Internal gravity waves feedback on a parallel mean flow: modelling of a boundary layer above a sinusoidal topography.

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Abstract :

We consider a boundary layer flow horizontally homogeneous in the presence of a vertical stratification and of a sinusoidal topography. We present a simple model describing the interaction between the mean flow and the packet of internal waves emitted at the bottom, assuming that it obeys to the laws of the refraction. We focus on the configurations where no critical layers develop and where the waves propagate upward. We show that an equilibrium state exists when the bottom boundary conditions are stationary. With numerical simulations and considering the analytical expression of equilibra, we show that the presence of the waves amplifies the mean flow evolutions.

Résumé :

Nous considérons un écoulement de couche limite horizontallement homogène en présence d'une stratification verticale et d'une topographie sinusoïdale. Nous présentons un modèle simple de l'interaction entre l'écoulement moyen et le paquet d'ondes émis au sol, en supposant qu'il obéit aux lois de la réfraction. Nous considérons des configurations telles qu'aucune couche critique ne se forme et telles que les ondes se propagent toujours vers le haut. Nous montrons l'existence d'états d'equilibre en présence des conditions aux limites stationnaires. A l'aide de simulations numériques et en considérant l'expression analytique des équilibres, nous montrons que la présence des ondes amplifie les évolutions du champ moyen.

Key-words :

stratified flow; internal gravity waves ; wave-mean flow interaction

1 Introduction

Gravity waves can be emitted by a wind flowing on a topography in a stratified media. They are often described by the Euler equations under the Boussinesq approximation (see for instance Lighthill (1978)). The effect of the mean flow on the wave propagation has been taken into account in the study of Bretherton (1966) through a ray theory describing the propagation of internal gravity wave packets. Propagation of waves in the mean flow has been successfully described by Bretherton & Garrett (1969) who introduced the action density variable and its conservation. The feedback of the waves on the mean flow due to energy transfer has been considered formerly by Grimshaw (1975). For internal waves generated by a topography, some recent results are found in Lott & Teitelbaum (1993) who provide an atmospheric non periodic picture of the phenomena discussed in this work. In the present paper, we focus on the feedback of the waves on the mean flow under the horizontal homogeneity assumption, e.g., a boundary layer on a sinusoidal topography.

If we denote $\overline{u}(z,t)$ and $\overline{\rho}(z,t)$ the components of, respectively, velocity and density mean fields, the internal waves propagating along the flow can be described by the real part of $(w_0, \rho_0)\exp\{i[k_1 x + \phi(z,t)]\}$, where $w_0(z,t)$ and $\rho_0(z,t)$ are the slowly varying complex amplitudes. In the linear approximation, the equations governing the evolution of the wave components w' and ρ' read:

$$\nabla.\mathbf{u}' = 0, \tag{1a}$$

$$\partial_t \nabla^2 w' - (\partial_{zz}\overline{u}\ \partial_x w' - \overline{u}\ \partial_x \nabla^2 w') = -(g/\rho_r)\partial_{xx}\rho', \tag{1b}$$

$$\partial_t \rho' + \overline{u}\ \partial_x \rho' = (\rho_r/g)\overline{N}^2 w - \partial_t \overline{\rho}. \tag{1c}$$

where $\overline{N}^2(z,t) = \frac{-g}{\rho_r}\frac{\partial\overline{\rho}}{\partial z}$ is the Brunt-Väisälä frequency, g is the gravity and ρ_r a reference density. Starting from these equations, it is then possible to use the WKB formalism to describe the refraction of a wave packet. We denote by $Z = \epsilon\, z$ and $T = \epsilon\, t$ the slow variables of the WKB method, where ϵ represents the relation between the vertical characteristic wavelength scale and the vertical variation scale of the mean fields, and we denote the slowly varying fields as functions of Z and T. At the first order (optical geometry approximation), we obtain the Eikonal equation and the dispersion relation of internal waves in the presence of the mean field becomes: $\Omega(k_3, \overline{u}, \overline{N}) = \overline{N}\frac{k_1}{\sqrt{k_1^2+k_3^2}} + k_1\ \overline{u}$. This leads to

