POD based Reduced Order Modelling of a compressible forced cavity flow

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Résumé :

L'objectif final de cette étude est d'utiliser un modèle réduit construit par POD pour contrôler un écoulement de cavité ouverte en régime compressible. Dans cette communication, nous utiliserons des méthodes de calibration pour améliorer la précision du modèle réduit et nous nous intéresserons à une stratégie de construction d'une base POD adaptée au contrôle.

Abstract :

The final objective of this work is to utilise a Reduced Order Model based on POD to control the compressible flow over an open cavity. In this communication we will employ some methods of calibration to improve the accuracy of the Reduced Order Model and will also present a strategy to extend the POD basis to include the effect of actuation.

Keywords : **Reduced Order Model, POD, Compressible Cavity**

1 Introduction

As far as solving an optimal control problem is concerned, the traditional approaches like Direct Numerical Simulation (DNS) and Large Eddy Simulation (LES) pose difficulties in terms of computational resources. Therefore, in these circumstances, Reduced Order Modelling (ROM) based on Proper Orthogonal Decomposition (POD) has been widely used as a predictive tool to model the flow dynamics. In the past, ROM has been successfully applied for various flow control problems like wake behind cylinder [1] and high lift configurations [2] to name a few. Flow past an open cavity has already been studied using ROM by [3] and [4] but without any application to flow control. More recently, ROM for controlled configurations has been proposed by [5] and [6]. In this work we propose a calibrated ROM for laminar cavity flow and give extensions to model actuated flow which will be used in the future for control purposes.

2 Reduced Order Modelling of Cavity Flows

2.1 Dynamical system

The basic idea of a POD ROM is to decompose the flow field into energy ranked coherent structures represented by mathematical modes and then use these modes within a Galerkin approach for reducing the order of the system. Let $\boldsymbol{q}(\boldsymbol{x},t)$ be any flow variable where $\boldsymbol{x}$ and $t \in [0,T]$ denote respectively the spatial variables and the time of the numerical simulation then we seek an expansion for $\boldsymbol{q}$ of the form

$$\boldsymbol{q}(\boldsymbol{x},t) = \overline{\boldsymbol{q}}(\boldsymbol{x}) + \sum_{i=1}^{N_{POD}} a_i(t)\boldsymbol{\phi_i}(\boldsymbol{x}) \tag{1}$$

where $\overline{\boldsymbol{q}}$ denotes the average of the flow snapshots. The spatial modes $\boldsymbol{\phi_i}(\boldsymbol{x})$ and the temporal coefficients $a_i(t)$ are determined by solving an eigenvalue problem that minimizes the average projection error. The Reduced Order Model is then obtained by performing a Galerkin projection of the governing dynamics (for instance the Navier-Stokes equations) onto the spatial modes. Two models for the ROM has been proposed in [4], one using the full compressible equations and the other with an isentropic approximation. In this work we use the isentropic equations for compressible flows as in [3]. Scaling the velocities u, v by the freestream velocity U_∞, the local sound speed c by the ambient sound speed c_∞, the lengths by the cavity depth D (see figure 1), and time by D/U_∞, the equations are given by

$$u_t + uu_x + vu_y + \frac{1}{M^2}\frac{2}{\gamma-1}cc_x = \frac{1}{Re}(u_{xx}+u_{yy})$$
$$v_t + uv_x + vv_y + \frac{1}{M^2}\frac{2}{\gamma-1}cc_y = \frac{1}{Re}(v_{xx}+v_{yy})$$
$$c_t + uc_x + vc_y + \frac{\gamma-1}{2}c(u_x+v_y) = 0$$

where $M = U_\infty/a_\infty$ is the Mach number and $Re = U_\infty D/\nu$ is the Reynolds number. If we denote $\boldsymbol{q} = (u,v,c)$ the vector of flow variables, the above equations can be recasted as

$$\dot{\boldsymbol{q}} = \frac{1}{Re}\boldsymbol{L}(\boldsymbol{q}) + \frac{1}{M^2}\boldsymbol{Q_1}(\boldsymbol{q},\boldsymbol{q}) + \boldsymbol{Q_2}(\boldsymbol{q},\boldsymbol{q}) \qquad \text{with} \tag{2}$$

