Shock Waves in Porous Media : A Variational Approach

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Résumé :

Dans la communication intitulée "Boundary Conditions in Porous Media : A Variational Approach" nous avons présenté un cadre cinématique adapté à la description d'un milieu continu présentant une surface de singularité. Ce cadre était appliqué au cas particulier d'un milieu poreux presentant une discontinuité matérielle dans la configuration du solide. Nous traitons içi des problèmes plus généraux dans lesquels la surface de singularité n'est attachée ni au squelette solide ni au fluide saturant. Cest en particulier le cas lors de la propagation d'ondes de choc dans un milieu poreux saturé.

Abstract :

In the paper " Boundary Conditions in Porous Media : A Variational Approach" we present a kinematical framework suitable for describing the motion of a continuum with a moving surface discontinuity. We also apply the obtained results to the particular case of a porous medium with a solid-material surface discontinuity. Nevertheless, more general problems such as the propagation of shock waves in porous media have not been treated in the quoted paper, and are then approached in the present work.

Mots clefs : Ondes de Choc dans le Milieux Poreux, Principe Variationel de Hamilton dans l'espace-temps, Conditions de Saut.

1 Introduction

In a recent paper (see [2]) we presented a four dimensional kinematical framework which is able to describe the motion of a continuum with a moving surface discontinuity. A four dimensional kinematical approach for fluid mixtures is presented in [3] . We applied the obtained results to the particular case of a fluid-filled porous solid with a solid-material surface discontinuity. In other words, we focused our attention on that class of problems in which the solid porous matrix has a fixed surface discontinuity which represents, for instance, a jump in the mechanical properties of the solid itself (elasticity, porosity, etc.). This is indeed the proper framework when one wants to study the motion of physical systems like two porous media which are in contact and which are filled by a fluid or like a fluid-filled porous medium surrounded by a pure fluid.
In this paper we use this four dimensional framework to study propagation of shock waves in porous media neglecting all dissipation effects. We derive governing equations and natural jump conditions by means of a four dimensional extended Hamilton principle.
The problem of propagation of shocks in fluid-filled porous materials is an open challenge in Mechanics and Engineering. Many authors studied practical problems associated to shock wave propagation in porous media (see e.g. [5]) such as wave propagation due to detonation of explosives. The model to which many of these authors refer is the Baer-Nunziato model (see [1]). The jump conditions usually used for shock waves are obtained from the conservation of mass, momentum, and energy for any single phase of the medium and do not stem from a variational principle.

2 Lagrangian Functions

In this section we introduce the Lagrangian functions which are needed to study the motion of a p-constituent porous medium with a moving surface discontinuity.

Notation 1. *We note by $\mathfrak{M}$ the set of all 4×4 matrices on $\mathbb{R}$ (i.e. second order tensors defined on $\mathbb{R}^4$). Moreover, we label by p the number of constituents of the considered porous medium. Finally, we use three indexes a, b and c which runs in the following ranges :*

$$a = 0, 1, ..., p-1; \qquad b = 0, 1, ..., p; \qquad c = 1, 2, ..., p.$$

Let us introduce two scalar functions $\mathcal{L}_*$ and $\mathcal{L}_0$ defined as

$$\begin{aligned} &\mathcal{L}_* : \mathbb{R}^{4(p+1)} \times \mathfrak{M}^{p+1} \to \mathbb{R} && \mathcal{L}_0 : \mathbb{R}^{4(p+1)} \times \mathfrak{M}^{p} \to \mathbb{R} \\ &({}^{b}\mathbb{V},\ {}^{b}\mathbb{M}) \mapsto \mathcal{L}_*({}^{b}\mathbb{V},\ {}^{b}\mathbb{M}) && ({}^{b}\mathbb{V},\ {}^{c}\mathbb{P}) \mapsto \mathcal{L}_0({}^{b}\mathbb{V},\ {}^{c}\mathbb{P}) \end{aligned} \tag{1}$$

