Sur la modélisation quasi-continue d'un modèle discret unidimensionnel de Cosserat

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Résumé :

Certaines limitations apparentées au verrouillage numérique de la discrétisation en éléments finis des milieux continus micropolaires (de Cosserat) linéairement élastiques ou, inversement, à la continualisation des milieux granulaires analogues, sont illustrées via l'analyse du comportement statique d'une simple chaîne monoatomique et la dérivation de ses modèles de poutres équivalentes qui peuvent être soit locaux (Timoshenko,Euler-Bernoulli) ou non-locaux (Eringen,Kunin), en tenant compte de ses propriétés spectrales et analytiques ambigües.

Abstract :

Some locking problems related to the discretization into finite elements of linearly elastic continuous micropolar (or Cosserat's) media or, conversely, to the "continualization" of analogous granular media are illustrated through the analysis of the static behavior of a simple monoatomic chain and the derivation of its equivalent beam models which can be either local (Timoshenko, Euler-Bernoulli) or non-local (Eringen, Kunin), while taking into account its analytically ambiguous spectral properties.

Mots clefs : milieux de Cosserat ; granularités ; élasticités locale et non-locale.

1 Introduction

Granular and cellular media that can locally undergo micro-displacements of translation as well as rotations are commonly thought to be more suitably modeled by the standard Cosserat micropolar continuum theory [3] for many mathematical and numerical reasons. There exists however a need for material constant inputs to establish the bridge between the two modelings. The experimental identification of the parameters required for the Cosserat material constitutive law still remains a tough challenging task [10]. That fact is due to our lack of analytical understanding of the nontrivial behaviour of those discrete materials as much in statics as in dynamics. Usually the static equilibrium behaviour is merely considered in order to determine those material parameters which are then re-used for the stationary dynamic analysis of the structures. However, as it is well known, Cosserat micropolar continuum models may degenerate into *strain-gradient* (also named *indeterminate couple-stress* [4] or *constrained-Cosserat* [5]) ones in the same manner as the Timoshenko's beam model may degenerate into Euler-Bernoulli's beam one [14]. When the elastic interaction constants of the discrete materials and/or its microstructure size dimensions vary and reach specific values, while still remaining in the allowed static stability domain of the generic discrete model, the recourse to the gradient model can be

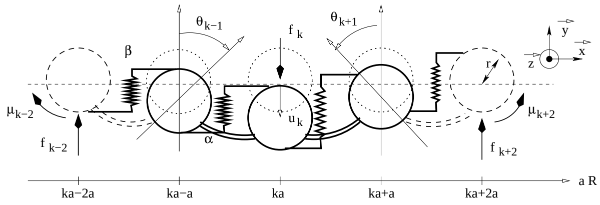

FIGURE 1 – *The granular model and its loading with respect to the direct orthonormal basis* $(\vec{x}, \vec{y}, \vec{z})$.

viewed as an incipient of instability (through the *apparent shear modulus* of the coarse, local, Cosserat description, that becomes negative) in the constitutive description of the elastic material.

2 The granular model

In order to illustrate the foregoing statement, the quasi-continuum modeling of the statics of a simple one-dimensional modeling of granular materials proposed by Pasternak and Mühlhaus [12, 13] has been revisited. The goal is to clarify certain aspects of the continuum properties and to exhibit barriers to overcome in order to bridge the gap between standard continuum model and granular (or atomic) physics. The current analysis aims at presenting different alternatives to Eringen/Kunin's "continualization" methodologies that are possible (and phenomenologically used) while following the analytic methodology used by Charlotte and Truskinovsky [1, 2] for one-dimensional homogeneous lattices with simple microstructures (as defined by Kunin [9]).