$$\frac{\partial k_3}{\partial T} + c_{g3}\frac{\partial k_3}{\partial Z} = -\left[\frac{\partial \overline{u}}{\partial Z}\frac{\partial}{\partial \overline{u}} + \frac{\partial \overline{N}}{\partial Z}\frac{\partial}{\partial \overline{N}}\right]\Omega\,(k_3, \overline{u}, \overline{N}), \tag{2}$$

where $k_3(Z,T) = \frac{\partial \phi}{\partial Z}$ is the local vertical wavenumber and $c_{g3}(Z,T) = -\overline{N}k_1k_3(k_1^2+k_3^2)^{(-3/2)}$ $= \frac{\partial\Omega}{\partial k_3}$ is the vertical group velocity. We assume that $k_1 > 0$ so that $c_{g3} > 0$ for $k_3 < 0$. At the second order, the action conservation relation (Bretherton & Garrett (1969)) reads:

$$\frac{\partial A}{\partial T} + \frac{\partial}{\partial Z}(c_{g3}A) = 0, \tag{3}$$

where $A(Z,T) = E/\Omega_r = \rho_r\ w_0^2\ (k_1^2+k_3^2)^{(3/2)}/(2\overline{N}k_1^3)$ is the action density, i.e. the ratio between energy wave density E and intrinsic frequency $\Omega_r = \Omega - k_1\,\overline{u}$. For such wave packets, one can show that the flux terms of the mean fields equations

$$\partial_T\overline{u} + \partial_Z\overline{u'w'} = 0, \tag{4}$$

$$\partial_T\overline{\rho} + \partial_Z\overline{\rho'w'} = 0, \tag{5}$$

are $\overline{\rho'w'} = 0$, from the polarization relation (see Lighthill (1978)), and $\overline{u'w'} = -k_3\ {w_0}^2/(2\,k_1)$. We thus have $\frac{\partial\overline{\rho}}{\partial t} = 0$ and $\overline{N}(Z)$ is time independent. A first analysis of this model has been performed by Grimshaw (1975) who was interested in the behavior of waves near critical layers and therefore to the energetic phenomena of dissipation. He assumed that the momentum-flux was strong enough to change $\overline{u}$ at the zeroth order and thus change the refraction properties of the medium. In the present paper, we focus on the response of the model to a change of the mean velocity $\overline{u}$.

2 The mean field feedback model

Equations (2), (3) and (4) define a coupled system representing the interactions between the waves (k_3, A) and the horizontal mean field $\overline{u}$.
In order to obtain dimensionless equations, we choose the following dimensionless variables: $Z^* = Z/L_r$, $T^* = T\ N_r$, $\overline{u}^* = \overline{u}/(N_r\ L_r)$, $w_0^* = w_0/(N_r\ L_r)$, $\overline{N}^* = \overline{N}/N_r$, $\mathbf{k}^* = \mathbf{k}\ L_r$, where L_r and N_r are the reference values of, respectively, a space length scale and the Brunt-Väisälä frequency (the dimensionless equations yield expressions identical to that dimensional:

in the following the asterisks will be dropped). With the definitions of the action density A and the vertical group velocity c_{g3} we can write $\frac{\rho_r}{k_1}\overline{u'w'} = c_{g3}\,A$ and therefore, combining (3) and (4),when $A(Z,0)$ and $\overline{u}(Z,0)$ are specified, a new simplified model can be written:

$$\frac{\partial k_3}{\partial T}(Z,T) + c_{g3}(Z,T)\,\frac{\partial k_3(Z,T)}{\partial Z} = -\left[\frac{\partial \overline{u}}{\partial Z}\frac{\partial}{\partial \overline{u}} + \frac{\partial \overline{N}}{\partial Z}\frac{\partial}{\partial \overline{N}}\right]\Omega\,(k_3,\overline{u},\overline{N}), \tag{6a}$$

$$\frac{\partial \overline{u}}{\partial T}(Z,T) + \frac{k_1}{\rho_r}\frac{\partial}{\partial Z}\left[c_{g3}(Z,T)\,A(Z,T)\right] = 0, \tag{6b}$$

$$\text{with} \qquad A(Z,T) = A(Z,0) + \frac{\rho_r}{k_1}\overline{u}(Z,T) - \frac{\rho_r}{k_1}\overline{u}(Z,0). \tag{6c}$$

