Boundary Conditions in Porous Media : a Variational Approach

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Résumé :

Nous utilisons une extension du principe d'Hamilton-Rayleigh pour établir un système général de conditions d'interface entre deux différents milieux poreux saturés par un même fluide. Ces conditions prennent en compte les effets d'inertie et de frottement. Leur version linéarisée permet de retrouver les conditions aux limites pour les équations de Darcy-Brinkman ainsi que de traiter les effets de dissipation de surface de Saffman-Beavers-Joseph.

Abstract :

A general set of boundary conditions at fluid-permeable interfaces between dissimilar fluid-filled porous matrices is established starting from an extended Hamilton-Rayleigh principle. These conditions do include friction and inertial effects. Once linearized, they encompass boundary conditions relative to volume Darcy-Brinkman and to surface Saffman-Beavers-Joseph dissipation effects.

Mots clefs : Milieux Poreux, Surface de Discontinuité, Principe Variationel de Hamilton-Rayleigh, Conditions au Limite pour une Surface Permeable

1 Introduction

In this work we study the mechanics of fluid-filled porous media by using a variational approach. We first introduce a kinematical scheme suitable for describing the motion of a fluid-filled porous medium with a solid-material discontinuity (see also [11], [22]). As a second step, in order to take into account some dissipative phenomena, we postulate a Hamilton-Rayleigh variational principle, extended to continuous systems. In particular, we model volume dissipation due to Darcy-Brinkman viscous fluid flow inside the matrix pores and due to Stokes-Navier viscosity of the fluid together with surface Saffman-Beavers-Joseph dissipation effects. We obtain governing bulk equations for the porous medium and jump conditions on the considered discontinuity. We particularize these equations and jump conditions to the limit case of a porous medium surrounded by a pure fluid.
The proposed theoretical scheme may be useful to model all those physical systems composed by two dissimilar porous media in contact, or by a porous medium surrounded by a pure fluid. Many authors proposed different jump conditions to be imposed at the interface between a porous medium and a pure fluid, mainly basing themselves on the empirical jump conditions originally considered by Beavers, Joseph and Saffman ([2], [23]). Deresiewicz ([12]) also proposed some jump conditions holding at the interface between two dissimilar porous systems. Nevertheless, they only describe phenomena related to the viscosity of the outflowing fluid with no consideration of inertial effects and Darcy-Brinkman dissipation.
The spirit of the approach adopted here for modeling porous systems is very similar to the one used to develop models for two fluid mixtures by [15], [16], [17], [20], [18], [19]. It also has some similarities with the treatment used to describe fluid saturated porous media by [14], [13] and [9].

2 Kinematics and Balances of Masses

In this section we set up a kinematical framework suitable for describing the motion of a porous medium with a solid-material surface discontinuity. Let B_s and B_f be two open subsets of $\mathbb{R}^3$ (usually referred to as the Lagrangian configurations of the two constituents), $(0,T)$ be a time interval and let

$$\chi_s : B_s \times (0,T) \to \mathbb{R}^3, \qquad \chi_f : B_f \times (0,T) \to \mathbb{R}^3 \tag{1}$$

be the maps which represent the placement of the solid and fluid constituents respectively. Moreover, we introduce the map $\phi_s : B_s \times (0,T) \rightarrow B_f$ which locates, for any instant $t \in (0,T)$, the fluid material particle $\mathbf{X}_f$ which is in contact with the solid material particle $\mathbf{X}_s$. The three introduced maps are related by $\phi_s = \chi_f^{-1} \circ \chi_s$. We also assume that χ_s and ϕ_s are continuous in B_s, while their space gradients $\nabla\chi_s$ and $\nabla\phi_s$ are continuous a.e. in B_s except on a surface S_s which is assumed to be fixed in the solid-Lagrangian configuration. This means that the fields $\nabla\chi_s$ and $\nabla\phi_s$ suffer a jump at the surface S_s. From now on we also assume that $\chi_s(B_s,t) = \chi_f(B_f,t)$ and we denote $B_e(t)$ this time-varying sub-domain of $\mathbb{R}^3$ usually referred to as Eulerian configuration of the porous system. This means that we are adopting a macroscopic kinematical model since each point $\mathbf{x}$ of the physical space is assumed to be simultaneously occupied both by a solid and a fluid material particle.

