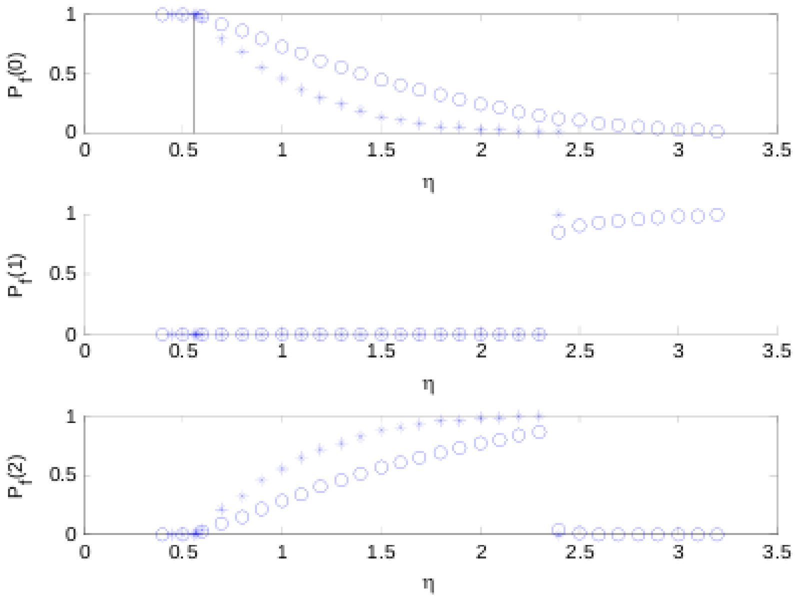

Figure 4.4: Fraction of surviving configurations $p(2)$, fraction of configurations with only defectors $p(0)$ and fraction of configurations with only cooperators $p(1)$, for infinite time using an asynchronous update in the regular one-dimensional ring network, with $z = 6$ and $M = 115$; $N = 40$ (o) and $N = 100$ (*). The vertical line is the critical value predicted, $\eta_c = \frac{14}{25}$.

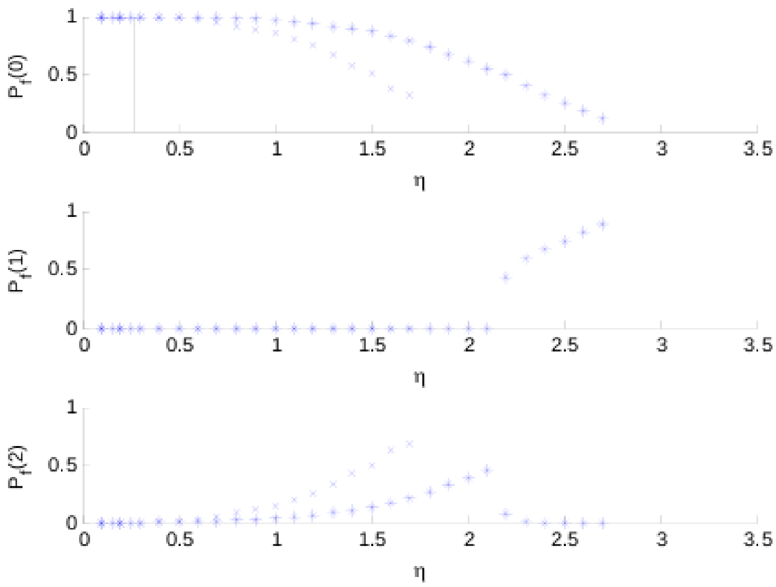

Figure 4.5: Fraction of surviving configurations $p(2)$, fraction of configurations with only defectors $p(0)$ and fraction of configurations with only cooperators $p(1)$, for infinite time using an asynchronous update in the regular one-dimensional ring network, with $z = 16$ and $M = 1550$; $N = 100$ (*) and $N = 500$ (x). The vertical line is the critical value predicted, $\eta_c = \frac{34}{125}$.

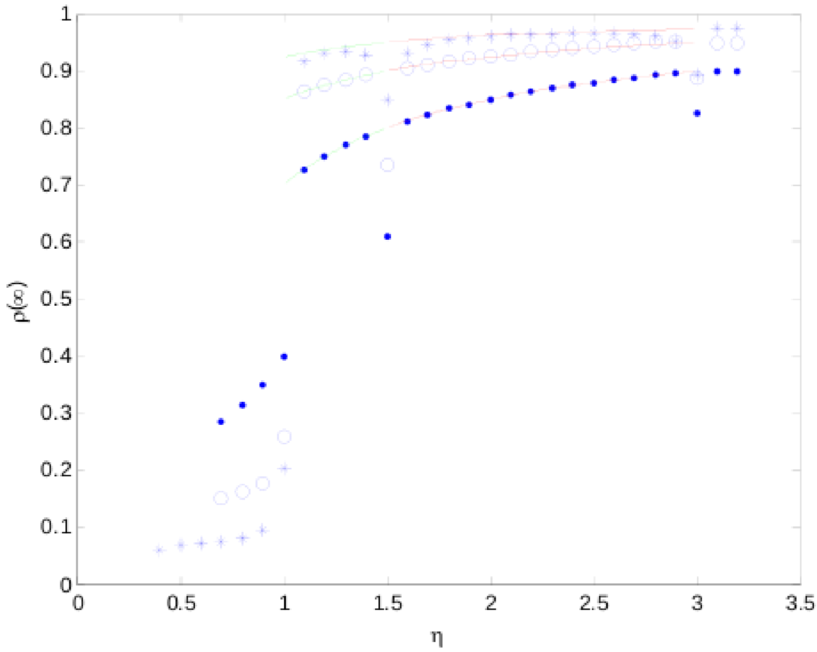

Figure 4.6: Cooperator's density in the surviving configurations for infinite time, using an asynchronous update in the regular one-dimensional ring network, with $z = 2$ and $M = 7.5$; $N = 10$ (.), $N = 20$ (o) and $N = 40$ (*). The vertical line is the critical value predicted, $\eta_c = 1$.

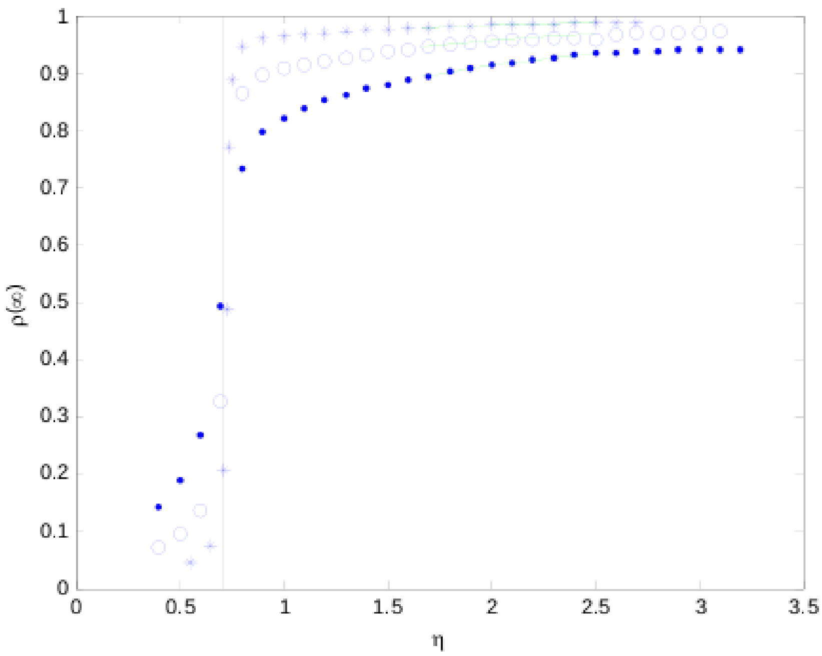

Figure 4.7: Cooperator's density in the surviving configurations for infinite time, using an asynchronous update in the regular one-dimensional ring network, with $z = 4$ and $M = 40$; $N = 20$ (.), $N = 40$ (o) and $N = 100$ (*). The vertical line is the critical value predicted, $\eta_c = \frac{5}{7}$.

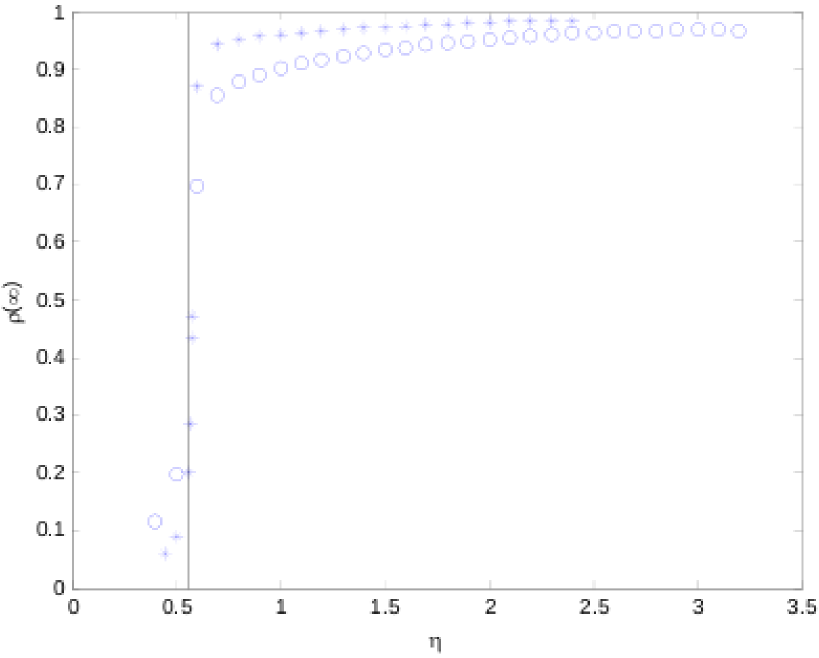

Figure 4.8: Cooperator's density in the surviving configurations for infinite time, using an asynchronous update in the regular one-dimensional ring network, with $z = 6$ and $M = 115$; $N = 40$ (o) and $N = 100$ (*). The vertical line is the critical value predicted, $\eta_c = \frac{14}{25}$.

