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On the sets of maximum points for generalized Takagi functions

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Abstract. Let φ be a continuous and periodic function on $\mathbb{R}$ with period 1 and $\varphi(0) = 0$. We consider the generalized Takagi function f_φ defined by $f_\varphi(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \varphi(2^n x)$ and the set M_φ of maximum points of f_φ in the interval $[0, 1]$. When $\varphi_0(x)$ is the function defined by the distance from x to the nearest integer, f_{φ_0} is just the Takagi function. Our aim is to seek a condition on φ in order that $M_\varphi \subset M_{\varphi_0}$.

1. Introduction and the result

The Takagi function is defined by

$$f_{\varphi_0}(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \varphi_0(2^n x), \quad x \in \mathbb{R}, \tag{1}$$

where $\varphi_0(x)$ is the function defined by the distance from x to the nearest integer (cf. [1, 6]). It is a pathological function in the sense that it is everywhere continuous but nowhere differentiable on $\mathbb{R}$. Let M_{φ_0} be the set of maximum points in the interval $[0, 1]$ for the Takagi function f_{φ_0}. By Kahane [4], we have

$$M_{\varphi_0} = \left\{ \sum_{j=1}^{\infty} \frac{a_j}{4^j} \mid a_j \in \{1, 2\} \right\}. \tag{2}$$

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Thus, M_{φ_0} is uncountable. It is important to note that $\min M_{\varphi_0} = \frac{1}{3}$ and $\max M_{\varphi_0} = \frac{2}{3}$.

As a generalization of the Takagi function, we consider the function f_φ of the form

$$f_\varphi(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \varphi(2^n x), \quad x \in \mathbb{R}, \tag{3}$$

where φ is a continuous and periodic function on $\mathbb{R}$ with period 1. We call the function of the form (3) a generalized Takagi function. The functions of the form (3) have been considered since the work of [5]. Behrend [3] studied non-differentiability of the functions of the form (3).

In the following, for the function f_φ of (3), we use the notations

$$M_\varphi = \{x \in [0,1] \mid f_\varphi(x) = m_\varphi\}, \quad m_\varphi = \max_{y \in [0,1]} f_\varphi(y). \tag{4}$$

Note that f_φ is also periodic with period 1.

Our aim is to compare M_φ with M_{φ_0}. In particular, we are interested in a condition on φ in order that $M_\varphi \subset M_{\varphi_0}$.

For a function $\varphi : \mathbb{R} \to \mathbb{R}$, we consider the following conditions:

(C1) φ is continuous and periodic on $\mathbb{R}$ with period 1 and $\varphi(0) = 0$. Furthermore, $\varphi \not\equiv 0$.

(C2) $\varphi(x) = \varphi(1-x), \quad x \in [0,1]$.

(C3) φ is concave on $[0,1]$, that is, for all $x, y \in [0,1]$ and $\theta \in [0,1]$,

$$\theta\varphi(x) + (1-\theta)\varphi(y) \le \varphi(\theta x + (1-\theta)y).$$

(C4) $\varphi(x) < \varphi\left(\frac{1}{3}\right), \quad x \in \left[0, \frac{1}{3}\right)$.

There exist many functions φ with (C1), (C2), (C3) and (C4).

Example 1.1. Let $\alpha \in \left[\frac{1}{3}, \frac{1}{2}\right]$. Define the periodic function $\eta_\alpha : \mathbb{R} \to [0,\infty)$ with period 1 by $\eta_\alpha(x) = \min\{\varphi_0(x), \alpha\}$ for $x \in \mathbb{R}$. Note that $\eta_{1/2} \equiv \varphi_0$. It is easy to see that, for each $\alpha \in \left[\frac{1}{3}, \frac{1}{2}\right]$, η_α fulfills (C1), (C2), (C3) and (C4).

Theorem 1.2. *Let φ be a function with* (C1), (C2), (C3) *and* (C4). *Then,* $M_\varphi \subset M_{\varphi_0}$.