Notation 1 *Given two fields $a(\mathbf{x},t)$ and $b(\mathbf{X}_f,t)$ defined on $B_e(t)$ and B_f respectively, we denote by $a^{\textcircled{s}}(\mathbf{X}_s,t)$ and $b^{\textcircled{s}}(\mathbf{X}_s,t)$ the corresponding fields transported on B_s as : $a^{\textcircled{s}} = a \circ \chi_s$ and $b^{\textcircled{s}} := b \circ \phi_s$.*

Let $\mathbf{v}_s := \partial\chi_s/\partial t$ and $\mathbf{v}_f := \partial\chi_f/\partial t$ be the velocity fields of the solid and fluid constituent respectively and let $\mathbf{u}_s := \dot{\phi}_s$. These three fields are related by

$$\mathbf{v}_f^{\textcircled{s}} = \mathbf{v}_s - \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s \tag{2}$$

where $\mathbf{F}_s := \nabla\chi_s$, $\mathbf{G}_s := \nabla\phi_s$ and the central dot indicates the single contraction between tensors.
Let now η_s and η_f be the solid and fluid Lagrangian apparent mass densities, defined on B_s and B_f respectively. We introduce the solid-Lagrangian fluid density as $m_f := \det(\nabla\phi_s)\ \eta_f^{\textcircled{s}}$ and, without loss of generality, we can assume that η_f is constant in space. We also assume that no creation (or dissolution) of mass occurs during the motion of the porous system so that $\partial\eta_s/\partial t = 0$ and $\partial\eta_f/\partial t = 0$.
As done in classical poromechanics (see e.g. [10], [11], [22]) the fluid conservation equation is pulled-back on the solid-Lagrangian configuration B_s and reads

$$\dot{m}_f + div\mathbf{D} = 0 \quad on \quad B_s, \tag{3}$$

$$[|\mathbf{D}|] \cdot \mathbf{N}_s = 0 \quad on \quad S_s, \tag{4}$$

where the superposed dot indicates partial time derivative, $\mathbf{D} := -m_f \mathbf{G}_s^{-1} \cdot \mathbf{u}_s$ and the symbol $[|\ |]$ indicates the jump of the given quantity through the surface S_s. The vector $\mathbf{D}$ is interpreted as the mass fluid flux through the porous medium in the Lagrangian configuration of the skeleton. If $\mathbf{N}_s$ is the unit normal vector to S_s, the quantity $\mathbf{D} \cdot \mathbf{N}_s$ is the flux (per unit area of S_s) of fluid flowing through the interface. Considering the Eulerian moving surface $S_e(t) := \chi_s(S_s,t)$, we can introduce the flux $d(\mathbf{x},t)$(per unit area of S_e) of fluid flowing through S_e, which is defined in terms of $\mathbf{D}$ as

$$d^{\textcircled{s}} := \frac{\mathbf{D} \cdot \mathbf{N}_s}{A_s}, \tag{5}$$

$A_s := \|J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s\|$ ($J_s := \det \mathbf{F}_s$) being the Eulerian-Solid Lagrangian changement of area.

3 Evolution Equations and Jump Conditions in Presence of Dissipation

3.1 Action and Rayleigh Functionals

We now introduce the solid-Lagrangian kinetic energy density

$$\Lambda\left(\eta_s, m_f, \mathbf{v}_s, \mathbf{v}_f^{\textcircled{s}}\right) = \frac{1}{2}\left(\eta_s \left(\mathbf{v}_s\right)^2 + m_f (\mathbf{v}_f^{\textcircled{s}})^2\right) \tag{6}$$