$$\boldsymbol{L}(\boldsymbol{q}) = \begin{bmatrix} u_{xx}+u_{yy} \\ v_{xx}+v_{yy} \\ 0 \end{bmatrix}, \qquad \boldsymbol{Q_1}(\boldsymbol{q^1},\boldsymbol{q^2}) = -\tfrac{2}{\gamma-1}\begin{bmatrix} c^1c^2_x \\ c^1c^2_y \\ 0 \end{bmatrix}, \qquad \boldsymbol{Q_2}(\boldsymbol{q^1},\boldsymbol{q^2}) = -\begin{bmatrix} u^1u^2_x + v^1u^2_y \\ u^1v^2_x + v^1v^2_y \\ u^1c^2_x + v^1c^2_y + \tfrac{\gamma-1}{2}c^1(u^2_x+v^2_y) \end{bmatrix}$$

To obtain the reduced order model by means of a Galerkin projection we define an inner product on the state space as

$$(\boldsymbol{q_1},\boldsymbol{q_2})_\Omega = \int_\Omega (u_1u_2 + v_1v_2 + \frac{2\alpha}{\gamma-1}c_1c_2)\, d\Omega$$

where α is a constant and γ is the ratio of specific heats. In this work we choose the value of $\alpha = 1$, which gives the definition of stagnation enthalpy while calculating the norm. The above definition ensures the stability of the origin of the attractor. We use the expansion (1) and the definition of the inner product given above to perform the Galerkin projection of the isentropic equations onto the first $n \ll N_{POD}$ spatial eigenfunctions. After some manipulation, we obtain the Reduced Order Model

$$\begin{aligned}\dot{a}^R_i(t) &= \frac{1}{Re}C^1_i + \frac{1}{M^2}C^2_i + C^3_i + \sum_{j=1}^{n}\left(\frac{1}{Re}L^1_{ij} + \frac{1}{M^2}L^2_{ij} + L^3_{ij}\right)a^R_j(t) + \sum_{j,k=1}^{n}\left(\frac{1}{M^2}Q^1_{ijk} + Q^2_{ijk}\right)a^R_j(t)a^R_k(t) \\ &= C_i + \sum_{j=1}^{n}L_{ij}a^R_j(t) + \sum_{j,k=1}^{n}Q_{ijk}a^R_j(t)a^R_k(t) = f_i(C_i,\boldsymbol{L_i},\boldsymbol{Q_i},\boldsymbol{a}^R(t))\end{aligned} \tag{3}$$

where f_i is a polynomial of degree 2 in $\boldsymbol{a}^R$ and where the coefficients are given by

$$C^1_i = (\boldsymbol{\phi_i},\boldsymbol{L}(\overline{\boldsymbol{q}}))_\Omega \qquad L^1_{ij} = (\boldsymbol{\phi_i},\boldsymbol{L}(\boldsymbol{\phi_j}))_\Omega \qquad Q^1_{ijk} = (\boldsymbol{\phi_i},\boldsymbol{Q_1}(\boldsymbol{\phi_j},\boldsymbol{\phi_k}))_\Omega$$
$$C^2_i = (\boldsymbol{\phi_i},\boldsymbol{Q_1}(\overline{\boldsymbol{q}},\overline{\boldsymbol{q}}))_\Omega \qquad L^2_{ij} = (\boldsymbol{\phi_i},\boldsymbol{Q_1}(\overline{\boldsymbol{q}},\boldsymbol{\phi_j}) + \boldsymbol{Q_1}(\boldsymbol{\phi_j},\overline{\boldsymbol{q}}))_\Omega \qquad Q^2_{ijk} = (\boldsymbol{\phi_i},\boldsymbol{Q_2}(\boldsymbol{\phi_j},\boldsymbol{\phi_k}))_\Omega$$
$$C^3_i = (\boldsymbol{\phi_i},\boldsymbol{Q_2}(\overline{\boldsymbol{q}},\overline{\boldsymbol{q}}))_\Omega \qquad L^3_{ij} = (\boldsymbol{\phi_i},\boldsymbol{Q_2}(\overline{\boldsymbol{q}},\boldsymbol{\phi_j}) + \boldsymbol{Q_2}(\boldsymbol{\phi_j},\overline{\boldsymbol{q}}))_\Omega$$

