

Experimental and theoretical study of the elliptic instability in a rotating stratified flow

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Abstract :

The combined effects of Coriolis force and buoyancy effects on the dynamics of a weakly elliptical bounded vortex are treated theoretically as well as experimentally. As predicted theoretically, stratification and rotation have antagonist contributions to the stability of an elliptical vortex. Thus if the stratification is strong enough ($N_b > \Omega_c$, N_b and Ω_c being respectively the Brunt-Väisälä frequency and the rotation rate of the flow in a frame rotating with the elliptical deformation at angular velocity Ω_t), we have observed that only anticyclones (such that $|W_a| < \Omega_c$ with $W_a = 2(\Omega_c + \Omega_t)$) are unstable, whereas the cyclones are always stable. In addition if the stratification is weak ($N_b < \Omega_c$), instability areas over change. These instability thresholds found theoretically have been observed experimentally with a good accuracy and the measured growth rate are in a good agreement with those predicted by a linear stability analysis in the limit of small deformation.

Résumé :

Les effets combinés de la force de Coriolis et des effets de flottabilité sur la dynamique d'un tourbillon elliptique confiné sont étudiés expérimentalement et théoriquement. Comme prévu théoriquement, la stratification et la rotation ont une contribution antagoniste sur la stabilité d'un tourbillon elliptique. Ainsi si la stratification est suffisamment forte ($N_b > \Omega_c$, N_b et Ω_c étant respectivement la fréquence de Brunt-Väisälä et la vitesse de rotation de l'écoulement dans un repère tournant avec la déformation elliptique à la vitesse angulaire Ω_t), nous observons que seuls les anticyclones (tels que $|W_a| < \Omega_c$ avec $W_a = 2(\Omega_c + \Omega_t)$) sont instables. Les cyclones étant toujours stables. Par ailleurs si la stratification est faible ($N_b < \Omega_c$), les zones d'instabilité s'inversent. Ces différents seuils d'instabilité prévus théoriquement ont été observés expérimentalement avec une grande précision et les taux de croissance mesurés sont en bon accord avec ceux calculés à l'aide d'une étude de stabilité linéaire effectuée dans la limite d'une faible déformation elliptique.

Key-words :

Elliptic Instability; rotation; stratification

1 Introduction

The stability of a weakly elliptical confined vortex in a rotating stratified fluid has been performed with local techniques such as WKB methods by several authors such as Kerswell (2002) or Miyazaki *et al.* (1992), but to the best of our knowledge this has not yet been performed with a global theory which is the purpose of the present work. This paper is organized as follows. In section 2, we focus on the theoretical study based on a global analysis in the general context. In section 3, we show some new observed resonances and we compare them with the normal mode theory. In the last section, we summarize the main results.

2 Theoretical study

In the present paper, we consider a confined vortex in solid body rotation. Variables are nondimensionalized with its radius R as a characteristic length scale and its velocity Ω_c^{-1} as a characteristic time scale. To perform our theoretical study of the linear stability of a confined weak elliptical vortex in a rotating stratified flow, we assume our vortex has a stationary strain field in the frame rotating at the global rotation rate Ω_t . So that at leading order in ε the stream function is:

$$\Psi = -\frac{r^2}{2}(1 - \varepsilon \sin 2\theta) \quad (1)$$

Following previous works on the global linear theory of the elliptical instability, we can show by taking wave solutions of the form (neutral modes):

$$(\mathbf{u}, b, p) = (u_r(r) \cos kz, u_\theta(r) \cos kz, u_z(r) \sin kz, b(r) \sin kz, p(r) \cos kz) e^{i(m\theta - \omega t)} \quad (2)$$

that the pressure is solution of the following Bessel equation of the first kind:

$$r \frac{d}{dr} \left(r \frac{dp}{dr} \right) + (\alpha^2 r^2 - m^2) p = 0, \quad (3)$$

where the axial wave number α is given by:

