Numerical analysis of ideal fluid flows through plane duct of finite length

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Abstract :

The results of numerical study of the non-stationary problem in which the governing equations are the 2D Euler equations are presented. On the boundary of the duct the normal velocity is prescribed everywhere and the vorticity is given on the inflow parts. The variant of vortex particles-in-cells method is proposed and is used for PDE approximation. Firstly we have studied dynamics of different types of initial vorticity patches for classical flows (uniform, Couette and Poiseuille flows). We have found that for sufficiently big perturbations of initial vorticity a new stable separated flows (which are consist of a trough flow zone and a recirculation zone) can be realized. We present a number of examples of separated flows with different structure. Investigation of vorticity behavior in time and structures of stable separated flows are also presented.

Key-words :

flowing through problem; vortex method; ideal incompressible fluid flow

1 Introduction

Flows through a given domain with the inflow and outflow of fluid through the certain parts of the boundary arise in the enormous number of applications, for example fluid flow in pipes with fittings, fluid flow in heat exchangers, blood flow in arterial stenoses, etc. The first theoretical result for this problem was considered by Kochin (1956). He specified the normal velocity of the fluid on the entire boundary of the flow domain and the vortex at the inlet of the flow domain. Later on, Yudovich (1963) has studied two-dimensional 'flowing-through' problem and proved that it always has a unique solution, well-defined on an arbitrarily long time interval. A survey of works on the connection in a flowing-through problem has been provided in Kazhikhov *et al*. (1990). Numerical methods for the solution of the non-stationary Euler equations of an ideal incompressible fluid flow through a bounded domain with the inflow and outflow parts of boundaries have not yet been considered in detail. The present study concentrates on a numerical method for the 'flowing-through' problem in which the governing equations are the non-stationary 2D Euler equations. The vortex particle-in-cell method is based on the classical stream-function vorticity formulation of the Euler equations. This method is used for numerical analysis of vortex patches dynamics and different steady flows generation.

2 Governing equations and numerical method

We consider a two dimensional flow of an inviscid incompressible and homogeneous fluid in a rectangular duct D: $D = \{(x, y) : 0 \le x \le l; 0 \le y \le 1\}$. The advection of vorticity $\omega = \omega(x, y, t)$ is described by the equations

$$\omega_t + \psi_y \omega_x - \psi_x \omega_y = 0; \quad -\Delta\psi = \omega, \tag{1}$$

where $\psi = \psi(x, y, t)$ is the stream function. Yudovich's boundary conditions are:

$$\psi|_{y=0} = 0, \quad \psi|_{y=1} = Q, \quad \psi|_{x=0} = \psi|_{x=l} = \psi^+; \; \omega\,|_{x=0} \,= \omega^+ \tag{2}$$

Here ω^+, ψ^+ are given functions of y; $Q = const > 0$, ψ^+ is the monotonically increasing function. The initial condition is: $\omega|_{t=0} = \omega_0$.

It is very important that inviscid fluid flow in a finite duct represents a non-conservative system: there are some 'pumping-in' and 'pumping-out' of energy due to the inflow and outflow of fluid. This property simplify some numerical problems wich are typical for conservative dynamics.

For the numerical solution of (1-2), we use the vortex method, see Cottet and Koumoutsakos (1999). The flow domain D was divided into $N_b = n_x \times n_y$ rectangular cells. At the initial time moment, $N_p(n)$ particles are placed in the $n-$th cell. The total number of particles is varied up to 200000. Every particle is equipped with the vorticity according to the given initial and boundary data. The equations for the instant coordinates of the particles are:

$$\dot{x}_i = \psi_y(x_i, y_i, t), \quad \dot{y}_i = -\psi_x(x_i, y_i, t); \; i = 1, \ldots, N_p. \tag{3}$$

To integrate Hamiltonian system (3), the pseudo-symplectic integrator (Aubry and Chartier (1998)) is employed.

