Size Distribution of Convective Thermals in an Unstable Stratified Layer

Vul'fson A. N. & Borodin O. O.

Oil and Gas Research Institute, Russian Academy of Sciences,
Laboratory of Mathematical Modeling of Heat and Mass Transfer and Physicochemical Processes
Gubkina Str. 3, 119991, Moscow, Russia
vulfson@ipng.ru; borodin@ipng.ru

Abstract:

An ensemble of convective thermals is considered in the layer of penetrative turbulent convection over a homogeneous heated horizontal surface. A model of an unsteady spontaneous jet is proposed to describe the dynamics of an isolated convective element. The model has an exact self-similar solution, which is used to derive a dynamic invariant that relates the velocity, temperature, and radius of the jet. The size distribution of spontaneous convective jets is derived, which agrees with the description of the fine structure of the atmospheric surface layer. The statistical size distribution of thermals is substantiated using the Fokker-Planck kinetic equation.

Key-words: convective thermals; unsteady spontaneous jets; gamma distribution

1 Introduction

Complex studies of the atmospheric boundary layer over a horizontally homogeneous land surface in the daytime, see Kaimal and Al (1976), have revealed the existence of a stochastic ensemble of isolated eddies whose temperature is warmer than the surroundings and the horizontal dimension is zero to three orders of magnitude less than the height of the convective boundary layer. Convective eddies with a horizontal size within this range are called thermals. The surface of rotation on which the temperature of a convective element equals the ambient temperature is the natural boundary of a thermal. Thermals of roughly equal vertical and horizontal sizes are called bubbles, and thermals whose vertical size far exceeds their horizontal size are called jets. More detailed information on the structure of thermals can be found in Hunt (1998). The role of thermals in the formation of the convective boundary layer was discussed in detail in studies of Scorer & Ludlam (1953); Frish & Businger (1973); Manton (1977); Lenschow & Stepens (1980). Vertical movements of thermals are chaotic in character. An empirical description of a system of convective thermals as a statistical ensemble through the size distribution function was first performed by N.I. Vul'fson (1961), see also Lord & Willis (1955).

In this study, we theoretically derive the size distribution function for convective thermals. In the framework of the proposed approach, classical Boltzmann statistics is supplemented with a hydrodynamic invariant characterizing the motion of isolated thermals. The gamma distribution thus obtained is consistent with earlier extensive empirical data on the size distribution of convective thermals.

2 Dynamics of an isolated spontaneous thermal in an unstably stratified layer

It is assumed that an unstable layer forms over a homogeneous heated horizontal surface, which is hereafter interpreted as a plane source of buoyancy of constant strength gQ_1, where $[gQ_1]=$ m^2/s^3. Let t denote time; r, φ, z be cylindrical coordinates with the z-axis directed oppositely to the acceleration of gravity g; Θ be the local potential temperature; and $\bar{\Theta}=\bar{\Theta}(z)$

be the background potential temperature. Assume that the background stratification of potential temperature in the unstable layer is determined by the Monin-Obukhov similarity relation

$$\frac{g}{\bar{\Theta}}\frac{d\bar{\Theta}}{dz} = g\Gamma(z) = -c_{\Gamma 0}\left(gQ_1\right)^{2/3} z^{-4/3} \qquad (1)$$

where $c_{\Gamma 0}$ is a constant coefficient. According to atmospheric field measurements, see Deacon (1959), $c_{\Gamma 0} \approx 0.9 - 1.0$.

A description of the dynamics of a convective isolated element is performed within the framework of the model of an unsteady convective jet under the Boussinesq assumption in the form of Ogura & Phillips (1962) and the vertical boundary-layer approximation, see Schlichting (1968):

$$\begin{aligned} &\frac{\partial}{\partial t}w + \frac{1}{r}\frac{\partial}{\partial r}uwr + \frac{\partial}{\partial z}ww = g\theta + \frac{1}{r}\frac{\partial}{\partial r}\left(\nu_w r\frac{\partial w}{\partial r}\right), \quad \theta = \frac{\Theta - \bar{\Theta}}{\bar{\Theta}} \\ &\frac{\partial}{\partial t}\theta + \frac{1}{r}\frac{\partial}{\partial r}u\theta r + \frac{\partial}{\partial z}w\theta + \Gamma(z)w = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu_\theta r\frac{\partial \theta}{\partial r}\right), \quad \frac{1}{r}\frac{\partial}{\partial r}ur + \frac{\partial}{\partial z}w = 0 \end{aligned} \qquad (2)$$

where u and w are the velocity components along the r and z axes, respectively; θ is a dimensionless fluctuation of potential temperature; ν_w, ν_θ are the eddy exchange coefficients for vertical velocity and dimensionless potential temperature.

