Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

Deformation of surfaces in 2D persistent homology

Tesi di laurea in Topologia Computazionale

Relatore:
Chiar.mo Prof.
Patrizio Frosini

Presentata da:
Francesca Tombari

III Sessione
Anno Accademico 2016/2017

Contents

Introduction

One of the greatest challenge of our time is to organize and analyze the huge amount of data collected in many scientific fields. In this context topological data analysis (TDA) has an important role. Its main goal consists in analyzing big datasets by means of topological tools. A common way used in this field to represent big datasets is the one of a cloud of points. The interpretation of a cloud of points containing data requires a multilevel analysis as the one used in persistent homology. The idea of it is that the topological properties which persist at different levels are relevant. Persistent homology tries to construct a bridge between topology and geometry, using homology groups, a central tool in algebraic topology, to study shapes.

In Figure 1 we can follow the evolution at different levels of a cloud of points. In this figure a cloud of points is represented, all the points are the centers of disks having different radius at every subfigure. In this case the relevant topological properties are the ones which persist during the increase of the radius.

Another example of the use of persistent homology is shown in Figure 2. Here the sublevel sets of a topological space $X \subseteq \mathbb{R}^3$ are considered varying the height. We notice that the homology group in degree one becomes zero when the height exceeds the value e.

Therefore, the theory of persistent homology is based on the study of homology groups of different degrees of the sublevel sets of a continuous function, called filtering function, informally speaking, the homology of degree k shows the k-dimensional holes of the object.

More precisely, the filtering function, f, is defined generally on a topological space and takes values in $\mathbb{R}^m$, where m is the number of properties

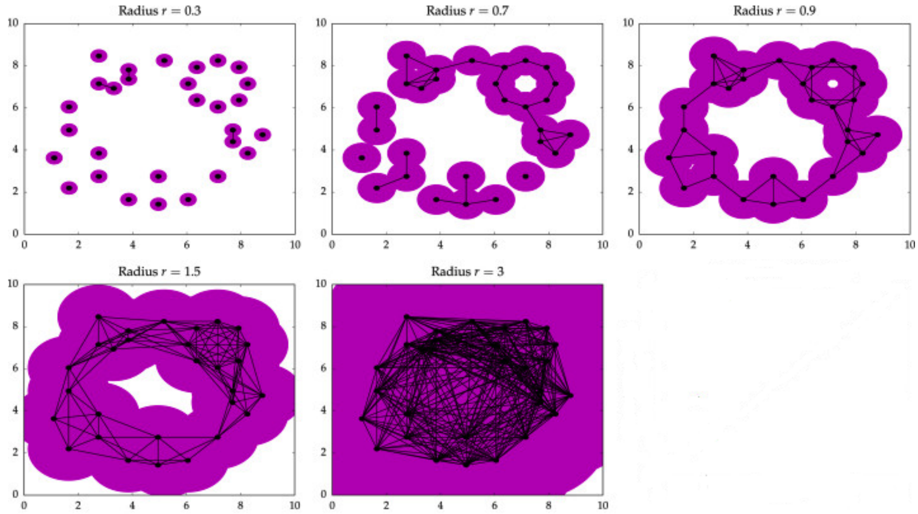

Figure 1: Disks centered in a cloud of points with increasing radius. (Image courtesy of the authors of [8])

we want to study simultaneously. In a cloud of points containing data each point represents different measurements on the sample; shape comparison of medical images also needs $\mathbb{R}^m$-valued filtering functions. When $\mathbb{R}^m$-valued filtering functions are involved, we speak of m-dimensional persistent homology. In the context of multidimensional persistence our objects of study will be continuous filtering functions with values in the real plane and defined on a smooth closed surface embedded in $\mathbb{R}^3$.

Persistent homology is proved to be much harder to study in the $2D$ case than in the $1D$ case. These difficulties require the development of new ideas and techniques. One of these new techniques is presented in [2], where the main idea is to take back our setting from the 2-dimensional to the 1-dimensional case by means of a specific family of functions depending on two parameters $(a, b) \in]0, 1[\times\mathbb{R}$. In particular if $f = (f_1, f_2) : X \longrightarrow \mathbb{R}^2$ is our filtering function, we can associate it with the family $f_{(a,b)} : X \longrightarrow \mathbb{R}$, where $f_{(a,b)}(x) := \max\left\{\frac{f_1(x)-b}{a}, \frac{f_2(x)+b}{1-a}\right\}$. Notice that every such a function can be associated with a line of the real plane having positive slope, denoted by $r_{(a,b)}$ and defined by the parametric equation $(u, v) = (at+b, (1-a)t-b)$. By means of this semplification it is possible to consider many 1-dimensional filtrations of the topological space X depending on (a, b), described by the

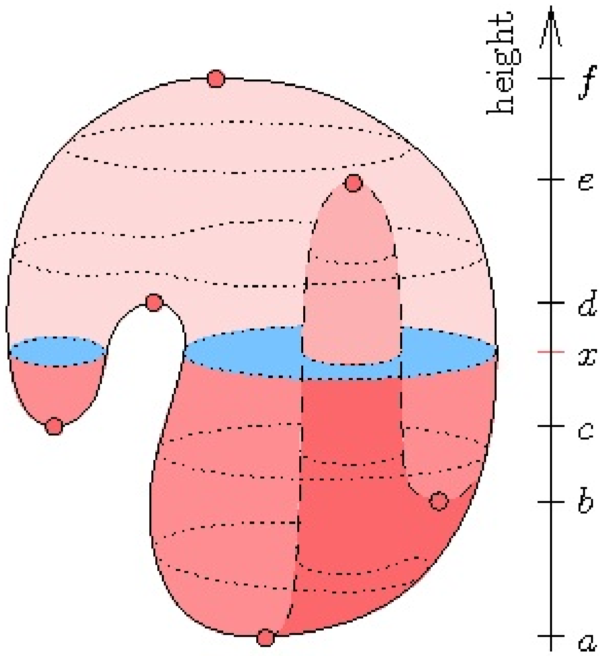

Figure 2: The sublevel sets of a topological space X. (Image courtesy of the authors of [6]

sets $\{x \in X \mid f_{(a,b)}(x) \leq t\}$, instead of one 2-dimensional filtration induced by f, i.e. $\{x \in X \mid f(x) \preceq (u,v)\}$. For technical reasons the function $f_{(a,b)}$ has to be normalized by multiplying $f_{(a,b)}$ by $\min\{a, 1-a\}$, and the new function is denoted by $f^*_{(a,b)}$. Figure 3 shows a filtration associated with a line having positive slope.

After fixing $k \in \mathbb{N}$, each filtration associated with the function $f^*_{(a,b)}$ defines a persistence diagram $\mathrm{Dgm}(f^*_{(a,b)})$ in degree k, while the collection of all 1-dimensional persistence diagrams is called a 2-dimensional persistence diagram. It encodes some topological properties of the filtered topological space we want to study.