$$\alpha = k \sqrt{\frac{W_a^2 - \lambda^2}{\lambda^2 - n^2}} \quad (4)$$

with $W_a = 2(1 + \tilde{\Omega}_t)$, $n = \tilde{N}_b$ and $\lambda = m - \omega$ (where the tildes represent nondimensionalized variables). Note that the vertical boundary conditions $u_z|_{z=0,H} = 0$ leads to the discretization of the axial wave-number $k = m\pi R/H$, m being an integer corresponding to the number of axial half-periods. By just looking at the expression of α and the formal expression of the neutral modes which are just combinations of Bessel functions of the first kind, we can easily verify that waves only exist if:

$$|W_a| < 1 \quad \text{and} \quad n > 1 \quad \text{or} \quad |W_a| > 1 \quad \text{and} \quad n < 1. \quad (5)$$

Moreover the dispersion relation of this problem is given by the radial boundary condition $u_r|_{r=1} = 0$ and reads simply in the general case as:

$$(\lambda + W_a) J_{m-1}(\alpha) = (\lambda - W_a) J_{m+1}(\alpha). \quad (6)$$

Waleffe (1990) and many other authors found that the mechanism of instability was a triadic resonance between two neutral modes (m_1, k_1, ω_1) , (m_2, k_2, ω_2) of the undeformed flow and the underlying strain field. In the context of the study the conditions of resonance simply are:

$$m_2 = m_1 + 2, \quad k_1 = k_2, \quad \omega_2 = \omega_1. \quad (7)$$

To clarify our notations the $(-1, 1, i)$ resonance corresponds to the resonance between the two modes $m_1 = -1$ and $m_2 = 1$ while the i refers to i^{th} root of the dispersion relation (6) giving the radial structure of those modes. As we will see in the experimental study we have only observed the stationary mode $(-1, 1, 1)$ resonance for large stratification and weak rotation ($n > 1$ and $|W_a| < 1$), so that in this region, natural tendency for the simplest mode regarding

its structure to take place alone has been verified. So we focused our analysis on this resonance and derived an explicit expression for the growth rate:

$$\sigma = \sqrt{\varepsilon^2 \sigma_{in}^2 - (\text{Im}(s_v) + A\Delta k)^2} - \frac{(\alpha^2 + k^2)B}{Re} - \frac{\text{Re}(s_v)}{\sqrt{Re}} \quad (8)$$

(see Figure 1), where σ_{in} is the inviscid growth rate, $Re = \Omega_c R^2 / \nu$ is the Reynolds number, $\text{Im}(s_v)$ and $\text{Re}(s_v)$ represent the imaginary and real part of the viscous frequency shift due to boundaries and calculated for $m = -1$, Δk represents the detuning of the axial wave number and every terms (also given in appendix) have explicit formulations depending on (α, n, W_a, k) . We can remark that s_v represent the viscous damping due to the presence of boundaries. Note finally that α is a function of W_a from (6) because $\omega = 0$ for the $(-1, 1, 1)$ resonance.