The stream function $\psi(x, y)$ was found from the second equation of system (1) under boundary condition (2). This boundary value problem was solved by the Galerkin method. We seek the solution in the form

$$\psi^{appr}(x, y, t) = \psi^+(y) + \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \psi_{i,j}(t) \sin\left(\frac{i\pi x}{l}\right) \sin(j\pi y), \tag{4}$$

The vorticity is interpolated for each value of t by piece-wise function:

$$\omega(x, y) \approx \sum_{k=1}^{N_{box}} \phi_k(x, y) = \sum_{k=1}^{N_{box}} \sum_{i,j=0, i+j\leq 3}^{3} a_{k,i,j} x^i y^j. \tag{5}$$

The coeffitients $a_{k,i,j}$ determine by the least squares method using the value of vorticity in particles currently located at the cell with number k. Substituting (4) into second equation of (1) and carrying out appropriate operations of projection is employed to derive and solve the Galerkin system for $\psi_{i,j}(t)$.

To evaluate the present numerical procedure and check validity of the numerical results a number of test cases was considered. The results of tests have shown that the current numerical procedures are capable of simulating 'flowing-through' problem with good accuracy. In order to test the adequacy of the calculations we considered different-order approximations and compared the results.

3 Numerical results

In this section, the numerical procedure described above is applied to the 'flowing-through' problem study. All numerical experiments was provided for duct D with length $l = 3$. The following boundary conditions was used:

$$\psi^+(y) = Q_1 y + Q_2 y^2 + Q_3 y^3, \quad \omega^+(y) = 2Q_2 + 6Q_3 y, \tag{6}$$

In this case we have stationary solution

$$\psi(x, y, t) = Q_1 y + Q_2 y^2 + Q_3 y^3, \quad \omega(x, y, t) = 2Q_2 + 6Q_3 y \tag{7}$$

of the problem (1-2) wich we call Pouseuille flow. If $Q_3 = 0$ we have Couette flow, and the case $Q_2 = Q_3 = 0$ gives the uniform flow with constant velocity Q_1. It was (see Morgulis and Yudovich (2002)) proved that these flows are asymptotically stable for small perturbations of vorticity.

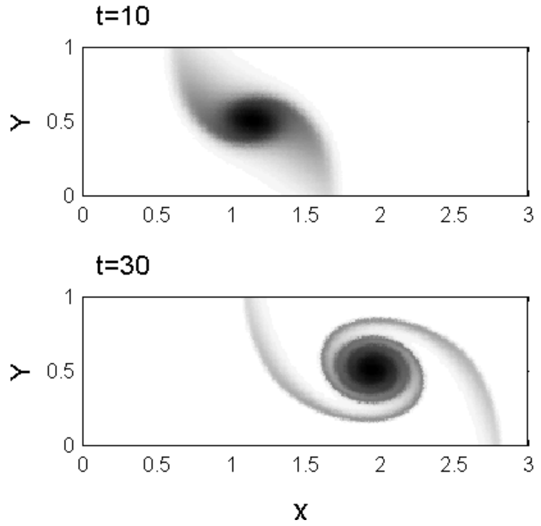

Figure 1: Vorticity distribution for different t and uniform flow with velocity $Q_1 = 0.04$ on the inlet and outlet of D. White color corresponds to minimal vorticity (zero) and black to maximal (one).

3.1 Dynamics of elleptical vortex patch

Here we study numerically dynamics of initial elleptical vortex patch

$$\omega_0(x, y) = e^{-45(x-3/4)^2 - 15(y-1/2)^2} \tag{8}$$

for different values of parameters Q_1 and $Q_2 = Q_3 = 0$. This case corresponds to uniform flow with constatnt horizontal velocity Q_1 on the inlet and outlet of the duct D.

If value of Q_1 is sufficiently big (for example $Q_1 = 0.1$) initial vortex patch (8) is quickly wash-out from the duct by uniform flow. The time of washing-out t_{out} is approximately equal to l/Q_1 in this case (for $Q_1 = 0.1, t_{out} = 30.9 \approx l/Q_1 = 30$) . It means that the initial vortex patch have not action on the velocity of the full washing-out of initial particles from the duct. As the result the asymptotic stable steady flow (7) is established. This fact completely complies with theoretical results of Morgulis and Yudovich (2002).