An important application of persistent homology is shape comparison. In this context persistent homology offers a method to compare different filtering functions and to measure “distances” between them. A common way used to compare $2D$ persistence diagrams, $\{\mathrm{Dgm}(f^*_{(a,b)})\}_{(a,b)\in]0,1[\times\mathbb{R}}$ and $\{\mathrm{Dgm}(g^*_{(a,b)})\}_{(a,b)\in]0,1[\times\mathbb{R}}$, is to compute the supremum of the classical bottle-

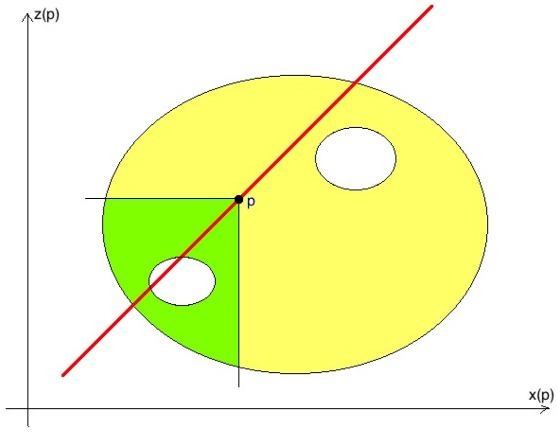

Figure 3: This picture shows the projection on the plane xz of a topological space $X \subseteq \mathbb{R}^3$. The red line has positive slope in $\mathbb{R}^2$ and induces a filtration on X. The green section is the sublevel associated with the point p of the line.

neck distance between $\mathrm{Dgm}(f^*_{(a,b)})$ and $\mathrm{Dgm}(g^*_{(a,b)})$ over (a, b). This metric is called *matching distance*, D_{match}. While the matching distance has a natural and simple definition, unfortunately it presents two main problems. First, the optimal matchings can greatly change when (a, b) changes. Secondly, the intrinsic discontinuity in the definition of matching distance makes studying its properties difficult.

For these reasons, a new metric for comparing $2D$ persistence diagrams has been introduced in [5] and further analyzed in [4] named the *coherent matching distance*. The main change of this distance with respect to the previous one is that it takes into account only matchings that change "coherently" with the filtrations.

The study of the coherent matching distance brings to light a phenomenon of monodromy. This comes out when a pair (a, b) at which a persistence diagram contains multiple points, called singular pair, is considered. Furthermore, turning around a singular pair can produce a permutation of the points of the persistence diagram, so that a link between the considered filtering function and a monodromy group appears.

In [4], some assumptions are made in order to define and study the coherent matching distance. The main assumption is to consider only a subset of continuous filtering functions with values in $\mathbb{R}^2$ and defined on a closed smooth manifold M of dimension n, i.e. the subset of the so called *normal functions*. This kind of functions are defined by four properties depending on the *extended Pareto grid*, a geometric construction in the real plain. The extended Pareto grid is a collection of arcs and half-lines of $\mathbb{R}^2$ intersecting each other and depending on the images of some arcs in M by the filtering function $f = (f_1, f_2)$. The idea is to consider only the images by f of the arcs on M at which the gradients of f_1 and f_2 are opposite, and half-lines starting at the image by f of critical points of f_1 and f_2. We refer to the arcs of the grid as *proper contours*, and to the half-lines as *improper contours*. The definition of the *Jacobi set* of an $\mathbb{R}^2$-valued function is also based on the linear dependence of the gradients: if they are linearly dependent and opposite at a point, this point will be called a *critical Pareto point*. As a consequence, the points of the extended Pareto grid belongs to the Jacobi set. Sets of these points were previously studied in the context of Morse theory for two functions. We refer the interested reader to [10] and [3].

The research presented in this thesis starts at this point with the question "Is the property of being normal for a filtering function generic in the set of continuous functions?" In order to face this problem, we consider a particular case of filtering function $f : M \longrightarrow \mathbb{R}^2$, defined by setting $f(p) = (x(p), z(p))$, where M is a closed smooth surface embedded in $\mathbb{R}^3$, since in this setting it is possible to make use of the classical geometry of surfaces. In this setting, we present a technical result for modifing the extended Pareto grid of a filtering function. The main idea is to deform the surface instead of the function on the surface, while preserving its smoothness. Furthermore, the diffeomorphism from the initial to the final surface will be local: in plain words it creates a "bump" on the surface. We hope the developed tool can be useful for approximating filtering functions with normal functions, avoiding situations in which at least one of the four properties fails. We also hope our research can pave the way to a proof for the genericity of being normal for a function in the general case, remaining aware that probably further ideas will be necessary.

The outline of this thesis is as follows. In the first chapter the mathematical setting is presented, focusing on the definitions of admissible line, Jacobi set and critical Pareto point, then we will construct the extended Pareto grid with its proper and improper contours, and we will finally give the definitions of normal function and contour-arc. The second chapter presents our main result, that is a technical theorem used for “moving” proper contours of a specific filtering function on a surface.

Introduzione

Una delle più grandi sfide del nostro tempo è quella di organizzare e analizzare l'ampia quantità di dati raccolti nei diversi ambiti scientifici. In questo contesto l'analisi topologica dei dati (TDA) ricopre un ruolo significativo. Il suo principale obiettivo è quello di analizzare grandi insiemi di dati per mezzo di strumenti topologici. Un modo molto comune in quest'ambito per rappresentare grandi insiemi di dati è quello di una nube di punti. Per interpretare l'informazione contenuta in una nube di punti è richiesta un'analisi multilivello come quella usata in omologia persistente. L'idea alla base di ciò è quella di considerare come rilevanti quelle proprietà che si mantengono in diversi livelli. L'omologia persistente cerca di costruire un ponte tra topologia e geometria utilizzando i gruppi di omologia, strumenti centrali in topologia algebrica, per studiare le forme geometriche.

Nella Figura 4 possiamo seguire l'evoluzione di una nube di punti a diversi livelli. In tale figura è infatti rappresentata una nube di punti, tutti i punti presenti sono i centri di dischi aventi diverso raggio in ogni sottofigura. In questo caso le proprietà topologiche rilevanti sono quelle che persistono all'aumentare del raggio.

Un altro esempio di utilizzo dell'omologia persistente viene mostrato in Figura 5. Qui sono considerati i sottolivelli di uno spazio topologico $X \subseteq \mathbb{R}^3$ al variare dell'altezza. Notiamo che il gruppo di omologia in grado uno diventa nullo quando l'altezza supera il valore e.

La teoria dell'omologia persistente dunque è basata sullo studio dei gruppi di omologia a diversi gradi dei sottolivelli di una funzione continua, chiamata funzione filtrante. Parlando informalmente i gruppi di omologia mostrano i buchi k-dimensionali di un oggetto.

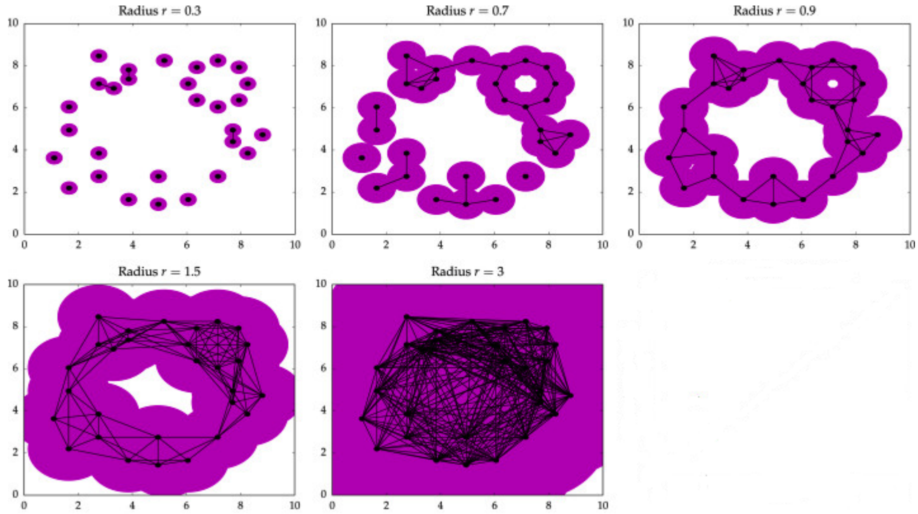

Figura 4: Dischi centrati nei punti di una nube aventi raggio crescente. (Ringraziamo gli autori di [8] per l'immagine)

Più precisamente, una funzione filtrante f è definita in generale su uno spazio topologico e ha valori in $\mathbb{R}^m$, dove m è il numero di proprietà che si vogliono studiare contemporaneamente. In una nube di punti contenente dei dati ogni punto rappresenta diverse misurazioni del campione; anche il confronto di forma nelle immagini mediche richiede funzioni filtranti a valori in $\mathbb{R}^m$. Quando sono coinvolte funzioni filtranti a valori in $\mathbb{R}^m$, si parla di omologia persistente m-dimensionale. Nel contesto della persistenza multidimensionale, il nostro oggetto di studio saranno le funzioni filtranti continue a valori nel piano reale e definite su una superficie liscia chiusa immersa in $\mathbb{R}^3$.

