Solving frictional contact problems within the bipotential framework

E. BOUDAIA[a], L. BOUSSHINE[b], G. DE SAXCE[b]

a. Départ génie mécanique, Faculté des sciences et techniques, Mghrila, BP 523, Beni-Mellal, 23000, Maroc
b. Laboratory of Technologies of Constructions and the Industrial Systems, ENSEM, BP.8118, Oasis, Casablanca, Morocco
c. LML, Université des Sciences et Techniques, Lille1, France

Abstract:

In this paper, we present a theoretical and numerical analysis of frictional contact problems for large deformation elastoplastic based on the finite element method (FEM) and the mathematical programming. The study is done on an elastoplastic material obeying to the von Mises criterion. The Coulomb's friction contact is used to implement the frictional boundary conditions and is formulated by the bipotential method, which leads to only one principle of minimum in displacement.

Keywords: **frictional contact, large deformation, FEM, mathematical programming, bipotential**

1 Introduction

A large number of algorithms for the modeling of contact problems by the finite element method have been presented in the literature.

See for example the monographs by Kikuchi and Oden [1], Zhong [2], Wriggers [3], Laursen [4] and the references therein. De Saxce and Feng [5] have proposed a bipotential method, in which an augmented Lagrangian formulation was developed. In this work, the boundary condition for contact law with Coulomb's friction in the interface is taken into account and is described by the bipotential concept leading us to minimize only one principle of minimum. On the other hand, the problem of the large deformation is solved by the updating of the geometrical configuration of the structure after each sequence. For reasons of clarity, this article is divided into six sections. We detailed the governing equations for problems of evolution elastoplastic coupled with Coulomb's frictional contact in section 2. The variational formulation for elastoplastic materials with frictional contact is presented in section 3, and the implementation of the finite element method discretization is discussed in section 4. The performance of the proposed methods is examined in section 5, and a conclusion is given in section 6.

2 Governing equations

Let $\Omega \subset \mathbb{R}^d$ $(d = 2 \text{ or } 3)$ be elastoplastic structure with a regular boundary $\Gamma = \partial\Omega$; submitted to a body force $\Delta\overline{f}$; imposed displacement increments $\Delta\overline{u}$ to a portion Γ_u ; imposed traction increments $\Delta\overline{t}$ to a portion Γ_t and on the part $\Gamma_c = \Gamma - \Gamma_u \cup \Gamma_t$ of boundary such as $\Gamma_c \cap \Gamma_t \cap \Gamma_u = \{\varnothing\}$, contact may occur. We have the following basic equations in domain $\Omega = \Omega_1 \cup \Omega_2$ such as Ω_1 and Ω_2 are the contacting bodies, one of which may be a rigid foundation (see [6]).

- Equilibrium equations

$$div(\Delta\sigma) + \Delta\overline{f} = 0 \quad in \quad \Omega \tag{1}$$

- Boundary conditions

$$\Delta u = \Delta\overline{u} \quad on\ the\ essential\ boundary\ \Gamma_u \tag{2}$$

$$\Delta t(\Delta\sigma) = \Delta\sigma n = \Delta\overline{t} \quad on\ the\ natural\ boundary\ \Gamma_t \tag{3}$$

in which n is the outward unit normal to domain Ω and $\Delta\sigma$ is the stress increment.

- Elastoplastic constitutive equations

$$\Delta\sigma = C^{ep}\Delta\varepsilon \tag{4}$$

where $\Delta\sigma$ is the Cauchy stress increment tensor, C^{ep} is called the tangential stiffness matrix and $\Delta\varepsilon$ is the strain increment tensor can be decomposed into elastic and plastic parts:

$$\Delta\varepsilon = \Delta\varepsilon^{e} + \Delta\varepsilon^{p} \tag{5}$$

The elastic constitutive relations:

$$\Delta\sigma_{ij} = C^{e}_{ijkl}\Delta\varepsilon^{e}_{kl} \tag{6}$$

where C^{e}_{ijkl} denotes elastic modulus tensor.

In this work, elastoplastic material model is considered. As a plasticity model, von Mises yield model is adopted. The yield function is written as

$$f(\sigma,\bar{\varepsilon}^{p}) = (\frac{3}{2}\bar{\sigma}_{ij}\bar{\sigma}_{ij})^{0.5} - \sigma_{Y}(\bar{\varepsilon}^{p}) \tag{7}$$

where $\bar{\sigma}_{ij}$ denotes deviatoric stress and σ_Y the yield stress. $\bar{\varepsilon}^{p}$ indicates equivalent (or effective) plastic strain, and its time rate is defined as

$$\dot{\bar{\varepsilon}}^{p} = (\frac{2}{3}\dot{\varepsilon}^{p}_{ij}\dot{\varepsilon}^{p}_{ij})^{0.5} \tag{8}$$

From the associative flow rule, plastic strain can be written as follows:

$$\dot{\varepsilon}^{p}_{ij} = \dot{\lambda}\frac{\partial f}{\partial\sigma_{ij}} \tag{9}$$

where λ denotes plastic multiplier.