The system of equations (2) is considered in the unbounded domain $V = \{0 \le r < \infty,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le h(t)\}$, which represents the convective surface layer. The initial conditions at $t = t_0$ correspond to the state of a static environment.

The momentum and heat fluxes are assumed to vanish at the side and the lower flat boundaries of the domain:

$$\lim_{z\to 0}\left[w \cdot w(r,z,t)\right] = 0, \quad \lim_{z\to 0}\left[w \cdot \theta(r,z,t)\right] = 0 \qquad (3)$$

Equations (1) – (3) form a closed system describing an unsteady convective spontaneous jet and are a natural generalization of the stationary equations in Batchelor (1954).

Let R be the radius of the jet and f_w, f_θ be the horizontal profiles of the vertical velocity and potential temperature specified from the experimental data of Rouse and Al (1952)

$$R = \alpha_R z, \quad f_w(\xi) = \exp\left(-\beta_w \xi^2\right), \quad f_\theta(\xi) = \exp\left(-\beta_\theta \xi^2\right), \quad \xi = r/R \qquad (4)$$

where $\beta_w/\alpha_R^2 = 96$, $\beta_\theta/\alpha_R^2 = 71$, and $\alpha_R = 0.1$ are constant coefficients.

An approximate solution to (1) – (3) should be sought within the framework of the integral von Karman-Pohlhausen method, for more details, see Schlichting (1968)

$$\begin{cases} u(r,z,t) = -\dfrac{\partial \tilde{w}}{\partial z}(z,t)\cdot\dfrac{1}{r}\displaystyle\int_0^r r\cdot f_w(r/R)\cdot dr \\ w(r,z,t) = \tilde{w}(z,t)\cdot f_w\left(r/R\right), \quad \theta(r,z,t) = \tilde{\theta}(z,t)\cdot f_\theta\left(r/R\right) \end{cases} \qquad (5)$$

Here, the functions $\tilde{w}(z,t)$ and $\tilde{\theta}(z,t)$ correspond to the vertical velocity and temperature on the axis of the jet and satisfy the system

$$\begin{cases} \dfrac{\partial}{\partial t}\tilde{w}R^2 + \dfrac{1}{2}\dfrac{\partial}{\partial z}\tilde{w}\tilde{w}R^2 = \alpha_g g\tilde{\theta}R^2, \quad R = \alpha_R z \\ \dfrac{\partial}{\partial t}\tilde{\theta}R^2 + \dfrac{1}{1+\alpha_g}\dfrac{\partial}{\partial z}\tilde{w}\tilde{\theta}R^2 + \dfrac{1}{\alpha_g}\Gamma(z)\tilde{w}R^2 = 0 \end{cases} \qquad (6)$$

where $\alpha_g = \beta_w/\beta_\theta = 1.35$ is a constant coefficient.

Equations (6) must be supplemented with boundary conditions at the underlying surface:

$$\lim_{z\to 0}\left[\tilde{w}\tilde{w}R^2(z,t)\right]=0, \quad \lim_{z\to 0}\left[\tilde{w}\tilde{\theta}R^2(z,t)\right]=0 \tag{7}$$

A self-similar solution to system (6), (7) can be sought in the universal form proposed by Vul'fson & Borodin (2003):

$$\begin{cases} \tilde{w}(z,t)=\dfrac{dh}{dt}w^*(z_*), \quad \tilde{\theta}(z,t)=\dfrac{1}{gh}\left(\dfrac{dh}{dt}\right)^2\theta^*(z_*), \quad R(z,t)=h\cdot R_*(z_*) \\ R_*(z_*)=\alpha_R z_*, \quad z_*=z/h(t), \quad h(t)=\dfrac{2}{3}\lambda_2^{-1}(gQ_1)^{1/2}t^{3/2}. \end{cases} \tag{8}$$

Here, z_* is a dimensionless parameter; w^*, θ^*, and R_* are dimensionless functions; $h = h(t)$ is the height of the rise of the convective jet, which depends on the flat heat source of constant strength Q_1; λ_2 is a constant coefficient.