Clearly, the map $\mathcal{L}_*$ associates to any set of $p+1$ vectors (or first order tensors on $\mathbb{R}^4$) ${}^{b}\mathbb{V}$ and of $p+1$ matrices ${}^{b}\mathbb{M}$ the real number $\mathcal{L}_*({}^{b}\mathbb{V},\ {}^{b}\mathbb{M})$ and analogously, the map $\mathcal{L}_0$ associates to any set of $p+1$ vectors ${}^{b}\mathbb{V}$ and of p matrices ${}^{c}\mathbb{P}$ the real number $\mathcal{L}_0({}^{b}\mathbb{V},\ {}^{c}\mathbb{P})$.
We call these functions Lagrangian functions, and we assume in the sequel that these two Lagrangians are not independent. Indeed, we assume that the following relationship holds

$$\mathcal{L}_*({}^{b}\mathbb{V},\ {}^{b}\mathbb{M}) = \det({}^{0}\mathbb{M})\ \mathcal{L}_0({}^{b}\mathbb{V},\ {}^{c}\mathbb{M} \cdot {}^{0}\mathbb{M}^{-1}), \quad \forall\ ({}^{b}\mathbb{V},\ {}^{b}\mathbb{M}) \in \mathbb{R}^{4(p+1)} \times \mathfrak{M}^{p+1} \tag{2}$$

We also assume that the variables ${}^{b}\mathbb{V}$ and ${}^{b}\mathbb{M}$ are all independent, if not differently specified.

Proposition 2. *(i) Given a Lagrangian $\mathcal{L}_0$ as defined in Eq. (1), if we introduce a Lagrangian $\mathcal{L}_*$ by means of Eq. (2), then the following identity holds*

$$\mathcal{L}_* \mathbb{I} - {}^{b}\mathbb{M}^T \cdot \frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\mathbb{M}} = 0, \tag{3}$$

where $\mathbb{I}$ denotes the identity tensor on $\mathbb{R}^4$.
(ii) Conversely, if a function $\mathcal{L}_*$ verifies Eq. (3) for any set of $p+1$ second order tensors ${}^{b}\mathbb{M}$, then there exists a function $\mathcal{L}_0$ for which Eq. (2) is verified.

3 Kinematics

We introduce here a 4D kinematical framework to describe the motion of a p-constituents porous medium with a moving surface discontinuity. We underline that [2] is generalized both by considering a multi-constituents porous medium and by introducing a moving surface discontinuity (shock wave) in the porous medium itself.
Let us introduce a fictive 3D configuration B_* with a fixed surface discontinuity S_*; we denote by $\mathbb{B}_* := B_* \times [0,T]$ the corresponding 4D fictive configuration and by $\mathbb{S}_* := S_* \times [0,T]$ its 4D discontinuity surface.
Analogously, let us introduce the 3D reference (Lagrangian) configurations B_a, $a = 0, 1, ..., p-1$, of the p constituents and let us denote by $\mathbb{B}_a := B_a \times [0,T]$ the corresponding 4D Lagrangian configurations [1].
Finally, if for any time t the 3D domain $B_p(t)$ represents the current (Eulerian) configuration of the porous medium we introduce the corresponding 4D Eulerian domain as $\mathbb{B}_p := \bigcup_{t\in[0,T]} B_p(t) \times \{t\}$.
According to the definitions given in [2], we introduce $p+1$ piecewise C^1 diffeomorphisms ${}^{b}\boldsymbol{\mathcal{X}}_*$, $b = 0, 1, ..., p$, as [2],

$$\begin{aligned} {}^{b}\boldsymbol{\mathcal{X}}_* &: \mathbb{B}_* \to \mathbb{B}_b, \\ \mathbf{X}_* &\mapsto {}^{b}\boldsymbol{\mathcal{X}}_*(\mathbf{X}_*)\,; \end{aligned}$$

these diffeomorphisms clearly have $\mathbb{S}_* := S_* \times [0,T]$ as singularity surface.
Consequently, the image surfaces $\mathbb{S}_b := {}^{b}\boldsymbol{\mathcal{X}}_*(\mathbb{S}_*)$ are singularity surfaces for the inverse maps ${}^{b}\boldsymbol{\mathcal{X}}_*^{-1}$.
We notice that, by composition of the $p+1$ maps ${}^{b}\boldsymbol{\mathcal{X}}_*$, it is possible to recover $p(p+1)$ other maps the domain and range of which can be chosen arbitrarily among the sets $\mathbb{B}_b$. These maps are also piecewise C^1 diffeomorphisms and include, for instance, the usual 4D placement maps

$${}^{p}\boldsymbol{\mathcal{X}}_a := {}^{p}\boldsymbol{\mathcal{X}}_* \circ {}^{a}\boldsymbol{\mathcal{X}}_*^{-1}. \tag{4}$$