È dimostrato che l'omologia persistente è più difficile da studiare nel caso $2D$ che nel caso $1D$. Queste difficoltà richiedono lo sviluppo di nuove idee e tecniche. Una di questa nuove tecniche è presentata in [2], dove l'idea principale è quella di ricondurre il caso 2-dimensionale al caso 1-dimensionale utilizzando una specifica famiglia di funzioni dipendente da due parametri $(a, b) \in]0, 1[\times\mathbb{R}$. In particolare, se $f = (f_1, f_2) : X \longrightarrow \mathbb{R}^2$ è la nostra funzione filtrante, possiamo associare a essa la famiglia $f_{(a,b)} : X \longrightarrow \mathbb{R}$, dove $f_{(a,b)}(x) := \max\left\{\frac{f_1(x)-b}{a}, \frac{f_2(x)+b}{1-a}\right\}$. Notiamo che ogni funzione di questa forma può essere associata a una retta del piano reale avente pendenza

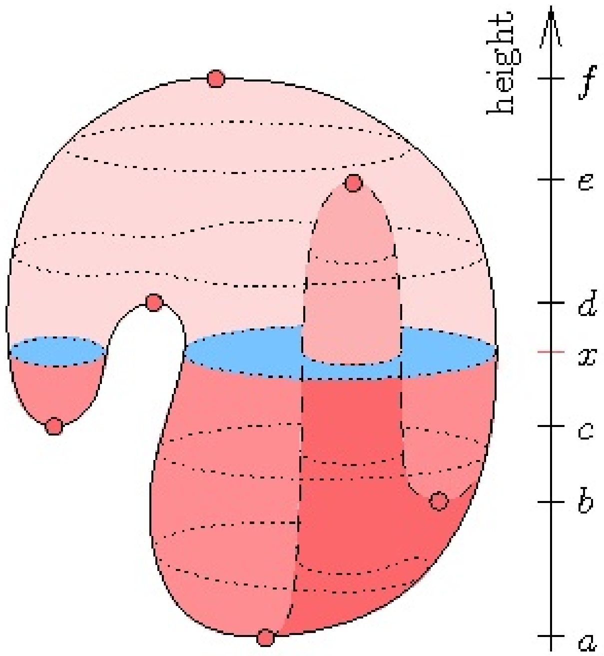

Figura 5: Sottolivelli di uno spazio topologico X. (Ringraziamo gli autori di [6] per l'immagine)

positiva, denotata con $r_{(a,b)}$ e definita dall'equazione parametrica $(u, v) = (at + b, (1 - a)t - b)$. Grazie a questa semplificazione, è possibile considerare tante filtrazioni 1-dimensionali di uno spazio topologico X dipendenti da (a, b), descritte dagli insiemi $\{x \in X \mid f_{(a,b)} \leq t\}$, al posto di un'unica filtrazione 2-dimensionale indotta da f, ovvero $\{x \in X \mid f(x) \preceq (u, v)\}$. Per ragioni tecniche la funzione $f_{(a,b)}$ deve essere normalizzata moltiplicandola per $\min\{a, 1 - a\}$, la nuova funzione sarà denotata con $f^*_{(a,b)}$. La Figura 6 mostra la filtrazione associata a una retta con pendenza positiva.

Dopo aver fissato $k \in \mathbb{N}$, ogni filtrazione associata a una funzione $f^*_{(a,b)}$ definisce un diagramma di persistenza $\mathrm{Dgm}(f^*_{(a,b)})$ in grado k, e la collezione di tutti questi diagrammi di persistenza 1-dimensionali è chiamata diagramma di persistenza 2-dimensionale. Esso contiene informazioni sulle proprietà topologiche dello spazio topologico filtrato che ci interessa studiare.

Un'applicazione importante dell'omologia persistente riguarda il confronto di forma. In questo ambito l'omologia persistente offre un metodo

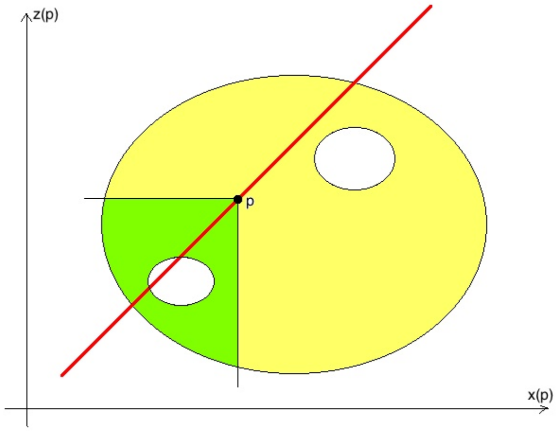

Figura 6: Questa figura mostra la proiezione sul piano xz di uno spazio topologico $X \subseteq \mathbb{R}^3$. La retta rossa ha pendenza positiva in $\mathbb{R}^2$ e induce una filtrazione su X. La sezione verde è il sottolivello associato al punto p della retta.

per comparare diverse funzioni filtranti e per misurare la "distanza" fra esse. Un modo molto comune per confrontare diagrammi di persistenza $2D$, $\{\mathrm{Dgm}(f^*_{(a,b)}\}_{(a,b)\in]0,1[\times\mathbb{R}}$ e $\{\mathrm{Dgm}(g^*_{(a,b)}\}_{(a,b)\in]0,1[\times\mathbb{R}}$, è quello di calcolare l'estremo superiore della classica distanza di bottleneck fra $\mathrm{Dgm}(f^*_{(a,b)})$ e $\mathrm{Dgm}(g^*_{(a,b)})$ su (a,b). Questa metrica è chiamata *matching distance*, D_{match}. Nonostante la matching distance abbia una definizione semplice e naturale, presenta due problemi rilevanti. Per prima cosa, i matching ottimali possono cambiare notevolmente in seguito a piccoli cambiamenti della retta usata per definire la filtrazione. Secondo, la definizione così intrinsecamente discontinua di matching distance rende difficile studiarne le proprietà.

Per queste ragioni, è stata introdotta in [5] e ulteriormente analizzata in [4] una nuova metrica per confrontare i diagrammi di persistenza $2D$, detta *coherent matching distance*. Ciò che più differenzia questa metrica dalla precedente è che questa prende in considerazione solo gli accoppiamenti che cambiano in modo "coerente" con la filtrazione.

Lo studio della coherent matching distance porta alla luce un fenomeno di monodromia. Questo emerge quando viene presa in considerazione una coppia (a,b) per la quale un diagramma di persistenza contiene punti multipli, chiamata coppia singolare. Inoltre, girando intorno a una coppia singolare si produce una permutazione dei punti del diagramma di persistenza, cosicché si crea un collegamento fra la funzione filtrante considerata e un gruppo di monodromia.