We also assume that the potential energy of the porous medium is characterized by a local density Ψ on B_s which depends on the kinematic descriptors χ_s and ϕ_s through the placement χ_s, the deformation tensor $\varepsilon := 1/2(\mathbf{F}_s^T \cdot \mathbf{F}_s - \mathbf{I})$ and the quantity of fluid contained in the porous medium $m_f = \det\nabla\phi_s\, \eta_f^{\textcircled{s}}$. As we do not intend to model surface tension phenomena, we do not consider any concentration of energy on the singularity surface S_s. Neither do we consider any dependence of Ψ on higher gradients of the kinematical fields as done for instance in [22].
Setting $\mathbb{B}_s := B_s \times (0,T)$, we define the action functional $\mathcal{A}$ for the porous system as

$$\mathcal{A} := \int_{\mathbb{B}_s} (\Lambda - \Psi)\,. \tag{7}$$

It is well known that, in absence of dissipation, imposing the stationarity of the action implies that the kinematic descriptors satisfy the virtual power principle i.e. a weak form of the balance of momentum. As we want to account for dissipation phenomena, we introduce a generalized Rayleigh dissipation pseudo-potential $\mathcal{R}$ in the Solid-Lagrangian configuration as

$$2\mathcal{R} = \int_{B_s} J_s \left[(\mathbf{v}_s - \mathbf{v}_f^{\textcircled{s}}) \cdot \kappa + \nabla \left((\mathbf{v}_s - \mathbf{v}_f^{\textcircled{s}}) \cdot \mathbf{F}_s^{-1} \right) : \Pi + \left(\nabla \mathbf{v}_f^{\textcircled{s}} \cdot \mathbf{F}_s^{-1} \right) : \Pi_f \right] + \int_{S_s} A_s \left[\left| \mathbf{v}_f^{\textcircled{s}} \right| \right] \cdot \sigma. \quad (8)$$

In this formula $\kappa := \mathbf{K} \cdot (\mathbf{v}_s - \mathbf{v}_f^{\textcircled{s}})$ is the Darcy friction force, $\Pi := \mathbf{B} : \nabla((\mathbf{v}_s - \mathbf{v}_f^{\textcircled{s}}) \cdot \mathbf{F}_s^{-1})$ is the Brinkman stress tensor, $\Pi_f := \mathbf{M} : (\nabla \mathbf{v}_f^{\textcircled{s}} \cdot \mathbf{F}_s^{-1})$ is the Stokes-Navier stress tensors and $\sigma := \mathbf{S} \cdot [\mid \mathbf{v}_f^{\textcircled{s}} \mid]$ is a friction surface force due to the viscosity of the fluid. Moreover, $\mathbf{K}$ and $\mathbf{S}$ are second order symmetric positive tensors, $\mathbf{M}$ and $\mathbf{B}$ are symmetric positive fourth order tensors and the symbol " : " stands for the double contraction product.

3.2 Equations of Motion

Let us denote by $\mathbf{q} := (\chi_s, \phi_s)$ the kinematic descriptor of the medium (a field defined on $\mathbb{B}_s$). Hence the action $\mathcal{A}$ is a functional of $\mathbf{q}$. Moreover, let us denote by $\mathbf{q}_t$, $\dot{\mathbf{q}}_t$ the fields defined at any instant t on B_s by $\mathbf{q}_t(\mathbf{X}_s) := \mathbf{q}(\mathbf{X}_s, t)$ and $\dot{\mathbf{q}}_t(\mathbf{X}_s) := \partial \mathbf{q}/\partial t(\mathbf{X}_s, t)$. The Rayleigh potential $\mathcal{R}$ is, at each instant t, a functional of $(\mathbf{q}_t, \dot{\mathbf{q}}_t)$.
The physical principle which determines the motion of a system can be alternatively stated in the framework of second Newton's law (balance of momentum), of D'Alembert principle (weak formulation of momentum balance) or of Rayleigh-Hamilton principle. We adopt this last approach which reads

$$\frac{\partial \mathcal{A}}{\partial \mathbf{q}} \mid \delta \mathbf{q} = \int_0^T \left(\frac{\partial \mathcal{R}}{\partial \dot{\mathbf{q}}_t} \mid \delta \mathbf{q}_t \right) dt. \quad (9)$$