From the study of the characteristics of the system (see Masi, Moulin & Thual (2007)) it is shown that the system is hyperbolic for $\frac{k_1^2}{k_3^2} < 2$ and that the wave information propagates upward for $\frac{w_0^2(k_1^2+k_3^2)^2}{2\overline{N}^2 k_1^2}\left(2 - \frac{k_1^2}{k_3^2}\right) < 1$. In this paper, we only explore regimes for which the model is everywhere hyperbolic with upward propagation of the waves.

3 Initial values: the equilibrium states

As stated in Section 1, $\overline{N}(Z)$ is time independent and fixed arbitrarily. Given a mean field profile $\overline{u}_e(Z)$, assumed to be stationary, we look for a family of stationary waves field $k_{3e}(Z)$ and $A_e(Z)$ for the coupled model ($6a$-$6b$). For such equilibria, Equation ($6a$) can be integrated into

$$\Omega\left[k_{3e}(Z),\overline{u}_e(Z),\overline{N}(Z)\right] = \Omega\left[k_{3e}(0),\overline{u}_e(0),\overline{N}(0)\right] = \Omega_0. \tag{7}$$

Equation (7) admits the solution

$$k_{3e}(Z) = \text{sign}\left[k_{3e}(0)\right]\,k_1\left[\left(\frac{\overline{N}(Z)}{\Omega_0 - k_1\,\overline{u}_e(Z)}\right)^2 - 1\right]^{(1/2)}, \tag{8}$$

defined on the interval $Z \in [0, Z_L]$ under the necessary and sufficient conditions:

$$0 < \left[\Omega_0 - k_1\,\overline{u}_e(Z)\right] < \overline{N}(Z)\ .$$

The breaking of those conditions are respectively associated to the formation of critical layers and the reflection of the waves.
The dimensionless value Z_L, in the framework of the WKB theory, is assumed as $Z_L = \epsilon\, z_L$ where z_L is the long-length vertical scale and ϵ is small. When these conditions are fulfilled, $c_{g3e}(Z)$ is also defined on the interval $Z \in [0, Z_L]$ and the integration of Equation ($6b$) lead to

$$A_e(Z) = \frac{c_{g3e}(0)\,A_e(0)}{c_{g3e}(Z)}. \tag{9}$$

Since $\overline{u}_e(Z)$ is given, the choice of $k_{3e}(0)$ will be restricted in order to obtain a stationary solution of Equations (8) and (9).

4 Analytical expression of final equilibria

Since we restrict our study to regimes for which the model ($6a$-$6b$) is hyperbolic with upward propagation, we can specify stationary bottom boundary conditions $\overline{u}(0,T) = \overline{u}_b$, $k_3(0,T) =$

k_{3b}, $A(0,T) = A_b$, and initial conditions $\overline{u}(Z,0) = \overline{u}_i(Z)$, $k_3(Z,0) = k_{3i}(Z)$, $A(Z,0) = A_i(Z)$, that will presumably evolve and reach a final equilibrium $\overline{u}_f$, k_{3f}, A_f. If we further assume that no critical layer develops and no reflection occurs during the evolution of the system, a final equilibrium has to be reached in finite time. In that case, we can infer the final configuration from an integration of the equations ($6a$-$6b$), yielding

$$k_{3f}(Z) = -k_1 \left[\left(\frac{\overline{N}(Z)}{\Omega_i - k_1\,\overline{u}_f(Z)} \right)^2 - 1 \right]^{(1/2)}, \tag{10a}$$

$$A_f(Z) = \frac{c_{g3b}\,A_b}{c_{g3f}(Z)}, \tag{10b}$$

$$\text{with} \qquad \overline{u}_f(Z) = \frac{k_1}{\rho_r} A_f(Z) - \frac{k_1}{\rho_r} A_i(Z) + \overline{u}_i(Z) \tag{10c}$$