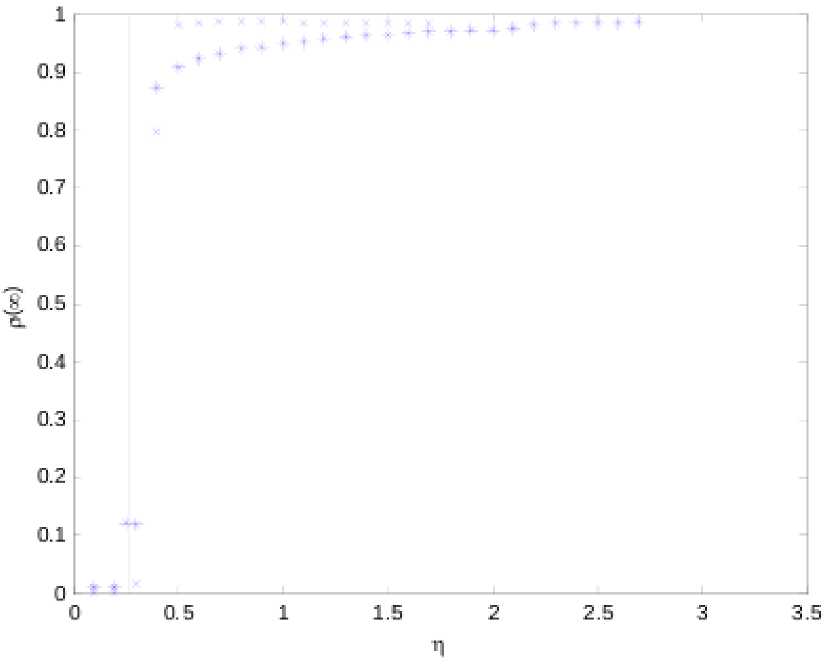

Figure 4.9: Cooperator's density in the surviving configurations for infinite time, using an asynchronous update in the regular one-dimensional ring network, with $z = 16$ and $M = 1550$; $N = 100$ (*) and $N = 500$ (x). The vertical line is the critical value predicted, $\eta_c = \frac{34}{125}$.

curves are different. Those feature can be easly explained, for example: one defector alone in one network with $N = 10$ represents $1/10$ of the network, while in one with $N = 20$ represents $1/20$.

4.2 Synchronous Update

4.2.1 One cooperator surrounded by defectors

The results are similar to the ones obtained in the same configuration for the asyncronous update calculated in section 4.1.1 and present in table 4.2, with exception that exists a possibility for $\eta > z+1$ to create configurations with only defectors. That probability is due to the creation of a colaborator in a position not adjacent to the first one, the defectors in between will always have a bigger gain so they can convert both cooperators in the same round as the update is synchronous.

4.2.2 One defector surrounded by cooperators

Some of the conclusions for the same configuration in the asyncronous are invalid for the sincronous update, because a group of up to z defectors could be converted simultaneously. Many configurations remain active for a critical value of η lower than for the synchronous case, with exception for $z = 2$ as show in figure 4.10. That value of η_c obtained will be explained in section 4.2.3.

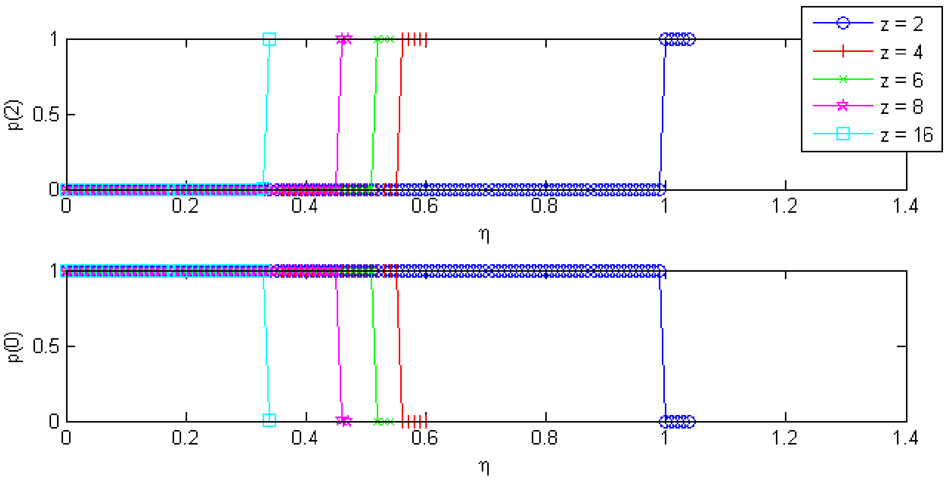

Figure 4.10: Fraction of surviving configurations $p(2)$ and fraction of configurations with only defectors $p(0)$, for one-dimensional regular ring networks with $n = 1000$ and $z = 2, 4, 6, 8$ and 16, using 10^5 samples and 10^5 rounds. The initial state was all cooperators with only one defector, and the update was synchronous

The minimum values for which it is possible to have configurations with only cooperators in the figure 4.11 are in agreement with the ones calculated in the section 4.1.5 and found in the table 4.7. They obey to relation $\eta > \frac{2(z+1)}{z}$. As it is a minimum value and the number of runs is not high enough, the exact value for higher values of z is not observed in plot.

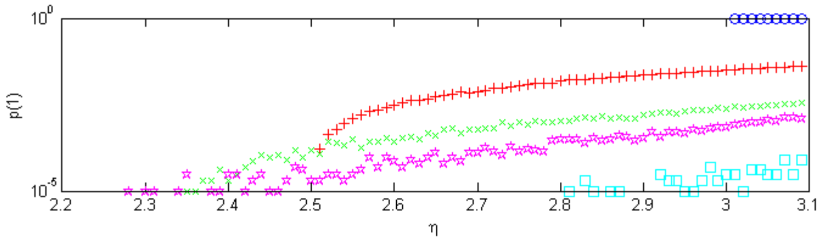

Figure 4.11: Fraction of configurations with only cooperators $p(1)$, for one-dimensional regular ring networks with $n = 1000$ and $z = 2, 4, 6, 8$ and 16, using 10^5 samples and 10^5 rounds. The initial state was all cooperators with only one defector, and the update was synchronous

4.2.3 Initial random distribution of cooperators and defectors

From the analysis of figure 4.12, which shows the fraction of cooperators averaged over all configurations, one can see that the results suggest that η_c do not depend on z (excluding $z = 2$).

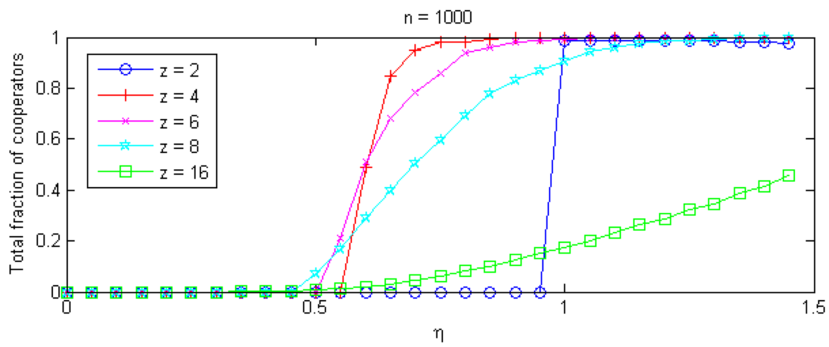

Figure 4.12: Total fraction of cooperators, for one-dimensional regular ring networks with $n = 1000$ and $z = 2, 4, 6, 8$ and 16, using 10^4 samples and 10^5 rounds. The initial state was a random distribution of cooperators and defectors, and the update was synchronous

When the fraction of cooperators is averaged only in the surviving configurations, as shown in figure 4.13, a clear dependence of z on η_c appears. However when those values are compared with the ones derived for the asynchronous update calculated in the section 4.1.9 and present in the table 4.12, they are not coincident except for $z = 2$. The small drift for lower values of η found may be due to the possibility from several defectors to be converted in the same round.

However, the main conclusions and dependences are still present. When analysing the fraction of final configurations shown in figures 4.14 and 4.15 the dependences on z and N are found. When analysing the figure 4.14 and comparing with the figures 4.2 4.3 4.4 and 4.5 obtained with asynchronous update, one can see that the interval where the surviving configurations dominate is smaller. The dependence on z of the values of η where the surviving configurations are more probable can be seen, but dependence seems only evident in the higher value of the interval as for the lower value they seem almost equal for all considered values of z except for 2 and 16. That is due to the possibility of up to z defectors be converted into cooperators in a single round. The analysis of the figure 4.15 shows a clear dependence on N of the number of configurations. For smaller values of N the fraction of surviving configurations decreases and the number of configurations that fall in an absorbing state increases.