Here, the symbol " | " is the scalar product between tensors, $\partial \mathcal{A}/\partial \mathbf{q}$ and $\partial \mathcal{R}/\partial \dot{\mathbf{q}}_t$ must be understood in the sense of functional differentiation.
Performing various integration by parts in space and time and assuming that the test functions $\delta\chi_s$ and $\delta\phi_s$ are suitably smooth (cfr [11] for extended calculations), we get the following system of equations valid in the regular points of $\mathbb{B}_s$

$$-\left(\eta_s \gamma_s + m_f \gamma_f^{\textcircled{s}} \right) + div \left(\mathbf{F}_s \cdot \frac{\partial \Psi}{\partial \varepsilon} \right) - \frac{\partial \Psi}{\partial \chi_s} = -div \left(J_s (\Pi_f^{\textcircled{s}})^T \cdot \mathbf{F}_s^{-T} \right), \quad (10)$$

$$m_f \left(\mathbf{F}_s^T \cdot \gamma_f^{\textcircled{s}} + \nabla \left(\frac{\partial \Psi}{\partial m_f} \right) \right) = \mathbf{F}_s^T \cdot \left(J_s \kappa^{\textcircled{s}} - div \left(J_s (\Pi^{\textcircled{s}} - \Pi_f^{\textcircled{s}})^T \cdot \mathbf{F}_s^{-T} \right) \right), \quad (11)$$

where $\gamma_s := \dot{\mathbf{v}}_s$ and $\gamma_f := \dot{\mathbf{v}}_f$ are the acceleration fields of the solid and fluid contituent respectively.
Moreover, assuming the arbitrariness of the test function $\delta\chi_s$ and the fact that $\mathbf{v}_s$ is continuous through the discontinuity surface S_s we get the following jump condition valid on $\mathbb{S}_s := S_s \times (0, T)$:

$$\left[\left| \mathbf{F}_s \cdot \frac{\partial \Psi}{\partial \varepsilon} - (\mathbf{v}_f^{\textcircled{s}} - \mathbf{v}_s) \otimes \mathbf{D} + J_s (\Pi_f^{\textcircled{s}})^T \cdot \mathbf{F}_s^{-T} \right| \right] \cdot \mathbf{N}_s = 0. \quad (12)$$

Finally, other jump conditions valid on $\mathbb{S}_s$ can be established using suitable regularity assumptions on the test function $\delta\phi_s$ (see [11] for details). In particular, we distinguish the case when S_s is permeable to fluid flow ($d \neq 0$) and when S_s is impermeable ($d = 0$).
If $d \neq 0$ the following scalar jump conditions hold on $\mathbb{S}_s$:

$$\tau_e^{\textcircled{s}} \cdot \left[\left| \mathbf{v}_f^{\textcircled{s}} - \mathbf{v}_s \right| \right] d^{\textcircled{s}} + \tau_e^{\textcircled{s}} \cdot \left[\left| (\Pi^{\textcircled{s}} - \Pi_f^{\textcircled{s}})^T \right| \right] \cdot \mathbf{N}_e^{\textcircled{s}} - A_s \, \tau_e^{\textcircled{s}} \cdot \mathbf{S} \cdot \left[\left| \mathbf{v}_f^{\textcircled{s}} - \mathbf{v}_s \right| \right] = 0, \quad (13)$$

$$\left[\left| \left(\frac{1}{2} \left(\mathbf{v}_f^{\textcircled{s}} - \mathbf{v}_s \right)^2 + \frac{\partial \Psi}{\partial m_f} \right) \right| \right] d^{\textcircled{s}} + \left[\left| (\mathbf{v}_f^{\textcircled{s}} - \mathbf{v}_s) \cdot \left(\left((\Pi^{\textcircled{s}} - \Pi_f^{\textcircled{s}})^T \right) \cdot \mathbf{N}_e^{\textcircled{s}} - \sigma^{\textcircled{s}} \right) \right| \right] = 0, \quad (14)$$

where τ_e is any vector tangent to the Eulerian surface S_e and $\mathbf{N}_e$ is the unit normal vector to S_e. $\mathbf{N}_e$ is related to $\mathbf{N}_s$ by $\mathbf{N}_e^{\textcircled{s}} = (J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s)/A_s$.
The situation is completely different when $d = 0$. In this case the jump conditions (13) and (14) valid on $\mathbb{S}_s$ have to be replaced by the three scalar conditions

$$\left(\tau_e^{\textcircled{s}} \cdot (\Pi^{\textcircled{s}} - \Pi_f^{\textcircled{s}})^T \cdot \mathbf{N}_e^{\textcircled{s}} \right)^- = 0, \qquad \left(\tau_e^{\textcircled{s}} \cdot (\Pi^{\textcircled{s}} - \Pi_f^{\textcircled{s}})^T \cdot \mathbf{N}_e^{\textcircled{s}} \right)^+ = 0, \quad (15)$$

and

$$\left[\left|\left(\frac{\partial\Psi}{\partial m_f}+\frac{J_s}{m_f}\mathbf{N}_s\cdot\mathbf{F}_s^T\cdot(\Pi^{\circledS}-\Pi_f^{\circledS})^T\cdot\mathbf{F}_s^{-T}\cdot\mathbf{N}_s\right)\right|\right]=0. \tag{16}$$