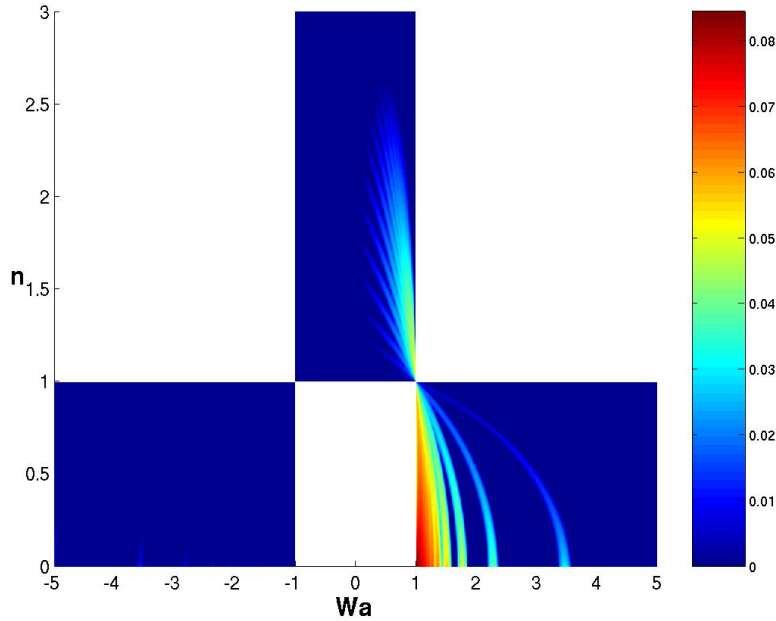


Figure 1: Contours of the viscous growth rate given by (8) of the elliptical instability mode $(-1, 1, 1)$ determined by global analysis as a function of the adimensionalized Brunt-Väisälä frequency n and the absolute vorticity W_a for a given radius $R = 2.75\text{cm}$, height $H = 19\text{cm}$, Brunt-Väisälä frequency $N_b = 2.70\text{rad/s}$ and eccentricity $\varepsilon = 0.085$. Note that the Reynolds number increases as n decreases.

Moreover the inviscid growth rate is explicitly given by:

$$\sigma_{in} = \left| \frac{(n^2 - 1)(W_a + 1)^2(\alpha^2 + (W_a - 1)^2)}{4(W_a - 1)^2(W_a + 1)(n^2 + W_a) + 4(W_a^2 - n^2)\alpha^2} \right| \quad (9)$$

which generalizes previous ones. The WKB result given by Kerswell (2002) is recovered when k tends to infinity, whereas for finite k Waleffe (1990) result in the absence of stratification ($n = 0$) and global rotation ($W_a = 2$) is obtained.

3 Experimental study

In the experiments we used a rotating deformable and transparent cylinder (with angular frequency Ω_c) filled with a solution of linear stratified salted water (with Brunt-Väisälä frequency

N_b and experimentally obtained with the classical two buckets method) which is elliptically deformed by two rollers. Moreover the all set-up (see Figure 2) is fixed on a rotating table of angular frequency Ω_t . The experimental control parameters are the Reynolds number based on the angular frequency of the cylinder Ω_c , the Brunt-Väisälä frequency N_b and the absolute vorticity of the flow $W_a = 2(1 + \hat{\Omega}_t)$.

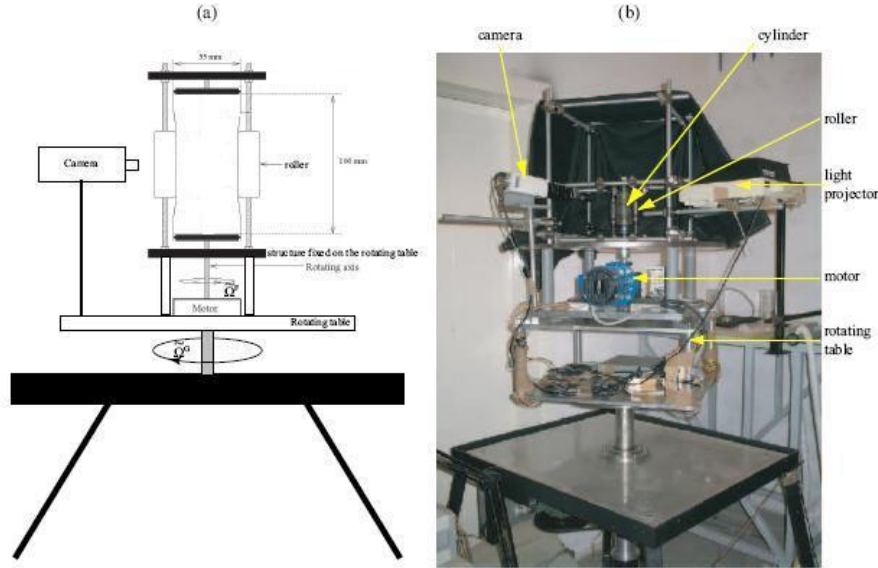


Figure 2: Experimental set-up with the deformed cylinder placed on the rotating table.