Like in asynchronous update the dependence on z on the intervals where a surviving state

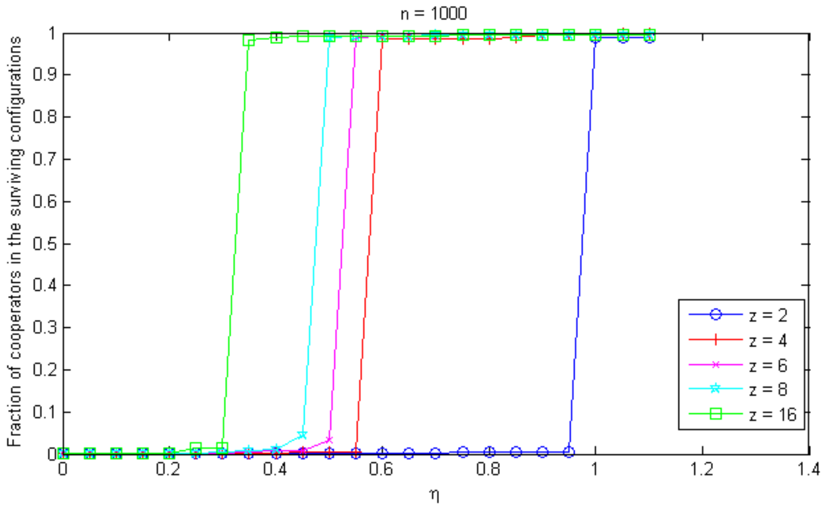

Figure 4.13: Fraction of cooperators in the surviving configuration, for one-dimensional regular ring networks with $n = 1000$ and $z = 2, 4, 6, 8$ and 16, using 10^4 samples and 10^5 rounds. The initial state was a random distribution of cooperators and defectors, and the update was synchronous

is more probable, and on the critical values of the efficiency paramenter η_c was observed. The dependence on N of the fraction of surviving configurations was also observed.

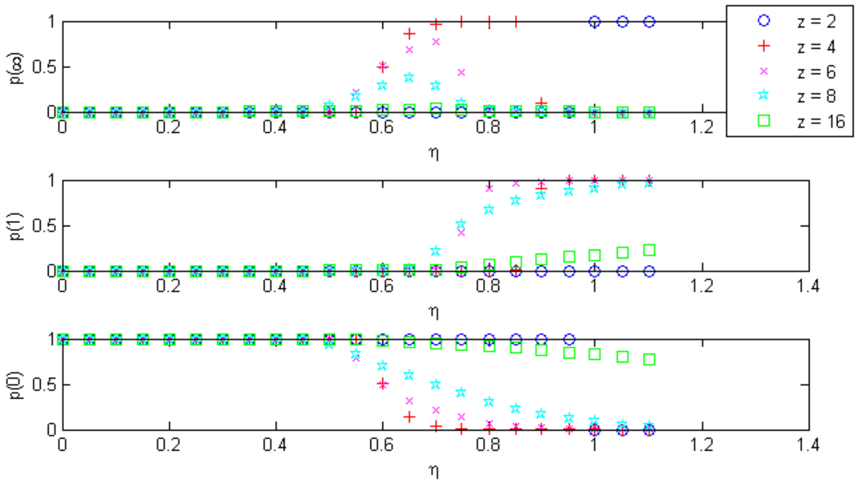

Figure 4.14: Fraction of surviving configurations $p(2)$, fraction of configurations with only cooperators $p(1)$ and fraction of configurations with only defectors $p(0)$, for one-dimensional regular ring networks with $n = 1000$ and $z = 2, 4, 6, 8$ and 16, using 10^4 samples and 10^5 rounds. The initial state was a random distribution of cooperators and defectors, and the update was synchronous

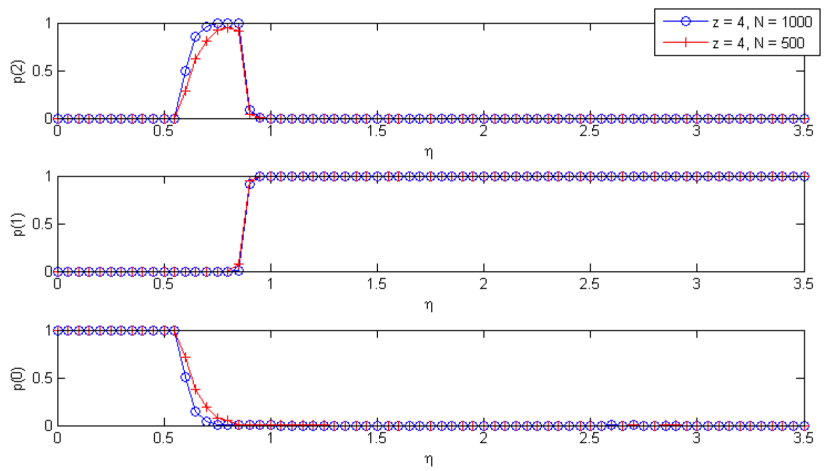

Figure 4.15: Fraction of surviving configurations $p(2)$, fraction of configurations with only cooperators $p(1)$and fraction of configurations with only defectors $p(0)$, for one-dimensional regular ring networks with $n = 1000$ and $z = 2, 4, 6, 8$ and 16, using 10^5 samples and 10^5 rounds. The initial state was a random distribution of cooperators and defectors, and the update was synchronous

Chapter 5

Results for two-dimensional lattice regular networks

Here public goods games defined on two-dimensional lattice with different number of neighbors are considered. A synchronous update was used in the simulations.

5.1 One cooperator surrounded by defectors

In order to understand more complex configurations one must first treat the simpler ones, therefore this chapter will start by considering the initial configuration of one cooperator surrounded by defectors. With the help of the representation of neighbors in figure 5.1 the gains from the different nodes can be calculated. The cooperator in all of those schemes is represented as a red square (A), and the colored squares are the first neighbors, for each of the 3 cases considered ($z = 4$, 8 and 12).

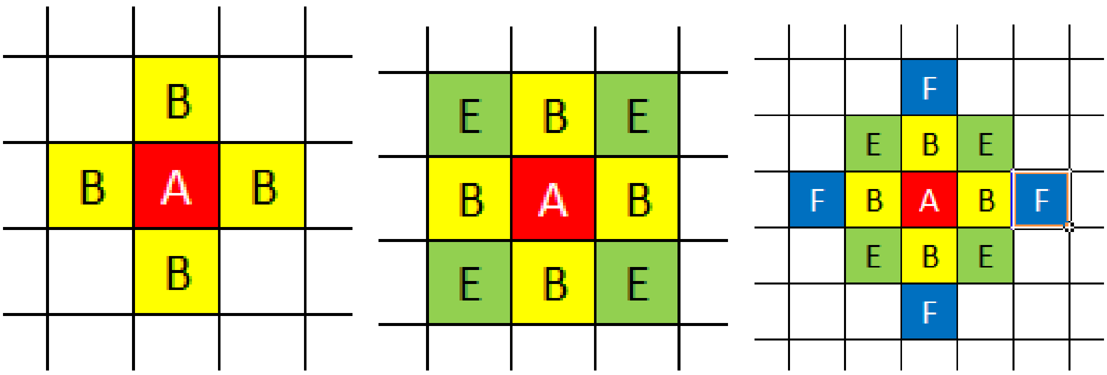

Figure 5.1: Neighbors of a given site A in regular two-dimensional networks: (left) z=4, (center) z=8 and (right) z=12 A cooperator is on site A and neighbors labelled with the same letter and color are equivalent.

In the case $z = 4$ all yellow sites (B) are equivalent, so the gains can be writen as:

$$\begin{cases} G(A) = \eta(z+1) - (z+1) \\ G(B) = 2\eta \end{cases} \tag{5.1}$$

In the case $z = 8$ the neighbors from the nodes in yellow (B) will have 6 nodes in common with the neighbors from the node in red (A) , while the green ones (E) only have 4 nodes in common, so the gains can be writen as:

$$\begin{cases} G(A) = \eta(z+1) - (z+1) \\ G(B) = 6\eta \\ G(E) = 4\eta \end{cases} \tag{5.2}$$

In the case $z = 12$, following the same method as in the case $z = 8$ one get three different gains for the neighbors of the red node (A):

$$\begin{cases} G(A) = \eta(z+1) - (z+1) \\ G(B) = 8\eta \\ G(E) = 7\eta \\ G(F) = 5\eta \end{cases} \tag{5.3}$$