In equation (15) the " $+$" and " $-$" superscripts label the traces of the given quantities on each side of S_s.
In conclusion, the motion of the porous medium is ruled by the equations (10) (11) valid far from S_s and, if $d\neq 0$, by the jump conditions (12), (13) and (14) valid on S_s. If $d=0$, the last two conditions have to be replaced by the three conditions stated in (15) and (16).

We remark that this system of equations respects Galilean invariance. Indeed, all equations (as well as the criterion $d=0$ or $d\neq 0$) involve only Galilean invariant physical quantities.
Equations (10) and (12) encompass the well known Lagrangian balance equation for the total stress and the corresponding jump condition : these equations only involve physical quantities. Equations (11), (13), (14), (15) and (16) are not available in the literature.

4 The Case of a Deformable Porous Medium Surrounded by a Pure Fluid

We now consider the case of a surface discontinuity S_s separating a porous medium (which occupies the volume B_s^+) from a pure fluid (which occupies the volume B_s^-). When the fluid is pure (and when external body forces are neglected), its Eulerian energy density, its chemical potential and its pressure are functions of its mass density only. These three real functions are denoted respectively by Ψ_f, μ_f and p_f and are related by

$$\mu_f(y)=\Psi_f'(y)\qquad and\qquad p_f(y)=-\Psi_f(y)+y\,\Psi_f'(y), \tag{17}$$

Let ρ_f be the Eulerian fluid mass density related to η_f by $\rho_f^{\circledS}=1/(\det\nabla\chi_f)^{\circledS}\,\eta_f^{\circledS}$. In the sequel, as no ambiguity can arise, we simply denote μ_f, p_f and Ψ_f the fields $\mu_f(\rho_f)$, $p_f(\rho_f)$ and $\Psi_f(\rho_f)$ defined on B_e.
The restriction Ψ^- of the potential Ψ in B_s^- is that of a pure fluid : transporting the Eulerian density $\Psi_f(\rho_f)$ on B_s, we get

$$\Psi^-(\varepsilon,m_f)=J_s\Psi_f\left(\rho_f^{\circledS}\right)=J_s\Psi_f\left(J_s^{-1}m_f\right), \tag{18}$$

Note that J_s is a function of ε only, as we have $J_s=\det\mathbf{F}_s=\sqrt{\det(2\varepsilon+\mathbf{I})}$ and that $\partial J_s/\partial\varepsilon=J_s\mathbf{F}_s^{-1}\cdot\mathbf{F}_s^{-T}$. Thus we have

$$\frac{\partial\Psi^-}{\partial\varepsilon}=(\Psi_f^{\circledS}-\mu_f^{\circledS}\rho_f^{\circledS})\frac{\partial J_s}{\partial\varepsilon}=-p_f^{\circledS}J_s\mathbf{F}_s^{-1}\cdot\mathbf{F}_s^{-T},\quad and\quad \frac{\partial\Psi^-}{\partial m_f}=\mu_f^{\circledS}.$$

As for the porous region, we simply denote Ψ^+ the potential density function in B_s^+.
In the pure fluid region B_s^- we clearly have $\eta_s=0$ and $\Pi=0$. Moreover, $\mathbf{K}=0$, $\mathbf{M}=0$.
Under these assumptions, using equations (17), we obtain the following expression for the equation of motion (10) in B_s^- :

$$m_f\gamma_f^{\circledS}=div\left(\mathbf{F}_s\cdot\frac{\partial\Psi^-}{\partial\varepsilon}\right)+div\left(J_s(\Pi_f^{\circledS})^T\cdot\mathbf{F}_s^{-T}\right)=div\left(J_s\left(-p_f^{\circledS}\,\mathbf{I}+(\Pi_f^{\circledS})^T\right)\cdot\,\mathbf{F}_s^{-T}\right). \tag{19}$$

