Internal wave breaking depending on stratification

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Abstract :

Euler equations for incompressible fluid stratified by a gravity field are investigated. It is found out that the system of Euler equations is not enough for statement of a correct generalized problem. Some auxiliary conditions are offered and justified. A numerical method is developed and applied for study of processes of whirl destruction and mixing in a stratified fluid. The dependence of vortex destruction on a stratification scale is investigated numerically and it is shown that the effect increases with the stratification scale. However, the effect of vortex destruction is absent when the fluid density is constant.

Key-words :

stratification; turbulence; mixing

1 Introduction

Internal wave breaking is one of interesting nonlinear effects, which is characterized by disintegration of waves and by formation of typical spots with an intensive small-scale convection inside them. The spots are often termed convective or turbulent ones because the convection inside them looks like turbulence. Static stability in the fluid becomes recovered with the course of time, but the density inside the spot remains different from the surrounding stratification. The phenomenon of internal wave breaking going with formation of a convective spot often is termed "internal wave mixing". The effect of internal wave breaking is often observed in the ocean Miropolskii (1981). The phenomenon is registered in the atmosphere Pfister *et al.* (1986). The broken down internal waves reorganize stratification with time. Therefore, it is impossible to understand and explain stratification of the ocean or the atmosphere without taking into account the effect.

In McEwan's experimental papers, the phenomenon has been studied by means of thin laser measurements. Internal waves were excited in a laboratory tank with sizes of $25\,\mathrm{cm} \times 50\,\mathrm{cm} \times 25$cm filled with a stratified fluid. Stratification is formed by dependence of water saltiness on height. The ratio $\delta = \frac{\Delta\rho}{\rho} = 0.04$, so the stratification is a weak one. Observation of small-scale structures had been based on dependence of refraction index of light on liquid density. McEwan has discovered and studied different stages of evolution of wave breaking and mixing McEwan (1971), McEwan (1983): overturning, development of interleaving microstructure, restoring of static stability.

Very large density gradients and velocity shears eventually are formed by mechanisms of the effect under study. These large density gradients essentially influence dynamics. The Boussinesq approximation is admissible before vortex destruction, but in time it influences the small-scale convection generated, and is not used. Also, it is wisely to presume the solution may contain discontinuities. It seems to be of interest to study nondifferentiable, generalized solutions of hydrodynamic equations of a stratified incompressible fluid placed into a gravity field.

2 Basic system of equations

We suppose the fluid behavior is described by Euler equations for an incompressible fluid

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0, \qquad \frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u w}{\partial z} = -\frac{\partial p}{\partial x},$$
$$\frac{\partial \rho w}{\partial t} + \frac{\partial \rho u w}{\partial x} + \frac{\partial \rho w^2}{\partial z} = -\frac{\partial p}{\partial z} - \rho g, \qquad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{1}$$

Here ρ is the density, p is the pressure, u and w are the horizontal and vertical mass velocities of the fluid, t is time, x, z are the horizontal and vertical coordinates respectively, g is the free fall acceleration.

It is supposed as well that the unperturbed fluid is stratified in density exponentially: $\rho_0(z) = \rho_{00} \exp\left(-\frac{z}{H}\right)$. Here $H = 6.23\,\mathrm{m}, \rho_{00} = 1000\,\mathrm{kg/m^3}$. The chosen value of H approximately corresponds to the one for McEwan's experiments McEwan (1971), McEwan (1983).

The boundary condition is $(\mathbf{v}, \mathbf{n}) = 0$ at the boundary $\partial\Omega$ of the domain Ω, where $\mathbf{v} = (u, w)$ and $\mathbf{n}$ is a normal line to the boundary. Keeping in mind McEwan's experiments, we take a rectangle with horizontal dimension 50 cm and vertical dimension $h = 25$ cm as the domain Ω.

3 Statement of the generalized problem

Theorem 1 *Let a solution to equations (1) be differentiable. Then the functional*

$$H_{nonl} = \int\limits_{\Omega} \left[\rho\, \frac{u^2 + w^2}{2} + \rho\, g\, z + + g\, H\, \left(\rho\, \ln\left(\frac{\rho}{\rho_0\,(0)}\right) + \left(\rho_0(z) - \rho\right)\right)\right]\, d\,\Omega \tag{2}$$

is conserved over solutions. The integrand in (2) is strictly non-negative at $\rho \geq 0$. Functional (2) turns into the functional of wave energy from a linear theory Miropolskii (1981), when we take a small-amplitude limit.