The analyzed material is illustrated in Fig. 1 and represents a homogeneous, linearly elastic, chain model of spherical rigid grains with indexes $k \in \mathcal{I} \subseteq \mathbb{Z}$, radius r and the respective mass/geometric center of which are separated by an inter-granular distance $a \geq 2r$. Each grain is kinematically limited to the transverse translation displacement $u_k\vec{y}$ and the in plane rotation angle $\theta_k\vec{z}$. The corresponding loading variables are the transverse forces $af_k\vec{y}$ and moments $a\mu_k\vec{z}$; for simplicity each sequence $\{af_k\}_{k\in\mathcal{I}}$ and $\{a\mu_k\}_{k\in\mathcal{I}}$ is assumed globally self-balanced and has a sufficiently decreasing rate for large $|k|$ if $\mathcal{I}$ is infinite. Moreover, the total elastic potential energy of the chain is postulated like

$$\mathcal{W}_d(\{\mathbf{u}_k\}_{k\in\mathcal{I}}) \overset{\text{def}}{=} \frac{a}{2}\sum_{k\in\mathcal{I}} \left\{ \alpha \left[\frac{u_{k+1}-u_k}{a} - \frac{\theta_{k+1}+\theta_k}{\xi} \right]^2 + \beta r^2 \left[\frac{\theta_{k+1}-\theta_k}{\xi a} \right]^2 \right\} \ , \text{ with } \mathbf{u}_k \overset{\text{def}}{=} \begin{bmatrix} u_k \\ \theta_k \end{bmatrix} . \quad (1)$$

Here $\alpha > 0$ and $\beta > 0$ are, respectively, shear-spring and bending-spring constant parameters and $\xi = 2$. The elastic energy in (1) is assumed to converge, what requires that both $\left|\frac{u_{k+1}-u_k}{a} - \frac{\theta_{k+1}+\theta_k}{\xi}\right|$ and $|\theta_{k+1} - \theta_k|$ tend sufficiently fast to zero at large $|k|$ (but without $\mathbf{u}_k$ necessarily tends to zero) if the chain (and so $\mathcal{I}$) is infinite. The total work of the external loadings reads like

$$\mathcal{P}_d(\{\mathbf{f}_k\}_{k\in\mathcal{I}}, \{\mathbf{u}_k\}_{k\in\mathcal{I}}) \overset{\text{def}}{=} a\sum_{k\in\mathcal{I}} \mathbf{f}_k^{\mathbf{t}}\,\mathbf{u}_k \equiv a\sum_{k\in\mathcal{I}} f_k\,u_k + a\sum_{k\in\mathcal{I}} \mu_k\,\theta_k \ , \text{ with } \mathbf{f}_k \overset{\text{def}}{=} \begin{bmatrix} f_k \\ \mu_k \end{bmatrix} \ , \ \sum_{k\in\mathcal{I}} \mathbf{f}_k \overset{\text{hyp}}{=} \mathbf{0} \quad (2)$$

the superscript symbol ($^{\mathbf{t}}$) denoting the matrix transposition operator.

In order to statically identify the continuum parameters of the considered mechanical systems, it is necessary to obtain the natural static equilibrium configurations that locally minimizes the lagrangian $\mathcal{W}_d - \mathcal{P}_d$ with respect to $\{\mathbf{u}_k\}_{k\in\mathcal{I}}$ at $\{\mathbf{f}_k\}_{k\in\mathcal{I}}$ given. If any exists, such a configuration satisfies then the following system of "bulk" equations of balance between the external and internal loadings

$$f_k = -\frac{\alpha}{a^2}\left[u_{k+1} + u_{k-1} - 2u_k\right] + \frac{\alpha}{a\xi}\left[\theta_{k+1} - \theta_{k-1}\right] \quad (3a)$$

$$\mu_k = -\frac{\alpha}{a\xi}\big[u_{k+1} - u_{k-1}\big] + \frac{\alpha}{\xi^2}\big[\theta_{k+1} + \theta_{k-1} + 2\theta_k\big] - \frac{\beta r^2}{\xi^2 a^2}\big[\theta_{k+1} + \theta_{k-1} - 2\theta_k\big]\ , \tag{3b}$$

with additionally the suitable natural boundary equations if the chain (and so $\mathcal{I}$) is bounded.