It is then clear that, since the introduced maps ${}^{b}\boldsymbol{\mathcal{X}}_*$ allow for deriving both the current placement of all the constituents of the porous medium and the current position of the singularity surface, they are suitable for the description of the motion of a p-constituents porous medium with a surface discontinuity.
From now on we assume that the maps ${}^{b}\boldsymbol{\mathcal{X}}_*$ are such that ${}^{b}\boldsymbol{\mathcal{X}}_*(\mathbb{B}_*) = \mathbb{B}_b$.
It is common practice in poromechanics to assume that one of the reference configurations $\mathbb{B}_a$ of the p constituents plays a predominant role with respect to the others (usually the reference configuration of a solid constituent).

1. From now on the index a runs from 0 to $p-1$ if not differently specified. Hence, a labels the Lagrangian configuration of each constituent.

2. From now on the index b runs from 0 to p if not differently specified. Thus, b includes the a reference configurations and also the current configuration of the porous medium.

Since we will also use this approach later on, it is worth to assume here that the configuration $\mathbb{B}_0$ represents the configuration of a solid constituent and we introduce the $p-1$ maps ${}^{c}\mathcal{X}_0 : \mathbb{B}_0 \to \mathbb{B}_c$, with $c = 1, 2, ..., p$, as [3]

$$ {}^{c}\mathcal{X}_0 := {}^{c}\mathcal{X}_* \circ {}^{0}\mathcal{X}_*^{-1}. \quad (5) $$

Notation 3. *Following notations introduced in [2], we denote*

$$ {}^{b}\mathbb{F}_* := \mathbb{V}\,{}^{b}\mathcal{X}_*, \quad {}^{c}\mathbb{F}_0 := \mathbb{V}\,{}^{c}\mathcal{X}_0; \qquad {}^{b}\mathbb{J}_* := \det\left({}^{b}\mathbb{F}_*\right), \qquad {}^{c}\mathbb{J}_0 := \det\left({}^{c}\mathbb{F}_0\right) \quad (6) $$

where $\mathbb{V}$ *is the 4D gradient operator.*
Moreover, if $\mathbb{t}_*$ *and* $\mathbb{t}_b$ *are tensor fields defined on* $\mathbb{B}_*$ *and* $\mathbb{B}_b$ *respectively, we denote*

$$ \mathbb{t}_*^{ⓑ} := \mathbb{t}_* \circ {}^{b}\mathcal{X}_*^{-1} \qquad \mathbb{t}_b^{⊛} := \mathbb{t}_b \circ {}^{b}\mathcal{X}_*. \quad (7) $$

Owing to Eq. (5) it is clear that [4]

$$ {}^{c}\mathbb{F}_0^{⊛} = {}^{c}\mathbb{F}_* \cdot {}^{0}\mathbb{F}_*^{-1}. \quad (8) $$

We also remark that, given any differentiable tensor fields $\mathbb{t}_*$ and $\mathbb{t}_b$ defined on $\mathbb{B}_*$ and $\mathbb{B}_b$ respectively, the following purely kinematical identities hold (see [2]) for any $b = 0, 1, ..., p$:

$$ \mathbb{DIV}\left({}^{b}\mathbb{J}_*\, \mathbb{t}_b^{⊛} \cdot {}^{b}\mathbb{F}_*^{-T}\right) = {}^{b}\mathbb{J}_*\left(\mathbb{DIV}\,\mathbb{t}_b\right)^{⊛} \quad \text{and} \quad \mathbb{DIV}\left({}^{b}\mathbb{J}_*^{-1}\, \mathbb{t}_* \cdot {}^{b}\mathbb{F}_*^{T}\right)^{ⓑ} = \left({}^{b}\mathbb{J}_*^{-1}\mathbb{DIV}\,\mathbb{t}_*\right)^{ⓑ}. \quad (9) $$