2.2 Calibration of ROM

The reduced order model has an intrinsic problem of converging to the wrong attractor when used to simulate the flow for long time periods and hence there is a need to identify the coefficients of the dynamical system so as to minimize the error between the actual time coefficients $a^P_i(t)$ and that obtained from the ROM $a^R_i(t)$ using a suitable norm for the error. Three definitions of error have been defined in [7]. The natural choice is in defining $\boldsymbol{e^{(1)}}(\boldsymbol{f},t) = \boldsymbol{a^P}(t) - \boldsymbol{a^R}(t)$ but this leads to a nonlinear constrained problem (see section 2.2.1). The other two choices being state calibration (error $\boldsymbol{e^{(1)}}$ without the dynamical constraint) and flow calibration $\boldsymbol{e^{(3)}}(\boldsymbol{f},t) = \dot{\boldsymbol{a}}^{\boldsymbol{P}}(t) - \boldsymbol{f}(\boldsymbol{a^P}(t))$ (see section 2.2.2) where $\boldsymbol{f}$ is characterized by the coefficients of (3).

2.2.1 Nonlinear constrained problem

In this method we have to find the coefficients of the dynamical system such that the error $\boldsymbol{e^{(1)}}$ is minimized under the constraints that the coefficients $C_i, \boldsymbol{L_i}, \boldsymbol{Q_i}$ ($i = 1,\cdots,n$) satisfy (3). We rather seek to minimize $\mathcal{I}^{(1)}(\boldsymbol{f}) = \langle\|\boldsymbol{e^{(1)}}(\boldsymbol{f},t)\|^2\rangle_T$ where $\langle\cdot\rangle_T$ is any time averaging operator and $\|\cdot\|$ is the norm defined as $\|\boldsymbol{e}\|^2 =$

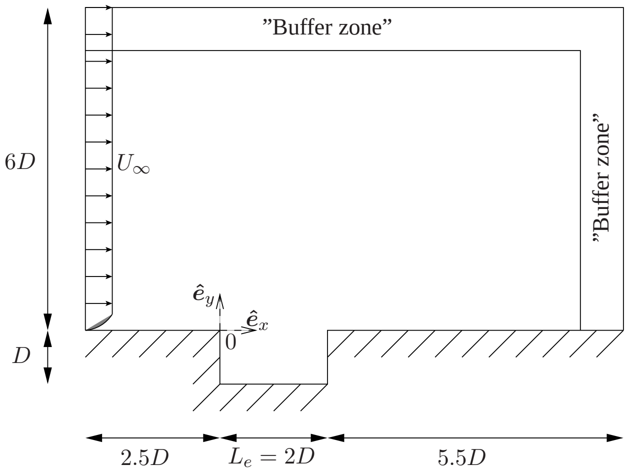

Figure 1: Schematic diagram of cavity configuration and computational domain.

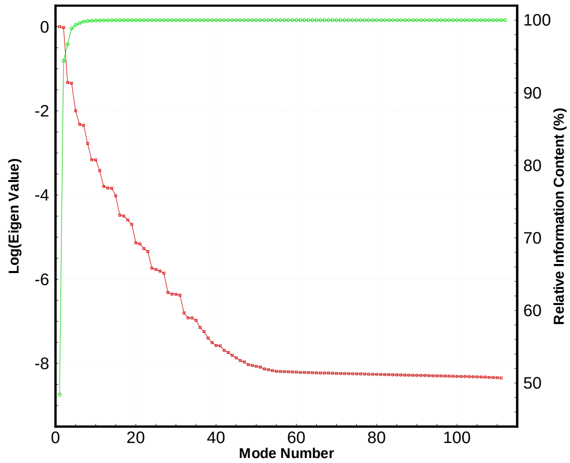

Figure 2: Eigenvalue spectrum and Relative Information Content.