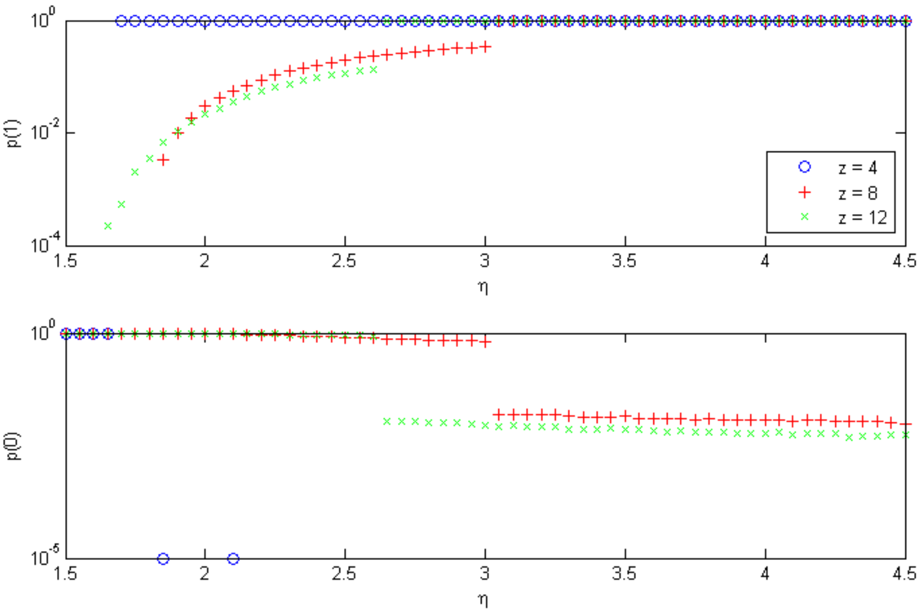

Figure 5.2: Fraction of configurations with only cooperators $p(1)$ and fraction of configurations with only defectors $p(0)$, for two-dimensional regular networks with $n = 1024$ and $z = 4, 8$ and 12, using 10^5 samples and 10^5 rounds. The initial state was a cooperator placed at random in a network with defectors, the update was synchronous, and it was normalized with a fixed value.

When comparing, for different z, the gains of the different defectors with the gains from the cooperator, the results are the η values present in table 5.1, which are the values where the cooperator is able to win over the defectors.

Table 5.1: Expected values of η for which one cooperator is able to win over the defectors for two-dimensional lattice networks

Number of neighbors	B	E	F
$z = 4$	$\frac{5}{3}$	–	–
$z = 8$	3	1.8	–
$z = 12$	2.6	$\frac{13}{6}$	1.625

The theoretical results found in table 5.1 and the simulation found in figure 5.2 agree once more. The smallest η values listed in table 5.1, for each case, are also the smallest η values observed in the figure that allow the system to reach the absorbing state with only cooperators. The bigger calculated value of η is present as a step in the probabilities which grow closer to reaching a value of 1. As in the results for the unidimensional ring network, the η which have transitions associated depends only on the initial state and on z.

5.2 One defector surrounded by cooperators

The next initial configuration to be analised is one defector surrounded by cooperators. One could use the representation of neighbors present in figure 5.1 again, but now the red square (A) represents the defector, while the other colors represent the cooperators with different gains. With the same logic one gets the following gains, for $z = 4$:

$$\begin{cases} G(A) = 20\eta \\ G(B) = 23\eta - 5 \end{cases} \tag{5.4}$$

Also, for $z = 8$ the gains are:

$$\begin{cases} G(A) = 71\eta \\ G(B) = 75\eta - 9 \\ G(E) = 77\eta - 9 \end{cases} \tag{5.5}$$

And finally the gains for $z = 12$:

$$\begin{cases} G(A) = 156\eta \\ G(B) = 161\eta - 13 \\ G(E) = 161\eta - 13 \\ G(F) = 164\eta - 13 \end{cases} \tag{5.6}$$

Using the same method, one get the values of η present in table 5.2. When those values are compared with figure 5.3 they disagree. The reason being that many defectors can be converted in the same round.

Table 5.2: Expected values of η for which one defector is defeated by the cooperators for two-dimensional lattice networks

Number of neighbors	B	E	F
$z = 4$	$\frac{5}{3}$	–	–
$z = 8$	$\frac{9}{4}$	$\frac{3}{2}$	–
$z = 12$	$\frac{13}{5}$	$\frac{13}{5}$	$\frac{13}{8}$

If one assumes two configurations for $z = 4$, one of them with only four defectors and another with three defectors, those results can be explained. Those configurations are pictured in figure 5.4. The yellow or blue squares are cooperators, while the red or green ones are defectors. Considering the configuration with four defectors, in figure 5.4 (left), the respective gains are:

$$\begin{cases} G(B) = 16\eta \\ G(E) = 24\eta - 5 \end{cases} \tag{5.7}$$

For the configuration with three defectors the gains are:

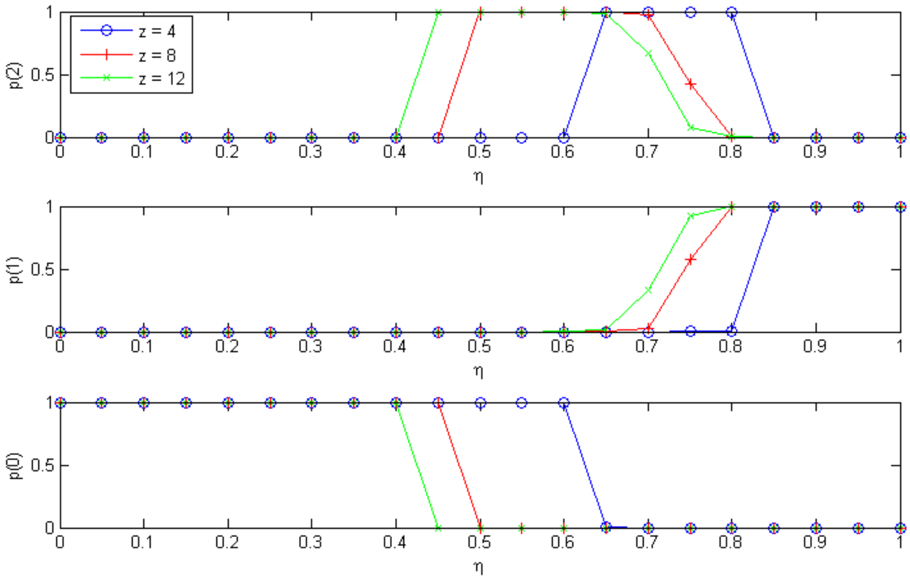

Figure 5.3: Fraction of surviving configurations $p(2)$, fraction of configurations with only cooperators $p(1)$ and fraction of configurations with only defectors $p(0)$, for two-dimensional regular networks with $n = 1024$ and $z = 4, 8$ and 12, using 10^5 samples and 10^5 rounds. The initial state was a cooperator placed at random in a network with defectors, the update was synchronous, and it was normalized with a fixed value.

$$\begin{cases} G(B) = 18\eta \\ G(A) = 17\eta \\ G(E) = 24\eta - 5 \end{cases} \tag{5.8}$$

Comparing the gains of the node B with those from the node E for the configuration with four defectors, the result $\eta = \frac{5}{8}$ is obtained. When comparing the gain from the node E with the node A for the configuration with three defectors, the result $\eta = \frac{5}{7}$; the node E compared with the node B, the result $\eta = \frac{5}{6}$ is obtained.

The conclusion for this kind of configuration, is that the number of defectors will increase for lower values of η eventually leading to the colapse of all the defectors in the group. Two specific cases have been shown, but there are many equivalent and similar configurations, but those will lead to the same values of η. For $z = 8$ and $z = 12$ the intervals are different, as they depend on z, but the same conclusions can be drawn.

5.3 Random initial distribution of defectors and cooperators

Stating from an initial configuration with a random distribution of cooperators and defectors, one gets the fraction of cooperators close to η_c present in figure 5.5. It is quite evident the dependence on z of the value of η_c, but in figure 5.6, which has the fraction of cooperators in the surviving configurations, it is even more evident.

When analysing the graphs present in the figures 5.6 and 5.7 which show the dependence of z and N in the η_c, again, the only one that have a noteworthy effect is z in figure 5.7. In both figures there are configurations far away from any absorbing state near to η_c. That effect can

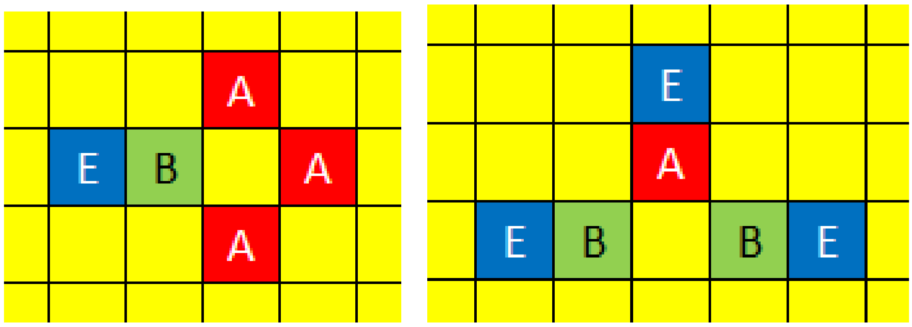

Figure 5.4: Classes of sites in regular two-dimensional networks with $z = 4$: (left) configuration with 4 defectors (right) configuration with 3 defectors. Red (A) and green (B) sites are defectors while blue (E) and yellow are cooperators.

be explained with the number of rounds used, for an infinite number of rounds, that lenght will be reduced to an abrupt transition.

In the plot of figures 5.8 and 5.9 the dependence on N, of the fraction of the configurations in the different states after 10^5 rounds is shown. Both figures show that the places where the transitions take place are the same, but the number of configurations in each state are different. Like the results for the regular one-dimensional ring network, with the bigger size of the system, more configurations are out of both absorbing states. These plots show that for an infinite size network there exists an interval in the values of η where the system remains out of both absorbing states. Those configurations are almost ocupied by cooperators, while only a small group of defectors survives, as show in figure 5.6. For lower values of η the system is in the absorbing state with only defectors, while for higher values the system is in the absorbing state with only cooperators, as presented in figures 5.8 and 5.9.