The material bulk properties of the granular model and its static configuration can be spectrally characterized in the sense of the *tempered* generalized functions with, on one hand, the Discrete Fourier's transform (**DFT**) [9] $\{w_k\}_{k\in\mathbb{Z}} \to w(\lambda) = a \sum_{k\in\mathbb{Z}} w_k\, \mathrm{e}^{-ki\lambda a}$ ($\forall\lambda \in a^{-1}\mathbb{R}$) while the arbitrary sequence $\big\{w_k\big\}_{k\in\mathbb{Z}}$ may eventually grow polynomially with $|k|$, and, on the other hand, the corresponding inverse transformation with the integral taken in the sense of Cauchy's principal values $w(\lambda) \to w_k = \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} w(\lambda)\,\mathrm{e}^{ki\lambda a}\, d\lambda$. The DFT provides from Eq.(3) then the spectral equation

$$\frac{4\alpha}{a^2}\,\Phi(\lambda)\,\mathbf{u}(\lambda) = \mathbf{f}(\lambda)\ ,\ \text{with}\ \Phi(\lambda) \stackrel{\text{def}}{=} \begin{bmatrix} \sin^2(\lambda a/2) & i\frac{a}{2\xi}\sin(\lambda a) \\ -i\frac{a}{2\xi}\sin(\lambda a) & \frac{a^2}{\xi^2}\Big\{\cos^2(\lambda a/2) + \frac{\beta r^2}{\alpha a^2}\sin^2(\lambda a/2)\Big\} \end{bmatrix} \tag{4}$$

and exhibits a $\frac{2\pi}{a}$-periodic spectral matrix $\frac{4\alpha}{a^2}\Phi(\lambda)$ that contains all the structural information about the elastic interactions in the system. In particular, besides the fact that the system is stable as both $\alpha > 0$ and $\beta > 0$ ($\Phi(\lambda)$ being both hermitian and positive definite for $\lambda \in a^{-1}(\mathbb{R}\setminus\mathbb{Z})$), it allows to state when the *strength-deformation constitutive relation* resumed by Eq.(4) is algebraically invertible in the Fourier complex space $a^{-1}\mathbb{C}$, which is not pointwisely possible at the λ-roots of the system characteristic equation $\det(\Phi(\lambda)) = \frac{\beta r^2}{\alpha\xi^2}\sin^4(\lambda a/2) = 0$. This determinant can also be expressed according to the Weierstrass's decomposition $\det(\Phi(\lambda)) \equiv \frac{\beta r^2}{16\alpha\xi^2}\lambda^4 a^4 \prod_{q=1}^{\infty}\Big(1 - \frac{\lambda^2}{\lambda_q^2}\Big)^4$, for $\lambda_q \stackrel{\text{def}}{=} 2q\pi a^{-1} \in 2\pi a^{-1}\mathbb{N}$ and $\lambda \in a^{-1}\mathbb{C}$. By resolving the spectral algebraic equation (4), one infers that one can formally write down then the general solution of the discrete static mechanical problem in Eq.(3) like

$$\mathbf{u}_k \equiv \sum_{p\in\mathcal{I}} \widetilde{\mathbf{G}}(\zeta, ka - pa)\mathbf{f}_p + \widetilde{\mathbf{u}}^h(\zeta, ka)\ ,\ \text{for}\ k\in\mathcal{I}\ \text{with}\ \zeta = 1\,. \tag{5a}$$

The first term of this possible form of interpolating functions for the discrete solution $\{\mathbf{u}_k\}_{k\in\mathcal{I}}$ represents the particular solution response to the prescribed loadings. It involves the fundamental solution matrix $\widetilde{\mathbf{G}}(\zeta, s)$ whose the values for $s \in a\mathbb{R}$ are

$$\widetilde{\mathbf{G}}(\zeta, s) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a^2\mathrm{e}^{is\lambda}}{4\alpha}\,\widetilde{\Phi}^{-1}(\zeta,\lambda)\,d\lambda \stackrel{\text{def}}{=} \frac{3|s|}{\alpha\ell^2}\begin{bmatrix} \frac{s^2}{3} + \frac{\zeta^2 a^2 - \ell^2}{6} & -\frac{\xi s}{2} \\ \frac{\xi s}{2} & -\frac{\xi^2}{2} \end{bmatrix}\ ,\ \text{with}\ \ell \stackrel{\text{def}}{=} r\sqrt{3\beta/\alpha}\,. \tag{5b}$$