We performed a series of experiments with a constant height $H = 19\text{cm}$ and Brunt-Väisälä frequency N_b around 2.70rad/s systematically changing W_a . We also changed the eccentricity from 0.085 to 0.179. New observations were made. First we saw the $(-1, 1, 1)$ resonance in the high stratified region ($n > 1$ and $|W_a| < 1$). It is in a good agreement with theory which predicts that this mode is always dominant except in a few narrow bands. The threshold in this region has been well observed and for $|W_a| \sim 1$ or $n \sim 1$ the instability disappears. We also observed the $(-1, 1, 1)$ resonance in the region of weak stratification ($n < 1$ and $|W_a| > 1$) and we also observed other resonance such as the $(0, 2, 1)$ one whereas in the high stratified region the only $(-1, 1, 1)$ resonance has been observed. The different thresholds were observed too in this region. Secondly the number of wave-length predicted by the theory in this region is in a very good agreement with the observations presented in Figure 3. However, we never saw instabilities in the region near $W_a = 0$ in contradiction with the theoretical predictions.

4 Conclusions

In this paper we presented briefly the global theory and shown some new analytical results about the growth rate of the $(-1, 1, i)$ resonance of the elliptical instability. These results recover some results already found with other techniques such as the WKB method or finite axial wave-number in the unstratified case. Moreover the shift in frequency of the growth rate due to boundaries has been calculated in the general case of a rotating stratified fluid for a cylinder. One can recover Kudlick (1966) or Kerswell *et al.* (1995) formulas for the special case $n = 0$. Furthermore the theoretically predicted regions where the elliptical instability occurs and its thresholds ($n < 1$ and $|W_a| > 1$ or $n > 1$ and $|W_a| < 1$) have been observed experimentally and are in good agreement with the theory. We can notice that in the region of high stratification

($n > 1$ and $|W_a| < 1$) the main resonance and the only resonance observed was the $(-1, 1, 1)$ one. This behavior is similar to the one noticed by Le Bars *et al.* (2007) for the destruction of elliptical anticyclones in the non stratified case.