Let $\mathbb{n}_*$ and $\mathbb{n}_b$ be any normal vectors to $\mathbb{S}_*$ and $\mathbb{S}_b$ respectively and let $\mathbb{N}_*$ and $\mathbb{N}_b$ be the corresponding *unit* normal vectors, then (see [2])

$$ \left[\left|{}^{b}\mathbb{J}_*^{-1}\,{}^{b}\mathbb{F}_*^{T}\right|\right] \cdot \mathbb{n}_b^{⊛} = 0 \text{ on } \mathbb{S}_*, \qquad \left[\left|\left({}^{b}\mathbb{J}_*\,{}^{b}\mathbb{F}_*^{-T}\right)^{ⓑ}\right|\right] \cdot \mathbb{n}_*^{ⓑ} = 0 \text{ on } \mathbb{S}_b \quad (10) $$

and

$$ \mathbb{N}_* = \frac{{}^{b}\mathbb{J}_*^{-1}\,{}^{b}\mathbb{F}_*^{T} \cdot \mathbb{n}_b^{⊛}}{\left\|{}^{b}\mathbb{J}_*^{-1}\,{}^{b}\mathbb{F}_*^{T} \cdot \mathbb{n}_b^{⊛}\right\|}, \qquad \mathbb{N}_b = \frac{\left({}^{b}\mathbb{J}_*\,{}^{b}\mathbb{F}_*^{-T} \cdot \mathbb{n}_*\right)^{ⓑ}}{\left\|\left({}^{b}\mathbb{J}_*\,{}^{b}\mathbb{F}_*^{-T} \cdot \mathbb{n}_*\right)^{ⓑ}\right\|}. \quad (11) $$

Here and in the sequel he symbol $[|\cdot|]$ indicates the jump of a quantity through a discontinuity surface.
Finally we remark that, since all the maps introduced here are assumed to be piecewise C^1 diffeomorphisms, then the Hadamard property holds for all of them. In particular, let $\mathbb{T}_*$ be any vector tangent to $\mathbb{S}_*$, then the vectors $\mathbb{T}_b := \left({}^{b}\mathbb{F}_* \cdot \mathbb{T}_*\right)^{ⓑ}$ are tangent to $\mathbb{S}_b$ and the Hadamard properties read

$$ \left[\left|{}^{b}\mathbb{F}_* \cdot \mathbb{T}_*\right|\right] = 0 \text{ on } \mathbb{S}_* \implies \left[\left|{}^{c}\mathbb{F}_0 \cdot \mathbb{T}_0\right|\right] = 0 \text{ on } \mathbb{S}_0. \quad (12) $$

4 Action Functional

4.1 Hamilton Principle for p-constituents media

It is now possible to introduce a functional $\mathcal{A}_*$ as

$$ \begin{aligned} &\mathcal{A}_* : \left(C^1\left(\mathbb{R}^4, \mathbb{R}^4\right)\right)^{p+1} \to \mathbb{R}, \\ &\mathbf{f} \mapsto \mathcal{A}_*(\mathbf{f}); \end{aligned} $$

and such that

$$ \mathcal{A}_*\left({}^{b}\mathcal{X}_*(\cdot)\right) = \int_{\mathbb{B}_*} \mathcal{L}_*\left({}^{b}\mathcal{X}_*(\mathbf{X}_*), {}^{b}\mathbb{F}_*(\mathbf{X}_*)\right), \quad (13) $$

where $\mathcal{L}_*$ is the Lagrangian function introduced in Eq. (1).

3. From now on the index c runs from 1 to p if not differently specified. Consequently, the index c labels those maps which have $\mathbb{B}_0$ as domain of definition and which go indifferently onto the Lagrangian configurations or the Eulerian one.

4. Given two tensors T and S of order k and h the components of which are $T_{i_1...i_k}$ and $S_{j_1...j_h}$ the tensor $T \cdot S$ is the $(k+h)-2$ order tensor with components $(T \cdot S)_{i_1...i_{k-1}j_2...j_h} = \sum_m T_{i_1...i_{k-1}m} S_{m\,j_2...,j_h}$.