Now, we consider the conditions (C1), (C2), (C3) and (C4). The condition (C1) is natural. The condition (C2) implies that φ is symmetric with respect to the line $x = \frac{1}{2}$. The condition (C3) is necessary in Theorem 1.2, since we have an example of a function $\hat{\varphi}$ with (C1), (C2) and (C4) such that $\hat{\varphi}$ does not fulfill (C3) and $M_{\hat{\varphi}} \not\subset M_{\varphi_0}$ (see Example 3.1 below). The condition (C4) is necessary in Theorem 1.2, since we have an example of a function $\hat{\eta}$ with (C1), (C2) and (C3) such that $\hat{\eta}$ does not fulfill (C4) and $M_{\hat{\eta}} \not\subset M_{\varphi_0}$ (see Example 3.2 below).

The contents of the present paper are as follows: In Section 2, we prove Theorem 1.2. In Section 3, we provide two examples which show that the conditions (C3) and (C4) for φ are necessary in order that $M_\varphi \subset M_{\varphi_0}$.

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We use the notations of Section 1.

Lemma 2.1. *Assume that φ fulfills* (C1) *and* (C3). *Then, $\varphi > 0$ in the open interval* $(0, 1)$.

Proof. We derive a contradiction by supposing that there exists an $x_0 \in (0, 1)$ such that $\varphi(x_0) = 0$. Since $\varphi \not\equiv 0$ by (C1), we find an $x_1 \in (0, 1) \setminus \{x_0\}$ such that $\varphi(x_1) \neq 0$. First, we consider the case that $0 < x_1 < x_0$. By (C3), we have

$$\varphi(x_1) = \varphi\left(\frac{x_1}{x_0}\, x_0 + \left(1 - \frac{x_1}{x_0}\right) \cdot 0\right) \geq \frac{x_1}{x_0}\, \varphi(x_0) + \left(1 - \frac{x_1}{x_0}\right) \varphi(0) = 0.$$

Thus, $\varphi(x_1) > 0$. On the other hand, let $\theta = \frac{1 - x_0}{1 - x_1}$. Then, $0 < \theta < 1$ and $\theta x_1 + (1 - \theta) \cdot 1 = x_0$. Thus, we have

$$0 = \varphi(x_0) = \varphi(\theta x_1 + (1 - \theta) \cdot 1) \geq \theta\varphi(x_1) + (1 - \theta)\, \varphi(1) = \theta\varphi(x_1) > 0.$$

This is a contradiction. When $x_0 < x_1 < 1$, we can derive a contradiction similarly. Therefore, $\varphi > 0$ in $(0, 1)$. □

In the following, we assume that φ fulfills (C1), (C2) and (C3), otherwise stated. We define the function $p_\varphi : \mathbb{R} \to \mathbb{R}$ of period 1 by

$$p_\varphi(x) = \varphi(x) + \frac{1}{2}\varphi(2x), \quad x \in \mathbb{R}. \tag{5}$$

Note that

$$f_\varphi(x) = \sum_{n=0}^{\infty} \frac{1}{4^n}\, p_\varphi(4^n x), \quad x \in \mathbb{R}. \tag{6}$$

Let

$$E_\varphi^{(n)} \;=\; \{x \in [0,1] \mid p_\varphi(4^n x) = \mu_\varphi\}\,, \quad n \in \mathbb{N} \cup \{0\}, \tag{7}$$

where $\mu_\varphi = \max_{y \in [0,1]} p_\varphi(y)$. By (6) and (7), it is clear to see that

$$\bigcap_{n=0}^{\infty} E_\varphi^{(n)} \subset M_\varphi. \tag{8}$$

It is also clear that if $\bigcap_{n=0}^{\infty} E_\varphi^{(n)} \neq \emptyset$, the inclusion relation in (8) is reduced to the equality.

Since the following two lemmas are clear, we omit their proofs.