In general, despite the different topology of the regular two-dimensional lattice network when compared with the regular one-dimensional ring network, in both the dependence of η_c on z and the dependence on N of the surviving configurations are observed.

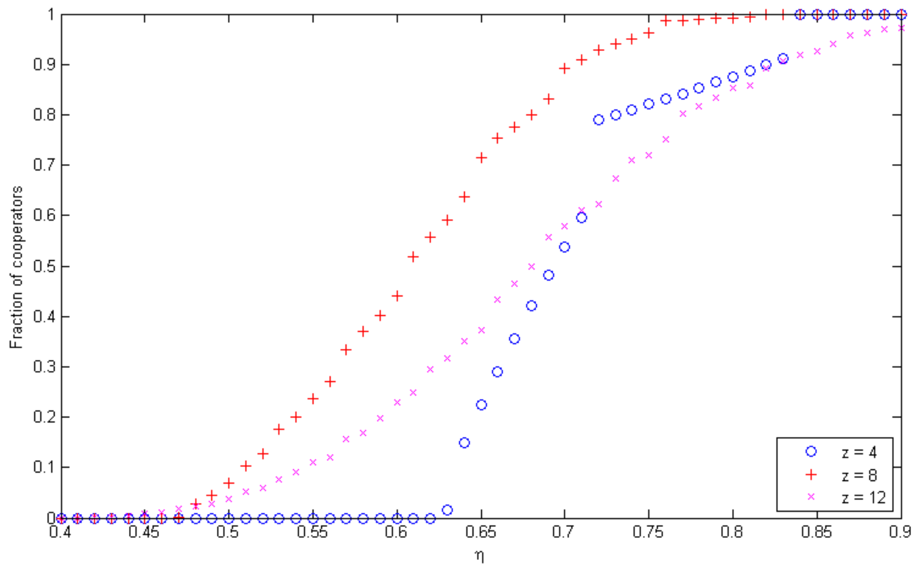

Figure 5.5: Total fraction of cooperators in regular two-dimensional lattice networks, with $N = 1024$ and $z = 4$, 8, and 12, using 10^4 samples and 10^5 rounds. The initial state was a random distribution of cooperators and defectors, the update was syncronous, and it was normalized with the maximum value.

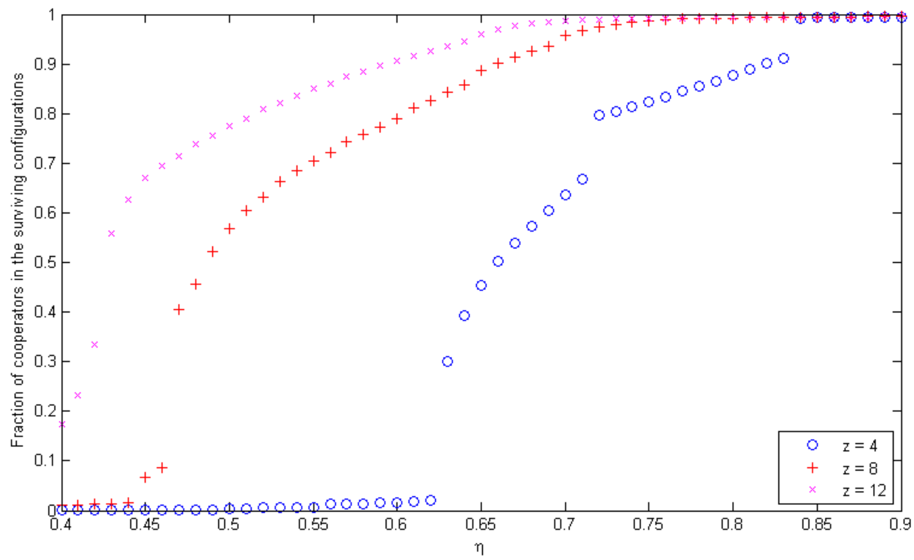

Figure 5.6: Fraction of cooperators in the surviving configurations for regular two-dimensional lattice network using synchronous update, using 10^4 samples and 10^5 rounds. The initial state was a random distribution of cooperators and defectors, it was normalized with the maximum value, with $N = 1024$ and $z = 4$, 8 and 12.

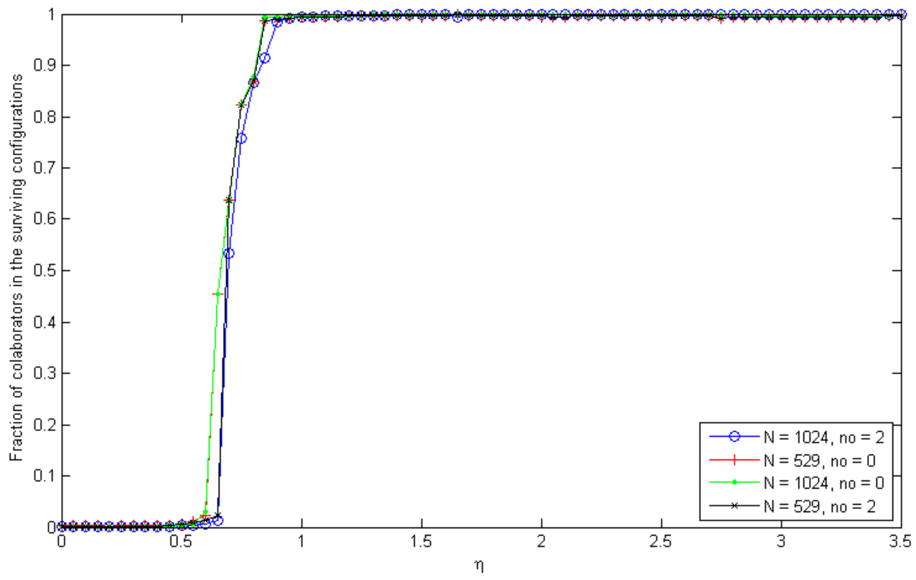

Figure 5.7: Fraction of cooperators in the surviving configurations for regular two-dimensional lattice network using synchronous update, using 10^4 samples and 10^5 rounds. The initial state was a random distribution of cooperators and defectors, it was normalized with the maximum value, with $N = 529$ and 1024; $z = 4$ and $(no = 0)$ normalized with the maximum value or $(no = 2)$ normalized with a fixed value.

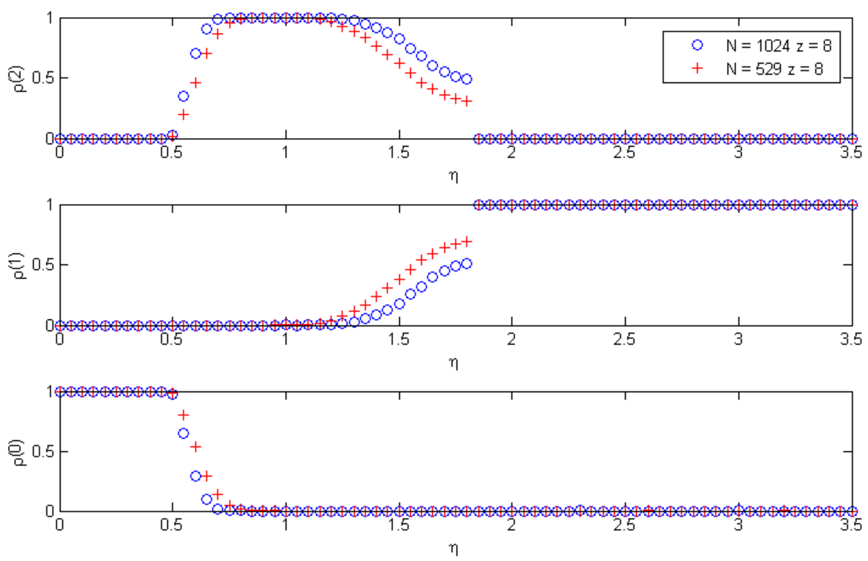

Figure 5.8: Fraction of surviving configurations $p(2)$, fraction of configurations with only cooperators $p(1)$ and fraction of configurations with only defectors for the regular two-dimensional lattice network with $N = 529$ and 1024; and $z = 8$. The simulations were made using synchronous update, with 10^4 samples and 10^5 rounds. The initial state was a random distribution of cooperators and defectors, it was normalized with the maximum value.

