

## Chaos and nonlinear resonances in the problems of geophysical fluid dynamics

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### Abstract :

*In the framework of background currents, we examine a dynamically consistent model of a periodic flow over an isolated submerged obstacle of Gaussian shape. Chaotic advection of passive scalars in the ocean is studied from the viewpoint of the dynamic system theory. Much attention is given to the nonlinear resonance role. The relationship between the rotation frequency of an unperturbed fluid particle and the appearance and disappearance of a nonlinear resonance is established. So, the frequency gives us the opportunity to explain the trajectories' chaotization via the Chirikov criteria. This simple estimation is verified by numerical calculations. A simplified model is used for more exact reasoning. The distance between nonlinear resonance and its width was established for large action, and accurate estimation of chaotic region boundary was calculated. Also, numerically and analytically, there are studied chaotization dependencies from some other flow parameters.*

**Key-words :** vortex flow; background currents; chaotic advection

### 1 Introduction

In this paper, in the framework of the conception of background currents on the  $f$ -plane (Kozlov 1995), we examine a dynamically consistent model of a periodic flow over an isolated submerged obstacle of Gaussian shape. The conception gives us opportunity to obtain models in which we have no problem with conservation of potential vorticity (Brown and Samelson 1994, Izrailsky et al 2004, Koshel and Prants 2006). The most popular way for studying the effects of dynamical chaos is the Melnikov method (Wiggins 2005) for the case of a small perturbation and numerical modelling. But we see that the nonlinear resonance analysis (Zaslavsky 1998, Koshel and Stepanov 2006) can give a good result in some aspects of dynamical chaos.

### 2 Formulation of the problem

The stream function may be written in the following form (Sokolovskiy et al 1998, Davies et al 2006):

$$\Psi = (1 + \mu \sin \omega_0 t) [(B \sin \Omega t + v_0)x - (A \cos \Omega t + u_0)y] + \int_0^r V(\rho) d\rho,$$

where  $A$ ,  $B$ ,  $\omega_0$ ,  $\Omega$ ,  $u_0$ ,  $v_0$ ,  $\mu$  are constants,  $r = \sqrt{x^2 + y^2}$ ,  $t$  - time,  $x$  and  $y$  - horizontal coordinates, directed to the East and North, respectively,  $V(r)$  is a radial distribution of azimuthal velocity. For an obstacle of Gaussian form, the velocity profile is

$$V = -\frac{\sigma}{r} \int_0^r h(\rho) \rho d\rho = \frac{\sigma}{2\alpha r} (e^{-\alpha r^2} - 1),$$

where  $\sigma \sim O(1)$  is a topographic parameter, and  $\alpha \approx 1.256$  has been chosen due to the condition  $dV(r)/dr|_{r=1} = 0$ .

Setting the average values for the ocean depth and current velocity equal to 4 km and 10 cm/s, respectively, we obtain  $\sigma = 3.511$ , what corresponds to the obstacle height of 1.021 km. In the general case the velocity vector of such model periodical current circumscribes an ellipse displaced from the obstacle center. Below we will show that any external disturbance will cause chaotization of trajectories of fluid particles. It is suitable to analyze the motion of a fluid particle in a coordinate system  $(x', y')$  which rotates with the current. The corresponding change of variables implies the following expression for the stream function

$$\Psi' = -Ay' + \delta \Psi(x', y', t) + \int_0^r V'(\rho) d\rho, \quad V'(r) = V(r) - \dot{\theta} r.$$

The non-stationary part of the stream function, which we assume to be a disturbance, has the form:

$$\delta \Psi = -A\mu y' \sin \omega_0 t + (1 + \mu \sin \omega_0 t) \left\{ x' \left[ (B - A) \sin \Omega t \cos \Omega t + v_0 \cos \Omega t - u_0 \sin \Omega t \right] - y' \left[ (B - A) \sin^2 \Omega t + v_0 \sin \Omega t + u_0 \cos \Omega t \right] \right\}.$$

When the disturbances are absent, the stream function is stationary, and, correspondingly, the fluid-particle trajectories are regular. In the rotating system of coordinates, the equations for a fluid particle can be written as:

$$\begin{aligned} \frac{dx'}{dt} &= -\frac{\partial \Psi'}{\partial y'} = A - V'(r) \frac{y'}{r} - \frac{\partial(\delta \Psi)}{\partial y'} \\ \frac{dy'}{dt} &= \frac{\partial \Psi'}{\partial x'} = V'(r) \frac{x'}{r} + \frac{\partial(\delta \Psi)}{\partial x'} \end{aligned}$$

### 3 Stationary case

Due to the numerical simulation we can examine some different scenarios of chaotization of fluid-particle trajectories. It is possible to understand all the effects by analyzing the rotational frequency as a function of the initial particle position.