Henceforth, we can deduce the Hamilton Principle for this functional by noting that [5, 6]

$$\delta \mathcal{A}_* = \int_{\mathbb{B}_*} \left(\frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\!\mathcal{X}_*} \mid \delta\, {}^{b}\!\mathcal{X}_* + \frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\mathbb{F}_*} \mid \mathbb{\nabla}\left(\delta\, {}^{b}\!\mathcal{X}_*\right) \right) = 0 \tag{14}$$

where the symbol δ indicates the variation operator. Assuming that the maps $\delta\, {}^{b}\!\mathcal{X}_*$ have compact support $\mathbb{K}_*$ included in $\mathbb{B}_*$, integrating by parts and using Gauss Theorem, Eq. (14) gives

$$\int_{\mathbb{K}_*} \left(\frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\!\mathcal{X}_*} - \mathbb{DIV}\left(\frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\mathbb{F}_*} \right) \right) \cdot \delta\, {}^{b}\!\mathcal{X}_* + \int_{\mathbb{K}_* \cap \mathbb{S}_*} \left[\left| \frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\mathbb{F}_*} \cdot \mathbb{N}_* \right|\right] \cdot \delta\, {}^{b}\!\mathcal{X}_* = 0,$$

where $\mathbb{N}_*$ is the unit normal vector to the surface $\mathbb{S}_*$ and $\mathbb{DIV}$ is the usual divergence operator on $\mathbb{R}^4$.
Owing to the arbitrariness of the set $\mathbb{K}_*$ and of the test functions $\delta\, {}^{b}\!\mathcal{X}_*$ one finally gets for any $b = 0, 1, ..., p$

$$\begin{aligned} &\frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\!\mathcal{X}_*} - \mathbb{DIV}\left(\frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\mathbb{F}_*} \right) = 0 \quad \text{in } \mathbb{B}_*, \\ &\left[\left| \frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\mathbb{F}_*} \cdot \mathbb{N}_* \right|\right] = 0 \quad \text{on } \mathbb{S}_*. \end{aligned} \tag{15}$$

We note that the use of Hamilton Principle on the configuration $\mathbb{B}_*$ leads to a set of $p+1$ equations of motion and of $p+1$ Rankine-Hugoniot conditions ($4\,(p+1)$ scalar conditions). We will show in the next section that, for a given choice of the function $\mathcal{L}_*$, only p of these equations of motion are independent ; moreover it will be shown that the $4\,(p+1)$ Rankine-Hugoniot scalar conditions reduces to $4p+1$ independent scalar equations.

4.2 Restrictions on the Action Functional $\mathcal{A}_*$

In the previous subsection we have shown how to obtain equations of motions and Rankine-Hugoniot conditions for a p-constituent porous medium by means of Hamilton Priciple applied to the action functional $\mathcal{A}_*$ given in Eq. (13). Nevertheless, equations of motions and jump conditions written on the domain $\mathbb{B}_*$ should not be all independent. In fact, this domain is just a fictive configuration which has been introduced for reasons connected to the study of the motion of the shock wave.
It is clear that proper equations of motion and jump conditions must be written on a physical domain : one of the Lagrangian configurations $\mathbb{B}_a$ or the Eulerian configuration $\mathbb{B}_p$.
Henceforth, we assume that the domain $\mathbb{B}_0$ represents the reference configuration of a solid constituent and that it plays a special role : following classical poromechanics we want to write all the equations and jump conditions on this configuration.
In order to do so, we introduce a functional $\mathcal{A}_0$ as

$$\begin{aligned} &\mathcal{A}_0 : \left(C^1\left(\mathbb{R}^4, \mathbb{R}^4\right)\right)^{p+1} \to \mathbb{R}, \\ &\mathbf{f} \mapsto \mathcal{A}_0\left(\mathbf{f}\right); \end{aligned}$$

and such that

$$\mathcal{A}_0\left(I\left(\cdot\right), {}^{c}\!\mathcal{X}_0\left(\cdot\right)\right) = \int_{\mathbb{B}_0} \mathcal{L}_0\left(\mathbf{X}_0, {}^{c}\!\mathcal{X}_0\left(\mathbf{X}_0\right), {}^{c}\mathbb{F}_0\left(\mathbf{X}_0\right)\right),$$