The expression of tensor C^{ep} can be written as

$$C^{ep} = C^{e} - \left(C^{e}N\right)\left(C^{e}\frac{\partial f}{\partial\sigma}\right)^{T} \Bigg/ \left(\frac{\partial f}{\partial\sigma}^{T} C^{e}N\right) \tag{10}$$

where $N(\sigma) = \dfrac{\partial f}{\partial\sigma} \Bigg/ \left\|\dfrac{\partial f}{\partial\sigma}\right\|$ is the unit flow direction.

- Frictional contact condition

The relative velocity $\dot{u}$ and the contact traction t can be decomposed in their normal and tangential components as follows:

$$\dot{u} = \dot{u}_t + \dot{u}_n n \quad and \quad t = t_t + t_n n \tag{11}$$

where $\dot{u}_n$ and $\dot{u}_t$ are, respectively, normal and tangential components of the relative velocity $\dot{u}$, t_n and t_t indicate normal and tangential components of the contact traction t.

The complete contact law with Coulomb's friction can be written in the following form:

$$\begin{cases} if \quad t_n = 0 \;\; then \;\; \dot{u}_n \geq 0 & > separating, \\ else\,if \quad t_n > 0 \;\; and \;\; \|t_t\| < \mu t_n \;\; then \;\; \dot{u} = 0 & > sticking, \\ else \quad (t_n > 0 \;\; and \;\; \|t_t\| = \mu t_n),\; (\dot{u}_n \geq 0 \;\; and \;\; \exists\bar{\lambda} \geq 0 \;\; such\ that \;\; \dot{u}_t = -\bar{\lambda}\, t_t / \|t_t\| & > sliding \\ endif & \end{cases} \tag{12}$$

where μ indicates the friction coefficient and $\bar{\lambda}$ is a positive multiplier.

The inverse law can be described as

$$\begin{cases} if \quad \dot{u}_n = 0 \;\; then \;\; t = 0 & > \; non\text{-}contact, \\ else\, if \;\; \dot{u} = 0 \;\; then \;\; \|t_t\| < \mu t_n & > \; contact \; with \; sticking, \\ else \quad \dot{u}_t \neq 0, \;\; \dot{u}_n = 0 \;\; and \;\; t_t = -\mu t_n . \dot{u}_t / \|\dot{u}_t\| & > \; contact \; with \; sliding \\ endif \end{cases} \quad (13)$$

In a compact form, the unilateral contact law with Coulomb's friction and its inverse will be rewritten by means of the contact bipotential, denoted $b_c(-\dot{u},t)$, which is defined as follows [7]:

$$b_c(-\dot{u},t) = \begin{cases} \mu t_n \|-\dot{u}_t\| \;\; if \;\; t \in K_\mu \;\; and \;\; \dot{u}_n \geq 0 \\ +\infty \qquad otherwise \end{cases} \quad (14)$$

where K_μ denote Coulomb's cone defined by

$$K_\mu = \left\{ (t_n, t_t) \in R^2 \;\; such \; that \;\; \|t_t\| - \mu\, t_n \leq \; 0 \right\} \quad (15)$$

Therefore, the complete contact law with friction becomes

$$-\dot{u} \in \partial_t b_c(-\dot{u},\, t) \;\; and \;\; t \in \partial_{-\dot{u}} b_c(-\dot{u},\, t) \quad (16)$$

The incremental formulation of the contact law with Coulomb's friction is expressed by the incremental bipotential:

$$\begin{aligned} \Delta b_c(-\Delta u, \Delta t) &= b_c(-\Delta u, t_0 + \Delta t) - (-t_0 \Delta u) \\ &= t_{n0}\Delta u_n + t_{t0}\Delta u_t + \mu (t_{n0} + \Delta t_n)\|\Delta u_t\| \end{aligned} \quad (17)$$

where Δu_n and Δu_t denote, respectively, normal and tangential components of the displacement increment Δu such that in the case of the application of the schema of integration implicit $\Delta u = u_1 - u_0 = \Delta \tau \dot{u}_1$ with $\Delta \tau$ is the increment of time, index 0 (resp. 1) is relative to beginning (resp. to the end) of the step ; t_{n0} and t_{t0} indicate initially normal and tangential components of contact traction *t*.
The corresponding incremental contact laws take the form

$$-\Delta u \in \partial_{\Delta t} \Delta b_c(-\Delta u, \Delta t) \;\; and \;\; \Delta t \in \partial_{-\Delta u} \Delta b_c(-\Delta u, \Delta t) \quad (18)$$