where $c_{g3b} = c_{g3}(0,T)$ and $\Omega_i = \Omega\left[k_{3i}(0), \overline{u}_i(0), \overline{N}(0)\right]$. From system ($10a$-$10c$) we derive an analytical solution for $\overline{u}_f(Z)$ as a solution of an eighth degree polynomia (see Masi, Moulin & Thual (2007)) :

$$(\overline{u}_f^2 + a^2 - 2a\overline{u}_f)\,(c^4 + \overline{u}_f^4 + 4c^2\overline{u}_f^2 - 4c^3\overline{u}_f - 4c\overline{u}_f^3 + 2\overline{u}_f^2 c^2)\,(\overline{N}^2 - k_1^2 c^2 - k_1^2\overline{u}_f^2 + 2ck_1^2\overline{u}_f) - d^2\overline{N}^6 = 0\,,$$

$$\text{with } a = -\frac{0.5\;w_{0i}^2(Z)\,[k_1^2 + k_{3i}^2(Z)]^{\frac{3}{2}}}{\overline{N}(Z)k_1^2} + \overline{u}_i(Z), \quad c = \overline{N}(0)\frac{1}{\sqrt{k_1^2 + k_{3b}^2}} + \overline{u}_b, \quad d = \frac{0.5\;k_{3b}\;w_{0b}^2}{\overline{N}(Z)k_1^2}.$$

From the eight real or complex roots obtained at each altitude Z, we can build a unique real and continuous profile $\overline{u}_f(Z)$ which satisfies the condition $\overline{u}_f(0) = \overline{u}_i(0)$.

5 Numerical Simulation - A boundary layer on a sinusoidal topography

We consider that $\overline{u}(Z,T)$ is flowing on a sinusoidal topography $h(x) = \frac{h_{max}}{2}\,cos(k_1, x)$ with $\overline{u}(0,T) = \overline{u}_b$ for all T. With this forcing, internal waves are emitted with the bottom wavenumber components k_1 and k_{3b} solutions of the dispersion relation $\Omega\left[k_{3b}, \overline{u}_b, \overline{N}(0)\right] = 0$, and they are amplified such as $w_{0b} = \frac{h_{max}}{2}\,(k_1\,\overline{u}_b)$. Imposing $\overline{u}_b$ and h_{max} is equivalent to a choice of k_{3b} and A_b. On Figure 1 an example of a physical configuration is shown. The numerical integration of model ($6a$-$6b$) is done by a difference finite explicit scheme, with an upwind space and an Euler temporal discretisation. In the simulation, we choose as boundary conditions $\overline{u}(0,T) = \overline{u}_b =: \overline{u}_e(0) + \overline{u}_p(0)$, $k_3(0,T) = k_{3b} =: k_{3e}(0)$, $A(0,T) = A_b =: A_e(0)$ for all T, and, as initial conditions, $\overline{u}_i(Z) = \overline{u}_e(Z) + \overline{u}_p(Z)$, $k_{3i}(Z) = k_{3e}(Z)$, $A_i(Z) = A_e(Z)$, where $\overline{u}_p$ is a perturbation term. Our intention is, in fact, to perturbate an equilibrium state by prescribing an initial condition of the mean flow ($\overline{u}_i$) that corresponds to a modification of the balanced mean flow profile ($\overline{u}_e$) by a perturbation term $\overline{u}_p$. The frequency profile is set to $\overline{N}(Z) = 1$. For a constant value $k_1 = 0.2$ and a value of A_b such that the amplitude bottom value is $w_{0b} = 0.2$, we choose a vertical wavenumber boundary value $k_{3b} = -0.45$ to fulfill the conditions of hyperbolicity and upward propagation at time $T = 0$. The initial condition $k_{3i}(Z) = k_{3e}(Z)$ is given by Equation (8) and the initial condition $A_i(Z) = A_e(Z)$ is inferred from Expression (9). In order to give an estimation of the physical (z,t) variables, the ratio $\epsilon = 0.2$ between characteristics lengths scales ($2\pi/|k_{3b}|$ and z_L) is proposed. We initialize our system with a wave field $[k_{3e}, A_e]$ in equilibrium with a mean flow $\overline{u}_e(Z) \;= \overline{u}_b + \; F\;(Z/Z_L)^{1/2}$. Since we deal with dimensionless equations, F is a mean flow Froude number based on L_r and N_r (in the simulation its value is set to $F = 1$). Then we perturbate this wind profile, by choosing