Substituting (8) into (6) gives the self-similar equations

$$\begin{cases} \dfrac{7}{3}w^*R_*^2 - z_*\dfrac{d}{dz_*}\left(w^*R_*^2\right)+\dfrac{1}{2}\dfrac{d}{dz_*}w^*w^*R_*^2=\alpha_g\theta^*R_*^2, \quad R_*=\alpha_R z_*. \\ \dfrac{5}{3}\theta^*R_*^2 - z_*\dfrac{d}{dz_*}\left(\theta^*R_*^2\right)+\dfrac{1}{1+\alpha_g}\dfrac{d}{dz_*}w^*\theta^*R_*^2=\left(\dfrac{2}{3}\lambda_2^2\right)^{2/3}\dfrac{c_{\Gamma 0}}{\alpha_g}z_*^{-4/3}\left(w^*R_*^2\right) \end{cases} \tag{9}$$

The boundary conditions for system (9) follow from (7) and have the form

$$\lim_{z\to 0}\left[w^*w^*R_*^2(z_*)\right]=0, \quad \lim_{z\to 0}\left[w^*\theta^*R_*^2(z_*)\right]=0 \tag{10}$$

The solution to system (9) that satisfies boundary conditions (10) is given by

$$\begin{gathered} w^*=\alpha_w z_*^{1/3}, \quad \theta^*=\alpha_\theta z_*^{-1/3}, \quad R_*=\alpha_R z_*, \\ \alpha_w=\left(\dfrac{2}{3}\lambda_2^2\right)^{1/3}\left[\dfrac{3}{8}(1+\alpha_g)c_{\Gamma 0}\right]^{1/2}, \quad \alpha_\theta=\left(\dfrac{2}{3}\lambda_2^2\right)^{2/3}\dfrac{1+\alpha_g}{2\alpha_g}c_{\Gamma 0} \end{gathered} \tag{11}$$

Using integration over the jet area, we can calculate the average squared velocity $\hat{w}$ and the average fluctuation of dimensionless potential temperature $\hat{\theta}$. Then, according to (4), we can write

$$\hat{w}^2=\frac{2}{R^2}\int_0^\infty w^2(r,z,t)rdr=\frac{1}{2\beta_w}\tilde{w}^2(z,t), \quad \hat{\theta}=\frac{2}{R^2}\int_0^\infty \theta(r,z,t)rdr=\frac{1}{\beta_\theta}\tilde{\theta}(z,t) \tag{12}$$

Combining (12) with (8) and (11) yields the following dynamic invariants, which hold in any cross section of the unsteady spontaneous jet:

$$\frac{1}{2}\frac{\hat{w}^2}{g\hat{\theta}z}=\frac{3}{16}, \quad \frac{1}{2}\frac{\hat{w}^2}{g\hat{\theta}R}=\frac{3}{16}\alpha_R^{-1} \tag{13}$$

The first relation in (13) represents a constant ratio of the kinetic to potential energy of the thermal. The second relation in (13) relates the thermal's dynamic parameters to its geometric size, see also Vul'fson (2001).

3. Ensemble of convective elements

Studies concerning the fine structure of penetrative convection over a heated surface suggest that an ensemble of convective elements represents a stochastic system of jets rising in an unstable stationary surrounding and having positive vertical velocities $\hat{w}>0$.

In the description of an ensemble of convective thermals, the following assumptions will be used.

- All the convective jets (which originate near the ground) have equal positive temperature excess at each level z at the same time t, i.e.,

$$\hat{\theta} = \theta_T(z,t) > 0 \tag{14}$$

where $\theta_T(z,t)$ is a known function.

Assumption (14) is confirmed by empirical data (Fig. 1) and rules out the random character of the temperature of convective thermals.

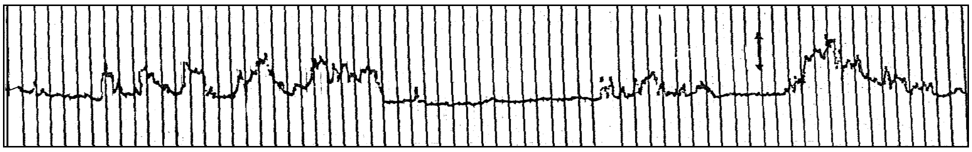

Fig. 1. Temperature fluctuations over Kyzyl-Kum sands at a height of 100 m.