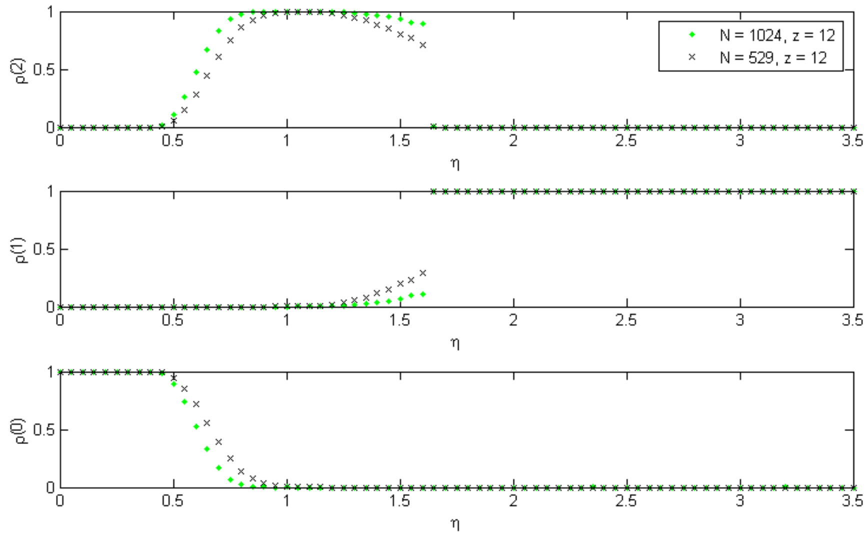

Figure 5.9: Fraction of surviving configurations $p(2)$, fraction of configurations with only cooperators $p(1)$ and fraction of configurations with only defectors $p(0)$ for the regular two-dimensional lattice network with $N = 1024$ and 529 and $z = 12$, using synchronous update, using 10^4 samples and 10^5 rounds. The initial state was a random distribution of cooperators and defectors, and it was normalized with the maximum value.

Chapter 6

Results for scale-free networks

Simulations using a random distribution of cooperators and defectors in the scale-free network were made. The update was synchronous and it was normalized with the maximum value.

Unlike the previous chapters, the gains will not be discussed, because the network structure is not regular, meaning that without a fixed number of neighbors, the number of possible cases that exist are large. As the scale-free networks have a random component in its creation, the conclusions taken for a particular case have to be averaged over the network structure itself.

6.1 Game A

When comparing the fraction of cooperators averaged over all configurations presented in figure 6.1 with the results from the regular network's results present in figure 4.12, the conclusion is that the values of η where the transition takes place are lower [13].

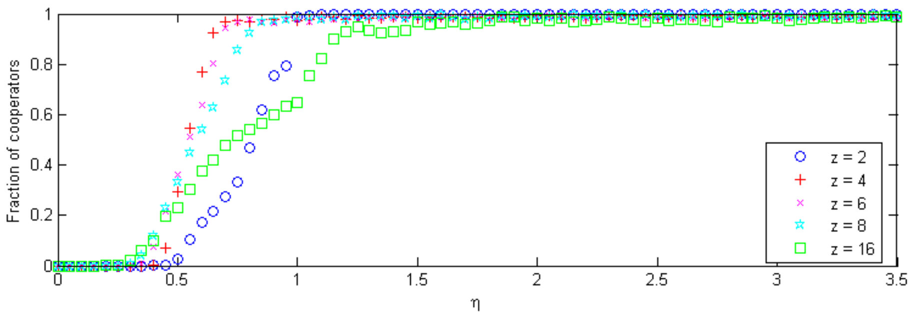

Figure 6.1: Total fraction of cooperators in scale-free networks in the game A, with $n = 1000$ and $m = 1, 2, 3, 4$ and 8, using 100 different realizations of the network, each of them simulated 100 times for 10^5 rounds. The initial state was a random distribution of cooperators and defectors, the update was syncronous, it was normalized with the maximum value.

The total fraction of cooperators as function of η and z is shown in figure 6.1. For $\eta > 1$, only $z = 16$ does not have almost total or total cooperation. When the size of the group where the games take place becomes bigger, the cooperators will colapse because of the high cost of cooperation.

The fraction of surviving configurations is presented in figure 6.2. From the analysis of the figure, a dependence on the average connectivity z can be seen. A region where many configurations stay in the active state can be found. In the figure these intervals are located in $0.5 \lessapprox \eta \lessapprox 1$ for $z = 2$, 4, 6 and 8. For bigger values of z the fraction is smaller when N and η

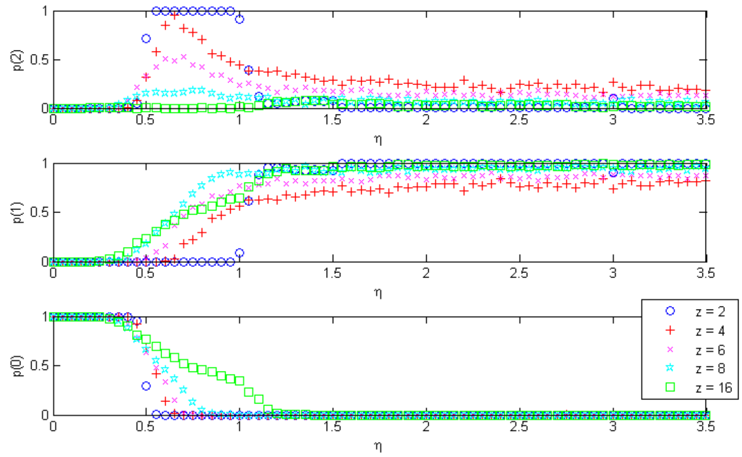

Figure 6.2: Fraction of surviving configurations $p(2)$, fraction of configurations with only cooperators $p(1)$ and fraction of configurations with only defectors $p(0)$, for scale-free network using the game A with synchronous update, using 100 different realizations of the network, each of them simulated 100 times for 10^5 rounds. The initial state was a random distribution of cooperators and defectors, it was normalized with the maximum value, with $N = 1000$ and $z = 2$, 4, 6, 8 and 16.

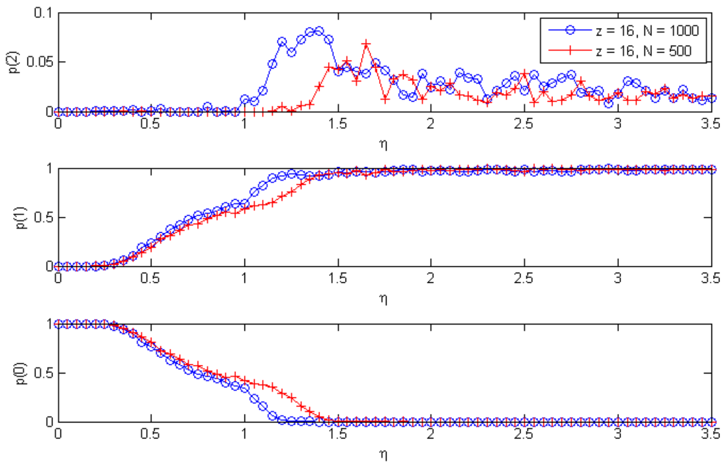

Figure 6.3: Fraction of surviving configurations $p(2)$, fraction of configurations with only cooperators $p(1)$ and fraction of configurations with only defectors $p(0)$, for scale-free network using the game A with synchronous update, using 100 different realizations of the network, each of them simulated 100 times for 10^5 rounds. The initial state was a random distribution of cooperators and defectors, it was normalized with the maximum value, with $N = 500$ and 1000; and $z = 16$.

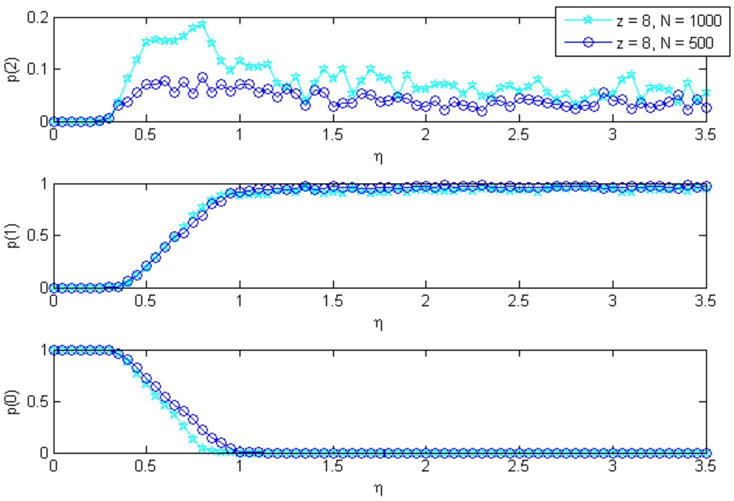

Figure 6.4: Fraction of surviving configurations $p(2)$, fraction of configurations with only cooperators $p(1)$ and fraction of configurations with only defectors $p(0)$ for scale-free network using the game A with synchronous update, using 100 different realizations of the network, each of them simulated 100 times for 10^5 rounds. The initial state was a random distribution of cooperators and defectors, it was normalized with the maximum value, with $N = 500$ and 1000; and $z = 8$.

are held constant.

In figures 6.3 and 6.4 the fraction of configurations is ploted for different N for $z = 16$ and $z = 8$ respectively. A dependence of the fraction of active configurations on N was found. For bigger values of N the number of configurations in the active state is bigger, the fraction of configurations with only cooperators is slightly bigger and the fraction of configurations with only defectors is smaller. From the analysis of both figures a dependence on z of the η intervals is more evident than in figure 6.2.