Let us generalize (2) to a case when solution may be nondifferentiable. The functionals of mass, hydrodynamic energy should be conserved over nondifferentiable solutions; it follows from their physical sense. However the functional $\int\limits_{\Omega} \left[g\, H\, \left(\rho\, \ln\left(\frac{\rho}{\rho_0(0)}\right)\right)\right]\, d\,\Omega$ is not conserved in case of discontinuities in the density. We demand

$$\frac{dH_{nonl}}{dt} \leq 0. \tag{3}$$

The sign "$\leq$" is chosen because conservation of (2) is impossible in case of discontinuities in density, but "$>$" in (3) leads to an unstable problem. We will use (2) and (3) as a basis of the theory of generalized solutions to nonlinear equations.

From the physical point of view, the set of equations (1) is incomplete because does not include the energy conservation law. Let $\omega \subset \Omega$ be an arbitrary star domain with piecewise smooth boundary, and let $S(\omega)$ be a boundary surface of the domain ω, $d\mathbf{S} = dS\,\mathbf{n}$, where dS is a surface element of $S(\omega)$, and $\mathbf{n}$ is a vector of an outer normal line to $S(\omega)$. We write an integral relation for energy

$$\int\limits_{\omega} e(x,z,t) d\Omega - \int\limits_{\omega} e(x,z,0) d\Omega - - \int\limits_{0}^{t} \oint\limits_{S(\omega)} e(x,z,t)\, \mathbf{v}(x,z,t)\, d\mathbf{S}\, dt = 0, \tag{4}$$

Here $e = \rho\left(\frac{u^2+w^2}{2} + gz\right)$. If the solution is differentiable, then (4) follows from (1); but if the solution does not belong to a class of differentiable functions, then relation (4) does not follow from (1). Therefore, permitting nondifferentiable solutions, we have to include (4) into the system of simulative equations as an individual equation.

Relations (1), (4) are not enough for statement of a correct problem. (3) should be fulfilled for stability of the nonlinear problem. As conservation of energy functional (4) is postulated, the relation

$$\frac{\partial}{\partial t}\left(f\left(\rho\right)\right) + \nabla\left(f\left(\rho\right)\mathbf{v}\right) = \nu, \tag{5}$$

follows from (3) and (4). Here $f\left(\rho\right) = \rho\ln\left(\rho\right)$, $\nu \le 0$. Requirement of compatibility of (5) with the equation of continuity in (1) results in conclusion that ν may be nonzero only at discontinuities. Evidently, ν is unknown. Coming from physical reasons, one can suggest to find ν from the condition of minimum of $|\nu|$ under the condition $\nu \le 0$.

Using the set of equations (1), we routinely build some integral relations for definition of a generalized solution

$$\int\limits_{\Omega} \rho_s(x,z,0)\rho(x,z,0)\,d\Omega - \int\limits_{Q_T}\left(\rho\frac{\partial\rho_s}{\partial t} + +\rho u\frac{\partial\rho_s}{\partial x} + \rho w\frac{\partial\rho_s}{\partial z}\right) dQ_T = 0, \tag{6}$$

$$\int\limits_{\Omega} u(x,z,0)\rho(x,z,0)\,u_s(x,z,0)\,d\Omega + \int\limits_{\Omega} w(x,z,0)\rho(x,z,0)w_s(x,z,0)\ d\Omega +$$

$$\int\limits_{Q_T}\left[\rho u\,\frac{\partial u_s(x,z,t)}{\partial t} + \rho u^2\,\frac{\partial u_s(x,z,t)}{\partial x} + +\rho uw\,\frac{\partial u_s(x,z,t)}{\partial z}\right]\ dQ_T$$

$$+\int\limits_{Q_T}\left[\rho w\frac{\partial w_s(x,z,t)}{\partial t} + \rho uw\frac{\partial w_s(x,z,t)}{\partial x} + +\rho w^2\frac{\partial w_s(x,z,t)}{\partial z} + \rho g w_s(x,z,t)\right] dQ_T = 0.$$

Relations (6) do not contain pressure $p(x,z,t)$. To calculate $p(x,z,t)$, it is possible to use an individual equation giving $p(x,z,t)$ through $\mathbf{v}(x,z,t)$.

