Lancaster University Management School

# Lancaster University Management School Working Paper 2005/040

# Smooth pasting as rate of return equalization

Mark Shackleton and Sigbjorn Sodal

The Department of Accounting and Finance
Lancaster University Management School
Lancaster LA1 4YX
UK



The LUMS Working Papers series can be accessed at http://www.lums.lancs.ac.uk/publications/
LUMS home page: http://www.lums.lancs.ac.uk/

# Smooth pasting as rate of return equalization

Mark B. Shackleton*& Sigbjørn Sødal†

18 April 2005

**Abstract**

We further elucidate the smooth pasting condition behind optimal early exercise of options. It is easy to show that smooth pasting implies rate of return equalization between the option and the levered position that results from exercise. This yields new economic insights into the optimal early exercise condition that the option holder faces.

Key Words: Smooth pasting, rates of return, option elasticity, real option.

JEL: G31; D92; D81; C61; Capital Budgeting; Investment Policy.

## 1 Introduction

The smooth pasting (or high–contact) condition associated with option and real options decisions has generated considerable interest because of the optimality of early exercise. It is well known that smooth pasting is a first–order condition for optimum; proposed by Samuelson (1965), proven by Merton (1973), discussed by Dumas (1991) and several others. Brekke and Øksendal (1991) also show that the condition is sufficient under weak constraints. Nonetheless, smooth pasting remains somewhat mysterious to both economists and practitioners and it is apparently not very useful for many except theorists. The popular real options introduction by Dixit and Pindyck (1994) saves the discussion of smooth pasting for a quite technical appendix, and no simple rules of thumb seem to exist for practitioners.

Dixit et al. (1999) bridged some of the gap between theory and practice using an analogy between optimal exercise of investment options of the

*Corresponding author. Department of Accounting and Finance, Lancaster University Management School, LA1 4YX, UK. +44(0)1524 594131 m.shackleton@lancaster.ac.uk.

†Agder University College, School of Management, Service Box 422, N-4604 Kristiansand, NORWAY. +47 38141522 sigbjorn.sodal@hia.no.

McDonald and Siegel (1986) type and application of standard market power models. Optimal investment can be characterized by an elasticity–based premium, analogous to the markup price chosen by a profit–maximizing monopolist.

We provide another, more intuitive and natural, explanation of the phenomenon; that of *rate of return equalization* between the option and its levered payoff. This allows a larger audience to appreciate and implement smooth pasting techniques in a wider variety of situations. We also relate results to the elasticity–based rules introduced by Dixit et al. (1999) and Sødal (1998). The results are illustrated here using geometric Brownian motion but are also valid for other diffusions.

## 2 Rates of return

Geometric Brownian diffusions can be written in the Risk Neutral $Q$ or Real World $P$, having drift $r-\delta$ or $\mu-\delta$ respectively ($r$, $\delta$, $\mu$, $\sigma$ represent the continuous risk free, dividend, project return and volatility rates)

$$\frac{dS}{S} = (r-\delta)\,dt + \sigma dW^Q \tag{1a}$$

$$\frac{dS}{S} = (\mu-\delta)\,dt + \sigma dW^P. \tag{1b}$$

Local changes $dC$ in the call price $C$ (puts can also be analyzed) are given by the Ito expansion, furthermore no arbitrage requires that Risk Neutral expectations $E^Q[dC]$ of these changes must be risk free (or the hedged position yields the risk free rate)

$$dC = \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2 + \frac{\partial C}{\partial t}dt \tag{2}$$

$$E^Q[dC] = \frac{\partial C}{\partial S}S(r-\delta)\,dt + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}dt + \frac{\partial C}{\partial t}dt = rCdt. \tag{3}$$

However, Real World returns depend on the premium $\mu - r$ through the expectation operator $E^P[dC]$, which itself can be simplified using the previous Risk Neutral condition

$$\begin{aligned} E^P[dC] &= \frac{\partial C}{\partial S}S(\mu-\delta)\,dt + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}dt + \frac{\partial C}{\partial t}dt \\ &= \frac{\partial C}{\partial S}S(\mu-r)\,dt + rCdt. \end{aligned} \tag{4}$$

Thus, the well known local expected rate of return of the call option is given by

$$r_C = \frac{1}{dt}\frac{E^P[dC]}{C} = r + \epsilon_C(\mu - r) \qquad \geqslant \mu \tag{5}$$

(see Merton (1973)), where the elasticity $\epsilon_C = \frac{\partial C}{\partial S}\frac{S}{C}$ is interpreted as the relative beta.