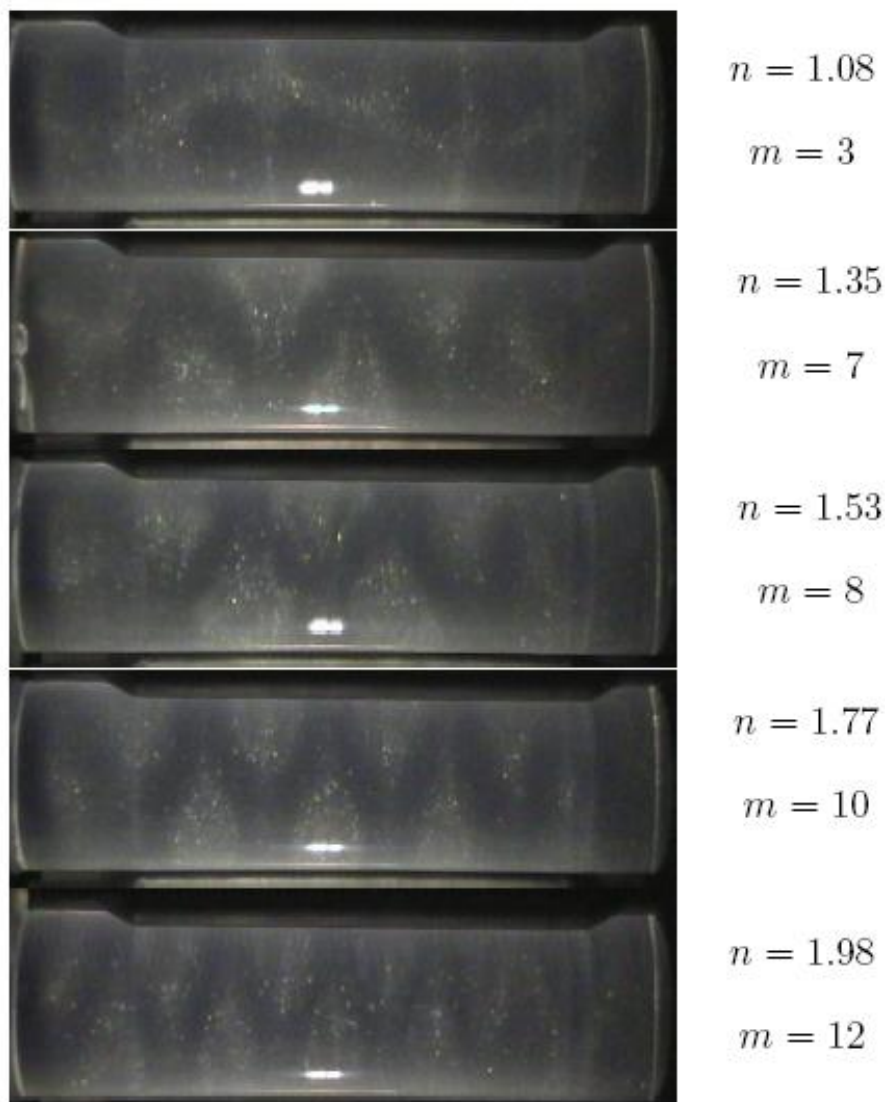


Figure 3: Variation of the wave-length of the $(-1, 1, 1)$ mode for a given cylinder of radius $R = 2.75\text{cm}$, height $H = 19\text{cm}$ and eccentricity $\varepsilon = 0.085$ filled with stratified salty water of Brunt-Väisälä frequency $N_b = 2.68\text{rad/s}$ for a fixed absolute vorticity $W_a = 0.7$. From the top to the bottom the value of n and m which is the number of axial half wave length increase.

5 Appendix. Mathematical expressions

$$A = \frac{(1 - n^2)(\alpha^2 + W_a^2 - 1)}{k(\alpha^2 + k^2 + (W_a - 1)(W_a + n^2))}$$

$$B = 1 - \frac{\alpha^2 n^2 (W_a^2 + \alpha^2 - 1)}{2(\alpha^2 + k^2 + (W_a - 1)(W_a + n^2))}$$

The viscous correction due to the presence of boundaries has two contributions. We write $s_v = I_r + I_z$ where I_r is the contribution of the surface $r = 1$ of the cylinder and I_z is the contribution of the surfaces $z = 0$ and $z = H$. The two expressions of I_r and I_z have been calculated using a classical approach based on the hypothesis that the effect of viscosity is taken into account in boundary layers of magnitude $Re^{-\frac{1}{2}}$. So we obtained:

$$I_{1r} = C \left[(1 - i) + (1 + i \operatorname{sgn}(n^2 - 1)) \frac{k^2}{|n^2 - 1|^{\frac{3}{2}}} \right]$$

$$I_{1z} = C \left[\frac{(\alpha^2 + (W_a - 1)^2)(1 + i \operatorname{sgn}(W_a - 1))}{|W_a - 1|^{\frac{3}{2}}} + \frac{(\alpha^2 + (W_a + 1)(W_a - 3))(1 - i \operatorname{sgn}(1 + W_a))}{|W_a + 1|^{\frac{3}{2}}} \right] \frac{R}{H}$$

with

$$C = \frac{(W_a^2 - 1)(1 - n^2)}{\sqrt{2}(\alpha^2 + k^2 + (W_a - 1)(W_a - n^2))}$$

References

- Kerswell, R.R. 2002 Elliptical Instability. *Ann. Rev. Fluid Mech.* **34** 83-113
- Kerswell, R.R., Barenghi, C.F. 1995 On the viscous decay rates of inertial waves in a rotating circular cylinder. *J. Fluid Mech.* **285** 203-214
- Kudlick, M. 1966 On the transient motions in a contained rotating fluid. *PhD thesis, MIT.*
- Miyazaki, T., Fukumoto, Y. 1992 Three-dimensional instability of strained vortices in a stably stratified fluid. *Phys. Fluids A* **4** 2515-2522
- Waleffe, F. A. 1990 On the three-dimensional instability of strained vortices. *Phys. Fluids* **2** 76-80
- Le Bars, M., Le Dizès, S., Le Gal, P. 2007 Coriolis effects on the elliptical instability in cylindrical and spherical rotating containers. *Under consideration for publication in J. Fluid Mech.*