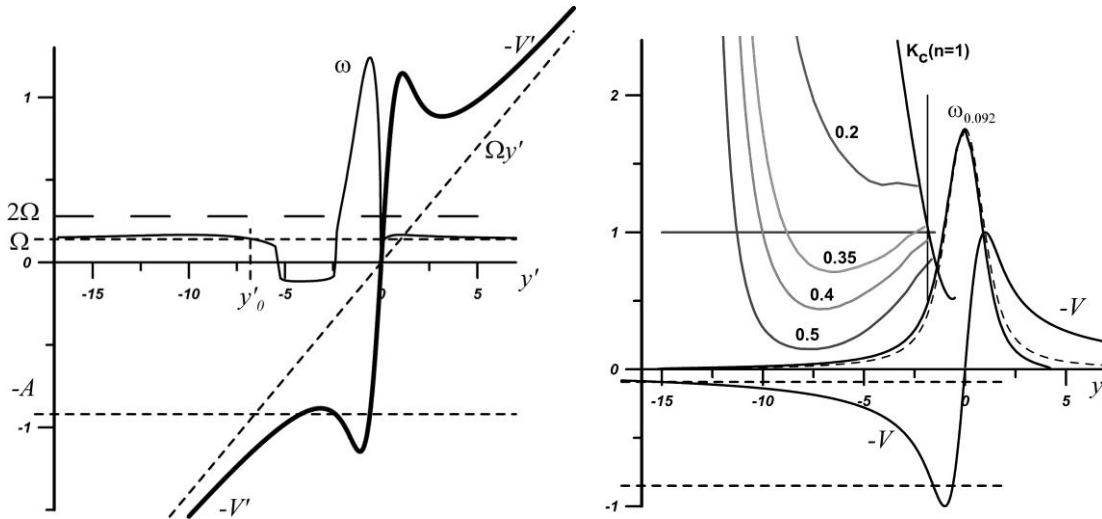


Figure 1. Azimuthal velocity, rotational frequency, and nonlinear resonance overlap criterion (only for nonrotational case). For  $\Omega = 0.14$  (left) and  $\Omega = 0$  (right).

On Figure 1 we can see the azimuthal velocity distribution in the rotating coordinate system for  $A = B$ ,  $\theta(t) = \Omega t$  (left), and without rotation for  $A = B = 0 = v_0 = 0$ ,  $\theta(t) = 0$  (right). The rotation frequency distribution  $\omega$  also presented here.

The analysis of these curves gives us important information about possible chaotic regime. The behavior in the inner region of the left figure was well studied in (Koshel and Stepanov 2006). The most interesting is the outer region. We can see that the rotation frequency  $\omega$  has a maximum near  $y' = -10$ . The other important feature is a very small frequency range. In the case of  $\Omega = 0.14$ , when  $\omega_{\max} = 0.16699$ , it's easy to estimate the frequency range:

$$0.595 = \frac{\omega_{\max}}{2\Omega} > \frac{m}{n} = \frac{[\omega]}{[\omega_0]} > \frac{\Omega}{2\Omega} = 0.5,$$

and the appropriate nonlinear resonance multiplicity. Here  $m$  is the multiplicity and  $n$  is the resonance order.

As it will be shown below, the nonlinear resonances with high multiplicity have very small width, and we can conclude that a chaotic region will appear near the maximum of rotational frequency, and we can expect reconnection with the inner and the outer resonances of the same multiplicity. We observe such effects at the Poincaré sections, which were calculated for the appropriated perturbation frequencies.