Lemma 2.2. *For each $m \in \mathbb{N}$, there exist integers $q_m \in \mathbb{N}$ and $r_m \in \mathbb{N} \cup \{0\}$ such that*

$$\frac{1}{3}(4^{2m} - 1) = 5q_m, \quad \frac{1}{3}(4^{2m-1} - 1) = 20r_m + 1.$$

Lemma 2.3. *Assume that φ fulfills* (C1)*,* (C2) *and* (C3)*. Then,*

$$f_\varphi(x) = f_\varphi(1-x), \quad p_\varphi(x) = p_\varphi(1-x), \quad x \in [0,1].$$

Lemma 2.4. *Assume that φ fulfills* (C1)*,* (C2) *and* (C3)*. Then,*

$$p_\varphi(x) \le p_\varphi\left(\frac{1}{3}\right) = p_\varphi\left(\frac{2}{3}\right), \quad x \in [0,1].$$

Proof. Let $0 \le x \le \frac{1}{2}$. By (C2) and (C3), we have

$$p_\varphi(x) = \frac{3}{2}\left[\frac{2}{3}\varphi(1-x) + \frac{1}{3}\varphi(2x)\right] \le \frac{3}{2}\,\varphi\left(\frac{2}{3}(1-x) + \frac{1}{3}(2x)\right)$$

$$= \frac{3}{2}\,\varphi\left(\frac{2}{3}\right) = \varphi\left(\frac{2}{3}\right) + \frac{1}{2}\,\varphi\left(2 \cdot \frac{1}{3}\right) = \varphi\left(\frac{1}{3}\right) + \frac{1}{2}\,\varphi\left(2 \cdot \frac{1}{3}\right) = p_\varphi\left(\frac{1}{3}\right).$$

By Lemma 2.3, we conclude the lemma. □

Proposition 2.5. *Assume that φ fulfills* (C1), (C2) *and* (C3). *Then,*

$$\frac{1}{3},\ \frac{2}{3} \in M_{\varphi}. \tag{9}$$

Proof. By Lemma 2.4, we have $\frac{1}{3} \in E_{\varphi}^{(0)}$. Let $n \in \mathbb{N}$. When n is even, it follows from Lemma 2.2 that there exists an integer $q_n \in \mathbb{N}$ such that $4^n \cdot \frac{1}{3} - 5q_n = \frac{1}{3} \in E_{\varphi}^{(0)}$. Thus, we have $\frac{1}{3} \in E_{\varphi}^{(n)}$. When n is odd, it follows from Lemma 2.2 that there exists an integer $r_n \in \mathbb{N} \cup \{0\}$ such that $4^n \cdot \frac{1}{3} - (20r_n + 1) = \frac{1}{3} \in E_{\varphi}^{(0)}$. Thus, we have $\frac{1}{3} \in E_{\varphi}^{(n)}$. Hence, $\frac{1}{3} \in E_{\varphi}^{(n)}$ for all $n \in \mathbb{N}$. Therefore, $\frac{1}{3} \in M_{\varphi}$ by (8). By Lemma 2.3, we see that $\frac{2}{3} \in M_{\varphi}$. □

Proposition 2.6. *Assume that φ fulfills (C1), (C2) and (C3). Then,* $M_{\varphi} = \bigcap_{n=0}^{\infty} E_{\varphi}^{(n)}$. *Furthermore,* $m_{\varphi} = 2\,\varphi\left(\frac{1}{3}\right)$, *where m_{φ} is the constant of* (4).

Proof. Note that $\frac{1}{3} \in \bigcap_{n=0}^{\infty} E_{\varphi}^{(n)}$ by the proof of Proposition 2.5. Since $\bigcap_{n=0}^{\infty} E_{\varphi}^{(n)} \neq \emptyset$, it is easy to see that $M_{\varphi} = \bigcap_{n=0}^{\infty} E_{\varphi}^{(n)}$ by (8). Furthermore, by (6), (8) and Proposition 2.5, we have

$$m_{\varphi} = \sum_{n=0}^{\infty} \frac{1}{4^n}\, p_{\varphi}\Big(\frac{1}{3}\Big) = 2\,\varphi\Big(\frac{1}{3}\Big).$$

Here, we used $p_{\varphi}\left(\frac{1}{3}\right) = \frac{3}{2}\,\varphi\left(\frac{1}{3}\right)$. □

Proposition 2.7. *Assume that φ fulfills* (C1), (C2), (C3) *and* (C4). *Then,* $\min M_{\varphi} = \frac{1}{3}$ *and* $\max M_{\varphi} = \frac{2}{3}$.