In [4], sono state necessarie alcune assunzioni di partenza per definire e studiare la coherent matching distance. L'assunzione principale è quella di considerare solo un sottoinsieme di funzioni filtranti continue a valori in $\mathbb{R}^2$ e definite su una varietà liscia chiusa di dimensione n, ovvero il sottoinsieme delle cosiddette *funzioni normali*. Questo genere di funzioni è definito da quattro proprietà dipendenti dalla *extended Pareto grid*, una costruzione geometrica nel piano reale. La extended Pareto grid è una collezione di archi e semirette di $\mathbb{R}^2$ che si intersecano a vicenda e sono dipendenti dalle immagini di particolari archi su M per mezzo della funzione filtrante $f = (f_1, f_2)$. L'idea è quella di considerare solo le immagini per mezzo di f di archi su M nei quali i gradienti di f_1 e f_2 sono opposti, e alcune semirette che iniziano nelle immagini tramite f dei punti critici delle funzioni f_1, f_2. Ci riferiremo agli archi della griglia come *proper contours*, e alle semirette come *improper contours*. Anche la definizione di *insieme di Jacobi* di una funzione a valori in $\mathbb{R}^2$ è basata sulla dipendenza lineare dei gradienti delle componenti f_1, f_2: se sono linearmente dipendenti e opposti in un punto, tale punto sarà chiamato *critical Pareto point*. Come conseguenza, i punti della extended Pareto grid appartengono all'insieme di Jacobi. Gli insiemi di questi punti sono stati precedentemente studiati nell'ambito della teoria di Morse per due funzioni. I principali riferimenti per i lettori interessati sono [10] e [3].

La ricerca presentata in questa tesi inizia con la domanda "Nell'insieme delle funzioni continue la proprietà di essere normale per una funzione filtrante è generica?" Per cominciare l'esame di questo problema, consideriamo un caso particolare di funzione filtrante $f : M \longrightarrow \mathbb{R}^2$, definita ponendo $f(p) = (x(p), z(p))$, dove M è una superficie liscia chiusa immersa in $\mathbb{R}^3$. In tal caso è possibile fare uso della teoria classica delle superfici. In questo contesto presentiamo un risultato tecnico per modificare la extended Pareto grid di una funzione filtrante. L'idea principale è quella di deformare la superficie, al posto di cambiare la funzione sulla superficie, preservando la sua regolarità. Inoltre, il diffeomorfismo fra la superficie di partenza e quella finale sarà locale, in altre parole esso creerà una protuberanza sulla superficie. Speriamo che lo strumento qui sviluppato possa essere utile per l'approssimazione di funzioni filtranti con funzioni normali, evitando situazioni in cui alcune delle quattro proprietà che le definiscono non siano verificate. Speriamo anche che la nostra ricerca apra la strada alla dimostrazione della proprietà di genericità per una qualsiasi funzione, rimanendo ben consapevoli che saranno necessarie ulteriori idee.

Di seguito riportiamo i contenuti principali della tesi. Nel primo capitolo viene presentato il setting matematico, focalizzando l'attenzione sulle definizioni di retta ammissibile, insieme di Jacobi e critical Pareto point, poi verrà costruita la extended Pareto grid con i suoi contour propri e impropri, e, per concludere, verranno illustrate le definizioni di funzione normale e di contour-arc. Il secondo capitolo contiene il nostro risultato centrale ovvero un teorema tecnico con lo scopo di "spostare" i contour propri di una specifica funzione filtrante su una superficie.

Capitolo 1

Mathematical setting

In this chapter the mathematical setting used in our research is described. First of all we will see what persistence diagrams are, how they are constructed in the 2-dimensional setting and their reduction to the study of a family of 1-dimensional persistence diagrams; in this context the concept of admissible line is also presented. Then the central construction for the study of $2D$ persistence is defined, i.e. the extended Pareto grid. In order to do this it is necessary to introduce the Jacobi set, already studied in [10] and [3]. Finally, normal functions are defined; they are functions with particular properties of regularity, taken as filtering functions in $2D$ persistent homology.

The interested reader can find more details about our setting in the papers [2], [4].

All the figures in this chapter are reproduced from [4], courtesy of the authors of the paper.

1.1 Persistence diagrams

For $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2$, we define the relation $\preceq$ in the following way: $u \preceq v$ (resp. $u \prec v$) if and only if $u_i \leq v_i$ (resp.$u_i < v_i$) for every index $i = 1, 2$.

We also denote by Δ^+ the set $\{(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid u \prec v\}$, and $M_{f \preceq u}$ will be the sublevel set $\{x \in M \mid f_i(x) \leq u_i, i = 1, 2\}$, where $u = (u_1, u_2) \in \mathbb{R}^2$

and $f : X \longrightarrow \mathbb{R}^2$, with Δ the boundary of Δ^+, and with Δ^* the union of Δ^+ and $\{(u, \infty) \mid u \in \mathbb{R}\}$.

Now, let $k \in \mathbb{Z}$. Let M be a topological space and $f : M \longrightarrow \mathbb{R}^2$ a continuous function. Let $i_*^k : H_k(M_{f \preceq u}) \longrightarrow H_k(M_{f \preceq v})$ be the homomorphism induced by the inclusion map $i^k : M_{f \preceq u} \hookrightarrow M_{f \preceq v}$ with $u \preceq v$, where H_k denotes the kth Čech homology group. If $u \prec v$, $i_*^k(H_k(M_{f \preceq u}))$ is called the *multidimensional kth persistent homology group of* (M, f) *at* (u, v) and it is denoted by $H_k^{(u,v)}(M, f)$.

For an explanation of the choice of working with Čech homology see [2], while for details about Čech Homology we refer to [7].

If we assume to work with coefficients in a field $\mathbb{K}$, the homology groups have the structure of vector spaces. Therefore, they are completely described by their dimension, so that the following definition has a central role.

Definition 1. The function $\beta_f(u, v) : \Delta^+ \longrightarrow \mathbb{N} \cup \{\infty\}$ defined by

$$\beta_f(u, v) := \dim H_k^{(u,v)}(M, f)$$

will be called the *persistent Betti numbers function of* f, or PBNs.

Obviously, the persistent Betti numbers function depends on k, but, for the sake of simplicity, we omit any reference to k in the notation. It has been proved that if M is a finitely triangulable space, β_f never attains the value ∞.

In the rest of Section 1.1 we will assume that $m = 1$. We have previously said that the persistent homology groups associated with a filtration of a topological space are completly described by suitable subsets of $\mathbb{R}^2$, that we will define soon, but first we give the statement of the following lemma:

Lemma 1. *Let* u_1, u_2, v_1, v_2 *be real numbers such that* $u_1 \leq u_2 < v_1 \leq v_2$. *It holds that*

$$\beta_f(u_2, v_1) - \beta_f(u_1, v_1) \geq \beta_f(u_2, v_2) - \beta_f(u_1, v_2).$$

In this way it is justified the following definition.

Definition 2. For every point $p = (u, v) \in \Delta^+$, we define the number $\mu(p)$ as the minimum over all the positive real numbers ϵ, with $u + \epsilon < v - \epsilon$, of

$$\beta_f(u + \epsilon, v - \epsilon) - \beta_f(u - \epsilon, v - \epsilon) - \beta_f(u + \epsilon, v + \epsilon) + \beta_f(u - \epsilon, v + \epsilon).$$

The number $\mu(p)$ is called the *multiplicity* of p for β_f. We will call every point $p \in \Delta^+$ with multiplicity $\mu(p) > 0$ a *proper cornerpoint for* β_f.

Definition 3. For every vertical line r, with equation $u = \bar{u}$, for $\bar{u} \in \mathbb{R}$, let us identify r with $(\bar{u}, \infty) \in \Delta^*$, and define the number $\mu(r)$ as the minimum over all the positive real numbers ϵ, with $\bar{u} + \epsilon < \frac{1}{\epsilon}$, of

$$\beta_f\left(\bar{u} + \epsilon, \frac{1}{\epsilon}\right) - \beta_f\left(\bar{u} - \epsilon, \frac{1}{\epsilon}\right).$$

The number $\mu(r)$ is called the *multiplicity of* r *for* β_f. When $\mu(r)$ is strictly positive, we call r a *cornerpoint at infinity for* β_f.

Thanks to the definition of multiplicity above, we can give a representation of PBNs as a persistence diagram.