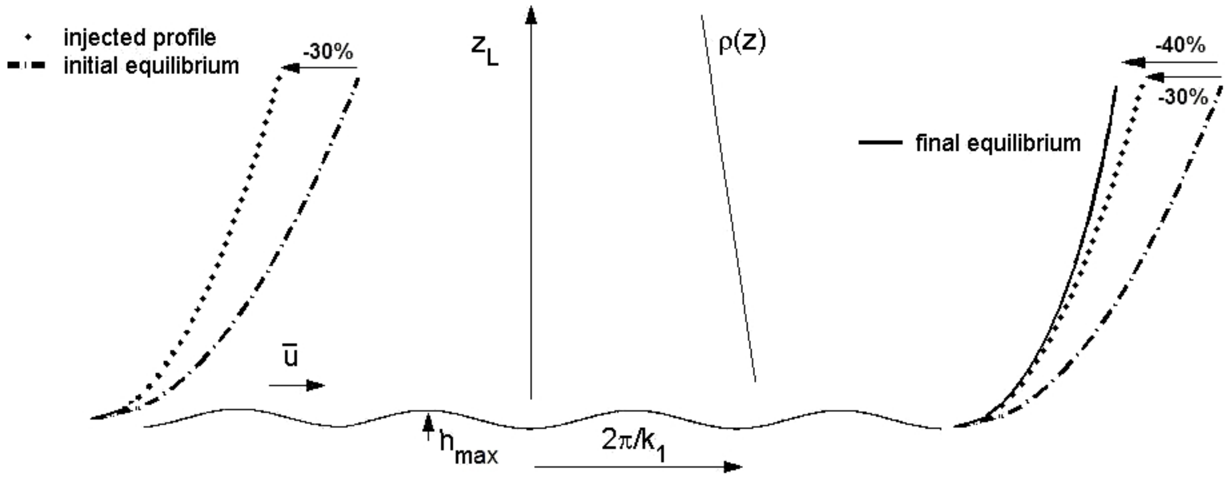

Figure 1: An example of a feedback on the mean flow above a sinusoidal topography.

an initial condition that corresponds at a 30% decrease of the maximum value of the variable velocity field component, i.e. $\overline{u}_i(Z) = \overline{u}_e(Z) + \overline{u}_p(Z) = \overline{u}_b + \ 0.7\,F\ (Z/Z_L)^{1/2}$. The other components of the system remain unchanged at time $T = 0$. The evolution of the principal variables in the transient time are plotted in Figure 2, along with the final equilibrium state reached by the system. As we can see, in the presence of a new profile of wind, the waves change their ray paths and, therefore, their influence the mean field which is modified. The final equilibrium state is reached in a finite time $T \simeq 60$ (corresponding to a physical time $t \simeq \frac{60}{\epsilon}\,N_r^{-1}$). The final profile of the mean flow represents a new equilibrium state where the velocity is lower than the perturbated initial mean flow. For example, at Z where $\overline{u}$ is maximum, an initial decrease of 30% of the mean flow, corresponds at the final decrease of 40%. This phenomenon can be related to a transfer of energy from the mean flow toward the wave field during the transient. In order to propose a realistic physical atmospheric configuration, we choose the dimensional values of length scale $L_r = 10^2 m$ and time scale $N_r = 10^{-2} s^{-1}$. The other variables become: the horizontal wavelength $l_1 = 2\pi/k_1 \simeq 3.1\,km$, the altitude of mean field variation $z_L \simeq 7\,km$, the mean field velocity at the bottom $\overline{u}_b = 2\,ms^{-1}$, the bottom value of vertical velocity fluctuations (wave amplitude) $w_{0b} = 0.2\,ms^{-1}$ and the bottom value of frequency $\omega_0 = 0.0041\,s^{-1}$. Therefore, the sinusoidal topography presents a maximum height $h_{max} = 2\,w_{0b}/\omega_0 \simeq 100\,m$. The physical time of transient to reach the new equilibrium is $t \simeq 8.3\,h$. The same numerical simulation proposed for a real oceanic configuration could be such as the choice is $L_r = 10\,m$ and $N_r = 10^{-1}\,s^{-1}$, obtaining a sinusoidal topography with a maximum height $h_{max} \simeq 10\,m$ and an horizontal wavelength $l_1 \simeq 310\,m$, with an altitude of mean flow variation $z_L \simeq 700\,m$ for the same mean field bottom velocity $\overline{u}_b = 2\,ms^{-1}$, the same bottom value of vertical velocity fluctuations $w_{0b} = 0.2\,ms^{-1}$ and a bottom value of frequency $\omega_0 = 0.041\,s^{-1}$. In this case, the new state of equilibrium after perturbation is reached to a physical time $t \simeq 50\,min$.