Regarding the spectral matrix integrand $\widetilde{\Phi}^{-1}(\zeta,\lambda)$ in (5b), the case $\zeta = 1$ provides the first rank term (involving the fundamental root $\lambda_0 = 0$) into the Mittag Leffler's meromorphic expansion of the components of $\Phi^{-1}(\lambda)$ according the roots $\{\pm\lambda_q\}_{q\in\mathbb{N}}$, while the case $\zeta = 0$ corresponds to the first approximation order term of the Taylor's polynomial expansion for $|\lambda|a \ll 1$. Besides, one distinguishes in (5a) the following field vector that solves the homogeneous equilibrium problem related to Eq.(3)

$$\widetilde{\mathbf{u}}^h(\zeta, s) \stackrel{\text{def}}{=} \begin{bmatrix} \tilde{u}^h(\zeta,s) \\ \tilde{\theta}^h(\zeta,s) \end{bmatrix} = \begin{bmatrix} A + \big\{2B + \frac{D}{6}\,(\zeta^2 a^2 - \ell^2)\big\}s + Cs^2 + \frac{D}{3}s^3 \\ \xi B + \xi C s + \xi D s^2 \end{bmatrix}\ ; \tag{5c}$$

here (A, B, C, D) are arbitrary real constants, the terms proportional to (A, B) in (5c) corresponding to a work-less rigid motion. The four constants are determined by considering the boundary conditions (if the chain is bounded) or the "*tempered*" growth conditions at the infinity to keep the elastic energy and the work finite. Lately, in addition to a and r, the previous expressions in (5b) also exhibit an intrinsic length ℓ that is related to the material constants (α, β) and the representative grain size r.

3 Analogous quasi-continuum models

The discrete solution field $\{\mathbf{u}_k\}_{k\in\mathcal{I}}$ given in (5a) possesses the same (static) kinematic features as those attributed to the motion of beam sections of the Euler-Bernoulli's (**EB**) and Timoshenko's (**T**) theories when described with both the following continuous interpolating field vector

$$\widetilde{\mathbf{u}}(\zeta,s) \stackrel{\text{def}}{=} \begin{bmatrix} u(\zeta,s) \\ \theta(\zeta,s) \end{bmatrix} = \int_{\mathcal{S}} \widetilde{\mathbf{G}}(\zeta, s-\hat{s})\,\widetilde{\mathbf{f}}(\hat{s})\,d\hat{s} + \widetilde{\mathbf{u}}^h(\zeta,s)\ ,\ \text{for } s\in\mathcal{S}\subseteq a\mathbb{R} \text{ with } \mathcal{I}\equiv\mathbb{Z}\cap a^{-1}\mathcal{S}, \quad (6a)$$

and the singular loading field vector that involves the Dirac's generalized function δ

$$\mathbf{f}(s) \equiv \begin{bmatrix} f(s) \\ \mu(s) \end{bmatrix} \stackrel{\text{def}}{=} \sum_{p\in\mathcal{I}} \mathbf{f}_p\ \delta(s/a-p) \quad , \text{ with } \delta(s/a) \stackrel{\text{def}}{=} a^2 D_s^2 \frac{|s|}{2a}\,. \quad (6b)$$

For $\zeta = 1$, those fields $(\widetilde{\mathbf{u}}, \mathbf{f})$ exactly *interpolate* (but in the tempered distribution sense for $\mathbf{f}$) their discrete counterpart of equilibrium. Moreover, they provide the generic expression of the equilibrium configurations associated to the following *local continuum* models of total external work and elastic energy

$$\widetilde{P}(\mathbf{f},\widetilde{\mathbf{u}}) \stackrel{\text{def}}{=} \int_{\mathcal{S}} \mathbf{f}^{\mathbf{t}}(s)\,\widetilde{\mathbf{u}}(\zeta,s)\,ds \equiv \int_{\mathcal{S}} \left[f(s)\,u(\zeta,s) + \mu(s)\,\theta(\zeta,s)\right] ds\ , \quad (7a)$$

$$\widetilde{W}(\widetilde{\mathbf{u}}) \stackrel{\text{def}}{=} \begin{cases} \frac{\tilde{\alpha}}{2}\int_{\mathcal{S}} \gamma^2(\zeta,s)\,ds + \frac{\beta r^2}{8}\int_{\mathcal{S}} \kappa^2(\zeta,s)\,ds & ,\ \text{if } \ell\neq\zeta a \quad (\mathbf{T}) \\ \frac{\beta r^2}{8}\int_{\mathcal{S}} \kappa^2(\zeta,s)\,ds \quad \text{with } \gamma(\zeta,s)\equiv 0 & ,\ \text{if } \ell=\zeta a>0 \quad (\mathbf{EB}) \end{cases} . \quad (7b)$$