where $\mathbf{X}_0 \in \mathbb{B}_0$, I is the identity function in $\mathbb{R}^4$ and $\mathcal{L}_0$ is the Lagrangian function introduced in Eq. (1).
The functional $\mathcal{A}_0$, when evaluated in the physical maps ${}^{c}\!\mathcal{X}_0$, is called the *action functional* of the p-constituents porous system.
Changing the variables and then recalling Eqs. (4) and (8), it is easy to get

$$\mathcal{A}_0\left({}^{b}\!\mathcal{X}_*\left(\cdot\right)\right) = \int_{\mathbb{B}_*} \mathcal{L}_0\left({}^{0}\!\mathcal{X}_*, {}^{c}\!\mathcal{X}_0 \circ {}^{0}\!\mathcal{X}_*, {}^{c}\mathbb{F}_0^{\circledast}\right) {}^{0}\mathbb{J}_* = \int_{\mathbb{B}_*} {}^{0}\mathbb{J}_* \mathcal{L}_0\left({}^{b}\!\mathcal{X}_*, {}^{c}\mathbb{F}_* \cdot {}^{0}\mathbb{F}_*^{-1}\right). \tag{16}$$

Comparing now expression (13) for $\mathcal{A}_*$ and expression (16) for $\mathcal{A}_0$, we can conclude that the restriction (2) assumed on the two functions $\mathcal{L}_*$ and $\mathcal{L}_0$ implies $\mathcal{A}_*({}^{b}\!\mathcal{X}_*\left(\cdot\right)) = \mathcal{A}_0({}^{b}\!\mathcal{X}_*\left(\cdot\right))$. This means that the assumption (2) which relates $\mathcal{L}_*$ and $\mathcal{L}_0$ implies the equivalence of the two functionals $\mathcal{A}_*$ and $\mathcal{A}_0$ when evaluated on the same argument.

5. We introduce here a slight abuse of notation. Let $\mathcal{L}\left(\mathbb{V}\right)$ be any function of the variable $\mathbb{V}$. If we evaluate the function $\mathcal{L}$ in $\mathbb{V} = \mathcal{X}\left(\mathbf{X}\right)$ it is well know that $\delta \mathcal{L} = \left(\partial \mathcal{L}/\partial \mathbb{V}\right)|_{\mathcal{X}\left(\mathbf{X}\right)} \cdot \delta \mathcal{X}\left(\mathbf{X}\right)$. In order to lighten notation we write $\delta \mathcal{L} = \partial \mathcal{L}/\partial \mathcal{X} \cdot \delta \mathcal{X}$.
6. Here, and from now on, the symbol $\mid$ indicates the scalar product between two tensors.

Remark 4. *Let, for any $b = 0, 1, ..., p$,*

$$\frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\boldsymbol{\mathcal{X}}_*} - \mathbb{DIV}\left(\frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\mathbb{F}_*}\right) = 0 \tag{17}$$

be the $p+1$ equations of motion, holding in $\mathbb{B}_*$, as obtained in Eq. (15). Assuming that the Lagrangians $\mathcal{L}_*$ and $\mathcal{L}_0$ are related by means of Eq. (2), then
i) Only p of the equations (17) are independent.
ii) The p independent equations

$$\frac{\partial \mathcal{L}_*}{\partial\, {}^{c}\boldsymbol{\mathcal{X}}_*} - \mathbb{DIV}\left(\frac{\partial \mathcal{L}_*}{\partial\, {}^{c}\mathbb{F}_*}\right) = 0, \tag{18}$$

imply the following system to hold in $\mathbb{B}_0$

$$\frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\boldsymbol{\mathcal{X}}_0} - \mathbb{DIV}\left(\frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\mathbb{F}_0}\right) = 0. \tag{19}$$

We remark that the system of p equations (19) is that one which would be obtained by applying the Hamilton principle to the Action density $\mathcal{L}_0\left(\mathbf{X}_0, {}^{c}\boldsymbol{\mathcal{X}}_0\left(\mathbf{X}_0\right), {}^{c}\mathbb{F}_0\left(\mathbf{X}_0\right)\right)$.