3 Variational Formulation

The use the incremental formulation with the bipotential method leads to the following bifunctional; more details can be seen in reference [8]:

$$\Delta\beta(\Delta u, \Delta\sigma) = \int_\Omega \left(\Delta\varepsilon(u)\Delta\sigma - \Delta\bar{f}\,\Delta u \right) d\Omega - \int_{\Gamma_u} \Delta t(\Delta\sigma)\Delta\bar{u} d\Gamma - \int_{\Gamma_t} \Delta\bar{t}\,\Delta u d\Gamma + \int_{\Gamma_c} \Delta b_c(-\Delta u, \Delta t) d\Gamma \quad (19)$$

The exact solution of boundary value problem, defined by Eqs. (1) to (3) and the contact laws (18), is also a solution to the kinematical variational principle:

$$\underset{\Delta u^k \;\; KA}{Inf}\; \Delta\beta(\Delta u^k, \Delta\sigma) \quad (20)$$

where Δu^k is the displacement field kinematically admissible (*KA*).
For the variational formulation in terms of displacements, the terms which do not depend on the incremental field Δu disappear and the Eq. (19) is reduced to

$$\Delta\Psi(\Delta u) = \int_\Omega \left[\Delta\varepsilon(u)^T C^{ep} \Delta\varepsilon(u) - \Delta\bar{f}\,\Delta u \right] d\Omega - \int_{\Gamma_t} \Delta\bar{t}\,\Delta u d\Gamma + \int_{\Gamma_c} \Delta b_c(-\Delta u, \Delta t) d\Gamma \quad (21)$$

Therefore, the kinematical variational principle becomes

$$\underset{\Delta u^k \;\; KA}{Inf}\; \Delta\Psi(\Delta u^k) \quad (22)$$

4 Finite Element Discretization

The displacement and strain increment fields are expressed with respect to an unknown nodal displacement increment vector ΔU as (see [9]):

$$\Delta u(x) = \phi(x)\Delta U \quad and \quad \Delta\varepsilon = B(x)\Delta U \tag{23}$$

where $\phi(x)$ is the matrix of the shape functions, $B(x) = \nabla_s(\phi(x))$ and ∇_s is the symmetric gradient operator.
Let us introduce the generalized nodal force increment vector:

$$\Delta F = \int_\Omega \phi^T \Delta\bar{f} d\Omega + \int_{\Gamma_t} \phi^T \Delta\bar{t} d\Gamma \tag{24}$$

The discretized form of the Eq. (21) is then a set of nonlinear equations:

$$\Delta\Psi(\Delta U) = \int_\Omega B^T C^{ep} B\Delta U d\Omega - \Delta F + \int_{\Gamma_c} \Delta b_c(-\phi\Delta U, \Delta t) d\Gamma \tag{25}$$

The bipotential of the contact with friction isn't differentiable everywhere which poses problems at the mathematical programming level. In order to overcome this difficulty, we suggest using the regularization method by penalization. For this purpose, we can introduce the following differentiable function, which will be added, by using the inf-convolution concept, to the incremental bipotential Δb_c proposed by Boudaia and al. [6].

$$\Delta b' = \frac{K_t}{2}(-\Delta u_t + \Delta u_t^f)^2 + \frac{K_n}{2}(-\Delta u_n + \Delta u_n^f)^2 \tag{26}$$

where K_t and K_n are the penalization factors, Δu_n^f and Δu_t^f are the fictitious increments computed from the actual displacement increment Δu and the previous contact forces increments Δt, so that

$$\Delta u_n = \Delta u_n^f + \frac{\Delta t_n}{K_n} \quad and \quad \Delta u_t = \Delta u_t^f + \frac{\Delta t_t}{K_t} \tag{27}$$

We show that Δb_c can be written as follows:

$$\Delta b_c = \Delta b_n + \Delta b_t \tag{28}$$

with

$$\begin{cases} \Delta b_n = \underset{-\Delta u_n^f}{Inf}\left(-t_{n0}(-\Delta u_n^f) + \frac{K_n}{2}(-\Delta u_n + \Delta u_n^f)^2\right), \\ \Delta b_t = \underset{-\Delta u_t^f}{Inf}\left(-t_{t0}(-\Delta u_t^f) + \mu(t_{n0} + \Delta t_n)\left\|-\Delta u_t^f\right\| + \frac{K_t}{2}(-\Delta u_t + \Delta u_t^f)^2\right) \end{cases} \tag{29}$$