That latter is expressed with the following *apparent shear modulus* $\tilde{\alpha}(\zeta a/\ell) \stackrel{\text{def}}{=} \frac{\alpha\ell^2}{\ell^2-\zeta^2a^2}$ and *apparent bending modulus* $\beta r^2/4$, as well as the standard Cosserat continuum measures of deformation

$$\gamma(\zeta,s) \stackrel{\text{def}}{=} D_s u(\zeta,s) - \frac{2}{\xi}\,\theta(\zeta,s) \ \text{ and } \ \kappa(\zeta,s) \stackrel{\text{def}}{=} \frac{2}{\xi} D_s\theta(\zeta,s) \quad \big(\text{while } D_s \stackrel{\text{def}}{=} \frac{d}{ds}\big)\,. \quad (7c)$$

If we consider the **T** beam modeling case, the bulk equations of balance for (7) read like

$$f(s) = -\tilde{\alpha}\left[D_s^2\tilde{u}(\zeta,s) - \frac{2}{\xi}\,D_s\tilde{\theta}(\zeta,s)\right]\ ,\ \mu(s) = -\tilde{\alpha}\left[\frac{2}{\xi}\,D_s\tilde{u}(\zeta,s) - \frac{4}{\xi^2}\,\tilde{\theta}(\zeta,s)\right] - \frac{\beta r^2}{\xi^2}\,D_s^2\tilde{\theta}(\zeta,s)\,. \quad (8)$$

In agreement with [12, 13], the case $\zeta = 0$ corresponds to the *long-wave approximating* quasi-continuum model. That case can straightforwardly be derived from Taylor's expanding Eq.(3) up to the first order in a while assuming a couple of *sufficiently smooth and slow-varying continuous* fields $(\widetilde{\mathbf{u}}(\zeta{=}0,s), \mathbf{f}(s))$ for $s\in\mathcal{S}\subseteq a\mathbb{R}$ that are approximatively capable of *interpolating* their discrete counterparts $(\mathbf{u}_k, \mathbf{f}_k)$ at $s = ka \in a\mathcal{I}$. From a numerical viewpoint, the energy model (1) with $\zeta = 0$ can also be interpreted as a Finite Element Approximation of (7b) while using a piecewise affine polynomial approximation of the displacement field $\widetilde{\mathbf{u}}(\zeta{=}0,s)$ and an one-point Gauss under-integration for the shear strain γ. One easily infers that $\widetilde{\mathbf{u}}(\zeta{=}0,ka)$ is a "good" approximation of $\widetilde{\mathbf{u}}(\zeta{=}1,ka)\equiv\mathbf{u}_k$ at the granular sites only if $\ell = r\sqrt{3\beta/\alpha} \gg a$, if one "tolerates" (as customarily done in the literature) that the derived model can still be used with the singular loading (6b) beyond of the required smooth hypotheses of Taylor's expansion derivation. However, in general a discrepancy can be observed between the fields derived by Taylor's expansions and by spectral-root expansions [1], mainly for the related shear strain measure in (7c) that notably reads then $\gamma(\zeta,s) \equiv \left(\frac{\zeta^2a^2}{\ell^2}-1\right)\left[\frac{\ell^2}{12\xi}\,D_s^2\tilde{\theta}^h(0) + \int_{\mathcal{S}} \operatorname{sgn}(s-\hat{s})\,\frac{\tilde{f}(\hat{s})}{2\alpha}\,d\hat{s}\right]$.