## 3 Smooth pasting

The rate of return can be investigated at the point of optimal early call exercise $\overline{S}$ (where $\overline{S} > X$, the exercise price). The two conditions necessary for this are value matching (payoff compensates for option termination) and smooth pasting (slope equality between option and payoff functions)

$$C\left(\overline{S}\right) = \overline{S} - X \tag{6a}$$

$$\left.\frac{\partial C}{\partial S}\right|_{S=\overline{S}} = 1. \tag{6b}$$

Thus at the critical exercise boundary $S = \overline{S}$ the call return $r_C$ is

$$r_C\left(\overline{S}\right) = r + \frac{\overline{S}}{C\left(\overline{S}\right)}(\mu - r) = \frac{\mu\overline{S} - rX}{\overline{S} - X} = r_{PO} \tag{7}$$

which is also $r_{PO}$, the return rate of the levered payoff $\overline{S} - X$ (as a fraction of the payoff value $PO$ itself).

At early exercise, not only do the option and payoff functions have the same value, but smooth pasting implies that expected rates of return on both positions are the same.

Risk neutral returns are not useful for determining early exercise since they are always equal (to $r$). Furthermore, any subjective estimate of the future risk premia, $\widehat{\mu}\ (> r)$, affects both return equations equally; overestimating or underestimating $\widehat{\mu}$ has the same effect on subjective returns $\widehat{r}_C$ and $\widehat{r}_{PO}$ at $\overline{S}$. This means that every investor will exercise early when their estimated rates of return on the option and the levered payoff are the same, no matter what their belief $\widehat{\mu}$

$$\widehat{r}_C - r = \left.\frac{\partial C}{\partial S}\right|_{S=\overline{S}} \frac{\overline{S}}{C\left(\overline{S}\right)}(\widehat{\mu} - r) = \frac{\overline{S}}{\overline{S} - X}(\widehat{\mu} - r) = \widehat{r}_{PO} - r. \tag{8}$$

The analysis works equally well for puts ($P(\underline{S})$ evaluated at a lower threshold) whose expected return can be negative

$$r_P(\underline{S}) = r - \frac{\underline{S}}{P(\underline{S})}(\mu - r) = \frac{rX - \mu\underline{S}}{X - \underline{S}} \qquad \gtrless 0. \tag{9}$$

# 4 Relationship to other approaches

The result that the return on the option equates the return on the net payoff is closely related to other findings on smooth pasting. Dixit et al. (1999) argues that the optimal exercise of a perpetual call option consists of maximizing the expected net present value

$$C = \max_{\overline{S}} D(S,\overline{S})(\overline{S} - X) \tag{10}$$

$$D(S,\overline{S}) = E^P\left[e^{-\rho T}\right] = E^Q\left[e^{-rT}\right] \tag{11}$$

where $T$ is the (random) first–hitting time from the current value of the project, $S$, up to the value $\overline{S}$ at which the option is exercised. The objective is to maximize the expected, discounted value of the net pay–off $S - X$, where $D(S,\overline{S})$, $E^P\left[e^{-\rho T}\right]$ or equivalently $E^Q\left[e^{-rT}\right]$ (depending on which approach is applied, (1a) or (1b)) represent expected discount factors that do not explicitly depend on time. Shackleton and Wojakowski (2002) use expected stopping times to show that perpetual calls have constant rates of return and that discounted expectations can be taken in either the Risk Neutral or Real World.

For perpetual call options, the discount rate $\rho$ (or option return $r_C$) is constant and consistent with prior analysis

$$\rho = r + (\mu - r)\,\epsilon_C(\infty) = r_C \tag{12}$$

$$\epsilon_C(\infty) = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad > 1. \tag{13}$$

Here the elasticity of the perpetual option is labelled $\epsilon_C(\infty)$ (section 5 details an analytic approximation to the finite case $\epsilon_C(T)$). Maximizing $C$ with respect to $\overline{S}$, optimal exercise is easily found through

$$\frac{\overline{S}}{\overline{S} - X} = \epsilon_D(\infty) = -\frac{\overline{S}}{D}\frac{\partial D}{\partial S} \tag{14}$$

where $\epsilon_D(\infty)$ is the magnitude of the elasticity of the discount factor with respect to $S$, evaluated at the optimal exercise point $\overline{S}$. The expected value of the project relative to the net payoff equals $C/(S-X) = D(S, \overline{S})$, so the relationship to the discount factor elasticity, $\epsilon_D(\infty)$ ($> 1$), is imminent; the elasticity of $D(S, \overline{S})$ measures the relative change in the net payoff following a marginal change in $S$. This is a measure of returns, but measured per unit of $S$ instead of time.