Decreasing of Q_1 leads to influence of initial vortex patch on full washing-out of duct. Figure 1 demonstrate the dynamics of elliptical vortex patch for $Q_1 = 0.04$. In this case the time of full washing-out of duct is $83.7 > l/Q_1 = 75$, so it takes longer time for all process. Movement of the vortex patch to the out of the duct is accompanied by attraction of the vortex to the horizontal boundary (see figure 1, t=45, 62). The vortice is retarded at the outlet, but the flow is still able to wash out them.

If velocity $Q_1 \leq 0.02$ the scenario of vortex patch dynamics changes radically (see Figure 2 for $Q_1 = 0.01$): the outlet partially rejects the vortice ($t \approx 185$); as the result the residual vorticity moves upstream practically up to inlet ($t \approx 400$). After that the patch turns and moves to the direction of outlet again (until $t = 704$), turns to the inlet and so on. These movements of the vortex patch from inlet to outlet repeat many times but with decreasing amplitude. As

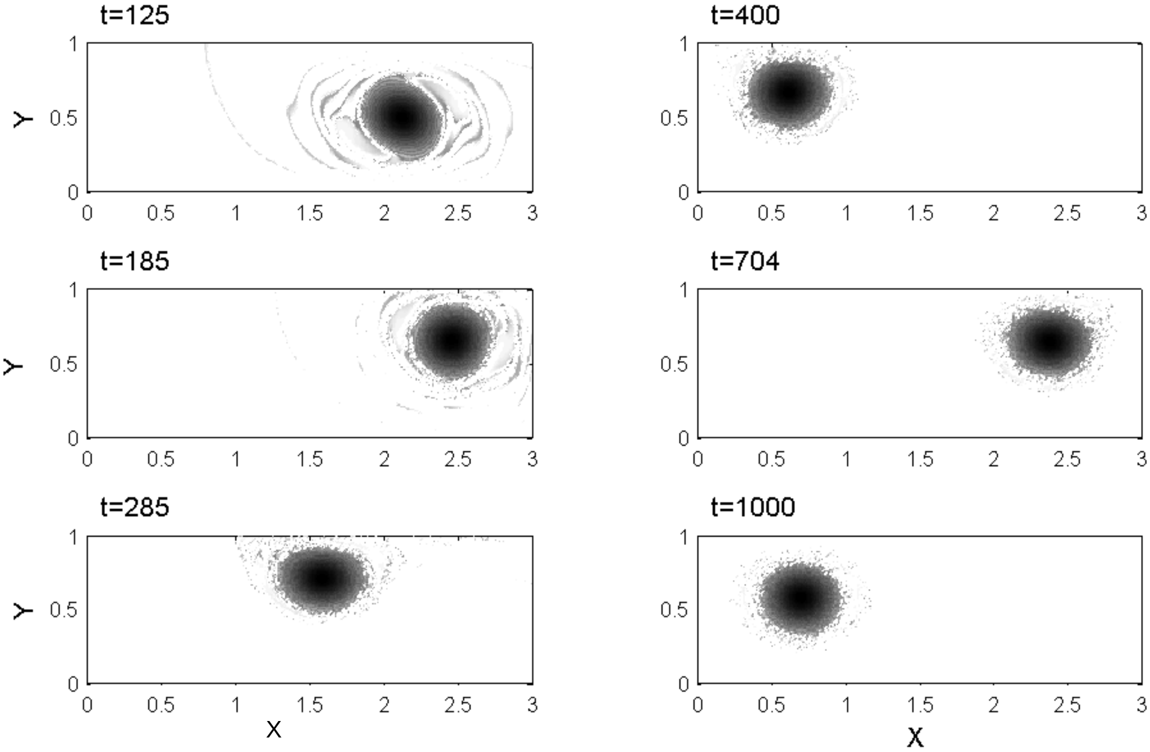

Figure 2: Vorticity distribution for different t and uniform flow with velocity $Q_1 = 0.01$ on the inlet and outlet of D. White color corresponds to minimal vorticity (zero) and black to maximal (one).

a result a big part of initial vortex patch (8) continues their stay in the duct for an uncertainly long time. For $Q_1 = 0.01$ and initial vorticity distribution (8) the resulting flow for long times t formes a single recirculation domain situated nearby the impermeable border.