However, the above integral relations are not enough for unambiguous definition of a generalized solution. To be sure, we should add relation (4) and equation (5) in an integral form to equations (6).

We construct here the definition of a generalized solution, which should be acceptable with physical point of view and correct with mathematical point of view. From the previous reviewing, it is evidently that (6) is not enough to find unique a physically justified generalized solution. If the solution is piecewise-continuous and restricted in Q_T, then all integrals in (4) exist; and from (5), (4) it follows

$$\int_{\Omega} e\left(x,z,t\right)\ d\Omega = \int_{\Omega} e\left(x,z,0\right)\ d\Omega, \tag{7}$$

$$\int\limits_{\Omega} f(x,z,t_1)d\Omega \le \int\limits_{\Omega} f(x,z,t_2)d\Omega \quad t_1 > t_2. \tag{8}$$

Definition 2 *Let* $\boldsymbol{v}(x,z,t) = (u(x,z,t), w(x,z,t))$*. We term* $\lambda(x,z,t) = \begin{pmatrix}\rho(x,z,t)\\ \boldsymbol{v}(x,z,t)\end{pmatrix}$ *a generalized solution of equations (1), if* $u(x,z,t) = \frac{\partial\psi(x,z,t)}{\partial z}$*,* $w(x,z,t) = -\frac{\partial\psi(x,z,t)}{\partial x}$*,* $\psi|_{\partial\Omega} = 0$ *and if (7), (8), (6) are fulfilled for any* $\rho_s(x,z,t) \in C_0^\infty\left(\Omega\right) \times C^\infty[0,T]$*,* $u_s(x,z,t) = \frac{\partial\psi_s}{\partial z}$*,* $w_s(x,z,t) = -\frac{\partial\psi_s}{\partial x}$*,* $\psi_s(x,z,t) \in C_0^\infty\left(\Omega\right) \times C^\infty[0,T]$*,* $\rho_s(x,z,T) = \psi_s(x,z,T) = 0$*.*

4 Simulation of vortex destruction

We solve Euler equations (1) by finite-difference method. The numerical method developed allow simulation of nonsmooth, generalized solutions, because finite-difference equations used satisfy all conditions of (7), (8), (6), 3 at $h \to 0$ Kshevetskii (2006).

We take the initial condition:

$$\psi\left(x,z,0\right) = A\, e^{\left\{-\left[\left(\frac{x-x_0}{l_x}\right)^2+\left(\frac{z-z_0}{l_z}\right)^2\right]\right\}}, \tag{9}$$

$\rho\left(x,z,0\right) = \rho_0\left(z\right)$, where $l_x = 0.16$ m, $l_z = 0.052$ m, $A = -0.0095$ m^2/s, $x_0 = 0.5$ m, $z_0 = 0.125$ m. This initial condition approximately corresponds to the disturbance created from an oscillating paddle in the tank in McEwan's experiments McEwan (1983), McEwan (1971). The horizontal scale is approximately twice more than the vertical one. Àmplitude of the vertical velocity is of 5 cm/s, amplitude U of the level velocity is of 16 cm/s. A scale of the initial wave $l = h/(2\pi) = 0.04\,$m. The Froude number characterizes the degree of nonlinearity, $Fr = U/\left(Nl\right) = 3$, the Richardson number is $Ri = \left(-\frac{1}{\rho}\frac{d\rho}{d}\right)/\left(\frac{\partial U}{\partial z}\right)^2 = 0.1$, where $N = \sqrt{\frac{g}{H}} = 1.2\,s^{-1}$. Known necessary condition of flow instability is $Ri < 0.25$.

The fluid density for $t = 3\,$s, $7\,$s is shown in fig. 1, 2.

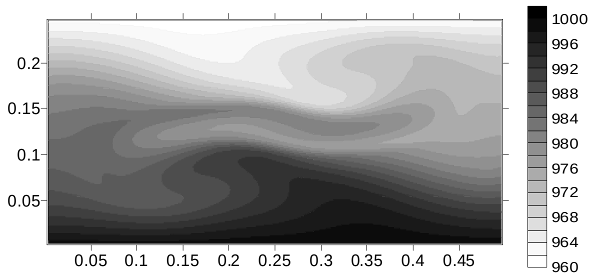

Figure 1: Fluid density at $t = 3\,$s. $H = 6.23\,$m.