Definition 4. The *persistence diagram* $\mathrm{Dgm}(f)$ is the multiset of all cornerpoints (both proper and at infinity) for β_f, counted with their multiplicities, union the points of Δ, counted with infinite multiplicity.

We refer to [2] for all the technical results concerning cornerpoints that are required to show that persistence diagrams actually describe PBNs.

1.2 The foliation method in the $2D$ setting

Let us consider the set Λ^+ of all lines of $\mathbb{R}^2$ that have positive slope. A parametrization of this set can be obtained by taking the parameter space $\mathcal{P}(\Lambda^+) =]0,1[\times\mathbb{R}$, where each line $r \in \Lambda^+$ is associated with the unique pair (a,b), with $0 < a < 1$ and $b \in \mathbb{R}$, such that $(a, 1-a)$ is a direction vector for r and $(b, -b) \in r$. The line r will be denoted by $r_{(a,b)}$. Λ^+ is referred to as the set of *admissible lines*. Each point $(u,v) = (u(t), v(t)) = t(a, 1-a) + (b, -b)$ of $r_{(a,b)}$ can be associated with the subset $M_t^{a,b} := M_{(u(t),v(t))}$, that is the set of points of M "whose image by f is under and on the left of $(u(t), v(t))$" while $(u(t), v(t))$ moves along the line $r_{(a,b)}$. This suggests to us that each admissible line $r_{(a,b)}$ defines a filtration $\{M_t^{a,b}\}$ of M and a persistence diagram associated with this filtration. The family of the persistence diagrams associated with the lines $r_{(a,b)}$ is called the $2D$ *persistence diagram of* f.

It is interesting to observe that the filtration $\{M_t^{a,b}\}$ can be also defined as the sublevel sets filtration induced by a suitable real-valued function. In fact, we have that $M_t^{a,b} = \{x \in M \mid f_{(a,b)}(x) \leq t\}$ where $f_{(a,b)} : M \to \mathbb{R}$ is defined by setting $f_{(a,b)}(x) := \max\left\{\frac{f_1(x)-b}{a}, \frac{f_2(x)+b}{1-a}\right\}$. The Reduction Theorem proved in [2] states that the persistent Betti numbers function β_f can be completely recovered by considering all and only the persistent Betti numbers function $\beta_{f_{(a,b)}}$ associated with the admissible lines $r_{(a,b)}$, which are in turn encoded in the corresponding persistence diagrams $\mathrm{Dgm}(f_{(a,b)})$.

In some sense, it is possible to bring back the study of $2D$ persistent homology to the study of $1D$ persistence by means of the filtrations defined by the lines $r_{(a,b)}$ varying $(a, b) \in \mathcal{P}(\Lambda^+)$.

For technical reasons we consider the persistence diagram $\mathrm{Dgm}(f_{(a,b)})$ associated with the admissible line $r_{(a,b)}$ and normalize it by multiplying its points by $\min\{a, 1-a\}$. This is equivalent to consider the *normalized persistence diagram* $\mathrm{Dgm}(f^*_{(a,b)})$, i.e. the persistence diagram of the function $f^*_{(a,b)} := \min\{a, 1-a\} \cdot f_{(a,b)}$.

1.3 The Jacobi set

In order to proceed we will assume that M is a closed smooth surface embedded in $\mathbb{R}^3$ and $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is defined by setting $f(p) = (f_1(p), f_2(p)) = (x(p), z(p))$. We will consider on M the filtering function given by the restriction of f to the surface M. This will be the context in which the main result of this thesis works. The redundant definition of f will be justified in the next chapter where we will restrict the function to different surfaces embedded in $\mathbb{R}^3$.

Let us choose a Riemannian metric on M so that we can define gradients for the two components of the function f.

Definition 5. The *Jacobi set* of f is the set

$$\mathbb{J}(f) = \{p \in M \mid \exists \lambda \in \mathbb{R} : \ \nabla f_1(p) = \lambda \nabla f_2(p) \text{ or } \nabla f_2(p) = \lambda \nabla f_1(p)\} \quad (1.1)$$

namely it is the set of all points at which the gradients of f_1 and f_2 are linearly dependent.

In particular, if $\nabla f_1(p) = \lambda \nabla f_2(p)$ or $\nabla f_2(p) = \lambda \nabla f_1(p)$ with $\lambda \leq 0$, the point $p \in M$ is said to be a *critical Pareto point* for f. The set of all critical Pareto points for f is denoted by $\mathbb{J}_P(f)$.

Notice that the first relation in (1.1) misses the cases in which $\nabla f_2(p) = 0$ and $\nabla f_1(p) \neq 0$, so the only reason for the second relation is to capture these cases. The points satisfying these conditions are the critical points of f_2 that are not critical points for f_1. They are obviously contained in $\mathbb{J}_P(f)$, as also the critical points of f_1.

We assume that

i) No point $p \in M$ exists such that both $\nabla f_1(p)$ and $\nabla f_2(p)$ vanish;

ii) $\mathbb{J}(f)$ is a smoothly embedded 1-manifold in M consisting of finitely many components, each one diffeomorphic to a circle;

iii) $\mathbb{J}_P(f)$ is a 1-dimensional closed submanifold of M, with boundary in $\mathbb{J}(f)$.

We consider the set $\mathbb{J}_C(f)$ of *cusp points* of f, that is, points of $\mathbb{J}(f)$ at which the restriction of f to $\mathbb{J}(f)$ fails to be an immersion. In other words $\mathbb{J}_C(f)$ is the subset of $\mathbb{J}(f)$ at which ∇f_1 and ∇f_2 are both orthogonal to $\mathbb{J}(f)$. Of course $\mathbb{J}_C(f) \subseteq \mathbb{J}_P(f)$.

We also assume that

iv) The connected components of $\mathbb{J}_P(f) \setminus \mathbb{J}_C(f)$ are finite in number, each one being diffeomorphic to an interval. With respect to any parametrization of each component, one of f_1 and f_2 is strictly increasing and the other is strictly decreasing. Each component can meet critical points for f_1, f_2 only at its endpoints.

In [10] it is proved the genericity of the previous properties in the set of smooth maps from M to $\mathbb{R}^2$.

Property iv) implies that the connected components of $\mathbb{J}_P(f) \setminus \mathbb{J}_C(f)$ are open, or closed, or semi-open arcs in M. Following the notation used in [10], they will be referred to as *critical intervals* of f. If an endpoint p of a critical interval actually belongs to that critical interval and hence is not a cusp point, then it is a critical point for either f_1 or f_2.

1.4 The extended Pareto grid

An important tool in 2D-persistent homology is the *extended Pareto grid*, a particular subset of $\mathbb{R}^2$ consisting of monotone arcs, along which y decreases when x increases, and half-lines. We will recall its definition in this section, together with a formal link between the position of points of the persistence diagram $\mathrm{Dgm}(f^*_{(a,b)})$ and the intersection between the admissible line $r_{(a,b)}$ with the extended Pareto grid.

Let us list the critical points $p_1, \ldots, p_h$ of f_1 and the critical points $q_1, \ldots, q_k$ of f_2 (our assumption i) guarantees that $\{p_1, \ldots, p_h\} \cap \{q_1, \ldots q_k\} = \emptyset$). Consider the following closed half-lines: for each critical point p_i of f_1 (resp. each critical point q_j of f_2), the half-line $\{(x, y) \in \mathbb{R}^2 \mid x = f_1(p_i), y \geq f_2(p_i)\}$ (resp. the half-line $\{(x, y) \in \mathbb{R}^2 \mid x \geq f_1(q_j), y = f_2(q_j)\}$).

Definition 6. The *extended Pareto grid* $\Gamma(f)$ is defined to be the union of $f(\mathbb{J}_P(f))$ with these closed half-lines. The closures of the images of critical intervals of f will be called *proper contours* of f, while the closed half-lines will be known as *improper contours* of f.