3.2 Steady flows with stagnation zones

We show that initial vortex patch for sufficiently small velocity on the inlet and outlet of the duct can partially repulsed by the outlet. As a result, some residual vorticity stays in the duct for an uncertainly long time. In the next step of our investigation we concentrate our attention on the study of the vortex structures, which can be the final stages in the relaxation process of different initial vortex patches with given Pouseuille and Couette flows on inlet and outlet of the duct.

We found different stable stationary separated flows (see Figure 3). The separated flow is divided into a running zone and a stagnation zone. The first zone consists of particles that reside in the flow domain D only for a finite period of time, while particles belonging to the second zone stay there forever. The stagnation zone may consists of several domains separated by the running zone (see Figure 3).

4 Conclusions and acknowledgments

Two-dimensional vortex motion in 'flowing-through' in the finit duct has been examined numerically. On the boundary of the duct the normal velocity is prescribed everywhere and the vorticity is given on the inflow parts. For the numerical solution of this problem the variant of

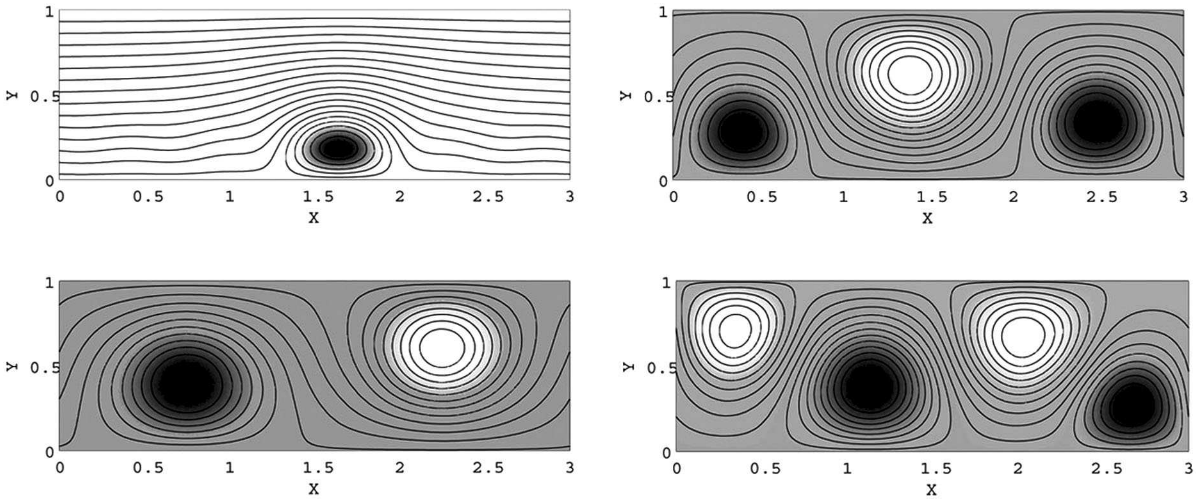

Figure 3: Vortex distribution and streamlines of steady regimes with different number of stagnation zones.

vortex particles-in-cells method was developed. Using this numerical method stationary steady flows with stagnation zones was found. The existence of complex steady vortex configurations in finite ducts with given velocity in inflow and outflow of the duct seems to be a new result in vortex dynamics. The formation of the stagnation zones near the straight walls of the duct is quite unexpected. Therefore, the calculations were additionally checked using special tests which will described in other articles.

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