Proof. Let $0 \leq x < \frac{1}{4}$. Then, by (C3) and (C4), we have

$$p_{\varphi}(x) = \frac{3}{2}\left[\frac{2}{3}\varphi(x) + \frac{1}{3}\varphi(2x)\right] \leq \frac{3}{2}\,\varphi\Big(\frac{2}{3}x + \frac{1}{3}(2x)\Big) = \frac{3}{2}\,\varphi\Big(\frac{4}{3}x\Big)$$

$$< \frac{3}{2}\,\varphi\Big(\frac{1}{3}\Big) = \varphi\Big(\frac{1}{3}\Big) + \frac{1}{2}\,\varphi\Big(2 \cdot \frac{1}{3}\Big) = p_{\varphi}\Big(\frac{1}{3}\Big).$$

Thus, $[0,\frac{1}{4})\cap E_\varphi^{(0)}=\emptyset$. By Proposition 2.6, $[0,\frac{1}{4})\cap M_\varphi=\emptyset$. For each $x\in[0,\frac{1}{4})$ and $n\in\mathbb{N}$, we have

$$p_\varphi\Big(4^n\big(\sum_{i=1}^{n}\frac{1}{4^i}+\frac{x}{4^n}\big)\Big)=p_\varphi(x)<p_\varphi\left(\frac{1}{3}\right).$$

Thus, $\Big[\sum_{i=1}^{n}\frac{1}{4^i},\sum_{i=1}^{n+1}\frac{1}{4^i}\Big)\cap E_\varphi^{(n)}=\emptyset$. By Proposition 2.6, $\Big[\sum_{i=1}^{n}\frac{1}{4^i},\sum_{i=1}^{n+1}\frac{1}{4^i}\Big)\cap M_\varphi$ $=\emptyset$. Since $\sum_{i=1}^{\infty}\frac{1}{4^i}=\frac{1}{3}$, we conclude that $[0,\frac{1}{3})\cap M_\varphi=\emptyset$. Since $\frac{1}{3}\in M_\varphi$ by Proposition 2.5, we have shown that $\min M_\varphi=\frac{1}{3}$. By Lemma 2.3, we have $\max M_\varphi=\frac{2}{3}$. □

Lemma 2.8. *Assume that φ and ψ fulfill* (C1), (C2) *and* (C3). *If $E_\varphi^{(0)}\subset E_\psi^{(0)}$, then $M_\varphi\subset M_\psi$.*

Proof. Let $x\in M_\varphi$. Then, by Proposition 2.6, $x\in E_\varphi^{(n)}$ for all $n\in\mathbb{N}\cup\{0\}$. Thus, $p_\varphi(4^nx)=p_\varphi\left(\frac{1}{3}\right)$ for all $n\in\mathbb{N}\cup\{0\}$, which implies that $4^nx-[4^nx]\in E_\varphi^{(0)}$ for all $n\in\mathbb{N}\cup\{0\}$, where $[y]$ denotes the greatest integer not exceeding $y\in\mathbb{R}$. Since $E_\varphi^{(0)}\subset E_\psi^{(0)}$, we have $4^nx-[4^nx]\in E_\psi^{(0)}$ for all $n\in\mathbb{N}\cup\{0\}$. Then, $x\in E_\psi^{(n)}$ for all $n\in\mathbb{N}\cup\{0\}$. Hence, $x\in M_\psi$ by Proposition 2.6. □

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. Let φ be a function with (C1), (C2), (C3) and (C4). By the proof of Proposition 2.7, we have $[0,\frac{1}{4})\cap M_\varphi=\emptyset$, so that $E_\varphi^{(0)}\subset\left[\frac{1}{4},\frac{3}{4}\right]$ by Lemma 2.3. Since $E_{\varphi_0}^{(0)}=\left[\frac{1}{4},\frac{3}{4}\right]$, we conclude the theorem by Lemma 2.8. □

3. Examples

In this section, we provide two examples which show that the conditions (C3) and (C4) for φ are indispensable in order that $M_\varphi\subset M_{\varphi_0}$.