To discuss the critical value of the efficiency paramenter η_c, one must analyse the fraction of cooperators in the surviving configuration present in the figure 6.5. The dependence on z of the critical value of η_c, is evident in the figure 6.5. The bigger values of z have lower values of η_c. This may seem contraditory to the conclusion taken from the figure 6.1, the defectors will take over easly in networks which the average number of connection of a node is bigger. The explanation for that is quite simple, if a given configuration with big z keeps itself out of an absorbing state for low values of η, the cooperators present will eventually take over. Meaning, as the value of z grows, that number of connected nodes required to win over the defectors will increase, so that in a given random configuration with a fixed N the probability to have those groups will deacrease as z increase. In the end, even for bigger values of z, if the required number of cooperators is present the fraction of cooperators can be almost 1 for lower values of η. Another feature present in that graph is the reduction of the fraction of cooperators, they hit a peak right after the η_c and then their fraction decrease. For larger values of z the effect the effect is more evident. That effect is due to the decrease in the number of the surviving configurations that more and more fall in the state with only cooperators, only the configurations far from falling in any absorbing state remain out of those state from values of $\eta > \eta_c$. That decrease in the number of surviving configurations can be observed in the figure 6.2, due to that the results for the fraction of cooperators in the surviving configurations have a big statistical error.

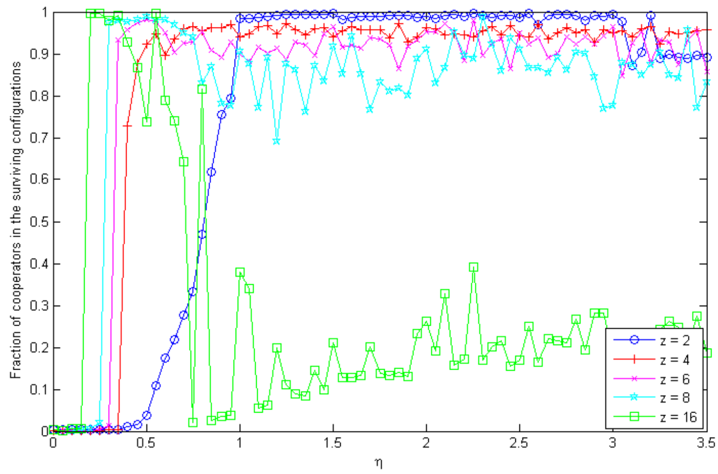

Figure 6.5: Fraction of cooperators in the surviving configurations for scale-free network using the game A with synchronous update, using 100 different realizations of the network, each of them simulated 100 times for 10^5 rounds, with for $N = 1000$ and $m = 1$, 2, 3, 4 and 8. The initial state was a random distribution of cooperators and defectors, it was normalized with the maximum value.

For the game A in the scale-free networks, using the synchronous update, the same dependence on η with the average number of neighbors and the dependende on N of the numbers of the configurations that fall in an absorbing state or remain out of it, had been found, similarly to the results found for the regular network.

6.2 Game B

In the figure 6.6 the total fraction of cooperators are shown as function of η and z. The values of η where the transition takes place are lower than the ones found for the same situation in the game A, regardeless of the considered z.

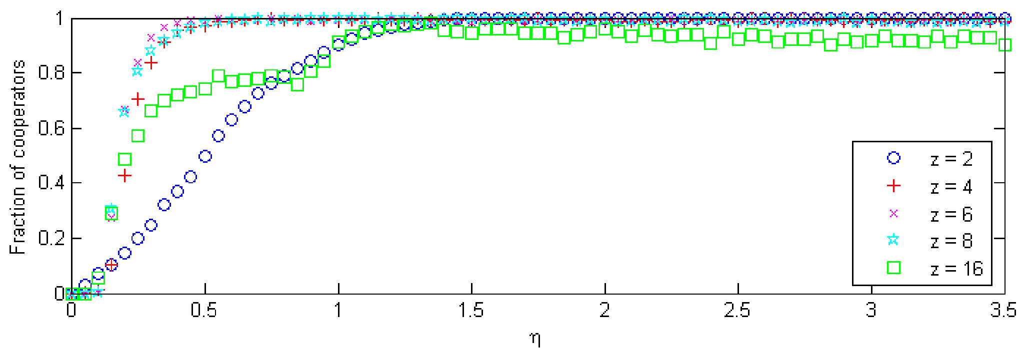

Figure 6.6: Total fraction of cooperators in scale-free networks in the game B, with $N = 1000$ and $z = 2$, 4, 6, 8 and 16, using 100 different realizations of the network, each of them simulated 100 times in 10^5 rounds. The initial state was a random distribution of cooperators and defectors, the update was syncronous, it was normalized with the maximum value.

The fraction of surviving configurations, shown in the figure 6.7, are equal or close to 1 in a bigger interval, stating for lower values of η and ending for bigger values, when compared with the intervals for the game A. This bigger fraction of configurations for a large η interval can be explained by the decrease of the cost of the cooperation. Many of the configurations that had collapsed in the game A into the state with only defectors can now remain out of it due to the decreased value for cooperation. After those peak in the value of the surviving configurations, the number of configurations with only cooperators stay even closer to 1 when compared with the values of the game A. The intervals where the two absorbing states dominate and where exists the active state depend upon the average connectivity of the network z.

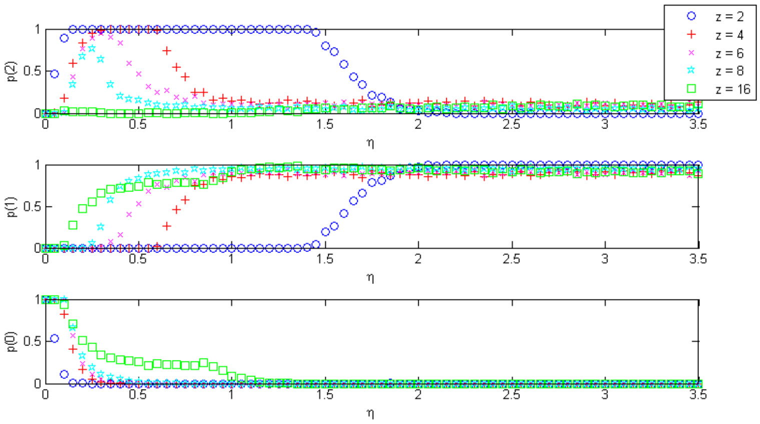

Figure 6.7: Fraction of surviving configurations $p(2)$, fraction of configurations with only cooperators $p(1)$ and fraction of configurations with only defectors $p(0)$ for scale-free network using the game B with synchronous update, using 100 different realizations of the network, each of them simulated 100 times in 10^5 rounds. The initial state was a random distribution of cooperators and defectors, it was normalized with the maximum value, with for $N = 1000$ and $z = 2$, 4, 6, 8 and 16.

Analysing the figures 6.8 and 6.9 the same features observed for the game A are present. A dependence on N in the fraction of configurations in each state is found. The bigger the size of the network, bigger the fraction of configurations in the active state.

In the figure 6.10 one sees that like the results in the game A, for $\eta > \eta_c$, the fraction of cooperators decrease after hitting a maximum. Also, the dependence on z in the value of η_c is present. In the interval, generaly the number of configurations with only defectors is smaller. The same conclusions for the η_c dependence on N can be drawn.

In general, both games A and B show the same dependences of N in the fraction of the different final states and of z in the value of η_c. A existence of an interval where for an infinite size network where the fraction of configurations in the active state is close to 1 is also present. Those effects are simillar to the ones found in the results for both one-dimensional regular ring network and two-dimensional lattice network.

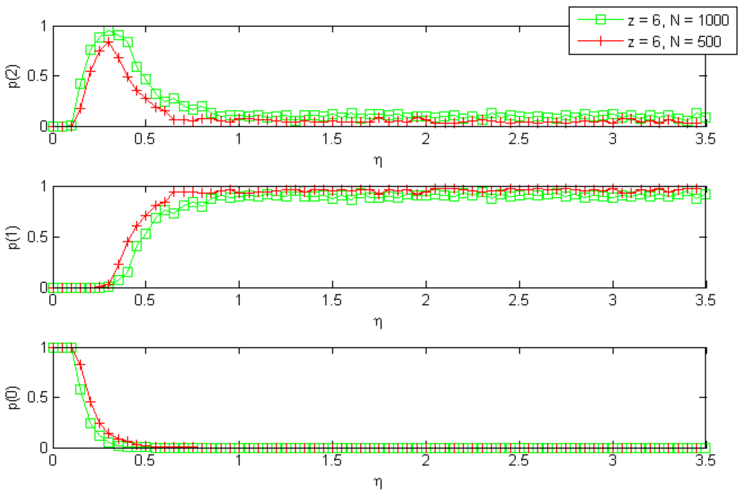

Figure 6.8: Fraction of surviving configurations $p(2)$, fraction of configurations with only cooperators $p(1)$ and fraction of configurations with only defectors $p(0)$ for scale-free network using the game B with synchronous update, using 100 different realizations of the network, each of them simulated 100 times in 10^5 rounds. The initial state was a random distribution of cooperators and defectors, it was normalized with the maximum value, with for $N = 500$ and 1000; and $z = 6$.