Observe that every contour is a closed subset of the real plane and the number of contours of f is finite because of property iv). Figure 1.2 shows an example of Pareto grid, where the manifold is a 2-dimensional torus and the filtering function is $f(p) = (x(p), z(p))$ (see Figure 1.1).

Let $\mathcal{S}(f)$ be the set of all points of $\Gamma(f)$ that belong to more than one contour. If $\mathcal{S}(f)$ contains only a finite number of points, it makes sense to define the *multiplicity* of p in $\Gamma(f)$ as the greatest k such that for every $\epsilon > 0$ a line $r_{(a,b)}$ with $(a, b) \in \mathcal{P}(\Lambda^+)$ exists, verifying these two properties: $r_{(a,b)}$ does not meet $\mathcal{S}(f)$ and the cardinality of $r_{(a,b)} \cap \Gamma(f) \cap B(p, \epsilon)$ is k, where $B(p, \epsilon)$ is the open ball of center p and radius ϵ with respect to the Euclidean distance. In plain words, we can say that the multiplicity of $p \in \Gamma(f)$ is the maximum k such that there exists a line with positive slope that does not intersect $\mathcal{S}(f)$ and contains k points of the extended Pareto grid close to p. (This definition should not be confused with multiplicity for points in persistence diagrams.)

Let $\mathcal{D}(f)$ be the set of all points $p \in \Gamma(f)$ that have multiplicity strictly greater than 1, still assuming that $\mathcal{S}(f)$ contains only a finite number of

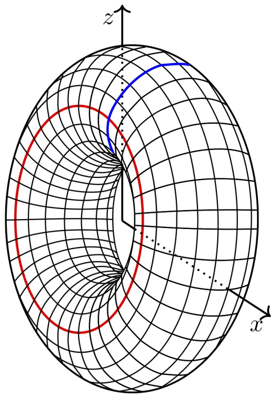

Figura 1.1: The torus endowed with the filtering function $f(p) := (x(p), z(p))$

points. We observe that $\mathcal{D}(f) \subseteq \mathcal{S}(f)$. Each connected component of $\Gamma(f) \setminus \mathcal{D}(f)$ will be called a *contour-arc* of f. Therefore, the contour-arcs do not contain their endpoints.

Figure 1.3 shows all the contour-arcs of the previous example.

1.5 Normal functions

Recently a new distance has been defined to compare different persistence diagrams of functions with value in $\mathbb{R}^2$, called *coherent matching distance*, and investigated in [4] and [5]. There the filtering function is required to be pretty regular in a precise sense described below.

Definition 7. We say that the function $f : M \longrightarrow \mathbb{R}^2$ verifying the properties i), ii), iii), iv) is *normal* if the following statements also hold:

(1) The set $\mathcal{S}(f)$ is finite;

(2) Each multiple point of $\Gamma(f)$ is double;

(3) No line $r_{(a,b)}$ exists containing more than two multiple points of $\Gamma(f)$;

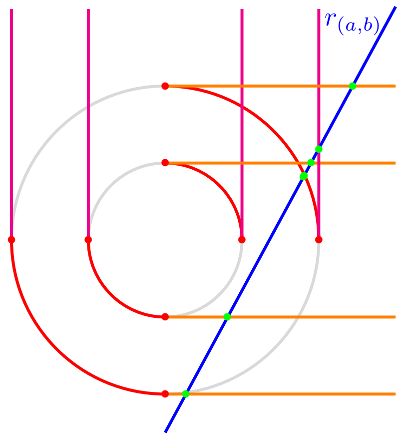

Figura 1.2: The extended Pareto grid for the example above. All the proper contours are in red, the red points are the images by f of the critical points of the two components of f, the purple half-lines are the improper contours associated with critical points of f_1 and the orange half-lines are the improper contours associated with critical points of f_2. A blue admissible line is also represented and the green points are its intersection with the extended Pareto grid.

(4) Every contour-arc γ of f is associated with a pair $(d(\gamma), s(\gamma)) \in \mathbb{Z} \times \{-1, 1\}$ such that at each point (u, v) of γ the following properties hold for every small enough $\epsilon > 0$, when i_*^k is the map $H_k(M_{(u-\epsilon,v-\epsilon)}) \to H_k(M_{(u+\epsilon,v+\epsilon)})$ induced by the inclusion $M_{(u-\epsilon,v-\epsilon)} \hookrightarrow M_{(u+\epsilon,v+\epsilon)}$:

- If $k \neq d(\gamma)$, i_*^k is an isomorphism;
- If $k = d(\gamma)$ and $s(\gamma) = 1$, i_*^k is injective and rank$(H_k(M_{(u+\epsilon,v+\epsilon)}) =$ rank$H_k(M_{(u-\epsilon,v-\epsilon)}) + 1$;
- If $k = d(\gamma)$ and $s(\gamma) = -1$, i_*^k is surjective and

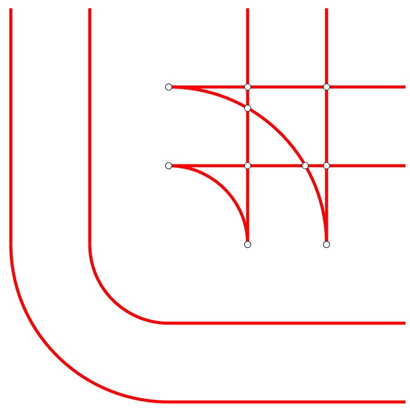

Figura 1.3: All the contour-arcs for the filtering function on the torus are in red and white points are the double points deleted from $\Gamma(f)$. Here are 20 contour-arcs.

$$\mathrm{rank}(H_k(M_{(u+\epsilon,v+\epsilon)}) = \mathrm{rank}H_k(M_{(u-\epsilon,v-\epsilon)}) - 1.$$

In plain words, Property (4) of the definition guarantees that the passage across a contour-arc γ along any direction just creates ($s(\gamma) = 1$) or destroys ($s(\gamma) = -1$) exactly one homological class in degree $d(\gamma)$, without producing any homological change in the other degrees. According to [1], this implies that the multiplicity of the points of each contour-arc is 1 in degree $d(\gamma)$.

Remark 1. The extended Pareto grid and the concept of normal function are introduced in this chapter in order to focus on the purpose of this work. It is a first step on the investigation of the genericity of the assumptions done; it should possibly be proved that properties describing normal functions are generic in the set of smooth maps from M to $\mathbb{R}^2$.

Capitolo 2

Main result

Definition 8. Let A be the boundary of an open subset U of $\mathbb{R}^n$. The *medial axis* of A is defined as follow:

$$Med(A) = \{z \in \mathbb{R}^n \mid \exists p, q \in A, p \neq q, \|p - z\| = \|q - z\| = d(z, A)\}.$$

In plain words, the medial axis of $A = \partial U$ represents the set of all points of $\mathbb{R}^n$ having more than one closest point on A. See Figure (2.1) for an example.

Furthermore, the *reach* of A is the number

$$\tau_A := \inf_{p \in A} d(p, Med(A)) = \inf_{z \in Med(A)} d(z, A).$$

In literature, the number $1/\tau_A$ is also studied, called the *condition number* of A.

For this definition we refer to [9].

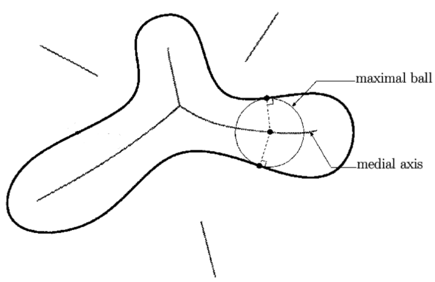

Figura 2.1: Example of medial axis for a planar curve.