Using value matching (6a) and smooth pasting (6b) as well as (14), at exercise, the discount factor elasticity coincides with the option elasticity itself, $(\partial C/\partial S)/(C/S)$

$$\epsilon_C(\infty) = \left(\frac{\partial C}{\partial S}\right)_{S=\overline{S}} \frac{\overline{S}}{\overline{C}} = 1 \cdot \frac{\overline{S}}{\overline{S}-X} = \epsilon_D(\infty). \tag{15}$$

Thus the elasticity $\epsilon_D(\infty)$ measures the return on the project relative to the net option payoff, just as $\epsilon_C(\infty)$, measures the return on the option itself. Where Dixit et al. (1999) represents an approach to optimal exercise of options that does not hinge on smooth pasting, Sødal (1998) also uses the discount factor methodology, but for *deriving* value matching (6a) and smooth pasting (6b) by direct optimization. Both references point out the equivalence of the elasticities, *but not the useful interpretation as measures of return*, which has motivated the writing of this note.

## 5 Dividend yields and an analytic approximation

The dividend yield $\delta$ is important since when zero, American calls become zero dividend Black Scholes calls; early exercise, smooth pasting and rate of return equalization are all ruled out ($r$ is key for the put). Without an opportunity cost of waiting, early exercise never occurs. Thus it is important to understand the yield, $\delta$ in determining option returns, since without it equalization is impossible.

The fractional amounts of stock and borrowing required in the replicating portfolio are often labelled $\Delta$, $\kappa$ (both positive functions of $\delta$ and $S, T$ dynamically)

$$C = \Delta S - \kappa X \tag{16}$$

$$\Delta = \frac{\partial C}{\partial S} = \frac{C}{S}\epsilon_C \qquad \kappa = -\frac{\partial C}{\partial X} > 0. \tag{17}$$

Although $\delta$ affects the respective stock and bond elasticities and also the cash yield/capital gain balance, it does not directly affect the option rate of total return. This is because when expected gains and flows are summed, $\delta$ cancels out of the total return (note also that replication requires simplification of the Ito expansion of $dC$ as a function of $\Delta$, $\kappa$)

$$\frac{1}{dt}E^{P}\left[dC\right] = \left(\mu-\delta\right)\Delta S+\delta\Delta S-r\kappa X = \mu\Delta S-r\kappa X. \tag{18}$$

If hedge ratios are homogenous functions in $S/X$ $\left(\Delta\left(S/X\right),\ -\kappa\left(S/X\right)\right)$, smooth pasting also implies $\kappa\left(\overline{S}/X\right)=1$ so that the values *and returns* of the hedge portfolio and the payoff converge irrespective of $\delta$.

If early exercise and smooth pasting become impossible as $\delta\to 0$, it is because the $\Delta$ and *elasticities are prevented from equalizing*, not because of a cashflow argument based on $\delta$.

The effective role of the dividend yield $\delta$ on elasticities ($\epsilon_C$), hedges ($\Delta$) and early exercise can be illustrated through the analytic approximation of Barone–Adesi, Whaley (1987) and MacMillan (1986) (hereafter BAWM). By decomposing the *premium* of an American option over its corresponding (dividend bearing) Black Scholes (1973) value into multiplicative functions ($j\left(T\right)$ of time $T$ and $k\left(S,1-e^{-rT}\right)$ of price $S$ and time $T$) an approximate form is obtained

$$\begin{aligned} C \;\approx\;& Se^{-\delta T}N\left(d_1\left(S\right)\right)-Xe^{-rT}N\left(d_2\left(S\right)\right) \\ &+\left(1-e^{-\delta T}N\left(d_1\left(\overline{S}\left(T\right)\right)\right)\right)\frac{\overline{S}\left(T\right)}{\epsilon_C\left(T\right)}\left(\frac{S}{\overline{S}\left(T\right)}\right)^{\epsilon_C\left(T\right)} \end{aligned} \tag{19}$$

where the special, maturity dependent, elasticity parameter $\epsilon_C\left(T\right)$ (which increases in $\delta$ and is greater than $\epsilon_C\left(\infty\right)$) is defined by

$$\epsilon_C\left(T\right)=\frac{1}{2}-\frac{\left(r-\delta\right)}{\sigma^2}+\sqrt{\left(\frac{\left(r-\delta\right)}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\left(1-e^{-rT}\right)\sigma^2}} \quad > \epsilon_C\left(\infty\right). \tag{20}$$

The maturity dependent critical stock price $\overline{S}\left(T\right)$ solves a value matching condition, which implies a modified form of the critical threshold in the perpetual case above

$$\frac{\overline{S}\left(T\right)}{X}=\frac{1-e^{-rT}N\left(d_2\left(\overline{S}\left(T\right)\right)\right)}{1-e^{-\delta T}N\left(d_1\left(\overline{S}\left(T\right)\right)\right)}\frac{\epsilon_C\left(T\right)}{\epsilon_C\left(T\right)-1}. \tag{21}$$

If the dividend yield increases, exercise will be earlier and less waiting will occur. The American option price itself is sensitive to $\delta$ (it decreases toward immediate payoff $\max\left(S-X,0\right)$ as $\delta$ increases, i.e. for $S>X$ there is a critical $\delta^*$ that triggers early exercise) but that opportunity cost of waiting is partially offset by the early exercise feature (the lowering of $\overline{S}(T)$ with $\delta$).