$\boldsymbol{e}^T\boldsymbol{e}$. This leads us to a constrained optimisation problem as found in [1] in which we solve backward in time with $\xi_i(T) = 0$ an adjoint equation given by

$$\dot{\xi}_i(t) = -\sum_{j=1}^{n} L_{ji}\xi_j(t) - \sum_{j,k=1}^{n} \xi_j(t)(Q_{jik} + Q_{jki})a_i^R(t) - 2\left(a_i^R(t) - a_i^P(t)\right) \quad (4)$$

where ξ_i is the adjoint variable. The gradients of the cost functional $\mathcal{I}^{(1)}$ with respect to the coefficients C_i and $\boldsymbol{L_i}$ of the ROM are

$$\nabla\mathcal{I}^{(1)}{}_{C_i} = \int_0^T \xi_i(t)\,dt, \qquad \nabla\mathcal{I}^{(1)}{}_{\boldsymbol{L_i}} = \int_0^T \xi_i(t)a_j^R(t)\,dt$$

and represent the sensitivity of the functional to the coefficients. The corresponding optimality system (state equations, adjoint equations, optimality conditions) is solved by a gradient based method to obtain the coefficients of the reduced order model.

2.2.2 Flow calibration with Tikhonov regularization

As demonstrated in [8] minimization of $\mathcal{I}^{(3)}(\boldsymbol{f}) = \langle \|\boldsymbol{e}^{(3)}(\boldsymbol{f}, t)\|^2 \rangle_T$ leads to solve a linear system for the coefficients $\mathbf{y}$ defining $\boldsymbol{f}$ (see [7] for the details). Unfortunately, this linear system is not well conditioned what may lead to numerical divergence when the calibrated coefficients are used to integrate in time (3). For that purpose, [8] introduced a new cost functional defined as a weighted sum of a term measuring the normalized error between the behavior of the model (3) with $\boldsymbol{f}$ and with the coefficients determined directly by Galerkin projection $\boldsymbol{f}^{GP}$ and another term linked to the distance between $\boldsymbol{f}$ and $\boldsymbol{f}^{GP}$. The value of the weighting factor which represents the cost of calibration was chosen rather arbitrarily and hence was user dependent. In [7], a Tikhonov regularization method was suggested to improve the conditioning of the linear system. The idea of this method is to seek the regularized solution $\mathbf{y}_\rho$ as the minimizer of the following weighted functional

$$\Phi_\rho(\mathbf{y}) = \|A\mathbf{y} - \mathbf{b}\|_2^2 + \rho\|\mathcal{L}\left(\mathbf{y} - \mathbf{y_0}\right)\|_2^2,$$

where the first term corresponds to the residual norm, and the second to a side constraint imposed on the solution. $\mathcal{L}$ represents the discrete approximation matrix of a differential operator of order d and $\mathbf{y_0}$ a reference solution. The value ρ is chosen so as to compromise between the minimization of the norm of the residual for the linear system and the semi-norm of the solution. Here, the regularization parameter ρ is determined by the classical L-curve method, see [7].

3 Cavity Flow Configuration

The cavity is of an L_e/D ratio of 2. The flow is initialized by a laminar boundary layer so as to have a thickness of $\delta/D = 0.28$ at the leading edge of the cavity. The Reynolds number of the flow based on the cavity depth

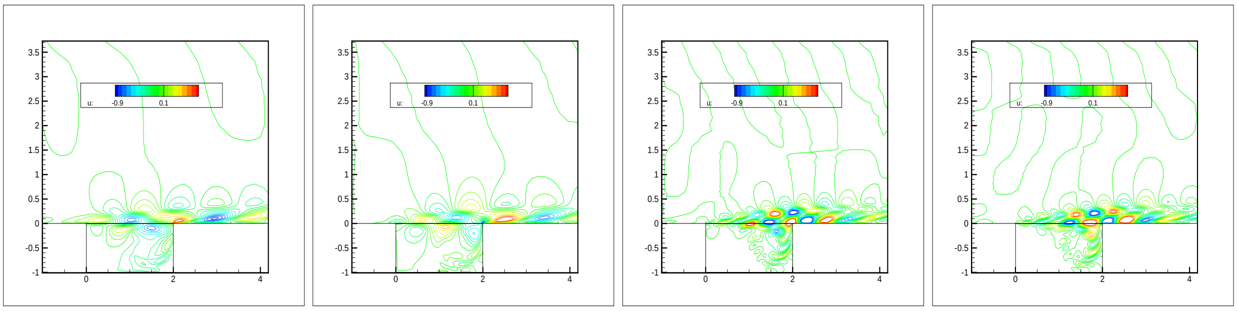

Figure 3: Representation of the u component of velocity for the first 4 spatial POD modes.