Now we introduce some notations we use in what follows. Different surfaces embedded in $\mathbb{R}^3$ will be considered, so for a more streamlined notation, we will consider the restriction of the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $f(p) = (x(p), z(p))$, on different surfaces, instead of defining a function for every surface we deal with. We will also use the notations $\mathbb{J}_M$ to denote the Jacobi set of the function f restricted to the surface M, and Γ_M to denote the extended Pareto grid of $f_{|M}$.

We can describe the Jacobi set of the filtering function above with a specific equation. Consider an open neighborhood S of the image of an arc $\alpha :]0,1[\longrightarrow \mathbb{R}^3$ of M, such that the closure of the set $f \circ \alpha(]0,1[)$ is a proper contour, and take a parametrization of S

$$\phi :]0,1[\times]-1,1[\longrightarrow S \quad (u,v) \mapsto (\phi_1(u,v), \phi_2(u,v), \phi_3(u,v)),$$

so that the filtering function f is given by (ϕ_1, ϕ_3), in terms of the parametrization. Let us also consider the path $\gamma = (\gamma_1, \gamma_2) : [0,1] \to \mathbb{R}^2$ that continuously extends the path $f \circ \alpha$. We can assume that γ_1 and γ_2 are respectively increasing and decreasing. We know that all the points of the surface at which the gradients $\left(\frac{\partial \phi_1}{\partial u}(u,v), \frac{\partial \phi_1}{\partial v}(u,v)\right)$ and $\left(\frac{\partial \phi_3}{\partial u}(u,v), \frac{\partial \phi_3}{\partial v}(u,v)\right)$ are parallel belong to the Jacobi set. If we write this condition of parallelism we obtain the equation:

$$\begin{vmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_1}{\partial v} \\ \frac{\partial \phi_3}{\partial u} & \frac{\partial \phi_3}{\partial v} \end{vmatrix} = 0, \text{ i.e. } \frac{\partial \phi_1}{\partial u}(u,v)\frac{\partial \phi_3}{\partial v}(u,v) - \frac{\partial \phi_1}{\partial v}(u,v)\frac{\partial \phi_3}{\partial u}(u,v) = 0. \quad (2.1)$$

Furthermore, we know that the normal vector at a point $\phi(u,v) \in M$ can be written as

$$\begin{aligned} N_M(\phi(u,v)) = \Big(& \frac{\partial \phi_2}{\partial u}\frac{\partial \phi_3}{\partial v} - \frac{\partial \phi_2}{\partial v}\frac{\partial \phi_3}{\partial u}, -\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_3}{\partial v} + \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_3}{\partial u}, \\ & \frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial v} - \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial u}\Big)(u,v). \end{aligned}$$

We can choose the parametrization ϕ in such a way that $\alpha(u) = \phi(u,0)$, for $u \in]0,1[$; it follows from (2.1) and the relation $\alpha(]0,1[) \subseteq \mathbb{J}_M$ that

$$N_M(\phi(u,0)) = \left(\frac{\partial \phi_2}{\partial u}\frac{\partial \phi_3}{\partial v} - \frac{\partial \phi_2}{\partial v}\frac{\partial \phi_3}{\partial u}, 0, \frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial v} - \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial u}\right)(u,0).$$

Therefore,

$$N_M(\phi(u,0)) \cdot (0,1,0) = 0.$$

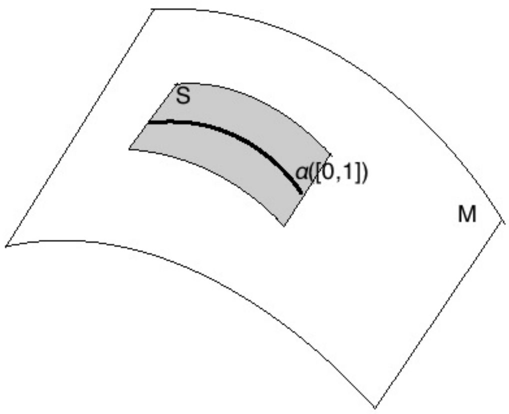

Figura 2.2: An example of the arc α considered on M and its neighbourhood S.

In other words, $(0,1,0)$ is a tangent vector to the surface at every point of $\alpha(]0,1[)$.

In the following proposition we will use a Fréchet-type distance, d_F. We set

$$d_F(\gamma,\gamma') := \inf_{f\in \mathrm{Diffeo}(\gamma,\gamma')} \sup_{t\in]0,1[} \|\gamma(t) - f(\gamma(t)\|$$

for two diffeomorphic arcs in $\mathbb{R}^2$, γ and γ', and

$$d_F(M,M') := \inf_{F\in \mathrm{Diffeo}(M,M')} \max_{x\in M} \|x - F(x)\|.$$

for two diffeomorphic surfaces, M and M', embedded in $\mathbb{R}^3$.

Here, $\mathrm{Diffeo}(M,M')$, with M and M' smooth closed surfaces embedded in $\mathbb{R}^3$, is the set of all diffeomorphisms from M to M'. Analogously, $\mathrm{Diffeo}(\gamma,\gamma')$, with γ and γ' arcs in $\mathbb{R}^2$ parametrized by $]0,1[$, is the set of all diffeomorphisms between γ and γ'.

Proposition 1. *Let $M \subseteq \mathbb{R}^3$ be a smooth closed surface. Let Γ_M be the extended Pareto grid of the filtering function $f_{|M}(p) = (x(p), z(p))$. Assume that $\gamma = (\gamma_1, \gamma_2) : [0,1] \to \mathbb{R}^2$ parametrizes a proper contour of $f_{|M}$, and that γ_1, γ_2 are increasing and decreasing, respectively. Finally, assume that $\delta \in]0, \tau_M[$ and $\gamma' = (\gamma'_1, \gamma'_2) : [0,1] \longrightarrow \mathbb{R}^2$ is a smooth arc verifying the following properties:*

- *γ'_1 and γ'_2 are respectively increasing and decreasing;*
- *$d_F(\gamma, \gamma') < \delta$;*
- *$\frac{\partial^k \gamma_1}{\partial t^k}(0) = \frac{\partial^k \gamma'_1}{\partial t^k}(0)$, $\frac{\partial^k \gamma_2}{\partial t^k}(0) = \frac{\partial^k \gamma'_2}{\partial t^k}(0)$, and*
  $\frac{\partial^k \gamma_1}{\partial t^k}(1) = \frac{\partial^k \gamma'_1}{\partial t^k}(1)$, $\frac{\partial^k \gamma_2}{\partial t^k}(1) = \frac{\partial^k \gamma'_2}{\partial t^k}(1)$, for every $k \in \mathbb{N}_0$.

Then there exists a smooth closed surface M' of $\mathbb{R}^3$ diffeomorphic to M such that:

- *$d_F(M, M') < \delta$;*
- *$\Gamma_{M'} = (\Gamma_M \smallsetminus \gamma([0,1])) \cup \gamma'([0,1])$.*

Proof

Let $\alpha :]0,1[\longrightarrow \mathbb{R}^2$ be a parametrization of an arc of M such that $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ continuously extends the path $f \circ \alpha$. Then, let as take an open neighborhood S of $\alpha(]0,1[)$, for which a parametrization

$$\phi :]0,1[\times]-1,1[\longrightarrow S \quad (u,v) \mapsto (\phi_1(u,v), \phi_2(u,v), \phi_3(u,v))$$

exists such that $\alpha(u) = \phi(u,0)$ for every $u \in]0,1[$.