6 Conclusion

In the present work we have studied a model for the interaction between internal gravity waves and a horizontal mean field in a two-dimensional slowly variable medium. We have developed an analytical solution which links the initial conditions to the resulting final equilibrium when it exists. This analytical solution has been used as a validation for numerical simulations. The present study has shown that the momentum-flux is strong enough to change the mean field

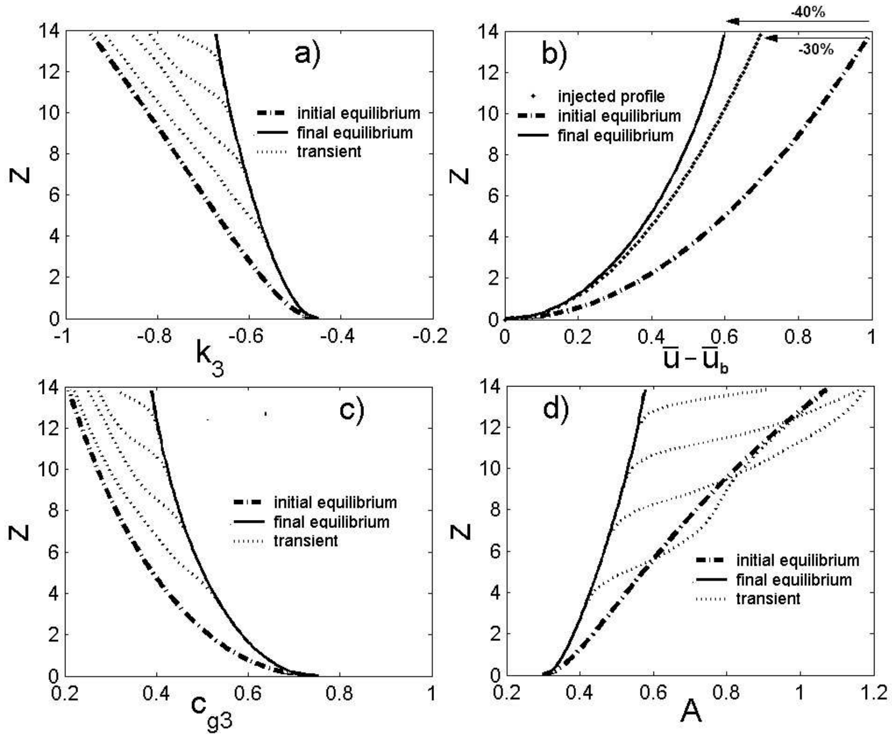

Figure 2: Evolution of: a) vertical wavenumber, b) mean flow, c) vertical group velocity, d) action density.

during a transient regime which duration can be predicted. Moreover, it has been shown that the waves have an "amplifying" behavior on the mean field by emphasizing its trend. We think that the present model could be used in a parametrization of momentum-flux transfer for the subgrid models of the oceanic and atmospheric circulations, in order to take into account the feedback phenomena between internal waves and the mean current.

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