- At each level *z*, the convective jets have a random squared vertical velocity $\hat{w}^2$ and a random diameter *D*. The motion of a thermal according to (13), (14) is governed by the dynamic invariant

$$\hat{w}^2 = \frac{3}{16}\alpha_R^{-1} g \theta_T D \tag{15}$$

where $\alpha_R = 0.1$ is a constant coefficient.

The ensemble of rising convective thermals is a quasi-Maxwell system, see Lovenda (1991). Thus,

$$\frac{N_w}{N_0} = \frac{a}{\Gamma(a)} \frac{1}{<\hat{w}^2>} \left\{ a\left(\frac{\hat{w}^2}{<\hat{w}^2>} \right) \right\}^{a-1} \exp\left\{ -a\left(\frac{\hat{w}^2}{<\hat{w}^2>} \right) \right\} \tag{16}$$

where $N_w d\hat{w}^2$ is the number of convective elements per unit area whose squared velocity varies from $\hat{w}^2$ to $\hat{w}^2 + d\hat{w}^2$; N_0 is the total number of elements per unit area; $<\hat{w}^2>$ is the effective squared velocity of the elements; $a > 0$ is a constant parameter; and $\Gamma(a)$ is the constant value of the gamma function at the point *a*[*].

4 Size distribution of thermals in the convective atmospheric layer

Using dynamic invariant (15), we can turn from the velocity distribution (16) to the diameter distribution. For this purpose, we introduce the effective diameter $<D>$ of a thermal assuming that

$$<\hat{w}^2> = \frac{3}{16}\alpha_R^{-1} g \theta_T <D> \tag{17}$$

Let $N_D dD$ be the number of convective jets per unit area with diameters between D and $D + dD$ and N_0 be the total number of elements per unit area. Substituting (15) and (17) into (16) yields

$$\frac{N_D}{N_0} = \frac{a}{\Gamma(a)} \frac{1}{<D>} \left\{ a\left(\frac{D}{<D>} \right) \right\}^{a-1} \exp\left\{ -a\left(\frac{D}{<D>} \right) \right\} \tag{18}$$

A typical profile of distribution (18) for the atmospheric convective surface layer is shown in Fig. 2.

[*] It can be shown that quasi-Maxwell distribution (16) is a particular stationary solution of the Fokker-Planck kinetic equation.

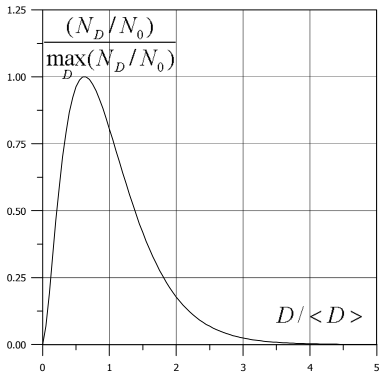

Fig. 2. Gamma distribution (18) with parameter $a - 1 = 5/3$.

A systematic experimental study of the statistical size distribution of thermals on the basis of 39895 aircraft measurements was first conducted by N.I. Vul'fson (1961). The values of the parameters obtained from this observations are listed in Table 1.

Table 1. Empirical values of gamma-distribution parameters and their dependence on height over the underlying surface.

Flying height, m	Number of measurements	Average diameter $< D >$, m	Number of jets N_0, 1/km^2	Distribution parameter, $a - 1$
30	2480	49	217	1.67
50	7611	55	138	1.67
100	8728	61	87	1.67
300	4748	68	52	2.13

An analysis of the data given in Table 1 shows that the parameters of the statistical distribution change sharply upon the transition from the surface layer to the mixed layer.

According to the model of a convective ensemble of thermals in the atmospheric convective surface layer, we set

$$N_0 < D >^2 = 0.42, \quad < D >= 2\alpha_R z \tag{19}$$

The first relation in (19) follows from the fact that the convective heat flux in the atmospheric surface layer is independent of height. This relation agrees (within some approximation) with the data listed in Table 1. The second relation in (19) follows directly from approximations of the model of an isolated spontaneous jet. Experiments fail to reveal this relation because of the nonlinear interaction between the thermals, which significantly deforms the dependence of the diameter on height.

In later field measurements Lenschow & Stepens (1980) it was found that

$$N_0 < D > z_i = 0.1, \qquad < D > / z_i = 0.16 \left(z / z_i \right)^{1/3} \tag{20}$$

where z_i is a height of the convective boundary layer.

The first relation in (19), the second relation in (20), and the parameter a–1 =5/3 entirely determine the statistical size distribution (18) of thermals in the atmospheric convective surface layer.