is 1500 and the flow Mach number is 0.6. The equations of Navier-Stokes are discretised using a 4^{th} order scheme in time and space (DNS code is provided by P. Comte, LEA Poitiers). Snapshots are taken once the flow has stabilized and 112 snapshots are uniformly sampled which corresponds to about 2 periods of the flow oscillation (5.6 in non dimensional time) corresponding to the first Rossiter mode [9]. Figure 2 demonstrates a degenerate eigen spectrum showing eigenvalues which occur in pairs. Also the first 4 eigenmodes capture around 98.5% of the total fluctuation energy as shown by the Relative Information Content (RIC) defined as $RIC(i) = \sum_{j=1}^{i} \lambda_j / \sum_{j=1}^{M} \lambda_j$. The representation of the first 4 spatial POD modes is shown in Fig. 3. Although the modes occur in pairs and their values are distinct the representation is topologically equivalent. Figure 4 presents for $n = 4$ the temporal modes before calibration and after calibration with the Tikhonov approach. The results of the iterative optimisation procedure are not plotted but are similar to the results obtained from Tikhonov approach. We observe that the un-calibrated ROM behaves well initially as compared to the DNS simulation and later starts diverging.

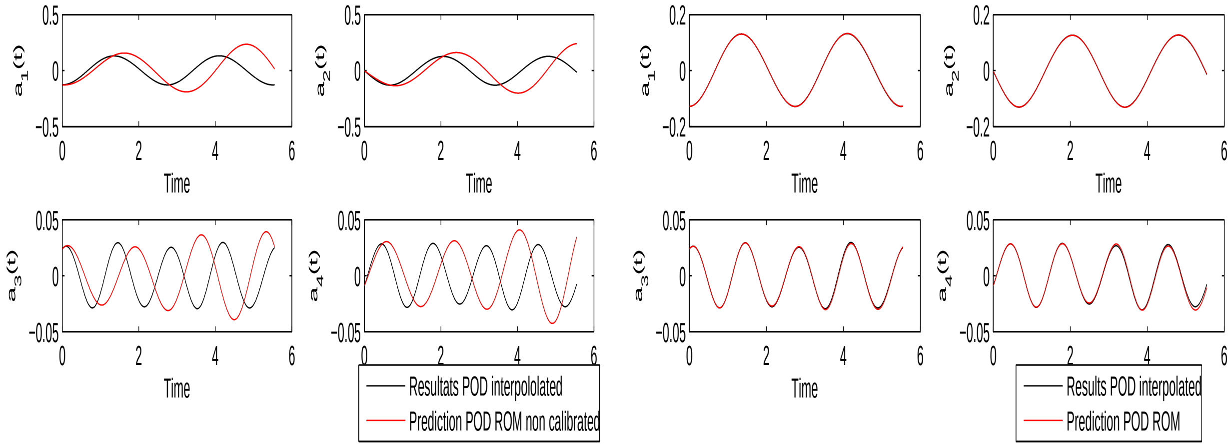

Figure 4: Comparison of the temporal modes before and after calibration ($n = 4$).

We also plot in Fig. 5 the error per mode for each method and we find that calibration by the iterative optimisation yields a better result. Also the error of calibration increases for the higher mode as, inter-modal transfers are not taken into account while truncating the terms in the ROM.

4 Actuated Flow ROM

The advantage of the reduced order models can be fully exploited when they are capable of being used in control studies, also we would like to have a dynamical system where the actuation effect is naturally embedded. The usual control input separation methods [1] have the disadvantage that one must be able to identify the control regions, while taking care to reproduce the un-actuated dynamics when the actuation value tends to zero. The method described in [6] solves an optimal problem to determine a suitable actuation mode to be included in the modal expansion. To start with, let the actuated snapshot sets be denoted as $\{\boldsymbol{q_k^{ac}}, \gamma_k\}_{k=1}^{m}$, where $\gamma_k = \gamma(t_k)$ is the value of the actuation, $\boldsymbol{q_k^{ac}} = \boldsymbol{q^{ac}}(\boldsymbol{x}, t_k)$ and m is the number of actuated snapshots. We subtract the mean $\boldsymbol{q_0}$ of the un-actuated base flow from the snapshot set. We define a new set of realizations by an innovation operator given as

$$\tilde{\boldsymbol{q_k}} = \boldsymbol{q_k^{ac}} - P_S \boldsymbol{q_k^{ac}} = \boldsymbol{q_k^{ac}} - \sum_{i=1}^{n} (\boldsymbol{q_k^{ac}}, \boldsymbol{\phi_i})_\Omega \, \boldsymbol{\phi_i}$$