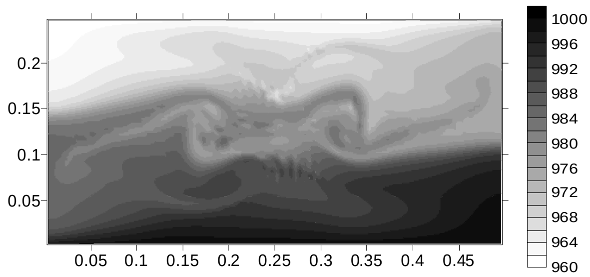

Figure 2: Field density at $t = 7\,$s. $H = 6.23\,$m.

The overturning and formation of small-scale structures is forestalled by formation of a tongue of a heavy liquid, penetrating into strata of a light liquid, and on the contrary. Some layer structure containing segments with inverse density is generated as a result. At $t = 5\,$s, abruptions arise in tongues, and isolated fragments of a light liquid arise inside a heavy liquid, and on the contrary. At $t = 7\,$s, the process of breaking becomes intensive. At $t = 9\,$s, about 10% of fluid in the tank is retracted in intensive small-scale convection. The vortex is atomized and the convective spot slowly grows. However, with the course of time, small-scale blobs of a heavy liquid subside downwards, and blobs of a light fluid go upward. As a result, stable stratification is practically restored to the moment $t = 14\,$s, but not for smooth density, and for density being averaged with the scale larger the scale of small-scale heterogeneities.

As a whole, the picture of wave breaking qualitatively coincides with circumscribed in McEwan (1971), McEwan (1983). However, the last stage of reestablishing of continuous stratification is absent. It is because the dissipative effects are not taken into account in the model.

At the first stage, the disturbance behavior is typical for internal waves and is perfectly explained by the theory of small-amplitude internal waves: left-hand and right-hand waves arise from this vortex. Because of the initial density perturbation is equal zero, the field of density perturbation is antisymmetric, and the field of a flow function is centrally symmetric. The symmetry is maintained in good approximation even when irregular movement is developed.

Irregular structures in the flow function appear later, than in the density. Probably it is explained by the fact that a flow function is as an integral of velocity, and is the most smooth of considered physical fields. Two opposite jet flows are formed at $t = 3$ s, and then they in unison create considerable velocity shift. The energy of small-scale waves is scooped from a kinetic energy of the vortex due to such a mechanism of instability development. It explains great intensity of the developed small-scale convection.

4.1 Investigation of dependence of vortex destruction on a stratification scale

Identical simulations for stratifications with doubled and diminished twice H have been carried out for investigation of dependence of phenomena on a stratification scale. Outcomes for $H = 3.1\,\mathrm{m}$ are shown in fig. 3. It is the case of $Fr = 2$, $Ri = 0.22$. We see that diminution of H essentially affects wave process. Fluid movement becomes more horizontal. The effect of wave breaking is starting later in spite of the fact that the inverse Vaisala-Brunt frequency is less. The wave breaking develops more slowly, and destruction runs languidly. Regular wave motion of the fluid is maintained as a whole, and only separate "winged nuts" are generated.

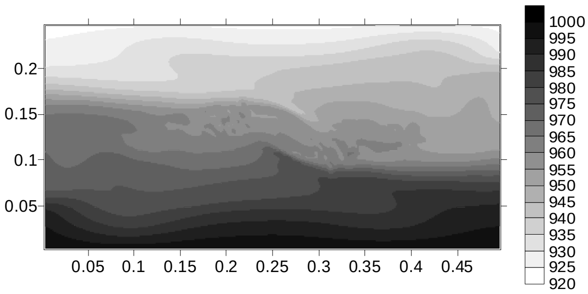

Figure 3: Fluid density at $t = 7\,\mathrm{s}$. $H = 3.1\,\mathrm{m}$.

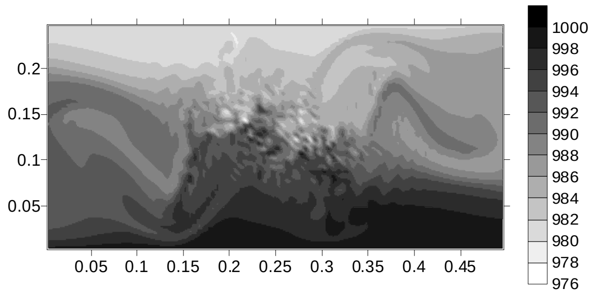

Figure 4: Fluid density at $t = 7\,\mathrm{s}$. $H = 12.4\,\mathrm{m}$.