The existence of a supplementary equation (11) may seem astonishing. Under the hypotheses we formulated, recalling (18) and (17), Eq. (11) can be rewritten

$$m_f\mathbf{F}_s^T\cdot\gamma_f^{\circledS}+m_f\nabla\mu_f^{\circledS}=\mathbf{F}_s^T\cdot div\left(J_s(\Pi_f^{\circledS})^T\cdot\,\mathbf{F}_s^{-T}\right),$$

which, multiplied on the left by $\mathbf{F}_s^{-T}$, gives

$$m_f\gamma_f^{\circledS}+m_f\mathbf{F}_s^{-T}\cdot\nabla\mu_f^{\circledS}=div\left(J_s(\Pi_f^{\circledS})^T\cdot\,\mathbf{F}_s^{-T}\right) \tag{20}$$

This equation is clearly equivalent to Eq. (19) owing to the identities $\nabla p_f^{\circledS}=\rho_f^{\circledS}\nabla\mu_f^{\circledS}$ and $div(J_s\mathbf{F}_s^{-T})=0$. Hence, as expected, the fluid is governed *only* by the usual Navier-Stokes equation.

Let us consider the Eulerian form of jump condition (12) on the surface S_s which divides the porous medium region B_s^+ from the pure fluid region B_s^- :

$$\left(J_s^{-1}\mathbf{F}_s\cdot\frac{\partial\Psi}{\partial\varepsilon}\cdot\mathbf{F}_s^T+(\Pi_f^{\circledS})^T\right)^+\cdot\mathbf{N}_e^{\circledS}=(p_f^{\circledS})^-\,\mathbf{N}_e^{\circledS}+\left[\left|\mathbf{v}_f^{\circledS}-\mathbf{v}_s\right|\right]d+((\Pi_f^{\circledS})^T)^-\cdot\mathbf{N}_e^{\circledS}. \tag{21}$$

The reader can recognize in the left hand side of this equation the total Cauchy stress tensor of the porous medium. It appears clearly that, as predictable, the action of the fluid on the surface do not depend on the extension of χ_s.
As for the jump conditions (13) or (15), they remain unchanged, as they do not involve Ψ^-. Since they are written in Eulerian form, they also do not depend on the extension of χ_s.
Considering the jump conditions (14) or (16), we remark that in this special case one just has to replace $\partial\Psi^-/\partial m_f$ by the chemical potential of the fluid $\mu_f^{\textcircled{s}}$. They are both written in Eulerian form if one recognizes that the term $[|(1/m_f\,\mathbf{N}_s\cdot\mathbf{F}_s^T\cdot J_s(\Pi^{\textcircled{s}}-\Pi_f^{\textcircled{s}})^T\cdot\mathbf{F}_s^{-T}\cdot\mathbf{N}_s)\,|]$ can be rewritten as $[|(1/\rho_f^{\textcircled{s}}\mathbf{N}_e^{\textcircled{s}}\cdot(\Pi^{\textcircled{s}}-\Pi_f^{\textcircled{s}})^T\cdot\mathbf{N}_e^{\textcircled{s}})\,|]$.
In conclusion, the motion of the porous medium is driven by two independent equations of the type (10)-(11), while the motion of the pure fluid is driven by a simple equation in the form (19). The differential system is completed by the jump conditions (13) and (14) in the case $d \neq 0$ and by the jump conditions (15) and (16) if $d = 0$.