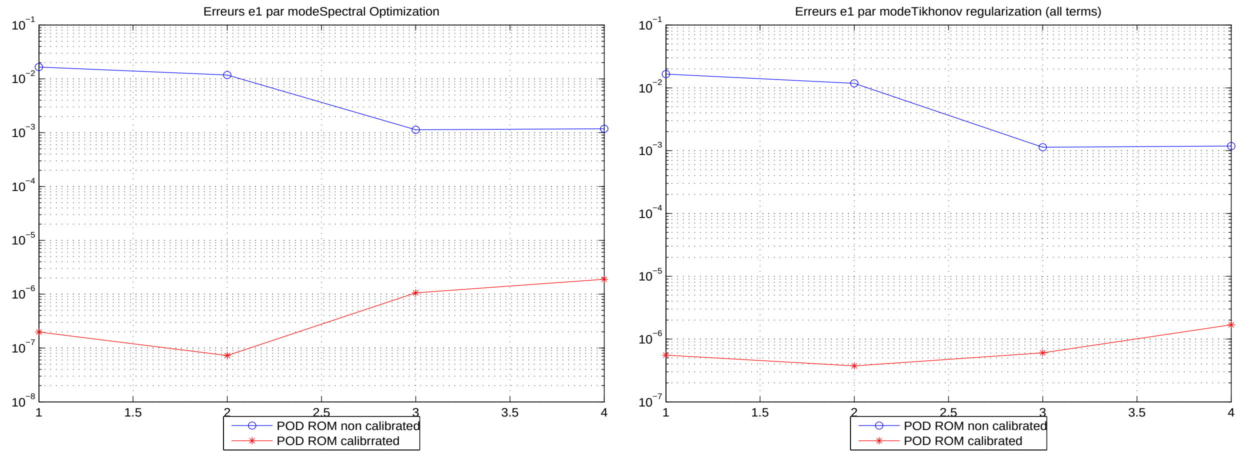

Figure 5: Comparison of the errors per mode for the two methods of calibrations. For the flow calibration, we have $d = 2$ and $\mathbf{y_0} = \mathbf{0}$.

to care of the part of the actuated mode which can be captured well by the un-actuated subspace. We then wish to construct an orthogonal subspace to the un-actuated space to capture the effect of actuation. This is done by solving an L_2 minimization problem for the functional given by

$$\mathcal{J}(\boldsymbol{\psi}) = E\left[\|\tilde{\boldsymbol{q}}_{\boldsymbol{k}} - \gamma_k \boldsymbol{\psi}\|^2\right]$$

where E is any averaging operator. The solution of the above minimization problem is given by

$$\boldsymbol{\psi} = \frac{E(\gamma_k \tilde{\boldsymbol{q}}_{\boldsymbol{k}})}{E(\gamma_k^2)}$$

The expansion for the flow field can now be written for the actuated case as

$$\boldsymbol{q^{ac}}(\boldsymbol{x}, t) = \overline{\boldsymbol{q}}^{\boldsymbol{ac}}(\boldsymbol{x}) + \sum_{i=1}^{n} a_i^{ac}(t)\boldsymbol{\phi_i}(\boldsymbol{x}) + \gamma(t)\boldsymbol{\psi}(\boldsymbol{x}) \tag{5}$$

With this expansion the Galerkin projection for equation (2) gives

$$\dot{a}_i^{ac}(t) = C_i + \sum_{j=1}^{n} L_{ij} a_j^{ac}(t) + \sum_{j,k=1}^{n} Q_{ijk} a_j^{ac}(t) a_k^{ac}(t) + h_{1i}\gamma(t) + \sum_{j=1}^{n} h_{2ij} a_j^{ac}(t)\gamma(t) + h_{3i}\gamma^2(t)$$