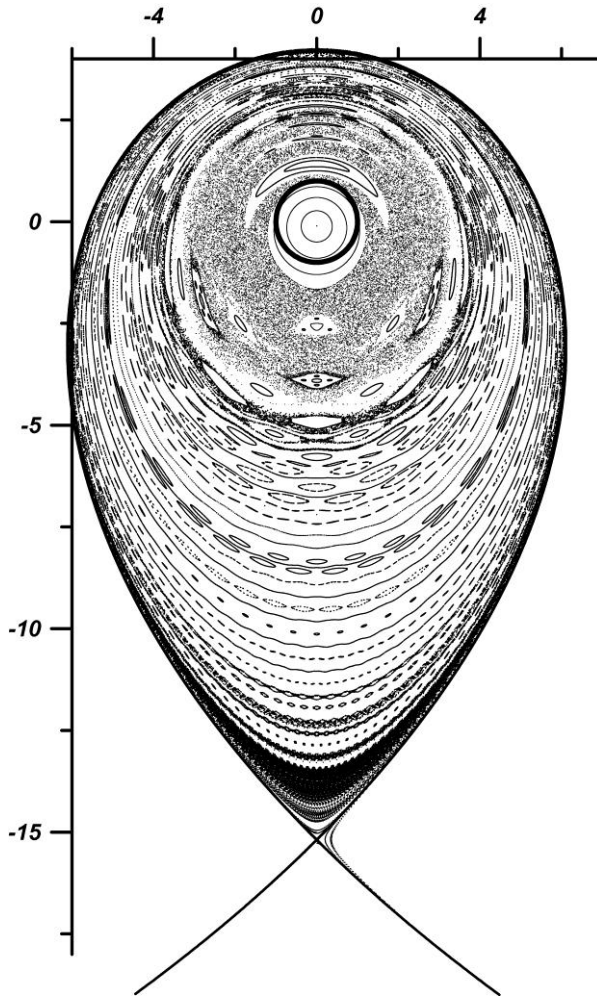


Figure 2. Poincaré sections at  $u_0 = 0.092$  and  $\omega_0 = 0.5$ .

#### 4. Asymptotic estimation of the Chirikov criterion

In the case of a non-rotational background current (right Figure 1) it's possible to obtain an asymptotic estimation which demonstrates the role of nonlinear resonances in the chaotic effects. In the case  $\theta(t) = 0$  we will consider only one perturbation  $A = B = v_0 = 0$ , and a small velocity  $u_0 \neq 0$ .

As a zero approximation with respect to  $u_0$ , the stream function depends only on  $r$ , and it is easy to obtain an action-angle variable  $(I, \varphi)$ , where  $\varphi$  is a polar angle, and  $I = r^2/2$ . In this approximation the frequency of a fluid particle rotation is

$$\omega(I) = \frac{V(I)}{\sqrt{2I}}.$$

The last relationship is also shown in Figure 1 (right), as a dotted line.

So we can use a "pendulum approximation" (Zaslavsky 1998) to estimate each of the main series nonlinear resonance width and position. The  $n$ -th resonance width is

$$\Delta\omega(J_1) \sim \sqrt{u_0\mu} \sqrt{\frac{J_1}{2} \frac{d\omega(J_1)}{dJ}}$$

and the distances between the centers of the neighboring resonance areas are

$$\omega(J_1 + \delta J) \approx \frac{\omega_0}{2} \approx \frac{\omega(J_1)}{2},$$

where the first order action was used. The n-th resonance angle width can be written as

$$\Delta J_n = \sqrt{u_0^n \mu \Phi_n(\mu) \frac{d^{n-1} \omega(J_n)}{dJ^{n-1}} \left[ \frac{\sqrt{2J_n}}{\omega(J_n)} \right]^n 2^{n-1} n! \frac{d\omega(J_n)}{dJ}},$$

and the angle distance is

$$|\delta J| \sim |2\omega(J_n)(n+2) \frac{d\omega(J_n)}{dJ}|.$$

Now we will introduce a modified Chirikov criterion (Zaslavsky 1998), which is constructed as a ratio between n-th resonance width and n-th to (n+2)-th distance

$$K_c = \frac{\Delta J}{2\delta J} \sim \sqrt{(n+2)^2 u_0^n \mu \Phi_n(\mu) \frac{d^{n-1} \omega(J_n)}{dJ^{n-1}} \frac{d\omega(J_n)}{dJ} \left[ \frac{\sqrt{2J_1}}{\omega(J_n)} \right]^n 2^{n-1} n! (2\omega(J_1))^2}.$$