Under this approximation, the $\Delta$ (and related elasticity $\epsilon_C$) of the American option is given by the sum of $\Delta's$ from the Black Scholes and early exercise premium. Even though BAWM is only an approximation, its analytic form smooth pastes at $\overline{S}(T)$

$$\frac{\partial C}{\partial S} \approx e^{-\delta T} N\left(d_1(S)\right) + \left(1 - e^{-\delta T} N\left(d_1\left(\overline{S}(T)\right)\right)\right)\left(\frac{S}{\overline{S}(T)}\right)^{\epsilon_C(T)-1} \quad (22)$$

$$\left.\frac{\partial C}{\partial S}\right|_{S=\overline{S}(T)} \approx 1.$$

Now as $\delta$ increases, the Black Scholes $\Delta$ (and elasticity in 22) decreases (both because of the exponent $e^{-\delta T}$ and the negative effect of $\delta$ on $d_1$). However, the early exercise premium's $\Delta$ *increases* because of the exponent $e^{-\delta T}$, $d_1$ term decrease and the critical threshold $\overline{S}(T)$ decreasing with increasing $\delta$ (even though $\epsilon_C(T)$ increases), rendering the "discount factor" dependence on $S$, greater. This is why (unlike Black Scholes) the American option $\Delta$ can reach unity. The effect of an increased dividend rate $\delta$ makes the $\Delta$ of the European element smaller but that of the early exercise element larger.

In summary, it is the early exercise premium that mitigates the opportunity cost of waiting and the same is true for the returns. It is the return on the early exercise premium that forces rate of return equalization as the underlying dividend yield increases.

# 6 Intuition and implications

The results above have economic implications and intuition, particularly for real option situations where it is difficult to evaluate the option value function explicitly. Optimal early exercise of real options is driven by two conditions; no loss (gain) of value on exercise and rate of return equalization.

This provides a second, more intuitive, condition to managers other than smooth pasting, which may be difficult to evaluate for some pricing problems. Do managers think that the *rate of return* on the project launch (call) has *fallen* to the same level as its underlying (levered) project? When complex modelling is not possible, managers may be able to assess this new condition heuristically.

If returns are near equalization (and the values of option and project are also close), they should exercise because this is equivalent to smooth pasting (and value matching). This raises the possibility that a market participant without a pricing model could determine his early exercise strategy *empirically* from the returns and pricing he experiences.

Thus, even if the value function and its derivative are theoretically unknown, *empirical rates of return* should be useful in determining the proximity of early exercise.

## 7 References


Brekke, K. A. and Øksendal, B. (1991): The high contact principle as a sufficiency condition for optimal stopping, in: D. Lund and B. Øksendal, eds., *Stochastic Models and Option Values*, North–Holland, pp. 187–208.

Barone–Adesi, G. and Whaley. R. E. (1987), Efficient analytic approximation of American option values, *Journal of Finance*, 42(2), 301–320.

Black, F. and Scholes, M. (1973), The pricing of options and corporate liabilities, *Journal of Political Economy*, 81, 637–659.

Dixit, A. and R. S. Pindyck (1994): *Investment Under Uncertainty*, Princeton University Press.

Dixit, A., Pindyck, R. S. and Sødal, S. (1999), A markup interpretation of optimal investment rules, *Economic Journal*, 109(455), 179–89.

Dumas, B. (1991), Super contact and related optimality conditions, *Journal of Economic Dynamics and Control*, 15(4), 675–685.

MacMillan, L. W. (1986), An analytic approximation for the American put price, *Advances in Futures and Options Research*, 1(1), 141–183.

McDonald, R. and D. Siegel (1986): The value of waiting to invest, *Quarterly Journal of Economics* 101, pp. 707-728.

Merton, R. C. (1973), The theory of rational option pricing, *Bell Journal of Economics*, 4(1), 141–183.

Samuelson, P. A. (1965): *Rational Theory of Warrant Pricing*, Industrial Management Review 6, pp. 13–31.

Shackleton, M. B. and Wojakowski, R. (2002), The expected return and exercise time of Merton style Real Options, *Journal of Business Finance and Accounting*, April 29(3&4), 541–555.

Sødal, S. (1998), A simplified exposition of smooth pasting, *Economics Letters*, 58, 217–233.