Now, the micro-rotation sequence $\{\theta_k\}_{k\in\mathcal{I}}$ and the translation one $\{u_k\}_{k\in\mathcal{I}}$ of the generic discrete model were assumed to be independent, as in the *couple-stress theory* or again *unconstrained Cosserat's pseudo-continuum* (of asymmetric elasticity). However, the independence of these kinematic

variables would fail when $\ell = r\sqrt{3\beta/\alpha} \equiv a$ because the macro-rotation field $D_s u$ becomes equal to the micro-rotation angle field $2\,\theta/\xi$, yielding so $\gamma(\zeta{=}1, s) \equiv 0$. That singular case corresponds to the *Euler-Bernoulli's hypotheses for thin beam modeling* and also to the so-called *indeterminate couple-stress theory* [5] or again *constrained Cosserat's theory* [11] corresponding to the strain-gradient theory.
It is obvious the previous **T** beam modeling violates the requirement of positive definiteness of the strain energy density as $\ell < a$ and so it is somewhat unsatisfactory. That suggests that such a phenomenological modeling is still far from giving a comprehensive treatment for the one-dimensional elastic Cosserat chain model and therefore a different modeling is necessary. Such a model can be derived by adopting a different spectral decomposition for the elastic kernel and analysis of the spectral equation in (4), without adopting Eringen/Kunin's "continualization" approaches [12, 13, 2]. Instead of Mittag-Leffler's expanding each component of the matrix $\Phi^{-1}(\lambda)$, one can also merely bound this to the determinant $\det(\Phi^{-1}(\lambda))$ and, accordingly with [1, 2], retain then the first rank term of the new expansion for $\Phi^{-1}(\lambda)$ as a weak approximation while keeping the ensuing matrix identity for $k \in \mathbb{Z}$

$$\int_{-\infty}^{\infty} \mathrm{e}^{ika\lambda} \left[\widehat{\Phi}(\lambda)\right]^{-1} d\lambda \equiv \int_{-\pi/a}^{\pi/a} \mathrm{e}^{ika\lambda}\, \Phi^{-1}(\lambda)\, d\lambda \;\text{ with }\; \widehat{\Phi}(\lambda) \overset{\text{def}}{=} \frac{3\beta r^2 \lambda^4 a^4}{8\alpha\xi^2\left(6+\lambda^2 a^2\right) \det\left(\Phi(\lambda)\right)}\, \Phi(\lambda)\,. \quad (9)$$

It results then a *strongly non-local* quasicontinnum model that, if $\mathcal{I} \equiv \mathbb{Z}$ (and so $\mathcal{S} \equiv a\mathbb{R}$) or else Dirichlet's kinematic boundary conditions are imposed at the chain end(s), corresponds to a lagrangian $\widehat{W}(\widehat{\mathbf{u}}) - \widetilde{P}(\mathbf{f}, \widehat{\mathbf{u}})$ involving the external load work $\widetilde{P}(\mathbf{f}, \widehat{\mathbf{u}})$ in (7a) and the ensuing elastic energy

$$\widehat{W}(\widehat{\mathbf{u}}) \overset{\text{def}}{=} 2\alpha \int_{\mathcal{S}}\int_{\mathcal{S}} \left[D_{\check{s}}\widehat{\mathbf{u}}(\check{s})\right]^{\mathbf{t}} \mathbf{K}(s-\check{s})\, D_s\widehat{\mathbf{u}}(s)\, d\check{s}\, ds\;,\; \text{with } \mathbf{K}(s) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\Phi}(\lambda)\, \frac{\mathrm{e}^{i\lambda s}}{a^2\lambda^2}\, d\lambda \quad (10)$$

(that elastic stiffness matrix $\mathbf{K}$ will be explicitly presented elsewhere). Subsequently, the related bulk equilibrium equations read so vectorially like

$$-4\alpha \int_{\mathcal{S}} \mathbf{K}(s-\check{s})\, D^2_{\check{s}}\widehat{\mathbf{u}}(\check{s})\, d\check{s} = \mathbf{f}(s)\;,\; \text{ for } s \in \mathcal{S}\,. \quad (11)$$