5 Conclusions

All the jump conditions available in the literature can be deduced from our jump condition (21), once assuming that inertial effects are negligible, that the solid matrix is suffering small deformation and that Stokes fluid-flow condition is verified.
In this paper the configuration of a fluid-filled porous medium is characterized by means of the placement fields χ_s and ϕ_s. Moreover, suitable action and dissipation functionals are postulated in order to account for the mechanical properties of a deformable *solid* matrix permeable to *fluid* flow. Therefore, the obtained Euler-Lagrange-Rayleigh equations naturally determine the time evolution of the fields χ_s, m_f and $\mathbf{u}_s$ which represent the solid placement, the solid-Lagrangian density of the fluid and the solid-fluid relative velocity, respectively.
In the presented model the "solid volume fraction" ν does not appear explicitly as a kinematical field : nevertheless, solid volume fraction plays a crucial role, for instance, in the determination of the macroscopic deformation energy of the fluid-filled solid matrix. Indeed, when the macroscopic constitutive equations for such a system need to be postulated, the most natural choice actually is to deduce them starting from the microscopic constitutive equations of the *pure* fluid and solid constituents. This is exactly what is done in the literature stemming from the papers of Fillunger, Terzaghi and Biot (see e.g. [3], [4], [5], [6], [7], [8], [2], [9], [23]). One suggestive reasoning to explain such an approach and its logical limits can be the following. Let us assume that a macroscopic solid material particle is in a state described by the macroscopic deformation gradient $\mathbf{F}_s$ and that the macroscopic fluid saturating particle has a (solid referential) density m_f. Then, the microscopic deformation energy ψ_{tot} of the fluid-filled porous matrix can be expressed, in terms of the fluid and solid volume fractions ν_f and ν, as follows :

$$\psi_{tot}\left(\mathbf{F}_s, m_f, \nu_f, \nu\right) = \psi_s\left(\mathbf{F}_s, \nu\right) + \psi_f\left(\frac{J_s^{-1} m_f}{\nu_f}\right) \tag{22}$$

where ψ_s is the microscopic deformation energy of a solid matrix when it experiences the macroscopic deformation $\mathbf{F}_s$ and its solid volume fraction is given by ν, while ψ_f is the microscopic deformation energy of the permeating fluid. In order to obtain from (22) a macroscopic energy density, one needs to assume a kind of "instantaneous local equilibrium hypothesis". In other words, it must be assumed that, for fixed $\mathbf{F}_s$ and m_f, with characteristic times much shorter than those characterizing macro-phenomena, the fluid and solid volume fractions adjust to a local equilibrium value. These equilibrium values are obtained by solving the following local minimization problem : find the functions $\nu_f(\mathbf{F}_s, m_f)$ and $\nu(\mathbf{F}_s, m_f)$ such that

$$\psi_{tot}\left(\mathbf{F}_s, m_f, \nu_f(\mathbf{F}_s, m_f), \nu(\mathbf{F}_s, m_f)\right) = \min_{\nu_f, \nu} \psi_{tot}\left(\mathbf{F}_s, m_f, \nu_f, \nu\right). \tag{23}$$

The minimization problem (23) is crucial and has been solved in very clever ways under physically acceptable assumptions. The resulting macro deformation energy density is thus obtained as follows

$$\Psi(\mathbf{F}_s, m_f) = \psi_{tot}\left(\mathbf{F}_s, m_f, \nu_f(\mathbf{F}_s, m_f), \nu(\mathbf{F}_s, m_f)\right). \tag{24}$$

In the present paper we refrain from any attempt of deducing any particular form for Ψ. Our aim is to find a logically consistent set of evolution equations and boundary conditions for models in which the independent kinematical descriptors are χ_s and ϕ_s.
To our knowledge, the inertia terms appearing in the jump conditions (13) and (14) are not found in the literature. Moreover, all presented boundary conditions are valid also when the solid matrix is suffering large deformations and when the Stokes fluid-flow condition is not applicable.
A deduction of a jump condition similar to (14) and valid in the particular case of absence of inertia, of Darcy-Brinkman and Beavers-Joseph dissipation is presented in ([1]).
As for our jump conditions (12), (13) and (14), they also allow for describing phenomena in which inertial effects are relevant. The inertia terms, which are here newly introduced, are at least quadratic in the relative

velocity fields at the interface : when Stokes fluid-flow conditions hold (and when the solid matrix is subjected to "small deformations") they may be negligible. Indeed, when the equations are linearized in the neighborhood of a state of rest (i.e. when all velocity fields and their gradients are vanishing) the aforementioned inertia terms do not produce, in the resulting boundary conditions, any term additional to those appearing in Beavers-Joseph-Saffman conditions. However, when the linearization procedure is performed in the neighborhood of a state in which some velocity fields are not vanishing, then inertia terms cannot be neglected.

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