Now, we can consider the function $s : [0,1] \longrightarrow \mathbb{R}$ that takes each value u to the distance between the point $\gamma(u)$ and the intersection of a line through $\gamma(u)$ with slope 1 and the arc $\gamma'([0,1])$ (see Figure 2.3). It is easy to check that the function s is smooth. We can define a smooth function r on $\mathbb{R}^2$ such that $r(u,0) = s(u)$, for every $u \in]0,1[$, r vanishes outside of $]0,1[\times]-1,1[$ and

$$\frac{\partial r}{\partial v}(u,v) > 0, \text{ for } u \in]0,1[\text{ and } v \in]-1,0[,$$

$$\frac{\partial r}{\partial v}(u,v) < 0, \text{ for } u \in]0,1[\text{ and } v \in]0,1[.$$

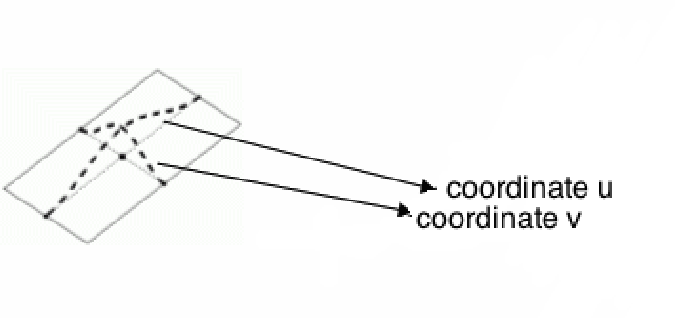

Figura 2.3: Here is an example of the function used in the proof of Proposition 1.

(In Figure 2.3 it is possible to visualize an exemplification of how the function r is constructed).

We observe that the function r verifies the following property:

$$\frac{\partial r}{\partial v}(u,0) = 0,\ \forall u \in]0,1[. \tag{2.2}$$

Let us consider a tangent vector field V on S, which extends the costant vector field $(0,1,0)$ defined along the arc $\alpha(]0,1[)$. Without loss of generality, we can assume that for $\bar{u} \in]0,1[$ the vertical segments $\{(\bar{u},v) \mid v \in]-1,1[\}$ are sent to the flow lines of V by the parametrization ϕ. In this way we have that

$$\frac{\partial \phi}{\partial v}(u,0) = \lambda(0,1,0) \Longrightarrow \frac{\partial \phi_1}{\partial v}(u,0) = \frac{\partial \phi_3}{\partial v}(u,0) = 0,\ \forall u \in]0,1[. \tag{2.3}$$

We can now define the smooth function

$$F : S \longrightarrow \mathbb{R}^3 \text{ by setting } F(\phi(u,v)) := \phi(u,v) + \frac{r(u,v)}{\sqrt{2}}(1,0,1).$$

Without loosing regularity, we can extend F to M by defining $F(p) = p$, $\forall p \in M \setminus S$.

Let us set $M' := F(M)$. Now we can corestrict F to its image so that $F : M \longrightarrow M'$ is a surjective smooth map. The injectivity of F is a consequence of the choice of δ, taken smaller than the reach τ_M of M. It is easy to check that also F^{-1} is smooth. Hence, F is a diffeomorphism.

It remains to prove that $F(\mathbb{J}_M) = \mathbb{J}_{M'}$, this will imply that γ' is actually a contour of $\Gamma_{M'}$. To do this we take the local parametrization

$$\phi' :]0,1[\times]-1,1[\longrightarrow M' \quad \phi'(u,v) = \phi(u,v) + \frac{r(u,v)}{\sqrt{2}}(1,0,1).$$

Thanks to the characterization of the Jacobi set given above, it suffices to see that each normal vector to M' at $\phi'(u,0)$ is orthogonal to $(0,1,0)$, i.e. the second component of the normal vector must be 0. We have that

$$\frac{\partial \phi_1'}{\partial u}(u,0) = \frac{\partial \phi_1}{\partial u}(u,0) + \frac{1}{\sqrt{2}}\frac{\partial r}{\partial u}(u,0);$$

$$\frac{\partial \phi_3'}{\partial u}(u,0) = \frac{\partial \phi_3}{\partial u}(u,0) + \frac{1}{\sqrt{2}}\frac{\partial r}{\partial u}(u,0);$$

$$\frac{\partial \phi_1'}{\partial v}(u,0) = \frac{\partial \phi_1}{\partial v}(u,0) + \frac{1}{\sqrt{2}}\frac{\partial r}{\partial v}(u,0) = 0;$$

$$\frac{\partial \phi_3'}{\partial v}(u,0) = \frac{\partial \phi_3}{\partial v}(u,0) + \frac{1}{\sqrt{2}}\frac{\partial r}{\partial v}(u,0) = 0.$$

As a direct consequence of (2.2) and (2.3), $\frac{\partial \phi_1'}{\partial v}(u,0)$ and $\frac{\partial \phi_3'}{\partial v}(u,0)$ vanish, and hence,

$$\left(\frac{\partial \phi_1'}{\partial u}\frac{\partial \phi_3'}{\partial v} - \frac{\partial \phi_3'}{\partial u}\frac{\partial \phi_1'}{\partial v}\right)(u,0) = 0 \quad \text{for every } u \in]0,1[.$$

Therefore, $N_{M'}(\phi(u,0) \perp (0,1,0)$ for $u \in]0,1[$, so that $F(\mathbb{J}_M) \subseteq \mathbb{J}_{M'}$. We can repeat the same argument for F^{-1}, obtaining $F^{-1}(\mathbb{J}_{M'}) \subseteq \mathbb{J}_M$, i.e. $\mathbb{J}_{M'} \subseteq F(\mathbb{J}_M)$. It follows that $F(\mathbb{J}_M) = \mathbb{J}_{M'}$. This concludes our proof. $\square$

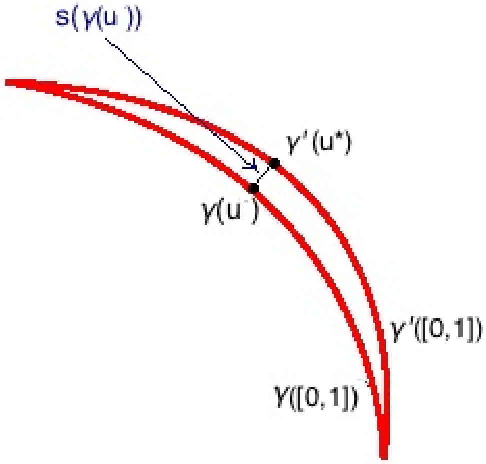

Figura 2.4: The initial contour $\gamma([0,1])$ and the final contour $\gamma'([0,1])$, corresponding to the two diffeomorphic surfaces M and M'. The function s is defined as the length of the segment with endpoints $\gamma(u)$ and $\gamma'(u^*)$.

Conclusions

In this thesis we have presented a geometrical framework concerning $2D$ persistent homology. We have considered a filtering function $f_{|M} : M \longrightarrow \mathbb{R}^2$, defined by $f(p) = (x(p), z(p))$, where M is a smooth closed surface embedded in $\mathbb{R}^3$, and we have proved that the extended Pareto grid of $f_{|M}$ can be modified by moving its proper contours. This result is expressed in Proposition 1, where a proper contour $\gamma([0,1])$ of $f_{|M}$ is fixed, and the surface M is modified in order to obtain a new smooth closed surface M', diffeomorphic to the previous one, such that the extended Pareto grid $\Gamma_{M'}$ equals Γ_M except for the contour $\gamma([0,1])$ which is replaced with a given contour $\gamma'([0,1])$.

In our view, the tool here developed could be used to make a first step in the direction of proving that the property of being normal is generic for regular filtering functions from M to $\mathbb{R}^2$. However, it is important to notice that our mathematical framework is particular, since it just refers to smooth closed surfaces embedded in $\mathbb{R}^3$ and endowed with the filtering function (x, z). We would like to extend our study by proving an analogous result for any regular filtering function defined on a closed regular surface. For this purpose, new ideas will be likely needed.

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