Remark 5. *Let, for any $b = 0, 1, ..., p$,*

$$\left[\left|\frac{\partial \mathcal{L}_*}{\partial\, {}^{b}\mathbb{F}_*} \cdot \mathbb{N}_*\right|\right] = 0$$

be the $p+1$ jump conditions, holding on $\mathbb{S}_*$, as obtained in Eq. (15). Assuming that the Lagrangians $\mathcal{L}_*$ and $\mathcal{L}_0$ are related by means of Eq. (2), then
(i) The p equations

$$\left[\left|\frac{\partial \mathcal{L}_*}{\partial\, {}^{c}\mathbb{F}_*} \cdot \mathbb{N}_*\right|\right] = 0 \tag{20}$$

are independent and imply the following system of equations holding on $\mathbb{S}_0$

$$\left[\left|\frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\mathbb{F}_0} \cdot \mathbb{N}_0\right|\right] = 0. \tag{21}$$

(ii) The remaining vectorial condition

$$\left[\left|\frac{\partial \mathcal{L}_*}{\partial\, {}^{0}\mathbb{F}_*} \cdot \mathbb{N}_*\right|\right] = 0 \tag{22}$$

when transported on $\mathbb{S}_0$ reduces to the simple scalar condition

$$\left[\left|\mathbb{N}_0 \cdot \left(\mathcal{L}_0 \mathbb{I} + {}^{c}\,\mathbb{F}_0^T \cdot \frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\mathbb{F}_0}\right) \cdot \mathbb{N}_0\right|\right] = 0. \tag{23}$$

As a consequence of Remark 4 we can conclude that the p vector equations of motion for a p constituents porous medium written on the Lagrangian configuration $\mathbb{B}_0$ of a solid constituents are given by (19).
Remark 5 states that there are p vector jump conditions valid on $\mathbb{S}_0$ given by (21) ; an additional scalar condition in the form of Eq. (23) also holds on $\mathbb{S}_0$.

5 Equations of Motion and Jump Conditions in the Space-Time

In this section, we separate the space and time components of the equations of motion (19) and of the jump conditions (21)-(23), by specifying the structure of the 4D placement maps ${}^{c}\boldsymbol{\mathcal{X}}_0$.
In order to do so we start by assuming that, for any instant t, there exist $p+1$ piecewise C^1 diffeomorphosms ${}^{b}\chi_*$ with singularity surface S_*

$$\begin{aligned} {}^{b}\chi_* &: \mathbb{B}_* \to B_b, \\ \mathbf{X}_* &:= \left(\mathbf{x}_*, t\right) \mapsto {}^{b}\chi_*\left(\mathbf{X}_*\right); \end{aligned}$$

such that ${}^{b}\boldsymbol{\mathcal{X}}_* = \left({}^{b}\chi_*, t\right)$
Let d be a new index running in $1, 2, ..., p-1$. Owing to Eqs. (4) and (5), we introduce the maps ${}^{c}\chi_0$ and ${}^{p}\chi_d$ as (recall that the indices 0 and p are fixed, while the remaining run in a given range)

$$ {}^{p}\chi_a = {}^{p}\chi_* \circ {}^{a}\chi_*^{-1} \qquad {}^{c}\chi_0 = {}^{c}\chi_* \circ {}^{0}\chi_*^{-1}.$$

We underline that the maps ${}^{p}\chi_{d}$ together with the map ${}^{p}\chi_{0}$ are the usual 3D placements maps wich determine, at any given instant t, the current placement of any constituent of the porous medium.
The maps ${}^{p}\chi_{d}$ and ${}^{c}\chi_{0}$ are, for any t, piecewise C^1 diffeomorphisms with singularities $S_a(t) := {}^{d}\chi_{*}(S_*, t)$ and $S_0(t) := {}^{0}\chi_{*}(S_*, t)$ respectively. It is easy to recover that ${}^{p}\mathcal{X}_d = ({}^{p}\chi_d, t)$ and ${}^{c}\mathcal{X}_0 = ({}^{c}\chi_0, t)$.
It is then possible to introduce the following second order tensors on $\mathbb{R}^3$

$$ {}^{b}\mathbf{F}_{*} := \nabla\, {}^{b}\chi_{*}, \qquad {}^{p}\mathbf{F}_{d} := \nabla\, {}^{p}\chi_{d}, \qquad {}^{c}\mathbf{F}_{0} := \nabla\, {}^{c}\chi_{0}, $$