Example 3.1. Let $\hat{\varphi} : \mathbb{R} \to \mathbb{R}$ be the periodic function with period 1 such that $\hat{\varphi}(x) = (\varphi_0(x))^2$ in $\mathbb{R}$. In this case, $\hat{\varphi}$ fulfills (C1), (C2) and (C4). However, $\hat{\varphi}$ does not fulfill (C3). We show that $M_{\hat{\varphi}} \not\subset M_{\varphi_0}$.

Indeed, we see that

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \big(\varphi_0(2^n x)\big)^2 = x - x^2, \quad x \in [0,1],$$

since $\psi^0(x) = x$ and $\psi^n(x) = 2\varphi_0(2^{n-1}x)$ $(n \in \mathbb{N})$ in the notations of the example of [8, p.335]. Thus, $f_{\hat{\varphi}}(x) = x - x^2$ in $[0,1]$. Hence, $M_{\hat{\varphi}} = \left\{\frac{1}{2}\right\}$.

Example 3.2. Define the periodic function $\hat{\eta} : \mathbb{R} \to [0,\infty)$ with period 1 by $\hat{\eta}(x) = \min\left\{\varphi_0(x), \frac{1}{15}\right\}$ for $x \in \mathbb{R}$. In this case, $\hat{\eta}$ fulfills (C1), (C2) and (C3). However, $\hat{\eta}$ does not fulfill (C4). We show that $M_{\hat{\eta}} \not\subset M_{\varphi_0}$.

Indeed, we have, on $\left[0, \frac{1}{2}\right]$,

$$p_{\hat{\eta}}(x) = \begin{cases} 2x, & 0 \le x \le \dfrac{1}{30}, \\ x + \dfrac{1}{30}, & \dfrac{1}{30} \le x \le \dfrac{1}{15}, \\ \dfrac{1}{10}, & \dfrac{1}{15} \le x \le \dfrac{7}{15}, \\ -x + \dfrac{17}{30}, & \dfrac{7}{15} \le x \le \dfrac{1}{2}. \end{cases}$$

Thus, $E_{\hat{\eta}}^{(0)} = \left[\frac{1}{15}, \frac{7}{15}\right] \cup \left[\frac{8}{15}, \frac{14}{15}\right]$.

We show that $\frac{1}{15} \in M_{\hat{\eta}}$. Note that $\frac{1}{15} \in E_{\hat{\eta}}^{(0)}$. Fix $n \in \mathbb{N}$ arbitrarily. When n is even, it follows from Lemma 2.2 that there exists a number $q_n \in \mathbb{N}$ such that $4^n \cdot \frac{1}{15} = q_n + \frac{1}{15}$. Since $\frac{1}{15} \in E_{\hat{\eta}}^{(0)}$, we have $\frac{1}{15} \in E_{\hat{\eta}}^{(n)}$. When n is odd, it follows from Lemma 2.2 that there exists a number $r_n \in \mathbb{N} \cup \{0\}$ such that $4^n \cdot \frac{1}{15} = 4r_n + \frac{4}{15}$. Since $\frac{4}{15} \in E_{\hat{\eta}}^{(0)}$, we have $\frac{1}{15} \in E_{\hat{\eta}}^{(n)}$. Therefore, $\frac{1}{15} \in E_{\hat{\eta}}^{(n)}$ for all $n \in \mathbb{N}$, and $\frac{1}{15} \in M_{\hat{\eta}}$. Since $\min M_{\varphi_0} = \frac{1}{3}$, we see that $\frac{1}{15} \notin M_{\varphi_0}$. Thus, we conclude the assertion.

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