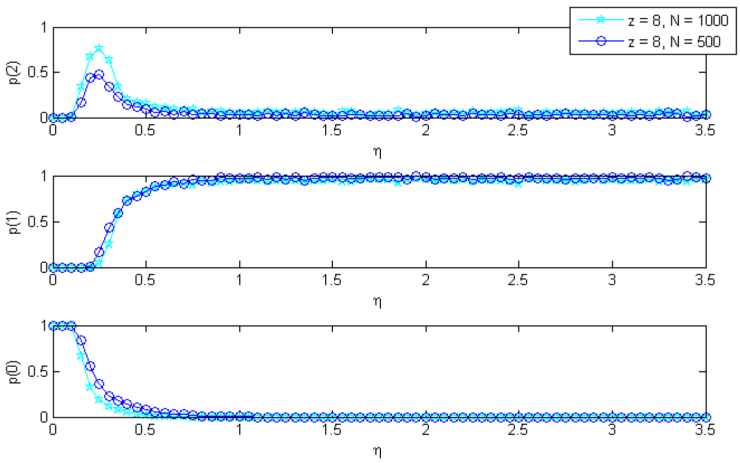

Figure 6.9: Fraction of surviving configurations $p(2)$, fraction of configurations with only cooperators $p(1)$ and fraction of configurations with only defectors $p(0)$ for scale-free network using the game B with synchronous update, using 100 different realizations of the network, each of them simulated 100 times in 10^5 rounds. The initial state was a random distribution of cooperators and defectors, it was normalized with the maximum value, with $N = 500$ and 1000; and $z = 8$.

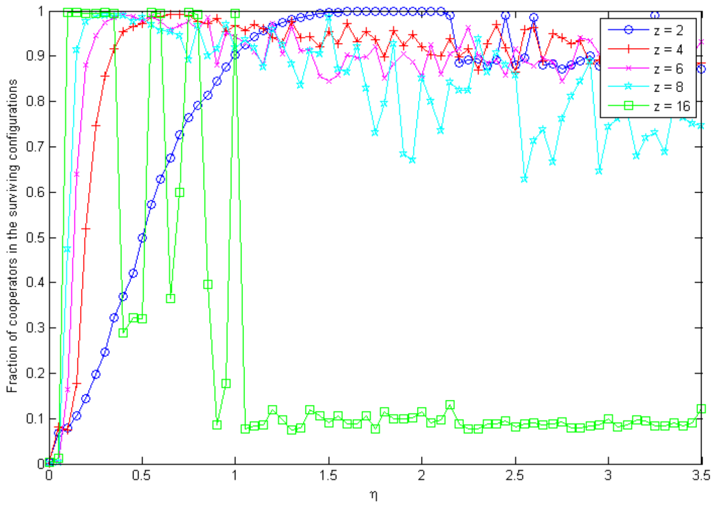

Figure 6.10: Fraction of cooperators in the surviving configurations for scale-free network using the game B with synchronous update, using 100 different realizations of the network, each of them simulated 100 times in 10^5 rounds. The initial state was a random distribution of cooperators and defectors, it was normalized with the maximum value, with $N = 1000$ and $z = 2$, 4, 6, 8 and 16.

Chapter 7

Future Work

The work presented in these pages is not complete. The next logical step is to complete the results with simulations from the asynchronous actualization for both regular two-dimensional lattice and scale-free networks.

The fitness distribution was not studied in this work.

There is a function already programed and tested for small-world networks [25]. In these kind of networks, each node connects $m = \frac{z}{2}$ times to other nodes chosen at random. Those networks when used with this model will hopefully contribute to provide a better understanding of the model.

A new type of game can be considered with a variable η for the game centered in each node depending on the number of nodes that take part in that game.

It would be interesting to consider a variable η dependent on time. Meaning, the value of η would be dependent on the round.

For small-world networks it would also be an interesting experiment to change some of the connections between each round.

Another possible inovation is allowing to have several moves in a round, and instead of, each node having one simple answer, they can have a stategy (eg. a TFT would cooperate if most of the nodes in the group in the last move had cooperate, instead it would defect).

A possible and interesting option, instead of introducing several moves in a round and create complex strategies, it would be to introduce the reciprocators, which only contribute if a given number of nodes are cooperators otherwise they will defect [26].

One must notice that most of these ideias are in this moment only ideias and no deep study about their usefulness was made.

Conclusion

The results of the simulations for the one-dimensional regular ring networks and for scale-free networks are in agreement with previous results [13]. The behavior of the total fraction of cooperators averaged over all configurations and starting with a random initial condition was similar in both cases. A boost in the cooperation was found for the scale-free network when compared with the regular one-dimensional ring network, meaning the total fraction of cooperators is bigger for lower values of the parameter of the efficiency of the contributions η. The same boost was found for the game B in scale-free network when compared with the game A. This effect was previously reported [13].

A different type of network was studied: a regular two-dimensional lattice network.

Different quantities were also studied for each of the networks mentioned above: the fraction of cooperators in the surviving configurations and the fraction of configurations that fall in one of the absorbing states for large amount times.

When analysing the number of configurations in each state it was found the existence of two absorbing states, one with only cooperators and other with only defectors. It was found that the number of configurations reaching the absorbing states depends on the size of the network N, and for an infinite size network it exists an interval where the the fraction of surviving configurations tends to 1 when the size of the network goes to infinity. These surviving configurations are dominated by cooperators, but a small group of defectors persists keeping that configuration away from the absorbing state with only cooperators. Consequentely it is important to study averages considering only the surviving configurations. When comparing the results from the asynchronous actualization with the ones from the synchronous actualization it was found that the η interval where the system remains active is smaller in the later case. It turns out that for a synchronous actualization a group up to z defectors can be converted simultaneously.

The study of the fraction of cooperators in the surviving configurations shows a phase transition value from an absorbing state full of defectors to a state dominated by cooperators as a function of the efficiency parameter η. It was found that this critical value depends on the average connectivity z of the network, decreasing as the connectivity z of the network increases. For the one-dimensional regular ring network it is possible, from an analysis of the gains of nodes in selected configurations to predict critical values of η.

Bibliography

Introduction

[1] Siegfried, Tom "A Beautiful Math - John Nash, Game Theory and the Modern quest for a Code of Nature", Joseph Henry Press (2006)

[2] Gould, Stephen Jay "The Structure of Evolutionary Theory", Harvard University Press, 2002

[3] Milgram, Stanley "The small world problem", Psychology Today **2**, 60-67, 1967

[4] Albert, R. & Barabási, A. L. "Emergence of scaling in random networks", Science **286** , 509-512. (1999)

[5] Einstein, Albert "On the Electrodynamics of Moving Bodies", Annalen der Physik **17** , 891-921. (1905)

[6] Einstein, Albert "Does the inertia of a Body Depend upon Its Energy Content?", Annalen der Physik **18** , 639-641. (1905)

[7] Hauert, C., De Monte, S., Hofbauer, J. & Sigmund, K. "Voluntering as Red Queen mechanism for cooperation in public goods games", Science **296** , 1129-1132. (2002)

[8] Szabo, G. L. & Hauert, C. "Phase Transitions and volunteering in spatial public goods games", Phys. Rev. Lett. **89** , (2002)

[9] Szabo, G. L. & Hauert, C. "Evolutionary prisoner's dilemma games with voluntary participation", Phys. Rev. E **66** , (2002)

[10] Santos, F. C. & Pacheco J. M. "Scale-free networks provide a unifying framework for the emergence of cooperation", Phys. Rev. Lett **95** , (2005)

[11] Santos, F. C., Pacheco J. M. & Lenaerts, T. "Evolutionary dynamics of social dilemmas in structured heterogeneous populations", Proc. Natl Acad. Sci. USA **10**, 3490-3494 (2006)

[12] Ohtsuki, H., Hauert, C., Lieberman, E. & Nowak, M. A. "A simple rule for the evolution of cooperation on graphs and social networks", Nature **441**, 502-505 (2006)

[13] Santos, Francisco C., Santos, Marta D. & Pacheco, Jorge M. "Social diversity promotes the emergence of cooperation in public goods games", Nature **454**, 213-216. (2008)

Evolutionary game theory

[14] Nowak, Martin "Evolutionary Dynamics - Exploring the equations of life", Harvard University Press (2006)

[15] Churchill, F.B. "William Johannsen and the genotype concept", Journal of the History of Biology **7**, 5-30 (1974)

[16] Hofbauer, Josef & Sigmund Karl "Evolutionary Games and Population Dynamics", Cambridge University Press (1998)

[17] Nash, John Forbes "Equilibrium Points in N-Person Games", Proceedings of the National Academy of Siences **36**, 48-49 (1950)

[18] Nash, John Forbes "The Bargaining Problem", Econometrica **18**, 155-162 (1950)

[19] Nash, John Forbes "Two-person Cooperative Games", Econometrica **21**, 128-140 (1953)

[20] Nash, John Forbes "Non-cooperative Games", Annals of Mathematics **54**, 256-295 (1951)

Public Goods Games

[21] Milinsky, M., Seemman, D., Krambeck, H. J. & Sigmund, K. "Stabilizing the Earth's climate is not a losing game: Supporting evidence from public goods experiments", Proc. Natl Acad. Sci. USA **103**' , 3994-3998. (2006)

[22] Albert, R. & Barabási, A. L. "Statistical mechanics of complex networks", Rev. Mod. Phys. **74**' , 47-97. (2002)

[23] Dorogovtsev, S. N. & Mendes, J. F. "Evolution of Networks: From Biological Nets to the Internet and WWW", Oxford: Claredon Press (2002)

Markov Chains

[24] Kemeny, John & Snell, J. Laurie "Finite Markov Chains", Springer-Verlag (1976)

Future Work

[25] Watts, D. J., Strogatz, S. H. "Colective dynamics of 'small-world' networks", Nature, **393** 409-410 (1998)

[26] Kurzban, Robert & Houser, Daniel "Experiments Investigating Cooperative Types in Humans: A Complement to Evolutionary Theory and Simulations", Proceedings of the National Academy of Sciences USA, **102** 1803-1807 (2005)