In fig. 4 the outcomes of simulation of the same wave, but for the stratification with doubled value of H, is shown. It is the case of $Fr = 4$, $Ri = 0.1$. Comparison with simulations relating to normal value of H and with simulations relating to diminished twice H displays that the phenomenon of vortex destruction, intensity of generation of small-scale convection is increased as H.

Because we have discovered the destruction effect increases with H, we have carried out simulation of evolution of a starting vortex for stratification with very large $H = 311,5\,\mathrm{m}$. The considered medium is almost homogeneous: $\delta = \frac{\Delta\rho}{\rho} = 0.0008$. Now we consider the case of $Fr = 15$, $Ri = 0.004$. The simulation outcomes are shown in Fig. 5. We see the wave collapses, and the time of starting of wave breakdown is the same as in case $H = 12.4\,\mathrm{m}$. Therefore, for large H, the time of wave breakdown starting is determined not so much by value of H, how much by starting conditions. By considering the flow function, we discover formation of oppositely directed jet flows. Originating of this velocity shears explains, probably, the instability development. Small-scale fluctuations appear in the density field, and with some retardation, they appear in the flow function.

Some late instant of evolution of the flow function for the same initial vortex, but in the fluid of a constant density, is shown in fig. 6. We see that the whirl breakdown and formation of small-scale convection is absent.

Numerical simulations reveal that the phenomenon of breakdown becomes more and more brightly with increase of stratification scale H. Nevertheless, the phenomenon of vortex destruction is absent, when the fluid density is strictly a constant. It points out that continuous limit from a stratified fluid into the case of a fluid of constant density is absent for our initial conditions. Approximation of density by a constant is out of physical sense in such cases, even

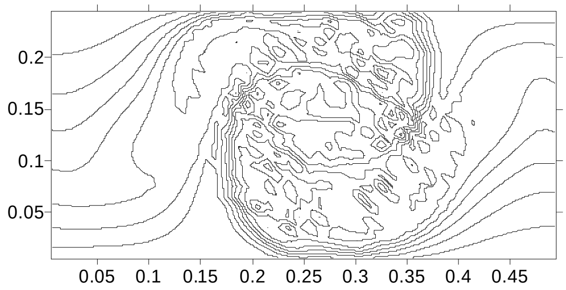

Figure 5: Lines of the flow function at $t = 7$ s. $H = 311.5$.

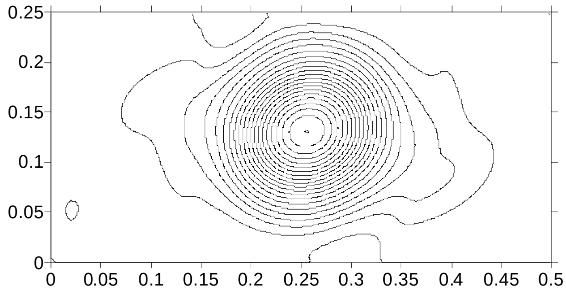

Figure 6: Lines of the flow function at $t = 7$ s in a homogeneous liquid.

if the density varies very little.

5 Conclusions

The Euler equations for an incompressible stratified fluid were studied. Non-negative nonincreasing functional extending wave energy functional of the theory of linearized Euler equations is suggested. The functional value is conserved over differentiable solutions, and diminishes, if density discontinuities are present. Functional properties have allowed using them for analysis of correctness of the generalized problem. It is shown that Euler equations are not enough for statement of a correct generalized problem for a stratified fluid. One auxiliary condition is an energy conservation law, and the second is a special condition for density (5). The statement of a generalized problem is formulated. The phenomenon of destruction of starting vortex in conditions of McEwan's laboratory experiments McEwan (1971), McEwan (1983) has been simulated and studied. Qualitative concurrence of the simulated effect with the one observed in laboratory experiments is good, but the last stage of restoring of smooth stratification is absent due to usage of an ideal fluid model. By means of numerical experiments, dependence of the solution on stratification scale H is studied. It is revealed that the effects of vortex destruction and formation of small-scale convection increases with H for strongly nonlinear waves. On the contrary, the effect weakens with diminution of H. The effect is not observed, if the fluid density is strictly a constant.

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