In addition, the problem of coupling of traction increments with those of displacements is solved by using an iterative procedure based on the fixed-point method. We note that the problem resolution of optimization (25) is realized using the Minos code [10].
Also, the problem of the large deformation is solved by the updating of the geometrical configuration of the structure. Let x_i be the vector position occupied by a particle P at the configuration $"i"$ of the body corresponding to the sequence number $"i"$ and x_{i+1} its position that will be occupied by the same particle at the next configuration $"i+1"$, the two successive vector positions of the same particle P are related via the velocity vector $\dot{u}_i$ and the time increment $\Delta\tau$ such as:

$$x_{i+1} = x_i + \dot{u}_i \Delta\tau. \tag{30}$$

5 Numerical Result

In this example, the size of the block is $L_0 \times D_0 = 10 \times 10 mm^2$ (height × diameter) as shown in Fig. 1. The yield stress is $\sigma_Y = 300 MPa$, Poisson's ratio is $\nu = 0.3$ and Young's modulus is $E = 21\times10^4 MPa$. The cylindrical block is compressed by the rigid punches. Because of the workpiece is axisymmetric, only a plane subdomain is considered for computational finite elements. For this, 100 quadratic quadriclateral elements are used for the mesh model, with seven Gauss's point integration.

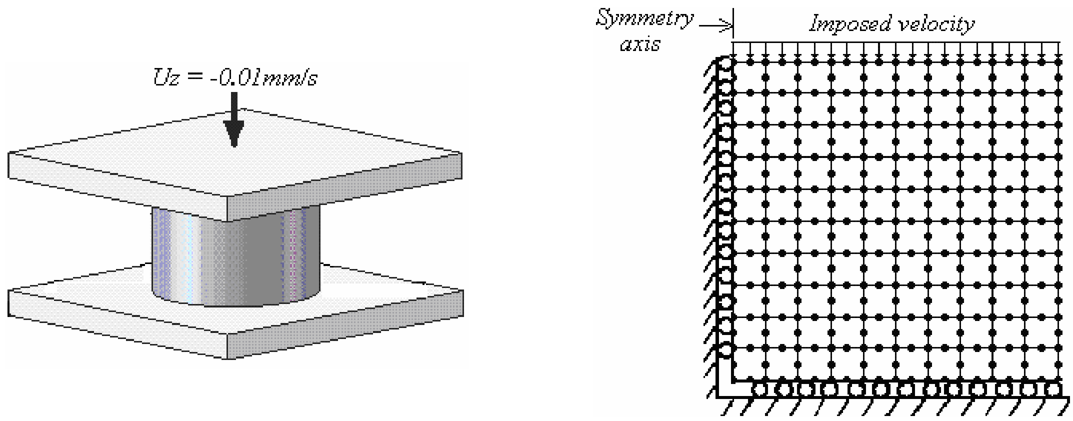

FIG. 1- Geometry and loading (left); Mesh model and boundary conditions (right).

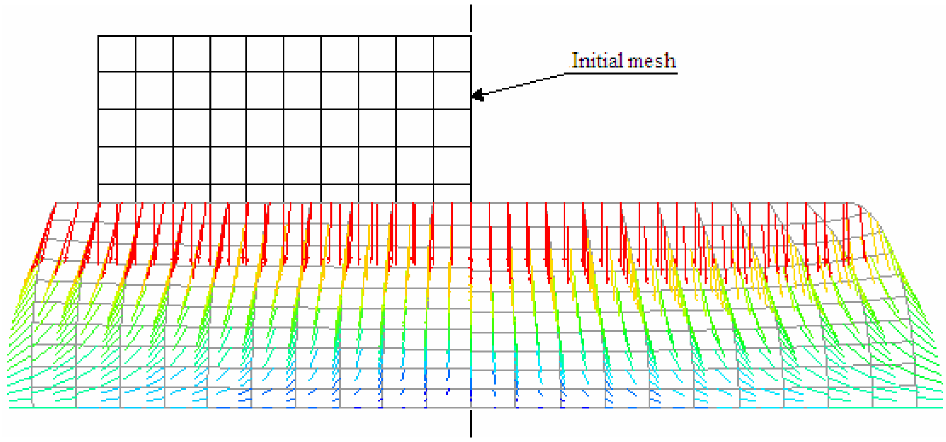

FIG. 2- Distribution of nodal velocity field at 45% reduction in height for different friction coefficients $\mu = 0.1$ (left) and $\mu = 0.3$ (right).

From Fig. 2, we can notice that the finite elements kept their initial geometrical form without distortion until 45% reduction in height was reached. For this reason, it is not necessary to remesh the structure.

6 Conclusion

The Finite Element Method formulation is presented with the theory of elastoplastic mechanics for the simulation of metal forming processes. Special emphasis is placed on the treatments of friction boundaries. However, the large deformation is introduced in a sequential way by updating of the geometry after each sequence of elastoplastic problem resolution. In addition, the non-differentiable of the bipotential representing the contact with friction is surmounted by the use of the regularization procedure by penalization. A second difficulty which does not miss importance is the presence of a term of coupling between the contact and friction in the bipotential function. This problem of coupling is solved by the use of an iterative procedure based on the fixed point method.

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