where the symbol ∇ represents the 3D space gradient operator.
We notice that, once restricted to the surface S_*, the maps ${}^{b}\chi_{*}$ represent, for any instant t, a parametric representation of the moving surfaces $S_b(t)$. Henceforth, if we denote by $\bar{\mathbf{x}}_b$ the points of S_b, the velocities of the moving surfaces $S_b(t)$ can be then introduced as $\mathbf{w}_b(\bar{\mathbf{x}}_b, t) := (\partial\, {}^{b}\chi_{*}/\partial t)\big|_{{}^{b}\chi_{*}^{-1}(\bar{\mathbf{x}}_b, t)}$. Consequently, if $\mathbf{N}_b$ is the unit normal vector to S_b, the celerities of these moving surfaces can be introduced as $c_b := \mathbf{w}_b \cdot \mathbf{N}_b$. As it is well known (see e.g. [4]), these quantities does not depend on the choice of the parametrization.
As we showed in detail in [2], 4D normal vectors $\mathbb{M}_b$ to $\mathbb{S}_b$ take the form $\mathbb{M}_b = (\mathbf{N}_b, -c_b)$.
We also introduce the velocities

$$ \mathbf{v}_a(\mathbf{x}_a, t) := \frac{\partial\, {}^{p}\chi_{a}}{\partial t}, \qquad {}^{c}\mathbf{u}_0(\mathbf{x}_0, t) := \frac{\partial\, {}^{c}\chi_{0}}{\partial t} $$

Notice that since a runs in $0, 1, ..., p-1$ and c runs in $1, 2, ..., p$, we have $\mathbf{v}_0 = {}^{p}\mathbf{u}_0$.
According to the introduced definitions, it is straightforward to recognize that

$$ {}^{c}\mathbb{F}_0 = \begin{pmatrix} {}^{c}\mathbf{F}_0 & ({}^{c}\mathbf{u}_0)^T \\ \mathbf{0} & 1 \end{pmatrix} \implies {}^{c}\mathbb{F}_0^T = \begin{pmatrix} {}^{c}\mathbf{F}_0^T & \mathbf{0} \\ {}^{c}\mathbf{u}_0 & 1 \end{pmatrix}. \tag{24} $$

At this point we can notice that the equations of motion (19) have the following space-time components : for any $c = 1, 2, ..., p$

$$ \frac{\partial \mathcal{L}_0}{{}^{c}\chi_{0}} - \operatorname{div}\left(\frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\mathbf{F}_0}\right) - \frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\mathbf{u}_0} = 0, \qquad \frac{\partial \mathcal{L}_0}{\partial t} = 0. $$

The last equation states that the Lagrangian $\mathcal{L}_0$ does not explicitely depend on time which is in agreement with the fact that the system we consider is conservative. The first p equations give the evolution of the p-constituent porous medium.
As for the jump conditions (21)-(23), they particularize into

$$ \left[\left| \frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\mathbf{F}_0} \cdot \mathbf{N}_0 - c_0 \frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\mathbf{u}_0} \right|\right] = 0, \qquad \left[\left| \mathcal{L}_0 + \mathbf{N}_0 \cdot {}^{c}\mathbf{F}_0^T \cdot \frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\mathbf{F}_0} \cdot \mathbf{N}_0 - c_0 \mathbf{N}_0 \cdot {}^{c}\mathbf{F}_0^T \cdot \frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\mathbf{u}_0} \right|\right] = 0 \tag{25} $$

$$ \left[\left| {}^{c}\mathbf{u}_0 \cdot \frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\mathbf{F}_0} \cdot \mathbf{N}_0 - c_0 \mathcal{L}_0 - c_0\, {}^{c}\mathbf{u}_0 \cdot \frac{\partial \mathcal{L}_0}{\partial\, {}^{c}\mathbf{u}_0} \right|\right] = 0 \tag{26} $$

6 Conclusions

We deduced bulk equations and Rankine-Hugoniot conditions governing the motion of shock waves in multi-constituents, deformable, fluid-filled porous media by using a four-dimensional variational principle on a fictive configuration. We transport these equations on the reference configuration of one solid constituent and we show that they are not all independent. Finally, we particularize this set of independent equations to the space-time, thus recovering bulk Euler-Lagrange equations and Rankine-Hugoniot jump conditions.

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