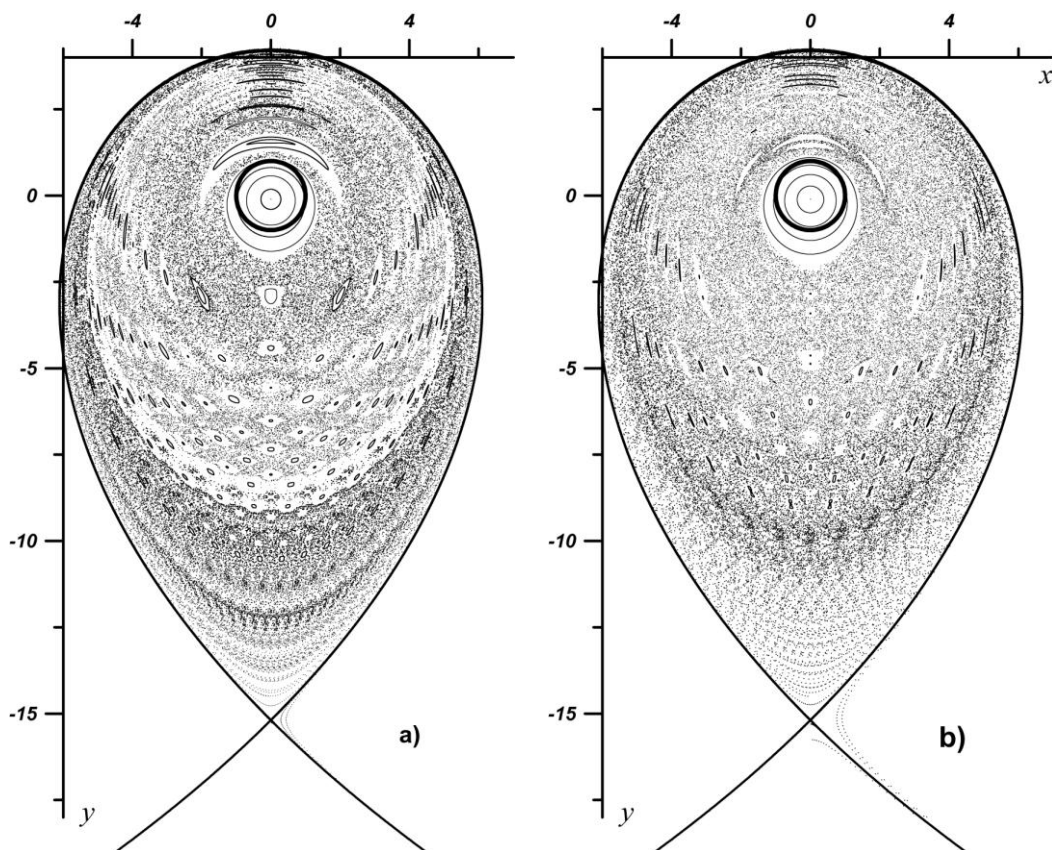


Figure 3. Poincaré sections at  $u_0 = 0.092$  and  $\omega_0 = 0.4$  - a);  $\omega_0 = 0.35$  - b).

The criterion  $K_c(r(n))$  is shown in Figure 1 (right) for few values of the perturbation frequency. Now we can examine the vortex area chaotization by analyzing these curves. For example, there are represented in Figure 2 the Poincaré sections at  $u_0 = 0.092$  and  $\omega_0 = 0.5$ . In Figure 1 we can see that the value of  $K_c$  is very small from  $y = -5$  to  $y = -10$ , and a regular region appears in this area at the Poincaré sections. The perturbation frequency value 0.4 corresponds to  $K_c \approx 0.5$  in the point of a minimum, and it gives us a partial overlapping of the nonlinear resonances. At the appropriate Poincaré (Figure 3, left) section we can see a small regular area near  $y = -8$ . This area is a barrier for marker transport from the central to the outer region.

The next  $K_c$  curve ( $\omega_0 = 0.35$ ) has a minimum value  $K_c \approx 0.8$ , which is close to the critical value 1. In this case all nonlinear resonances were partially destroyed, and we have particle transport from the inner to the outer region.

#### 4 Conclusions

We note that the evolution of nonlinear odd-multiplicity resonances seems to be a characteristic property of this dynamical system. Even resonances, which arise at some values of parameters, disintegrate much quickly. In the examples discussed above we demonstrated only some characteristic properties of the chaos initialisation and nonlinear resonances in a barotropic non-stationary flow over a submerged hill with non-zero parameters  $\mu, u_0$ . The effect of influence on the system of other external parameter both separately and under their common action will be given in a more detailed study.

The area of the topographic vortex, as a well mixed spot, may be much bigger than the non-perturbed region. Also it may have a satellite vortex (nonlinear resonances), or a complex structure, which is defined by main series of the nonlinear resonances.

The main features of the vortex area chaotization are well defined by nonlinear resonance widths and distributions. So, the particle mixing and the property transport were established by the main series of the nonlinear resonances.

The mentioned above main futures of the phase space chaotization are defined by the frequency of the particle rotation. The same qualitative effects occur in many vortex systems with analogous rotation frequency dependencies.

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