where

$$h_{1i} = \frac{1}{Re}\left(\boldsymbol{\phi_i}, \boldsymbol{L}(\boldsymbol{\psi})\right)_\Omega + \left(\boldsymbol{\phi_i}, \boldsymbol{Q}(\overline{\boldsymbol{q}}, \boldsymbol{\psi})\right)_\Omega + \left(\boldsymbol{\phi_i}, \boldsymbol{Q}(\boldsymbol{\psi}, \overline{\boldsymbol{q}})\right)_\Omega,$$
$$h_{2ij} = \left(\boldsymbol{\phi_i}, \boldsymbol{Q}(\boldsymbol{\phi_j}, \boldsymbol{\psi})\right)_\Omega + \left(\boldsymbol{\phi_i}, \boldsymbol{Q}(\boldsymbol{\psi}), \boldsymbol{\phi_j})\right)_\Omega, h_{3i} = \left(\boldsymbol{\phi_i}, \boldsymbol{Q}(\boldsymbol{\psi}, \boldsymbol{\psi})\right)_\Omega$$

with $\boldsymbol{Q}(\cdot,\cdot) = \frac{1}{M^2}\boldsymbol{Q_1}(\cdot,\cdot) + \boldsymbol{Q_2}(\cdot,\cdot)$.

The actuated temporal modes are obtained from (5) as

$$a_i^{ac}(t) = \left(\boldsymbol{\phi_i}, \boldsymbol{q^{ac}} - \overline{\boldsymbol{q}}^{\boldsymbol{ac}} - \gamma(t)\boldsymbol{\psi}\right)_\Omega$$

Here we make the assumption that the average of the mean flow in unactuated and actuated cases are equal ($\overline{\boldsymbol{q}}^{\boldsymbol{ac}} = \overline{\boldsymbol{q}}$). It can induce some numerical error, so calibration is also required for the forced dynamical system. For calibration of the above system we use a two fold approach first by calibrating the coefficients of the un-actuated flow to represent the temporal modes of the un-actuated system $a_i^R(t)$ and then to calibrate the additional coefficients of the actuated system above to match the temporal behavior for the actuated case $a_i^{ac}(t)$. The actuated spatial mode are presented in Fig. 6. The snapshots are obtained from the DNS by introducing an actuation of the form $|A\sin(\omega t)|$ just before the leading edge of the cavity ($x \in [-0.15; -0.05]$ and $y = 0$) where the flow is more sensitive to actuation. The spatial modes exhibit a local behavior capturing the effect of actuation.

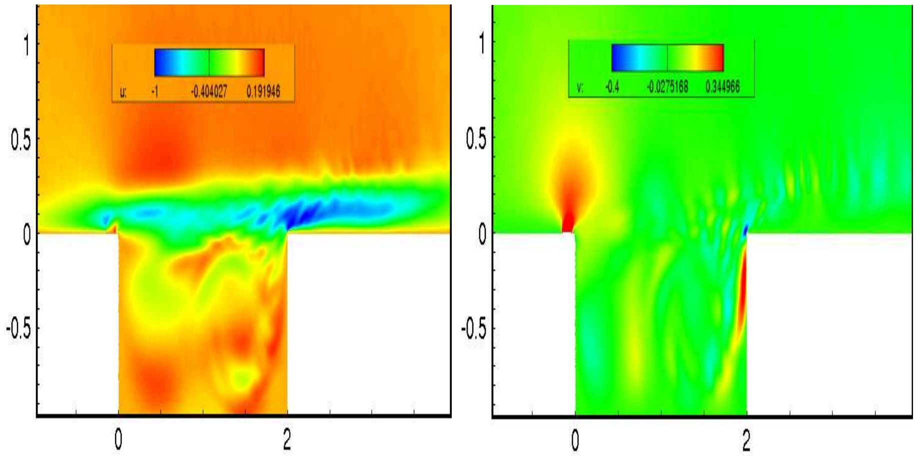

Figure 6: u and v velocity components of the actuation mode ψ corresponding to an actuation defined by $v_{wall} = |0.2\ \sin(0.4t)|$.

5 Conclusion

In this work we have presented an approach for the reduced order modelling of isentropic compressible flows. The various methods for the calibration of the ROM has also been discussed. Results for the cavity flow configuration has been presented. Also the extension to include actuation effect, and its calibration, which distinguishes two steps of calibration one for the un-actuated coefficients and the second for the actuated coefficients has been outlined. The current work lies in applying the method to perform optimal control of cavity flows.

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