For small diameters $D / < D > << 1$, the gamma function (18) becomes a power function:

$$\frac{N_D}{N_0} = \frac{a}{\Gamma(a)} \frac{1}{< D >} \left\{ a \left(\frac{D}{< D >} \right) \right\}^{a-1} \tag{21}$$

Essentially, the exponent a–1 = 1.67 for the spectrum of spontaneous jets is nearly equal to the Obukhov exponent 5/3 for squared temperature fluctuations. It should be especially stressed that the experiments by N.I. Vul'fson (1961) were conducted nearly simultaneously with publishing the –5/3 power law, see Obukhov (1960), which entirely rules out the possibility of artificial agreement created between the theory and the observations.

5 Conclusions

The self-similar solution obtained for a spontaneous jet and the dynamic invariant based on it were used to theoretically estimate the role of thermals in the formation of the convective boundary layer and its fine structure. Combining the dynamic invariant with the stationary solution of the Fokker-Planck kinetic equation, we constructed a size distribution of thermals in the convective boundary layer that agrees with available experimental data. In the light of these results, the model describing an ensemble of thermals becomes constructive in character.

References

Kaimal, J.C., Wyngaard, J.C., Haugen, D.A., Cote, O.R., Izumi, Y., Caughey, S.J. & Readings, C.J. 1976 Turbulence structure in the convective boundary layer *J. Atmos. Sci.* **33**, 2152-2169.

Hunt, J. C. R. 1998 Eddy dynamics and kinematics of convective turbulence. (ed Plate E. J., Fedorovich. E.E., Viegas D. X. & Wyngaard, J. C. – Buoyant convection in geophysical flows. NATO ASI Series C: Mathematical and physical science.) **513**, pp. 41–82. Kluwer Academic Publishers, Dorcdrecht–Boston–London.

Scorer, R.S. & Ludlam, F. H. 1953 Bubble theory of penetrative convection. *Quart. J. Roy Metor. Soc.* **79**, 94–106.

Frish, A.S., Businger, J. A. 1973 A study of convective elements in the atmospheric surface layer. *Boundary-Layer Meteorol.* **3**, 301–328.

Manton, M. J. 1977 On the structure of convection *Boundary-Layer. Meteorol.* **12**, 491–509.

Lenschow, D. H. & Stepens, P. L. 1980 The role of thermals in the convective boundary layer. *Boundary–Layer Meteorol*. **19**, 509–532

Vul'fson, N.I. 1961 Convective motions in a free atmosphere. Israel Program for Scientific Translation. Jerusalem (translated 1964) Clearing house U. S. Dept. of Commerce 188 p.

Lord, G. W. & Willis, T. F. 1955 Calculation of air bubble size distribution from results of Rossiaval traverse of aerated concrete. *Bull. Amer. Soc. Test Mater.* **177**.

Deacon, E. L. 1959 Atmospheric diffusion and air pollution. (ed. Frenkel F. N., Sheppard P. A.) pp. 236 –250. Academic Press, New York–London.

Ogura, Y. & Phillips, N.A. 1962 Scale analysis of deep and shallow convection in the atmosphere. *J. Atmos. Sci.* **19,** 173-179.

Batchelor, G. K. 1954 Heat convection and buoyancy effects in fluids. Quart. J. Roy. Meteor. Soc. **80**, 339–358.

Rouse H., Yih C.-S. & Humphreys H.W. 1952 Gravitational convection from a boundary source. *Tellus.* 1952. **4**, 201–210.

Schlichting, H. 1968 Boundary layer theory. 6th ed. New York. Mc Graw-Hill Book Co. 455 p.

Vul'fson, A. N. & Borodin, O. O. The contribution to the integral theory of a turbulent jet above a point heat source of constant strength. *Journal of Engineering Physics and Thermophysics.* **76**, 187–192.

Vul'fson, A. N. 2001 Algebraic relationships among the velocity, temperature and pressure on the axis of symmetry of a stationary convective turbulent jet in a stratified atmosphere. *Izvestiya, Atmospheric and oceanic physics.* **37**, 298–305.

Lovenda, B. H. 1991 Statistical physics. A probabilistic approach. Wiley Inc. New York – Toronto – Singapore. 346 p.

Obukhov, A. M. 1960 On the structure of the temperature and velocity fields in the conditions of free convection. *Izvestiya Akad. Nauk SSSR, Ser. Geopfiz.* **9**, 1392–1396.