The equilibrium configuration minimizing the foregoing nonlocal lagrangian model satisfies $u_k = \widehat{\mathbf{u}}(ka)$ (for $k \in \mathbb{Z}$) and reads formally like

$$\widehat{\mathbf{u}}(s) \equiv \int_{\mathcal{S}} \widehat{\mathbf{G}}(s-\hat{s})\, \mathbf{f}(\hat{s})\, d\hat{s} \;+\; \widetilde{\mathbf{u}}^h(\zeta{=}1, s)\;,\; \text{with } \widehat{\mathbf{G}}(s) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a^2\mathrm{e}^{is\lambda}}{4\alpha}\widehat{\Phi}^{-1}(\lambda)\, d\lambda \text{ for } s \in a\mathbb{R} \quad (12a)$$

while knowing that $\det\left(\widehat{\Phi}(\lambda)\right) = \left(\frac{\ell^2\lambda^4 a^4}{8\xi^2\left(6+\lambda^2 a^2\right)}\right)^2 \frac{1}{\det\left(\Phi(\lambda)\right)}$ has only $\lambda = 0$ as a fourth-order root.
As a result, the fundamental solution matrix $\widehat{\mathbf{G}}$ given hereabove reads more explicitly like

$$\begin{aligned}\widehat{\mathbf{G}}(s) \;=\; & \frac{1}{\alpha\ell^2}\begin{bmatrix} |s|^3 + \frac{\ell^2-3a^2}{12a^2}\{2|s|^3 - |s+a|^3 - |s-a|^3\} & -\frac{\xi}{4a}\{|s+a|^3 - |s-a|^3\} \\ \frac{\xi}{4a}\{|s+a|^3 - |s-a|^3\} & \frac{\xi^2}{4a^2}\{2|s|^3 - |s+a|^3 - |s-a|^3\} \end{bmatrix} \\ & -\frac{a^2}{\alpha\ell^2}\begin{bmatrix} |s| + \frac{\ell^2-3a^2}{12a^2}\{2|s| - |s+a| - |s-a|\} & -\frac{\xi}{4a}\{|s+a| - |s-a|\} \\ \frac{\xi}{4a}\{|s+a| - |s-a|\} & \frac{\xi^2}{4a^2}\{2|s| - |s+a| - |s-a|\} \end{bmatrix} \end{aligned} \quad (12b)$$

(while reminding that $\ell \overset{\text{def}}{=} r\sqrt{3\beta/\alpha}$) and is different from $\widetilde{\mathbf{G}}(\zeta{=}1, s)$ in (5b) only as $s \in a(\mathbb{R}\setminus\mathbb{Z})$.

4 Concluding remarks

As predicted for the continuum models [10], size effects also occur for the discrete Cosserat's model through the exact value of the apparent shear modulus $\widetilde{\alpha}(a/\ell)$, which notably tends to infinity as the spheric grain radius $r \equiv \ell\sqrt{\alpha/3\beta}$ tends to $a\sqrt{\alpha/3\beta}$. However, when $\ell < a$, the observed micro-displacements of the grains can no longer be accurately modeled with a mathematically sound, standard (*i.e.* local) Cosserat/Timoshenko's beam model. It is likely that a more refined continuum theory model than the standard Cosserat's elasticity would be required to deal with the relevant micro-phenomena that can be observed at the lengthscale a, but also in dynamics. For an accurate and mathematically consistent description of the discrete equilibrium configuration, a *de-localization* of the balance and constitutive continuum laws of the material becomes necessary. Already in statics, the delocalization corresponding to a nonlocal model [7, 8] obviously constitutes the last recourse only when the apparent shear modulus of the unconstrained-Cosserat model becomes negative, or high-frequency dynamical effects are to be taken into account. In their former attempt, [12, 13] proposed a nonlocal quasi-continuum model based on Kunin's viewpoint [9]. That one assumes that $\{\mathbf{f}_k\}_{k\in\mathcal{I}}$ and $\{\mathbf{u}_k\}_{k\in\mathcal{I}}$ can be interpolated by sinus cardinal series and is in principle valid for unbounded domains (*i.e.* $\mathcal{I} \equiv \mathbb{Z}$). However, as the series coefficients $\{\mathbf{u}_k\}_{k\in\mathcal{I}}$ do not necessary tend to zero for large $|k| \in \mathbb{N}$, such a constraining hypothesis must be weakened for the continuous interpolating displacement field by imposing to its Fourier image to be *only band-limited* to the first Brillouin's zone $[-\pi/a, \pi/a]$ as $\lambda \in a^{-1}\mathbb{R}$. This statement is presented elsewhere with a new analysis of the granular chain dynamics.

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